Appendices

A THE TRANSFORMATION MATRIX $\Sigma(T)$

This appendix presents the elements of the transformation matrix $\Sigma(t) = [A(t) + A_2 B(t)]$, where $A_2 = 3J_2 R_e^2$. All the orbital elements are the elements of the chief; for brevity, the subscript "0" has been dropped. The following quantities will be needed:

$$
\begin{aligned}
r &= p/\left(1 + q_1 \cos\theta + q_2 \sin\theta\right) \\
p &= a\left(1 - q_1^2 - q_2^2\right) \\
v_t &= \sqrt{\frac{\mu}{p}}\left(1 + q_1 \cos\theta + q_2 \sin\theta\right) \\
v_r &= \sqrt{\frac{\mu}{p}}\left(q_1 \sin\theta - q_2 \cos\theta\right) \\
v_{rt} &= \frac{v_r}{v_t} \\
n &= \sqrt{\frac{\mu}{a^3}}
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\Sigma_{11} &= \frac{r}{a} \\
\Sigma_{12} &= v_{rt} r \\
\Sigma_{13} &= 0 \\
\Sigma_{14} &= -\frac{r}{p}\left(2aq_1 + r\cos\theta\right) \\
\Sigma_{15} &= -\frac{r}{p}\left(2aq_2 + r\sin\theta\right) \\
\Sigma_{16} &= 0
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\Sigma_{21} &= -\frac{1}{2}\frac{v_r}{a} \\
\Sigma_{22} &= \left(1 - \frac{r}{p}\right) v_t \\
\Sigma_{23} &= 0 \\
\Sigma_{24} &= \left[v_{rt}\left(\frac{a}{p}\right) q_1 + \left(\frac{r}{p}\right)\sin\theta\right] v_t \\
\Sigma_{25} &= \left[v_{rt}\left(\frac{a}{p}\right) q_2 - \left(\frac{r}{p}\right)\cos\theta\right] v_t \\
\Sigma_{26} &= 0
\end{aligned}
\tag{A.3}
$$

$$\begin{aligned}
\Sigma_{31} &= 0 \\
\Sigma_{32} &= r \\
\Sigma_{33} &= 0,\ \Sigma_{34} = 0,\ \Sigma_{35} = 0 \\
\Sigma_{36} &= r\cos i
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\Sigma_{41} &= -\frac{3}{2}\frac{v_t}{a} \\
\Sigma_{42} &= -v_r \\
\Sigma_{43} &= -\left(\frac{A_2}{pr}\right)\left(\sin i \cos i \sin^2\theta\right) v_t \\
\Sigma_{44} &= \left(\frac{r}{p}\right)\left[3\left(\frac{a}{r}\right) q_1 + 2\cos\theta\right] v_t \\
\Sigma_{45} &= \left(\frac{r}{p}\right)\left[3\left(\frac{a}{r}\right) q_2 + 2\sin\theta\right] v_t \\
\Sigma_{46} &= v_r \cos i + \left(\frac{A_2}{pr}\right)\left(\sin^2 i \cos i \sin\theta\cos\theta\right) v_t
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\Sigma_{51} &=,\ \Sigma_{52} = 0 \\
\Sigma_{53} &= r\sin\theta \\
\Sigma_{54} &= 0,\ \Sigma_{55} = 0 \\
\Sigma_{56} &= -r\sin i\cos\theta
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
\Sigma_{61} &= 0 \\
\Sigma_{62} &= \left(\frac{A_2}{pr}\right)(\sin i \cos i \sin\theta)\, v_t \\
\Sigma_{63} &= v_t\cos\theta + v_r\sin\theta \\
\Sigma_{64} &= 0,\ \Sigma_{65} = 0 \\
\Sigma_{66} &= v_t\,(\sin\theta - v_{rt}\cos\theta)\sin i + \left(\frac{A_2}{pr}\right)\left(\sin i\cos^2 i\sin\theta\right) v_t
\end{aligned} \tag{A.7}$$

B THE TRANSFORMATION MATRIX $\Sigma(T)^{-1}$

This appendix presents the elements of the transformation matrix $\Sigma(t)^{-1} = [A(t) + A_2 B(t)]^{-1}$.

$$\begin{aligned}
\Sigma_{11}^{-1} &= -\left(\frac{2a}{r}\right)\left[3\left(\frac{a}{r}\right)\left(1-\frac{p}{r}\right) - 2\left(1+\left(\frac{ap}{r^2}\right)v_{rt}^2\right)\right] \\
\Sigma_{12}^{-1} &= \frac{2p}{v_t}\left(\frac{a}{r}\right)^2 v_{rt} \\
\Sigma_{13}^{-1} &= \left(\frac{2a}{r}\right)\left[2\left(\frac{a}{r}\right)\left(1-\frac{p}{r}\right) - \left(1+\left(\frac{ap}{r^2}\right)v_{rt}^2\right)\right] v_{rt}
\end{aligned} \tag{B.1}$$

$$\Sigma_{14}^{-1} = -\left(\frac{2a}{v_t}\right)\left[2\left(\frac{a}{r}\right)\left(1-\frac{p}{r}\right)-\left(1+\left(\frac{ap}{r^2}\right)v_{rt}^2\right)\right]$$
$$\Sigma_{15}^{-1} = -\left(\frac{A_2}{r^2}\right)\left(\frac{a}{p}\right)(2\sin i\cos i\sin\theta)\left[\left(1-\frac{p}{r}\right)-\left(1+\left(\frac{ap}{r^2}\right)v_{rt}^2\right)\right]$$
$$\Sigma_{16}^{-1} = 0$$

$$\begin{aligned}
\Sigma_{21}^{-1} &= 0,\ \Sigma_{22}^{-1} = 0\\
\Sigma_{23}^{-1} &= \frac{1}{r} + \frac{A_2}{r^2}\left(\frac{\cos^2 i\sin^2\theta}{p}\right)\\
\Sigma_{24}^{-1} &= 0\\
\Sigma_{25}^{-1} &= (\cos\theta + v_{rt}\sin\theta)\left(\frac{\cos i}{r\sin i}\right)\\
\Sigma_{26}^{-1} &= -\left(\frac{\sin\theta}{v_t}\right)\left(\frac{\cos i}{\sin i}\right)
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
\Sigma_{31}^{-1} &= 0,\ \Sigma_{32}^{-1} = 0\\
\Sigma_{33}^{-1} &= -\left(\frac{A_2}{r^2}\right)\left(\frac{\sin i\cos i\sin\theta\cos\theta}{p}\right)\\
\Sigma_{34}^{-1} &= 0\\
\Sigma_{35}^{-1} &= \frac{(\sin\theta - v_{rt}\cos\theta)}{r}\\
\Sigma_{36}^{-1} &= \frac{\cos\theta}{v_t}
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\Sigma_{41}^{-1} &= \left(\frac{p}{r^2}\right)(3\cos\theta + 2v_{rt}\sin\theta)\\
\Sigma_{42}^{-1} &= \left(\frac{p}{rv_t}\right)\sin\theta\\
\Sigma_{43}^{-1} &= -\left(\frac{1}{r}\right)\left[\left(\frac{p}{r}-1\right)\sin\theta + \left(\frac{p}{r}\right)v_{rt}(\cos\theta + v_{rt}\sin\theta)\right]\\
&\quad + \left(\frac{A_2}{r^2}\right)\left(\frac{\cos^2 i\sin^2\theta}{r}\right)\left[\left(\frac{r}{p}-1\right)\sin\theta + v_{rt}\cos\theta\right]\\
\Sigma_{44}^{-1} &= \left(\frac{p}{rv_t}\right)(2\cos\theta + v_{rt}\sin\theta)\\
\Sigma_{45}^{-1} &= (\cos\theta + v_{rt}\sin\theta)\left[\left(1-\frac{p}{r}\right)\sin\theta + \left(\frac{p}{r}\right)v_{rt}\cos\theta\right]\left(\frac{\cos i}{r\sin i}\right)\\
&\quad + \left(\frac{A_2}{r^2}\right)\left(\frac{\sin i\cos i\sin\theta}{r}\right)(2\cos\theta + v_{rt}\sin\theta)\\
\Sigma_{46}^{-1} &= -\left(\frac{\sin\theta}{v_t}\right)\left[\left(\frac{p}{r}\right)v_{rt}\cos\theta + \left(1-\frac{p}{r}\right)\sin\theta\right]\left(\frac{\cos i}{\sin i}\right)
\end{aligned} \tag{B.4}$$

$$
\begin{aligned}
\Sigma_{51}^{-1} &= \left(\frac{p}{r^2}\right)(3\sin\theta - 2v_{rt}\cos\theta)\\
\Sigma_{52}^{-1} &= -\left(\frac{p}{rv_t}\right)\cos\theta\\
\Sigma_{53}^{-1} &= \left(\frac{1}{r}\right)\left[\left(\frac{p}{r}-1\right)\cos\theta + \left(\frac{p}{r}\right)v_{rt}\,(v_{rt}\cos\theta - \sin\theta)\right]\\
&\quad + \left(\frac{A_2}{r^2}\right)\left(\frac{\cos^2 i\sin^2\theta}{r}\right)\left[\left(1-\frac{r}{p}\right)\cos\theta + v_{rt}\sin\theta\right]\\
\Sigma_{54}^{-1} &= \left(\frac{p}{rv_t}\right)(2\sin\theta - v_{rt}\cos\theta)\\
\Sigma_{55}^{-1} &= \left(\frac{1}{r}\right)(\cos\theta + v_{rt}\sin\theta)\left[\left(\frac{p}{r}-1\right)\cos\theta + \left(\frac{p}{r}\right)v_{rt}\sin\theta\right]\left(\frac{\cos i}{\sin i}\right)\\
&\quad + \left(\frac{A_2}{r^2}\right)\left(\frac{\sin i\cos i\sin\theta}{r}\right)(2\sin\theta - v_{rt}\cos\theta)\\
\Sigma_{56}^{-1} &= \left(\frac{\sin\theta}{rv_t}\right)\left[\left(1-\frac{p}{r}\right)\cos\theta - \frac{p}{r}v_{rt}\sin\theta\right]\left(\frac{\cos i}{\sin i}\right)
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
&\Sigma_{61}^{-1} = 0,\ \Sigma_{62}^{-1} = 0\\
&\Sigma_{63}^{-1} = -\left(\frac{A_2}{r^2}\right)\left(\frac{\cos i\sin^2\theta}{p}\right)\\
&\Sigma_{64}^{-1} = 0\\
&\Sigma_{65}^{-1} = -\left(\frac{1}{r\sin i}\right)(\cos\theta + v_{rt}\sin\theta)\\
&\Sigma_{66}^{-1} = \frac{\sin\theta}{v_t\sin i}
\end{aligned}
\tag{B.6}
$$

C THE MATRIX $\bar{B}(T)$

$$
\bar{B}_{1j} = \bar{B}_{3j} = \bar{B}_{5j} = 0, \quad j = 1-6 \tag{C.1}
$$

$$
\begin{aligned}
\bar{B}_{21} &= \left(\frac{5nrv_{rt}}{8ap^2}\right)\left(5\cos^2 i - 1\right)\\
\bar{B}_{22} &= \left(\frac{nr}{4p^2}\right)\left(1-5\cos^2 i\right)\left[\left(1-\frac{r}{p}\right) + 2v_{rt}^2\right]\\
\bar{B}_{23} &= \left(\frac{5nrv_{rt}}{2p^2}\right)(\sin i\cos i)\\
\bar{B}_{24} &= \left(\frac{nr}{4p^2}\right)\left(5\cos^2 i - 1\right)\left[2v_{rt}\left(\frac{r}{p}\cos\theta - \frac{a}{p}q_1\right) - \left(\frac{r}{p}\right)\sin\theta\right]\\
\bar{B}_{25} &= \left(\frac{nr}{4p^2}\right)\left(5\cos^2 i - 1\right)\left[2v_{rt}\left(\frac{r}{p}\sin\theta - \frac{a}{p}q_2\right) + \left(\frac{r}{p}\right)\cos\theta\right]\\
\bar{B}_{26} &= 0
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\bar{B}_{41} &= \left(\frac{7}{4}\frac{nr}{ap^2}\right)\cos^2 i \\
\bar{B}_{42} &= \left(\frac{1}{4}\frac{nrv_{rt}}{p^2}\right)\left(1-5\cos^2 i\right) \\
\bar{B}_{43} &= \left(\frac{1}{2}\frac{nr}{p^2}\right)\sin i\cos i \\
\bar{B}_{44} &= -\left(\frac{2nraq_1}{p^3}\right)\cos^2 i \\
\bar{B}_{45} &= -\left(\frac{2nraq_2}{p^3}\right)\cos^2 i \\
\bar{B}_{46} &= \left(\frac{1}{4}\frac{nrv_{rt}\cos i}{p^2}\right)\left(1-5\cos^2 i\right)
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
\bar{B}_{61} &= -\left(\frac{7}{4}\frac{nr}{ap^2}\right)\cos\theta\sin i\cos i \\
\bar{B}_{62} &= 0 \\
\bar{B}_{63} &= \left(\frac{1}{4}\frac{nr}{p^2}\right)\left[v_{rt}\sin\theta\left(1-5\cos^2 i\right)-2\cos\theta\sin^2 i\right] \\
\bar{B}_{64} &= \left(\frac{2nraq_1}{p^3}\right)\cos\theta\sin i\cos i \\
\bar{B}_{65} &= \left(\frac{2nraq_2}{p^3}\right)\cos\theta\sin i\cos i \\
\bar{B}_{66} &= \left(\frac{1}{4}\frac{nrv_{rt}}{p^2}\right)\cos\theta\sin i\left(5\cos^2 i-1\right)
\end{aligned}
\tag{C.4}
$$

D THE STATE TRANSITION MATRIX FOR RELATIVE MEAN ELEMENTS

In this appendix, all the variables are mean variables, that is, those that result from the averaged Hamiltonian. The subscript “0” indicates the value at the initial time, t_0, except for the quantities $G_{\gamma_0}, \gamma = \{\theta, q_1, q_2\}$, for which the definition is given at the end of the Appendix. We use this notation convention to be consistent with the notation in Ref. [75].

$$
\begin{aligned}
\bar{\phi}_{\overline{\text{œ}}11} &= 1 \\
\bar{\phi}_{\overline{\text{œ}}1j} &= 0, \quad j = 2, \ldots, 6
\end{aligned}
\tag{D.1}
$$

$$
\begin{aligned}
\bar{\phi}_{\overline{\text{œ}}21} = &-\frac{(t-t_0)}{G_\theta}\left[\left(\frac{3}{2}\frac{n}{a}\right)+\left(\frac{7A_2}{8p^2}\right)\left(\frac{n}{a}\right)\left[\eta\left(3\cos^2 i-1\right)\right.\right. \\
&\left.\left.+K\left(5\cos^2 i-1\right)\right]\right] \\
\bar{\phi}_{\overline{\text{œ}}22} = &-\frac{G_{\theta_0}}{G_\theta}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\phi}_{\text{œ}23} &= -\frac{(t-t_0)}{G_\theta}\left[\left(\frac{A_2}{2p^2}\right) n\,(\sin i \cos i)\,(3\eta + 5K)\right] \\
\bar{\phi}_{\text{œ}24} &= -\frac{1}{G_\theta}\left(G_{q_{10}} + G_{q_1}\cos(\Delta\omega) + G_{q_2}\sin(\Delta\omega)\right) + \frac{(t-t_0)}{G_\theta} \\
&\quad \times \left(\frac{A_2}{4p^2}\right)\left(\frac{anq_{10}}{p}\right)\left[3\eta\left(3\cos^2 i - 1\right) + 4K\left(5\cos^2 i - 1\right)\right] \\
\bar{\phi}_{\text{œ}25} &= -\frac{1}{G_\theta}\left(G_{q_{20}} - G_{q_1}\sin(\Delta\omega) + G_{q_2}\cos(\Delta\omega)\right) + \frac{(t-t_0)}{G_\theta} \\
&\quad \times \left(\frac{A_2}{4p^2}\right)\left(\frac{anq_{20}}{p}\right)\left[3\eta\left(3\cos^2 i - 1\right) + 4K\left(5\cos^2 i - 1\right)\right] \\
\bar{\phi}_{\text{œ}26} &= 0
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
\bar{\phi}_{\text{œ}33} &= 1 \\
\bar{\phi}_{\text{œ}3j} &= 0, \quad j = 1, \ldots, 6
\end{aligned}
\tag{D.3}
$$

$$
\begin{aligned}
\bar{\phi}_{\text{œ}41} &= \left(\frac{7A_2}{8p^2}\right)\left(\frac{n}{a}\right)(q_{10}\sin(\Delta\omega) + q_{20}\cos(\Delta\omega))\left(5\cos^2 i - 1\right)(t-t_0) \\
\bar{\phi}_{\text{œ}42} &= 0 \\
\bar{\phi}_{\text{œ}43} &= \left(\frac{5}{2}\frac{A_2}{p^2}\right) n\,(q_{10}\sin(\Delta\omega) + q_{20}\cos(\Delta\omega))\,(\sin i\cos i)\,(t-t_0) \\
\bar{\phi}_{\text{œ}44} &= \cos(\Delta\omega) - \left(\frac{A_2}{p^2}\right)\left(\frac{nq_{10}}{\eta^2}\right)(q_{10}\sin(\Delta\omega) + q_{20}\cos(\Delta\omega)) \\
&\quad \times \left(5\cos^2 i - 1\right)(t-t_0) \\
\bar{\phi}_{\text{œ}45} &= -\sin(\Delta\omega) - \left(\frac{A_2}{p^2}\right)\left(\frac{nq_{20}}{\eta^2}\right)(q_{10}\sin(\Delta\omega) + q_{20}\cos(\Delta\omega)) \\
&\quad \times \left(5\cos^2 i - 1\right)(t-t_0) \\
\bar{\phi}_{\text{œ}46} &= 0
\end{aligned}
\tag{D.4}
$$

$$
\begin{aligned}
\bar{\phi}_{\text{œ}51} &= -\left(\frac{7A_2}{8p^2}\right)\left(\frac{n}{a}\right)(q_{10}\cos(\Delta\omega) - q_{20}\sin(\Delta\omega))\left(5\cos^2 i - 1\right)(t-t_0) \\
\bar{\phi}_{\text{œ}52} &= 0 \\
\bar{\phi}_{\text{œ}53} &= -\left(\frac{5}{2}\frac{A_2}{p^2}\right) n\,(q_{10}\cos(\Delta\omega) - q_{20}\sin(\Delta\omega))\,(\sin i\cos i)\,(t-t_0) \\
\bar{\phi}_{\text{œ}54} &= \sin(\Delta\omega) + \left(\frac{A_2}{p^2}\right)\left(\frac{nq_{10}}{\eta^2}\right)(q_{10}\cos(\Delta\omega) \\
&\quad - q_{20}\sin(\Delta\omega))\left(5\cos^2 i - 1\right)(t-t_0) \\
\bar{\phi}_{\text{œ}55} &= \cos(\Delta\omega) + \left(\frac{A_2}{p^2}\right)\left(\frac{nq_{20}}{\eta^2}\right)(q_{10}\cos(\Delta\omega) \\
&\quad - q_{20}\sin(\Delta\omega))\left(5\cos^2 i - 1\right)(t-t_0) \\
\bar{\phi}_{\text{œ}56} &= 0
\end{aligned}
\tag{D.5}
$$

$$\begin{aligned}
\bar{\phi}_{\overline{\text{œ}}61} &= \left(\frac{7}{4}\frac{A_2}{p^2}\right)\left(\frac{n\cos i}{a}\right)(t-t_0) \\
\bar{\phi}_{\overline{\text{œ}}62} &= 0 \\
\bar{\phi}_{\overline{\text{œ}}63} &= \left(\frac{A_2}{2p^2}\right)(n\sin i)\,(t-t_0) \\
\bar{\phi}_{\overline{\text{œ}}64} &= -\left(\frac{2A_2}{p^2}\right)\left(\frac{nq_{10}\cos i}{\eta^2}\right)(t-t_0) \\
\bar{\phi}_{\overline{\text{œ}}65} &= -\left(\frac{2A_2}{p^2}\right)\left(\frac{nq_{20}\cos i}{\eta^2}\right)(t-t_0) \\
\bar{\phi}_{\overline{\text{œ}}66} &= 1
\end{aligned} \tag{D.6}$$

where

$$\begin{aligned}
\Delta\omega &= \dot{\omega}^{(s)}\,(t-t_0) \\
\dot{\omega}^{(s)} &= 0.75J_2 n\left(\frac{R_e}{p}\right)^2\left(5\cos^2 i - 1\right) \\
K &= 1 + G_{q1}\left[q_{10}\sin(\Delta\omega) + q_{20}\cos(\Delta\omega)\right] \\
&\quad - G_{q2}\left[q_{10}\cos(\Delta\omega) - q_{20}\sin(\Delta\omega)\right]
\end{aligned} \tag{D.7}$$

$$\begin{aligned}
G &= \lambda - \lambda(t_0) = \dot{\lambda}^{(s)}\,(t-t_0) \\
\dot{\lambda}^{(s)} &= n\left[1 + 0.75J_2\left(\frac{R_e}{a}\right)^2\left(\frac{1}{\eta^4}\right)\left[\eta\left(3\cos^2 i - 1\right) + \left(5\cos^2 i - 1\right)\right]\right] \\
G_\gamma &= \frac{\partial G}{\partial \gamma}
\end{aligned}$$

where γ denotes any variable, and the superscript $(\cdot)^{(s)}$ denotes the secular value.

$$\begin{aligned}
G_\theta &= \frac{rn}{v_t} \\
G_{\theta_0} &= -\frac{r_0 n_0}{v_{t0}}
\end{aligned} \tag{D.8}$$

$$\begin{aligned}
G_{q_1} &= \frac{q_2}{\eta(1+\eta)} + \frac{q_1 v_{rt}}{\eta} - \eta\left(\frac{r}{p}\right)^2\left(1+\frac{a}{r}\right)(q_2 + \sin\theta) \\
G_{q_{10}} &= -\frac{q_{20}}{\eta_0(1+\eta_0)} - \frac{q_{10}v_{rt0}}{\eta_0} + \eta_0\left(\frac{r_0}{p_0}\right)^2\left(1+\frac{a_0}{r_0}\right)(q_{20} + \sin\theta_0) \\
G_{q_2} &= -\frac{q_1}{\eta(1+\eta)} + \frac{q_2 v_{rt}}{\eta} - \eta\left(\frac{r}{p}\right)^2\left(1+\frac{a}{r}\right)(q_1 + \cos\theta) \\
G_{q_{20}} &= \frac{q_{10}}{\eta_0(1+\eta_0)} - \frac{q_{20}v_{rt0}}{\eta_0} - \eta_0\left(\frac{r_0}{p_0}\right)^2\left(1+\frac{a_0}{r_0}\right)(q_{10} + \cos\theta_0)
\end{aligned} \tag{D.9}$$

E TRANSFORMATION FROM MEAN TO OSCULATING ELEMENTS

$$a^{(lp)} = 0 \tag{E.1}$$

$$\lambda^{(lp)} = \left[\frac{q_1 q_2 \sin^2 i}{8a^2\eta^2(1+\eta)}\right]\left(1 - 10\Theta\cos^2 i\right) + \left(\frac{q_1 q_2}{16a^2\eta^4}\right) \times \left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right) \tag{E.2}$$

$$\theta^{(lp)} = \lambda^{(lp)} - \left(\frac{\sin^2 i}{16a^2\eta^4}\right)\left(1 - 10\Theta\cos^2 i\right) \times \left\{q_1 q_2\left[3 + \frac{2\eta^2}{(1+\eta)}\right] + 2\left(q_1\sin\theta + q_2\cos\theta\right) + \frac{\varepsilon_1 \sin 2\theta}{2}\right\} \tag{E.3}$$

$$i^{(lp)} = \left(\frac{\sin 2i}{32a^2\eta^4}\right)\left(1 - 10\Theta\cos^2 i\right)\left(q_1^2 - q_2^2\right) \tag{E.4}$$

$$q_1^{(lp)} = -\left(\frac{q_1\sin^2 i}{16a^2\eta^2}\right)\left(1 - 10\Theta\cos^2 i\right) - \left(\frac{q_1 q_2^2}{16a^2\eta^4}\right)\left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right) \tag{E.5}$$

$$q_2^{(lp)} = \left(\frac{q_2\sin^2 i}{16a^2\eta^2}\right)\left(1 - 10\Theta\cos^2 i\right) + \left(\frac{q_1^2 q_2}{16a^2\eta^4}\right)\left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right) \tag{E.6}$$

$$\Omega^{(lp)} = \left(\frac{q_1 q_2\cos i}{8a^2\eta^4}\right)\left(11 + 80\Theta\cos^2 i + 200\Theta^2\cos^4 i\right) \tag{E.7}$$

$$a^{(sp1)} = \left[\frac{\left(1 - 3\cos^2 i\right)}{2a\eta^6}\right]\left[(1+\varepsilon_2)^3 - \eta^3\right] \tag{E.8}$$

$$\lambda^{(sp1)} = \left[\frac{\varepsilon_3\left(1 - 3\cos^2 i\right)}{4a^2\eta^4(1+\eta)}\right]\left[(1+\varepsilon_2)^2 + (1+\varepsilon_2) + \eta^2\right] + \left[\frac{3\left(1 - 5\cos^2 i\right)}{4a^2\eta^4}\right](\theta - \lambda + \varepsilon_3) \tag{E.9}$$

$$\theta^{(sp1)} = \lambda^{(sp1)} - \left[\frac{\varepsilon_3\left(1 - 3\cos^2 i\right)}{4a^2\eta^4(1+\eta)}\right]\left[(1+\varepsilon_2)^2 + \eta(1+\eta)\right] \tag{E.10}$$

$$i^{(sp1)} = 0 \tag{E.11}$$

$$\begin{aligned} q_1^{(sp1)} &= \left[\frac{\left(1 - 3\cos^2 i\right)}{4a^2\eta^4(1+\eta)}\right]\Big\{\left[(1+\varepsilon_2)^2 + \eta^2\right][q_1 + (1+\eta)\cos\theta] \\ &\quad + (1+\varepsilon_2)\left[(1+\eta)\cos\theta + q_1(\eta - \varepsilon_2)\right]\Big\} \\ &\quad - \left[\frac{3q_2\left(1 - 5\cos^2 i\right)}{4a^2\eta^4}\right](\theta - \lambda + \varepsilon_3) \end{aligned} \tag{E.12}$$

$$\begin{aligned} q_2^{(sp1)} &= \left[\frac{\left(1 - 3\cos^2 i\right)}{4a^2\eta^4(1+\eta)}\right]\Big\{\left[(1+\varepsilon_2)^2 + \eta^2\right][q_2 + (1+\eta)\sin\theta] \\ &\quad + (1+\varepsilon_2)\left[(1+\eta)\sin\theta + q_2(\eta - \varepsilon_2)\right]\Big\} \\ &\quad + \left[\frac{3q_1\left(1 - 5\cos^2 i\right)}{4a^2\eta^4}\right](\theta - \lambda + \varepsilon_3) \end{aligned} \tag{E.13}$$

$$\Omega^{(sp1)} = \left(\frac{3\cos i}{2a^2\eta^4}\right)[(\theta - \lambda) + \varepsilon_3] \tag{E.14}$$

$$a^{(sp2)} = -\left(\frac{3\sin^2 i}{2a\eta^6}\right)(1+\varepsilon_2)^3\cos 2\theta \tag{E.15}$$

$$\begin{aligned} \lambda^{(sp2)} &= -\left[\frac{3\varepsilon_3\sin^2 i\cos 2\theta}{4a^2\eta^4(1+\eta)}\right](1+\varepsilon_2)(2+\varepsilon_2) - \left[\frac{\sin^2 i}{8a^2\eta^2(1+\eta)}\right] \\ &\quad \times \left[3\left(q_1\sin\theta + q_2\cos\theta\right) + \left(q_1\sin 3\theta - q_2\cos 3\theta\right)\right] \\ &\quad - \left[\frac{\left(3 - 5\cos^2 i\right)}{8a^2\eta^4}\right]\left[3\left(q_1\sin\theta + q_2\cos\theta\right) + 3\sin 2\theta\right. \\ &\quad \left.+ \left(q_1\sin 3\theta - q_2\cos 3\theta\right)\right] \end{aligned} \tag{E.16}$$

$$\begin{aligned} \theta^{(sp2)} &= \lambda^{(sp2)} - \left[\frac{\sin^2 i}{32a^2\eta^4(1+\eta)}\right] \\ &\quad \times \left\{\begin{array}{l} 36q_1q_2 - 4\left(3\eta^2 + 5\eta - 1\right)\left(q_1\sin\theta + q_2\cos\theta\right) \\ + 12\varepsilon_2 q_1 q_2 - 32(1+\eta)\sin 2\theta \\ - (\eta^2 + 12\eta + 39)(q_1\sin 3\theta - q_2\cos 3\theta) \\ + 36q_1q_2\cos 4\theta - 18\left(q_1^2 - q_2^2\right)\sin 4\theta \\ - 3\left(q_1^2 - q_2^2\right)q_1\sin 5\theta + 3\left(3q_1^2 - q_2^2\right)q_2\cos 5\theta \end{array}\right\} \end{aligned} \tag{E.17}$$

$$i^{(sp2)} = -\left(\frac{\sin 2i}{8a^2\eta^4}\right)[3\,(q_1\cos\theta - q_2\sin\theta) \\ +3\cos 2\theta + (q_1\cos 3\theta + q_2\sin 3\theta)] \tag{E.18}$$

$$\begin{aligned} q_1^{(sp2)} = {} & \left[\frac{q_2\left(3-5\cos^2 i\right)}{8a^2\eta^4}\right][3\,(q_1\sin\theta + q_2\cos\theta) + 3\sin 2\theta \\ & + (q_1\sin 3\theta - q_2\cos 3\theta)] \\ & + \left(\frac{\sin^2 i}{8a^2\eta^4}\right)\left[3\left(\eta^2 - q_1^2\right)\cos\theta + 3q_1q_2\sin\theta\right. \\ & \left. - \left(\eta^2 + 3q_1^2\right)\cos 3\theta - 3q_1q_2\sin 3\theta\right] - \left(\frac{3\sin^2 i\cos 2\theta}{16a^2\eta^4}\right) \\ & \times \begin{bmatrix} 10q_1 + \left(8 + 3q_1^2 + q_2^2\right)\cos\theta + 2q_1q_2\sin\theta \\ +6\,(q_1\cos 2\theta + q_2\sin 2\theta) \\ + \left(q_1^2 - q_2^2\right)\cos 3\theta + 2q_1q_2\sin 3\theta \end{bmatrix} \end{aligned} \tag{E.19}$$

$$\begin{aligned} q_2^{(sp2)} = {} & -\left[\frac{q_1\left(3-5\cos^2 i\right)}{8a^2\eta^4}\right][3\,(q_1\sin\theta + q_2\cos\theta) \\ & + 3\sin 2\theta + (q_1\sin 3\theta - q_2\cos 3\theta)] \\ & - \left(\frac{\sin^2 i}{8a^2\eta^4}\right)\left[3\left(\eta^2 - q_2^2\right)\sin\theta + 3q_1q_2\cos\theta\right. \\ & \left. + \left(\eta^2 + 3q_2^2\right)\sin 3\theta + 3q_1q_2\cos 3\theta\right] - \left(\frac{3\sin^2 i\cos 2\theta}{16a^2\eta^4}\right) \\ & \times \begin{bmatrix} 10q_2 + \left(8 + q_1^2 + 3q_2^2\right)\sin\theta + 2q_1q_2\cos\theta \\ +6\,(q_1\sin 2\theta - q_2\cos 2\theta) \\ + \left(q_1^2 - q_2^2\right)\sin 3\theta - 2q_1q_2\cos 3\theta \end{bmatrix} \end{aligned} \tag{E.20}$$

$$\Omega^{(sp2)} = -\left(\frac{\cos i}{4a^2\eta^4}\right)[3\,(q_1\sin\theta + q_2\cos\theta) \\ + 3\sin 2\theta + (q_1\sin 3\theta - q_2\cos 3\theta)] \tag{E.21}$$

$$\lambda_{q_1} = \left(\frac{\partial\lambda}{\partial q_1}\right) = \frac{q_2}{\eta(1+\eta)} + \frac{q_1}{\eta}v_{rt} - \frac{\eta r(a+r)}{p^2}\,(q_2 + \sin\theta) \tag{E.22}$$

$$\lambda_{q_2} = \left(\frac{\partial\lambda}{\partial q_2}\right) = -\frac{q_1}{\eta(1+\eta)} + \frac{q_2}{\eta}v_{rt} + \frac{\eta r(a+r)}{p^2}\,(q_1 + \cos\theta) \tag{E.23}$$

$$\Theta = \frac{1}{\left(1 - 5\cos^2 i\right)} \tag{E.24}$$

$$\varepsilon_1 = \sqrt{q_1^2 + q_2^2} \tag{E.25}$$

$$\varepsilon_2 = q_1 \cos\theta + q_2 \sin\theta \tag{E.26}$$

$$\varepsilon_3 = q_1 \sin\theta - q_2 \cos\theta \tag{E.27}$$

F JACOBIAN FOR MEAN TO OSCULATING ELEMENTS

This appendix contains the Jacobian for the mean to osculating transformation. The variables $\varepsilon_1, \varepsilon_2, \varepsilon_3$ were defined in Eqs. (E.25)–(E.27). The Jacobian D is defined as

$$D = \frac{\partial \boldsymbol{\text{œ}}}{\partial \overline{\boldsymbol{\text{œ}}}} = I - J_2 R_e^2 \left(D^{(lp)} + D^{(sp1)} + D^{(sp2)} \right) \tag{F.1}$$

$$D_{11}^{(lp)} = -\left(\frac{1}{a}\right) a^{(lp)},\ D_{12}^{(lp)} = D_{13}^{(lp)} = D_{14}^{(lp)} = D_{15}^{(lp)} = D_{16}^{(lp)} = 0 \tag{F.2}$$

$$\begin{aligned}
D_{21}^{(lp)} &= -\left(\frac{2}{a}\right)\theta^{(lp)} \\
D_{22}^{(lp)} &= -\left(\frac{\sin^2 i}{16a^2\eta^4}\right)\left(1 - 10\Theta\cos^2 i\right)\left[2\left(q_1\cos\theta - q_2\sin\theta\right) + \varepsilon_1\cos 2\theta\right] \\
D_{23}^{(lp)} &= \left(\frac{\sin 2i}{16a^2\eta^4}\right)\Big\{5q_1q_2\Big(11 + 112\Theta\cos^2 i \\
&\quad + 520\Theta^2\cos^4 i + 800\Theta^3\cos^6 i\Big) \\
&\quad - \left[2q_1q_2 + (2+\varepsilon_2)(q_1\sin\theta + q_2\cos\theta)\right] \\
&\quad \times\left[\left(1 - 10\Theta\cos^2 i\right) + 10\Theta\sin^2 i\left(1 + 5\Theta\cos^2 i\right)\right]\Big\} \\
D_{24}^{(lp)} &= \left(\frac{1}{16a^2\eta^6}\right)\Big\{\left(\eta^2 + 4q_1^2\right) \\
&\quad \times\Big[q_2\left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right) \\
&\quad - \sin^2 i\left(1 - 10\Theta\cos^2 i\right)(3q_2 + 2\sin\theta)\Big] \\
&\quad -2\sin^2 i\left(1 - 10\Theta\cos^2 i\right)\left[4q_2 + \sin\theta(1+\varepsilon_1)\right]q_1\cos\theta\Big\} \\
D_{25}^{(lp)} &= \left(\frac{1}{16a^2\eta^6}\right)\Big\{\left(\eta^2 + 4q_2^2\right) \\
&\quad \times\Big[q_1\left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right) \\
&\quad - \sin^2 i\left(1 - 10\Theta\cos^2 i\right)(3q_1 + 2\cos\theta)\Big] \\
&\quad - 2\sin^2 i\left(1 - 10\Theta\cos^2 i\right)\left[4q_1 + \cos\theta(1+\varepsilon_1)\right]q_2\sin\theta\Big\} \\
D_{26}^{(lp)} &= 0
\end{aligned} \tag{F.3}$$

$$
\begin{aligned}
D_{31}^{(lp)} &= -\left(\frac{2}{a}\right) i^{(lp)},\ D_{32}^{(lp)} = 0 \\
D_{33}^{(lp)} &= \left(\frac{q_1^2 - q_2^2}{16a^2\eta^4}\right)\left[\cos 2i\left(1 - 10\Theta\cos^2 i\right) + 5\Theta\sin^2 2i\left(1 + 5\Theta\cos^2 i\right)\right] \\
D_{34}^{(lp)} &= \left(\frac{q_1 \sin 2i}{16a^2\eta^6}\right)\left(1 - 10\Theta\cos^2 i\right)\left[\eta^2 + 2\left(q_1^2 - q_2^2\right)\right] \\
D_{35}^{(lp)} &= -\left(\frac{q_2 \sin 2i}{16a^2\eta^6}\right)\left(1 - 10\Theta\cos^2 i\right)\left[\eta^2 - 2\left(q_1^2 - q_2^2\right)\right] \\
D_{36}^{(lp)} &= 0
\end{aligned}
\tag{F.4}
$$

$$
\begin{aligned}
D_{41}^{(lp)} &= -\left(\frac{2}{a}\right) q_1^{(lp)},\ D_{42}^{(lp)} = 0 \\
D_{43}^{(lp)} &= -\left(\frac{q_1 \sin 2i}{16a^2\eta^4}\right)\left[\eta^2\left[\left(1 - 10\Theta\cos^2 i\right) + 10\Theta\sin^2 i\left(1 + 5\Theta\cos^2 i\right)\right]\right. \\
&\quad \left. + 5q_2^2\left(11 + 112\Theta\cos^2 i + 520\Theta^2\cos^4 i + 800\Theta^3\cos^6 i\right)\right] \\
D_{44}^{(lp)} &= -\left(\frac{1}{16a^2\eta^6}\right)\left[\eta^2\sin^2 i\left(1 - 10\Theta\cos^2 i\right)\left(\eta^2 + 2q_1^2\right)\right. \\
&\quad \left. + q_2^2\left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right)\left(\eta^2 + 4q_1^2\right)\right] \\
D_{45}^{(lp)} &= -\left(\frac{q_1 q_2}{8a^2\eta^6}\right)\left[\eta^2\sin^2 i\left(1 - 10\Theta\cos^2 i\right)\right. \\
&\quad \left. + \left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right)\left(\eta^2 + 2q_2^2\right)\right] \\
D_{46}^{(lp)} &= 0
\end{aligned}
\tag{F.5}
$$

$$
\begin{aligned}
D_{51}^{(lp)} &= -\left(\frac{2}{a}\right) q_2^{(lp)},\ D_{52}^{(lp)} = 0 \\
D_{53}^{(lp)} &= \left(\frac{q_2 \sin 2i}{16a^2\eta^4}\right)\left[\eta^2\left[\left(1 - 10\Theta\cos^2 i\right) + 10\Theta\sin^2 i\left(1 + 5\Theta\cos^2 i\right)\right]\right. \\
&\quad \left. + 5q_1^2\left(11 + 112\Theta\cos^2 i + 520\Theta^2\cos^4 i + 800\Theta^3\cos^6 i\right)\right] \\
D_{54}^{(lp)} &= \left(\frac{q_1 q_2}{8a^2\eta^6}\right)\left[\eta^2\sin^2 i\left(1 - 10\Theta\cos^2 i\right)\right. \\
&\quad \left. + \left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right)\left(\eta^2 + 2q_1^2\right)\right] \\
D_{55}^{(lp)} &= \left(\frac{1}{16a^2\eta^6}\right)\left[\eta^2\sin^2 i\left(1 - 10\Theta\cos^2 i\right)\left(\eta^2 + 2q_2^2\right)\right. \\
&\quad \left. + q_1^2\left(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i\right)\left(\eta^2 + 4q_2^2\right)\right] \\
D_{56}^{(lp)} &= 0
\end{aligned}
\tag{F.6}
$$

$$
\begin{aligned}
D_{61}^{(lp)} &= -\left(\frac{2}{a}\right)\Omega^{(lp)},\ D_{62}^{(lp)} = 0 \\
D_{63}^{(lp)} &= -\left(\frac{q_1 q_2 \sin i}{8a^2\eta^4}\right)\left[\left(11 + 80\Theta\cos^2 i + 200\Theta^2\cos^4 i\right)\right. \\
&\qquad \left. +160\Theta\cos^2 i\left(1 + 5\Theta\cos^2 i\right)^2\right] \\
D_{64}^{(lp)} &= \left(\frac{q_2 \cos i}{8a^2\eta^6}\right)\left(11 + 80\Theta\cos^2 i + 200\Theta^2\cos^4 i\right)\left(\eta^2 + 4q_1^2\right) \\
D_{65}^{(lp)} &= \left(\frac{q_1 \cos i}{8a^2\eta^6}\right)\left(11 + 80\Theta\cos^2 i + 200\Theta^2\cos^4 i\right)\left(\eta^2 + 4q_2^2\right) \\
D_{66}^{(lp)} &= 0
\end{aligned}
\tag{F.7}
$$

$$
\begin{aligned}
D_{11}^{(sp1)} &= -\left(\frac{1}{a}\right)a^{(sp1)} \\
D_{12}^{(sp1)} &= -\left(\frac{3\varepsilon_3}{2a\eta^6}\right)\left(1 - 3\cos^2 i\right)(1+\varepsilon_2)^2 \\
D_{13}^{(sp1)} &= \left(\frac{3\sin 2i}{2a\eta^6}\right)\left[(1+\varepsilon_2)^3 - \eta^3\right] \\
D_{14}^{(sp1)} &= \left[\frac{3\left(1 - 3\cos^2 i\right)}{2a\eta^8}\right] \\
&\quad \times\left[2q_1(1+\varepsilon_2)^3 + \eta^2(1+\varepsilon_2)^2\cos\theta - \eta^3 q_1\right] \\
D_{15}^{(sp1)} &= \left[\frac{3\left(1 - 3\cos^2 i\right)}{2a\eta^8}\right] \\
&\quad \times\left[2q_2(1+\varepsilon_2)^3 + \eta^2(1+\varepsilon_2)^2\sin\theta - \eta^3 q_2\right] \\
D_{16}^{(sp1)} &= 0
\end{aligned}
\tag{F.8}
$$

$$
\begin{aligned}
D_{21}^{(sp1)} &= -\left(\frac{2}{a}\right)\theta^{(sp1)} \\
D_{22}^{(sp1)} &= \left[\frac{\left(1 - 3\cos^2 i\right)}{4a^2\eta^4(1+\eta)}\right]\left[\varepsilon_2(1+\varepsilon_2-\eta) - \varepsilon_3^2\right] \\
&\quad + \left[\frac{3\left(1 - 5\cos^2 i\right)}{4a^2\eta^4(1+\varepsilon_2)^2}\right]\left[(1+\varepsilon_2)^3 - \eta^3\right] \\
D_{23}^{(sp1)} &= \left[\frac{3\varepsilon_3\sin 2i}{4a^2\eta^4(1+\eta)}\right]\left[(1+\varepsilon_2) + (5+4\eta)\right] + \left(\frac{15\sin 2i}{4a^2\eta^4}\right)(\theta - \lambda)
\end{aligned}
\tag{F.9}
$$

$$
\begin{aligned}
D_{24}^{(sp1)} &= \left[\frac{\left(1-3\cos^2 i\right)}{4a^2\eta^6(1+\eta)^2}\right]\Big\{\eta^2\left[\varepsilon_1\sin\theta+(1+\eta)(\varepsilon_2\sin\theta+\varepsilon_3\cos\theta)\right] \\
&\quad + q_1\varepsilon_3\left[4(\varepsilon_1+\varepsilon_2)+\eta(2+5\varepsilon_2)\right]\Big\} \\
&\quad + \left[\frac{3\left(1-5\cos^2 i\right)}{4a^2\eta^6}\right]\Big\{4q_1\left[(\theta-\lambda)+\varepsilon_3\right]+\eta^2\sin\theta\Big\} \\
&\quad - \left[\frac{3\left(1-5\cos^2 i\right)}{4a^2\eta^4}\right]\left(\lambda_{q_1}\right) \\
D_{25}^{(sp1)} &= -\left[\frac{\left(1-3\cos^2 i\right)}{4a^2\eta^6(1+\eta)^2}\right]\Big\{\eta^2\left[\varepsilon_1\cos\theta+(1+\eta)(\varepsilon_2\cos\theta-\varepsilon_3\sin\theta)\right] \\
&\quad - q_2\varepsilon_3\left[4(\varepsilon_1+\varepsilon_2)+\eta(2+5\varepsilon_2)\right]\Big\} \\
&\quad + \left[\frac{3\left(1-5\cos^2 i\right)}{4a^2\eta^6}\right]\Big\{4q_2\left[(\theta-\lambda)+\varepsilon_3\right]-\eta^2\cos\theta\Big\} \\
&\quad - \left[\frac{3\left(1-5\cos^2 i\right)}{4a^2\eta^4}\right]\left(\lambda_{q_2}\right) \\
D_{26}^{(sp1)} &= 0 \\
D_{31}^{(sp1)} &= -\left(\frac{2}{a}\right) i^{(sp1)}, \\
D_{32}^{(sp1)} &= D_{33}^{(sp1)} = D_{34}^{(sp1)} = D_{35}^{(sp1)} = D_{36}^{(sp1)} = 0 \\
D_{41}^{(sp1)} &= -\left(\frac{2}{a}\right) q_1^{(sp1)} \\
D_{42}^{(sp1)} &= -\left[\frac{\left(1-3\cos^2 i\right)}{4a^2\eta^4}\right]\Big[(1+\varepsilon_2)(2\sin\theta+\varepsilon_2\sin\theta+2\varepsilon_3\cos\theta) \\
&\quad + \varepsilon_3(q_1+\cos\theta)+\eta^2\sin\theta\Big] \\
&\quad - \left[\frac{3q_2\left(1-5\cos^2 i\right)}{4a^2\eta^4(1+\varepsilon_2)^2}\right]\Big[(1+\varepsilon_2)^3-\eta^3\Big] \\
D_{43}^{(sp1)} &= \left[\frac{3q_1\sin 2i}{4a^2\eta^2(1+\eta)}\right]+\left(\frac{3\sin 2i}{4a^2\eta^4}\right)\Big\{(1+\varepsilon_2)\left[q_1+(2+\varepsilon_2)\cos\theta\right] \\
&\quad - 5q_2\varepsilon_3+\eta^2\cos\theta\Big\} - \left(\frac{15q_2\sin 2i}{4a^2\eta^4}\right)(\theta-\lambda)
\end{aligned}
\tag{F.10}
$$

$$\begin{aligned}
D_{44}^{(sp1)} &= \left[\frac{\left(1-3\cos^2 i\right)}{4a^2\eta^2(1+\eta)}\right] + \left[\frac{\left(1-3\cos^2 i\right)}{8a^2\eta^6}\right] \\
&\times \left\{\eta^2\left[5+2(5q_1\cos\theta+2q_2\sin\theta)+(3+2\varepsilon_2)\cos 2\theta\right]\right. \\
&+ \left. 2q_1\left[4(1+\varepsilon_2)(2+\varepsilon_2)\cos\theta+(3\eta+4\varepsilon_2)q_1\right]\right\} \\
&+ \left[\frac{\left(1-3\cos^2 i\right)q_1^2(4+5\eta)}{4a^2\eta^6(1+\eta)^2}\right] \\
&- \left[\frac{3q_2\left(1-5\cos^2 i\right)}{4a^2\eta^6}\right]\left(4q_1\varepsilon_3+\eta^2\sin\theta\right) \\
&- \left[\frac{3q_1q_2\left(1-5\cos^2 i\right)}{a^2\eta^6}\right](\theta-\lambda) + \left[\frac{3q_2\left(1-5\cos^2 i\right)}{4a^2\eta^4}\right]\left(\lambda_{q_1}\right)
\end{aligned} \tag{F.11}$$

$$\begin{aligned}
D_{45}^{(sp1)} &= \left[\frac{\left(1-3\cos^2 i\right)}{8a^2\eta^6}\right]\left\{\eta^2\left[2(q_1\sin\theta+2q_2\cos\theta)+(3+2\varepsilon_2)\sin 2\theta\right]\right. \\
&+ \left. 2q_2\left[4(1+\varepsilon_2)(2+\varepsilon_2)\cos\theta+(3\eta+4\varepsilon_2)q_1\right]\right\} \\
&+ \left[\frac{\left(1-3\cos^2 i\right)q_1q_2(4+5\eta)}{4a^2\eta^6(1+\eta)^2}\right] \\
&- \left[\frac{3\left(1-5\cos^2 i\right)}{4a^2\eta^6}\right]\left[\varepsilon_3\left(\eta^2+4q_2^2\right)-\eta^2q_2\cos\theta\right] \\
&- \left[\frac{3\left(1-5\cos^2 i\right)}{4a^2\eta^6}\right]\left[\left(\eta^2+4q_2^2\right)(\theta-\lambda)\right] \\
&+ \left[\frac{3q_2\left(1-5\cos^2 i\right)}{4a^2\eta^4}\right]\left(\lambda_{q_2}\right)
\end{aligned}$$

$$D_{46}^{(sp1)} = 0$$

$$\begin{aligned}
D_{51}^{(sp1)} &= -\left(\frac{2}{a}\right)q_2^{(sp1)} \\
D_{52}^{(sp1)} &= \left[\frac{\left(1-3\cos^2 i\right)}{4a^2\eta^4}\right]\left[(1+\varepsilon_2)(2\cos\theta+\varepsilon_2\cos\theta-2\varepsilon_3\sin\theta)\right. \\
&- \left.\varepsilon_3(q_2+\sin\theta)+\eta^2\cos\theta\right] \\
&+ \left[\frac{3q_1\left(1-5\cos^2 i\right)}{4a^2\eta^4(1+\varepsilon_2)^2}\right]\left[(1+\varepsilon_2)^3-\eta^3\right] \\
D_{53}^{(sp1)} &= \left[\frac{3q_2\sin 2i}{4a^2\eta^2(1+\eta)}\right] \\
&+ \left(\frac{3\sin 2i}{4a^2\eta^4}\right)\left\{(1+\varepsilon_2)\left[q_2+(2+\varepsilon_2)\sin\theta\right]+5q_1\varepsilon_3+\eta^2\sin\theta\right\} \\
&- \left(\frac{15q_1\sin 2i}{4a^2\eta^4}\right)(\theta-\lambda)
\end{aligned} \tag{F.12}$$

$$
\begin{aligned}
D_{54}^{(sp1)} &= \left[\frac{\left(1-3\cos^2 i\right)}{8a^2\eta^6}\right]\Big\{\eta^2\left[2(2q_1\sin\theta+q_2\cos\theta)+(3+2\varepsilon_2)\sin 2\theta\right] \\
&\quad +\ 2q_1\left[4(1+\varepsilon_2)(2+\varepsilon_2)\sin\theta+(3\eta+4\varepsilon_2)q_2\right]\Big\} \\
&\quad +\left[\frac{\left(1-3\cos^2 i\right)q_1q_2(4+5\eta)}{4a^2\eta^6(1+\eta)^2}\right] \\
&\quad +\left[\frac{3\left(1-5\cos^2 i\right)}{4a^2\eta^6}\right]\left[\varepsilon_3\left(\eta^2+4q_1^2\right)+\eta^2 q_1\sin\theta\right] \\
&\quad +\left[\frac{3\left(1-5\cos^2 i\right)}{4a^2\eta^6}\right]\left[\left(\eta^2+4q_1^2\right)(\theta-\lambda)\right] \\
&\quad -\left[\frac{3q_1\left(1-5\cos^2 i\right)}{4a^2\eta^4}\right]\left(\lambda_{q_1}\right) \\
D_{55}^{(sp1)} &= \left[\frac{\left(1-3\cos^2 i\right)}{4a^2\eta^2(1+\eta)}\right]+\left[\frac{\left(1-3\cos^2 i\right)}{8a^2\eta^6}\right]\Big\{\eta^2[5+2(2q_1\cos\theta+5q_2\sin\theta) \\
&\quad -(3+2\varepsilon_2)\cos 2\theta]+2q_2\left[4(1+\varepsilon_2)(2+\varepsilon_2)\sin\theta+(3\eta+4\varepsilon_2)q_2\right]\Big\} \\
&\quad +\left[\frac{\left(1-3\cos^2 i\right)q_2^2(4+5\eta)}{4a^2\eta^6(1+\eta)^2}\right] \\
&\quad +\left[\frac{3q_1\left(1-5\cos^2 i\right)}{4a^2\eta^6}\right]\left(4q_2\varepsilon_3-\eta^2\cos\theta\right) \\
&\quad +\left[\frac{3q_1q_2\left(1-5\cos^2 i\right)}{a^2\eta^6}\right](\theta-\lambda)-\left[\frac{3q_1\left(1-5\cos^2 i\right)}{4a^2\eta^4}\right]\left(\lambda_{q_2}\right) \\
D_{56}^{(sp1)} &= 0
\end{aligned}
$$

$$
\begin{aligned}
D_{61}^{(sp1)} &= -\left(\frac{2}{a}\right)\Omega^{(sp1)} \\
D_{62}^{(sp1)} &= \left[\frac{3\cos i}{2a^2\eta^4(1+\varepsilon_2)^2}\right]\left[(1+\varepsilon_2)^3-\eta^3\right] \\
D_{63}^{(sp1)} &= -\left(\frac{3\varepsilon_3\sin i}{2a^2\eta^4}\right)-\left(\frac{3\sin i}{2a^2\eta^4}\right)(\theta-\lambda) \\
D_{64}^{(sp1)} &= \left(\frac{3\cos i}{2a^2\eta^6}\right)\left(4q_1\varepsilon_3+\eta^2\sin\theta\right)+\left(\frac{6q_1\cos i}{a^2\eta^6}\right)(\theta-\lambda) \\
&\quad -\left(\frac{3\cos i}{2a^2\eta^4}\right)\left(\lambda_{q_1}\right) \\
D_{65}^{(sp1)} &= \left(\frac{3\cos i}{2a^2\eta^6}\right)\left(4q_2\varepsilon_3-\eta^2\cos\theta\right)+\left(\frac{6q_2\cos i}{a^2\eta^6}\right)(\theta-\lambda) \\
&\quad -\left(\frac{3\cos i}{2a^2\eta^4}\right)\left(\lambda_{q_2}\right) \\
D_{66}^{(sp1)} &= 0
\end{aligned}
\tag{F.13}
$$

$$
\begin{aligned}
D_{11}^{(sp2)} &= -\left(\frac{1}{a}\right) a^{(sp2)} \\
D_{12}^{(sp2)} &= \left(\frac{3\sin^2 i}{2a\eta^6}\right)(1+\varepsilon_2)^2\left[2(1+\varepsilon_2)\sin 2\theta + 3\varepsilon_3\cos 2\theta\right] \\
D_{13}^{(sp2)} &= -\left(\frac{3\sin 2i\cos 2\theta}{2a\eta^6}\right)(1+\varepsilon_2)^3 \\
D_{14}^{(sp2)} &= -\left(\frac{9\sin^2 i\cos 2\theta}{2a\eta^8}\right)(1+\varepsilon_2)^2\left[2q_1(1+\varepsilon_2)+\eta^2\cos\theta\right] \\
D_{15}^{(sp2)} &= -\left(\frac{9\sin^2 i\cos 2\theta}{2a\eta^8}\right)(1+\varepsilon_2)^2\left[2q_2(1+\varepsilon_2)+\eta^2\sin\theta\right] \\
D_{16}^{(sp2)} &= 0
\end{aligned}
\tag{F.14}
$$

$$
\begin{aligned}
D_{21}^{(sp2)} &= -\left(\frac{2}{a}\right)\theta^{(sp2)} \\
D_{22}^{(sp2)} &= -\left(\frac{1}{8a^2\eta^4}\right) \\
&\quad\times\left\{\begin{array}{l}
3\left(3-5\cos^2 i\right)\left[(q_1\cos\theta - q_2\sin\theta) + 2\cos 2\theta\right. \\
\left.+\,(q_1\cos 3\theta + q_2\sin 3\theta)\right] - \sin^2 i\left[5\,(q_1\cos\theta - q_2\sin\theta)\right. \\
\left.+\,16\cos 2\theta + 9\,(q_1\cos 3\theta + q_2\sin 3\theta)\right]
\end{array}\right\} \\
D_{23}^{(sp2)} &= -\left(\frac{\sin 2i}{8a^2\eta^4}\right)\left[10\,(q_1\sin\theta + q_2\cos\theta)\right. \\
&\quad\left.+\,7\sin 2\theta + 2\,(q_1\sin 3\theta - q_2\cos 3\theta)\right]
\end{aligned}
\tag{F.15}
$$

$$
\begin{aligned}
D_{24}^{(sp2)} &= -\left[\frac{\left(3-5\cos^2 i\right)}{8a^2\eta^6}\right]\left\{4q_1\left[3\sin 2\theta + q_2\,(3\cos\theta - \cos 3\theta)\right]\right. \\
&\left.+\left(\eta^2+4q_1^2\right)(3\sin\theta + \sin 3\theta)\right\} - \left[\frac{\sin^2 i}{8a^2\eta^2(1+\eta)}\right] \\
&\times(3\sin\theta + \sin 3\theta) - \left[\frac{\sin^2 i}{32a^2\eta^4(1+\eta)}\right] \\
&\times\left\{\begin{array}{l}
36q_2 - 4(2+3\eta)\sin\theta - \left(39+12\eta+\eta^2\right)\sin 3\theta + 9\varepsilon_1\sin 5\theta \\
+\,12q_2\,(2q_1\cos\theta + q_2\sin\theta) + 9q_1\,(q_1\sin 3\theta - q_2\cos 3\theta) \\
+\,18\,(3q_1\sin 4\theta + 2q_2\cos 4\theta) - 3q_1\,(q_1\sin 5\theta - 11q_2\cos 5\theta) \\
+\,24\left[(1+\varepsilon_2)(2+\varepsilon_2)\sin\theta + \varepsilon_3(3+2\varepsilon_2)\cos\theta\right]\cos 2\theta
\end{array}\right\} \\
&-\left[\frac{3\sin^2 i}{32a^2\eta^4(1+\eta)^2}\right]\left[4\sin\theta - 6q_1\sin 4\theta - q_1\,(q_1\sin 5\theta + q_2\cos 5\theta)\right] \\
&+\left[\frac{q_1\sin^2 i}{8a^2\eta^6(1+\eta)}\right]\left[\begin{array}{l}
20(1+\eta)\,(q_1\sin\theta + q_2\cos\theta) + 32(1+\eta)\sin 2\theta \\
+\,3(4+3\eta)\,(q_1\sin 3\theta - q_2\cos 3\theta)
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\left[\frac{q_1 \sin^2 i\,(4+5\eta)}{32a^2\eta^6(1+\eta)^2}\right] \\
&\times \left\{\begin{array}{l} 24\,(q_1 \sin\theta + q_2\cos\theta) + 24\varepsilon_3(1+\varepsilon_2)(2+\varepsilon_2)\cos 2\theta \\ -\,(27+3\eta)\,(q_1\sin 3\theta - q_2\cos 3\theta) - 18\sin 4\theta \\ -\,3\,(q_1\sin 5\theta + q_2\cos 5\theta) + 12q_2[(3+\varepsilon_2)q_1 \\ +\,3\,(q_1\cos 4\theta + q_2\sin 4\theta) + q_1\,(q_1\cos 5\theta + q_2\sin 5\theta)] \end{array}\right\} \\
D_{25}^{(sp2)} &= -\left[\frac{\left(3-5\cos^2 i\right)}{8a^2\eta^6}\right]\Big\{4q_2\,[3\sin 2\theta + q_1\,(3\sin\theta + \sin 3\theta)] \\
&\quad + \left(\eta^2 + 4q_2^2\right)(3\cos\theta - \cos 3\theta)\Big\} - \left[\frac{\sin^2 i}{8a^2\eta^2(1+\eta)}\right] \\
&\quad \times (3\cos\theta - \cos 3\theta) - \left[\frac{\sin^2 i}{32a^2\eta^4(1+\eta)}\right] \\
&\quad \times \left\{\begin{array}{l} 36q_1 - 4(2+3\eta)\cos\theta + \left(39 + 12\eta + \eta^2\right)\cos 3\theta + 9\varepsilon_1\cos 5\theta \\ +\,12q_1\,(q_1\cos\theta + 2q_2\sin\theta) + 9q_2\,(q_1\sin 3\theta - q_2\cos 3\theta) \\ +\,18\,(2q_1\cos 4\theta + 7q_2\sin 4\theta) + 3q_2\,(11q_1\sin 5\theta - q_2\cos 5\theta) \\ +\,24\,[\varepsilon_3(3+2\varepsilon_2)\sin\theta - (1+\varepsilon_2)(2+\varepsilon_2)\cos\theta]\cos 2\theta \end{array}\right\} \\
&\quad - \left[\frac{3\sin^2 i}{32a^2\eta^4(1+\eta)^2}\right][4\cos\theta - 6q_2\sin 4\theta - q_2\,(q_1\sin 5\theta + q_2\cos 5\theta)] \\
&\quad + \left[\frac{q_2\sin^2 i}{8a^2\eta^6(1+\eta)}\right]\left[\begin{array}{l} 20(1+\eta)\,(q_1\sin\theta + q_2\cos\theta) + 32(1+\eta)\sin 2\theta \\ +\,3(4+3\eta)\,(q_1\sin 3\theta - q_2\cos 3\theta) \end{array}\right] \\
&\quad - \left[\frac{q_1\sin^2 i\,(4+5\eta)}{32a^2\eta^6(1+\eta)^2}\right] \\
&\quad \times \left\{\begin{array}{l} 24\,(q_1 \sin\theta + q_2\cos\theta) + 24\varepsilon_3(1+\varepsilon_2)(2+\varepsilon_2)\cos 2\theta \\ -\,(27+3\eta)\,(q_1\sin 3\theta - q_2\cos 3\theta) - 18\sin 4\theta \\ -\,3\,(q_1\sin 5\theta + q_2\cos 5\theta) + 12q_2[(3+\varepsilon_2)q_1 \\ +\,3\,(q_1\cos 4\theta + q_2\sin 4\theta) + q_1\,(q_1\cos 5\theta + q_2\sin 5\theta)] \end{array}\right\} \\
D_{26}^{(sp2)} &= 0
\end{aligned}
$$

$$
\begin{aligned}
D_{31}^{(sp2)} &= -\left(\frac{2}{a}\right) i^{(sp2)} \\
D_{32}^{(sp2)} &= \left(\frac{3\sin 2i}{8a^2\eta^4}\right)[(q_1\sin\theta + q_2\cos\theta) + 2\sin 2\theta \\
&\quad + (q_1\sin 3\theta - q_2\cos 3\theta)] \\
D_{33}^{(sp2)} &= -\left(\frac{\cos 2i}{4a^2\eta^4}\right)[3\,(q_1\cos\theta - q_2\sin\theta) + 3\cos 2\theta \\
&\quad + (q_1\cos 3\theta + q_2\sin 3\theta)]
\end{aligned} \tag{F.16}
$$

$$
\begin{aligned}
D_{34}^{(sp2)} &= -\left(\frac{\sin 2i}{8a^2\eta^6}\right)\Big\{4q_1\left[3\cos 2\theta - q_2\left(3\sin\theta - \sin 3\theta\right)\right] \\
&\quad + \left(\eta^2 + 4q_1^2\right)\left(3\cos\theta + \cos 3\theta\right)\Big\} \\
D_{35}^{(sp2)} &= -\left(\frac{\sin 2i}{8a^2\eta^6}\right)\Big\{4q_2\left[3\cos 2\theta + q_1\left(3\cos\theta + \cos 3\theta\right)\right] \\
&\quad - \left(\eta^2 + 4q_2^2\right)\left(3\sin\theta - \sin 3\theta\right)\Big\} \\
D_{36}^{(sp2)} &= 0
\end{aligned}
$$

$$
\begin{aligned}
D_{41}^{(sp2)} &= -\left(\frac{2}{a}\right) q_1^{(sp2)} \\
D_{42}^{(sp2)} &= \left[\frac{3q_2\left(3 - 5\cos^2 i\right)}{8a^2\eta^4}\right]\left[\left(q_1\cos\theta - q_2\sin\theta\right)\right. \\
&\quad \left. + 2\cos 2\theta + \left(q_1\cos 3\theta + q_2\sin 3\theta\right)\right] \\
&\quad + \left(\frac{3\sin^2 i}{32a^2\eta^4}\right)\left\{\begin{array}{l} 2\left[\begin{array}{l} 2q_2\varepsilon_2 - 9q_2\left(q_1\cos 3\theta + q_2\sin 3\theta\right) \\ +12\left(q_1\sin 4\theta - q_2\cos 4\theta\right) \\ -5q_2\left(q_1\cos 5\theta + q_2\sin 5\theta\right) \end{array}\right] \\ +\Big[4\left(1 + 3q_1^2\right)\sin\theta + 40q_1\sin 2\theta \\ +(28 + 17\varepsilon_1)\sin 3\theta + 5\varepsilon_1\sin 5\theta\Big] \end{array}\right\}
\end{aligned}
$$

$$
D_{43}^{(sp2)} = -\left(\frac{\sin 2i}{32a^2\eta^4}\right)\left\{\begin{array}{l} 2\left[\begin{array}{l} 36q_1\left(q_1\cos\theta - q_2\sin\theta\right) \\ + 30\left(q_1\cos 2\theta - q_2\sin 2\theta\right) \\ - q_2\left(q_1\sin 3\theta - q_2\cos 3\theta\right) \\ + 9\left(q_1\cos 4\theta + q_2\sin 4\theta\right) \\ + 3q_2\left(q_1\sin 5\theta - q_2\cos 5\theta\right) \end{array}\right] \\ + \left[6q_1\left(3 + 2q_1\cos\theta\right) + 12(1 - 4\varepsilon_1)\cos\theta\right. \\ \left. + (28 + 17\varepsilon_1)\cos 3\theta + 3\varepsilon_1\cos 5\theta\right] \end{array}\right\}
$$

$$
\begin{aligned}
D_{44}^{(sp2)} &= \left[\frac{q_2\left(3 - 5\cos^2 i\right)}{8a^2\eta^6}\right]\Big\{4q_1\left[3\sin 2\theta + q_2\left(3\cos\theta - \cos 3\theta\right)\right] \\
&\quad + \left(\eta^2 + 4q_1^2\right)\left(3\sin\theta + \sin 3\theta\right)\Big\} \\
&\quad - \left(\frac{\sin^2 i}{8a^2\eta^4}\right)\left\{\begin{array}{l} \left[8q_1\cos 3\theta - 3q_2\left(\sin\theta - \sin 3\theta\right)\right] \\ +3\left[5 + \varepsilon_2 + 3\cos 2\theta + 3\left(q_1\cos 3\theta\right.\right. \\ \left.\left. +q_2\sin 3\theta\right)\right]\cos 2\theta \end{array}\right\} \\
&\quad - \left(\frac{3q_1\sin^2 i}{4a^2\eta^6}\right)\left\{\begin{array}{l} 2q_1\left[\left(q_1\cos\theta - q_2\sin\theta\right)\right. \\ \left. + \left(q_1\cos 3\theta + q_2\sin 3\theta\right)\right] \\ + \left[\begin{array}{l} 9\cos\theta - \cos 3\theta + 2q_1(5 + \varepsilon_2) \\ +6\left(q_1\cos 2\theta + q_2\sin 2\theta\right) \\ +2q_1\left(q_1\cos 3\theta + q_2\sin 3\theta\right) \end{array}\right]\cos 2\theta \end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
D_{45}^{(sp2)} &= \left[\frac{\left(3-5\cos^2 i\right)}{8a^2\eta^6}\right]\left\{\left(\eta^2+4q_2^2\right)\left[3\sin 2\theta+q_1\left(3\sin\theta+\sin 3\theta\right)\right]\right. \\
&\quad \left.+2q_2\left(\eta^2+2q_2^2\right)\left(3\cos\theta-\cos 3\theta\right)\right\} \\
&\quad +\left(\frac{\sin^2 i}{16a^2\eta^4}\right)\left[6\left(q_1\sin\theta+2q_2\cos\theta\right)-\left(9q_1\sin 3\theta+q_2\cos 3\theta\right)\right. \\
&\quad \left.-9\sin 4\theta-3\left(q_1\sin 5\theta+q_2\cos 5\theta\right)\right] \\
&\quad -\left(\frac{3q_2\sin^2 i}{8a^2\eta^6}\right)\left\{\begin{array}{l} 2q_1\left[\begin{array}{l} 3+2\left(2q_1\cos\theta-q_2\sin\theta\right) \\ +10\cos 2\theta \\ +3\left(q_1\cos 3\theta+q_2\sin 3\theta\right) \\ +\left(q_1\cos 5\theta+q_2\sin 5\theta\right) \end{array}\right] \\ +\left[8\cos\theta+9\cos 3\theta\right. \\ \left.+6\left(q_1\cos 4\theta+q_2\sin 4\theta\right)-\cos 5\theta\right] \end{array}\right\}
\end{aligned}
$$

$$
D_{46}^{(sp2)} = 0 \tag{F.17}
$$

$$
\begin{aligned}
D_{51}^{(sp2)} &= -\left(\frac{2}{a}\right)q_2^{(sp2)} \\
D_{52}^{(sp2)} &= -\left[\frac{3q_1\left(3-5\cos^2 i\right)}{8a^2\eta^4}\right]\left[\left(q_1\cos\theta-q_2\sin\theta\right)\right. \\
&\quad \left.+2\cos 2\theta+\left(q_1\cos 3\theta+q_2\sin 3\theta\right)\right] \\
&\quad +\left(\frac{3\sin^2 i}{32a^2\eta^4}\right)\left\{\begin{array}{l} 2\left[\begin{array}{l} 2q_1\varepsilon_2+9q_1\left(q_1\cos 3\theta+q_2\sin 3\theta\right) \\ -12\left(q_1\cos 4\theta+q_2\sin 4\theta\right) \\ -5q_1\left(q_1\cos 5\theta+q_2\sin 5\theta\right) \end{array}\right] \\ +\left[4\left(1+3q_2^2\right)\cos\theta+40q_2\sin 2\theta\right. \\ \left.-\left(28+17\varepsilon_1\right)\cos 3\theta+5\varepsilon_1\cos 5\theta\right] \end{array}\right\} \\
D_{53}^{(sp2)} &= -\left(\frac{\sin 2i}{32a^2\eta^4}\right)\left\{\begin{array}{l} 2\left[\begin{array}{l} 36q_1\left(q_1\sin\theta+q_2\cos\theta\right) \\ +30\left(q_1\sin 2\theta+q_2\cos 2\theta\right) \\ +q_1\left(q_1\sin 3\theta-q_2\cos 3\theta\right) \\ +9\left(q_1\sin 4\theta-q_2\cos 4\theta\right) \\ +3q_1\left(q_1\sin 5\theta-q_2\cos 5\theta\right) \end{array}\right] \\ -\left[6q_2\left(3+2q_2\sin\theta\right)+12(1+2\varepsilon_1)\sin\theta\right. \\ \left.-\left(28+17\varepsilon_1\right)\sin 3\theta+3\varepsilon_1\sin 5\theta\right] \end{array}\right\} \\
D_{54}^{(sp2)} &= -\left[\frac{\left(3-5\cos^2 i\right)}{8a^2\eta^6}\right]\left\{\left(\eta^2+4q_1^2\right)\left[3\sin 2\theta+q_2\left(3\cos\theta-\cos 3\theta\right)\right]\right. \\
&\quad \left.+2q_1\left(\eta^2+2q_1^2\right)\left(3\sin\theta+\sin 3\theta\right)\right\} \\
&\quad -\left(\frac{\sin^2 i}{16a^2\eta^4}\right)\left[6\left(2q_1\sin\theta+q_2\cos\theta\right)+\left(q_1\sin 3\theta+9q_2\cos 3\theta\right)\right. \\
&\quad \left.+9\sin 4\theta-3\left(q_1\sin 5\theta+q_2\cos 5\theta\right)\right]
\end{aligned}
$$

$$
+\left(\frac{3q_1\sin^2 i}{8a^2\eta^6}\right)\left\{\begin{array}{l} 2q_2\left[\begin{array}{l} 3-2\left(2q_1\cos\theta-2q_2\sin\theta\right)\\ -10\cos 2\theta-3\left(q_1\cos 3\theta+q_2\sin 3\theta\right)\\ +\left(q_1\cos 5\theta+q_2\sin 5\theta\right)\end{array}\right]\\ +\left[8\sin\theta-9\sin 3\theta-6\left(q_1\sin 4\theta\right.\right.\\ \left.\left.-q_2\cos 4\theta\right)-\sin 5\theta\right]\end{array}\right\}
$$

$$
\begin{aligned}
D_{55}^{(sp2)} &= -\left[\frac{q_1\left(3-5\cos^2 i\right)}{8a^2\eta^6}\right]\Big\{4q_2\left[3\sin 2\theta+q_1\left(3\sin\theta+\sin 3\theta\right)\right]\\
&\quad+\left(\eta^2+4q_2^2\right)\left(3\cos\theta-\cos 3\theta\right)\Big\}\\
&\quad-\left(\frac{\sin^2 i}{8a^2\eta^4}\right)\left\{\begin{array}{l}\left[8q_2\sin 3\theta+3q_1\left(\cos\theta+\cos 3\theta\right)\right]\\ +3\left[5+\varepsilon_2-3\cos 2\theta\right.\\ \left.-\left(q_1\cos 3\theta-q_2\sin 3\theta\right)\right]\cos 2\theta\end{array}\right\}\\
&\quad-\left(\frac{3\sin^2 i}{4a^2\eta^6}\right)\left[\begin{array}{l}9\sin\theta-\sin 3\theta+2q_2(5+\varepsilon_2)\\ +6\left(q_1\sin 2\theta-q_2\cos 2\theta\right)\\ +2q_1\left(q_1\sin 3\theta-q_2 cos 3\theta\right)\end{array}\right]\left(q_2\cos 2\theta\right)\\
D_{56}^{(sp2)} &= 0
\end{aligned}
\tag{F.18}
$$

$$
\begin{aligned}
D_{61}^{(sp2)} &= -\left(\frac{2}{a}\right)\Omega^{(sp2)}\\
D_{62}^{(sp2)} &= -\left(\frac{3\cos i}{4a^2\eta^4}\right)\left[\left(q_1\cos\theta-q_2\sin\theta\right)+2\cos 2\theta\right.\\
&\quad\left.+\left(q_1\cos 3\theta+q_2\sin 3\theta\right)\right]\\
D_{63}^{(sp2)} &= \left(\frac{\sin i}{4a^2\eta^4}\right)\left[3\left(q_1\sin\theta+q_2\cos\theta\right)+3\sin 2\theta\right.\\
&\quad\left.+\left(q_1\sin 3\theta-q_2\cos 3\theta\right)\right]\\
D_{64}^{(sp2)} &= -\left(\frac{\cos i}{4a^2\eta^6}\right)\Big\{4q_1\left[3\sin 2\theta+q_2\left(3\cos\theta-\cos 3\theta\right)\right]\\
&\quad+\left(\eta^2+4q_1^2\right)\left(3\sin\theta+\sin 3\theta\right)\Big\}\\
D_{65}^{(sp2)} &= -\left(\frac{\cos i}{4a^2\eta^6}\right)\Big\{4q_2\left[3\sin 2\theta+q_1\left(3\sin\theta+\sin 3\theta\right)\right]\\
&\quad+\left(\eta^2+4q_2^2\right)\left(3\cos\theta-\cos 3\theta\right)\Big\}\\
D_{66}^{(sp2)} &= 0
\end{aligned}
\tag{F.19}
$$

G SMALL ECCENTRICITY THEORY

This appendix contains the equations for a theory that is valid for small eccentricities. First-order eccentricity terms are included for the terms that do not have J_2 as a factor. In the terms that include J_2 the eccentricity is set to zero, i.e., $e = q_1 = q_2 = 0$. The Σ and Σ^{-1} matrices do not simplify much by retaining only $\mathcal{O}(e)$ terms in the non-J_2 terms and $\mathcal{O}(e^0)$ terms in the terms multiplied by J_2. Therefore, they are not changed. If it is desired to change them substitute

$$\begin{aligned}
p &= a \\
\eta &= 1 \\
r &= a\,(1 - q_1 \cos\theta - q_2 \sin\theta) \\
r^{-1} &= a^{-1}\,(1 + q_1 \cos\theta + q_2 \sin\theta) \\
v_r &= \sqrt{\frac{\mu}{a}}\,(q_1 \sin\theta - q_2 \cos\theta) \\
v_t &= \sqrt{\frac{\mu}{a}}\,(1 + q_1 \cos\theta + q_2 \sin\theta) \\
v_{rt} &= 0
\end{aligned} \tag{G.1}$$

in the non-J_2 terms and

$$\begin{aligned}
p &= a \\
\eta &= 1 \\
r &= a \\
v_r &= 0 \\
v_t &= an \\
v_{rt} &= (q_1 \sin\theta - q_2 \cos\theta)
\end{aligned} \tag{G.2}$$

in the terms multiplied by J_2. In addition, as shown below in Eq. (G.12), the long-periodic variations of the elements are multiplied by e, so they are zero. In addition,

$$\begin{aligned}
\Delta\omega &= \dot{\omega}^{(s)}\,(t - t_0) \\
\dot{\omega}^{(s)} &= 0.75\left(\frac{R_e}{a}\right)^2 n\left(5\cos^2 i - 1\right)
\end{aligned} \tag{G.3}$$

$\bar{B}$ Matrix

The non-zero terms in the $\bar{B}(t)$ matrix are

$$\begin{aligned}
\bar{B}_{24} &= -\left(\frac{n}{4a}\right)\left(5\cos^2 i - 1\right)\sin\theta \\
\bar{B}_{25} &= \left(\frac{n}{4a}\right)\left(5\cos^2 i - 1\right)\cos\theta \\
\bar{B}_{41} &= \left(\frac{7}{4}\frac{n}{a^2}\right)\cos^2 i \\
\bar{B}_{43} &= \left(\frac{n}{2a}\right)\sin i \cos i \\
\bar{B}_{61} &= -\left(\frac{7}{4}\frac{n}{a^2}\right)\cos\theta \sin i \cos i \\
\bar{B}_{63} &= -\left(\frac{n}{2a}\right)\cos\theta \sin^2 i
\end{aligned} \tag{G.4}$$

Mean Element State Transition Matrix $\bar{\phi}_{\overline{œ}}$

$$
\begin{aligned}
G_\theta &= \frac{rn}{v_t} \approx (1 - 2q_1 \cos\theta - 2q_2 \sin\theta),\ G_{\theta_0} = -G_\theta(t_0) \\
G_{q_1} &= -2\sin\theta + 0.5\,(q_2 + 3q_1 \sin 2\theta - 3q_2 \cos 2\theta),\ G_{q_{10}} = -G_{q_1}(t_0) \\
G_{q_2} &= 2\cos\theta + 0.5\,(q_1 - 3q_1 \cos 2\theta - 3q_2 \sin 2\theta),\ G_{q_{20}} = -G_{q_2}(t_0)
\end{aligned}
\tag{G.5}
$$

The non-zero elements of $\bar{\phi}_{\overline{œ}}$ are

$$\bar{\phi}_{\overline{œ}11} = 1 \tag{G.6}$$

$$
\begin{aligned}
\bar{\phi}_{\overline{œ}21} &= -\frac{3n(t-t_0)}{2aG_\theta}\left[1 + \left(\frac{7A_2}{6a^2}\right)\left(4\cos^2 i - 1\right)\right] \\
\bar{\phi}_{\overline{œ}22} &= -\frac{G_{\theta_0}}{G_\theta} \\
\bar{\phi}_{\overline{œ}23} &= -\left(\frac{2A_2}{a^2}\right)\frac{(t-t_0)}{G_\theta}\sin 2i \\
\bar{\phi}_{\overline{œ}24} &= -\frac{1}{G_\theta}\left[G_{q_{10}} + G_{q_1}\cos(\Delta\omega) + G_{q_2}\sin(\Delta\omega)\right] \\
\bar{\phi}_{\overline{œ}25} &= -\frac{1}{G_\theta}\left[G_{q_{20}} - G_{q_1}\sin(\Delta\omega) + G_{q_2}\cos(\Delta\omega)\right]
\end{aligned}
\tag{G.7}
$$

$$\bar{\phi}_{\overline{œ}33} = 1 \tag{G.8}$$

$$
\begin{aligned}
\bar{\phi}_{\overline{œ}44} &= \cos(\Delta\omega) \\
\bar{\phi}_{\overline{œ}45} &= -\sin(\Delta\omega)
\end{aligned}
\tag{G.9}
$$

$$
\begin{aligned}
\bar{\phi}_{\overline{œ}54} &= \sin(\Delta\omega) \\
\bar{\phi}_{\overline{œ}55} &= \cos(\Delta\omega)
\end{aligned}
\tag{G.10}
$$

$$
\begin{aligned}
\bar{\phi}_{\overline{œ}61} &= \left(\frac{7}{4}\frac{A_2}{p^2}\right)\left(\frac{n\cos i}{a}\right)(t-t_0) \\
\bar{\phi}_{\overline{œ}63} &= \left(\frac{A_2}{2p^2}\right)(n\sin i)(t-t_0) \\
\bar{\phi}_{\overline{œ}66} &= 1
\end{aligned}
\tag{G.11}
$$

Mean-to-Osculating Transformation

$$a^{(lp)} = \theta^{(lp)} = i^{(lp)} = q_1^{(lp)} = q_2^{(lp)} = \Omega^{(lp)} = 0 \tag{G.12}$$

$$
\begin{aligned}
a^{(sp1)} &= \theta^{(sp1)} = i^{(sp1)} = \Omega^{(sp1)} = 0 \\
q_1^{(sp1)} &= \frac{3\left(1 - 3\cos^2 i\right)}{4a^2}\cos\theta \\
q_2^{(sp1)} &= \frac{3\left(1 - 3\cos^2 i\right)}{4a^2}\sin\theta
\end{aligned}
\tag{G.13}
$$

$$
\begin{aligned}
a^{(sp2)} &= -\left(\frac{3\sin^2 i}{2a}\right)\cos 2\theta \\
\lambda^{(sp2)} &= \frac{3\left(3 - 5\cos^2 i\right)}{8a^2}\sin 2\theta \\
\theta^{(sp2)} &= \lambda^{(sp2)} + \left(\frac{\sin^2 i}{a^2}\right)\sin 2\theta \\
i^{(sp2)} &= -\left(\frac{3\sin 2i}{8a^2}\right)\cos 2\theta \\
q_1^{(sp2)} &= -\left(\frac{\sin^2 i}{8a^2}\right)(3\cos\theta + 7\cos 3\theta) \\
q_2^{(sp2)} &= \left(\frac{\sin^2 i}{8a^2}\right)(3\sin\theta - 7\sin 3\theta) \\
\Omega^{(sp2)} &= -\left(\frac{3\cos i}{4a^2}\right)\sin 2\theta
\end{aligned}
\tag{G.14}
$$

Mean-to-Osculating Jacobian D

The non-zero elements of D are:

$$
\begin{aligned}
D_{24}^{(lp)} &= -\left(\frac{\sin^2 i}{8a^2}\right)\left(1 - 10\Theta\cos^2 i\right)\sin\theta \\
D_{25}^{(lp)} &= -\left(\frac{\sin^2 i}{8a^2}\right)\left(1 - 10\Theta\cos^2 i\right)\cos\theta \\
D_{44}^{(lp)} &= -\left(\frac{\sin^2 i}{16a^2}\right)\left(1 - 10\Theta\cos^2 i\right) \\
D_{55}^{(lp)} &= \left(\frac{\sin^2 i}{16a^2}\right)\left(1 - 10\Theta\cos^2 i\right) \\
\Theta &= \left(1 - 5\cos^2 i\right)^{-1}
\end{aligned}
\tag{G.15}
$$

$$
D_{14}^{(sp1)} = \frac{3\left(1 - 3\cos^2 i\right)}{2a}\cos\theta
$$

$$D_{15}^{(sp1)} = \frac{3\left(1-3\cos^2 i\right)}{2a}\sin\theta \tag{G.16}$$

$$\begin{aligned} D_{24}^{(sp1)} &= \frac{9\left(1-5\cos^2 i\right)}{4a^2}\sin\theta \\ D_{25}^{(sp1)} &= -\frac{9\left(1-5\cos^2 i\right)}{4a^2}\cos\theta \end{aligned} \tag{G.17}$$

$$\begin{aligned} D_{41}^{(sp1)} &= -\frac{3\left(1-3\cos^2 i\right)}{2a^3}\cos\theta \\ D_{42}^{(sp1)} &= -\frac{3\left(1-3\cos^2 i\right)}{4a^2}\sin\theta \\ D_{43}^{(sp1)} &= \left(\frac{9\sin 2i}{4a^2}\right)\cos\theta \\ D_{44}^{(sp1)} &= \frac{3\left(1-3\cos^2 i\right)}{8a^2}(2+\cos 2\theta) \\ D_{45}^{(sp1)} &= \frac{3\left(1-3\cos^2 i\right)}{8a^2}\sin 2\theta \end{aligned} \tag{G.18}$$

$$\begin{aligned} D_{51}^{(sp1)} &= -\frac{3\left(1-3\cos^2 i\right)}{2a^3}\sin\theta \\ D_{52}^{(sp1)} &= \frac{3\left(1-3\cos^2 i\right)}{4a^2}\cos\theta \\ D_{53}^{(sp1)} &= \left(\frac{9\sin 2i}{4a^2}\right)\sin\theta \\ D_{54}^{(sp1)} &= \frac{3\left(1-3\cos^2 i\right)}{8a^2}\sin 2\theta \\ D_{55}^{(sp1)} &= \frac{3\left(1-3\cos^2 i\right)}{8a^2}(2-\cos 2\theta) \end{aligned} \tag{G.19}$$

$$\begin{aligned} D_{64}^{(sp1)} &= \left(\frac{9\cos i}{4a^2}\right)\sin\theta \\ D_{65}^{(sp1)} &= -\left(\frac{9\cos i}{4a^2}\right)\cos\theta \end{aligned} \tag{G.20}$$

$$\begin{aligned} D_{11}^{(sp2)} &= \left(\frac{3\sin^2 i}{2a^2}\right)\cos 2\theta \\ D_{12}^{(sp2)} &= \left(\frac{3\sin^2 i}{a}\right)\sin 2\theta \end{aligned} \tag{G.21}$$

$$
\begin{aligned}
D_{13}^{(sp2)} &= -\left(\frac{3\sin 2i}{2a}\right)\cos 2\theta \\
D_{14}^{(sp2)} &= -\left(\frac{9\sin^2 i}{4a}\right)(\cos\theta + \cos 3\theta) \\
D_{15}^{(sp2)} &= \left(\frac{9\sin^2 i}{4a}\right)(\sin\theta - \sin 3\theta)
\end{aligned}
$$

$$
\begin{aligned}
D_{21}^{(sp2)} &= -\left(\frac{6-7\sin^2 i}{4a^3}\right)\sin 2\theta \\
D_{22}^{(sp2)} &= \left(\frac{6-7\sin^2 i}{4a^2}\right)\cos 2\theta \\
D_{23}^{(sp2)} &= -\left(\frac{7\sin 2i}{8a^2}\right)\sin 2\theta \\
D_{24}^{(sp2)} &= \left(\frac{24-47\sin^2 i}{32a^2}\right)\sin\theta + \left(\frac{\cos^2 i}{4a^2}\right)\sin 3\theta \\
D_{25}^{(sp2)} &= \left(\frac{24-47\sin^2 i}{32a^2}\right)\cos\theta - \left(\frac{\cos^2 i}{4a^2}\right)\cos 3\theta
\end{aligned}
\tag{G.22}
$$

$$
\begin{aligned}
D_{31}^{(sp2)} &= \left(\frac{3\sin 2i}{4a^3}\right)\cos 2\theta \\
D_{32}^{(sp2)} &= \left(\frac{3\sin 2i}{4a^2}\right)\sin 2\theta \\
D_{33}^{(sp2)} &= -\left(\frac{3\cos 2i}{4a^2}\right)\cos 2\theta \\
D_{34}^{(sp2)} &= -\left(\frac{\sin 2i}{8a^2}\right)(3\cos\theta + \cos 3\theta) \\
D_{35}^{(sp2)} &= \left(\frac{\sin 2i}{8a^2}\right)(3\sin\theta - \sin 3\theta)
\end{aligned}
\tag{G.23}
$$

$$
\begin{aligned}
D_{41}^{(sp2)} &= \left(\frac{\sin^2 i}{4a^3}\right)(3\cos\theta + 7\cos 3\theta) \\
D_{42}^{(sp2)} &= \left(\frac{3\sin^2 i}{8a^2}\right)(\sin\theta + 7\sin 3\theta) \\
D_{43}^{(sp2)} &= -\left(\frac{\sin 2i}{8a^2}\right)(3\cos\theta + 7\cos 3\theta)
\end{aligned}
\tag{G.24}
$$

$$D_{44}^{(sp2)} = -\left(\frac{3\sin^2 i}{16a^2}\right)(3 + 10\cos 2\theta + 3\cos 4\theta)$$
$$D_{45}^{(sp2)} = \frac{3\left(3 - 5\cos^2 i\right)}{8a^2}\sin 2\theta - \left(\frac{9\sin^2 i}{16a^2}\right)\sin 4\theta$$

$$\begin{aligned}
D_{51}^{(sp2)} &= -\left(\frac{\sin^2 i}{4a^3}\right)(3\sin\theta - 7\sin 3\theta) \\
D_{52}^{(sp2)} &= \left(\frac{3\sin^2 i}{8a^2}\right)(\cos\theta - 7\cos 3\theta) \\
D_{53}^{(sp2)} &= \left(\frac{\sin 2i}{8a^2}\right)(3\sin\theta - 7\sin 3\theta) \\
D_{54}^{(sp2)} &= -\frac{3\left(3 - 5\cos^2 i\right)}{8a^2}\sin 2\theta - \left(\frac{9\sin^2 i}{16a^2}\right)\sin 4\theta \\
D_{55}^{(sp2)} &= \left(\frac{3\sin^2 i}{16a^2}\right)(3 - 10\cos 2\theta + 3\cos 4\theta)
\end{aligned} \tag{G.25}$$

$$\begin{aligned}
D_{61}^{(sp2)} &= \left(\frac{3\cos i}{2a^3}\right)\sin 2\theta \\
D_{62}^{(sp2)} &= -\left(\frac{3\cos i}{2a^2}\right)\cos 2\theta \\
D_{63}^{(sp2)} &= \left(\frac{3\sin i}{4a^2}\right)\sin 2\theta \\
D_{64}^{(sp2)} &= -\left(\frac{\cos i}{4a^2}\right)(3\sin\theta + \sin 3\theta) \\
D_{65}^{(sp2)} &= -\left(\frac{\cos i}{4a^2}\right)(3\cos\theta - \cos 3\theta)
\end{aligned} \tag{G.26}$$

H YAN-ALFRIEND NONLINEAR THEORY COEFFICIENTS

This appendix contains the coefficients used in the Yan-Alfriend nonlinear theory used in Section 8.2.

$$\begin{aligned}
a_{10} &= -\frac{3}{L^4} \\
a_{11} &= -\left(\frac{21}{4L^8\eta^4}\right)\left[(1+\eta) - (5+3\eta)\cos^2 i\right] \\
a_{12} &= -\left(\frac{33}{64L^{12}\eta^8}\right)\Big[\left(-35 + 9\eta + 41\eta^2 + 25\eta^3\right) \\
&\quad + \left(90 - 162\eta - 222\eta^2 - 90\eta^3\right)\cos^2 i \\
&\quad + \left(385 + 465\eta + 189\eta^2 + 25\eta^3\right)\cos^4 i\Big]
\end{aligned} \tag{H.1}$$

$$
\begin{aligned}
a_{21} &= -\left(\frac{3}{4L^7\eta^5}\right)\left[(4+3\eta)-(20+9\eta)\cos^2 i\right] \\
a_{22} &= -\left(\frac{3}{64L^{11}\eta^9}\right)\Big[\left(280-63\eta-246\eta^2-125\eta^3\right) \\
&\quad +\left(-720+1134\eta+1332\eta^2+450\eta^3\right)\cos^2 i \\
&\quad -\left(3080+3255\eta+1134\eta^2+125\eta^3\right)\cos^4 i\Big]
\end{aligned}
\tag{H.2}
$$

$$
\begin{aligned}
a_{31} &= \left(\frac{3\sin 2i}{4L^7\eta^4}\right)(5+3\eta) \\
a_{32} &= -\left(\frac{3\sin 2i}{32L^{11}\eta^8}\right)\Big[\left(45-81\eta-111\eta^2-45\eta^3\right) \\
&\quad +\left(385+465\eta+189\eta^2+25\eta^3\right)\cos^2 i\Big]
\end{aligned}
\tag{H.3}
$$

$$
\begin{aligned}
a_{40} &= \frac{12}{L^5} \\
a_{41} &= -\frac{8}{L}a_{11} = \left(\frac{42}{L^9\eta^4}\right)\left[(1+\eta)-(5+3\eta)\cos^2 i\right] \\
a_{42} &= -\frac{12}{L}a_{12} = \left(\frac{99}{16L^{13}\eta^8}\right)\Big[\left(-35+9\eta+41\eta^2+25\eta^3\right) \\
&\quad +\left(90-162\eta-222\eta^2-90\eta^3\right)\cos^2 i \\
&\quad +\left(385+465\eta+189\eta^2+25\eta^3\right)\cos^4 i\Big]
\end{aligned}
\tag{H.4}
$$

$$
\begin{aligned}
a_{51} &= \left(\frac{3}{L^7\eta^6}\right)\left[(5+3\eta)-(25+9\eta)\cos^2 i\right] \\
a_{52} &= -\left(\frac{3}{32L^{11}\eta^{10}}\right)\Big[\left(1260-252\eta-861\eta^2-375\eta^3\right) \\
&\quad -\left(3240-4536\eta-4662\eta^2-1350\eta^3\right)\cos^2 i \\
&\quad -\left(13860+13020\eta+3969\eta^2+375\eta^3\right)\cos^4 i\Big]
\end{aligned}
\tag{H.5}
$$

$$
\begin{aligned}
a_{61} &= -\left(\frac{3\cos 2i}{2L^7\eta^4}\right)(5+4\eta) \\
a_{62} &= -\left(\frac{3}{32L^{11}\eta^9}\right)\Big[\left(280-63\eta-246\eta^2-125\eta^3\right) \\
&\quad \left(-720+1134\eta+1332\eta^2+450\eta^3\right)\cos^2 i \\
&\quad -\left(3080+3255\eta+1134\eta^2+125\eta^3\right)\cos^4 i\Big]
\end{aligned}
\tag{H.6}
$$

$$
\begin{aligned}
a_{71} &= -\frac{7}{L}a_{21} = \left(\frac{21}{4L^8\eta^5}\right)\left[(4+3\eta)-(20+9\eta)\cos^2 i\right] \\
a_{72} &= -\frac{11}{L}a_{22} = -\left(\frac{33}{64L^{12}\eta^9}\right)\left[\left(280-63\eta-246\eta^2-125\eta^3\right)\right. \\
&\quad -\left(720-1134\eta-1332\eta^2-450\eta^3\right)\cos^2 i \\
&\quad \left.-\left(3080+3255\eta+1134\eta^2+125\eta^3\right)\cos^4 i\right]
\end{aligned}
\tag{H.7}
$$

$$
\begin{aligned}
a_{81} &= -\frac{7}{L}a_{31} = -\left(\frac{21\sin 2i}{4L^8\eta^4}\right)(5+3\eta) \\
a_{82} &= -\frac{11}{L}a_{32} = \left(\frac{33\sin 2i}{32L^{12}\eta^8}\right)\left[\left(45-81\eta-111\eta^2-45\eta^3\right)\right. \\
&\quad \left.+\left(385+465\eta+189\eta^2+25\eta^3\right)\cos^2 i\right]
\end{aligned}
\tag{H.8}
$$

$$
\begin{aligned}
a_{91} &= -\left(\frac{3}{4L^7\eta^5}\right)(20+9\eta)\sin 2i \\
a_{92} &= -\left(\frac{3\sin 2i}{32L^{12}\eta^9}\right)\left[\left(360+567\eta+666\eta^2+225\eta^3\right)\right. \\
&\quad \left.+\left(3080+3255\eta+1134\eta^2+125\eta^3\right)\cos^2 i\right]
\end{aligned}
\tag{H.9}
$$

$$
b_{11} = -\left(\frac{21}{4L^8\eta^4}\right)\left(1-5\cos^2 i\right) \tag{H.10}
$$

$$
\begin{aligned}
b_{12} &= \left(\frac{33}{64L^{12}\eta^8}\right)\left[\left(35-24\eta-25\eta^2\right)+\left(-90+192\eta+126\eta^2\right)\right. \\
&\quad \left.\cos^2 i-\left(385+360\eta+45\eta^2\right)\cos^4 i\right]
\end{aligned}
$$

$$
b_{21} = -\left(\frac{3}{L^7\eta^5}\right)\left(1-5\cos^2 i\right) \tag{H.11}
$$

$$
\begin{aligned}
b_{22} &= \left(\frac{3}{32L^{11}\eta^9}\right)\left[\left(140-84\eta-75\eta^2\right)+\left(-360+672\eta+378\eta^2\right)\cos^2 i\right. \\
&\quad \left.-\left(1544+1260\eta+135\eta^2\right)\cos^4 i\right]
\end{aligned}
$$

$$
b_{31} = \left(\frac{15}{4L^7\eta^4}\right)\sin 2i \tag{H.12}
$$

$$
b_{32} = \left(\frac{3\sin 2i}{32L^{11}\eta^8}\right)\left[\left(-45+96\eta+63\eta^2\right)-\left(385+360\eta+45\eta^2\right)\cos^2 i\right]
$$

$$
b_{41} = -\frac{8}{L}b_{11} = \left(\frac{42}{L^9\eta^4}\right)\left(1-5\cos^2 i\right) \tag{H.13}
$$

$$b_{42} = -\frac{12}{L} c_{12} = -\left(\frac{99}{16L^{13}\eta^8}\right)\Big[\left(35 - 24\eta - 25\eta^2\right) + \left(-90 + 192\eta + 126\eta^2\right)\cos^2 i - \left(385 + 360\eta + 45\eta^2\right)\cos^4 i\Big]$$

$$b_{51} = \left(\frac{15}{L^7\eta^6}\right)\left(1 - 5\cos^2 i\right) \tag{H.14}$$

$$b_{52} = \left(\frac{9}{32L^{11}\eta^{10}}\right)\Big[\left(420 - 224\eta - 175\eta^2\right) + \left(-1080 + 1792\eta + 882\eta^2\right)\cos^2 i - \left(4632 + 3360\eta + 315\eta^2\right)\cos^4 i\Big]$$

$$b_{61} = \left(\frac{15}{2L^7\eta^4}\right)\cos 2i \tag{H.15}$$

$$b_{62} = \left(\frac{3}{16L^{11}\eta^8}\right)\Big[\left(45 - 96\eta - 63\eta^2\right) + 4\left(555 + 588\eta + 99\eta^2\right)\cos^2 i + 8\left(385 + 360\eta + 45\eta^2\right)\cos^4 i\Big]$$

$$b_{71} = -\frac{7}{L} b_{21} = \left(\frac{21}{L^8\eta^5}\right)\left(1 - 5\cos^2 i\right) \tag{H.16}$$

$$b_{72} = -\frac{11}{L} b_{22} = -\left(\frac{33}{32L^{12}\eta^9}\right)\Big[\left(140 - 84\eta - 75\eta^2\right) + \left(-360 + 672\eta + 378\eta^2\right)\cos^2 i - \left(1544 + 1260\eta + 135\eta^2\right)\cos^4 i\Big]$$

$$b_{81} = -\frac{7}{L} b_{31} = -\left(\frac{105}{4L^7\eta^4}\right)\sin 2i \tag{H.17}$$

$$b_{82} = -\frac{11}{L} b_{32} = -\left(\frac{33\sin 2i}{32L^{11}\eta^8}\right)\Big[\left(-45 + 96\eta + 63\eta^2\right) - \left(385 + 360\eta + 45\eta^2\right)\cos^2 i\Big]$$

$$b_{91} = -\left(\frac{15}{L^7\eta^5}\right)\sin 2i \tag{H.18}$$

$$b_{92} = -\left(\frac{3\sin 2i}{16L^{11}\eta^8}\right)\Big[\left(-180 + 336\eta + 189\eta^2\right) - \left(1540 + 1260\eta + 135\eta^2\right)\cos^2 i\Big]$$

$$c_{11} = -\left(\frac{21}{2L^8\eta^4}\right)\cos i \tag{H.19}$$

$$c_{12} = \left(\frac{33\cos i}{16L^{12}\eta^8}\right)\Big[\left(5 - 12\eta - 9\eta^2\right) + \left(35 + 36\eta + 5\eta^2\right)\cos^2 i\Big]$$

$$c_{21} = -\left(\frac{6}{L^7\eta^5}\right)\cos i \tag{H.20}$$

$$c_{22} = \left(\frac{3\cos i}{8L^{11}\eta^9}\right)\left[\left(2042\eta - 27\eta^2\right) + \left(140 + 126\eta + 15\eta^2\right)\cos^2 i\right]$$

$$c_{31} = -\left(\frac{3}{2L^7\eta^4}\right)\sin i \tag{H.21}$$

$$c_{32} = \left(\frac{3\sin i}{16L^{11}\eta^8}\right)\left[\left(5 - 12\eta - 9\eta^2\right) + 3\left(35 + 36\eta + 5\eta^2\right)\cos^2 i\right]$$

$$c_{41} = -\frac{8}{L}c_{11} = \left(\frac{84}{L^9\eta^4}\right)\cos i \tag{H.22}$$

$$\begin{aligned} c_{42} &= -\frac{12}{L}c_{12} \\ &= -\left(\frac{99\cos i}{4L^{13}\eta^8}\right)\left[\left(5 - 12\eta - 9\eta^2\right) + \left(35 + 36\eta + 5\eta^2\right)\cos^2 i\right] \end{aligned}$$

$$c_{51} = \left(\frac{30}{L^7\eta^6}\right)\cos i \tag{H.23}$$

$$\begin{aligned} c_{52} = -\left(\frac{9\cos i}{8L^{11}\eta^{10}}\right)&\left[\left(60 - 112\eta - 63\eta^2\right)\right. \\ &\left.+\left(420 + 336\eta + 35\eta^2\right)\cos^2 i\right] \end{aligned}$$

$$c_{61} = -\left(\frac{3}{2L^7\eta^4}\right)\cos i \tag{H.24}$$

$$\begin{aligned} c_{62} = \left(\frac{3\cos i}{16L^{11}\eta^8}\right)&\left[-\left(205 + 228\eta + 39\eta^2\right)\right. \\ &\left.+9\left(35 + 36\eta + 5\eta^2\right)\cos^2 i\right] \end{aligned}$$

$$c_{71} = \left(\frac{42}{L^5\eta^6}\right)\cos i \tag{H.25}$$

$$c_{72} = -\left(\frac{33\cos i}{8L^{12}\eta^9}\right)\left[\left(20 - 42\eta - 27\eta^2\right) + \left(140 + 126\eta + 15\eta^2\right)\cos^2 i\right]$$

$$c_{81} = -\frac{7}{L}c_{31} = -\left(\frac{21}{2L^8\eta^4}\right)\sin i \tag{H.26}$$

$$c_{82} = -\frac{11}{L}c_{32}$$

$$c_{82} = -\left(\frac{33\sin i}{16L^{12}\eta^8}\right)\left[\left(5 - 12\eta - 9\eta^2\right) + 3\left(35 + 36\eta + 5\eta^2\right)\cos^2 i\right]$$

$$c_{91} = \left(\frac{6}{L^7\eta^5}\right)\sin i \tag{H.27}$$

$$c_{92} = -\left(\frac{3\sin i}{8L^{11}\eta^9}\right)\left[\left(20 - 42\eta - 27\eta^2\right) + 3\left(140 + 126\eta + 15\eta^2\right)\cos^2 i\right]$$