APPENDIX A:

AUTONOMOUS SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

A.1 OVERVIEW

Here we summarize important results pertaining to the solution of the following five types of matrix differential equations:

Type 1: $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ (Section A.2)

Type 2: $A\dot{\mathbf{y}} + B\mathbf{y} = \mathbf{f}$ (Section A.3)

Type 3: $M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{f}$ (Section A.4)

Type 4: $\dot{X} = AX + XB + C$ (Section A.5)

Type 5: $\dot{P} = -A^TP - PA + PCP - Q$ (Section A.6)

In the above, $(\dot{\cdot})$ denotes time differentiation, lower case bold symbols denote time varying column vectors, upper case symbols denote matrices. Unless specified, all coefficient matrices are assumed to be constant. Specific additional restrictions on symmetry, positive-definitness, etc. are introduced in context.

These are the most common systems of differential equations encountered in analysis of linear dynamical systems and modern feedback control calculations. Except for certain new results in Sections A.5 and A.6, the discussion presented here represents a collection/synthesis of well-known (although rather diffuse) results. Given the central role played by these differential equations in applications, these results are of particular significance.

A.2 SOLUTIONS OF $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$

A.2.1 Homogeneous Solution

Consider $\mathbf{x}$ to be a real n-vector, $\mathbf{u}$ to be a real m-vector, and A, B to be n x n and n x m constant real matrices, respectively. We first consider the $\mathbf{u}=0$ case:

$$\dot{x} = Ax \quad , \quad x(t_o) \text{ given} \tag{A.1}$$

Upon expanding $x(t)$ as a Taylor series about time t_o, we have

$$x(t) = x(t_o) + \sum_{k=1}^{\infty} \left.\frac{d^k x}{dt^k}\right|_{t_o} \frac{(t-t_o)^k}{k!} \tag{A.2}$$

Taking repeated derivatives of Eq. A.1, we can readily establish

$$\begin{aligned} \frac{dx}{dt} &= Ax & \left.\frac{dx}{dt}\right|_{t_o} &= Ax(t_o) \\ \frac{d^2x}{dt^2} &= A\frac{dx}{dt} = A^2x & \left.\frac{d^2x}{dt^2}\right|_{t_o} &= A^2x(t_o) \\ &\vdots & &\vdots \\ \frac{d^kx}{dt^k} &= A\frac{d^{k-1}x}{dt^{k-1}} = A^kx & \left.\frac{d^kx}{dt^k}\right|_{t_o} &= A^kx(t_o) \end{aligned}$$

We immediately see that the solution for $x(t)$ is

$$x(t) = [I + A\frac{(t-t_o)}{1!} + A^2\frac{(t-t_o)^2}{2!} + \ldots + A^k\frac{(t-t_o)^k}{k!} + \ldots]x(t_o) \tag{A.3}$$

Recalling the scalar exponential series

$$e^{at} = 1 + (at)/1! + (at)^2/2! + \ldots + (at)^k/k!) + \ldots = \sum_{k=0}^{\infty} (at)^k/k!$$

It is natural to define the matrix series in Eq. A.3 as the n x n *matrix exponential*

$$\Phi(t,t_o) = I + \sum_{k=1}^{\infty} A^k \frac{(t-t_o)^k}{k!} \equiv [e^{A(t-t_o)}] \tag{A.4}$$

From Eq. A.3, we see that the matrix exponential $\Phi(t,t_o)$ is the *state transition matrix* which linearly maps the initial conditions $x(t_o)$ into the instantaneous solution $x(t)$ according to

$$x(t) = \Phi(t,t_o)x(t_o) \quad , \quad \Phi(t_o,t_o) = I \tag{A.5}$$

In fact, Eq. A.5 holds for $A \neq$ constant, however Eq. A.4 is no longer valid; instead we find by direct substitution of Eq. A.5 into Eq. A.1 that $\Phi(t,t_o)$ in general must satisfy

$$\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0) \quad , \quad \Phi(t_0,t_0) = I \tag{A.6}$$

As can be readily established by substitution, the constant A case solution of Eq. A.4 is a special case solution of Eq. A.6. For most non-constant A(t), Eq. A.6 must be solved numerically to determine $\Phi(t,t_0)$. However, for special cases of non-constant A(t), the successive decomposition method of Wu (ref. 11) can frequently be used to obtain the state transition matrix in the product form solution

$$\Phi(t,t_0) = \prod_{i=1}^{m} \phi_i(t,t_0) = \phi_1(t,t_0)\phi_2(t,t_0) \ldots \phi_m(t,t_0)$$

where A(t) is decomposed into

$$A(t) = \sum_{i=1}^{m} A_i(t)$$

and the differential equation for $\phi_i(t,t_0)$ follows as

$$\dot{\phi}_i(t,t_0) = F_i(t)\phi_i(t,t_0) \quad , \quad \phi_i(t_0,t_0) \doteq I \quad , \quad i = 1,2,\ldots,m$$

where

$$F_i(t) = T_{i-1}^{-1}(t)A_i(t)T_{i-1}(t) \quad , \quad i = 1,2,\ldots,m$$

$$T_i(t) = \phi_i(t,t_0)T_{i-1}(t) \quad , \quad T_0(t) = I \quad , \; i = 1,2,\ldots,(m-1)$$

It is evident from Eq. A.5 that $\Phi(t,t_0)$ satifies the following additional group properties

$$[\Phi(t_2,t_1)]^{-1} = \Phi(t_1,t_2) \tag{A.7}$$

$$\Phi(t_3,t_1) = \Phi(t_3,t_2)\Phi(t_2,t_1) \tag{A.8}$$

for all t_1, t_2, t_3.

Equation A.4 does not immediately provide a practical algorithm for general computation of $\Phi(t,t_0)$; it is obviously limited to small $(t-t_0)$ unless A^k rapidly approaches zero. However, direct, high-precision approximations of this matrix series can be obtained using the identity

$$[e^{A\tau}] = [e^{(A\tau/2^m)}]^{2^m} \tag{A.9}$$

and the Padé approximation technique of ref. A.1. This method has been found practical for "stiff" problems (where the A matrix has a wide eigenvalue spectrum), as well as for the case of repeated or near-repeated eigenvalues. Nineteen alternative methods for calculating the matrix exponential are discussed in ref. A.2.

To gain some insight, let us proceed down the conventional path and assume an exponential solution of Eq. A.1 of the form $\mathbf{x}(t) = \mathbf{r}e^{\lambda t}$. Direct substitution yields the eigenvalue/eigenvector condition

$$[A - \lambda I]\mathbf{r} = 0 \tag{A.10}$$

We conclude that n sets of λ's and generally non-zero $\mathbf{r}$'s satisfy Eq. A.10, the n λ's *(eigenvalues)* are the roots of the *characteristic equation*

$$\det[A - \lambda I] = 0 \rightarrow \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \tag{A.11}$$

which is an nth degree polynomial (this amounts to a solvability condition). For each eigenvalue (λ_i), we can determine the corresponding *eigenvector* ($\mathbf{r}_i$) from Eq. A.10, to within an arbitrary scale factor. We now consider the relationship of this eigenvalue problem to the linear coordinate transformation defined by

$$\mathbf{x} = R\eta \tag{A.12}$$

Substitution of Eq. A.12 into Eq. A.1 yields the transformed differential equation

$$\dot{\eta} = R^{-1}AR\eta \tag{A.13}$$

In particular, we seek a transformation matrix R which diagonalizes A in the sense

$$R^{-1}AR = \text{Diag}\,(\alpha_{11}\ \alpha_{22}\ \cdots\ \alpha_{nn})$$

Upon pre-multiplying by R and equating columns, we see that the diagonalization constraint is equivalent to

$$A\mathbf{R}_j = \alpha_{jj}\mathbf{R}_j \quad , \quad j = 1,2,\ldots,n$$

or

$$[A - \alpha_{jj}I]\mathbf{R}_j = 0 \quad , \quad j = 1,2,\ldots,n \tag{A.14}$$

Where $\mathbf{R}_j$ is the jth column of R. By comparison of Eqs. A.14 and A.10, we conclude $\alpha_{jj} = \lambda_j$ and $\mathbf{R}_j \equiv \mathbf{r}_j$. Thus solving the eigenvalue problem of Eq. A.10 for $R = [\mathbf{r}_1 \ldots \mathbf{r}_n]$ and $\{\lambda_1 \ldots \lambda_n\}$, we can perform the transformation to modal coordinates $\eta = R^{-1}\mathbf{x}$, from which it follows that the elements of η then satisfy the uncoupled equations

$$\dot{\eta}_j = \lambda_j \eta_j \quad , \quad j = 1,2,\ldots,n \tag{A.15}$$

having the solutions

$$\eta_j = \eta_j(t_o)e^{\lambda_j(t-t_o)} \quad , \quad j = 1,2,\ldots,n$$

or, in vector-matrix form

$$\eta(t) = [\mathrm{Diag}(e^{\lambda_1(t-t_o)},\ldots,e^{\lambda_n(t-t_o)})]\eta(t_o) \tag{A.16}$$

Moreover, from Eqs. A.5 and A.12, the state transition matrix is given by

$$\Phi(t,t_o) = R[\mathrm{Diag}(e^{\lambda_1(t-t_o)},\ldots,e^{\lambda_n(t-t_o)})]R^{-1} \tag{A.17}$$

The above results require modification for the case of repeated eigenvalues, and special economies can be achieved in the calculations if A is symmetric and positive-definite (for this case, $R^{-1} = R^T$, the real unit eigenvectors constitute an orthonormal set); details are given in Refs. A.3, A.4.

A.2.2 Forced Solution

The forced response of

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{A.18}$$

is sought by the method of variation of parameters. To begin, we replace $\mathbf{x}(t_o)$ in the homogeneous solution of Eq. A.5 by a variable vector $\mathbf{g}(t)$ satisfying

$$\mathbf{x}(t) = \Phi(t,t_o)\mathbf{g}(t) \quad , \quad \mathbf{g}(t_o) = \mathbf{x}(t_o) \tag{A.19}$$

where $\mathbf{g}(t)$ is to be determined.

Direct substitution of Eq. A.19 into Eq. A.18 and making use of Eq. A.6 leads to

$$\dot{g}(t) = \Phi^{-1}(t,t_o)Bu(t) \tag{A.20}$$

from which it follows that

$$g(t) = x(t_o) + \int_{t_o}^{t} \Phi(t_o,\tau)Bu(\tau)d\tau \tag{A.21}$$

where $\Phi^{-1}(\tau,t_o) = \Phi(t_o,\tau)$ from Eq. A.7. Writing $\Phi(t_o,\tau) = \Phi(t_o,t)\Phi(t,\tau) = \Phi^{-1}(t,t_o)\Phi(t,\tau)$ and substituting into Eqs. A.21 and A.19, we arrive at the *convolution integral* solution of Eq. A.18

$$x(t) = \Phi(t,t_o)x(t_o) + \int_{t_o}^{t} \Phi(t,\tau)Bu(\tau)d\tau \tag{A.22}$$

For the most general cases, the above integral must be preformed numerically, where the group properties of Eq. A.8 can be exploited for efficient computation of $\Phi(t,t_o)$ and $\Phi(t,\tau)$ in Eq. A.22.

A.3 SOLUTIONS OF $A\dot{y} + By = f$

A.3.1 Homogeneous Solutions

We initially consider the pair of homogeneous differential equations

$$A\dot{y} + By = 0 \tag{A.23}$$

and

$$A^T\dot{y} + B^Ty = 0 \tag{A.24}$$

For Eqs. A.23 and A.24, we assume exponential solutions of the $y = ue^{\lambda t}$ and $y = ve^{\lambda t}$, respectively, and obtain the following eigenvalue/eigenvector conditions:

$$[\lambda A + B]u = 0 \tag{A.25}$$

and

$$[\lambda A^T + B^T]v = 0 \tag{A.26}$$

Nontrivial solutions exist if and only if λ satisfies the characteristic equation (solvability condition):

$$\det[\lambda A + B] = 0 \tag{A.27}$$

which is an nth order polynomial having n generally complex eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. For each λ_j, we solve for the corresponding left and right eigenvectors from

$$\lambda_k A\mathbf{u}_k + B\mathbf{u}_k = 0 \quad , \quad k = 1,2,\ldots,n \tag{A.28}$$

$$\lambda_j A^T\mathbf{v}_j + B^T\mathbf{v}_j = 0 \quad , \quad j = 1,2,\ldots,n \tag{A.29}$$

Upon pre-multiplying Eq. A.28, by $\mathbf{v}_j^T$, and post-multiplying the transpose of Eq. A.29 by $\mathbf{u}_k$, we obtain the pair of equations

$$\lambda_k \mathbf{v}_j^T A\mathbf{u}_k + \mathbf{v}_j^T B\mathbf{u}_k = 0 \tag{A.30}$$

$$\lambda_j \mathbf{v}_j^T A\mathbf{u}_k + \mathbf{v}_j^T B\mathbf{u}_k = 0 \tag{A.31}$$

and, upon subtracting, we have

$$(\lambda_k - \lambda_j)\mathbf{v}_j^T A\mathbf{u}_k = 0 \quad \text{for all } j,k \tag{A.32}$$

For the case of $\lambda_k \neq \lambda_j$, Eqs. A.32 and A.31 establish the *orthogonality conditions*

$$\mathbf{v}_j^T A\mathbf{u}_k = 0 \quad , \quad j \neq k \tag{A.33a}$$

$$\mathbf{v}_j^T B\mathbf{u}_k = 0 \quad , \quad j \neq k \tag{A.33b}$$

Notice that Eq. A.33b follows upon adding Eqs. A.30 and A.31, and recalling Eq. A.32. The non-distinct eigenvalue case is considered in Reference A.13. We elect to normalize the eigenvectors with respect to the A matrix so that

$$\mathbf{v}_j^T A\mathbf{u}_j = 1 \quad , \quad j = 1,2,\ldots,n \tag{A.34a}$$

which leads to

$$\mathbf{v}_j^T B\mathbf{u}_j = -\lambda_j \quad , \quad j = 1,2,\ldots,n \tag{A.34b}$$

where we made use of Eq. A.30. If we let $U \equiv [\mathbf{u}_1, \ldots, \mathbf{u}_n]$, $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$, Eqs. A.33 and A.34, are summarized in matrix form as

$$V^T A U = I \quad , \quad V^T B U = -\mathrm{Diag}(\lambda_1, \ldots, \lambda_n) \tag{A.35}$$

From Eq. A.35, we observe that the inverse of V and U can be simply obtained, as follows:

$$U^{-1}U = I \quad , \quad U^{-1} = V^T A$$

$$V^{-1}V = I \quad , \quad V^{-1} = U^T A^T$$

Thus inversion of U and V is obtained efficiently from simple matrix multiplication and is not a source of numerical anxiety.

With the developments above, we are now prepared to address the problem of solving the trajectory **y**(t) satisfying Eq. A.23. To this end, let us consider the linear coordinate transformation

$$\mathbf{y}(t) = U\eta(t) \tag{A.36}$$

Observe that substituting Eq. A.36 into Eq. A.23 yields

$$AU\dot{\eta} + BU\eta = 0 \tag{A.37}$$

Motivated by the properties of Eq. A.35, we premultiply Eq. A.37 by V^T to verify that the components of η satisfy the n uncoupled equations

$$\dot{\eta}_j = -\lambda_j \, \eta_j \quad , \quad j = 1,\ldots,n \tag{A.38}$$

Thus the solution of Eqs. A.37 and A.38 is given by

$$\eta(t) = [\mathrm{Diag}(e^{-\lambda_1(t-t_o)},\ldots,e^{-\lambda_n(t-t_o)})]\eta(t_o) \tag{A.39}$$

Using the inverse transformation of Eq. A.36, we obtain

$$\mathbf{y}(t) = U[\mathrm{Diag}(e^{-\lambda_1(t-t_o)},\ldots,e^{-\lambda_n(t-t_o)})]V^T A\mathbf{y}(t_o) \tag{A.40}$$

or

$$\mathbf{y}(t) = \Phi(t,t_o)\mathbf{y}(t_o), \tag{A.41}$$

where the identity $U^{-1} = V^T A$ has been employed. In this case, the state transition matrix can be shown to satisfy the differential equation

$$A\dot{\Phi} + B\Phi = 0 \quad , \quad \Phi(t_o,t_o) = I$$

In Ref. A.13, the above results are extended to the more general case in which A and/or B can be singular.

A.3.2 Forced Solution

Using the method of variation of parameters, we can replace $\mathbf{y}(t_o)$ in Eq. A.41 by a variable vector $\mathbf{g}(t)$, and upon substituting the resulting equation into

$$A\dot{\mathbf{y}} + B\mathbf{y} = \mathbf{f} \tag{A.43}$$

we find that $\mathbf{g}(t)$ satisfies the following differential equation

$$A\Phi(t,t_o)\dot{\mathbf{g}}(t) = \mathbf{f}(t) \tag{A.44}$$

Upon substituting $\Phi(t,t_o) = U[\mathrm{Diag}(e^{-\lambda_1(t-t_o)},\ldots,e^{-\lambda_n(t-t_o)})]U^{-1}$, premultiplying by V^T, and making use of $V^TAU = I$, the solution for $\mathbf{g}(t)$ follows as

$$\mathbf{g}(t) = \mathbf{y}(t_o) + U\int_{t_o}^{t} [\mathrm{Diag}(e^{\lambda_1(\tau-t_o)},\ldots,e^{\lambda_n(\tau-t_o)})]V^T\mathbf{f}(\tau)d\tau$$

and therefore, using Eq. A.41, the general solution for $\mathbf{y}(t)$ is

$$\mathbf{y}(t) = \Phi(t,t_o)\mathbf{y}(t_o) + \Phi(t,t_o)U\int_{t_o}^{t} [\mathrm{Diag}(e^{\lambda_1(\tau-t_o)},\ldots,e^{\lambda_n(\tau-t_o)})]V^T\mathbf{f}(\tau)d\tau \tag{A.45}$$

A.4 SOLUTIONS OF $M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{f}$

Initially, we consider M, C, and K to be non-zero, and will not restrict the discussion to take advantage of the common event that the matrices are symmetric and positive-definite. Various economies can be realized for these special cases; these special cases are also discussed.

For $\mathbf{x}$ an n-vector, we introduce a 2n-vector $\tilde{\mathbf{y}}$ defined as

$$\tilde{\mathbf{y}} = \begin{Bmatrix} \tilde{\mathbf{y}}_1 \\ \tilde{\mathbf{y}}_2 \end{Bmatrix}$$

where $\tilde{\mathbf{y}}_1$ and $\tilde{\mathbf{y}}_2$ are the n-vectors defined as

$$\tilde{\mathbf{y}}_1 = \mathbf{x} \quad , \quad \tilde{\mathbf{y}}_2 = \dot{\mathbf{x}}$$

It follows that

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{f} \tag{A.46}$$

can be written in the equivalent first-order form

$$\tilde{A}\dot{\mathbf{y}} + \tilde{B}\mathbf{y} = \tilde{\mathbf{f}} \tag{A.47}$$

where

$$\tilde{A} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} , \quad \tilde{B} = \begin{bmatrix} 0 & K \\ K & C \end{bmatrix} , \quad \tilde{\mathbf{f}} = \begin{Bmatrix} 0 \\ \mathbf{f} \end{Bmatrix} \tag{A.48}$$

where all of the submatrices of $\tilde{A}$, $\tilde{B}$ are n x n and the zero submatrix of $\tilde{\mathbf{f}}$ is n x 1.

Since Eq. A.47 has the identical structure of Eq. A.43, all of the results of Section A.3 immediately apply. However, it is useful to particularize these results to reflect the zero sub-matrices in Eq. A.48. For convenience, we index the 2n eigenvalues such that $\{0 \leq Im(\lambda_1) \leq Im(\lambda_2) \ldots \leq Im(\lambda_n)\}$ and $\lambda_{n+k} = \overline{\lambda_k}$ for $k = 1,2,\ldots,n$. If we assumed the following exponential solution

$$\tilde{\mathbf{y}}_1 = \mathbf{x} = \mathbf{u}e^{\lambda t}$$

then

$$\tilde{\mathbf{y}}_2 = \dot{\mathbf{x}} = \lambda \mathbf{u}e^{\lambda t} \tag{A.49}$$

Thus the eigenvalues and right eigenvectors satisfying

$$(\lambda_k \tilde{A} + \tilde{B})\tilde{\mathbf{u}}_k = 0$$

have the special structure

$$\tilde{\mathbf{u}}_k = \begin{Bmatrix} \mathbf{u}_k \\ \lambda_k \mathbf{u}_k \end{Bmatrix} \tag{A.50}$$

Similarly, the eigenvalues and left eigenvectors satisfying

$$(\lambda_k \tilde{A}^T + \tilde{B}^T)\mathbf{v}_k = 0$$

have the special structure

$$\tilde{\mathbf{v}}_k = \begin{Bmatrix} \mathbf{v}_k \\ \lambda_k \mathbf{v}_k \end{Bmatrix} \tag{A.51}$$

where $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are normalized so that $\tilde{\mathbf{v}}^T \tilde{A} \tilde{\mathbf{u}} = 1$.

If we consider Eq. A.45, the particular solution of Eq. A.47 can be written down by analogy

$$\tilde{\mathbf{y}}_p = \tilde{\Phi}(t,t_o)\tilde{U}\int_{t_o}^{t} [\text{Diag}(e^{\lambda_1(t-t_o)},\ldots,e^{\lambda_{2n}(t-t_o)})]\tilde{V}^T\tilde{\mathbf{f}}(\tau)d\tau \tag{A.52}$$

The integrand of Eq. A.52 can be manipulated to obtain the column vector

$$\begin{Bmatrix} \lambda_1 e^{\lambda_1(t-t_o)} & v_1^T f(\tau) \\ \vdots & \\ \lambda_{2n} e^{\lambda_{2n}(t-t_o)} & v_{2n}^T f(\tau) \end{Bmatrix}$$

where $\tilde{U} = [\tilde{u}_1 \ldots \tilde{u}_{2n}]$, $\tilde{V}_1 = [\tilde{v}_1 \ldots \tilde{v}_{2n}]$, $\tilde{f} = \{{0 \atop f}\}$.

Upon substituting Eq. A.53 and making use of Eq. A.50, and noting

$$\tilde{\phi}(t,t_o) = \tilde{U}[\text{Diag}(e^{-\lambda_1(t-t_o)},\ldots,e^{-\lambda_{2n}(t-t_o)})]\tilde{U}^{-1} \tag{A.54}$$

The particular solution to Eq. A.52 can be written as

$$\tilde{y}_p = \begin{Bmatrix} \tilde{y}_{1p} \\ \cdots \\ \tilde{y}_{2p} \end{Bmatrix} = \begin{Bmatrix} x_p \\ \cdots \\ \dot{x}_p \end{Bmatrix} = \begin{Bmatrix} \sum_{k=1}^{2n} \lambda_k u_k e^{\lambda_k(t-t_o)} \int_{t_o}^{t} e^{-\lambda_k(\tau-t_o)} v_k^T f(\tau)\,d\tau \\ \cdots\cdots\cdots \\ \sum_{k=1}^{2n} \lambda_k^2 u_k e^{\lambda_k(t-t_o)} \int_{t_o}^{t} e^{-\lambda_k(\tau-t_o)} v_k^T f(\tau)\,d\tau \end{Bmatrix} \tag{A.55}$$

and, further, imposing the requirement that $\tilde{y}_{1p} = x_p$, $\tilde{y}_{2p} = \dot{x}_p$, we find upon differentiating the top partition of Eq. A.55 that

$$\dot{\tilde{y}}_{1p} = \tilde{y}_{2p} + \sum_{k=1}^{2n} \lambda_k u_k v_k^T f(t) \tag{A.56}$$

But, since $\dot{\tilde{y}}_{1p} \equiv \tilde{y}_{2p}$, it follows that the eigenvectors must satisfy the condition

$$\sum_{k=1}^{2n} \lambda_k u_k v_k^T = 0 \tag{A.57}$$

since $f(t)$ is arbitrary.

Example A.1

As the simplest numerical example to illustrate the above, consider the differential equation

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix}$$

This equation has the form of Eq. A.47. The eigenvalue problem of Eq. A.49 leads to the characteristic equation

$$\lambda^2 + \lambda + 1 = 0$$

with the eigenvalues

$$\lambda_1 = \frac{-1 + i\sqrt{3}}{2} \quad , \quad \lambda_2 = \frac{-1 - i\sqrt{3}}{2}$$

and eigenvectors (unormalized):

$$u_1 = \begin{Bmatrix} 1 \\ \lambda_1 \end{Bmatrix} \quad , \quad u_2 = \begin{Bmatrix} 1 \\ \lambda_2 \end{Bmatrix}$$

Scaling u_1 by α_1 and u_2 by α_2 to impose the normalization

$$V^T AU = I = \begin{Bmatrix} \alpha_1^2(\lambda_1^2 - 1) & \alpha_1\alpha_2(\lambda_1\lambda_2 - 1) \\ \alpha_1\alpha_2(\lambda_1\lambda_2 - 1) & \alpha_2^2(\lambda_2^2 - 1) \end{Bmatrix}$$

gives

$$\alpha_1 = (-1 + \lambda_1^2)^{-1/2} = 0.19665995 + i(0.73394491)$$
$$\alpha_2 = (-1 + \lambda_2^2)^{-1/2} = 0.19665995 - i(0.73394491)$$

so that the normalized eigenvectors become

$$\begin{Bmatrix} \alpha_1 \\ \lambda_1\alpha_1 \end{Bmatrix} = \begin{Bmatrix} 0.1966\ldots + i(0.7339\ldots) \\ -0.7339\ldots - i(0.1966\ldots) \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_2 \\ \lambda_2\alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0.1966\ldots - i(0.7339\ldots) \\ -0.7339\ldots + i(0.1966\ldots) \end{Bmatrix}$$

By direct expansion, it can be verified that Eq. A.35 is satisfied and

$$V^T BU = - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where

$$V = U^T = \begin{bmatrix} \alpha_1 & \lambda_1\alpha_1 \\ \alpha_2 & \lambda_2\alpha_2 \end{bmatrix} , \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

Also direct substitution verifies that the eigenvalues and eigenvectors satisfy Eq. A.57.

More significant information on the above solution can be obtained efficiently by using Laplace transform theory (ref. A.6). Taking the Laplace transform of the original Eq. A.46, with zero initial conditions (to obtain only the particular, or steady state, solution), we obtain

$$\mathcal{L}\{\mathbf{x}_p(t)\} = [Ms^2 + Cs + K]^{-1}\mathcal{L}\{\mathbf{f}(t)\} \tag{A.58}$$

Also, making use of

$$\mathcal{L}\{\int_0^t F_1(t - \tau)F_2(\tau)d\tau\} = f_1(s)f_2(s) \tag{A.59}$$

with $f_i(s) = \mathcal{L}\{f_i(t)\}$ and taking the Laplace transform of $\mathbf{x}_p = \tilde{\mathbf{y}}_{1p}$ from the top partition of Eq. A.55, we obtain

$$\mathcal{L}\{\mathbf{x}_p(t)\} = \sum_{k=1}^{2n} \frac{\mathbf{u}_k\mathbf{v}_k^T}{s-\lambda_k} \mathcal{L}\{\mathbf{f}(t)\} \tag{A.60}$$

Equating Eqs. A.58 and A.60, the *transfer function* H(s) can be written in either of the two forms

$$H(s) \equiv [Ms^2 + Cs + K]^{-1} \equiv \sum_{k=1}^{2n} \frac{\mathbf{u}_k\mathbf{v}_k^T}{s-\lambda_k} \tag{A.61}$$

Next we determine some particular solutions of Eq. A.46 for some simple vector functions f(t). Set $\mathbf{f}(t) = \mathbf{e}_j$ and denote the step response to $\mathbf{e}_j h(t)$ (where h(t) is the unit step function) by $H_j(t)$, leading to

$$\mathbf{H}_j(t) = \sum_{k=1}^{2n} \mathbf{u}_k\mathbf{v}_k^T\beta_k(t)\mathbf{e}_j \tag{A.62}$$

where

$$\beta_k(t) = \begin{cases} (\exp(\lambda_k t) - 1)/\lambda_k & , \quad \lambda_k \neq 0 \\ t & , \quad \lambda_k = 0 \end{cases}$$

By differentiating the step response above with respect to time, we obtain the impulse response, as follows

$$\mathbf{I}_j(t) = \frac{d}{dt}\,(\mathbf{H}_j(t)) = \sum_{k=1}^{2n} \mathbf{u}_k\mathbf{v}_k^T\gamma_k(t)\mathbf{e}_j \tag{A.63}$$

where

$$\gamma_k(t) = \begin{cases} \exp(\lambda_k t) & , \quad \lambda_k \neq 0 \\ 1 & , \quad \lambda_k = 0 \end{cases}$$

$\mathbf{e}_j$ denotes the vector whose jth component is 1 and all other components zero. Also, let $f_j(t)$ denote the jth component of the vector forcing function $\mathbf{f}(t)$. Then $\mathbf{f}(t)$ can be written as

$$\mathbf{f}(t) = \mathbf{e}_1 f_1(t) + \ldots + \mathbf{e}_n f_n(t).$$

Replacing $\mathbf{f}(\tau)$ in Eq. A.55 with this expression, we obtain

$$\mathbf{x}_1(t) = \sum_{j=1}^{n} \int_0^t \sum_{k=1}^{2n} \exp\lambda_k(t - \tau)\mathbf{u}_k\mathbf{v}_k^T\mathbf{e}_j f_j(\tau)d\tau$$

From Eq. A.63 the response can be written as

$$\mathbf{x}_1(t) = \sum_{j=1}^{n} \int_0^t \mathbf{I}_j(t - \tau)f_j(\tau)d\tau \tag{A.64}$$

Next we consider the harmonic excitation $\mathbf{f}(t) = \mathbf{r}\exp(i\omega t)$, where $\mathbf{r}$ denotes a real constant vector and obtain from Eq. A.55

$$\mathbf{x}_1(t) = \sum_{k=1}^{2n} \frac{\mathbf{u}_k\mathbf{v}_k^T\mathbf{r}}{i\omega - \lambda_k}\,\mathbf{r}\exp(i\omega t) - \sum_{k=1}^{2n} \frac{\mathbf{u}_k\mathbf{v}_k^T}{i\omega - \lambda_k}\,\mathbf{r}\exp(\lambda_k t) \tag{A.65}$$

For complex λ_k with negative real part, the second summation in Eq. A.65 goes to zero as t becomes large. Hence

$$\mathbf{y}\exp(i\omega t) = \sum_{k=1}^{2n} \frac{\mathbf{u}_k\mathbf{v}_k^T}{i\omega - \lambda_k}\, \mathbf{r}\exp(i\omega t) \tag{A.66}$$

is the steady state response to the harmonic excitation $\mathbf{r}\exp(i\omega t)$.

From Eq. A.66, it follows that

$$\mathbf{y} = \sum_{k=1}^{2n} \frac{\mathbf{u}_k\mathbf{v}_k^T}{i\omega - \lambda_k}\mathbf{r} = \left[\sum_{k=1}^{2n} \frac{\mathbf{u}_k\mathbf{v}_k^T}{i\omega - \lambda_k}\right]\mathbf{r} = H(i\omega)\mathbf{r} \tag{A.67}$$

The matrix $H(i\omega) = \sum_{k=1}^{2n} \mathbf{u}_k\mathbf{v}_k^T/(i\omega - \lambda k)$ is the so-called steady state *frequency response function*; it can also be obtained by taking Fourier transform of Eq. A.46. Observe that it differs from the transfer function of Eq. A.61, only by the replacement of s by $i\omega$ in the denominator.

Substituting $\mathbf{x}_p = \mathbf{y}\exp(i\omega t)$ into $M\ddot{\mathbf{x}}_p + C\dot{\mathbf{x}}_p + K\mathbf{x}_p = \mathbf{r}\exp(i\omega t)$ yields the condition

$$[-\omega^2 M + i\omega C + K]\mathbf{y} = \mathbf{r} \tag{A.68}$$

from which we can conclude that the steady state frequency response function satisfies the following ("spectral decomposition") relationship, completely analogous to the transfer function identity (A.61):

$$H(i\omega) = [-\omega^2 M + i\omega C + K]^{-1} = \sum_{k=1}^{2n} \frac{\mathbf{u}_k\mathbf{v}_k^T}{i\omega - \lambda_k} \tag{A.69}$$

Several methods (see Refs. A.5, A.6) for structural identification are based upon recovering the characteristic vectors and eigenvalues by fitting Eq. A.67 to steady state response for a large number of harmonic excitation frequencies. Equation A.69 can then be used, in principle, to recover the M, C, and K matrices from these characteristic vectors and eigenvalues.

The above developments can be specialized to take advantage of various special cases; for example, the zero damping (C = 0) case is treated in Ref. A.7, which also imposes symmetry and positivity assumptions upon M and K. The most significant impact is upon the structure of the eigenvalue problem. In particular, for the undamped, unforced special case for M and K positive-definite symmetric:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = 0 \tag{A.70}$$

the associated eigenvalue problem has the structure

$$[-\omega_k^2 M + K]\mathbf{u}_k = 0 \quad , \quad k = 1,2,\ldots,n \tag{A.71}$$

and we adopt the usual normalizations for the eigenvectors

$$\mathbf{u}_k^T \; M \; \mathbf{u}_k = 1 \quad , \quad k = 1,2,\ldots,n \tag{A.72}$$

$$\mathbf{u}_k^T \; K \; \mathbf{u}_k = \omega_k^2 \quad , \quad k = 1,2,\ldots,n \tag{A.73}$$

The ω_k^2 and the components of $\mathbf{u}_k$ are real. The right and left eigenvectors are clearly identical for M and K. symmetric. Thus, it is clear that imposing zero damping and symmetry introduces significant analytical and numerical simplifications. For systems with light damping, Ref. A.8 presents useful perturbation solutions for the undamped eigenvalues and eigenvectors.

A.5 SOLUTIONS OF $\dot{X} = A_1X + XA_2 + B$, $X(t_o) = C$

A.5.1 Preliminary Remarks

A number of engineering applications require the solution for a linear matrix (Lyapunov) differential equation of the form

$$\dot{X}(t) = A_1X(t) + X(t)A_2 + B \quad , \quad X(0) = C \tag{A.74}$$

where $X(t)$ is $n_1 \times n_2$, A_1 is $n_1 \times n_1$, A_2 is $n_2 \times n_2$, and B is $n_1 \times n_2$.

The solution presented in this section extends the results of Lacoss and Shakel (Ref. A.9) by (1) permitting A_1 and A_2 to be nonsymmetric; (2) eliminating the eigenvalue constraint:

$$\lambda_i(A_1) + \lambda_j(A_2) \neq 0 \quad ; \quad i = 1,\ldots,n_1 \quad , \quad j = 1,\ldots,n_2$$

on the solution for $X(t)$, where $\lambda_i(B)$ denotes the ith eigenvalue of B; and (3) eliminating the requirement for computing the inverse of complex matrices in the solution for $X(t)$, by exploiting the *adjoint* eigenvalue solutions for A_1 and A_2.

The solution is obtained by (1) transforming the dependent variable $X(t)$ in terms of the eigenvectors of A_1 and A_2; and (2) applying a variation of parameters technique on the transformed equation. However, if either A_1 or A_2 possess repeated eigenvalues then the exponential matrix solution technique of Serbin and Serbin (ref. 14) is particularly useful.

A.5.2 Coordinate Transformation for the Differential Equation

In order to transform Eq. A.74 into a simpler form, we first solve for the right and left eigenvectors of A_1 and A_2, leading to

$$A_1R_1 = R_1\Lambda_1 \tag{A.75}$$

$$A_1^TL_1 = L_1\Lambda_1 \tag{A.76}$$

$$A_2R_2 = R_2\Lambda_2 \tag{A.77}$$

$$A_2^TL_2 = L_2\Lambda_2 \tag{A.78}$$

where R_i is the matrix of right eigenvectors of A_i,

L_i is the matrix of left eigenvectors of A_i^T.

Λ_i is the diagonal matrix of eigenvalues of A_i and A_i^T,

The right and left eigenvectors can be shown to be biorthogonal and can be normalized to satisfy the following relationships:

$$L_1^TR_1 = I_1 \quad (I_1 \text{ is a } n_1 \times n_1 \text{ identity matrix}) \tag{A.79}$$

and

$$L_2^TR_2 = I_2 \quad (I_2 \text{ is a } n_2 \times n_2 \text{ identity matrix}) \tag{A.80}$$

from which we conclude that

$$L_1^T = R_1^{-1} \quad , \quad L_2^T = R_2^{-1} \tag{A.81}$$

Taking the transpose of Eqs. A.79 and A.80, we find the reciprocal relationships

$$R_1^T = L_1^{-1} \quad , \quad R_2^T = L_2^{-1} \tag{A.82}$$

To express A_1 and A_2 in terms of R_1, L_1, R_2, and L_2 respectively, we post-multiply Eq. A.75 by R_1^{-1} and Eq. A.77 by R_2^{-1} and recall Eq. A.81, leading to

$$A_1 = R_1\Lambda_1R_1^{-1} = R_1\Lambda_1L_1^T \tag{A.83}$$

$$A_2 = R_2\Lambda_2R_2^{-1} = R_2\Lambda_2L_2^T \qquad (A.84)$$

Introducing Eqs. A.83 and A.84 into Eq. A.74, we find

$$\dot{X}(t) = (R_1\Lambda_1L_1^T)X(t) + X(t)(R_2\Lambda_2L_2^T) + B \qquad (A.85)$$

By pre-multiplying Eq. A.85 L_1^T and post-multiplying the resulting equation by R_2, we find

$$\dot{T}(t) = \Lambda_1T(t) + T(t)\Lambda_2 + \beta \qquad (A.86)$$

where

$$T(t) = L_1^TX(t)R_2$$

$$\beta = L_1^TBR_2$$

As shown in what follows, a closed form solution can be obtained for T(t) because Λ_1 and Λ_2 are diagonal.

A.5.3 Variation of Parameters Solution of the Transformed Differential Equation

The homogeneous solution for Eq. A.86 is given by:

$$T_H(t) = e^{\Lambda_1(t-t_o)}T(t_o)e^{\Lambda_2(t-t_o)} \qquad (A.87)$$

where $T(t_o)$ is the initial condition matrix for T(t), $e^{\Lambda_i(t-t_o)}$ is given by

$$e^{\Lambda_i(t-t_o)} = \text{Diag}[e^{\Lambda_{i1}(t-t_o)},\ldots,e^{\Lambda_{in_i}(t-t_o)}] \qquad (A.88)$$

for (i = 1,2), $e^{\Lambda_1(t-t_o)}$ is $n_1 \times n_1$, $e^{\Lambda_2(t-t_o)}$ is $n_2 \times n_2$, Λ_{1i} is the ith eigenvalue of A_1, and Λ_{2j} is the jth eigenvalue of A_2.

To solve Eq. A.86, we use Lagrange's variation of parameters technique and assume a solution of the form

$$T(t) = e^{\Lambda_1(t-t_o)}Q(t)e^{\Lambda_2(t-t_o)} \qquad (A.89)$$

Differentiating Eq. A.89 and comparing the resulting equation with Eq. A.86, we find the following differential equation for Q(t):

$$e^{\Lambda_1(t-t_o)}\dot{Q}(t)e^{\Lambda_2(t-t_o)} = \beta \tag{A.90}$$

or

$$\dot{Q}(t) = e^{-\Lambda_1(t-t_o)}\beta e^{-\Lambda_2(t-t_o)} \tag{A.91}$$

The solution for Q(t) follows as:

$$Q(t) = T(t_o) + \int_{t_o}^{t} e^{-\Lambda_1(r-t_o)}\beta e^{-\Lambda_2(r-t_o)}dr \tag{A.92}$$

where $T(t_o) = Q(t_o)$. Introducing Eq. A.92 into Eq. A.89, we find

$$T(t) = e^{\Lambda_1(t-t_o)}[T(t_o) + D(t,t_o)]e^{\Lambda_2(t-t_o)}$$

$$\text{where } D(t,t_o) = \int_0^t e^{-\Lambda_1(r-t_o)}\beta e^{-\Lambda_2(r-t_o)}dr \tag{A.93}$$

The closed form solution for the i-jth element of the integral in Eq. A.93 follows as

$$\left[\int_{t_o}^{t} e^{-\Lambda_1(r-t_o)}\beta e^{-\Lambda_2(r-t_o)}dr\right]_{ij} = \int_{t_o}^{t} \beta_{ij}e^{-\rho_{ij}(r-t_o)}dr \tag{A.94}$$

where $\rho_{ij} = \Lambda_{1i} + \Lambda_{2j}$ for $i = 1,\ldots,n_1$ and $j = 1,\ldots,n_2$. There are two cases of interest in the solution of Eq. A.94.

Case I: $\rho_{ij} \neq 0$, then Eq. A.94 becomes

$$\int_{t_o}^{t} \beta_{ij}e^{-\rho_{ij}(r-t_o)}dr = \frac{\beta_{ij}}{\rho_{ij}}\left(1 - e^{-\rho_{ij}(t-t_o)}\right) \tag{A.95}$$

Case II: $\rho_{ij} = 0$, then Eq. A.94 becomes

$$\int_{t_o}^{t} \beta_{ij}e^{-\rho_{ij}(r-t_o)}dr = \beta_{ij}(t-t_o) \tag{A.96}$$

Defining the i-jth element of Eq. A.93 subject to Eqs. A.95 and A.96, we obtain

$$D_{ij}(t,t_o) = \begin{cases} \dfrac{\beta_{ij}}{\rho_{ij}}\left(1 - e^{-\rho_{ij}(t-t_o)}\right) & , \quad \rho_{ij} \neq 0 \\ \beta_{ij}(t-t_o) & , \quad \rho_{ij} = 0 \end{cases} \tag{A.97}$$

and introducing Eq. A.97 into the equation above Eq. A.93, we obtain the complete solution as

$$T(t) = e^{\Lambda_1(t-t_o)}[T(t_o) + D(t,t_o)]e^{\Lambda_2(t-t_o)} \tag{A.98}$$

From Eqs. A.81, A.82, and A.86 the solution for X(t) follows as

$$X(t) = R_1 T(t) L_2^T \tag{A.99}$$

$$= R_1 e^{\Lambda_1(t-t_o)}[T(t_o) + D(t,t_o)]e^{\Lambda_2(t-t_o)} L_2^T \tag{A.100}$$

A.5.4 Solution for the Initial Condition Matrix $T(t_o)$

To find the initial condition for $T(t_o)$, we recall that $X(t_o) = C$; thus setting $t = t_o$ in Eq. A.100, we find

$$C = R_1 T(t_o) L_2^T \tag{A.101}$$

or

$$T(t_o) = L_1^T C R_2 \tag{A.102}$$

Introducing Eq. A.102 into Eq. A.98 the i-jth element of T(t) follows as

$$T_{ij}(t) = \begin{cases} \left[T_{ij}(t_o) + \dfrac{\beta_{ij}}{\rho_{ij}}\left(1 - e^{-\rho_{ij}(t-t_o)}\right)\right]e^{\rho_{ij}(t-t_o)} & , \quad \rho_{ij} \neq 0 \\ [T_{ij}(t_o) + \beta_{ij}(t - t_o)] & , \quad \rho_{ij} = 0 \end{cases} \tag{A.103}$$

or

$$T_{ij}(t) = \begin{cases} \left[T_{ij}(t_o) + \dfrac{\beta_{ij}}{\rho_{ij}}\right]e^{\rho_{ij}(t-t_o)} - \dfrac{\beta_{ij}}{\rho_{ij}} & , \quad \rho_{ij} \neq 0 \\ [T_{ij}(t_o) + \beta_{ij}(t - t_o)] & , \quad \rho_{ij} = 0 \end{cases} \tag{A.104}$$

for $i = 1,\ldots,n_1$ and $j = 1,\ldots,n_2$.

A.5.6 Summary of Solution

Step 1: Compute the right and left eigenvalues and eigenvectors of A_1 and A_2, yielding R_1, L_1, Λ_1, R_2, L_2, and Λ_2 and normalize the right and left eigenvectors subject to $L_1^T R_1 = I_1$ and $L_2^T R_2 = I_2$ (see Eqs. A.75- A.82).

Step 2: Compute
$\rho_{ij} = \Lambda_{1i} + \Lambda_{2j}$ (see Eq. A.97)
$\beta = L_1^T B R_2$ (see Eq. A.91)
$T(t_o) = L_1^T C R_2$ (see Eq. A.105)

Step 3: For each time t compute T(t) (see Eq. A.104)

$$T_{ij}(t) = \begin{cases} a_{ij}e^{\rho_{ij}(t-t_o)} - b_{ij} \quad , & \rho_{ij} \neq 0 \\ T_{ij}(t_o) + \beta_{ij}(t-t_o) \quad , & \rho_{ij} \neq 0 \end{cases}$$

where $a_{ij} = T_{ij}(t_o) + b_{ij}$ and $b_{ij} = \dfrac{\beta_{ij}}{\rho_{ij}}$, for $i = 1,\ldots,n_1$ and $j = 1,\ldots,n_2$.

Step 4: Compute X(t).

$X(t) = R_1 T(t) L_2^T$

Note: When $A_1 = A_2^T$ and X(t) is $n_1 \times n_1$ then $R_2 = L_1$ and $L_2 = R_1$.

A.6 SOLUTION OF THE MATRIX RICCATI EQUATION

A.6.1 Preliminary Remarks

The matrix Riccati differential equation

$$\dot{P}(t) = -A^T P(t) - P(t)A + P(t)BR^{-1}B^T P(t) - Q, \ (n \times n) \qquad (A.105)$$

arises in the solution of the optimal control problem for the linear time-invariant system (see Chapter 6 and Chapter 11)

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t_o) = x_o \qquad (A.106)$$

where the performance index is given by

$$2J = x^T(t_f)Sx(t_f) + \int_{t_o}^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \qquad (A.107)$$

In Eqs. A.105, A.106, and A.107, x(t) is the state, u(t) is the control, A is the system dynamics matrix, B is the control influence matrix, S is the terminal state penalty matrix, Q is the state weighting matrix, and R is the control weighting matrix. It can be shown (Ref. A.14), that if R, Q, and $P(t_f)$ = S are symmetric and positive definite, then P(t) is symmetric and positive-definite if the system is controllable.

The optimal control of Eq. A.106 is the linear state feedback

$$u(t) = -R^{-1}B^TP(t)x(t) \qquad (A.108)$$

for $t_o \le t \le t_f$, where P(t) is the solution for Eq. A.105 with the terminal boundary condition

$$P(t_f) = S \qquad (A.109)$$

A.6.2 Steady State Solution (Potter's Method)

Consider Eq. A.105 for the $\dot{P}(t) = 0$ (steady state) case. Clearly this is a system of algebraic equations of second degree; a number of methods exist for solving it. Among the most attractive and most widely used is a method developed by Potter (ref. A.10). We summarize this method as follows:

Step 1: Form the 2n x 2n matrix

$$M = \begin{bmatrix} A & Q \\ BR^{-1}B & -A \end{bmatrix}$$

Step 2: Solve for the eigenvalues and eigenvectors of M. Note, if $\lambda_j = \sigma_j + i\omega_j$ is an eigenvalue, so is $-\bar{\lambda}_j = -\sigma_j + i\omega_j$. Denote the 2n x 1 eigenvectors as

$$\mathbf{a}_j = \left\{ \begin{array}{c} \mathbf{b}_j \\ \hline \mathbf{c}_j \end{array} \right\} \quad , \quad j = 1,2,\ldots,2n$$

where $\mathbf{b}_j$ and $\mathbf{c}_j$ are n x 1 vectors. It is not necessary to normalize the eigenvectors.

Step 3: Discard the n eigenvectors having corresponding eigenvalues with negative real parts, re-number the remaining n eigenvectors and collect them in two n x n matrices as

$$[\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_n] \quad , \quad [\mathbf{c}_1\mathbf{c}_2 \ldots \mathbf{c}_n]$$

Step 4: The steady state solution (P_{ss}) (of Eq. A.108) with $\dot{P} = 0$) is obtained by solving for the columns of P_{ss} from

$$[\mathbf{c}_1\mathbf{c}_2 \ldots \mathbf{c}_n]^T P_{ss} = [\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_n]^T$$

Letting $\{P_{ss}\}_i$ denote the ith column of P_{ss}, we have the n linear systems

$$[\mathbf{c}_1\mathbf{c}_2 \ldots \mathbf{c}_n]^T \{P_{ss}\}_i = \{[\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_n]^T\}_i \quad , \quad i = 1,2,\ldots,n \qquad (A.110)$$

To solve for $\{P_{ss}\}_i$, since each of the n equations of Eq. A.110 has the same coefficient matrix, it is evident that all reduction/elimination calculations depending only upon $[\mathbf{c}_1\mathbf{c}_2 \ldots \mathbf{c}_n]^T$ can be done once and reused n-times. For example, we can use the classical Gaussian elimination or householder reduction methods to reduce $[\mathbf{c}_1\mathbf{c}_2 \ldots \mathbf{c}_n]$ to upper triangular form so that $\{P_{ss}\}_i$ can be solved by back-substitution.

The above algorithm is relatively well known, and P_{ss} has been calculated for many systems with $n > 100$. Numerical difficulties are often encountered in Step 2, depending upon the particular matrices A, B, Q, R. A more numerically stable version of this algorithm is obtained by modifying the above discussion using the Schur decomposition as discussed in Reference A.19. The calculation of P(t) (for other than the steady state case) has usually been calculated by using numerical methods for solving differential equations. In the following section, we show how to determine $P(t)$ from a judicious variable change and avoid numerical integration.

A.6.3 Change of Variables for the Matrix Riccati Differential Equation

By assuming that the solution for $P(t)$ can be written (Refs. A.14-A.18) as

$$P(t) = P_{ss} + Z^{-1}(t) \tag{A.111}$$

where P_{ss} is the solution to the algebraic matrix Riccati equation

$$-A^T P_{ss} - P_{ss}A + P_{ss}EP_{ss} - Q = 0, \tag{A.112}$$

$Z(t)$ is a matrix function which we determine below, $E = BR^{-1}B^T$ and P_{ss} is computed from Eq. A.112 using Potter's method or another method.

To find the differential equation for $Z(t)$ we differentiate Eq. A.111, leading to

$$\dot{P}(t) = -Z^{-1}(t)\dot{Z}(t)Z^{-1}(t) \tag{A.113}$$

Introducing Eqs. A.111 and A.113 into Eq. A.105 yields

$$-Z^{-1}(t)\dot{Z}(t)Z^{-1}(t) = -A^T(P_{ss} + Z^{-1}(t)) - (P_{ss} + Z^{-1}(t))A$$
$$+ (P_{ss} + Z^{-1}(t))E(P_{ss} + Z^{-1}(t)) - Q \tag{A.114}$$

or, after collecting terms

$$0 = (-A^T P_{ss} - P_{ss}A + P_{ss}EP_{ss} - Q)$$
$$+ Z^{-1}(t)[\dot{Z}(t) - \bar{A}Z(t) - Z(t)\bar{A}^T + E]Z^{-1}(t) \tag{A.115}$$

where $\bar{A} = A - EP_{ss}$ is the steady state closed-loop system matrix, and P_{ss} is the stabilizing solution of Eq. A.112 such that all the eigenvalues of $\bar{A}$ have negative real parts.

From Eq. A.112 it follows that the first term in parenthesis in Eq. A.115 vanishes. As a result, the second term in Eq. A.115 provides the following linear constant coefficient matrix differential equation for $Z(t)$:

$$\dot{Z}(t) = \bar{A}Z(t) + Z(t)\bar{A}^T - E \tag{A.116}$$

Thus we have reduced the problem to a special case of Eq. A.87 which we solved in Section A.5.3. Since certain economies result from $A_2 = A_1^T = \bar{A}$, we carry through the solution in detail.

A.6.3.1 Coordinate Transformation

To solve Eq. A.116 we first solve for the right and left eigenvectors of $\bar{A}$, leading to

$$\bar{A}R = R\Lambda \tag{A.117}$$

and

$$\bar{A}^T L = L^T \Lambda \tag{A.118}$$

where R is the matrix of right eigenvectors of $\bar{A}$, L is the matrix of left eigenvectors of $\bar{A}^T$, and Λ is the diagonal matrix of eigenvalues for $\bar{A}$ and $\bar{A}^T$. The right and left eigenvectors can be shown to be biorthogonal and they can be normalized to satisfy the relationship

$$R^T L = I \tag{A.119}$$

from which we conclude that

$$R^T = L^{-1} \tag{A.120}$$

Taking the transpose of Eq. A.119, we find the reciprocal relationship

$$L^T R = I \quad , \quad L^T = R^{-1} \tag{A.121}$$

To express $\bar{A}$ in terms of R and L, we post-multiply Eq. A.117 by R^{-1} and recall Eq. A.120, leading to

$$\bar{A} = R\Lambda R^{-1} = R\Lambda L^T \tag{A.122}$$

Taking the transpose of $\bar{A}$, yields,

$$\bar{A}^T = L\Lambda R^T \tag{A.123}$$

Introducing Eqs. A.122 and A.123 into Eq. A.116 we find

$$\dot{Z}(t) = (R\Lambda L^T)Z(t) + Z(t)(L\Lambda R^T) - E \tag{A.124}$$

By pre-multiplying Eq. A.124 by L^T and post-multiplying the resulting equation by L, we obtain

$$\dot{T}(t) = \Lambda T(t) + T(t)\Lambda - \Sigma \tag{A.125}$$

where $T(t) = L^T Z(t) L$

$$\Sigma = L^T E L$$

As shown in Section A.5.3, the fact that Λ is diagonal permits a closed form solution to be obtained for T(t).

Alternatively, if it proves difficult to obtain the right and left eigenvector solutions for $\bar{A}$ in Eqs. A.117 and A.118, then the matrix exponential solution technique of Serbin and Serbin (ref. A.14) can be applied for solving Eq. A.116.

A.6.3.2 Variation of Parameters Solution of the Transformed Differential Equation

The homogeneous solution for T(t) is given by

$$T_H(t) = e^{\Lambda(t-t_o)} T(t_o) e^{\Lambda(t-t_o)} \tag{A.126}$$

where

$$e^{\Lambda(t-t_o)} = \mathrm{Diag}[e^{\Lambda_1(t-t_o)}, \ldots, e^{\Lambda_n(t-t_o)}] \tag{A.127}$$

and Λ_i is the ith eigenvalue of $\bar{A}$.

To solve Eq. A.125 we use Lagrange's variation of parameters technique and assume a solution of the form

$$T(t) = e^{\Lambda(t-t_o)} Q(t) e^{\Lambda(t-t_o)} \tag{A.128}$$

By differentiating Eq. A.128 and comparing the resulting equation with Eq. A.125, we find the following differential equation for Q(t):

$$e^{\Lambda(t-t_o)} \dot{Q}(t) e^{\Lambda(t-t_o)} = -\Sigma \tag{A.129}$$

or

$$\dot{Q}(t) = -e^{-\Lambda(t-t_o)} \Sigma \, e^{-\Lambda(t-t_o)} \tag{A.130}$$

The solution for Q(t) follows as

$$Q(t) = T(t_o) - \int_{t_o}^{t} e^{-\Lambda(r-t_o)} \Sigma \, e^{-\Lambda(r-t_o)} dr \tag{A.131}$$

where $T(t_o) = Q(t_o)$. Introducing Eq. A.131 into Eq. A.128, we obtain

$$T(t) = e^{\Lambda(t-t_o)}[T(t_o) - \int_{t_o}^{t} e^{-\Lambda(r-t_o)}\Sigma e^{-\Lambda(r-t_o)}dr]e^{\Lambda(t-t_o)} \qquad (A.132)$$

The closed form solution for the i-jth element of the inegral in Eq. A.132 follows as

$$[\int_{t_o}^{t} e^{-\Lambda(r-t_o)}\Sigma e^{-\Lambda(r-t_o)}dr]_{ij} = \int_{t_o}^{t} \Sigma_{ij}e^{-\rho_{ij}(r-t_o)}dr$$

$$= \frac{\Sigma_{ij}}{\rho_{ij}}(1 - e^{-\rho_{ij}(t-t_o)}) \qquad (A.133)$$

where $\rho_{ij} = \Lambda_i + \Lambda_j \neq 0$ for $i,j = 1,2,\ldots,n$ because the eigenvalues of $\bar{A}$ all have negative real parts. Defining the i-jth value of $D(t,t_o)$ as

$$D_{ij}(t,t_o) = \frac{\Sigma_{ij}}{\rho_{ij}}(1 - e^{-\rho_{ij}(t-t_o)}) \qquad (A.134)$$

Equation A.132 can be written as

$$T(t) = e^{\Lambda(t-t_o)}[T(t_o) - D(t,t_o)]e^{\Lambda(t-t_o)} \qquad (A.135)$$

From Eqs. A.120, A.121, and A.125 the solution for $Z(t)$ follows as

$$Z(t) = RT(t)R^T$$

$$= Re^{\Lambda(t-t_o)}[T(t_o) - D(t,t_o)]e^{\Lambda(t-t_o)}R^T \qquad (A.136)$$

Substituting Eq. A.136 into Eq. A.111 yields the solution for $P(t)$:

$$P(t) = P_{ss} + [Re^{\Lambda(t-t_o)}[T(t_o) - D(t,t_o)]e^{\Lambda(t-t_o)}R^T]^{-1} \qquad (A.137)$$

A.6.3.3 Solution for the Initial Condition Matrix

To find the initial condition for $T(t_o)$ we recall that $P(t_f) = S$, thus setting $t = t_f$ in Eq. A.137, we find

$$S = P_{ss} + [Re^{\Lambda(t_f-t_o)}[T(t_o) - D(t_f,t_o)]e^{\Lambda(t_f-t_o)}R^T]^{-1} \qquad (A.138)$$

and solving Eq. A.138 for $T(t_o)$, leads to

$$T(t_o) = D(t_f,t_o) + e^{-\Lambda(t_f-t_o)}He^{-\Lambda(t_f-t_o)} \tag{A.139}$$

where

$$H = L^T(S - P_{ss})^{-1}L \tag{A.140}$$

The i-jth element of Eq. A.139 is given by

$$[T(t_o)]_{ij} = \frac{\Sigma_{ij}}{\rho_{ij}} (1 - e^{-\rho_{ij}(t_f-t_o)}) + H_{ij}e^{-\rho_{ij}(t_f-t_o)} \tag{A.141}$$

A.6.3.4 Solution for the Transformed Matrix Elements

Subject to Eq. A.141, the i-jth element of the difference between $T(t_o)$ and $D(t,t_o)$ in Eq. A.137 is given by

$$[T(t_o) - D(t,t_o)]_{ij} = \frac{\Sigma_{ij}}{\rho_{ij}} (1 - e^{-\rho_{ij}(t_f-t_o)}) + H_{ij}e^{-\rho_{ij}(t_f-t_o)} - \frac{\Sigma_{ij}}{\rho_{ij}} (1 - e^{-\rho_{ij}(t-t_o)}) \tag{A.142}$$

or

$$[T(t_o) - D(t,t_o)]_{ij} = (H_{ij} - \frac{\Sigma_{ij}}{\rho_{ij}})e^{-\rho_{ij}(t_f-t_o)} + \frac{\Sigma_{ij}}{\rho_{ij}} e^{-\rho_{ij}(t-t_o)} \tag{A.143}$$

Recalling that $T(t)$ in Eq. A.135 is

$$T(t) = e^{\Lambda(t-t_o)}[T(t_o) - D(t,t_o)]e^{\Lambda(t-t_o)}, \tag{A.144}$$

the i-jth element of $T(t)$ follows as

$$[T(t)]_{ij} = (H_{ij} - \frac{\Sigma_{ij}}{\rho_{ij}})e^{-\rho_{ij}(t_f-t)} + \frac{\Sigma_{ij}}{\rho_{ij}} \tag{A.145}$$

Thus the solution for $P(t)$ can be written as

$$P(t) = P_{ss} + [RT(t)R^T]^{-1} \tag{A.146}$$

A.6.5 Evaluation of the Matrix Riccati Equation at Discrete Time Steps

The discrete time form of the matrix Riccati equation is established by replacing t with $n\Delta t$ in Eq. A.146, leading to

$$P(n\Delta t) = P_{ss} + [RT(n\Delta t)R^T]^{-1} \tag{A.147}$$

where $n = 0,1\ldots,m$ and $\Delta t = (t_f - t_o)/m$. The equation for i-jth element of $T(n\Delta t)$ follows as

$$[T(n\Delta t)]_{ij} = F_{ij}(n\Delta t) - \frac{\Sigma_{ij}}{\rho_{ij}}\ , \quad n = 0,1,\ldots,m \tag{A.148}$$

where

$$F_{ij}(n\Delta t) = F_{ij}[(n-1)\Delta t]e^{\rho_{ij}\Delta t} \tag{A.149}$$

and

$$F_{ij}[0] = (H_{ij} - \frac{\Sigma_{ij}}{\rho_{ij}})e^{-\rho_{ij}t_f} \tag{A.150}$$

A.6.6 Summary of Solution

<u>Step 1</u>: Compute the solution for the algebraic matrix Riccati equation for P_{ss}, e.g., Eq. A.110, such that the eigenvalues of $\bar{A} = A - EP_{ss}$ have all negative real parts where $E = BR^{-1}B^T$.

<u>Step 2</u>: Compute the right and left eigenvectors of $\bar{A}$, yielding L and R and normalize subject to $L^TR = I$ (see Eq. A.117 through A.121).

<u>Step 3</u>: Compute $\rho_{ij} = \Lambda_i + \Lambda_j$ (see Eq. A.133)

$\Sigma = L^TEL$ (see Eq. A.125)

$H = L^T(S - P_{ss})^{-1}L$ (see Eq. A.140)

<u>Step 4</u>: For each time t compute F(t) (see Eq. A.146)

$$[T(t)]_{ij} = F_{1_{ij}} e^{-\rho_{ij}(t_f - t)} + F_{2_{ij}}$$

where $F_{1_{ij}} = H_{ij} - \frac{\Sigma_{ij}}{\rho_{ij}}\ ,\quad F_{2_{ij}} = \frac{\Sigma_{ij}}{\rho_{ij}}$,

and taking advantage of symmetry in T(t).

Step 5: Compute P(t) (see Eq. A.148)

$$P(t) = P_{ss} + [RT(t)R^T]^{-1} \quad (A.150)$$

while taking advantage of structure of $RT(t)R^T$ and observing that the product is real; thus yielding a real symmetric matrix to be inverted.

A.6.7 Calculation of the Optimal Control

Introducing Eq. A.111 into Eq. A.108, we have

$$u(t) = -R^{-1}B^T\{P_{ss} + Z^{-1}(t)\}x(t)$$

or

$$u(t) = -R^{-1}B^TP_{ss}x(t) - R^{-1}B^TZ^{-1}(t)x(t) \quad (A.151)$$

We observe, however, that the second term above can be rearranged in order to eliminate the explicit calculation of $Z^{-1}(t)$. This follows on defining the new variable

$$\xi(t) = Z^{-1}(t)x(t) \quad (A.152)$$

where the solution for $\xi(t)$ can be obtained from the linear algebraic equation

$$Z(t)\xi(t) = x(t) \quad (A.153)$$

using Gaussian elimination.

As a result, the optimal control can be written as

$$u(t) = -C_1x(t) - C_2\xi(t) \quad (A.154)$$

where $C_1 = R^{-1}B^TP_{ss}$ and $C_2 = R^{-1}B^T$

REFERENCES

A.1 Ward, R. C., "Numerical Computation of the Matrix Exponential with Accuracy Estimate," **SIAM J. Numerical Analysis**, Vol. 14, pp. 600-610, 1977.

A.2 Moler, C. B., and C. Van Loan, "Nineteen Dubious Ways to Compute the Exponential of a Matrix," **SIAM Review**, Vol. 20, NO. 4, pp. 801-832, Oct. 1978.

A.3 Wilkinson, J. H., **The Algebraic Eigenvalue Problem**, Oxford Univ. PRess, Oxford, 1965.

A.4 Van Loan, C., "A General Matrix Eigenvalue Algorithm," **SIAM J. Numerical Analysis**, Vol. 12, No. 6, Dec. 1975.

A.5 Keller, C. L., "Determination of Complex Characteristic Values and Vectors," Air Force Wright Aeronautical Laboratories, Technical Report No. AFWAL-TR-80-3136, Wright-Patterson Air Force Base, Ohio, June 1981.

A.6 Keller, C. L., "Methods for Determining Modal Parameters and Mass, Stiffness, and Damping MaTrices," Air Force Wright Aeronautical Laboratories," Technical Report No. AFFDL-TR-78-59, Wright-Patterson Air Force Base, Ohio, June 1978.

A.7 Meirovitch, L., **Elements of Vibration Analysis**, McGraw-Hill Book Co., New York, 1975.

A.8 Ryland, G., **"Perturbation Methods for Slightly Damped Systems,"** Ph.D. Dissertation, Engineering Mechanics, VPI, Blacksburg, VA, Sept., 1982.

A.9 Lacoss, R. T., and Shakal, A. F., "More $A_1E + EA_2 = -D$ and $\dot{X} = A_1X + XA_2 + D$, $X(0) = C$", **IEEE Trans. on Auto. Control**, June 1976, pp. 405-406.

A.10 Potter, J. E., "Matrix Quadratic Solutions," **J. SIAM Applied Math.**, Vol. 14, No. 3, pp. 496-501, May, 1966.

A.11 Watson, G. N., **A Treatise on the Theory of Bessel Functions**, Cambridge Univ. Press, Cambridge, MA, 1966.

A.12 Min-Yem Wu, "A Successive Decomposition Method for the Solution of Linear Time-Varying Systems," **Int. J. Control**, 1981, Vol. 33, No. 1, pp. 181-186.

A.13 Wilkerson, J. H., "The Differential System $B\dot{x} = Ax$ and the Generalized Eigenvalue Problem $Au = \lambda Bu$," National Physical Laboratory, England, NPL Report NAC73, Jan. 1977.

A.14 Serbin, S. M., and Serbin, C. A., "A Time-Stepping Procedure for $\dot{X} = A_1X + XA_2 + D, X(0) = C$," **IEEE Transactions on Automatic Control**, Vol. AC-25, No. 6, December 1980, pp. 1138-1141.

A.15 Potter, J. E., "A Matrix Equation Arising in Statistical Filter Theory," NASA CR-270, 1965.

A.16 Brockett, R. W., **Finite Dimensional Linear Systems**, John Wiley and Sons, Inc., New York, 1970.

A.17 Potter, J. E. and Vander Velde, W. E., "Optimal Mixing of Gyroscope and Star Tracker Data," **Journal of Spacecraft and Rockets**, Vol. 5, May 1968, pp. 536-540.

A.18 Prussing, J. E., "A Simplified Method for Solving the Matrix Riccati Equation," **International Journal of Control**, Vol. 15, No. 5, 1972, pp. 995-1000.

A.19 Laub, A., "A Schur Method for Solving Algebraic Riccati Equations," IEEE Transactions on Automatic Control, AC-24, 1979, pp. 913-925.