Satellite Dynamics Equations

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1 Overview

This document summarizes the equations used for a satellite rotational dynamics simulation. The state vector is defined as

$$\mathbf{x} = \begin{bmatrix} p \\ q \\ r \\ \phi \\ \theta \\ \psi \\ \delta_1 \\ \vdots \\ \delta_n \end{bmatrix},$$

where

- p, q, r are the body-frame angular velocities,
- ϕ, θ, ψ are the Euler angles (roll, pitch, yaw),
- δ_i are optional control-moment-gyroscope (CMG) gimbal angles.

2 Rotational Dynamics

Let $\boldsymbol{\omega} = \begin{bmatrix} p & q & r \end{bmatrix}^T$ be the angular velocity vector in the body frame, and let $\mathbf{I}_b$ be the inertia matrix of the satellite. The rotational dynamics are governed by

$$\dot{\boldsymbol{\omega}} = \mathbf{I}_b^{-1}\Big(\mathbf{T}_{\text{ext}} + \mathbf{T}_{\text{CMG}} - \boldsymbol{\omega} \times \big(\mathbf{I}_b\,\boldsymbol{\omega}\big)\Big).$$

Here,

- $\mathbf{T}_{\text{ext}}$ is the external torque, e.g. from gravity-gradient effects,
- $\mathbf{T}_{\text{CMG}}$ is the torque generated by CMGs (if any),
- the term $\boldsymbol{\omega} \times (\mathbf{I}_b\,\boldsymbol{\omega})$ captures the effect of angular momentum within the rotating body.

3 Gravity-Gradient Torque

For a satellite in a circular orbit at radius r from the central body (Earth), the gravity-gradient torque in the body frame is modeled as

$$\mathbf{T}_{\text{ext}} = 3\,n^2 \begin{bmatrix} (I_{yy} - I_{zz})\,\sin(\theta)\,\cos(\theta) \\ (I_{zz} - I_{xx})\,\sin(\phi)\,\cos(\phi) \\ (I_{xx} - I_{yy})\,\sin(\phi)\,\sin(\theta) \end{bmatrix},$$

where

$$n = \sqrt{\frac{\mu}{r^3}},$$

- μ is the gravitational parameter ($\mu \approx 3.986 \times 10^{14}\,\mathrm{m}^3/\mathrm{s}^2$ for Earth),
- r is the orbital radius (the distance from Earth's center, assuming a circular orbit),
- I_{xx}, I_{yy}, I_{zz} are the principal moments of inertia of the satellite (or the diagonal terms of $\mathbf{I}_b$),
- ϕ, θ are the roll and pitch Euler angles, respectively.

4 Euler Angle Kinematics

The Euler angles (ϕ, θ, ψ) evolve according to

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \sin(\phi)\,\tan(\theta) & \cos(\phi)\,\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \frac{\sin(\phi)}{\cos(\theta)} & \frac{\cos(\phi)}{\cos(\theta)} \end{bmatrix}}_{\mathbf{M}(\phi,\theta)} \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

5 CMG Gimbal Dynamics (Optional)

If the satellite includes n CMGs, each with a gimbal angle δ_i, their dynamics may be expressed as:

$$\dot{\delta}_i = (\mathbf{L}^+)\,\big(\text{some function of } \boldsymbol{\omega}\big) + u_i,$$

where

- $\mathbf{L}$ is the Jacobian relating gimbal rates to torque,
- $\mathbf{L}^+$ may be the pseudoinverse of $\mathbf{L}$,
- u_i are control inputs for the i-th gimbal.

When no gimbal actuation is present or gimbal angles are fixed, set $\dot{\delta}_i = 0$.

6 State Derivative

Combining the above, the state derivative is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\delta}_1 \\ \vdots \\ \dot{\delta}_n \end{bmatrix}.$$

Explicitly, in terms of the sub-blocks:

$$\dot{p} = \big[\dot{\boldsymbol{\omega}}\big]_x, \quad \dot{q} = \big[\dot{\boldsymbol{\omega}}\big]_y, \quad \dot{r} = \big[\dot{\boldsymbol{\omega}}\big]_z,$$

$$\dot{\phi} = \big[\mathbf{M}(\phi, \theta)\, \boldsymbol{\omega}\big]_x, \quad \dot{\theta} = \big[\mathbf{M}(\phi, \theta)\, \boldsymbol{\omega}\big]_y, \quad \dot{\psi} = \big[\mathbf{M}(\phi, \theta)\, \boldsymbol{\omega}\big]_z,$$

$$\dot{\delta}_i = 0 \quad \text{(or a function if CMGs are used)}.$$

7 Numerical Integration Using RK4

The dynamics can be integrated in time using the classical Runge-Kutta 4 (RK4) method:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{\Delta t}{6} \Big(\mathbf{k}_1 + 2\,\mathbf{k}_2 + 2\,\mathbf{k}_3 + \mathbf{k}_4\Big),$$

where

$$\begin{aligned} \mathbf{k}_1 &= f\big(t_n, \mathbf{x}_n\big), \\ \mathbf{k}_2 &= f\Big(t_n + \frac{\Delta t}{2}, \mathbf{x}_n + \frac{\Delta t}{2} \mathbf{k}_1\Big), \\ \mathbf{k}_3 &= f\Big(t_n + \frac{\Delta t}{2}, \mathbf{x}_n + \frac{\Delta t}{2} \mathbf{k}_2\Big), \\ \mathbf{k}_4 &= f\big(t_n + \Delta t,\ \mathbf{x}_n + \Delta t\, \mathbf{k}_3\big). \end{aligned}$$

Here, $f(t, \mathbf{x})$ represents the system of ordinary differential equations from Sections **??**–**??**.

8 Summary

These equations describe the rotational motion of a satellite under gravity-gradient effects (and optionally CMGs). By forming $\dot{\mathbf{x}} = f(\mathbf{x})$ and using an ODE integrator such as RK4, one obtains time histories of the satellite's orientation and angular velocities.

References:

1. J. Wertz, *Spacecraft Attitude Determination and Control*, D. Reidel Publishing, 1978.

2. H. Schaub and J. L. Junkins, *Analytical Mechanics of Space Systems*, 4th edition, AIAA, 2018.