APPENDIX C:

CLOSED-FORM SOLUTIONS FOR CONVOLUTION MATRIX INTEGRALS AND SENSITIVITY CALCULATIONS

C.1 CONVOLUTION MATRIX INTEGRAL CALCULATIONS

In order to evaluate the terms in the residual state trajectory of the tracking and disturbance accommodating closed-loop formulation of Section 11.5.5, we must compute eleven matrix integrals which can be expressed as one of the following two forms:

$$G_1(t) = \int_0^t e^{A_1(t-s)} A_2 e^{A_3 s}\, ds \tag{C.1}$$

or

$$H_1(t) = \int_0^t \int_0^s e^{A_1(t-s)} A_2 e^{A_3(s-r)} A_4 e^{A_5 r}\, dr\, ds \tag{C.2}$$

where the constant matrices A_1, A_2, A_3, A_4, and A_5 are specified. This problem, and its solution, are analogous to that discussed in Appendix B. In order to compute $G_1(t)$ and $H_1(t)$, we define the following matrix function

$$\Gamma(t) = \Gamma(A_1,A_2,A_3,A_4,A_5,t) = e^{\Pi t} \tag{C.3}$$

where Π is the constant matrix

$$\begin{array}{cc} \Pi = \begin{bmatrix} A_1 & A_2 & 0 \\ 0 & A_3 & A_4 \\ 0 & 0 & A_5 \end{bmatrix} & \begin{matrix} \}n_1 \\ \}n_2 \\ \}n_3 \end{matrix} \\ \quad\;\; \begin{matrix} n_1 & n_2 & n_3 \end{matrix} & \end{array} \tag{C.4}$$

As shown in Ref. C.1, $\Gamma(t)$ has the representation:

$$\begin{array}{cc} \Gamma(t) = \begin{bmatrix} F_1(t) & G_1(t) & H_1(t) \\ 0 & F_2(t) & G_2(t) \\ 0 & 0 & F_3(t) \end{bmatrix} & \begin{matrix} \}n_1 \\ \}n_2 \\ \}n_3 \end{matrix} \\ \qquad\quad \begin{matrix} n_1 & \quad n_2 & \quad n_3 \end{matrix} & \end{array} \tag{C.5}$$

which includes the desired integrals $G_1(t)$ and $H_1(t)$. In addition,

$$F_j(t) = e^{A_{2j-1}t} \quad , \quad j = 1,2,3 \tag{C.6}$$

$$G_2(t) = \int_0^t e^{A_3(t-s)} A_4 e^{A_5 s} ds \tag{C.7}$$

Using the semi group properties of matrix exponentials, the partitions of $\Gamma(t)$ can be recursively generated from the following formulas:

$$F_j(t+\Delta t) = F_j(\Delta t)F_j(t) \quad ; \quad F_j(0) = I \quad (j = 1,2,3) \tag{C.8}$$

$$G_j(t+\Delta) = F_j(\Delta t)G_j(t) + G_j(\Delta t)F_{j+1}(t) \quad ; \quad G_j(0) = 0 \quad (j = 1,2) \tag{C.9}$$

$$H_1(t+\Delta) = F_1(\Delta t)H_1(t) + G_1(\Delta t)G_2(t) + H_1(\Delta t)F_3(t) \quad ; \quad H_1(0) = 0 \tag{C.10}$$

When the recursion relationships above are post-multiplied by vectors, the following modified propagation equations result:

$$x_j(t + \Delta t) = F_j(t + \Delta t)\sigma = F_j(\Delta t)x_j(t) \quad (j = 1,2,3) \tag{C.11}$$

$$x_{j+3}(t + \Delta) = G_j(t + \Delta t)\sigma = F_j(\Delta t)x_{j+3}(t) + G_j(\Delta t)x_{j+1}(t) \quad (j = 1,2) \tag{C.12}$$

$$x_6(t + \Delta) = H_1(t + \Delta t)\sigma = F_1(\Delta t)x_6(t) + G_1(\Delta t)x_5(t) + H_1(\Delta t)x_3(t) \tag{C.13}$$

where

$$x_j(0) = \begin{cases} \sigma & (j = 1,2,3) \\ 0 & (j = 4,5,6) \end{cases} \tag{C.14}$$

In generating the matrix-vector products involving matrix integrals of the form of Eq. C.1, only $x_2(t)$ and $x_4(t)$ in Eq. C.11 and C.12 must be propagated. Similarly, in generating matrix-vector products involving matrix integrals of the form of Eq. C.2, only $x_3(t)$, $x_5(t)$, and $x_6(t)$ must be propagated.

C.2 CONVOLUTION MATRIX INTEGRAL SENSITIVITY CALCULATIONS

In order to compute the sensitivity partial derivatives appearing in Eq. 11.168, we must deal with expressions of the form

$$F_1(t) = e^{A_1 t} \tag{C.15}$$

$$G_1(t) = \int_0^t e^{A_1(t-s)} A_2 e^{A_3 s} ds \tag{C.16}$$

$$I_1(t) = \int_0^t e^{A_1 s} A_2 e^{A_3 s} ds \tag{C.17}$$

where the constant matrices A_1, A_2, and A_3 are specified. Let γ and σ be the free parameters and the first- and second-order partial derivatives of Eqs. C.15-17 will be given in what follows.

C.2.1 Sensitivity Calculation for $e^{A_1 t}$

The differential equation for Eq. C.15 is given by

$$\dot{F}_1(t) = A_1 F_1(t) \quad , \quad F_1(0) = I \tag{C.18}$$

The partial derivative of Eq. C.18 w.r.t. γ follows as

$$\dot{F}_{1,\gamma} = A_1 F_{1,\gamma} + A_{1,\gamma} F_1 \quad ; \quad F_{1,\gamma}(0) = I \tag{C.19}$$

where the time dependence is suppressed for convenience and $\frac{\partial}{\partial\gamma}[*] = [*]_{,\gamma}$. By defining an augmented state, Eqs. C.18 and C.19 can be written as

$$\dot{X}(t) = AX(t) \tag{C.20}$$

where

$$X(t) = \begin{bmatrix} F_{1,\gamma}(t) \\ F_1(t) \end{bmatrix} , \quad X(0) = \begin{bmatrix} 0 \\ I \end{bmatrix} , \quad A = \begin{bmatrix} A_1 & A_{1,\gamma} \\ 0 & A_1 \end{bmatrix}$$

leading to the well known solution

$$X(t) = e^{At} X(0) \tag{C.21}$$

Defining

$$\phi(t) = e^{At} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \tag{C.22}$$

it follows that

$$F_1(t) = \phi_{22}(t) \tag{C.23}$$

$$F_{1,\gamma}(t) = \phi_{12}(t) \tag{C.24}$$

where Eq. C.24 is the desired sensitivity partial derivative and $\phi_{ij}(t)$ $(i,j = 1,2)$ are the nxn partitions of $\phi(t)$.

To compute the second-order partial derivatives of Eq. C.15, we take the partial derivative of Eq. C.19 w.r.t σ, leading to

$$\dot{F}_{1,\gamma,\sigma} = A_1 F_{1,\gamma,\sigma} + A_{1,\sigma} F_{1,\gamma} + A_{1,\gamma} F_{1,\sigma} + A_{1,\gamma,\sigma} F_1; \; F_{1,\gamma,\sigma}(0) = 0 \tag{C.25}$$

By defining an augmented state matrix, Eqs. C.18, C.19, and C.25 can be written as

$$\dot{Y}(t) = BY(t) \tag{C.26}$$

where

$$Y(t) = \begin{bmatrix} F_{1,\gamma,\sigma} \\ F_{1,\sigma} \\ F_{1,\gamma} \\ F_1 \end{bmatrix}, \quad Y(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}$$

$$B = \begin{bmatrix} A & A_{,\sigma} \\ 0 & A \end{bmatrix} = \begin{bmatrix} A_1 & A_{1,\gamma} & A_{1,\sigma} & A_{1,\gamma,\sigma} \\ 0 & A_1 & 0 & A_{1,\sigma} \\ 0 & 0 & A_1 & A_{1,\gamma} \\ 0 & 0 & 0 & A_1 \end{bmatrix}$$

leading to the well known solution

$$Y(t) = e^{Bt} Y(0) \tag{C.27}$$

Defining

$$\phi(t) = e^{Bt} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) & \phi_{13}(t) & \phi_{14}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \phi_{23}(t) & \phi_{24}(t) \\ \phi_{31}(t) & \phi_{32}(t) & \phi_{33}(t) & \phi_{34}(t) \\ \phi_{41}(t) & \phi_{42}(t) & \phi_{43}(t) & \phi_{44}(t) \end{bmatrix} \tag{C.28}$$

it follows that

$$F_1(t) = \phi_{44}(t) \tag{C.29}$$

$$F_{1,\gamma}(t) = \phi_{34}(t) \tag{C.30}$$

$$F_{1,\sigma}(t) = \phi_{24}(t) \tag{C.31}$$

$$F_{1,\gamma,\sigma}(t) = \phi_{14}(t) \tag{C.32}$$

where the $\phi_{ij}(t)$ $(i,j = 1,\ldots,4)$ are the nxn partitions of $\phi(t)$.

Higher-order partial derivatives follow a similar pattern.

It is of interest to note that the analytic solutions for $F_{1,\gamma}(t)$, $F_{1,\sigma}(t)$, and $F_{1,\gamma,\sigma}(t)$ can be shown to be

$$F_{1,\gamma}(t) = \int_0^t e^{A_1(t-s)} A_{1,\gamma} e^{A_1 s} ds$$

$$F_{1,\sigma}(t) = \int_0^t e^{A_1(t-s)} A_{1,\sigma} e^{A_1 s} ds$$

$$F_{1,\gamma,\sigma}(t) = \int_0^t e^{A_1(t-s)} A_{1,\gamma,\sigma} e^{A_1 s} ds + \int_0^t \int_0^s e^{A_1(t-s)} A_{1,\gamma} e^{A_1(s-r)} A_{1,\sigma} e^{A_1 r} dr\, ds$$

$$+ \int_0^t \int_0^s e^{A_1(t-s)} A_{1,\sigma} e^{A_1(s-r)} A_{1,\gamma} e^{A_1 r} dr\, ds$$

Comparing the equations above with Eq. C.27, it is clear that one advantage of the augmented state matrix approach for computing the sensitivities is that simple linear systems provide the solutions. It should be observed, however, that the integrals have the same structure as Eqs. C.1 and C.2.

C.2.2 Sensitivity Partial Derivatives for $G_1(t)$

From Ref. C.1 it follows that the solution for $G_1(t)$ in Eq. C.16 is obtained by integrating the system of equations

$$\dot{\psi}_1(t) = A_1\psi_1(t) + A_2\psi_2(t) \quad ; \quad \psi_1(0) = 0 \tag{C.33}$$

$$\dot{\psi}_2(t) = A_3\psi_2(t) \quad ; \quad \psi_2(0) = I \tag{C.34}$$

or

$$\dot{X}(t) = AX(0)$$

where

$$X(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \quad , \quad X(0) = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

where A_1 is $n_1 \times n_1$, A_2 is $n_1 \times n_2$, A_3 is $n_2 \times n_2$, $\psi_1(t) = G(t)$ is $n_1 \times n_2$, and $\psi_2(t) = e^{A_3 t}$ is $n_2 \times n_2$.

To compute the partial derivative of $G_1(t)$ w.r.t. γ, we partially differentiate Eqs. C.33 and C.34 w.r.t. γ, leading to (Ref. C.2)

$$\dot{\psi}_{1,\gamma} = A_{1,\gamma}\psi_1 + A_1\psi_{1,\gamma} + A_{2,\gamma}\psi_2 + A_2\psi_{2,\gamma} \quad ; \quad \psi_{1,\gamma}(0) = 0 \tag{C.35}$$

$$\dot{\psi}_{2,\gamma} = A_{3,\gamma}\psi_2 + A_3\psi_{2,\gamma} \quad ; \quad \psi_{2,\gamma}(0) = 0 \tag{C.36}$$

By defining an augmented state matrix, Eqs. C.33-.36 can be written as

$$\dot{Y}(t) = BY(t) \tag{C.37}$$

where

$$Y(t) = [\psi_{1,\gamma}(t), \psi_{2,\gamma}(t), \psi_1(t), \psi_2(t)]^T$$

$$Y(0) = [0 \ , \ 0 \ , \ 0 \ , \ I]^T$$

$$B = \begin{bmatrix} A & A_{,\gamma} \\ 0 & A \end{bmatrix} = \begin{array}{cccccl} & & & & & \\ \left[\begin{array}{cccc} A_1 & A_2 & A_{1,\gamma} & A_{2,\gamma} \\ 0 & A_3 & 0 & A_{3,\gamma} \\ 0 & 0 & A_1 & A_2 \\ 0 & 0 & 0 & A_3 \end{array}\right] & \begin{array}{l} \}n_1 \\ \}n_2 \\ \}n_1 \\ \}n_2 \end{array} \\ \begin{array}{cccc} n_1 & n_2 & n_1 & n_2 \end{array} & \end{array}$$

leading to

$$Y(t) = e^{Bt}Y(0) \tag{C.38}$$

Defining

$$\phi(t) = e^{Bt} = \begin{array}{cl} \left[\begin{array}{cccc} \phi_{11}(t) & \phi_{12}(t) & \phi_{13}(t) & \phi_{14}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \phi_{23}(t) & \phi_{24}(t) \\ \phi_{31}(t) & \phi_{32}(t) & \phi_{33}(t) & \phi_{34}(t) \\ \phi_{41}(t) & \phi_{42}(t) & \phi_{43}(t) & \phi_{44}(t) \end{array}\right] & \begin{array}{l} \}n_1 \\ \}n_2 \\ \}n_1 \\ \}n_2 \end{array} \\ \begin{array}{cccc} n_1 & n_2 & n_1 & n_2 \end{array} & \end{array}$$

it follows that

$$Y(t) = [\phi_{14}(t), \phi_{24}(t), \phi_{34}(t), \phi_{44}(t)]^T \tag{C.39}$$

where from Eq. C.37 it is seen that $\phi_{14}(t)$ is the desired partial derivative for $G_1(t)$ in Eq. C.16.

To compute the second-order partial derivatives of Eq. C.16, we take the partial derivative of Eqs. C.35 and C.36 w.r.t. σ, leading to

$$\begin{aligned} \dot{\psi}_{1,\gamma,\sigma} = {} & A_{1,\gamma,\sigma}\psi_1 + A_{1,\gamma}\psi_{1,\sigma} + A_{1,\sigma}\psi_{1,\gamma} + A_1\psi_{1,\gamma,\sigma} \\ & + A_{2,\gamma,\sigma}\psi_2 + A_{2,\gamma}\psi_{2,\sigma} + A_{2,\sigma}\psi_{2,\gamma} + A_2\psi_{2,\gamma,\sigma} \quad ; \quad \psi_{1,\gamma,\sigma}(0) = 0 \end{aligned} \tag{C.40}$$

$$\dot{\psi}_{2,\gamma,\sigma} = A_{3,\gamma,\sigma}\psi_2 + A_{3,\gamma}\psi_{2,\sigma} + A_{3,\sigma}\psi_{2,\gamma} + A_3\psi_{2,\gamma,\sigma} \quad ; \quad \psi_{2,\gamma,\sigma}(0) = 0 \tag{C.41}$$

By defining an augmented state matrix, Eqs. C.33-.36, C.40, and C.41 can be written as

$$\dot{Z}(t) = CZ(t) \tag{C.42}$$

where

$$Z(t) = [\psi_{1,\gamma,\sigma}(t),\psi_{2,\gamma,\sigma}(t),\psi_{1,\sigma}(t),\psi_{2,\sigma}(t),\psi_{1,\gamma}(t),\psi_{2,\gamma}(t),\psi_1(t),\psi_2(t)]^T$$

$$Z(0) = [0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ I]^T$$

$$C = \begin{array}{cc} \begin{bmatrix} A_1 & A_2 & A_{1,\gamma} & A_{2,\gamma} & A_{1,\sigma} & A_{2,\sigma} & A_{1,\gamma,\sigma} & A_{2,\gamma,\sigma} \\ 0 & A_3 & 0 & A_{3,\gamma} & 0 & A_{3,\sigma} & 0 & A_{3,\gamma,\sigma} \\ 0 & 0 & A_1 & A_2 & 0 & 0 & A_{1,\sigma} & A_{2,\sigma} \\ 0 & 0 & 0 & A_3 & 0 & 0 & 0 & A_{3,\sigma} \\ 0 & 0 & 0 & 0 & A_1 & A_2 & A_{1,\gamma} & A_{2,\gamma} \\ 0 & 0 & 0 & 0 & 0 & A_3 & 0 & A_{3,\gamma} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_1 & A_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_3 \end{bmatrix} & \begin{matrix} \}n_1 \\ \}n_2 \\ \}n_1 \\ \}n_2 \\ \}n_1 \\ \}n_2 \\ \}n_1 \\ \}n_2 \end{matrix} \\ \begin{matrix} n_1 & n_2 & n_1 & n_2 & n_1 & n_2 & n_1 & n_2 \end{matrix} & \end{array} = \begin{bmatrix} B & B,\sigma \\ 0 & B \end{bmatrix}$$

leading to

$$Z(t) = e^{Ct}Z(0) \tag{C.43}$$

Defining

$$\phi(t) = e^{Ct} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) & \cdots & \phi_{18}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \cdots & \phi_{28}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{81}(t) & \phi_{82}(t) & \cdots & \phi_{88}(t) \end{bmatrix}$$

it follows that

$$Z(t) = [\phi_{18}(t),\phi_{28}(t),\ldots,\phi_{88}(t)]^T \tag{C.44}$$

From Eq. C.42 it follows that $\phi_{18}(t) = G,\gamma,\sigma(t)$, $\phi_{38}(t) = G,\sigma(t)$, $\phi_{58}(t) = G,\gamma(t)$, and $\phi_{78}(t) = G(t)$.

Moreover, from Eqs. C.33, C.34, C.37, C.38, C.42 and C.43 it is of interest to note that the matrix exponential solutions for e^{At}, e^{Bt}, and e^{Ct}

can be written as (Ref. C.1)

$$e^{At} = \exp\left(\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} t\right) = \begin{bmatrix} e^{A_1 t} & e^{A_1 t}\int_{0^-}^{t} e^{-A_1 r} A_2 e^{A_3 r}\,dr \\ 0 & e^{A_3 t} \end{bmatrix}$$

$$e^{Bt} = \exp\left(\begin{bmatrix} A & A,_\gamma \\ 0 & A \end{bmatrix} t\right) = \begin{bmatrix} e^{At} & e^{At}\int_{0}^{t} e^{-Ar} A,_\gamma e^{Ar}\,dr \\ 0 & e^{At} \end{bmatrix}$$

$$e^{Ct} = \exp\left(\begin{bmatrix} B & B,_\sigma \\ 0 & B \end{bmatrix} t\right) = \begin{bmatrix} e^{Bt} & e^{Bt}\int_{0}^{t} e^{-Br} B,_\sigma e^{Br}\,dr \\ 0 & e^{Bt} \end{bmatrix}$$

It should be observed that the simple structure of the equations above results from the special ordering of terms in the X(t), Y(t), and Z(t) matrix state vectors.

We observe that the calculations for e^{Bt} and e^{Ct} above can be greatly simplified if the associated A and B matrices can be expressed in terms of right and left eigenvectors. For example, if we assume that the following eigenvalue problems can be solved:

$$AR = R\Lambda$$

$$A^T L = L\Lambda$$

subject to $L^T R = I$, where R is the matrix of right eigenvectors, L is the matrix of left eigenvectors, and $\Lambda = \mathrm{Diag}[\lambda_1, \ldots, \lambda_n]$ is the matrix of the eigenvalues of the matrix A. Then A can be expressed as

$$A = R\Lambda R^{-1} = R\Lambda L^T$$

from which it follows that

$$e^{At} = Re^{\Lambda t}L^T$$

Accordingly, we can write the convolution matrix integral appearing in the expression for e^{Bt} as

$$e^{At}\int_0^t e^{-Ar}A_{,\gamma}e^{Ar}dr = Re^{\Lambda t}L^T\int_0^t Re^{-\Lambda r}L^TA_{,\gamma}Re^{\Lambda r}L^Tdr$$

$$= Re^{\Lambda t}[\int_0^t e^{-\Lambda r}L^TA_{,\gamma}Re^{\Lambda r}dr]L^T = Re^{\Lambda t}H(t)L^T$$

where the closed-form solution for the ijth element of H(t) is given by

$$[H(t)]_{ij} = \begin{cases} [L^TA_{,\gamma}R]_{ij}(e^{(\lambda_j-\lambda_i)t}-1)/(\lambda_j-\lambda_i), & \text{if } \lambda_j-\lambda_i\neq 0 \\ [L^TA_{,\gamma}R]_{ij}t \quad , & \text{if } \lambda_j-\lambda_i = 0 \end{cases}$$

Accordingly, for each new value of γ only the $L^TA_{,\gamma}R$ matrix triple product must be recomputed; thereby greatly reducing the associated computational costs. Similar results can be obtained for the convolution matrix integral appearing in e^{Ct}, when the B matrix can be expressed in terms of a right and left eigenvector description.

C.2.3 Sensitivity Calculation for $I_1(t)$

From Eqs. C.16 and C.17 it follows that the relationship between $G_1(t)$ and $I_1(t)$ is given by

$$I_1(t) = F_3(t)G_1(t) \qquad \text{(C.45)}$$

where $F_3(t) = e^{-A_1t}$. As a result, the partial derivative of Eq. C.45 follows as

$$I_{1,\gamma} = F_{3,\gamma}G_1 + F_3G_{1,\gamma} \qquad \text{(C.46)}$$

where $F_{3,\gamma}$ can be obtained from Eqs. C.18-.24 by replacing A_1 with $-A_1$ and $G_{1,\gamma}$ can be obtained from Eqs. C.33-.39.

To obtain the second-order partial derivatives we partially differentiate Eq. C.56 w.r.t. σ, leading to

$$I_{1,\gamma,\sigma} = F_{3,\gamma,\sigma}G_1 + F_{3,\gamma}G_{1,\sigma} + F_{3,\sigma}G_{1,\gamma} + F_3G_{1,\gamma,\sigma} \qquad (C.47)$$

where $F_{3,\gamma}$ and $F_{3,\sigma}$ can be obtained from Eqs. C.18-.24 by replacing A_1 with $-A_1$, $G_{1,\gamma}$ and $G_{1,\sigma}$ can be obtained from Eqs. C.33-.39, $F_{3,\gamma,\sigma}$ can be obtained from Eqs. C.26-.32, and $G_{1,\gamma,\sigma}$ can be obtained from Eqs. C.42-.44.

References

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C.2 Turner, J. D., Messac, A., and Junkins, J. L., "Finite Time Matrix Convolution Integral Sensitivity Calculations," **AIAA Journal of Guidance, Dynamics, and Control**, to appear.