APPENDIX E:

AN ANALYTIC FOURIER TRANSORM FOR A CLASS OF FINITE-TIME CONTROL PROBLEMS

A common method for investigating the effectiveness of various control designs consists of studying frequency domain characteristics of the control, by numerically evaluting the required Fourier transform. For finite-time open- and closed-loop control problems, this can be accomplished by either numerically integrating the integral definition of Fourier transform for each frequency of interest, or using a fast Fourier transform algorithm. Alternatively, we present in this appendix a computationally efficient closed-form solution for the Fourier transform of finite-time open- and closed-loop control problems, where the dynamics of the control is governed by matrix exponentials.

E.1 PROBLEM FORMULATION

The fundamental definition of the complex Fourier transform follows as

$$\bar{u}(\omega) = \int_0^{\tau} u(t)e^{-i\omega t}dt \qquad (n\times 1) \tag{E.1}$$

where u(t) is assumed to be given by (refs. 1-4)

$$u(t) = Ae^{Bt}b \qquad (n\times 1) \tag{E.2}$$

A is nxm, B is mxm, $e^{(\cdot)}$ is the matrix exponential, and b is mx1.

Introducing Eq. E.2 into Eq. E.1 yields

$$\bar{u}(\omega) = A\xi(\omega) \tag{E.3}$$

where

$$\xi(\omega) = \int_0^{\tau} e^{Bt}be^{-i\omega t}dt \tag{E.4}$$

As shown in ref. (5), the integral appearing in Eq. 4 can be evaluated by defining the complex matrix

$$D(\omega) = \begin{array}{cc} \left[\begin{array}{cc} -B & b \\ 0 & -i\omega \end{array}\right] & \begin{array}{l} \}n \\ \}1 \end{array} \\ \begin{array}{cc} n & 1 \end{array} & \end{array} \tag{E.5}$$

and computing the matrix exponential

$$e^{D\tau} = \begin{bmatrix} F_1 & G_1 \\ 0 & F_2 \end{bmatrix} = \begin{bmatrix} e^{-B\tau} & e^{-B\tau}\xi(\omega) \\ 0 & e^{-i\omega\tau} \end{bmatrix} \tag{E.6}$$

from which it follows that

$$\xi(\omega) = e^{B\tau}G_1(\omega) \tag{E.7}$$

Since the numerical effort required to compute G_1 for each desired value of ω is prohibitive, we present in the next section a coordinate transformation which exploits the special structure of Eq. E.5.

E.2 REDUCING SUBSPACE COORDINATE TRANSFORMATION

In this section we present an algorithm for reducing the complex matrix D in Eq. E.5 to block diagonal form by a similarity transformation (ref. 6). In particular, we seek a complex nonsingular matrix Φ, such that $\Phi^{-1}D\Phi$ has the form

$$\Phi^{-1}D\Phi = \bar{D} = \mathrm{Diag}(-B, -i\omega) \tag{E.8}$$

The transformation matrix Φ is assumed to have the special form

$$\Phi = \begin{bmatrix} I & -p \\ 0 & 1 \end{bmatrix}$$

where the inverse of Φ can be shown to be

$$\Phi^{-1} = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

Hence,

$$\Phi^{-1}D\Phi = \begin{bmatrix} -B & Bp-i\omega p + b \\ 0 & -i\omega \end{bmatrix} \tag{E.9}$$

and the problem of determining Φ becomes that of solving the following linear equation for p:

$$[B - i\omega I]p(\omega) = -b \tag{E.10}$$

where the solution for p is well defined provided that $i\omega$ is not an eigenvalue of B. From Eq. E.8 it follows that the matrix exponential of Eq. E.6 can be written as

$$e^{D\tau}=\Phi e^{\bar{D}\tau}\Phi^{-1} = \begin{bmatrix} e^{-B\tau} & e^{-B\tau}p-pe^{-i\omega\tau} \\ 0 & e^{-i\omega\tau} \end{bmatrix} \tag{E.11}$$

Comparing Eqs. E.6 and E.11 it follows that the desired integral for $\xi(\omega)$ in Eq. E.7 is given by

$$\xi(\omega) = p - e^{B\tau}pe^{-i\omega\tau} \tag{E.12}$$

where the entire solution follows after determining p from Eq. E.10 for each frequency of interest. The significant feature of Eq. E.12 is that the computationally intensive solution for $e^{B\tau}$ must be carried out only one time, thus greatly reducing the labor required to produce $\xi(\omega)$.

E.3 SOLUTION FOR THE UNCOUPLING TRANSFORMATION VECTOR

Since the B matrix in Eq. E.10 is constant and generally fully populated, we seek a solution technique that minimizes the computational effort. However, we recognize that there are two classes of solutions possible: 1) systems where B is diagonalizable; and 2) systems where the eigensystem for B is ill-conditioned.

The solutions for both classes of problems are obtained via a "transformation method." In particular, such methods are based upon the equivalence of the problems (refs. 7 and 8)

$$[B - i\omega I]p(\omega) = -b \quad , \quad [\Lambda - i\omega I]\gamma(\omega) = -\beta \tag{E.13}$$

The solution algorithms for both classes of problems are listed in Table E.1. However, if $i\omega = \lambda_j$, then Eq. E.6 can be used to obtain the solution. The desired solution for $u(\omega)$ follows on introducing Eq. E.12 into Eq. E.3, yielding

$$\bar{u}(\omega) = A\{p - e^{B\tau}pe^{-i\omega\tau}\} \tag{E.14}$$

In order to efficiently evaluate Eq. E.14, it is necessary to recast the equation in the form

$$\bar{u}(\omega) = A_1\gamma(\omega) - A_2\gamma(\omega)e^{-i\omega\tau} \tag{E.15}$$

where $A_1 = AR$ and $A_2 = Ae^{B\tau}R$ if B is diagonalizable, and $A_1 = AU$ and $A_2 = Ae^{B\tau}U$ if B is ill-conditioned (ref. E.9).

Table E.1 Solution techniques for p(ω)

B Diagonalizable	B Ill-conditioned eigensystem
$\Lambda=\text{Diag}[\lambda_1,\ldots,\lambda_n]$	Λ=upper quasitriangular[a]
$\beta=L^Tb$	$\beta=U^Tb$
$\gamma_j=-\beta_j/(\lambda_j-i\omega),(j=1,\ldots,n)$	$[\Lambda-i\omega I]\gamma(\omega)=-\beta$ (easy to solve)
$p(\omega)=R\gamma(\omega)$	$p(\omega)=U\gamma(\omega)$
R=right eigenvector of B	$U^TBU=\Lambda$(real Schur decomposition)
L=left eigenvector of B	$U^TU=I$ (orthogonality)
$L^TR=I$(biorthogonality)	

[a]A quasitriangular matrix is triangular with possible nonzero 2x2 blocks on the diagonal.

E.4 EXAMPLE APPLICATION

Given the first-order system

$$\dot{x} = -x + u \quad ; \quad \text{given } x(0) = 0 \quad x(\tau) = 1$$

we seek the control u to minimize

$$J = \frac{1}{2}\int_0^\tau u^2dt$$

The open-loop control can be shown to be

$$u(t) = -\lambda(t) \qquad (\lambda = \text{co-state}) \tag{E.16}$$

where

$$\begin{Bmatrix} x(t) \\ \lambda(t) \end{Bmatrix} = e^{Bt} \begin{Bmatrix} x(0) \\ \lambda(0) \end{Bmatrix}, \quad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \tag{E. 17}$$

$$e^{Bt} = \begin{bmatrix} e^{-t} & -\sinh t \\ 0 & e^{t} \end{bmatrix} \; ; \quad \lambda(0) = -1/\sinh\tau \tag{E.18}$$

Thus the control of Eq. E.16 can be written as

$$u(t) = Ae^{Bt}b = e^{t}/\sinh\tau \tag{E.19}$$

where $A = [0 \quad -1]$, e^{Bt} is defined by Eq. E.18 and $b = (0, -1/\sinh\tau)^T$. The analytic transform of $u(t)$ follows as

$$\bar{u}(\omega) = (\int_0^\tau e^{(1-i\omega)t}dt)/\sinh\tau = [e^{(1-i\omega)\tau} - 1]/[(1 - i\omega)\sinh\tau] \tag{20}$$

Since B is diagonalizable, we use the right and left eigenvector transformation method to solve Eq. E.13, leading to

$$R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 2/\sqrt{2} \end{bmatrix}, \quad L = \begin{bmatrix} 2\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

from which it follows that

$$\beta = L^T b = -[1/(\sqrt{2}\sinh\tau), \; 1/(\sqrt{2}\sinh\tau)]^T$$

$$\gamma = (-1/[(1 + i\omega)\sqrt{2}\sinh\tau] \;\; , \;\; 1/[(1 - i\omega)\sqrt{2}\sinh\tau])^T$$

$$A_1 = AR = [0, \; -2/\sqrt{2}] \;\; , \;\; A_2 = Ae^{B\tau}R = [0, \; -2e^{\tau}/\sqrt{2}]$$

Thus, from Eq. E.15 we have

$$\bar{u}(\omega) = [e^{(1-i\omega)\tau} - 1]/[(1 - i\omega)\sinh\tau] \tag{E.21}$$

where Eq. E.21 agrees with Eq. E.20.

E.5 CONCLUSIONS

A computationally efficient algorithm has been presented for obtaining the complex Fourier transform of a class of vector functions that frequently occur in modern control theory. The basic algorithm requires 1) the evaluation of a single matrix exponential for the dynamics of the time-varying control; 2) the solution for either the right and left eigenvectors or a real Schur decomposition of the constant control dynamics matrix; 3) the sequential solution for the reducing subspace transformation vector p; and 4) the evaluation of a single scalar complex exponential.

References

E.1 Turner, J. D. and Junkins, J. L., "Optimal Large Angle Single-Axis Rotational Maneuvers of Flexible Spacecraft," **Journal of Guidance and Control**, Vol. 3, No. 6, Nov.-Dec. 1980, pp. 578-585.

E.2 Breakwell, J. A., "Optimal Feedback Slewing of Flexible Spacecraft," **Journal of Guidance and Control**, Vol. 4, No. 5, Sept.-Oct. 1981, pp. 472-479.

E.3 Turner, J. D. and Chun, H. M., "Optimal Distributed Control of a Flexible Spacecraft Using Control-Rate Penalties in the Controller Design," AIAA Paper 82-1438, 1982.

E.4 Turner, J. D., Chun, H. M., and Juang, J. N., "An Analytic Solution for the State Trajectorires of a Feedback Control System," **Journal of Guidance, Control, and Dynamics**, Vol. 8, Jan.-Feb. 1984, pp. 147-148.

E.5 Van Loan, C. F., "Computing Integrals Involving Matrix Exponentials," **IEEE Transactions on Automatic Control**, Vol. AC-23, No. 3, June 1978, pp. 395-404.

E.6 Bavely, C. C. and Stewart, G. W., "An Algorithm for Computing Reducing Subspaces by Block Diagonalization," **SIAM Journal of Numerical Analysis**, Vol. 16, No. 2, Apr. 1979, pp. 359-367.

E.7 Bartels, R. H. and Stewart, G. W., "Solution of the Matrix Equation AX + XB = C," Commun. ACM, Vol. 15, 1972, pp. 820-826.

E.8 Golub, G. H., Nash, S., and Van Loan, C., "A Hessenberg-Schur Method for the Problem AX + XB = C," **IEEE Transactions on Automatic Control**, Vol. AC-24, No. 6, Dec. 1979, pp. 909-913.

E.9 Turner, J. D. and Chun, H. M., "An Analytic Fourier Transform for a Class of Finite-Time Control Problems," **Journal of Guidance, Control, and Dynamics**, Vol. 8, No. 4, July-August 1985, pp. 539-541.