Chapter | four

Nonlinear Models of Relative Dynamics

wheels repeating the same gesture
remain relatively stationary
rails forever parallel
return on themselves infinitely.
The dance is sure.

William Carlos Williams (1883–1963)

We present commonly used nonlinear models for relative motion in this chapter, for both perturbed and unperturbed motion. For unperturbed Keplerian motion, this chapter suggests a rather simple method for generating bounded relative motion between any two spacecraft [90]. The underlying methodology relies on the concept of orbital-period *commensurability*: Two nonzero real numbers c and d are said to be commensurable if and only if c/d is a rational number; in other words, there exists some real number g, and integers m and n, such that $c = mg$ and $d = ng$. Thus, two elliptic orbits with orbital periods T_1 and T_2 are said to be $m : n$ commensurable if

$$\frac{T_1}{T_2} = \frac{m}{n}, \tag{4.1}$$

Based on our discussion of Keplerian orbits in Chapter 2, it is obvious that the corresponding relationship between the orbital energies, $\mathcal{E}_1$ and $\mathcal{E}_2$, and the semimajor axes, a_1 and a_2, derived based on Eq. (2.33), satisfies

$$\frac{T_1}{T_2} = \left(\frac{\mathcal{E}_2}{\mathcal{E}_1}\right)^{3/2} = \left(\frac{a_1}{a_2}\right)^{3/2}, \tag{4.2}$$

which implies that

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{a_1}{a_2} = \left(\frac{m}{n}\right)^{2/3} \tag{4.3}$$

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Thus, although m/n is rational, $\mathcal{E}_2/\mathcal{E}_1$ and a_1/a_2 may not be so. Nevertheless, orbital commensurability necessarily constrains the ratio between orbital energies and between the semimajor axes of the orbits. This constraint is therefore referred to as the *energy matching condition*. When $m = n = 1$, which is of a particular interest in our forthcoming discussion, we have $a_1 = a_2$ and $\mathcal{E}_1 = \mathcal{E}_2$.

The concept of commensurability will help us model the general problem of Keplerian relative motion. We will do so by formulating the energy matching condition in the rotating reference-orbit-fixed frame. The formulation using orbital commensurability will give us a single, simple, algebraic constraint on initial conditions guaranteeing bounded relative motion between spacecraft flying on elliptic, perturbation-free orbits.

The approach for generating bounded relative motion presented in this chapter is a convenient stepping stone for developing an optimal *formation-keeping* scheme. In real-world scenarios, spacecraft formation initialization will almost always entail an initialization error [91]. A formation-keeping maneuver is required for arresting the relative spacecraft drift. As relative spacecraft position control is sometimes performed utilizing relative measurements, it is important to design formation-keeping maneuvers utilizing the relative state variables.

We will discuss in detail formation control concepts in Chapter 10. In this chapter, we develop a *single-impulse formation-keeping* method by using the inherent freedom of the energy matching condition. We also provide an insight into the resulting formation-keeping maneuver by utilizing the classical orbital elements discussed in Chapter 2.

The bottom line is that we design a simple framework for both initialization and initialization-error correction for spacecraft formation flying. This framework yields bounded relative motion between any two spacecraft flying on arbitrary elliptic orbits – at this stage, without utilizing any simplifying assumptions regarding the relative dynamics – except that it is perturbation-free. We will relieve this assumption, however, in Section 4.7, where we present a model for relative motion under the influence of J_2.

4.1 EQUATIONS OF RELATIVE MOTION IN THE UNPERTURBED CASE

Consider two spacecraft orbiting a common primary. One of the spacecraft will be termed *chief* and the other will be referred to as *deputy* (see our discussion of the chief/deputy notation vs. the leader/follower notation on p. 2). We wish to develop the equations of relative motion under the setup of the Keplerian two-body problem. We will start with an inertial description of the relative motion in $\mathscr{I}$ and then transform the equations into a chief-fixed, LVLH rotating frame, $\mathscr{L}$, called the *Euler–Hill frame*, as shown in Fig. 4.1 (both $\mathscr{I}$ and $\mathscr{L}$ were discussed in Chapter 2). This operation is required due to the fact that control of relative position and velocity, to be discussed in Chapters 10 and 11, most often utilizes measurements taken in the chief rotating frame.

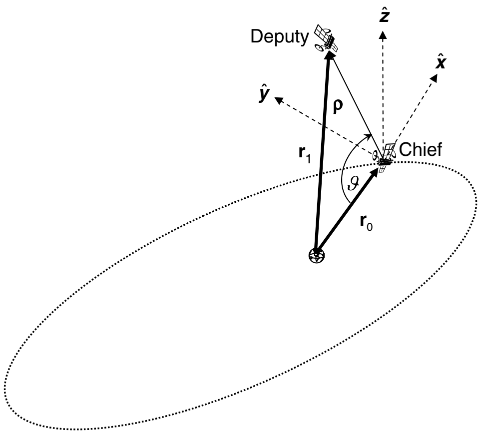

FIGURE 4.1 Rotating Euler–Hill frame, centered at the chief spacecraft. This figure also shows the deputy spacecraft, whose position vectors in the rotating and inertial reference frames are denoted by $\boldsymbol{\rho}$ and $\mathbf{r}_1$, respectively.

Recalling Eq. (2.1), the inertial equations of motion of the chief are given by

$$\ddot{\mathbf{r}}_0 = -\frac{\mu}{r_0^3}\mathbf{r}_0 \tag{4.4}$$

where

$$r_0 = \|\mathbf{r}_0\| = \frac{a_0(1-e_0^2)}{(1+e_0\cos f_0)}, \tag{4.5}$$

and a_0, e_0, f_0 are the chief's orbit semimajor axis, eccentricity and true anomaly, respectively. In a similar fashion, the inertial equations of motion of the deputy are

$$\ddot{\mathbf{r}}_1 = -\frac{\mu}{r_1^3}\mathbf{r}_1 \tag{4.6}$$

where

$$r_1 = \|\mathbf{r}_1\| = \frac{a_1(1-e_1^2)}{(1+e_1\cos f_1)}, \tag{4.7}$$

and a_1, e_1, f_1 are the deputy's orbit semimajor axis, eccentricity and true anomaly, respectively. Let

$$\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_0 \tag{4.8}$$

denote the position of the deputy relative to the chief. Subtracting Eq. (4.4) from Eq. (4.6) yields

$$\ddot{\boldsymbol{\rho}} = -\frac{\mu(\mathbf{r}_0 + \boldsymbol{\rho})}{\|\mathbf{r}_0 + \boldsymbol{\rho}\|^3} + \frac{\mu}{r_0^3}\mathbf{r}_0 \tag{4.9}$$

In order to express the relative acceleration in frame $\mathscr{L}$, we recall that

$$\begin{aligned}\ddot{\boldsymbol{\rho}} = {}& \frac{d^{2\mathscr{L}}\boldsymbol{\rho}}{dt^2} + 2\,{}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times \frac{d^{\mathscr{L}}\boldsymbol{\rho}}{dt} \\ &+ \frac{d^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}}}{dt} \times \boldsymbol{\rho} + {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times ({}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times \boldsymbol{\rho})\end{aligned} \tag{4.10}$$

where ${}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}}$ denotes the angular velocity vector of frame $\mathscr{L}$ relative to frame $\mathscr{I}$.

As ${}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}}$ is normal to the orbital plane, we may write

$${}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} = [0, 0, \dot{\theta}_0]^T \tag{4.11}$$

The position vector of the chief can be written as

$$\mathbf{r}_0 = [r_0, 0, 0]^T \tag{4.12}$$

Also, let

$$[\boldsymbol{\rho}]_{\mathscr{L}} = [x, y, z]^T \tag{4.13}$$

Substituting Eqs. (4.9), (4.11), and (4.13) into Eq. (4.10) yields the following component-wise equations for relative motion:

$$\ddot{x} - 2\dot{\theta}_0\dot{y} - \ddot{\theta}_0 y - \dot{\theta}_0^2 x = -\frac{\mu(r_0 + x)}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + \frac{\mu}{r_0^2} \tag{4.14}$$

$$\ddot{y} + 2\dot{\theta}_0\dot{x} + \ddot{\theta}_0 x - \dot{\theta}_0^2 y = -\frac{\mu y}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} \tag{4.15}$$

$$\ddot{z} = -\frac{\mu z}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} \tag{4.16}$$

Equations (4.14)–(4.16) together with Eqs. (2.5) and (2.6),

$$\ddot{r}_0 = r_0\dot{\theta}_0^2 - \frac{\mu}{r_0^2},\ \ddot{\theta}_0 = -\frac{2\dot{r}_0\dot{\theta}_0}{r_0}$$

constitute a 10-dimensional system of nonlinear differential equations. For $\ddot{\theta}_0 \neq 0$, these equations admit a single relative equilibrium at $x = y = z = 0$, meaning that the deputy spacecraft will appear stationary in the chief frame if and only if their positions coincide on a given elliptic orbit. We will later see that the single relative equilibrium is transformed into infinitely many relative equilibria if the chief is assumed to follow a circular reference orbit.

If there are external (differential) perturbations, denoted by $\mathbf{d} = [d_x, d_y, d_z]^T$, and (differential) control forces, $\mathbf{u} = [u_x, u_y, u_z]^T$, they are introduced into Eqs. (4.14)–(4.16) in the following manner:

$$\ddot{x} - 2\dot{\theta}_0\dot{y} - \ddot{\theta}_0 y - \dot{\theta}_0^2 x = -\frac{\mu(r_0 + x)}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + \frac{\mu}{r_0^2} + d_x + u_x \quad (4.17)$$

$$\ddot{y} + 2\dot{\theta}_0\dot{x} + \ddot{\theta}_0 x - \dot{\theta}_0^2 y = -\frac{\mu y}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + d_y + u_y \quad (4.18)$$

$$\ddot{z} = -\frac{\mu z}{\left[(r_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + d_z + u_z \quad (4.19)$$

Let us mention some mathematical formalities related to Eqs. (4.14)–(4.16). We say that the configuration space for the relative spacecraft dynamics of Eqs. (4.14)–(4.16) is $\mathbb{R}^3$, and that $T_S(\mathbb{R}^3) = \mathbb{R}^3 \times \mathbb{R}^3$ is the tangent space of $\mathbb{R}^3$. We use $(\boldsymbol{\rho}, \dot{\boldsymbol{\rho}})$ as coordinates for $T_S(\mathbb{R}^3)$.

To get a more compact form of Eqs. (4.14)–(4.16), we can use the formalism developed by Szebehely and Giacaglia [92] for simplifying the equations of motion obtained in the elliptic restricted three-body problem. To that end, we define normalized position components,

$$\bar{x} = x/r_0, \bar{y} = y/r_0, \bar{z} = z/r_0 \quad (4.20)$$

and use the true anomaly, f, as our independent variable. By the chain rule, derivatives with respect to f, to be denoted by $(\cdot)'$, satisfy

$$\frac{d(\cdot)}{dt} = (\cdot)'\dot{f} \quad (4.21)$$

In the Keplerian setup, we can use the auxiliary relations

$$\dot{\theta}_0 = \dot{f}_0 = \frac{h_0}{r_0^2}, h_0 = \sqrt{\mu p_0}, r_0 = \frac{p_0}{1 + e_0\cos f_0}, r_0' = \frac{p_0 e_0 \sin f_0}{(1 + e_0\cos f_0)^2} \quad (4.22)$$

Thus,

$$\dot{x} = \dot{f}_0(r_0'\bar{x} + r_0\bar{x}') = \frac{h_0}{p_0}[(e_0\sin f_0)\bar{x} + (1 + e_0\cos f_0)\bar{x}'] \quad (4.23)$$

Equivalent relations hold for $\dot{y}$ and $\dot{z}$. Application of the chain rule again provides us with the expression

$$\ddot{x} = \frac{h_0^2}{p_0^2}[(1 + e_0\cos f_0)\bar{x}'' + e_0(\cos f_0)\bar{x}](1 + e_0\cos f_0)^3 \quad (4.24)$$

Again, similar relations hold for $\ddot{y}$ and $\ddot{z}$. Now, we define the non-dimensional potential function

$$\mathcal{U} = -\frac{1}{\left[(1+\bar{x})^2 + \bar{y}^2 + \bar{z}^2\right]^{\frac{1}{2}}} + 1 - \bar{x} \tag{4.25}$$

and the *pseudo-potential* function

$$\mathcal{W} = \frac{1}{1 + e_0 \cos f_0}\left[\frac{1}{2}(\bar{x}^2 + \bar{y}^2 - e_0 \bar{z}^2 \cos f_0) - \mathcal{U}\right] \tag{4.26}$$

This enables us to write

$$\bar{x}'' - 2\bar{y}' = \frac{\partial \mathcal{W}}{\partial \bar{x}} \tag{4.27}$$

$$\bar{y}'' + 2\bar{x}' = \frac{\partial \mathcal{W}}{\partial \bar{y}} \tag{4.28}$$

$$\bar{z}'' = \frac{\partial \mathcal{W}}{\partial \bar{z}} \tag{4.29}$$

4.2 THE ENERGY MATCHING CONDITION

It is natural to inquire whether Eqs. (4.14)–(4.16) provide bounded solutions. This question has a straightforward answer which is derived from the physics of the orbits: If both vehicles follow Keplerian elliptic orbits, then their separation cannot grow unboundedly. However, it is intuitively clear that if the periods of the spacecraft orbits are not commensurate, periodicity of the relative motion will not be exhibited, and hence, on shorter time scales, the relative motion may appear to be "locally" unbounded. The relative motion between two non-commensurable elliptic orbits is said to be *quasi-periodic*.

Since the periods of elliptic orbits uniquely determine the energy of the orbit, we can use the energy matching conditions to find periodic relative orbits. In spacecraft formation flying, the only interesting case is 1:1 commensurability, guaranteeing that the semimajor axes and energies are equal. Other commensurability ratios are of interest in problems such as interplanetary travel, orbital transfer and Lambert's problem [33].

To implement the energy matching condition for finding periodic relative orbits in the formation flying scenario, we will first write the velocity of the deputy in the rotating frame:

$$\mathbf{v}_1 = \frac{{}^{\mathcal{L}}d}{dt}\boldsymbol{\rho} + \frac{{}^{\mathcal{L}}d}{dt}\mathbf{r}_0 + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}} \times \mathbf{r}_0 + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{L}} \times \boldsymbol{\rho} \tag{4.30}$$

Substitution of Eqs. (4.11)–(4.13) into Eq. (4.30) yields

$$\mathbf{v}_1 = \begin{bmatrix} \dot{x} - \dot{\theta}_0 y + \dot{r}_0 \\ \dot{y} + \dot{\theta}_0(x + r_0) \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{4.31}$$

where

$$\dot{r}_0 = \frac{d}{dt}\left[\frac{a_0(1-e_0^2)}{1+e_0\cos f_0}\right] = \dot{f}_0\frac{d}{df_0}\left[\frac{a_0(1-e_0^2)}{1+e_0\cos f_0}\right]$$
$$= \dot{\theta}_0\frac{a_0e_0(1-e_0^2)\sin f_0}{(1+e_0\cos f_0)^2} \tag{4.32}$$

Substituting for $\dot{\theta}_0$ from Eq. (2.29) we obtain

$$\dot{r}_0 = e_0\sin f_0\sqrt{\frac{\mu}{a_0(1-e_0^2)}} \tag{4.33}$$

The total specific energy of the deputy spacecraft comprises the kinetic and potential energies,

$$\mathcal{E}_1 = \frac{1}{2}v_1^2 - \frac{\mu}{r_1} = \frac{1}{2}v_x^2 + v_y^2 + v_z^2 - \frac{\mu}{r_1}$$
$$= \frac{1}{2}\{(\dot{x}-\dot{\theta}_0 y+\dot{r}_0)^2 + [\dot{y}+\dot{\theta}_0(x+r_0)]^2 + \dot{z}^2\}$$
$$- \frac{\mu}{\sqrt{(r_0+x)^2+y^2+z^2}} \tag{4.34}$$

The total energy of the chief is given by (cf. Eq. (2.25))

$$\mathcal{E}_0 = -\frac{\mu}{2a_0} \tag{4.35}$$

The energy matching condition, guaranteeing a 1:1 commensurable relative motion, is therefore

$$\frac{1}{2}\{(\dot{x}-\dot{\theta}_0 y+\dot{r}_0)^2 + [\dot{y}+\dot{\theta}_0(x+r_0)]^2 + \dot{z}^2\}$$
$$- \frac{\mu}{\sqrt{(r_0+x)^2+y^2+z^2}} = -\frac{\mu}{2a_0} \tag{4.36}$$

In order to design a 1:1 bounded formation, we require the following constraint on the initial conditions of Eqs. (4.14)–(4.16):

$$\frac{1}{2}\{[\dot{x}(0)-\dot{\theta}_0(0)y(0)+\dot{r}_0(0)]^2 + \{\dot{y}(0)+\dot{\theta}_0(0)[x(0)+r_0(0)]\}^2 + \dot{z}(0)^2\}$$
$$- \frac{\mu}{\sqrt{[r_0(0)+x(0)]^2+y^2(0)+z^2(0)}} = -\frac{\mu}{2a_0} \tag{4.37}$$

Most often, Eq. (4.37) is normalized, with the distances being measured in units of a_0 and angular velocities in units of $\sqrt{\mu/a_0^3}$. We will denote normalized quantities by $\overline{(\cdot)}$ and differentiation with respect to normalized time by $(\cdot)'$.

Example 4.1. *Consider a chief spacecraft on an elliptic orbit. Normalize positions by a_0 and angular velocities by $\sqrt{\mu/a_0^3}$ so that $a_0 = \mu = 1$. Using these normalized units, let*

$$\begin{aligned}\bar{y}(0) = 0, \bar{z}(0) = 0.1, \bar{x}'(0) = 0.02\\ \bar{y}'(0) = 0.02, \bar{z}'(0) = 0, f_0(0) = 0, e_0 = 0.1\end{aligned} \tag{4.38}$$

Find $\bar{x}(0)$ that guarantees a 1:1 bounded relative motion.

We first need to calculate $\bar{r}_0(0)$, $\bar{r}_0'(0)$ and $\theta_0'(0)$:

$$\bar{r}_0(0) = \frac{1 - e_0^2}{1 + e_0 \cos f_0(0)} = \frac{1 - 0.1^2}{1 + 0.1} = 0.9$$

$$\bar{r}_0'(0) = e_0 \sin f_0(0)\sqrt{\frac{1}{(1 - e_0^2)}} = 0$$

$$\theta_0'(0) = \sqrt{\frac{1}{(1 - e_0^2)^3}}(1 + e_0 \cos f_0(0))^2 = 1.22838$$

Upon substitution into (4.37), we obtain a sixth-order equation for $\bar{x}_0$:

$$\begin{aligned}&2.2768\bar{x}^6(0) + 12.4431\bar{x}^5(0) + 31.3762\bar{x}^4(0) + 45.46062\bar{x}^3(0)\\ &\quad + 39.5905\bar{x}^2(0) + 19.5344\bar{x}(0) + 0.2151 = 0\end{aligned}$$

There are two real solutions:

$$\bar{x}_1(0) = -0.01127, \bar{x}_2(0) = -1.8059$$

These initial conditions on the radial separation of the spacecraft will guarantee 1:1 bounded motion. To illustrate the resulting orbits, we simulated Eqs. (4.14)–(4.16) using the initial conditions (4.38) and $\bar{x}(0) = \bar{x}_1(0)$. The results are depicted in Figs. 4.2 and 4.3. Figure 4.2 shows that the relative position components are periodic, with period equal to the chief's orbital period. This is not surprising, as we have obtained the 1:1 commensurability solution. Fig. 4.3 shows the relative orbits in the configuration space. We note that the constraint (4.37) can be satisfied in many other ways, e.g., by selecting $\dot{y}(0)$, with the other initial conditions specified.

4.3 IMPULSIVE FORMATION-KEEPING

Equation (4.37) constitutes a necessary and sufficient condition for 1:1 commensurable relative motion. However, in practice, due to initialization errors, this constraint cannot be satisfied exactly. The ideal initial conditions required for 1:1 boundedness are violated, and a "drift" in relative position results. To compensate for such errors, the deputy spacecraft must maneuver. In most cases, the maneuver will be carried out using the on-board propulsion system,

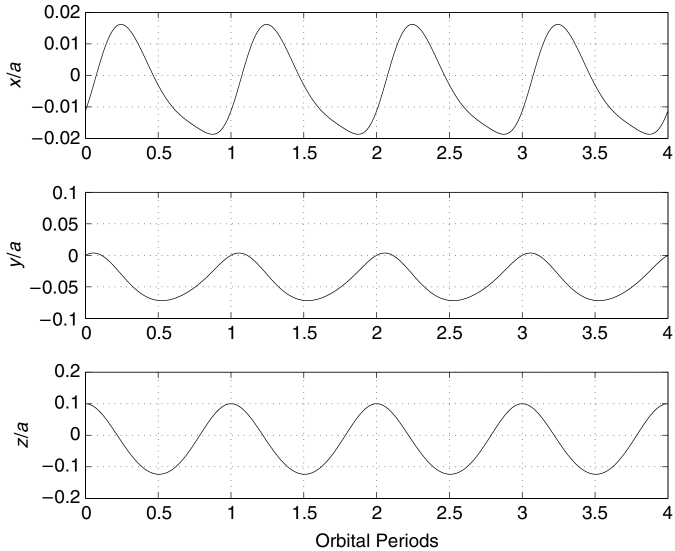

FIGURE 4.2 Time histories of the normalized relative position components show periodic behavior with period equal to the chief's orbital period.

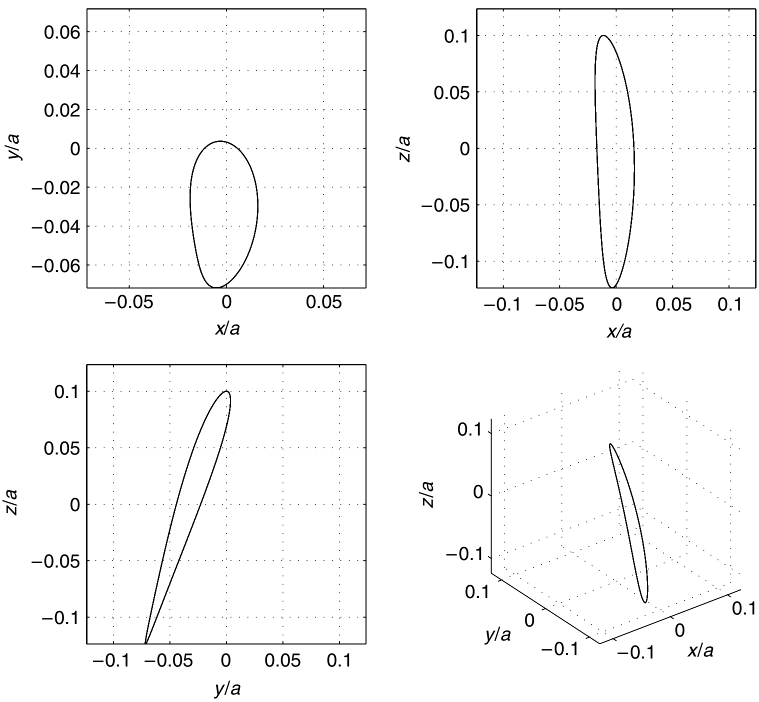

FIGURE 4.3 The relative motion in the configuration space exhibits bounded orbits.

providing short thrust pulses. Such maneuvers are called *impulsive* and the relative position regulation is called *formation-keeping*. Ideal impulsive maneuvers provide instantaneous velocity changes only; in other words, while the velocity of an impulsively maneuvering spacecraft is immediately changed, the position remains intact at impulse application.

We will develop an impulsive formation-keeping maneuver strategy aimed at correcting the relative position drift while consuming minimum fuel. To this end, let the position initialization errors be denoted by

$$\delta\boldsymbol{\rho}(0) = [\delta x(0), \delta y(0), \delta z(0)]^T$$

and the velocity initialization errors be

$$\delta\mathbf{v}(0) = [\delta\dot{x}(0), \delta\dot{y}(0), \delta\dot{z}(0)]^T$$

The actual initial conditions are thus

$$\begin{aligned}\boldsymbol{\rho}^{\delta}(0) &= \boldsymbol{\rho}(0) + \delta\boldsymbol{\rho}(0)\\ \mathbf{v}^{\delta}(0) &= \mathbf{v}(0) + \delta\mathbf{v}(0)\end{aligned} \tag{4.39}$$

The impulsive maneuver will be represented by a relative velocity correction performed by the deputy in the rotating Euler–Hill frame, $[\Delta\mathbf{v}]_{\mathscr{L}} = [\Delta v_x, \Delta v_y, \Delta v_z]^T$. We will subsequently omit the subscript $\mathscr{L}$ to facilitate the notation. If the impulsive maneuver is initiated at time t_i, then the required velocity correction components can be determined by the energy matching condition applied on the deputy's total energy after the correction, $\mathcal{E}_1^+$ (recall that under the ideal impulsive maneuver assumption, the position remains unchanged due to an impulsive correction):

$$\begin{aligned}\mathcal{E}_1^+ &= \frac{1}{2}\left[(v_x^- + \Delta v_x)^2 + (v_y^- + \Delta v_y)^2 + (v_z^- + \Delta v_z)^2\right] - \frac{\mu}{r_1}\\ &= -\frac{\mu}{2a_0}\end{aligned} \tag{4.40}$$

where

$$v_x^- = \dot{x}^-(t_i) - \dot{\theta}_0^-(t_i)y(t_i) + \dot{r_0}^-(t_i) \tag{4.41}$$

$$v_y^- = \dot{y}^-(t_i) + \dot{\theta}_0^-(t_i)[x(t_i) + r_0(t_i)] \tag{4.42}$$

$$v_z^- = \dot{z}^-(t_i) \tag{4.43}$$

$$r_1 = \sqrt{[r_0(t_i) + x(t_i)]^2 + y^2(t_i) + z^2(t_i)} \tag{4.44}$$

and $(\cdot)^-(t_i)$ indicates values prior to the impulsive maneuver, applied at $t = t_i$. Apparently, we have three degrees of freedom for choosing the velocity correction components but only one constraint (4.40). This means that we are free to choose two degrees of freedom. As we are interested in saving fuel consumption, the extra freedom will be used for minimizing the required fuel consumption by solving an optimization problem. We note that if t_i is given, the underlying optimization problem is static, as it involves parameters defined at a single time instant, t_i. Otherwise, a dynamic optimization problem must be solved or approximated by successive static optimization procedures for each t_i. We will assume hereafter that the t_i is pre-determined based upon operational consideration, and subsequently solve a static optimization problem. Recall that we described a method for solving such problems in Section 3.5.

Static parameter optimization problems with equality constraints can be straightforwardly solved utilizing the well-known concept of Lagrange multipliers, described in Section 3.5. In our case, the optimization problem may be stated as follows: Find an optimal impulsive maneuver, $\Delta\mathbf{v}^*$, satisfying

$$\begin{aligned}&\Delta\mathbf{v}^* = \arg\min_{\Delta\mathbf{v}} \|\Delta\mathbf{v}\|^2\\ &\text{s. t.}\\ &\mathcal{E}_1^+ = -\frac{\mu}{2a_0}\end{aligned} \tag{4.45}$$

Augmenting the objective function with the equality constraint using the Lagrange multiplier λ yields the Lagrangian

$$\mathcal{L} = \|\Delta\mathbf{v}\|^2 + \lambda\left(\mathcal{E}_1^+ + \frac{\mu}{2a_0}\right) \tag{4.46}$$

The necessary condition for the existence of a stationary point is

$$\frac{\partial\mathcal{L}}{\partial(\Delta\mathbf{v})} = \mathbf{0} \tag{4.47}$$

This stationary point is a minimum if the bordered Hessian of $\mathcal{L}$ satisfies the test of Eq. (3.34). Equation (4.47) together with the constraint (4.40) establishes a system of four quadratic algebraic equations for the four variables Δv_x, Δv_y, Δv_z and λ. These equations are

$$2\Delta v_x + \lambda(v_x^- + \Delta v_x) = 0, \tag{4.48}$$

$$2\Delta v_y + \lambda(v_y^- + \Delta v_y) = 0, \tag{4.49}$$

$$2\Delta v_z + \lambda(v_z^- + \Delta v_z) = 0, \tag{4.50}$$

$$\frac{1}{2}\left[(v_x^- + \Delta v_x)^2 + (v_y^- + \Delta v_y)^2 + (v_z^- + \Delta v_z)^2\right] - \frac{\mu}{r_1} - \frac{\mu}{2a_0} = 0 \tag{4.51}$$

A solution thereof yields

$$\frac{\Delta v_x}{v_x^-} = \frac{\Delta v_y}{v_y^-} = \frac{\Delta v_z}{v_z^-} = -1 \pm \frac{1}{v_1^-}\sqrt{\frac{\mu(2a_0 - r_1)}{a_0 r_1}} \tag{4.52}$$

$$\lambda = -2 \pm 2v_1^-\sqrt{\frac{a_0 r_1}{\mu(2a_0 - r_1)}} \tag{4.53}$$

Real and finite solutions are obtained provided that $r_1 \leq 2a_0$. The bordered Hessian test of Eq. (3.34) shows that a sufficient condition for a minimum is $\lambda > -2$. Since $v_1^- > 0, r_1 > 0, \mu > 0, a > 0$, we conclude that only the first of solutions (4.53) corresponds to a minimum,

$$\lambda = -2 + 2v_1^- \sqrt{\frac{a_0 r_1}{\mu(2a_0 - r_1)}} \tag{4.54}$$

The corresponding optimal velocity corrections are therefore

$$\frac{\Delta v_x^*}{v_x^-} = \frac{\Delta v_y^*}{v_y^-} = \frac{\Delta v_z^*}{v_z^-} = -1 + \frac{1}{v_1^-}\sqrt{\frac{\mu(2a_0 - r_1)}{a_0 r_1}} \tag{4.55}$$

and the minimum velocity correction is

$$\Delta v^* = \sqrt{(\Delta v_x^*)^2 + (\Delta v_y^*)^2 + (\Delta v_z^*)^2} = v_1^- - \sqrt{\frac{\mu(2a_0 - r_1)}{a_0 r_1}} \tag{4.56}$$

Example 4.2. *Consider a chief spacecraft on an elliptic orbit. Normalize positions by a_0 and angular velocities by $\sqrt{\mu/a_0^3}$ so that $a_0 = \mu = 1$. Let the nominal normalized initial conditions be as in Eq. (4.38) with $\bar{x}(0) = -0.01127$. Assume that the initialization errors are*

$$\begin{gathered} \delta\bar{x}(0) = 0.001, \delta\bar{y}(0) = 0.001, \delta\bar{z}(0) = 0.01 \\ \delta\bar{x}'(0) = 0, \delta\bar{y}'(0) = 0, \delta\bar{z}'(0) = 0 \end{gathered} \tag{4.57}$$

Compute the minimum-fuel maneuver required to obtain a $1 : 1$ bounded relative motion assuming that the maneuver is to be applied after one orbital period of the chief.

After a single orbital period, $\bar{t}_i = 1$ (in normalized units). Utilizing the initial conditions (4.38) and Eqs. (4.14)–(4.16) we have, at $\bar{t}_i = 1$,

$$\begin{gathered} \bar{x} = -0.015374, \bar{y} = -0.084596, \bar{z} = 0.109547 \\ (\bar{x}')^- = 0.00994, (\bar{y}')^- = 0.021792, (\bar{z}')^- = 0.011765 \\ (\theta_0')^- = 1.22838, \bar{r}_0 = 0.9, (\bar{r}_0')^- = 0 \end{gathered} \tag{4.58}$$

Substituting into Eqs. (4.41)–(4.44) yields the normalized values

$$\begin{gathered} (\bar{v}_x)^- = 0.11386, (\bar{v}_y)^- = 1.10845, \\ (\bar{v}_z)^- = 0.01177, \bar{r}_1 = 0.89538 \end{gathered} \tag{4.59}$$

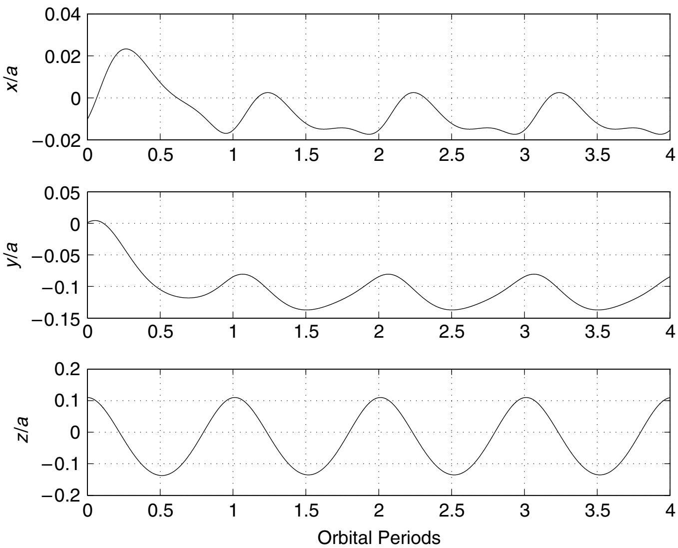

FIGURE 4.4 Time histories of the normalized relative position components show periodic behavior following the formation-keeping maneuver initiated after one orbital period.

Substituting again into Eq. (4.55) yields the optimal formation-keeping maneuver components (in normalize units)

$$\begin{aligned}\Delta\bar{v}_x^* &= -0.00037144,\ \Delta\bar{v}_y^* = -0.00361606,\\ \Delta\bar{v}_z^* &= -0.00003838\end{aligned} \tag{4.60}$$

which results in the total normalized Δv

$$\Delta\bar{v}^* = \sqrt{(\Delta\bar{v}_x^*)^2 + (\Delta\bar{v}_y^*)^2 + (\Delta\bar{v}_z^*)^2} = 0.0036353 \tag{4.61}$$

Figures 4.4–4.6 depict the results of a simulation performed utilizing the above values. Fig. 4.4 shows the time histories of the normalized relative position components. The time is normalized by the chief's orbital period. The impulsive formation-keeping maneuver is carried out at $\bar{t} = 1$. Following this maneuver, the position components converge to a periodic motion reflecting the 1:1 relative motion commensurability. Note the transient response leading to periodicity.

Figure 4.5 shows the three-dimensional relative orbit in the Euler–Hill frame utilizing normalized position components and the projections of the orbit on the xy, xz and yz planes. The initial relative drift is arrested at $\bar{t} = 1$, and a bounded relative motion results.

Figure 4.6 exhibits a magnification of the time history of the normalized relative velocity components $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ at the vicinity of the impulsive maneuver, and the total energy of the deputy. The discontinuity in the relative velocity component is a result of the impulsive velocity change. Note that the initial

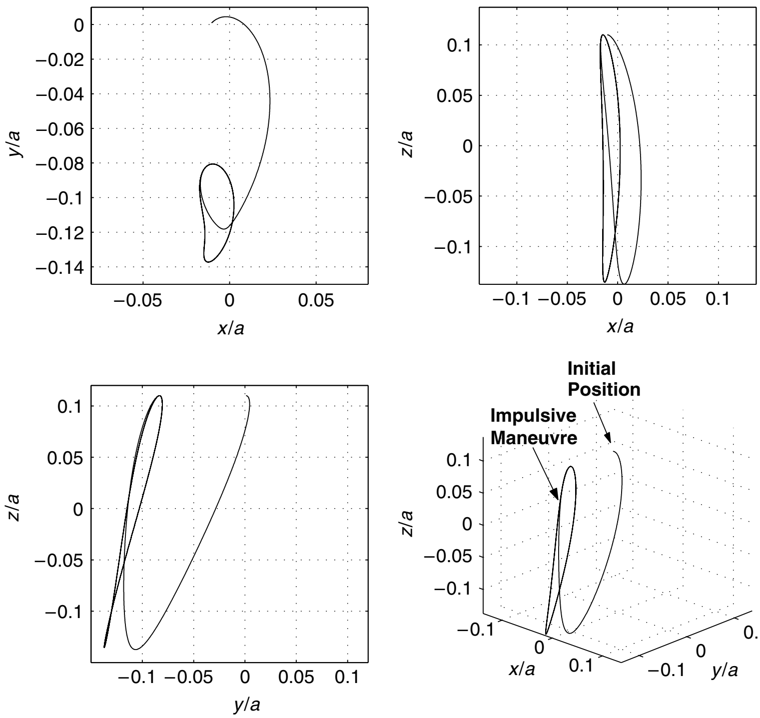

FIGURE 4.5 The relative motion in the configuration space exhibits bounded orbits following the formation-keeping maneuver.

energy of the deputy in normalized units is $\bar{\mathcal{E}}_1 = -0.496$, which differs from the total energy of the chief, $\bar{\mathcal{E}}_0 = -0.5$. The impulsive maneuver then decreases the total energy of the deputy by 0.004, matching it to the chief's energy and establishing a 1:1 bounded motion.

4.4 ANOTHER OUTLOOK ON OPTIMAL FORMATION-KEEPING

In the previous section, we developed an optimal formation-keeping maneuver for the deputy spacecraft using an impulsive velocity correction, $[\Delta\mathbf{v}]_{\mathscr{L}}$, formulated in a chief-fixed Euler–Hill frame. This maneuver has straightforward meaning if formulated in a *deputy-fixed frame*. To see this, we utilize the GVE (2.108). Denoting $[u_r, u_\theta, u_h]^T$ as the deputy's thrust acceleration components in the radial, along-track and cross-track directions, respectively, in its own $\mathscr{L}$ frame, the variational equation for the semimajor axis of the deputy is

$$\dot{a}_1 = \frac{2a_1^2}{h_1}\left(u_r e_1 \sin f_1 + \frac{p_1}{r_1}u_\theta\right) \tag{4.62}$$

where $h_1 = \sqrt{\mu a_1(1 - e_1^2)}$ is the deputy's orbital angular momentum. An

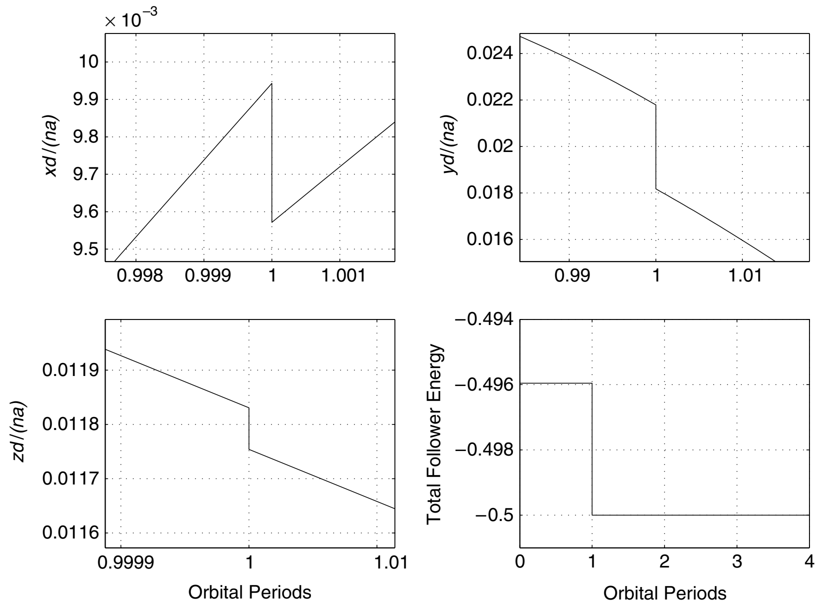

FIGURE 4.6 Magnification of the relative position components at the vicinity of the formation-keeping maneuver shows the applied velocity impulse, decreasing the total energy of the deputy to match the total energy of the chief (bottom right graph).

impulsive maneuver aimed at matching the semimajor axis of the deputy to the chief's semimajor axis can therefore be expressed as

$$\Delta a_1 = \frac{2a_1^2}{h_1}\left(\Delta v_r e_1 \sin f_1 + \frac{p_1}{r_1}\Delta v_\theta\right) \tag{4.63}$$

where $\Delta \mathbf{v}_1 = [\Delta v_r, \Delta v_\theta, \Delta v_h]^T$ is the impulsive velocity correction vector in a deputy-fixed $\mathscr{L}$ frame.

Let us find a minimum-energy impulsive maneuver using an optimization problem formulated similarly to Eq. (4.45),

$$\Delta \mathbf{v}_1^* = \arg \min_{\Delta v_r, \Delta v_\theta} \Delta v_r^2 + \Delta v_\theta^2 \tag{4.64}$$

s. t.

$$\Delta a_1 = \frac{2a_1^2}{h_1}\left(\Delta v_r e_1 \sin f_1 + \frac{p_1}{r_1}\Delta v_\theta\right) \tag{4.65}$$

Using a Lagrange multiplier λ, the Lagrangian becomes

$$\mathcal{L}_1 = \Delta v_r^2 + \Delta v_\theta^2 + \lambda_1\left[\frac{2a_1^2}{h_1}\left(\Delta v_r e \sin f_1 + \frac{p_1}{r_1}\Delta v_\theta\right) - \Delta a_1\right] \tag{4.66}$$

Note that $\mathcal{L}_1$ is time-dependent through the deputy's true anomaly, f_1. The necessary conditions for an optimum are therefore

$$\frac{\partial \mathcal{L}_1}{\partial(\Delta \mathbf{v}_1)} = 0,$$
$$\frac{\partial \mathcal{L}_1}{\partial f_1} = 0, \tag{4.67}$$

yielding

$$\Delta v_r + \frac{2\lambda_1 a_1^2 e_1 \sin f_1}{h_1} = 0 \tag{4.68}$$
$$\Delta v_\theta + \frac{\lambda_1 a_1^2 (1 + e_1 \cos f_1)}{h_1} = 0 \tag{4.69}$$
$$\lambda_1 a_1^2 (\Delta v_r e_1 \cos f_1 - e_1 \sin f_1 \Delta v_\theta) = 0 \tag{4.70}$$

Equations (4.68)–(4.70) together with constraint (4.65) constitute a system of four algebraic equations for the unknowns $\Delta v_r, \Delta v_\theta, \lambda_1, f_1$. There are two possible solutions:

$$\Delta v_r = 0,\ \Delta v_\theta = \frac{h_1 \Delta a_1}{2a_1^2(1+e_1)},\ \lambda_1 = -\frac{h_1^2 \Delta a_1}{2a_1^4(e_1+1)^2},\ f_1 = 0 \tag{4.71}$$
$$\Delta v_r = 0,\ \Delta v_\theta = \frac{h_1 \Delta a_1}{2a_1^2(1-e_1)},\ \lambda_1 = -\frac{h_1^2 \Delta a_1}{2a_1^4(e_1-1)^2},\ f_1 = \pi \tag{4.72}$$

Evaluation of the bordered Hessian shows that Eq. (4.71) provides the minimum solution. It implies that the optimal impulsive maneuver should be performed at periapsis ($f_1 = 0$):

$$\Delta \mathbf{v}_1^* = [0, \Delta v_\theta(f_1 = 0), 0]^T \tag{4.73}$$

Hence, we have obtained an indirect indication of the optimal maneuver initiation time, t_i, discussed in the preceding section. The magnitude of the optimal maneuver equals the magnitude of Δv_θ.

Note that the magnitude of the velocity correction in the deputy's frame should be equal to the magnitude of the velocity correction as viewed in the chief's frame, as magnitude of vectors are invariant to a particular selection of reference frames.

4.5 CIRCULAR CHIEF ORBIT

In Section 4.1 we derived the general nonlinear equations of relative motion for arbitrary chief orbits.[1] A simpler, autonomous, form of the relative motion

[1] Generally, the chief's orbit can be hyperbolic, and not necessarily elliptic; but this is of little practical value.

equations can be derived, however, if we assume that the chief follows a circular orbit. In many practical cases this is a realistic assumption.

In the circular chief orbit case, $\dot{\theta}_0 = n_0 = \text{const.}$, $\ddot{\theta}_0 = 0$ and $r_0 = a_0 =$ const. Substituting into Eqs. (4.14)–(4.16) results in

$$\ddot{x} - 2n_0\dot{y} - n_0^2 x = -\frac{\mu(a_0 + x)}{\left[(a_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} + \frac{\mu}{a_0^2} \tag{4.74}$$

$$\ddot{y} + 2n_0\dot{x} - n_0^2 y = -\frac{\mu y}{\left[(a_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} \tag{4.75}$$

$$\ddot{z} = -\frac{\mu z}{\left[(a_0 + x)^2 + y^2 + z^2\right]^{\frac{3}{2}}} \tag{4.76}$$

These equations admit an *equilibria continuum* ($\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0$) given by

$$z = 0, \, (x + a_0)^2 + y^2 = a_0^2 \tag{4.77}$$

It can be straightforwardly shown that, not unexpectedly, the equilibria (4.77) conform to the energy matching condition.

Equation (4.77) defines a circle that coincides with the chief's orbit: It is centered at ($x = -a_0$, $y = 0$), which are the coordinates of the primary in frame $\mathscr{L}$. This result reflects the trivial physical observation that the deputy spacecraft will appear stationary in a chief-fixed frame if the deputy is co-located on the circular orbit of the chief. This type of in-line relative motion is referred to as *co-orbital motion*. From the dynamical systems perspective, we expect that there exist small perturbations near equilibria that will generate periodic orbits about the equilibria. The resulting periodic motion is called *libration*. Libration orbits can be found from the energy matching condition and hence can be viewed as a subset of the 1:1 commensurability periodic orbits.

A similar terminology is used for describing the relative motion between celestial bodies. The dynamic structure of the problem in this case is much more involved, as it includes the mutual gravitation of the bodies [93].

Equations (4.74)–(4.76) can be normalized, so that the distances are measured in units of a_0 and the angular velocities are written in units of $\sqrt{\mu/a_0^3}$. In this case, Eq. (4.77) defines a *unit circle*.[2] A libration about a point moving in a circular orbit is a form of *epicyclic motion*.[3] We will also discuss epicyclic motion in Chapter 6. To illustrate epicyclic motion, we consider the following example.

---

[2] In this context, we should also mention that the triangular equilibrium points in the restricted three body problem, known as Lagrange's points L_4 and L_5, are also equilibria associated to motion in the vicinity of the unit circle.

[3] In Ptolemaic astronomy, a small circle, the center of which moves on the circumference of a larger circle at whose center is the Earth and the circumference of which describes the orbit of one of the planets around Earth.

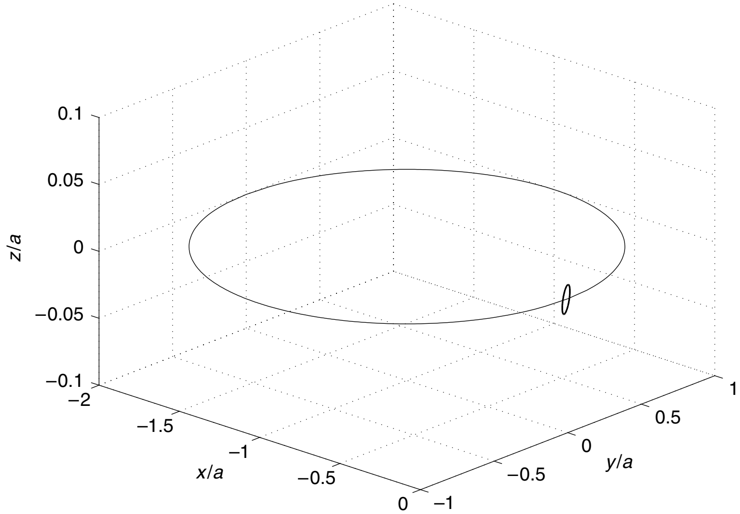

FIGURE 4.7 Libration orbits about a unit circle constitute epicyclic motion.

Example 4.3. *Simulate the libration orbits obtained for the normalized initial conditions*

$$\bar{x}(0) = -0.005019, \bar{y}(0) = 0.01, \bar{z}(0) = 0.01$$
$$\bar{x}'(0) = 0.01, \bar{y}'(0) = 0.01, \bar{z}'(0) = 0 \tag{4.78}$$

The energy matching condition with a circular chief orbit, using normalized units, simplifies into (cf. Eq. (4.34)):

$$\begin{aligned}\bar{\mathcal{E}}_1 &= \frac{1}{2}\{(\bar{x}' - \bar{y})^2 + [\bar{y}' + (\bar{x} + 1)]^2 + (\bar{z}')^2\} - \frac{1}{\sqrt{(1+\bar{x})^2 + \bar{y}^2 + \bar{z}^2}} \\ &= -\frac{1}{2}\end{aligned} \tag{4.79}$$

It can be readily verified that initial conditions (4.78) satisfy this condition. Utilizing these initial conditions, we simulate Eqs. (4.74)–(4.76) for a single orbital period. The results of the simulation are depicted in Fig. 4.7, showing the libration orbit (thick line) with respect to the unit circle (thin line). As previously explained, the unit circle constitutes a continuum of relative equilibria representing the chief's orbit. A libration orbit represents a small oscillation about each fixed point.

4.6 LAGRANGIAN AND HAMILTONIAN DERIVATIONS

We will now show how to obtain the relative motion equations using Lagrangian and Hamiltonian formalisms, following a treatment similar to the one proposed by Kasdin *et al.* [94]. This will be used as a stepping stone for the derivation of linear equations of motion in the next chapter, and for incorporating perturbations (Chapter 8). For simplicity, we treat the case of a relative motion with respect to a circular reference orbit. We treat this case because of its simplicity, allowing us to focus attention on the details of the method.

The first step is to develop the Lagrangian of relative motion in the rotating frame $\mathscr{L}$. In the circular reference orbit case, the velocity of the deputy in $\mathscr{L}$ is derived from Eq. (4.30):

$$\mathbf{v}_1 = {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times \mathbf{r}_0 + \frac{d^{\mathscr{L}}}{dt}\boldsymbol{\rho} + {}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} \times \boldsymbol{\rho} \tag{4.80}$$

where, as before, $\mathbf{r}_0 \in \mathbb{R}^3$ is the inertial position vector of the chief spacecraft along the reference orbit, $\boldsymbol{\rho} = [x, y, z]^T \in \mathbb{R}^3$ is the relative position vector in the rotating frame, and ${}^{\mathscr{I}}\boldsymbol{\omega}^{\mathscr{L}} = [0, 0, n]^T$ is the angular velocity of the rotating frame $\mathscr{L}$ with respect to the inertial frame $\mathscr{I}$. Assuming a circular reference orbit, denoting $\|\mathbf{r}_0\| = a$ and substituting into Eq. (4.80), we can write the velocity in a component-wise notation:

$$\mathbf{v}_1 = \begin{bmatrix} \dot{x} - n_0 y \\ \dot{y} + n_0 x + n_0 a_0 \\ \dot{z} \end{bmatrix} \tag{4.81}$$

The kinetic energy per unit mass is given by

$$\mathcal{K} = \frac{1}{2}\|\mathbf{v}_1\|^2 \tag{4.82}$$

The potential energy (for a spherical attracting body) of the deputy, whose position vector is $\mathbf{r}_1$, is the usual gravitational potential written in terms of $\rho = \|\boldsymbol{\rho}\|$:

$$\mathcal{U} = -\frac{\mu}{\|\mathbf{r}_1\|} = -\frac{\mu}{\|\mathbf{r}_0 + \boldsymbol{\rho}\|} = -\frac{\mu}{a_0\left[1 + 2\frac{\mathbf{r}_0 \cdot \boldsymbol{\rho}}{a_0^2} + \left(\frac{\rho}{a_0}\right)^2\right]^{1/2}} \tag{4.83}$$

The last term in Eq. (4.83) can be expanded by using Legendre polynomials, P_k, which we introduced in Subsection 2.5.3:

$$\mathcal{U} = -\frac{\mu}{a_0}\sum_{k=0}^{\infty} P_k(\cos\vartheta)\left(\frac{\rho}{a_0}\right)^k \tag{4.84}$$

where ϑ is the angle between the reference orbit radius vector and the relative position vector (Fig. 4.1), so that

$$\cos\vartheta = -\frac{\boldsymbol{\rho} \cdot \mathbf{r}_0}{a_0\rho} = \frac{-x}{\sqrt{x^2 + y^2 + z^2}} \tag{4.85}$$

As per the definition appearing in Eq. (3.1), the Lagrangian $\mathcal{L} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is now found by subtracting the potential energy from the kinetic energy:

$$\mathcal{L} = \frac{1}{2}\left\{(\dot{x} - n_0 y)^2 + (\dot{y} + n_0 x + n_0 a_0)^2 + \dot{z}^2\right\} + n_0^2 a_0^2 \sum_{k=0}^{\infty} P_k(\cos\vartheta)\left(\frac{\rho}{a_0}\right)^k \tag{4.86}$$

The equations of relative motion, Eqs. (4.14)–(4.16), can now be obtained by applying the Euler–Lagrange equations (3.3). However, due to the use of the Legendre polynomials, one can obtain arbitrary-order approximations to the full nonlinear equations of relative motion by truncating the Legendre polynomials at some desired degree. This exercise is performed in Section 5.3.

We can now proceed with the Hamiltonian formalism. Finding the Hamiltonian for the Cartesian system is straightforward. First, the canonical momenta are found from the definition (3.4):

$$\begin{aligned} p_x &= \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} - n_0 y \\ p_y &= \frac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{y} + n_0 x + n_0 a_0 \\ p_z &= \frac{\partial \mathcal{L}}{\partial \dot{z}} = \dot{z} \end{aligned} \tag{4.87}$$

and then, using the Legendre transformation $\mathcal{H} = \mathbf{p}^T \dot{\mathbf{q}} - \mathcal{L}$, as given in Eq. (3.5), the Hamiltonian for relative motion is found:

$$\mathcal{H} = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + n_0 y p_x - (n_0 x + n_0 a_0) p_y - n_0^2 a_0^2 \sum_{k=0}^{\infty} P_k(\cos\vartheta)\left(\frac{\rho}{a_0}\right)^k \tag{4.88}$$

It is worthwhile to note that the Hamiltonian is a constant of motion. We will use expressions (4.86) and (4.88) in Chapter 5 to derive linear equations of relative motion. A Hamiltonian description of relative motion between satellites in similar Keplerian orbits, satisfying the principles of conservation of angular momentum and the Hamiltonian, has been provided by Palmer and İmre [95], who derive their results in a perigee-fixed coordinate system of the chief, not the $\mathscr{L}$ frame.

4.7 EQUATIONS OF RELATIVE MOTION UNDER THE INFLUENCE OF J_2

The equations of motion of a satellite under the influence of gravitational and thrust perturbations can be written in frame $\mathscr{I}$ as (see our discussion of orbital perturbations in Section 2.5)

$$\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}}\mathcal{V} + \mathbf{u} \tag{4.89}$$

where $\mathbf{r}$ is the position vector, $\mathcal{V} = \mathcal{U} - \mathcal{R}$ is the *total gravitational potential* (the nominal potential plus the J_2 potential), and $\nabla_{\mathbf{r}}\mathcal{V}$ is its gradient. The thrust acceleration is denoted by $\mathbf{u}$. The gravitational potential, including the contribution of J_2, is given by (see Eq. (2.98) and Ref. [96]):

$$\mathcal{V} = -\frac{\mu}{r}\left\{1 - \frac{J_2}{2}\frac{{R_e}^2}{r^2}\left[\frac{3}{r^2}(\mathbf{r}\cdot\hat{\mathbf{K}})^2 - 1\right]\right\} \tag{4.90}$$

where r is the magnitude of $\mathbf{r}$ and $\hat{\mathbf{K}}$ is the unit vector along the polar axis of the $\mathscr{I}$ frame; it can be expressed in any desired frame of reference, $\mathscr{I}$ or $\mathscr{L}$. In the $\mathscr{L}$ frame we have:

$$\hat{\mathbf{K}} = \sin\theta_0 \sin i_0 \hat{\mathbf{i}} + \cos\theta_0 \sin i_0 \hat{\mathbf{j}} + \cos i_0 \hat{\mathbf{k}} \tag{4.91}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, are respectively, the unit vectors along the radial, along-track, and cross-track directions of the $\mathscr{L}$ frame of the satellite. Note that the term $(\mathbf{r}\cdot\hat{\mathbf{K}})$ in Eq. (4.90) is nothing but Z, the polar component of the satellite's position vector, $\mathbf{r}$. The gravitational acceleration, $\mathbf{F}_g$, obtained from Eq. (4.90) is

$$\begin{aligned}\mathbf{F}_g = -\nabla_{\mathbf{r}}\mathcal{V} = -\frac{\mu}{r^3}\mathbf{r} - \frac{1}{2r^5}\mu J_2 {R_e}^2 \Big\{ & 6(\mathbf{r}\cdot\hat{\mathbf{K}})\hat{\mathbf{K}} \\ & + \left[3 - \frac{15}{r^2}(\mathbf{r}\cdot\hat{\mathbf{K}})^2\right]\mathbf{r}\Big\}\end{aligned} \tag{4.92}$$

Equations (4.91) and (4.92) can be substituted into Eq. (4.89) to obtain the equations of motion. A very convenient $\mathscr{I}$ frame representation of the perturbed motion, including the effects of $J_2 - J_6$, can be found in Ref. [44]; see also our discussion of zonal perturbations in Subsection 2.5.3. These equations are presented below for the special case of J_2-perturbed motion:

$$\ddot{X} = -\frac{\mu X}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2\left(5\frac{Z^2}{r^2} - 1\right)\right] \tag{4.93a}$$

$$\ddot{Y} = -\frac{\mu Y}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2\left(5\frac{Z^2}{r^2} - 1\right)\right] \tag{4.93b}$$

$$\ddot{Z} = -\frac{\mu Z}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{R_e}{r}\right)^2\left(5\frac{Z^2}{r^2} - 3\right)\right] \tag{4.93c}$$

where X,Y, and Z are the components of $\mathbf{r}$ expressed in the $\mathscr{I}$ frame. We note that the Hamiltonian, which, in this case, is identical to the total energy, is a constant of motion,

$$\mathcal{H} = \mathcal{K} + \mathcal{V} = \frac{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}{2} - \frac{\mu}{r}\left\{1 - \frac{J_2}{2}\frac{R_e{}^2}{r^2}\left[\frac{3Z^2}{r^2} - 1\right]\right\}$$
$$= \text{const.} \tag{4.94}$$

and so is the polar component of the angular momentum,[4]

$$\mathbf{h} \cdot \hat{\mathbf{K}} = X\dot{Y} - Y\dot{X} = \text{const.} \tag{4.95}$$

The relative displacement and velocity vectors, expressed in the $\mathscr{I}$ frame, are defined as

$$\delta \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0 \tag{4.96a}$$
$$\delta \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_0 \tag{4.96b}$$

where $\mathbf{v}$ denotes the inertial velocity vector of a satellite. We remind the reader that variables relating to the chief are indicated with the subscript 0 and those connected with the deputy are denoted by the subscript 1.

A nonlinear simulation for multiple satellites is carried out by numerically integrating copies of Eqs. (4.93a)–(4.93c), one for each satellite, with different sets of initial conditions. The results thus obtained can be transformed into the relative motion states given by Eqs. (4.96a)–(4.96b). For many applications, the relative motion states are typically expressed in the $\mathscr{L}$ frame of the chief. Transformation of the results of Eqs. (4.96a)–(4.96b) into the relative motion variables in the $\mathscr{L}$ frame is discussed next.

4.7.1 Relative motion states in the $\mathscr{L}$ frame

The relative motion between two satellites in general elliptic orbits, written using a slightly different representation compared to previous chapters, can be expressed in the chief's $\mathscr{L}$ frame [99]:

$$x = \frac{\delta \mathbf{r}^T \mathbf{r}_0}{r_0} \tag{4.97a}$$
$$y = \frac{\delta \mathbf{r}^T (\mathbf{h}_0 \times \mathbf{r}_0)}{\|\mathbf{h}_0 \times \mathbf{r}_0\|} \tag{4.97b}$$
$$z = \frac{\delta \mathbf{r}^T \mathbf{h}_0}{h_0} \tag{4.97c}$$

where, as before, x, y, and z indicate, respectively, the radial, along-track,

[4]The total energy and the polar component of the angular momentum are the *only* two integrals of the J_2-perturbed problem [97,98]. Since there are three degrees-of-freedom and only two integrals, the J_2-perturbed problem constitutes a *non-integrable Hamiltonian system*, implying that there are phase-space regions in which chaotic motion is likely to exist. However, these regions are extremely small for actual Earth orbits.

and cross-track displacements; and h_0 is the magnitude of $\mathbf{h}_0$, the angular momentum vector of the chief.

The relative velocities in the $\mathscr{L}$ frame can be obtained by differentiating the expressions (4.97a)–(4.97c) with respect to time:

$$\dot{x} = \frac{\delta\mathbf{v}^T\mathbf{r}_0 + \delta\mathbf{r}^T\mathbf{v}_0}{r_0} - \frac{(\delta\mathbf{r}^T\mathbf{r}_0)(\delta\mathbf{r}_0^T\mathbf{v}_0)}{r_0^3} \tag{4.98a}$$

$$\begin{aligned}\dot{y} = &\frac{\delta\mathbf{v}^T(\mathbf{h}_0 \times \mathbf{r}_0) + \delta\mathbf{r}^T(\dot{\mathbf{h}}_0 \times \mathbf{r}_0 + \mathbf{h}_0 \times \mathbf{v}_0)}{\|\mathbf{h}_0 \times \mathbf{r}_0\|} \\ &- \frac{\delta\mathbf{r}^T(\mathbf{h}_0 \times \mathbf{r}_0)(\mathbf{h}_0 \times \mathbf{r}_0)^T(\dot{\mathbf{h}}_0 \times \mathbf{r}_0 + \mathbf{h}_0 \times \mathbf{v}_0)}{\|\mathbf{h}_0 \times \mathbf{r}_0\|^3}\end{aligned} \tag{4.98b}$$

$$\dot{z} = \frac{\delta\mathbf{v}^T\mathbf{h}_0 + \delta\mathbf{r}^T\dot{\mathbf{h}}_0}{h_0} - \frac{\delta\mathbf{r}^T\mathbf{h}_0(\mathbf{h}_0^T\dot{\mathbf{h}}_0)}{h_0^3} \tag{4.98c}$$

Notice that the relative velocities as computed from Eqs. (4.98a)–(4.98c) depend on $\dot{\mathbf{h}}_0 = \mathbf{r}_0 \times \dot{\mathbf{v}}_0$. Hence, the inertial acceleration vector of the chief, including the contributions due to the perturbations acting upon it, is required in the above transformation. However, this requirement is not problematic if the motion of the chief (real or virtual) is assumed to be known accurately.

4.7.2 Coordinate transformation between the $\mathscr{L}$ and $\mathscr{I}$ frames

Many applications involving formation control may require that a vector be transformed from $\mathscr{L}$ to $\mathscr{I}$ or vice versa. Given the $\mathscr{I}$ frame representation of the position and velocity vectors of the chief, the transformation from $\mathscr{L}$ to $\mathscr{I}$ is given by

$$T_{\mathscr{L}}^{\mathscr{I}} = \left[\hat{\mathbf{r}}_0 \;\; (\hat{\mathbf{h}}_0 \times \mathbf{r}_0) \;\; \hat{\mathbf{h}}_0\right] \tag{4.99}$$

where $(\hat{\cdot})$ represents a unit vector. The inverse transformation can easily be obtained by using the matrix transpose operation; this leads to the expression

$$T_{\mathscr{I}}^{\mathscr{L}}(\Omega, i, \theta) = \begin{bmatrix} \mathrm{c}_\Omega \mathrm{c}_\omega - \mathrm{s}_\Omega \mathrm{s}_\theta \mathrm{c}_i & \mathrm{s}_\Omega \mathrm{c}_\theta + \mathrm{c}_\Omega \mathrm{s}_\theta \mathrm{c}_i & \mathrm{s}_\theta \mathrm{s}_i \\ -\mathrm{c}_\Omega \mathrm{s}_\theta - \mathrm{s}_\Omega \mathrm{c}_\theta \mathrm{c}_i & -\mathrm{s}_\Omega \mathrm{s}_\theta + \mathrm{c}_\Omega \mathrm{c}_\theta \mathrm{c}_i & \mathrm{c}_\theta \mathrm{s}_i \\ \mathrm{s}_\Omega \mathrm{s}_i & -\mathrm{c}_\Omega \mathrm{s}_i & \mathrm{c}_i \end{bmatrix} \tag{4.100}$$

where $\theta = \omega + f$ (cf. also our development in Chapter 2).

4.7.3 Initial conditions

It is convenient for numerical simulations to specify the initial conditions of the satellites in terms of their mean orbital elements. These initial mean elements can be transformed into the respective osculating elements via the use of Brouwer theory [66] (for the classical elements) or the procedure of Gim and Alfriend [75,100], for nonsingular and equinoctial elements. The osculating elements for each satellite can be transformed into the $\mathscr{I}$ frame position and velocity coordinates using the principles of orbital mechanics detailed in Chapter 2. The initial conditions thus obtained, for each satellite, are applicable to Eqs. (4.93a)–(4.93c). A discussion of the process of mean-to-osculating orbit element conversion is presented in Section 3.4.

SUMMARY

This chapter presented a variety of models for simulating relative motion in perturbed or unperturbed elliptic orbits. In the absence of perturbations, bounded relative motion between any two spacecraft in elliptic Keplerian orbits can be straightforwardly found from the energy matching condition. By dealing with the full nonlinear problem, global conditions for bounded motion can be found which naturally accommodate the nonlinearity of the Keplerian relative motion equations as well as the reference orbit eccentricity.

Orbital commensurability, guaranteeing bounded relative motion, can be re-established in spite of initialization errors by applying a single thrust impulse, which can be calculated and optimized using relative state variables and has a simple interpretation in terms of the deputy's orbital elements. This single-impulse maneuver also has considerable operational advantages over multiple impulses.

If the chief's orbit is circular, commensurability conditions can be considerably simplified. The relative motion in this case has a simple geometric interpretation: Epicyclic motion is generated if orbital commensurability is violated, and librations are obtained if the motions are commensurable.

A set of useful transformations for converting the relative motion states between the $\mathscr{L}$ and $\mathscr{I}$ frames was introduced. These transformations have been applied in many instances in the chapters to follow. The direct derivation of the perturbed equations of motion in the $\mathscr{L}$ frame is treated in the next chapter.