Chapter | two

Fundamental Astrodynamics

'Twas noontide of summer,
And midtime of night,
And stars, in their orbits,
Shone pale, through the light.

Edgar Allan Poe (1809–1849)

In this chapter, we review some notions in orbital mechanics, assuming that the reader has some previous knowledge of astrodynamics; a good source for acquiring such knowledge is, e.g., Ref. [44]. We will briefly explain the main coordinate systems used in the following chapters, and discuss the Keplerian *two-body problem*. We start by recalling the following terminology:

- *Orbital plane* – Newton's inverse-square law of gravitation implies that in the absence of any additional internal and/or external forces, two perfectly-spherical masses, whose shape remains unchanged under mutual gravitation, will perform planar motion. The plane of motion is the *orbital plane*. This plane contains the position and velocity vectors of the orbiting bodies. Planar motion can also be obtained in the presence of some perturbations.
- *Ecliptic plane* – A plane containing the mean orbit of the Earth around the Sun.
- *Periapsis* – When viewed from one of the gravitating masses, the *primary*, the point on the orbit of an orbiting body closest to the primary is the periapsis.
- *Orbital angular momentum* – A vector quantity related to the rotation of an orbiting body about a gravitating primary, defined as the cross product of the position and linear angular momentum vectors.
- *Vernal equinox* – The date when night and day are nearly the same length, and Sun crosses the celestial equator moving northward. The vernal equinox marks the first day of spring, and is used as the reference line for inertial measurements (with a proper fixture of some

Spacecraft Formation Flying; ISBN: 9780750685337

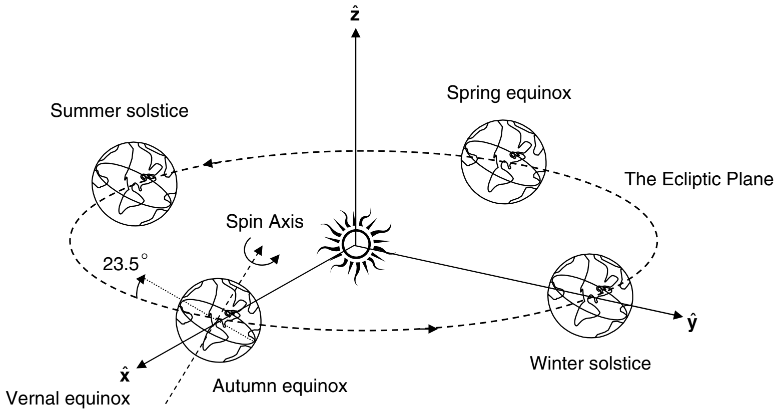

FIGURE 2.1 The heliocentric-ecliptic inertial coordinate system. Seasons refer to the northern hemisphere. Also shown is the vernal equinox vector, used as the inertial $\hat{\mathbf{x}}$-axis.

reference date); it is usually denoted by the symbol ϒ, since the vernal equinox once pointed to the constellation Aries. For example, the vernal equinox of 2009 occurred on March 20, 11 hr 44 m (Universal Time).

2.1 COORDINATE SYSTEMS

Problems that involve *kinematics*, or *rates of change*, of physical quantities, require a definition of *reference frames*, giving rise to *coordinate systems*, which the rates can be referred to.

The study of problems in orbital mechanics usually requires the definition of a few coordinate systems. The most common coordinate systems are:

- $\mathscr{I}$, a Cartesian, rectangular, dextral (CRD) *inertial coordinate system*, centered at the gravitational body (primary).
  - A *heliocentric* system is centered at the Sun, the fundamental plane is the ecliptic plane, the unit vector $\hat{\mathbf{x}}$ is directed from the Sun's center along the vernal equinox, $\hat{\mathbf{z}}$ is normal to the fundamental plane, positive in the direction of the celestial north, and $\hat{\mathbf{y}}$ completes the setup (see Fig. 2.1).
  - A *geocentric* system is centered at the Earth, the fundamental plane is the equator, the unit vector $\hat{\mathbf{x}}$ is directed from the Earth's center along the vernal equinox, $\hat{\mathbf{z}}$ is normal to the fundamental plane, positive in the direction of the geographic north pole, and $\hat{\mathbf{y}}$ completes the setup (see Fig. 2.2). This reference frame is usually referred to as *Earth-centered inertial* (ECI).

  An inertial system is used to define a satellite's position and velocity vectors, denoted by $\mathbf{r}$ and $\mathbf{v}$, respectively, as well as the right ascension, α_r, and the declination, δ_d (see Fig. 2.2). The angle α_r is measured from the vernal equinox to the projection of $\mathbf{r}$ onto the

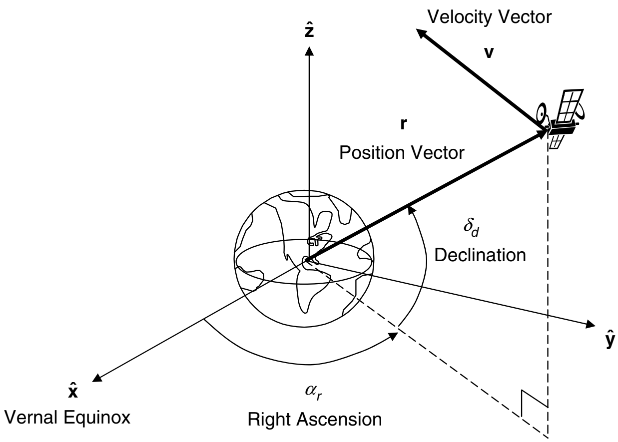

FIGURE 2.2 The Earth-centered inertial (ECI) coordinate system. Also shown are $\mathbf{r}$ and $\mathbf{v}$, the position and velocity vectors, respectively, as well as the right ascension, α_r, and the declination, δ_d.

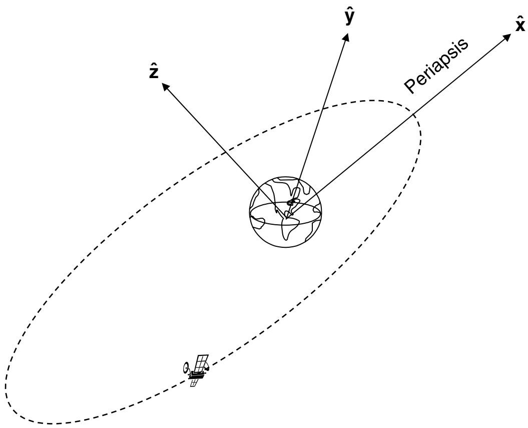

FIGURE 2.3 A perifocal coordinate system. The $\hat{\mathbf{x}}$-axis points to the (instantaneous) periapsis.

equatorial plane, whereas δ_r is measured from the same projection to $\mathbf{r}$.

- $\mathscr{P}$, a CRD *perifocal* coordinate system, centered at the primary. The fundamental plane is the (instantaneous) orbital plane. The unit vector $\hat{\mathbf{x}}$ is directed from the primary's center to the (instantaneous) periapsis, $\hat{\mathbf{z}}$ is normal to the fundamental plane, positive in the direction of the (instantaneous) orbital angular momentum vector, and $\hat{\mathbf{y}}$ completes the setup (see Fig. 2.3). Note that in the presence of orbital perturbations, this coordinate system is non-inertial.
- $\mathscr{F}$, a CRD, *Earth-centered Earth-fixed* (ECEF) rotating coordinate system, centered at the Earth. The unit vector $\hat{\mathbf{z}}$ is parallel to the

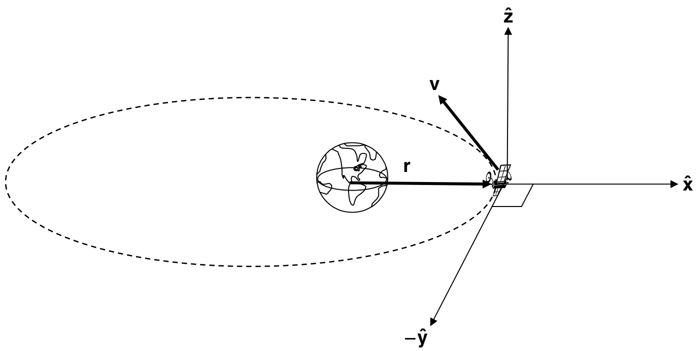

FIGURE 2.4 Local-vertical, local-horizon rotating coordinate system, centered at the spacecraft. The unit vector $\hat{\mathbf{x}}$ is directed from the spacecraft radially outward.

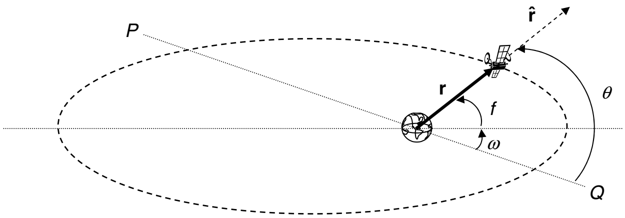

FIGURE 2.5 A polar coordinate system centered at the primary. The unit vector $\hat{\mathbf{r}}$ points radially outward. The angle θ is measured counterclockwise from some reference line, PQ, to the instantaneous radius-vector, $\mathbf{r}$. Also shown are the angles f and ω.

Earth's geographic north, $\hat{\mathbf{x}}$ intersects the Earth's sphere at the 0° latitude, 0° longitude, and $\hat{\mathbf{y}}$ completes the setup.

- $\mathscr{L}$, a CRD *local-vertical, local-horizontal* (LVLH) rotating coordinate system, centered at the spacecraft. The fundamental plane is the orbital plane. The unit vector $\hat{\mathbf{x}}$ is directed from the spacecraft radially outward, $\hat{\mathbf{z}}$ is normal to the fundamental plane, positive in the direction of the (instantaneous) angular momentum vector, and $\hat{\mathbf{y}}$ completes the setup (see Fig. 2.4).
  Occasionally, $\mathscr{L}$ is also called an RSW frame. This name is used in analysis of orbital perturbations in a frame co-rotating with the satellite. In this context, the unit vector $\hat{\mathbf{R}}$ is directed radially outwards, $\hat{\mathbf{S}}$ is perpendicular to $\hat{\mathbf{R}}$ in the direction of the instantaneous velocity, and $\hat{\mathbf{W}}$ completes the right-hand triad.
- $\mathscr{R}$, a *polar rotating coordinate system*, centered at the primary. The fundamental plane is the orbital plane. The unit vector $\hat{\mathbf{r}}$ is directed from the primary radially outward, and the angle θ is measured in the counterclockwise direction from some reference line, PQ, to $\mathbf{r}$ (see Fig. 2.5).

2.2 THE KEPLERIAN TWO-BODY PROBLEM

The equations of motion of the Keplerian two-body problem are obtained under the following assumptions:

- There are no external or internal forces except gravity.
- The gravitating bodies are spherical.
- There are no tidal forces.
- The primary's mass is much larger than the orbiting body's mass.
- The gravitational force is Newtonian.

Under these assumptions, the Keplerian two-body equations of motion can be written as follows:[1]

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = \mathbf{0} \tag{2.1}$$

where $\mathbf{r} = [X, Y, Z]^T$ is the position vector in the ECI frame $\mathscr{I}$, μ is the gravitational constant, and $r = \|\mathbf{r}\|$. In order to solve Eq. (2.1), it is useful to transform the inertial equations of motion into polar coordinates, $\mathscr{R}$. To that end, we recall that

$$\mathbf{r} = r\hat{\mathbf{r}} \tag{2.2}$$

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \tag{2.3}$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} \tag{2.4}$$

where θ is the *argument of latitude*. Substituting Eq. (2.1) into Eq. (2.4) yields the equations of motion in $\mathscr{R}$:

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} \tag{2.5}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} \tag{2.6}$$

It is immediately seen from Eq. (2.6) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = r(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{2.7}$$

Let us show that $r^2\dot{\theta}$ is the magnitude of the orbital angular momentum vector per unit mass, $\mathbf{h}$:

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{v} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{bmatrix} = r^2\dot{\theta}\hat{\mathbf{z}} = h\hat{\mathbf{z}} \tag{2.8}$$

where $\hat{\mathbf{z}}$ is a unit vector normal to the orbital plane.

[1] This version of the Keplerian two-body problem is sometimes referred to as the *restricted two-body problem*, implying that the particular shape factors of each body are neglected, and hence the inter-gravitation does not induce gravitational torques.

Hence, Eq. (2.7) implies *conservation of angular momentum*:

$$\mathbf{h} = \text{const.} \tag{2.9}$$

In fact, $\mathbf{h}$ is conserved as a vector quantity, meaning that not only is its magnitude, but also each of its components in the inertial space is constant. To see this, we can differentiate $\mathbf{h}$ to get

$$\dot{\mathbf{h}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{r} = \mathbf{0} \tag{2.10}$$

We now examine Eq. (2.5). Note that

$$\ddot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \dot{r}\frac{\mathrm{d}}{\mathrm{d}r}(\dot{r}) = d\left(\frac{\dot{r}^2}{2}\right) \tag{2.11}$$

Substituting into Eq. (2.5) yields

$$d\left(\frac{\dot{r}^2}{2}\right) = \left(\frac{h^2}{r^3} - \frac{\mu}{r^2}\right) \tag{2.12}$$

Integrating both sides of Eq. (2.12), we obtain

$$\mathcal{E} = \frac{\dot{r}^2}{2} + \frac{h^2}{2r^2} - \frac{\mu}{r} = \underbrace{\frac{\dot{r}^2}{2} + \frac{(r\dot{\theta}^2)}{2}}_{\text{kinetic energy}} \quad \underbrace{-\frac{\mu}{r}}_{\text{potential energy}} \quad = \text{const.} \tag{2.13}$$

where $\mathcal{E}$, the constant of integration, is the *total energy* per unit mass, viz. the *kinetic energy* plus the *potential energy*. Equation (2.13) thus implies *conservation of energy*. It is sometime written in the *vis-viva* [2] form

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} \tag{2.14}$$

where v is the magnitude of the velocity vector.

We are now able to solve the equations of motion in polar coordinates. Utilizing both constants of motion, we write

$$\dot{r} = \sqrt{2\left(\mathcal{E} + \frac{\mu}{r}\right) - \frac{h^2}{r^2}} \tag{2.15}$$

$$\dot{\theta} = \frac{h}{r^2} \tag{2.16}$$

Although Eqs. (2.15) and (2.16) appear with the time as the independent variable, it is more convenient to obtain a relationship of the form $r = r(\theta)$.

[2] Latin for "living force".

To that end, we divide Eq. (2.15) by Eq. (2.16) to obtain

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r^2\sqrt{2\left(\mathcal{E}+\mu/r\right)-h^2/r^2}}{h} \tag{2.17}$$

Equation (2.17) is a separable differential equation which can be solved by direct integration utilizing the initial condition $\theta_0 = \omega$, where ω, shown in Fig. 2.5, is the *argument of periapsis*:

$$\theta = \int \frac{h dr}{r^2\sqrt{2(\mathscr{E}+\mu/r)-h^2/r^2}} + \omega = \cos^{-1}\frac{1/r-\mu/h^2}{\sqrt{2\mathscr{E}/h+\mu^2/h^4}} + \omega \tag{2.18}$$

Solving for r yields

$$r = \frac{h^2/\mu}{1+\sqrt{1+2\mathcal{E}h^2/\mu^2}\cos(\theta-\omega)} \tag{2.19}$$

This is the well-known equation of *conic sections* in polar coordinates, and is sometimes referred to as the *conic equation*, as elaborated below. The orbits determined by Eq. (2.19) are called *Keplerian orbits*. An alternative formulation of Eq. (2.19) is

$$r = \frac{p}{1+e\cos f} \tag{2.20}$$

where

$$p = h^2/\mu \tag{2.21}$$

is called the *semilatus rectum* or *the parameter*,

$$e = \sqrt{1+2\mathcal{E}h^2/\mu^2} \tag{2.22}$$

is the *eccentricity*, and

$$f = \theta - \omega \tag{2.23}$$

is the *true anomaly*, shown in Fig. 2.5.

Equation (2.20) yields two families of non-degenerate orbits, shown in Fig. 2.6. By defining

$$a = \frac{p}{1-e^2} \tag{2.24}$$

these families can be categorized as follows:

- An *ellipse*, which is a *closed orbit* constituting the locus of all points whose sum of distances from two fixed points, called the *foci*, is

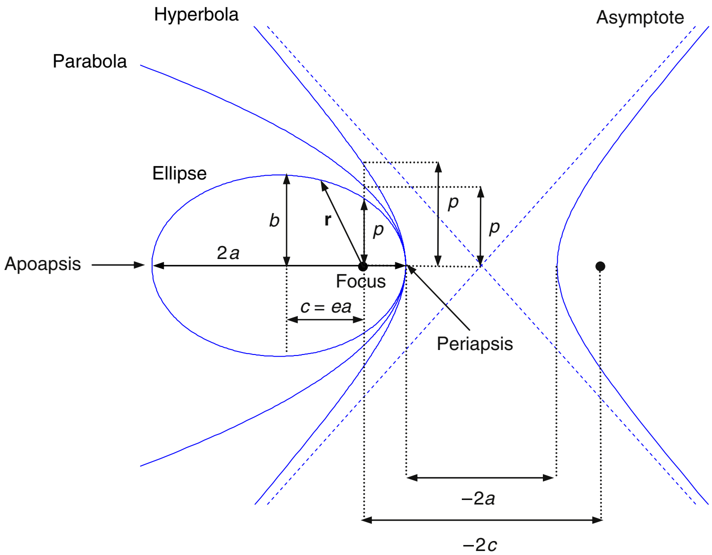

FIGURE 2.6 The conic sections: Ellipse, parabola and hyperbola.

constant. In this case, $a > 0$ is the *semimajor axis*, $0 < e < 1$, and $b = a\sqrt{1-e^2}$ is the *semiminor axis*. We distinguish between the following special cases of an ellipse:
– The case $e = 0$ is a closed orbit called a *circle*, and
– The case $e = 1$, $a = \infty$ is an *open orbit* called a *parabola*

- A *hyperbola*, which is an open orbit constituting the locus of all points whose difference of distances from two fixed points, the foci, is constant. In this case, the semimajor axis is negative, $a < 0$, and equals half the minimum distance between the two hyperbola branches. The eccentricity satisfies $e > 1$.

In orbital mechanics, the gravitating body is located at one of the foci and the additional focus is vacant. The closest point to the gravitating body is called periapsis (see above) and the farthest point is called *apoapsis*. The line connecting the periapsis and apoapsis is called the *line of apsides*. Open orbits do not posses an apoapsis.

The Keplerian orbits can be similarly categorized in terms of the *total orbital energy*, $\mathcal{E}$. Substitution of Eqs. (2.21), (2.22) and (2.24) into Eq. (2.13) yields

$$\mathcal{E} = -\frac{\mu}{2a} \tag{2.25}$$

Thus, for an ellipse $(0 < a < \infty)$ $\mathcal{E} < 0$, for a parabola $(a = \infty)$ $\mathcal{E} = 0$ and for a hyperbola $(-\infty < a < 0)$ $\mathcal{E} > 0$.

Expression (2.20) provides an elegant closed-form expression for the radius vector as a function of the true anomaly. However, it is sometimes required to express the position as a function of time. To that end, a relationship between the

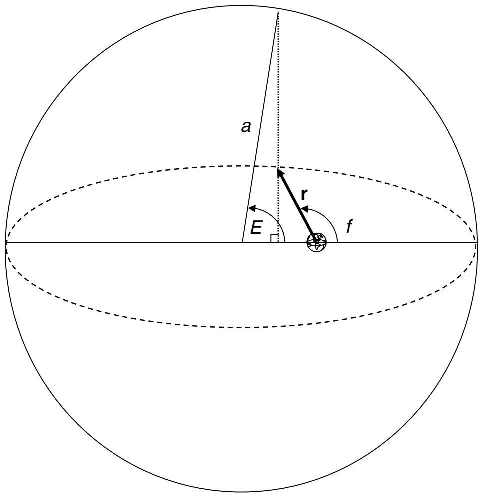

FIGURE 2.7 The eccentric anomaly. A bounding circle of radius a is plotted around the elliptic orbit. At a given point on the orbit, a line perpendicular to the ellipse major axis is plotted. Connecting the circle–line intersection with the center of the circle will yield a line whose angle with respect to the major axis is the eccentric anomaly, E.

true anomaly and time must be found. However, a closed-form relationship of the form $f = f(t)$ does not exist. Time is therefore introduced into the problem by using an auxiliary variable called the *eccentric anomaly*, $E(f)$, defined as the angle between the perifocal unit vector $\hat{\mathbf{x}}$ and the radius of a bounding circle at a point normal to the line of apsides at a given f, as depicted in Fig. 2.7. *Kepler's equation* states that

$$M = M_0 + n(t - t_0) = E - e \sin E \tag{2.26}$$

where $n = \sqrt{\mu/a^3}$ is the *mean motion*, M is the *mean anomaly*, t_0 is the *epoch* and M_0 is the *mean anomaly at epoch*. We note that due to Kepler's equation, the true anomaly is a function of a, e, t and M_0.

We say that the mean anomaly at epoch, M_0, is a *constant of motion* for the Keplerian two-body problem. It joins the other constants of motion: The total energy, $\mathcal{E}$, and the three components of the angular momentum in inertial space, $\mathbf{h}$; so we are one constant-of-motion short of solving the three-degrees-of-freedom equations (2.1). In fact, we can find three such constants of motion, by recalling that motion in a conservative field yields a constant vector called the *Laplace–Runge–Lenz vector*. In the Keplerian two-body problem, the Laplace–Runge–Lenz vector is usually referred to as the *eccentricity vector*. For $\mathbf{v} = \dot{\mathbf{r}}$, $v = \|\mathbf{v}\|$, this vector is defined as follows:

$$\mathbf{e} = \frac{v^2\mathbf{r}}{\mu} - \frac{(\mathbf{r} \cdot \mathbf{v})\mathbf{v}}{\mu} - \frac{\mathbf{r}}{r} \tag{2.27}$$

or, alternatively,

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r} \tag{2.28}$$

By taking the time derivative of (2.28), keeping in mind that $\mathbf{h} =$ const. and recalling that $\mathbf{r} \times (\mathbf{r} \times \mathbf{v}) = \mathbf{r}(\mathbf{r} \cdot \mathbf{v}) - \mathbf{v}(\mathbf{r} \cdot \mathbf{r})$, it can be shown that $\dot{\mathbf{e}} = \mathbf{0}$. Furthermore, $\mathbf{r} \cdot \mathbf{e} = r\, e \cos f$, and $e = \|\mathbf{e}\|$. Thus, the unit vector $\hat{\mathbf{e}} = \mathbf{e}/e$ points to the periapsis, and is therefore identical to the $\hat{\mathbf{x}}$-axis in our perifocal frame, $\mathscr{P}$.

To conclude this section, we will mention that the angular velocity along a Keplerian orbit is obtained by differentiating the true anomaly with respect to time. This operation yields [33]:

$$\dot{f} = \sqrt{\frac{\mu}{a^3(1-e^2)^3}}(1 + e\cos f)^2 \tag{2.29}$$

For Keplerian orbits $\omega =$ const. and hence

$$\dot{f} = \dot{\theta} \tag{2.30}$$

Based on Eq. (2.8), Eq. (2.30) implies that $h = r^2 \dot{f}$, or, equivalently, $dt = r^2/h df$. Letting the radius-vector of an elliptic orbit sweep an element area $dA = r^2 df/2$, we are convinced that $dt = 2dA/h$. Hence, if T stands for the orbital period and πab is the ellipse's area,

$$\int_0^T dt = \frac{2}{h}\int_0^{\pi ab} dA \tag{2.31}$$

which, upon substitution of $h = \sqrt{\mu a(1-e^2)}$ and $b = a\sqrt{1-e^2}$, provides us with an expression for the orbital period on elliptic (and circular) orbits:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{2.32}$$

Substituting from Eq. (2.25) yields an expression for the orbital period in terms of the total energy, $\mathcal{E}$:

$$T = \frac{\pi\mu}{\sqrt{2(-\mathcal{E})^3}} \tag{2.33}$$

2.3 SOLUTION OF THE INERTIAL EQUATIONS OF MOTION

We have obtained a solution to the differential equations of motion written in polar coordinates. Recall, however, that the original equations of motion (2.1) were formulated in inertial coordinates. In order to solve the inertial differential equations, it is customary to express the position vector in the perifocal coordinate system, $\mathscr{P}$:

$$[\mathbf{r}]_{\mathscr{P}} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} \tag{2.34}$$

where r is given in Eq. (2.20) with $p = a(1 - e^2)$. An alternative expression for the perifocal position vector can be written in terms of the eccentric anomaly, E [33]:

$$[\mathbf{r}]_{\mathscr{P}} = \begin{bmatrix} a(\cos E - e) \\ b\sin E \\ 0 \end{bmatrix} \tag{2.35}$$

This expression will be used in Chapter 6.

The velocity vector in the perifocal system, $\mathscr{P}$, can be calculated by writing

$$[\dot{\mathbf{r}}]_{\mathscr{P}} = \dot{f}\left[\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}f}\right]_{\mathscr{P}} = \sqrt{\frac{\mu}{a^3(1-e^2)^3}}(1 + e\cos f)^2\left[\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}f}\right]_{\mathscr{P}} \tag{2.36}$$

which yields

$$[\dot{\mathbf{r}}]_{\mathscr{P}} = \sqrt{\frac{\mu}{a(1-e^2)}}\begin{bmatrix} -\sin f \\ e + \cos f \\ 0 \end{bmatrix} \tag{2.37}$$

In order to obtain $\mathbf{r}$, the inertial position vector, and $\dot{\mathbf{r}}$, the inertial velocity vector, we need to find a rotation matrix from perifocal to inertial coordinates, $T_{\mathscr{P}}^{\mathscr{I}}$. Recall that frame $\mathscr{P}$ is defined by the triad $\hat{\mathbf{e}}$, pointing to the periapsis; $\hat{\mathbf{h}}$, coinciding with the angular momentum vector; and a unit vector $\hat{\mathbf{p}}$, which completes the right-hand system (this vector actually lies along the orbit's semilatus rectum). Both $\mathscr{P}$ and $\mathscr{I}$ are shown in Fig. 2.8.

Now, assume that the spacecraft is rotating counterclockwise in its orbit when viewed from the primary's north pole. If the orbital plane of the spacecraft and the fundamental plane of frame $\mathscr{I}$ (which coincides with the equatorial plane if the inertial system is geocentric) intersect, then we define two points of interest on the line of intersection, as shown in Fig. 2.8: The first is the *ascending node*, denoted by ☊. This point marks the spacecraft's location on the line of intersection when moving to the north; the second is the *descending node*, denoted by ☋. This point marks the spacecraft's location on the line of intersection when moving to the south. The line connecting ☋ to ☊ is called the *line of nodes* (LON); we will use the notation $\hat{\mathbf{l}}$ to denote a unit vector that lines along the LON.

We can now transform from $\mathscr{P}$ to $\mathscr{I}$ using three consecutive clockwise rotations by Euler angles conforming to the 3–1–3 sequence. The rotation

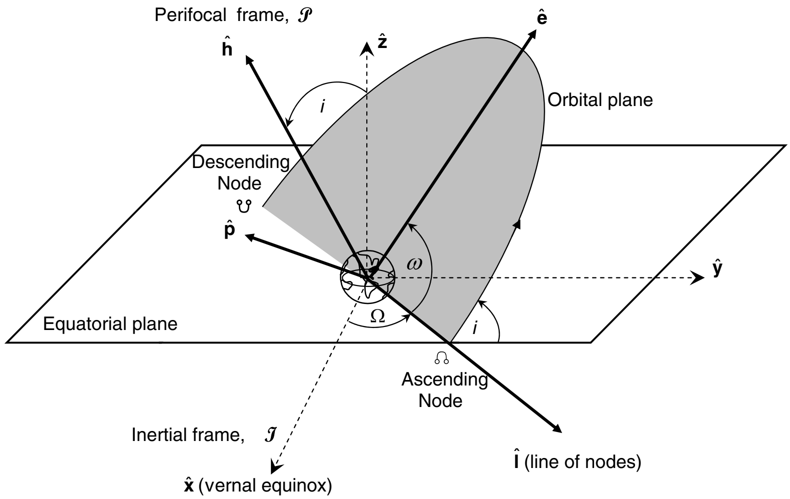

FIGURE 2.8 Definition of the classical orbital elements: The right ascension of the ascending node, Ω, the argument of periapsis, ω, and the inclination, i. These three Euler angles are used when transforming from a perifocal frame, $\mathscr{P}$, to an inertial frame, $\mathscr{I}$.

sequence is as follows:

- $T_1(\omega, \hat{\mathbf{h}})$, a rotation about $\hat{\mathbf{h}}$ by $0 \le \omega \le 2\pi$, mapping $\hat{\mathbf{e}}$ onto $\hat{\mathbf{l}}$.
- $T_2(i, \hat{\mathbf{l}})$, a rotation about $\hat{\mathbf{l}}$ by $0 \le i \le \pi$, mapping $\hat{\mathbf{h}}$ onto $\hat{\mathbf{z}}$.
- $T_3(\Omega, \hat{\mathbf{z}})$, a rotation about $\hat{\mathbf{z}}$ by $0 \le \Omega \le 2\pi$, mapping $\hat{\mathbf{l}}$ onto $\hat{\mathbf{x}}$.

The composite rotation, $T = T_3 \circ T_2 \circ T_1 \in \mathrm{SO}(3)$, transforming any vector in $\mathscr{P}$ into the inertial frame, is given by[3]

$$T_{\mathscr{P}}^{\mathscr{I}}(\omega, i, \Omega) = T_3(\Omega, \hat{\mathbf{z}}) T_2(i, \hat{\mathbf{l}}) T_1(\omega, \hat{\mathbf{h}}) \tag{2.38}$$

Evaluating Eq. (2.38) gives the directional cosines matrix (DCM)

$$T_{\mathscr{P}}^{\mathscr{I}}(\omega, i, \Omega) = \begin{bmatrix} \mathrm{c}_\Omega \mathrm{c}_\omega - \mathrm{s}_\Omega \mathrm{s}_\omega \mathrm{c}_i & -\mathrm{c}_\Omega \mathrm{s}_\omega - \mathrm{s}_\Omega \mathrm{c}_\omega \mathrm{c}_i & \mathrm{s}_\Omega \mathrm{s}_i \\ \mathrm{s}_\Omega \mathrm{c}_\omega + \mathrm{c}_\Omega \mathrm{s}_\omega \mathrm{c}_i & -\mathrm{s}_\Omega \mathrm{s}_\omega + \mathrm{c}_\Omega \mathrm{c}_\omega \mathrm{c}_i & -\mathrm{c}_\Omega \mathrm{s}_i \\ \mathrm{s}_\omega \mathrm{s}_i & \mathrm{c}_\omega \mathrm{s}_i & \mathrm{c}_i \end{bmatrix} \tag{2.39}$$

where we used the compact notation $\mathrm{c}_x = \cos x$, $\mathrm{s}_x = \sin x$. The Euler angles used for the transformation are

- Ω, the *right ascension of the ascending node*, an angle measured from the vernal equinox to the LON;
- ω, the argument of periapsis, which we already used in Eq. (2.18), an angle measured from the LON to the eccentricity vector; and

[3]The subgroup of 3×3 orthogonal matrices with determinant $+1$ is called the *special orthogonal group*, denoted SO(3).

- i, the *inclination*, an angle measured from $\hat{\mathbf{z}}$ to $\hat{\mathbf{h}}$.

These angles are shown in Fig. 2.8.

Transforming into inertial coordinates using Eqs. (2.34) and (2.39), we obtain the general solution to Eq. (2.1):[4]

$$\begin{aligned}\mathbf{r} &= T_{\mathscr{P}}^{\mathscr{I}}(\omega, i, \Omega)\mathbf{r}_{\mathscr{P}}(a, e, M_0, t) = \mathbf{r}(a, e, i, \Omega, \omega, M_0, t) \\ &= \frac{a(1-e^2)}{1+e\cos f}\begin{bmatrix} c_{f+\omega}c_\Omega - c_i s_{f+\omega}s_\Omega \\ c_i c_\Omega s_{f+\omega} + c_{f+\omega}s_\Omega \\ s_i s_{f+\omega}\end{bmatrix}\end{aligned} \tag{2.40}$$

In a similar fashion, the expression for the inertial velocity is given by

$$\begin{aligned}\mathbf{v} = \dot{\mathbf{r}} &= T_{\mathscr{P}}^{\mathscr{I}}(\omega, i, \Omega)\dot{\mathbf{r}}_{\mathscr{P}}(a, e, M_0, t) = \mathbf{v}(a, e, i, \Omega, \omega, M_0, t) \\ &= \sqrt{\frac{\mu}{a(1-e^2)}}\begin{bmatrix} -c_\Omega s_{f+\omega} - s_\Omega c_i c_{f+\omega} - e(c_\Omega s_\omega + s_\Omega c_\omega c_i) \\ c_\Omega c_i c_{f+\omega} - s_\Omega s_{f+\omega} - e(s_\Omega s_\omega - c_\Omega c_\omega c_i) \\ s_i(c_{f+\omega} + e c_\omega)\end{bmatrix}\end{aligned} \tag{2.41}$$

Thus, the inertial position and velocity depend upon time and the *classical orbital elements*, given by

$$\mathbf{œ} = \{a, e, i, \Omega, \omega, M_0\} \tag{2.42}$$

While the Euler angles Ω, i, ω may become degenerate in some cases (for instance, ω is undefined for circular orbits; both ω and Ω are undefined for equatorial orbits), the position and velocity vectors are always well-defined. However, occasionally alternative orbital elements are used to alleviate these deficiencies. These alternative elements are collectively referred to as *nonsingular orbital elements*. A good survey of these elements can be found in Ref. [45]. The next section provides a brief overview of the nonsingular elements used in this book (e.g. in Chapter 10).

2.4 NONSINGULAR ORBITAL ELEMENTS

To alleviate the singularity of the classical elements for circular orbits, Deprit and Rom [46] suggested replacing e, M and ω by the following elements:

$$q_1 = e\cos\omega, \quad q_2 = e\sin\omega, \quad \lambda = \omega + M \tag{2.43}$$

Here, λ is the *mean argument of latitude*. However, the set $\{a, q_1, q_2, i, \Omega, \lambda\}$ will still be singular for equatorial orbits. Removal of all singularities associated with the classical elements is possible by defining the *equinoctial elements* [47–49]:

$$\left\{a, e\sin(\omega+\Omega), e\cos(\omega+\Omega), \tan\frac{i}{2}\sin\Omega, \tan\frac{i}{2}\cos\Omega, \omega+\Omega+M\right\} \tag{2.44}$$

[4] When resolving some vector $\mathbf{w}$ in inertial coordinates, we will usually write $\mathbf{w}$ instead of $[\mathbf{w}]_{\mathscr{I}}$.

In Eqs. (2.43) and (2.44), M may be replaced by M_0.

Another set of nonsingular elements, to be used in Section 7.2, was proposed by Gurfil [50]. The main idea is to replace the three Euler angles Ω, i, ω, transforming from the perifocal to the ECI frame, by *Euler parameters*, denoted by $\beta_1, \beta_2, \beta_3, \beta_0$, subject to the constraint

$$\sum_{i=0}^{3} \beta_i^2 = 1 \tag{2.45}$$

Another variation of such nonsingular representations can be found in Sengupta and Vadali [31]. Note that the Euler parameters, presented herein as a means for regularizing orbital motion, are more common in the context of attitude dynamics; indeed, they are used in this context in Chapter 9. An analogy between orbital motion and rigid body attitude dynamics is drawn by using the Euler parameters as in Ref. [51].

The new set of elements is given by

$$\text{œ} = \left\{a, \sqrt{1-e^2}, \beta_1, \beta_2, \beta_3, M_0\right\} \tag{2.46}$$

where

$$\beta_1 = \sin\frac{i}{2}\cos\frac{\Omega-\omega}{2} \tag{2.47}$$

$$\beta_2 = \sin\frac{i}{2}\sin\frac{\Omega-\omega}{2} \tag{2.48}$$

$$\beta_3 = \cos\frac{i}{2}\sin\frac{\Omega+\omega}{2} \tag{2.49}$$

and

$$\beta_0 = \cos\frac{i}{2}\cos\frac{\Omega+\omega}{2} \tag{2.50}$$

The Euler parameters form a *quaternion*,[5] given by $\beta_0 + i\beta_1 + j\beta_2 + k\beta_3$, where i, j, k satisfy the relations

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,\ i^2 = j^2 = k^2 = -1$$

The transformation from the perifocal frame to the ECI frame can be written as [52]

$$\begin{aligned} ix + jy + kz = {} & (\beta_0 + i\beta_1 + j\beta_2 + k\beta_3)(iX + jY + kZ) \\ & \times (\beta_0 - i\beta_1 - j\beta_2 - k\beta_3) \end{aligned} \tag{2.51}$$

[5] Quaternions form a number system that extends the complex numbers. The quaternions were first described by Hamilton in 1843.

This transformation represents a single rotation of magnitude $\wp$ about the *Euler vector*. The relationships between the Euler parameters and the rotation angle $\wp$ are given by [52]

$$\beta_1 = \sin\frac{\wp}{2}\cos\phi_1 \tag{2.52}$$

$$\beta_2 = \sin\frac{\wp}{2}\cos\phi_2 \tag{2.53}$$

$$\beta_3 = \sin\frac{\wp}{2}\cos\phi_3 \tag{2.54}$$

where ϕ_i, $i = 1, 2, 3$ is the angle between the Euler vector and the inertial unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, respectively, and

$$\beta_0 = \cos\frac{\wp}{2} \tag{2.55}$$

The DCM rotating from $\mathscr{P}$ to $\mathscr{I}$ can now be written in terms of Euler parameters instead of the Euler angles. This DCM, replacing the Euler angles-based DCM of Eq. (2.39), is [52]

$$T_{\mathscr{P}}^{\mathscr{I}}(\beta_1, \beta_2, \beta_3) = \begin{bmatrix} \beta_1^2 - \beta_2^2 - \beta_3^2 + \beta_0^2 & 2(\beta_1\beta_2 - \beta_3\beta_0) & 2(\beta_1\beta_3 + \beta_2\beta_0) \\ 2(\beta_1\beta_2 + \beta_3\beta_0) & \beta_2^2 + \beta_0^2 - \beta_1^2 - \beta_3^2 & 2(\beta_2\beta_3 - \beta_1\beta_0) \\ 2(\beta_1\beta_3 - \beta_2\beta_0) & 2(\beta_2\beta_3 + \beta_1\beta_0) & \beta_3^2 + \beta_0^2 - \beta_1^2 - \beta_2^2 \end{bmatrix} \tag{2.56}$$

where β_0 is given by Eq. (2.45).

Using the new DCM of Eq. (2.56), the expressions for the inertial position and velocity read

$$\mathbf{r} = T_{\mathscr{P}}^{\mathscr{I}}(\beta_1, \beta_2, \beta_3)\mathbf{r}_{\mathscr{P}}(a, e, M_0, t) = \mathbf{r}(a, e, \beta_1, \beta_2, \beta_3, M_0, t) = \frac{a(1-e^2)}{1+e\cos f}\begin{bmatrix} (\beta_1^2 - \beta_2^2 - \beta_3^2 + \beta_0^2)\mathrm{c}_f + 2(\beta_1\beta_2 - \beta_3\beta_0)\mathrm{s}_f \\ 2(\beta_1\beta_2 + \beta_3\beta_0)\mathrm{c}_f + (\beta_2^2 + \beta_0^2 - \beta_1^2 - \beta_3^2)\mathrm{s}_f \\ 2(\beta_1\beta_3 - \beta_2\beta_0)\mathrm{c}_f + 2(\beta_2\beta_3 + \beta_1\beta_0)\mathrm{s}_f \end{bmatrix} \tag{2.57}$$

and

$$\mathbf{v} = \dot{\mathbf{r}} = T_{\mathscr{P}}^{\mathscr{I}}(\omega, i, \Omega)\mathbf{v}_{\mathscr{P}}(a, e, M_0, t) = \mathbf{v}(a, e, i, \Omega, \omega, M_0, t) = \sqrt{\frac{\mu}{a(1-e^2)}} \times \begin{bmatrix} 2(\beta_1\beta_2 - \beta_3\beta_0)\mathrm{c}_f + (2\beta_2^2 + 2\beta_3^2 - 1)\mathrm{s}_f + 2e(\beta_1\beta_2 - \beta_3\beta_0) \\ (1 - 2\beta_1^2 - 2\beta_3^2)\mathrm{c}_f - 2(\beta_1\beta_2 + \beta_3\beta_0)\mathrm{s}_f - e(2\beta_1^2 + 2\beta_3^2 - 1) \\ 2(\beta_2\beta_3 + \beta_1\beta_0)\mathrm{c}_f + 2(\beta_2\beta_0 - \beta_1\beta_3)\mathrm{s}_f + 2e(\beta_2\beta_3 + \beta_1\beta_0) \end{bmatrix} \tag{2.58}$$

A set of nonsingular elements that also form a quaternion, or a *spinor*, are the *Kustaanheimo–Stiefel orbital elements* [53,54], which are designed to regularize collision orbits in addition to the common singularities using a Hamiltonian framework [55].

2.5 NON-KEPLERIAN MOTION AND ORBITAL PERTURBATIONS

The solutions in Section 2.3 were obtained for the nominal, undisturbed Keplerian motion. When perturbations act upon the body, the motion is no longer Keplerian. In order to solve for the resulting non-Keplerian motion, Euler [56] and Lagrange [57] have developed the *variation-of-parameters* (VOP) procedure, a general and powerful method for the solution of nonlinear differential equations. Before applying this method to non-Keplerian motion, we will illustrate it in the next subsection.

2.5.1 Variation of parameters

In essence, the VOP method suggests to turn the constants of the unperturbed motion, resulting from the homogenous solution of a given differential equation, into functions of time. In order to illustrate the VOP method, consider the following example:

Example 2.1 (*Newman's example [58]*). *Solve the differential equation*

$$\ddot{x} + x = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \tag{2.59}$$

using the VOP method.

The homogenous solution of Eq. (2.59) is given by

$$x = s \sin t + c \cos t \tag{2.60}$$

where s and c are constants. According to the VOP method, we need to seek a solution of the form

$$x = s(t) \sin t + c(t) \cos t \tag{2.61}$$

Differentiation of Eq. (2.61) results in

$$\dot{x} = \dot{s}(t) \sin t + \dot{c}(t) \cos t + s(t) \cos t - c(t) \sin(t) \tag{2.62}$$

Note that an additional differentiation, required to substitute for $\ddot{x}$, will yield a fourth-order system whereas the original differential equation is only of second order. Obviously, there is an excess of freedom in the system stemming from the transformation to the new state variables, $(x, \dot{x}) \mapsto (s, \dot{s}, c, \dot{c})$.

In order to solve for the excess freedom, it is customary to impose the constraint

$$\dot{s}(t) \sin t + \dot{c}(t) \cos t = 0 \tag{2.63}$$

This constraint will simplify the resulting differential equations, but is otherwise completely arbitrary. A more general form is

$$\dot{s}(t)\sin t + \dot{c}(t)\cos t = \Xi(t) \tag{2.64}$$

where $\Xi(t)$ is arbitrary. Substituting Eq. (2.64) into Eq. (2.62) and differentiating the resulting expression yields

$$\ddot{x} = \dot{\Xi} + \dot{s}(t)\cos(t) - \dot{c}(t)\sin t - s(t)\sin(t) - c(t)\cos(t) \tag{2.65}$$

Hence

$$\ddot{x} + x = \dot{\Xi} + \dot{s}(t)\cos(t) - \dot{c}(t)\sin(t) = f(t) \tag{2.66}$$

The resulting system of differential equations are

$$\dot{\Xi} + \dot{s}(t)\cos(t) - \dot{c}(t)\sin t = f(t) \tag{2.67}$$

$$\dot{s}(t)\sin(t) + \dot{c}(t)\cos(t) = \Xi(t) \tag{2.68}$$

This system can be re-written in the form

$$\dot{s}(t) = f(t)\cos(t) - \frac{\mathrm{d}}{\mathrm{d}t}(\Xi\cos t) \tag{2.69}$$

$$\dot{c}(t) = -f(t)\sin t + \frac{\mathrm{d}}{\mathrm{d}t}(\Xi\sin t) \tag{2.70}$$

Integration of Eqs. (2.69)–(2.70) yields

$$s(t) = \int_0^t f(\tau)\cos\tau d\tau - \Xi\cos t + s(0) \tag{2.71}$$

$$c(t) = \int_0^t f(\tau)\sin\tau d\tau + \Xi\sin t + c(0) \tag{2.72}$$

Substituting Eqs. (2.71)–(2.72) into Eq. (2.61) entails

$$x = -\cos t\int_0^t f(\tau)\sin\tau d\tau + \sin t\int_0^t f(\tau)\cos\tau d\tau + s(0)\sin t + c(0)\cos t \tag{2.73}$$

Not unexpectedly, the Ξ-dependent terms cancel out. Thus, while the state-space representation using the new state variables s and c depends upon the constraint function Ξ, the solution in terms of x is invariant to a particular selection of Ξ. This phenomenon is called *symmetry*. The analogy to orbital elements is straightforward, and will be discussed next.

2.5.2 Lagrange's planetary equations

When a perturbing specific force, $\mathbf{d}$, is introduced into Eq. (2.1), we have[6]

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = \mathbf{d} \tag{2.74}$$

Applying the VOP formalism requires re-defining the classical orbital elements as functions of time, yielding a modified solution of the form

$$\mathbf{r} = \mathbf{f}[\boldsymbol{œ}(t), t] \tag{2.75}$$

Taking the time derivative of Eq. (2.75) yields the relationship

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{f}}{\partial t} + \frac{\partial \mathbf{f}}{\partial \boldsymbol{œ}} \dot{\boldsymbol{œ}} \tag{2.76}$$

The differential equations describing the temporal change of the classical orbital elements for a conservative, position-only dependent perturbing potential are known as *Lagrange's planetary equations* (LPE). The *Gauss variational equations* (GVE) model the time evolution of the orbital elements due to an arbitrary perturbing acceleration resolved in the spacecraft's own LVLH frame.

To derive the LPE, Eq. (2.76) is differentiated and substituted into Eq. (2.74). This operation results in a 12-dimensional system of differential equations for $\boldsymbol{œ}$ and $\dot{\boldsymbol{œ}}$. However, there are only three degrees of freedom. Hence, the resulting system will be underdetermined, meaning that three extra conditions can be imposed. Lagrange chose to impose the constraint

$$\frac{\partial \mathbf{f}}{\partial \boldsymbol{œ}} \dot{\boldsymbol{œ}} = \mathbf{0} \tag{2.77}$$

which is also known as the *Lagrange constraint* or *osculation constraint*. Mathematically, this restriction confines the dynamics of the orbital state space to a 9-dimensional submanifold of the 12-dimensional manifold spanned by the orbital elements and their time derivatives. More importantly, this freedom reflects an internal symmetry in the mapping $(\mathbf{r}, \dot{\mathbf{r}}) \mapsto (\boldsymbol{œ}, \dot{\boldsymbol{œ}})$, which is inherent to Lagrange's planetary equations.

Physically, the Lagrange constraint postulates that the trajectory in the inertial configuration space is always tangential to an "instantaneous" ellipse (or hyperbola) defined by the "instantaneous" values of the time-varying orbital elements $\boldsymbol{œ}(t)$, meaning that the perturbed physical trajectory would coincide with the Keplerian orbit that the body would follow if the perturbing force was to cease instantaneously. This instantaneous orbit is called the *osculating orbit*. Accordingly, the orbital elements which satisfy the Lagrange constraint

[6] The external specific force can also be a control (thrust) input. In this case we will denote it by $\mathbf{u}$.

are called *osculating orbital elements*. The Lagrange constraint, however, is completely arbitrary. The generalized form of the Lagrange constraint is

$$\frac{\partial \mathbf{f}}{\partial \boldsymbol{\oe}}\dot{\boldsymbol{\oe}} = \mathbf{v}_g(\boldsymbol{\oe}, \dot{\boldsymbol{\oe}}, t) \tag{2.78}$$

where the velocity $\mathbf{v}_g$ is an arbitrary, user-defined function of the classical orbital elements, their time derivatives and possibly time. Equation (2.76) then becomes

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{f}}{\partial t} + \mathbf{v}_g \tag{2.79}$$

where

$$\frac{\partial \mathbf{f}}{\partial t} = \mathbf{g}[\boldsymbol{\oe}(t), t] = \mathbf{v} \tag{2.80}$$

This important observation has been made by a few researchers [59–61]. Recently, Efroimsky et al. have published key works on planetary equations with a generalized Lagrange constraint [58,62–64]. They termed the constraint function $\mathbf{v}_g$ *gage function* or *gage velocity*, which are terms taken from the field of electrodynamics.[7] The zero gage $\mathbf{v}_g = 0$ was termed the *Lagrange gage*.

The use of a generalized Lagrange constraint gives rise to *non-osculating orbital elements*. Thus, although the description of the physical orbit in the inertial Cartesian configuration space remains invariant to a particular selection of a gage velocity, its description in the orbital elements space depends on whether osculating or non-osculating orbital elements are used. However, to avoid complications, we will adhere to the Lagrange gage, and assume from now on that the orbital elements are osculating, i.e., $\mathbf{v}_g \equiv 0$. In this case

$$\dot{\mathbf{r}} = \mathbf{v} \tag{2.81}$$

We can now derive the differential equations for the orbital elements utilizing the VOP method as illustrated in Subsection 2.5.1. The time derivative of Eq. (2.79), using Eq. (2.81), is

$$\ddot{\mathbf{r}} = \frac{\mathrm{d}\dot{\mathbf{r}}}{\mathrm{d}t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{r}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \boldsymbol{\oe}}\dot{\boldsymbol{\oe}} = \frac{\partial^2 \mathbf{r}}{\partial^2 t} + \frac{\partial \mathbf{v}}{\partial \boldsymbol{\oe}}\dot{\boldsymbol{\oe}} \tag{2.82}$$

Substituting for $\ddot{\mathbf{r}}$ from Eq. (2.74) yields

$$\frac{\partial^2 \mathbf{r}}{\partial^2 t} + \frac{\mu}{r^3}\mathbf{r} + \frac{\partial \mathbf{v}}{\partial \boldsymbol{\oe}}\dot{\boldsymbol{\oe}} = \mathbf{d} \tag{2.83}$$

[7] Gage theory is an approach for developing a unified theory of the fundamental forces based on the concept of symmetry. Gage theory gets its name from the fact that measurements can be "re-gaged", yielding the same results. Gage theory is of great importance in electromagnetic interactions; mathematical models of gage symmetry have facilitated the discovery of *bosons*, which are counterparts of the photon. Gage symmetry is also used in cosmology to establish the theory of inflation.

Since the Keplerian solution satisfies

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0} \tag{2.84}$$

Thus, Eq. (2.83) reduces to the perturbed Keplerian motion equation:

$$\frac{\partial \mathbf{v}}{\partial \boldsymbol{\text{œ}}}\dot{\boldsymbol{\text{œ}}} = \mathbf{d} \tag{2.85}$$

Together with Eq. (2.77), we have six differential equations for the classical osculating orbital elements. These equations can be written in a compact form utilizing the following formalism, originally conceived by Lagrange: Multiply Eq. (2.77) by $[\partial \mathbf{v}/\partial \boldsymbol{\text{œ}}]^T$, multiply Eq. (2.85) by $[\partial \mathbf{r}/\partial \boldsymbol{\text{œ}}]^T$, and subtract the two expressions. This procedure results in

$$\mathfrak{L}\dot{\boldsymbol{\text{œ}}} = \left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\text{œ}}}\right]^T \mathbf{d} \tag{2.86}$$

where $\mathfrak{L}$ is called the *Lagrange matrix*,

$$\mathfrak{L} \triangleq \left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\text{œ}}}\right]^T \frac{\partial \mathbf{v}}{\partial \boldsymbol{\text{œ}}} - \left[\frac{\partial \mathbf{v}}{\partial \boldsymbol{\text{œ}}}\right]^T \frac{\partial \mathbf{r}}{\partial \boldsymbol{\text{œ}}} \tag{2.87}$$

The entries of this matrix are called *Lagrange brackets*,

$$[\text{œ}_i, \text{œ}_j] \triangleq \left[\frac{\partial \mathbf{r}}{\partial \text{œ}_i}\right]^T \frac{\partial \mathbf{v}}{\partial \text{œ}_j} - \left[\frac{\partial \mathbf{r}}{\partial \text{œ}_j}\right]^T \frac{\partial \mathbf{v}}{\partial \text{œ}_i} \tag{2.88}$$

It can be shown that [33] $[\text{œ}_i, \text{œ}_j] = -[\text{œ}_j, \text{œ}_i]$ and $[\text{œ}_i, \text{œ}_i] = 0$, so the Lagrange matrix is skew symmetric,

$$\mathfrak{L}^T = -\mathfrak{L} \tag{2.89}$$

Also, $\partial[\text{œ}_i, \text{œ}_j]/\partial t = 0$, meaning that the Lagrange matrix is time-invariant,

$$\frac{\partial \mathfrak{L}}{\partial t} = 0 \tag{2.90}$$

The Lagrange matrix is nonsingular $\forall \boldsymbol{\text{œ}} \setminus \{e = 0, i = 0\}$, and therefore can be inverted to yield the explicit differential equation

$$\dot{\boldsymbol{\text{œ}}} = \mathfrak{L}^{-1}\left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{\text{œ}}}\right]^T \mathbf{d} \tag{2.91}$$

The inverse of the Lagrange matrix is denoted by

$$\mathfrak{P}^T = \mathfrak{L}^{-1} \tag{2.92}$$

where $\mathfrak{P}$ is a skew symmetric matrix called the *Poisson matrix*. Its entries are called the *Poisson brackets*,

$$\left\{œ_i, œ_j\right\} \triangleq \frac{\partial œ_i}{\partial \mathbf{r}}\left[\frac{\partial œ_j}{\partial \mathbf{v}}\right]^T - \frac{\partial œ_i}{\partial \mathbf{v}}\left[\frac{\partial œ_j}{\partial \mathbf{r}}\right]^T \tag{2.93}$$

The Poisson matrix is given by

$$\mathfrak{P}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dfrac{2}{na} \\ 0 & 0 & 0 & 0 & -\dfrac{\sqrt{1-e^2}}{na^2 e} & \dfrac{1-e^2}{na^2 e} \\ 0 & 0 & 0 & -\dfrac{1}{na^2\sqrt{1-e^2}\sin i} & \dfrac{\cot i}{na^2\sqrt{1-e^2}} & 0 \\ 0 & 0 & \dfrac{1}{na^2\sqrt{1-e^2}\sin i} & 0 & 0 & 0 \\ 0 & \dfrac{\sqrt{1-e^2}}{na^2 e} & -\dfrac{\cot i}{na^2\sqrt{1-e^2}} & 0 & 0 & 0 \\ -\dfrac{2}{na} & -\dfrac{1-e^2}{na^2 e} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.94}$$

An important special case arises when the orbital perturbations are conservative and depend only upon position, viz.

$$\mathbf{d} = \nabla_{\mathbf{r}}\mathcal{R} = \frac{\partial \mathcal{R}}{\partial \mathbf{r}} \tag{2.95}$$

where $\mathcal{R}$ is a perturbing potential. Substituting

$$\left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{œ}}\right]^T \mathbf{d} = \left[\frac{\partial \mathbf{r}}{\partial \boldsymbol{œ}}\right]^T \frac{\partial \mathcal{R}}{\partial \mathbf{r}} = \frac{\partial \mathcal{R}}{\partial \boldsymbol{œ}} \tag{2.96}$$

into Eq. (2.91) yields *Lagrange's planetary equations* (LPE),

$$\dot{\boldsymbol{œ}} = \mathfrak{L}^{-1}\frac{\partial \mathcal{R}}{\partial \boldsymbol{œ}} \tag{2.97}$$

2.5.3 Zonal harmonics

The perturbing potential $\mathcal{R}$ appearing in Eq. (2.97) is due to any conservative perturbation; for example, *zonal gravitational harmonics*, accounting for an axially-symmetric non-spherical primary with equatorial radius R_e, yield the perturbing potential[8]

$$\mathcal{R} = -\frac{\mu}{r} \sum_{k=2}^{\infty} J_k \left(\frac{R_e}{r}\right)^k P_k(\cos\phi) \tag{2.98}$$

where ϕ is the *colatitude angle*, satisfying

$$\cos\phi = \sin i \sin(f + \omega) \tag{2.99}$$

and P_k is a *Legendre polynomial* of the first kind of order k. The Legendre polynomials, denoted for some argument x by $P_k(x)$, are useful when expanding functions such as

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \sum_{k=0}^{\infty} \frac{(r')^k}{r^{k+1}} P_k(\cos\gamma) \tag{2.100}$$

where $\|\mathbf{r}\|$ and $\|\mathbf{r}'\|$ are the Euclidean norms of the vectors $\mathbf{r}$ and $\mathbf{r}'$, respectively, so that $\mathbf{r} > \mathbf{r}'$, and γ is the angle between those two vectors. Each Legendre polynomial, $P_k(x)$, is an k th-degree polynomial. It may be expressed using the *Rodrigues formula*:

$$P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} \left[(x^2 - 1)^k\right] \tag{2.101}$$

For example, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = 0.5(3x^2 - 1)$ and $P_3(x) = 0.5(5x^3 - 3x)$. It is worth noting that the Legendre polynomials are orthogonal, and that the most dominant term in Eq. (2.98) is the J_2 term. The values of the second and third zonal harmonics for the Earth are $J_2 = 1082.63 \times 10^{-6}$ and $J_3 = -2.52 \times 10^{-6}$.

2.5.4 Gauss' variational equations

Gauss' variational equations (GVE), describing the time evolution of the orbital elements in the presence of perturbations or control inputs, can be obtained by the use of the chain rule:

$$\dot{\mathbf{œ}} = \frac{\partial \mathbf{œ}}{\partial t} + \frac{\partial \mathbf{œ}}{\partial \mathbf{r}} \left(\frac{\partial \mathbf{r}}{\partial t} + \frac{\partial \mathbf{r}}{\partial \mathbf{œ}} \dot{\mathbf{œ}}\right) + \frac{\partial \mathbf{œ}}{\partial \mathbf{v}} \left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{œ}} \dot{\mathbf{œ}}\right) \tag{2.102}$$

Substituting Eq. (2.77) into Eq. (2.102) using the fact that for the unperturbed problem

$$\frac{\partial \mathbf{œ}}{\partial \mathbf{r}} \frac{\partial \mathbf{r}}{\partial t} + \frac{\partial \mathbf{œ}}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial t} = 0 \tag{2.103}$$

[8] Equation (2.98) is written using the *Vinti representation* [65]. Brouwer [66] and Kozai [67, 68] use different representations by introducing functions of both J_k and R_e as the zonal series constants.

and that œ_i lacks direct time-dependence will yield

$$\dot{\boldsymbol{\text{œ}}} = \frac{\partial \boldsymbol{\text{œ}}}{\partial \mathbf{v}} \mathbf{d} \tag{2.104}$$

The second step is to express $\mathbf{r}$, $\mathbf{v}$, $\partial \boldsymbol{\text{œ}} / \partial \mathbf{v}$ and $\mathbf{d}$ in an LVLH coordinate system fixed to the spacecraft. In this coordinate system, denoted by $\mathscr{L}$ in Section 2.1, the unit vector $\hat{\mathbf{R}}$ is directed radially outwards, $\hat{\mathbf{S}}$ is perpendicular to $\hat{\mathbf{R}}$ in the direction of the instantaneous velocity, and $\hat{\mathbf{W}}$ completes the right-hand triad.

An explicit form of the GVE is derived by choosing the classical orbital elements as the state variables, i.e., $\boldsymbol{\text{œ}} = [a, e, i, \Omega, \omega, M_0]^T$, where, as before, a is the semimajor axis, e is the eccentricity, i is the inclination, Ω is the right ascension of the ascending node, ω is the argument of periapsis, and M_0 is defined as [69]

$$M_0 = M - \int_{t_0}^{t} n dt \tag{2.105}$$

where M is the mean anomaly, t_0 is a reference time (which may differ from the periapsis passage time) and $n = \sqrt{\mu / a^3}$ is the mean motion.

In addition, we write the position and velocity vectors in frame $\mathscr{L}$ using a polar representation, so that

$$\mathbf{r} = r\hat{\mathbf{R}}, \quad \mathbf{v} = \frac{\partial r}{\partial t}\hat{\mathbf{R}} + r\frac{\partial f}{\partial t}\hat{\mathbf{S}} \tag{2.106}$$

where f is the true anomaly. By collecting all expressions and substituting into Eq. (2.104), we can obtain GVE. Writing $\mathbf{d} = d_r\hat{\mathbf{R}} + d_\theta\hat{\mathbf{S}} + d_h\hat{\mathbf{W}} = [d_r, d_\theta, d_h]^T$, provides the following equations:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 2\frac{d_r a^2 e \sin f}{h} + 2\frac{d_\theta a^2 p}{hr} \tag{2.107a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{d_r p \sin f}{h} + \frac{d_\theta\left[(p+r)\cos f + re\right]}{h} \tag{2.107b}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{d_h r \cos(f+\omega)}{h} \tag{2.107c}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{d_h r \sin(f+\omega)}{h \sin i} \tag{2.107d}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{d_r p \cos f}{he} + \frac{d_\theta (p+r)\sin f}{he} - \frac{d_h r \sin(f+\omega)\cos i}{h \sin i} \tag{2.107e}$$

$$\frac{\mathrm{d}M_0}{\mathrm{d}t} = d_r\left[\frac{\left(-2e + \cos f + e\cos^2 f\right)\left(1-e^2\right)}{e\,(1+e\cos f)\,na}\right] + d_\theta\left[\frac{\left(e^2-1\right)(e\cos f + 2)\sin f}{e\,(1+e\cos f)\,na}\right] \tag{2.107f}$$

where as before $p = a(1 - e^2)$ is the semilatus rectum and $h = \sqrt{\mu p}$ is the magnitude of the angular momentum vector. Another useful relationship is the variational equation for the true anomaly, obtained by using Eqs. (2.16), (2.23) and (2.107e):

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{h}{r^2} + \frac{1}{eh}\left[d_r p \cos f - d_\theta (p + r) \sin f\right] \tag{2.108}$$

2.6 AVERAGING THEORY

A very powerful and useful tool for dealing with the orbital elements variational equations is *averaging*. Classical averaging theory was originally developed in order to simplify nonlinear nonautonomous systems. The standard form of the equations of motion for averaging is

$$\dot{\mathbf{x}} = \epsilon \mathbf{F}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.109}$$

where $\mathbf{x} \in \mathbb{M} \subset \mathbb{R}^n$, $\mathbf{F}$ is T-periodic, $\mathbf{F}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, t + T)$. The *averaging operator* on $\mathbf{F}$, denoted by $\langle \mathbf{F} \rangle$ yields a *mean value*, $\bar{\mathbf{F}}$, via the quadrature

$$\bar{\mathbf{F}} = \langle \mathbf{F} \rangle \triangleq \frac{1}{T} \int_t^{t+T} \mathbf{F}(\mathbf{x}, \tau) d\tau \tag{2.110}$$

The averaged differential equations are

$$\dot{\bar{\mathbf{x}}} = \epsilon \mathbf{F}(\bar{\mathbf{x}}) \tag{2.111}$$

The primary goal of averaging is removing the time-dependence of the original differential equations, thus permitting a considerable simplification of the resulting dynamics. However, one should determine under what conditions the averaged and the original systems coincide, in order for the averaging to be meaningful. To that end, we consider the following theorem.

Theorem 2.1 (*First-order averaging [70]*). *Consider the dynamical systems* (2.110) and (2.111). *If* $\mathbf{F}(\mathbf{x}, t)$ *is continuous in* $\mathbf{x}$ *and* t *and in addition* $\bar{\mathbf{x}} \in \mathbb{M}, \forall t_0 \leq t \leq t_0 + T/\epsilon$, *then* $\mathbf{x} = \bar{\mathbf{x}} + \mathcal{O}(\epsilon), \forall t_0 \leq t \leq t_0 + T/\epsilon$ *as* $\epsilon \to 0$.

Theorem 2.1 determines the conditions under which the averaged and original dynamical systems are identical to first-order, meaning that taking $\mathbf{x} = \bar{\mathbf{x}}$ and $\dot{\mathbf{x}} = \dot{\bar{\mathbf{x}}}$ is correct to first-order in ϵ. In this case, the averaging error will be $\mathcal{O}(\epsilon^2)$.

Application of averaging to the orbital variational equations is straightforward. For some perturbing potential $\mathcal{R}$, we average using [33]:

$$\bar{\mathcal{R}} = \langle \mathcal{R} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R} dM \tag{2.112}$$

or, alternatively,

$$\bar{\mathcal{R}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{n}{h}\mathcal{R}r^2 df \tag{2.113}$$

Battin [33] provides an expression for the first-order averaged potential due to J_2 (cf. Subsection 2.5.3):

$$\bar{\mathcal{R}} = \frac{\bar{n}^2 J_2 R_e^2}{4(1-\bar{e}^2)^{\frac{3}{2}}}(3\cos^2\bar{i} - 1) \tag{2.114}$$

Upon substitution into the LPE (2.97), one obtains the linear differential equations for the *mean classical orbital elements*:

$$\frac{d\bar{a}}{dt} = 0 \tag{2.115a}$$

$$\frac{d\bar{e}}{dt} = 0 \tag{2.115b}$$

$$\frac{d\bar{i}}{dt} = 0 \tag{2.115c}$$

$$\frac{d\bar{\Omega}}{dt} = -\frac{3}{2}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}\cos\bar{i} \tag{2.115d}$$

$$\frac{d\bar{\omega}}{dt} = \frac{3}{4}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}(5\cos^2\bar{i} - 1) \tag{2.115e}$$

$$\frac{d\bar{M}_0}{dt} = \frac{3}{4}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}\bar{\eta}(3\cos^2\bar{i} - 1) \tag{2.115f}$$

where $\bar{n} = \sqrt{\frac{\mu}{\bar{a}^3}}$ and $\bar{\eta} = \sqrt{1-\bar{e}^2}$. These equations predict secular growths of Ω, ω and M_0. To first order in J_2, there are no long-periodic and no secular variations in the remaining elements.

Gurfil [50] presented the first-order averaged LPE for the Euler-parameters-based elements defined by Eqs. (2.46) due to the J_2-perturbation as an alternative for Eqs. (2.115). The alternative equations are given below:

$$\dot{\bar{a}} = 0 \tag{2.116a}$$

$$\dot{\bar{\eta}} = 0 \tag{2.116b}$$

$$\dot{\bar{\beta}}_1 = \frac{3}{4}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}\bar{\beta}_2[3 + 10(\bar{\beta}_1^2 + \bar{\beta}_2^2)^2 - 12(\bar{\beta}_1^2 + \bar{\beta}_2^2)] \tag{2.116c}$$

$$\dot{\bar{\beta}}_2 = -\frac{3}{4}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}\bar{\beta}_1[3 + 10(\bar{\beta}_1^2 + \bar{\beta}_2^2)^2 - 12(\bar{\beta}_1^2 + \bar{\beta}_2^2)] \tag{2.116d}$$

$$\dot{\bar{\beta}}_3 = \frac{3}{4}J_2\left(\frac{R_e}{\bar{p}}\right)^2 \bar{n}\bar{\beta}_4[1 + 10(\bar{\beta}_1^2 + \bar{\beta}_2^2)^2 - 8(\bar{\beta}_1^2 + \bar{\beta}_2^2)] \tag{2.116e}$$

$$\dot{\bar{M}}_0 = \frac{3}{2} J_2 \left(\frac{R_e}{\bar{p}} \right)^2 \bar{n}\bar{\eta}[1 - 6\bar{\beta}_1^2 - 6\bar{\beta}_2^2 + 6(\bar{\beta}_1^2 + \bar{\beta}_2^2)^2] \tag{2.116f}$$

Example 2.2. *Determine the impulse required to suppress the nodal precession accumulated over one orbit period for a mean circular orbit with* $\bar{a} = 7100$ km *and* $\bar{i} = 70°$.

Assuming that $R_e = 6378.1363$ km and $\mu = 3.98604415 \times 10^5\ \mathrm{km^3/s^2}$, the nodal precession accumulated over one orbit period can be calculated from Eq. (2.115d) to be $\delta\Omega = -0.00282$ rad. The cross-track impulse required to cancel this accumulation is obtained from Eq. (2.107d) using the impulsive thrust assumption as

$$\Delta v_h = \left| \frac{\delta\Omega\sqrt{\mu/\bar{a}}\,\sin\bar{i}}{\sin\theta} \right| = 19.829\ \mathrm{m/s} \tag{2.117}$$

We made the tacit assumption in performing the above calculation that the impulse is applied at $\theta = \pi/2$. This example also illustrates the expense involved in "fighting with nature".

Example 2.3. *Determine the impulse required to suppress the differential nodal precession accumulated over one orbit period for a mean circular orbit with* $\bar{a} = 7100$ km *and* $\bar{i} = 70°$ *due to a differential inclination* $\delta i = 1/7100$ rad.

The differential nodal precession for the example of a circular orbit over one period, using Eq. (2.115d), is $\delta\Omega = 3\pi J_2 \left(R_e/\bar{a}\right)^2 (\sin\bar{i})\delta i$. Substituting for the given data into Eqs. (2.107d) and (2.115d), we obtain

$$\Delta v_h = 7.6732 \times 10^{-3}\ \mathrm{m/s} \tag{2.118}$$

Over a period of one year the impulse required to perform this operation is 41 m/s.

SUMMARY

This chapter presented a brief treatment of the vast field of astrodynamics. We touched upon the important topics that serve as the building blocks for the material to follow in the subsequent chapters. The orbital mechanics of Keplerian motion as well as perturbed motion were introduced. Of prime significance are the Gauss variational equations, the concept of averaging, first-order secular rates in the orbital elements due to the J_2 perturbation, and the estimates of impulse requirements to mitigate the absolute and differential effects of the perturbations.