Final Project Report

Alexander Little

November 30, 2022

Introduction

Harkening back to the Tower of Babel from the book of Genesis and inspiring science fiction authors for decades, the idea of the space elevator long evoked an image of a beneficial structure, yet one which could not be built with known materials. This worry was put to rest upon the 1991 discovery of carbon nanotubes, whose tensile strength would technically allow for the elevator's construction. Once assembled, the elevator would provide unparalleled ease of access to Earth orbit, one not reliant on expensive chemical rockets. By skipping the need to escape Earth's gravity well, spacecraft can readily be deployed from the elevator to Earth orbit, the other planets, and to depart the solar system. This deployment is the focus of this project: it seeks to analyze and calculate the dynamics of launching spacecraft from a space elevator.

Mission Objectives

Modeling the dynamics of a spacecraft being launched from a space elevator is the primary objective of this project. This launch will be tested at, below, and above geostationary orbit, with the spacecraft utilizing both the angular velocity given by an elevator fixed to the rotation of the Earth, as well as its radial velocity accumulated during its journey up the elevator [3]. Peet [4] showed that a necessary constraint is that of the spacecraft launching along the Earth's equatorial plane, as the space elevator is necessarily fixed to the equator of the Earth. As a result, any launches that wish to escape Earth's sphere of influence must do so at a significant angle to the ecliptic plane with the Sun. Modeling the dynamics of the elevator itself is also not within this project's scope. Spacecraft dynamics will be simulated using Matlab.

Mission Performance Metrics

This project's performance will be measured by following a schedule and its associated Gantt chart.

Project Section	**Target Finish Date**	**Weeks to Complete**
Project proposal	September 28	3
Problem formulation	October 26	4
Numerical modelling	November 16	3
Final report	November 30	9

Figure 1: The plan for the project by week allotment.

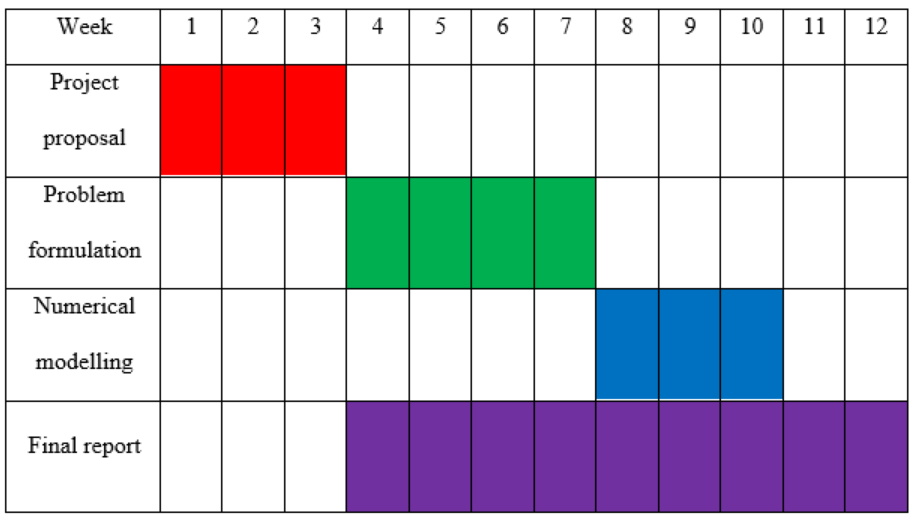


Figure 2: A Gantt chart detailing the amount of time spent on each project section.

The project should be able to successfully mathematically analyze and numerically simulate the dynamics of a spacecraft after launching from a space elevator.

System Model

Assumptions

In reality, a constructed elevator faces a host of structural challenges. These challenges show themselves in the elevator's tension, which is a two-fold problem. For one, the elevator needs to be tapered in design such that at the extreme ends of it, the elevator is at its thinnest, and at GEO, it is at its thickest. When these conditions are met, the system-wide tension will be considerably lowered. [2]. Also, the question of the choice of material for the elevator is a long-documented one. That choice was long thought to be an insurmountable roadblock until scientists became optimistic upon the 1991 discovery of carbon nanotubes [1]. Whether carbon nanotubes remain feasible as a choice of material is still in debate. For this project's scope, we are assuming the elevator to be a straight-line object that stretches from the surface of the Earth to well above GEO, regardless of tapering or choice of material.

Coordinate Frames

With the help of Peet [4], a coordinate system is outlined in this project's scope, namely the Space Elevator Inertial frame. This coordinate system has its origin at Earth's center, with the $\hat{x}_{SEI}$ unit vector pointing radially outwards from the elevator from the spacecraft's launch point, $\hat{z}_{SEI}$ pointing towards the Earth's north pole, and $\hat{y}_{SEI}$ pointing tangentially outwards from the spacecraft's launch point, where $\hat{y}_{SEI} = \hat{z}_{SEI} \times \hat{x}_{SEI}$.

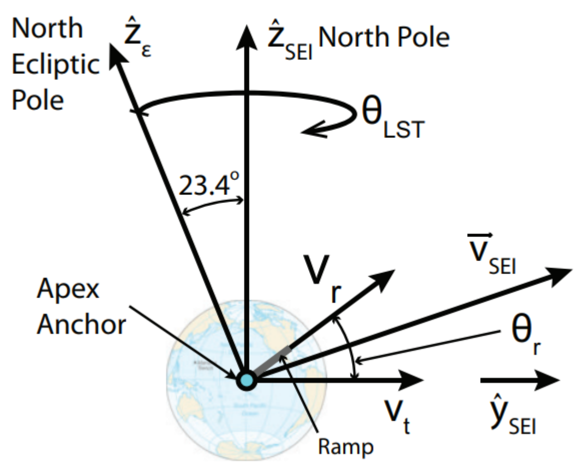


Figure 3: An equatorial view of the geometry of the launch system, where $\hat{x}_{SEI}$ points out of the page. v_t is the tangential velocity at the apex of the elevator, and v_r is initially redirected along $\hat{y}_{SEI}$ until rotated in the direction of $\hat{z}_{SEI}$ by θ_r. Through a combination of these terms, we obtain the velocity vector $\vec{v}_{SEI}$.

Satellites and Planetary Bodies

Bodies considered in this project include (1) the launching spacecraft in question, considered to be a point mass with velocity, (2) the space elevator, considered to be a fixed straight-line structure spanning from the surface of the Earth to greater than or equal to GEO, and (3) the Earth itself, considered a point mass for simplification. The equations of motion for the spacecraft are as follows: in this project's elevator system, a spacecraft will ride up the elevator, building up radial velocity as it goes. Accounting for both the gravitational and centripetal accelerations that the spacecraft faces, the equation of motion for a spacecraft launching from a space elevator at a certain distance from the Earth's centre is given by

$$a(r) = -\frac{\mu}{r^2} + \omega_e^2 r$$

where $a(r)$ is the spacecraft's acceleration as a function of its distance r from Earth's centre, μ is Earth's gravitational parameter, and ω_e is the Earth's rotation rate. The final velocity of the spacecraft is further obtained by

$$v(r_f, r_0) = \sqrt{v_0^2 + \frac{2\mu}{r_f} - \frac{2\mu}{r_0} + \omega_e^2(r_f^2 - r_0^2)}$$

where r_0 is the initial radius from Earth's centre, r_f the final (launch) radius, v_0 the initial radial velocity of the spacecraft, and v the final launch velocity of the spacecraft as a function of r_f and r_0.

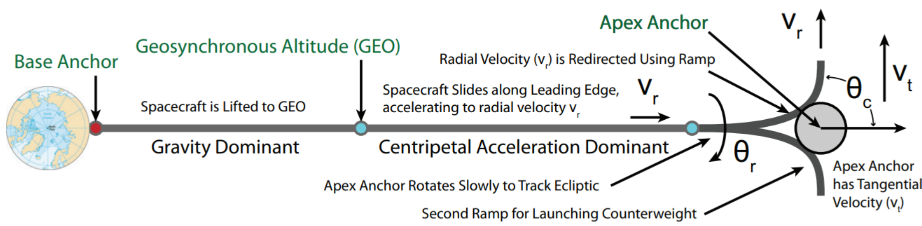


Figure 4: An overview of the spacecraft launch configuration for the described system.

External Disturbances

Edwards [2] demonstrated that in a real-life elevator system, there would be at least four significant external disturbances, namely micrometeorite impacts, LEO spacecraft impacts, radiation damage, heating from electrical currents induced by the magnetic field, and oscillations in the elevator. Furthermore, Pearson [3] also calculated that the elevator would be susceptible to wind loads at its base. These natural disturbances, however, are not included in the scope of the model used in this project.

Desired Trajectory and Orientation

The spacecraft's velocity vector at departure from the elevator can be calculated in the Space Elevator Inertial frame as

$$[\vec{v_0}, r]_{SEI} = \begin{bmatrix} v_r \\ \omega_e r_p \\ 0 \end{bmatrix}$$

where $\vec{v}_0$ is the initial velocity vector, r is the distance the spacecraft will travel up the elevator, v_r is the radial velocity of the spacecraft at launch, ω_e is the rotation rate of the Earth (and therefore the elevator which the spacecraft is launching from), and r_p is the total height of the elevator from Earth's centre to its apex.

Mathematical Analysis

To enable numerical simulation, the given system dynamics equation

$$a(r) = -\frac{\mu}{r^2} + \omega_e^2 r$$

is to be converted into a series of first-order differential equations by letting

$$x = \begin{bmatrix} r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

whose derivative is given by

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{\mu}{x_1^2} + \omega_e^2 x_1 \end{bmatrix}$$

Combining this derivation with the predictions from *Figure 4*, a numerical simulation should demonstrate that launch distances below geostationary height will result in negative velocity due to the more potent gravitational force. In comparison, distances above it will increase velocity due to the more potent centripetal force.

Numerical Simulation Results

This section presents the results from simulating the described dynamical system in MATLAB code with zero initial spacecraft velocity.

Listing 1: planet.m

```
%% Earth-based constants

% Earth's gravitational parameter (km^3/s^2)
mu = 398500;

% Radius of geostationary orbit from Earth's centre (
   km)
r_GEO = 42164;
```

```
% Rotation rate at geostationary orbit (rad/s)
omega_e = sqrt(mu/r_GEO^3);
```

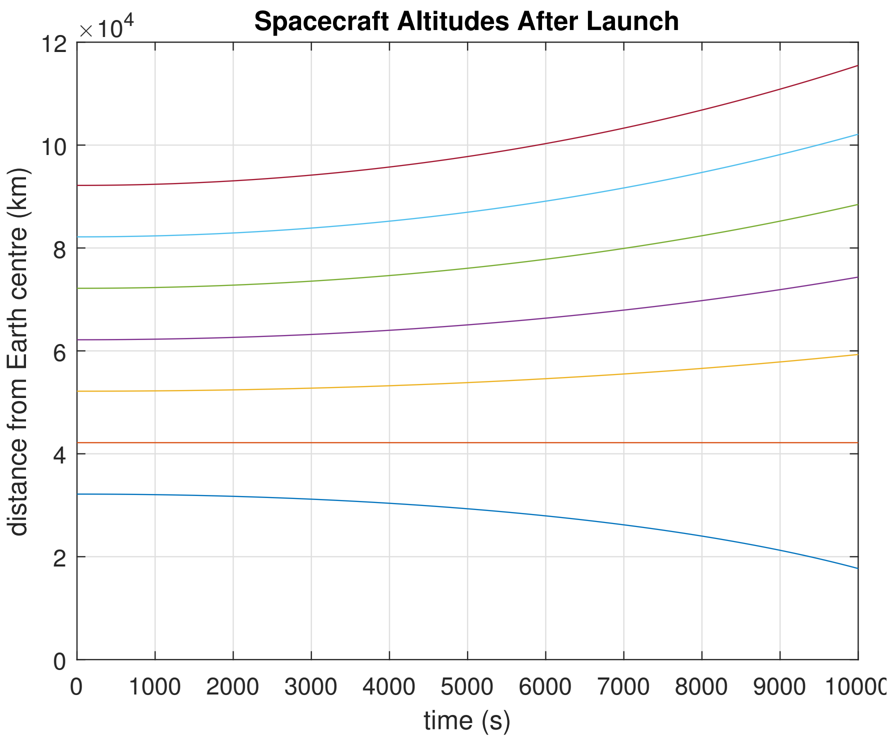
Listing 2: spacecraftODE.m

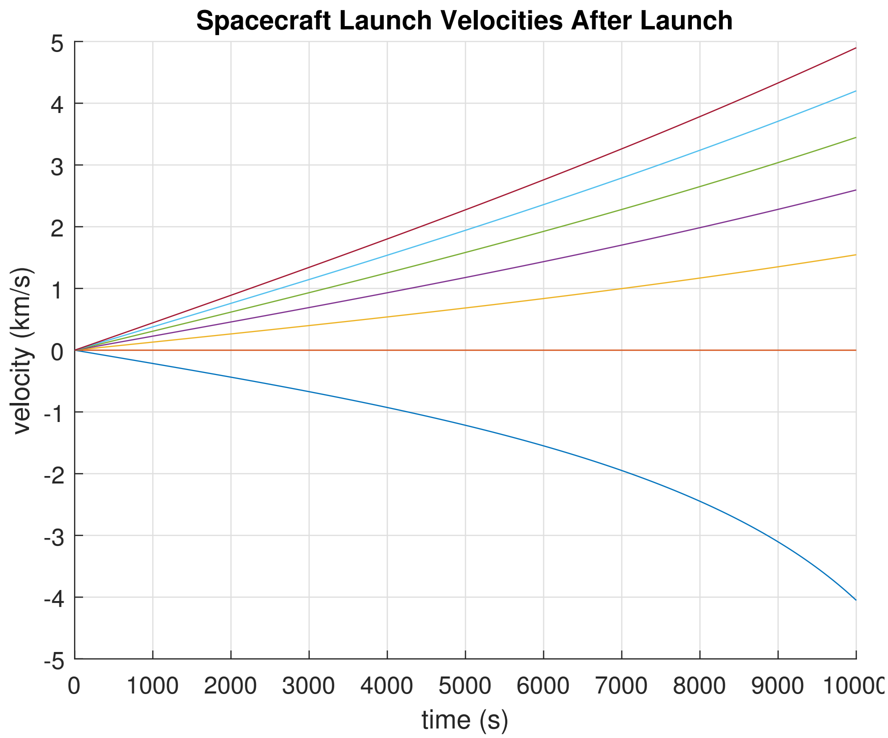
```
%% Spacecraft dynamics model
function dxdt = spacecraftODE(t,x)

% Import Earth-based constants
planet

% Initialize matrix of return values
dxdt = zeros(size(x));

% Distance
dxdt(1) = x(2);

% Velocity
dxdt(2) = -(mu/x(1)^2) + omega_e^2*x(1);

end
```

Listing 3: simulation.m

```
%% Simulation of spacecraft launching from a space
    elevator at increasing altitudes

% Import Earth-based constants
planet

% Initial conditions
y0 = r_GEO - 10000:10000:100000;

% Time span
tspan = 0:1:10000;

% Plot distance
figure(1)
for k = 1:length(y0)
    [t,x] = ode45(@spacecraftODE,tspan,[y0(k) 0]);
    plot(t,x(:,1))
    hold on
end
title('Spacecraft Altitudes After Launch')
xlabel('time (s)')
```

```
ylabel('distance from Earth centre (km)')
grid
hold off

% Plot velocity
figure(2)
for k = 1:length(y0)
    [t,x] = ode45(@spacecraftODE,tspan,[y0(k) 0]);
    hold on
    plot(t,x(:,2))
end
title('Spacecraft Launch Velocities After Launch')
xlabel("time (s)")
ylabel("velocity (km/s)")
grid
hold off
```





As we can see from the plots, the spacecraft's launch velocities fall into three categories. First, if the launch distance from Earth's centre is below geostationary orbit, the gravitational force overpowers the centripetal force, and the spacecraft loses its velocity over time. Second, if the distance is precisely at geostationary orbit, the spacecraft will be perfectly balanced between both forces and remain at its current velocity. Third, and finally, we see that increasing the launch distance of a spacecraft beyond geostationary orbit will result in increasingly higher final velocities. Therefore, the system dynamics described in the previous mathematical analysis correctly predicted the results from this numerical simulation.

Conclusions

This project's treatment of a spacecraft's dynamics launching from a space elevator is that of a greatly simplified model: it ignores structural tapering, choice of material, and avoidance measures for all external disturbances inflicted on the elevator, to name a few. Still, it underpins the topical mission to significantly reduce the cost of transporting materials to Earth orbit and beyond. This mission is the space industry's most significant hurdle. The reaped benefits would be boundless if it can successfully create an elevator of length sufficiently more prolonged than that of geostationary orbit. If we suppose the project was to treat the elevator topic further, it might consider the opposite scenario of capturing spacecraft instead of launching them: a feature that would provide a safe and direct transportation line back to Earth's surface.

References

1. Aravind, P. K. (2006). The physics of the space elevator. *American Journal of Physics, 75*(2), 125-130. https://doi.org/10.1119/1.2404957

2. Edwards, B. C. (2000). Design and deployment of a space elevator. *Acta Astronautica, 47*(10), 735-744. https://doi.org/10.1016/S0094-5765(00)00111-9

3. Pearson, J. (1975). The orbital tower: A spacecraft launcher using the earth's rotational energy. *Acta Astronautica, 2*(9-10), 785–799. https://doi.org/10.1016/0094-5765(75)90021-1

4. Peet, M. M. (2021). The orbital mechanics of Space Elevator Launch Systems. *Acta Astronautica, 179*, 153–171. https://doi.org/10.1016/j.actaastro.2020.10.032