Appendix A

Motion dynamics

by

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A.1 Equations of relative motion for circular orbits

This section intends to provide the details and intermediate calculations required for the derivation of the general equations of the relative motion for circular orbits. The results of this derivation will be shown as a set of differential equations and in a closed form of the state transition matrix for the system.

A.1.1 General system of differential equations

The general assumption for this derivation is, at this point, that the motion of a body is subject to the effects of a central spherical gravity field and to forces from thruster actuation or disturbances. The spacecraft are considered as point masses for this work.

The position vectors in inertial space are defined in figure A.1 for the chaser ($\mathbf{r}_{\mathrm{c}}$) and target ($\mathbf{r}_{\mathrm{t}}$). Their relative position is denoted by $\mathbf{s}$. The equations of motion will be derived conveniently in the target local orbital frame $\boldsymbol{F}_{\mathrm{lo}}$. In the following, scalars will be in normal type and vectors and matrices will be in bold, and it should be clear from the context what is what. Vectors are defined as column vectors.

The general equation for motion under the influence of a central force is Newton's law of gravitation (Newton 1713); see also Eq. (3.1):

$$\mathbf{F}_{\mathrm{g}}(\mathbf{r}) = -G\frac{Mm}{r^2}\frac{\mathbf{r}}{r} = -\mu\frac{m}{r^3}\mathbf{r} \tag{A.1}$$

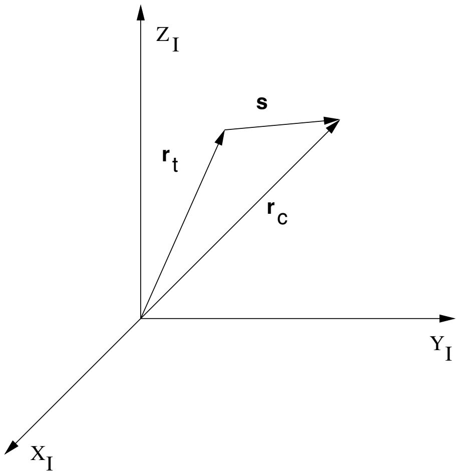

Figure A.1. Definition of the position vectors to the chaser and target as well as the relative vector in the inertial frame.

where

$\mathbf{F}_g$	=	gravitational force
G	=	universal gravitational constant
M	=	mass of the central body (e.g. Earth)
m	=	mass of the spacecraft (second mass)
$\mathbf{r}$	=	the radius vector, $r = \|\mathbf{r}\|$
μ	=	GM

Dividing by the mass on both sides of Eq. (A.1) to normalise the equation, one obtains for the general motion:

$$\mathbf{f}_g(\mathbf{r}) = -\mu \frac{\mathbf{r}}{r^3} \tag{A.2}$$

The target motion from Eq. (A.1) is

$$\mathbf{F}_g(\mathbf{r}_t) = m_t \ddot{\mathbf{r}}_t = -\mu \frac{m_t}{r_t^3} \mathbf{r}_t$$

$$\mathbf{f}_g(\mathbf{r}_t) = \ddot{\mathbf{r}}_t = -\mu \frac{\mathbf{r}_t}{r_t^3} \tag{A.3}$$

The chaser motion from Eq. (A.1) and from the non-gravitational force is

$$m_c \ddot{\mathbf{r}}_c = \mathbf{F}_g(\mathbf{r}_c) + \mathbf{F} = -\mu \frac{m_c}{r_c^3} \mathbf{r}_c + \mathbf{F}$$

Inserting Eq. (A.2) yields

$$\ddot{\mathbf{r}}_c = \mathbf{f}_g(\mathbf{r}_c) + \frac{\mathbf{F}}{m_c} \tag{A.4}$$

The relative motion $\mathbf{s}$ is defined as follows, and the relative accelerations become directly the derivatives in inertial space:

$$\begin{aligned} \mathbf{r}_{\mathrm{t}} + \mathbf{s} &= \mathbf{r}_{\mathrm{c}} \\ \mathbf{s} &= \mathbf{r}_{\mathrm{c}} - \mathbf{r}_{\mathrm{t}} \\ \ddot{\mathbf{s}} &= \ddot{\mathbf{r}}_{\mathrm{c}} - \ddot{\mathbf{r}}_{\mathrm{t}} \end{aligned} \tag{A.5}$$

Inserting Eqs. (A.3) and (A.4) into Eq. (A.5), one obtains

$$\ddot{\mathbf{s}} = \mathbf{f}_{\mathrm{g}}(\mathbf{r}_{\mathrm{c}}) - \mathbf{f}_{\mathrm{g}}(\mathbf{r}_{\mathrm{t}}) + \frac{\mathbf{F}}{m_{\mathrm{c}}} \tag{A.6}$$

We will now linearise $\mathbf{f}_{\mathrm{g}}(\mathbf{r}_{\mathrm{c}})$ around the vector $\mathbf{r}_{\mathrm{t}}$ by means of a Taylor expansion to first order:

$$\mathbf{f}_{\mathrm{g}}(\mathbf{r}_{\mathrm{c}}) = \mathbf{f}_{\mathrm{g}}(\mathbf{r}_{\mathrm{t}}) + \left.\frac{\mathrm{d}\mathbf{f}_{\mathrm{g}}(\mathbf{r})}{\mathrm{d}\mathbf{r}}\right|_{\mathbf{r}=\mathbf{r}_{\mathrm{t}}} (\mathbf{r}_{\mathrm{c}} - \mathbf{r}_{\mathrm{t}}) \tag{A.7}$$

Since vectors are defined as column vectors, the Jacobian matrix becomes (Wie 1998)

$$\frac{\mathrm{d}\mathbf{g}(\mathbf{x})}{\mathrm{d}\mathbf{x}} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_3} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_3}{\partial x_1} & \cdots & \frac{\partial g_3}{\partial x_3} \end{bmatrix}$$

To obtain the elements of the Jacobian of Eq. (A.7), we will initially find the diagonal elements of the Jacobian, where we define

$$\mathbf{r} = [r_x, r_y, r_z]^T \ and \ r = |\mathbf{r}| = \sqrt{(r_x^2 + r_y^2 + r_z^2)}$$

For element (i, j), where $i = j$, and using Eq. (A.2):

$$\begin{aligned} \frac{\partial f_{\mathrm{g}}(r_i)}{\partial r_i} &= -\mu \left[r^{-3} + r_i \left(-\frac{3}{2} \right) (r_x^2 + r_y^2 + r_z^2)^{-\frac{5}{2}} 2 r_i \right] \\ &= -\mu \left[r^{-3} - 3 r^{-5} r_i^2 \right] \\ &= -\frac{\mu}{r^3} \left[1 - 3 \frac{r_i^2}{r^2} \right] \end{aligned} \tag{A.8}$$

For element (i, j), where $i \neq j$ and it shall be noticed that r_i is not a function of r_j,

$$\begin{aligned} \frac{\partial f_{\mathrm{g}}(r_i)}{\partial r_i} &= -\mu \left[-\frac{3}{2} (r_x^2 + r_y^2 + r_z^2)^{-\frac{5}{2}} 2 r_j r_i \right] \\ &= -\mu \left[-3 r^{-5} r_i r_j \right] \\ &= -3 \frac{\mu}{r^3} \frac{r_i r_j}{r^2} \end{aligned} \tag{A.9}$$

Rewriting Eq. (A.7) and inserting Eqs. A1.1 (A.8) and (A.9), and defining $\mathbf{r} = \mathbf{r}_\mathrm{t}$ we obtain

$$\mathbf{f}_\mathrm{g}(\mathbf{r}_\mathrm{c}) - \mathbf{f}_\mathrm{g}(\mathbf{r}_\mathrm{t}) = -\frac{\mu}{r_\mathrm{t}^3}\mathbf{M}\mathbf{s}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 - 3\frac{r_x^2}{r_\mathrm{t}^2} & 3\frac{r_x r_y}{r_\mathrm{t}^2} & 3\frac{r_x r_z}{r_\mathrm{t}^2} \\ 3\frac{r_y r_x}{r_\mathrm{t}^2} & 1 - 3\frac{r_y^2}{r_\mathrm{t}^2} & 3\frac{r_y r_z}{r_\mathrm{t}^2} \\ 3\frac{r_z r_x}{r_\mathrm{t}^2} & 3\frac{r_z r_y}{r_\mathrm{t}^2} & 1 - 3\frac{r_z^2}{r_\mathrm{t}^2} \end{bmatrix}$$

and Eq. (A.6) becomes

$$\ddot{\mathbf{s}} = -\frac{\mu}{r_\mathrm{t}^3}\mathbf{M}\mathbf{s} + \frac{\mathbf{F}}{m_\mathrm{c}} \tag{A.10}$$

The objective is to represent the chaser motion in the rotating target local orbital frame $\mathbf{F}_\mathrm{lo}$, which has its origin at the COM of the target spacecraft and is defined in section 3.1.3.

From a general kinematic equation for translation and rotating systems, we can obtain the chaser acceleration in the rotating target frame. The translation is trivial and part of the equations (Symon 1979). Generally one obtains the following, where the starred frame $(*)$ is rotating with the orbital rate $\boldsymbol{\omega}$:

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}^{*2}\mathbf{x}^*}{\mathrm{d}t^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}^*) + 2\boldsymbol{\omega} \times \frac{\mathrm{d}^*\mathbf{x}^*}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \mathbf{x}^* \tag{A.11}$$

We now define $\mathbf{s} = \mathbf{x}$ and $\mathbf{s}^* = [x, y, z]^\mathrm{T}$ in the rotating starred system, and inserting Eq. (A.10) yields

$$\frac{\mathrm{d}^{*2}\mathbf{s}^*}{\mathrm{d}t^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}^*) + 2\boldsymbol{\omega} \times \frac{\mathrm{d}^*\mathbf{s}^*}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \mathbf{s}^* + \frac{\mu}{r_\mathrm{t}^3}\mathbf{M}\mathbf{s}^* = \frac{\mathbf{F}}{m_\mathrm{c}} \tag{A.12}$$

Expressed in the target frame, we obtain for $\mathbf{r}_\mathrm{t}$ and $\boldsymbol{\omega}$

$$\mathbf{r}_\mathrm{t} = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix} \quad \text{and} \quad \boldsymbol{\omega} = \begin{bmatrix} 0 \\ -\omega \\ 0 \end{bmatrix}$$

The terms of Eq. (A.12) become

$$\boldsymbol{\omega} \times \mathbf{s}^* = \begin{bmatrix} -\omega z \\ 0 \\ \omega x \end{bmatrix}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}^*) = \begin{bmatrix} -\omega^2 x \\ 0 \\ -\omega^2 z \end{bmatrix}$$

$$\boldsymbol{\omega} \times \frac{\mathrm{d}^*\mathbf{s}^*}{\mathrm{d}t} = \begin{bmatrix} -\omega\dot{z} \\ 0 \\ \omega\dot{x} \end{bmatrix}$$

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \mathbf{s}^* = \begin{bmatrix} -\dot{\omega}z \\ 0 \\ \dot{\omega}x \end{bmatrix}$$

$$\mathbf{Ms}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{s}^* = \begin{bmatrix} x \\ y \\ -2z \end{bmatrix}$$

As the angular momentum $\mathbf{L}$ is constant for fixed elliptical orbits and, for the special case of circular orbits the angular rate ω is also constant, it can be expressed as

$$\omega^2 = \frac{\mu}{r_\mathrm{t}^3} \tag{A.13}$$

From Eq. (A.13) we see that $\mathrm{d}\omega/\mathrm{d}t = 0$. We will now insert Eq. (A.13) and the terms for Eq. (A.12) into Eq. (A.12) to obtain the general linear equations for the relative motion, Known as Hill's equations (Hill 1878); see also section 3.21:

$$\begin{aligned} \ddot{x} - 2\omega\dot{z} &= \frac{1}{m_\mathrm{c}} F_x \\ \ddot{y} + \omega^2 y &= \frac{1}{m_\mathrm{c}} F_y \\ \ddot{z} + 2\omega\dot{x} - 3\omega^2 z &= \frac{1}{m_\mathrm{c}} F_z \end{aligned} \tag{A.14}$$

It shall be noted that the system of linear time varying differential equations in Eqs. (A.14) is the general system valid for an arbitrary relative trajectory between a chaser spacecraft and a target spacecraft, where the latter moves under the influence of a central gravity field only. Hence the validity of Eq. (A.14) for the target spacecraft.

For convenience we will now represent Eq. (A.14) in state space form. In order to reduce the size of the matrices it is convenient to have two systems: one for the out-of-plane and one for the in-plane dynamics. The general form is

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{A.15}$$

where $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the input vector, $\mathbf{A}$ is the transition matrix and $\mathbf{B}$ is the input matrix with matching dimensions.

For in-plane motion, the coupled dynamics from Eq. (A.14) is as follows, with the state vector $\mathbf{x} = [x, z, \dot{x}, \dot{z}]^\mathrm{T}$:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \\ \ddot{x}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2\omega \\ 0 & 3\omega^2 & -2\omega & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \\ \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_\mathrm{c}} & 0 \\ 0 & \frac{1}{m_\mathrm{c}} \end{bmatrix} \begin{bmatrix} F_x \\ F_z \end{bmatrix} \tag{A.16}$$

The out-of-plane dynamics from Eq. (A.14) is as follows, with the state vector $\mathbf{x}_o = [y, \dot{y}]^{\mathrm{T}}$:

$$\begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_c} \end{bmatrix} \begin{bmatrix} F_y \end{bmatrix} \tag{A.17}$$

The equation system described in Eqs. (A.16) and (A.17) is known as Hill's equations (Hill 1878). In the literature, these equations are sometimes also referred to as the Clohessy–Wiltshire (CW) equations, though the CW equations are here presented in Eq. (A.22).

A.1.2 Homogeneous solution

In practice, the forces on the right hand side of Eq. (A.14) are not regular well-behaved functions of time, and therefore it is complex to find a general particular solution to Eq. (A.14), if possible at all in the general case. In this section we will instead concentrate on arriving at an analytical homogeneous solution which will give a good insight into the behaviour of the relative trajectories when treated as an initial value problem.

A Laplace transformation method will be used to find the solution, recalling that the transformation of a derivative in the initial value case is

$$\mathcal{L}(f'(t)) = sF(s) - f(0_+) \tag{A.18}$$

and

$$\mathcal{L}(f''(t)) = s^2F(s) - sf(0_+) - f'(0_+) \tag{A.19}$$

Applying Eqs. (A.18) and (A.19) to Eq. (A.14), we obtain the following expressions in the Laplace domain:

$$\begin{aligned} s^2X(s) - sx_0 - \dot{x}_0 - 2\omega sZ(s) + 2\omega z_0 &= 0 \\ s^2Y(s) - sy_0 - \dot{y}_0 + \omega^2Y(s) &= 0 \\ s^2Z(s) - sz_0 - \dot{z}_0 + 2\omega sX(s) - 2\omega x_0 - 3\omega^2Z(s) &= 0 \end{aligned} \tag{A.20}$$

Solving for $X(s)$ and $Z(s)$ in Eq. (A.20) and re-arranging, the Laplace transformation of the components can be written as

$$\begin{aligned} X(s) &= x_0\frac{1}{s} + (\dot{x}_0 - 2\omega z_0)\frac{1}{s^2} + 2\omega z_0\frac{1}{s^2+\omega^2} + 2\omega\dot{z}_0\frac{1}{s(s^2+\omega^2)} \\ &\quad + 2\omega(4\omega^2 z_0 - 2\omega\dot{x}_0)\frac{1}{s^2(s^2+\omega^2)} \\ Y(s) &= y_0\frac{s}{s^2+\omega^2} + \dot{y}_0\frac{1}{s^2+\omega^2} \\ Z(s) &= z_0\frac{s}{s^2+\omega^2} + \dot{z}_0\frac{1}{s^2+\omega^2} + (4\omega^2 z_0 - 2\omega\dot{x}_0)\frac{1}{s(s^2+\omega^2)} \end{aligned} \tag{A.21}$$

The inverse Laplace transformation is performed to obtain the time domain solution. Taking the inverse Laplace transformation of each term in Eqs. (A.21) and collecting terms, the final homogeneous solution can be found to be, where the initial time is t_0 and $\tau = t - t_0$:

$$\begin{aligned} x(t) &= \left(\frac{4\dot{x}_0}{\omega} - 6z_0\right)\sin(\omega\tau) - \frac{2\dot{z}_0}{\omega}\cos(\omega\tau) + (6\omega z_0 - 3\dot{x}_0)\tau + \left(x_0 + \frac{2\dot{z}_0}{\omega}\right) \\ y(t) &= y_0\cos(\omega\tau) + \frac{\dot{y}_0}{\omega}\sin(\omega\tau) \\ z(t) &= \left(\frac{2\dot{x}_0}{\omega} - 3z_0\right)\cos(\omega\tau) + \frac{\dot{z}_0}{\omega}\sin(\omega\tau) + \left(4z_0 - \frac{2\dot{x}_0}{\omega}\right) \end{aligned} \tag{A.22}$$

From Eqs. (A.22) one sees that $y(t)$ and $z(t)$ are oscillating, whereas $x(t)$ progresses with time t. The equations in (A.22) are the CW equations (Clohessy & Wiltshire 1960) without input forces.

As for the differential equations in (A.16) and (A.17), we will find a state space representation of the solution to the equations. First we will find the Laplace transformation of Eq. (A.15) which becomes:

$$s\mathbf{x}(s) - \mathbf{x}(0_+) = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s) \tag{A.23}$$

Re-arranging Eq. (A.23) gives

$$(s\mathbf{I} - \mathbf{A})\mathbf{x}(s) = \mathbf{x}(0_+) + \mathbf{B}\mathbf{u}(s) \tag{A.24}$$

and the resolvant matrix is defined as

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \tag{A.25}$$

Re-arranging Eq. (A.24) and inserting Eq. (A.25), we obtain the frequency domain solution as

$$\mathbf{x}(s) = \mathbf{\Phi}(s)[\mathbf{x}(0_+) + \mathbf{B}\mathbf{u}(s)] \tag{A.26}$$

From Eq. (A.26) it is easy to obtain the solution in the time domain:

$$\mathbf{x}(t) = \underbrace{\mathcal{L}^{-1}[\mathbf{\Phi}(s)\mathbf{x}(0_+)]}_{\text{zero input component}} + \underbrace{\mathcal{L}^{-1}[\mathbf{\Phi}(s)\mathbf{B}\mathbf{u}(s)]}_{\text{zero state component}} \tag{A.27}$$

Equation (A.27) gives the desired solution. The first component depends only on the initial state $\mathbf{x}(0_+)$, and the second component depends only on the input. The first term of Eq. (A.27) is actually equivalent to the equation set in Eqs. (A.22), but in matrix notation.

Another form of the general solution to the state space system in Eq. (A.15) is

$$\mathbf{x}(t) = \boldsymbol{\phi}(t, t_0)\mathbf{x}(t_0) + \mathbf{x}_\mathrm{p} \tag{A.28}$$

where $\mathbf{x}_\mathrm{p}$ is the particular solution and $\boldsymbol{\phi}(t,t_0)$ is the transition matrix which maps the initial state vector to the final state vector at time t. The general form of the particular solution is

$$\mathbf{x}_\mathrm{p} = \int_{t_0}^{t} \boldsymbol{\phi}(t,a)\mathbf{B}\mathbf{u}(a)\mathrm{d}a \tag{A.29}$$

There exist several methods of obtaining the transition matrix in Eq. (A.28). The easiest method in this case, as we already have the general solution in Eqs. (A.22), is to find the derivatives and re-arrange with respect to the initial values. A more formal methodology is to calculate it according to Eq. (A.30), which describes the general way of calculating the transition matrix for state space systems:

$$\boldsymbol{\phi}(t,t_0) = \mathrm{e}^{\mathbf{A}(t-t_0)} = \mathrm{e}^{\mathbf{A}(\tau)} \tag{A.30}$$

From Eqs. (A.27) and (A.28), we see the equivalence of the matrices:

$$\phi(\tau) \leftrightarrow \boldsymbol{\Phi}(s) \tag{A.31}$$

The transition matrix in terms of τ now follows for the in-plane and the out-of-plane motion, respectively. For in-plane motion,

$$\phi(\tau) = \begin{bmatrix} 1 & 6(\omega\tau - \sin(\omega\tau)) & \frac{4}{\omega}\sin(\omega\tau) - 3\tau & \frac{2}{\omega}(1-\cos(\omega\tau)) \\ 0 & 4 - 3\cos(\omega\tau) & \frac{2}{\omega}(\cos(\omega\tau)-1) & \frac{1}{\omega}\sin(\omega\tau) \\ 0 & 6\omega(1-\cos(\omega\tau)) & 4\cos(\omega\tau) - 3 & 2\sin(\omega\tau) \\ 0 & 3\omega\sin(\omega\tau) & -2\sin(\omega\tau) & \cos(\omega\tau) \end{bmatrix} \tag{A.32}$$

And for out-of-plane:

$$\phi_o(\tau) = \begin{bmatrix} \cos(\omega\tau) & \frac{1}{\omega}\sin(\omega\tau) \\ -\omega\sin(\omega\tau) & \cos(\omega\tau) \end{bmatrix} \tag{A.33}$$

A.1.3 Particular solution

In this section we will find an analytical solution for $\mathbf{x}_\mathrm{p}$ in Eq. (A.28). We will consider the special case where the input $\mathbf{u}(t)$ is a super-position of step functions, and the resulting pulses are assumed to be of constant amplitude. This is the solution for a special case, but is nevertheless of interest since the actuators for trajectory manoeuvres provide forces as pulses.

For the development of a solution, we will consider only a single principal pulse, which will later be generalised for an arbitrary number of pulses. The input function is defined as follows:

$$f(t) \triangleq k u_{t_1}(t) - k u_{t_2}(t) \tag{A.34}$$

where k is the amplitude of the pulse and $u_a(t)$ is the unit step function defined as

$$u_a(t) = \begin{cases} 1 & \text{for} \quad t \geq a \\ 0 & \text{for} \quad t < a \end{cases} \tag{A.35}$$

For Eq. (A.34) the following inequality must be fulfilled, where t_0 is the initial time:

$$t_0 \leq t_1 < t_2 \tag{A.36}$$

In order to find the zero state component in Eq. (A.27), we will now find the Laplace transformation of Eq. (A.34), yielding

$$\mathcal{L}(f(t)) = F(s) = \frac{k}{s}(\mathrm{e}^{-t_1 s} - \mathrm{e}^{-t_2 s}) \tag{A.37}$$

The product $\mathbf{\Phi}(s)\mathbf{B}$ in Eq. (A.27) consists only of the last two columns of $\mathbf{\Phi}(s)$ as the first two rows of $\mathbf{B}$ in Eq. (A.16) are zero. For the out-of-plane equation it concerns only the last row using Eq. (A.17).

The principles of this particular solution will be based on the in-plane equation, which will be similar for the out-of-plane solution except the dimensions are smaller. From Eq. (A.16) we find the product $\mathbf{BF}(s)$, yielding

$$\mathbf{BF}(s) = \frac{1}{m_\mathrm{c}} \begin{bmatrix} 0 \\ 0 \\ F_x(s) \\ F_z(s) \end{bmatrix} \tag{A.38}$$

What we seek is the zero state component of Eq. (A.27) and, using Eq. (A.38), we can write the form of that term for the in-plane motion as

$$\mathcal{L}^{-1}[\mathbf{\Phi}(s)\mathbf{Bu}(s)] = \frac{1}{m_\mathrm{c}}\mathcal{L}^{-1} \begin{bmatrix} \Phi_{1,3}(s)F_x(s) + \Phi_{1,4}(s)F_z(s) \\ \Phi_{2,3}(s)F_x(s) + \Phi_{2,4}(s)F_z(s) \\ \Phi_{3,3}(s)F_x(s) + \Phi_{3,4}(s)F_z(s) \\ \Phi_{4,3}(s)F_x(s) + \Phi_{4,4}(s)F_z(s) \end{bmatrix} \tag{A.39}$$

The Laplace transformation of the transition matrix can be found from Eq. (A.32). We will not calculate all the elements in detail, but we will look at the first non-zero one, which is element $\Phi(s)_{1,3}$ multiplied by the third element of Eq. (A.38):

$$\Phi_{1,3}(s)F_x(s) = \frac{1}{m_\mathrm{c}} \left[\frac{4}{s^2 + \omega^2} - \frac{3}{s^2} \right] \frac{k}{s}(\mathrm{e}^{-t_1 s} - \mathrm{e}^{-t_2 s}) \tag{A.40}$$

and on re-arranging terms, it yields

$$\Phi_{1,3}(s)F_x(s) = \frac{k}{m_\mathrm{c}} \left[\frac{4}{s(s^2 + \omega^2)} - \frac{3}{s^3} \right] (\mathrm{e}^{-t_1 s} - \mathrm{e}^{-t_2 s}) \tag{A.41}$$

The inverse Laplace transformation of Eq. (A.41) yields

$$\frac{k}{m_\mathrm{c}}\left[\frac{4}{\omega^2}[\cos(\omega(t-t_2))-\cos(\omega(t-t_1))]+\frac{3}{2}[(t-t_2)^2-(t-t_1)^2]\right] \tag{A.42}$$

The procedure is the same for the other elements for both in-plane and out-of-plane equations. Considering *one* pulse only, we can formulate the particular solution $\mathbf{x}_\mathrm{p}$ as

$$\mathbf{x}_{\mathrm{p}_\mathrm{one}} = \frac{1}{m_\mathrm{c}}\mathbf{H}\mathbf{u} \tag{A.43}$$

where $\mathbf{H}$ is a 4×2 matrix for the in-plane motion and a 2×1 matrix for the out-of-plane motion, both consisting of the terms in Eq. (A.39) and illustrated in Eq. (A.41). The mass m_c is kept separate in Eq. (A.43) and the input vector $\mathbf{u}$ is re-defined to contain only the amplitude of the pulses applied, being 2×1 for the in-plane and a scalar for the out-of-plane.

To write this in a more compact form, we will now separate out the columns of Eq. (A.39) as follows:

$$\mathbf{H} = [\mathbf{h}_1\ \mathbf{h}_3] \tag{A.44}$$

for the in-plane motion and directly one column $\mathbf{h}_2$ for the out-of-plane motion.

We are now in a position to find all the elements of the column vectors for the particular solution; this yields

$$\mathbf{h}_1 = \begin{bmatrix} \frac{4}{\omega^2}\left[\cos(\omega(t-t_2))-\cos(\omega(t-t_1))\right]+\frac{3}{2}\left[(t-t_2)^2-(t-t_1)^2\right] \\ \frac{2}{\omega^2}\left[\sin(\omega(t-t_1))-\sin(\omega(t-t_2))+\omega(t_1-t_2)\right] \\ \frac{4}{\omega}\left[\sin(\omega(t-t_1))-\sin(\omega(t-t_2))\right]+3(t_1-t_2) \\ \frac{2}{\omega}\left[\cos(\omega(t-t_1)-\cos(\omega(t-t_2))\right] \end{bmatrix} \tag{A.45}$$

$$\mathbf{h}_2 = \begin{bmatrix} \frac{1}{\omega^2}\left[\cos(\omega(t-t_2))-\cos(\omega(t-t_1))\right] \\ \frac{1}{\omega}\left[\sin(\omega(t-t_1))-\sin(\omega(t-t_2))\right] \end{bmatrix} \tag{A.46}$$

and

$$\mathbf{h}_3 = \begin{bmatrix} \frac{2}{\omega^2}\left[\sin(\omega(t-t_2))-\sin(\omega(t-t_1))+\omega(t_2-t_1)\right] \\ \frac{1}{\omega^2}\left[\cos(\omega(t-t_2))-\cos(\omega(t-t_1))\right] \\ \frac{2}{\omega}\left[\cos(\omega(t-t_2)-\cos(\omega(t-t_1))\right] \\ \frac{1}{\omega}\left[\sin(\omega(t-t_1))-\sin(\omega(t-t_2))\right] \end{bmatrix} \tag{A.47}$$

We recall that the result in Eqs. (A.45), (A.46) and (A.47) is for one single pulse. This result can now be generalised for an arbitrary number of pulses, by making a summation of particular solutions, where the corresponding start time t_1 and finish time t_2 must be inserted for each pulse. We can now write the general expressions.

For the in-plane expression, there is a summation where index i and k refer to the x-axis and the z-axis, respectively:

$$\mathbf{x}_{\mathrm{p}} = \frac{1}{m_{\mathrm{c}}} \sum_i \sum_k \left[(\mathbf{h}_1 u_1)_i + (\mathbf{h}_3 u_3)_k \right] \tag{A.48}$$

For the out-of-plane expression, there is a summation index j which refers to the y-axis:

$$\mathbf{x}_{\mathrm{p_o}} = \frac{1}{m_{\mathrm{c}}} \sum_j \left[(\mathbf{h}_2 u_2)_j \right] \tag{A.49}$$

Equations (A.48) and (A.49) describe the general form of the particular solutions, whether it is a series of pulses or a constant thrust. Cases with simple initial conditions are presented in section 3.3.3.

A.1.4 Discrete time state space system

For the design of controllers in continuous time, the models in Eqs. (A.16) and (A.17) are convenient, but discrete time controllers are most convenient and efficient when designed directly in the discrete time domain. For that purpose a discrete time model is required.

To obtain the discrete model we use a step invariant Z-transformation, where the input signal is considered constant during the sample time T. This fits very well with this type of system, where the inputs are pulses of constant amplitude. The time t is now considered as the discrete time, and time $t+1$ means the current time plus the sampling time T. The state space model is defined as

$$\mathbf{x}(t+1) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) \tag{A.50}$$

The coefficient matrix $\mathbf{F}$ can be found from Eq. (A.32) for the in-plane motion and from Eq. (A.33) for the out-of-plane motion by replacing the independent variable τ with the sampling time T, and formally computed as

$$\mathbf{F} = \mathrm{e}^{\mathbf{A}T} = \phi(T) \tag{A.51}$$

The input matrix in Eq. (A.50), $\mathbf{G}$, is defined as follows:

$$\mathbf{G} = \int_0^T \mathrm{e}^{\mathbf{A}t}\mathbf{B}\,\mathrm{d}t \tag{A.52}$$

The product $\mathrm{e}^{\mathbf{A}t}\mathbf{B}$ can be found from Eqs. (A.32) and (A.33) for the in- and out-of-plane motion, respectively. From Eqs. (A.16) and (A.17), it can be seen that it gives the last two, respectively one, columns of Eqs. (A.32) and (A.33) divided by the mass.

These elements are then integrated according to Eq. (A.52) from zero to the sampling time, with the following result for in-plane motion:

$$\mathbf{G} = \frac{1}{m_c}\begin{bmatrix} \frac{4}{\omega^2}(1-\cos(\omega T)) - \frac{3}{2}T^2 & \frac{2}{\omega^2}(\omega T - \sin(\omega T)) \\ \frac{2}{\omega^2}(\sin(\omega T) - \omega T) & \frac{1}{\omega^2}(1-\cos(\omega T)) \\ \frac{4}{\omega}\sin(\omega T) - 3T & \frac{2}{\omega}(1-\cos(\omega T)) \\ \frac{2}{\omega}(\cos(\omega T) - 1) & \frac{1}{\omega}\sin(\omega T) \end{bmatrix} \tag{A.53}$$

And for the out-of-plane motion,

$$\mathbf{G}_o = \frac{1}{m_c}\begin{bmatrix} \frac{1}{\omega^2}(1-\cos(\omega T)) \\ \frac{1}{\omega}\sin(\omega T) \end{bmatrix} \tag{A.54}$$

It should be recalled that the discrete state space model is not an approximation of the continuous model, but gives the exact values at the sampling times. For designs where the implementation is in discrete time, meaning a computer controlled system, the direct design in discrete time should always be performed. It ensures a better design with larger stability margins for the same sampling time. The sampling time should be selected such that it is seven to ten times faster than the fastest mode in the closed loop system.

A.1.5 Travelling ellipse formulation

This formulation will only be for the in-plane motion, where a cycloid motion exists. The out-of-plane motion is a pure oscillator and is decoupled, as we know from previous sections. The motivation for this formulation is that it is easier to work with than Eq. (A.22) because the cycloid centre coordinates are expressed explicitly and the influence of velocities appears clearer.

The formulation of the solution to the CW equations in Eqs. (A.22) and (A.28) may take several forms. Here we consider an elliptic formulation, chosen because it is very practical for analytical work with the in-plane guidance and calculation of ΔV manoeuvres.

The form of a general ellipse can be formulated as

$$\begin{Bmatrix} x \\ z \end{Bmatrix} = \begin{Bmatrix} x_c \\ z_c \end{Bmatrix} + \begin{Bmatrix} a\cos(\theta) \\ b\sin(\theta) \end{Bmatrix} \tag{A.55}$$

Equation (A.55) is the parametric equation for a general ellipse with the centre at (x_c, y_c), and with semi major axis a and semi minor axis b. The period of the ellipse is in this case the orbital period, and the θ in Eq. (A.55) is equivalent to the eccentric anomaly. We will now rewrite Eqs. (A.22) as follows:

$$\begin{aligned} x(t) &= 2[A\sin(\omega t) - B\cos(\omega t)] + (6\omega z_0 - 3\dot{x}_0)t + (x_0 + \tfrac{2\dot{z}_0}{\omega}) \\ z(t) &= A\cos(\omega t) + B\sin(\omega t) + (4z_0 - \tfrac{2\dot{x}_0}{\omega}) \end{aligned} \tag{A.56}$$

In Eqs. (A.56) we recognise the vertical component of the centre to be

$$z_{\mathrm{c}} = 4z_0 - \frac{2\dot{x}_0}{\omega} \tag{A.57}$$

From Eqs. (A.56) and (A.57) the horizontal component of the centre of the ellipse yields

$$x_{\mathrm{c}} = x_0 + \frac{2\dot{z}_0}{\omega} + \frac{3}{2}\omega\left(4z_0 - \frac{2\dot{x}_0}{\omega}\right)t \tag{A.58}$$

$$x_{\mathrm{c}} = x_0 + \frac{2\dot{z}_0}{\omega} + \frac{3}{2}z_{\mathrm{c}}\omega t \tag{A.59}$$

From Eqs. (A.22) the semi minor axis can be expressed as

$$b = \sqrt{A^2 + B^2} = \sqrt{\left(\frac{2\dot{x}_0}{\omega} - 3z_0\right)^2 + \left(\frac{\dot{z}_0}{\omega}\right)^2} \tag{A.60}$$

and on adding and subtracting z_0 to A we obtain a more compact form:

$$b = \sqrt{\left(\frac{2\dot{x}_0}{\omega} - 4z_0 + z_0\right)^2 + \left(\frac{\dot{z}_0}{\omega}\right)^2} \tag{A.61}$$

$$b = \sqrt{(z_0 - z_c)^2 + \left(\frac{\dot{z}_0}{\omega}\right)^2} \tag{A.62}$$

From Eqs. (A.56) we recognise that the semi major axis is twice the size of the semi minor axis, leading to

$$a = 2b \tag{A.63}$$

As the coefficients for the trigonometric functions in Eqs. (A.56) are opposite in x and z, and with opposite sign for the cosine function, the same phase angle will be obtained in θ, where

$$\theta = \omega t + \varphi \tag{A.64}$$

and

$$\varphi = \arctan\left(\frac{A}{B}\right) \tag{A.65}$$

$$\varphi = \arctan\left(\frac{\frac{2\dot{x}_0}{\omega} - 3z_0}{\frac{\dot{z}_0}{\omega}}\right) \tag{A.66}$$

$$\varphi = \arctan\left(\frac{z_0 - z_c}{\frac{\dot{z}_0}{\omega}}\right) \tag{A.67}$$

$$\varphi = \arctan\left(\frac{\omega}{\dot{z}_0}(z_0 - z_{\mathrm{c}})\right) \tag{A.68}$$

The correct quadrant has to be taken care of by performing the inverse trigonometric function in Eq. (A.65). Summarising the results for the in-plane motion in parameterised elliptic form yields

$$x(t) = x_\mathrm{c}(t) + 2b\cos(\omega t + \varphi) \tag{A.69}$$

$$z(t) = z_\mathrm{c} + b\sin(\omega t + \varphi) \tag{A.70}$$

where

$$x_\mathrm{c}(t) = x_0 + 2\frac{\dot{z}_0}{\omega} + \frac{3}{2}z_\mathrm{c}\omega t \tag{A.71}$$

$$z_\mathrm{c} = 4z_0 - 2\frac{\dot{x}_0}{\omega} \tag{A.72}$$

$$b = \sqrt{(z_0 - z_c)^2 + \left(\frac{\dot{z}_0}{\omega}\right)^2} \tag{A.73}$$

$$\varphi = \arctan\left(\frac{\omega}{\dot{z}_0}(z_0 - z_\mathrm{c})\right) \tag{A.74}$$

It is seen from Eq. (A.71) that the centre of the ellipse travels with a constant velocity $\dot{x}_\mathrm{c}$ proportional to the altitude of the centre of the ellipse z_c. If z_c is zero and no disturbances are present, it is theoretically possible to station keep without any expenditure of fuel.

An important consideration in the guidance for controlling the in-plane motion is that of minimising the performance index for fuel expenditure, which consists of the sum of ΔV pulses. From Eq. (A.72) it is seen that a change in altitude is only affected by x-axis velocity corrections. Similarly in Eq. (A.71) it is seen that $x_\mathrm{c}(t)$ is affected only by z-axis velocity corrections. It is therefore clear that, in order to control the drift of the ellipse, it will be sufficient to apply forces only in the vertical direction.

A.2 Attitude dynamics and kinematics

In this section, the details and intermediate calculations required for the derivation of the general equations of the attitude dynamics and kinematics are provided.

A.2.1 Direction cosine matrix (DCM)

This section summarises the direction cosine matrix (DCM) for an Euler (3,2,1) rotation, as the individual matrices will be needed. Recall that 1 is x-axis, 2 is y-axis and 3 is z-axis. The rotation is from a frame a to a frame b, such that

$$\mathbf{v}_b = \mathbf{R}_{ba}\mathbf{v}_a \tag{A.75}$$

where $\mathbf{v}_a$ is a vector projected on the axes of the a frame and $\mathbf{v}_b$ is the same vector projected on the axes of the b frame. The DCM is derived by a rotation around the third axis of a followed by rotations around the second and first axes of the resulting intermediate frames. We can therefore write the three individual matrices as follows:

$$\mathbf{R}_{ba}(\boldsymbol{\theta}) = \mathbf{R}_1(\theta_1)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_3) \tag{A.76}$$

$$\mathbf{R}_{ba}(\boldsymbol{\theta}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & \sin(\theta_1) \\ 0 & -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) \\ 0 & 1 & 0 \\ \sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \begin{bmatrix} \cos(\theta_3) & \sin(\theta_3) & 0 \\ -\sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.77}$$

$$\mathbf{R}_{ba}(\boldsymbol{\theta}) = \begin{bmatrix} c(\theta_3)c(\theta_2) & c(\theta_2)s(\theta_3) & -s(\theta_2) \\ s(\theta_1)s(\theta_2)c(\theta_3) - c(\theta_1)s(\theta_3) & s(\theta_1)s(\theta_2)s(\theta_3) + c(\theta_1)c(\theta_3) & s(\theta_1)c(\theta_2) \\ c(\theta_1)s(\theta_2)c(\theta_3) + s(\theta_1)s(\theta_3) & c(\theta_1)s(\theta_2)s(\theta_3) - s(\theta_1)c(\theta_3) & c(\theta_1)c(\theta_2) \end{bmatrix} \tag{A.78}$$

where $c(\theta_i) = \cos(\theta_i)$ and $s(\theta_i) = \sin(\theta_i)$, and where $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^{\mathrm{T}}$ is the rotation angle about the respective axis. The inverse rotation is found from the transpose of the orthonormal matrix $\mathbf{R}_{ba}$ such that $\mathbf{R}_{ab} = \mathbf{R}_{ba}^{\mathrm{T}}$.

A.2.2 Nonlinear dynamics

We can write the angular momentum of a rigid body as

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \tag{A.79}$$

where $\mathbf{I}$ is the inertia matrix and $\boldsymbol{\omega}$ is the inertial angular velocity vector. The torque vector $\mathbf{N}$ can be expressed as (Symon 1979)

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{N} \tag{A.80}$$

Expressing a vector in a rotating (starred) system we get for Eq. (A.79)

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}^*\mathbf{L}^*}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{L}^* \tag{A.81}$$

$$\mathbf{N} = \frac{\mathrm{d}^*(\mathbf{I}\boldsymbol{\omega}^*)}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega}^* \tag{A.82}$$

and as $\boldsymbol{\omega}$ is also the angular velocity of the rotating frame, $\boldsymbol{\omega}^* = \boldsymbol{\omega}$. If we also consider the rotating frame fixed to the body, the inertia matrix is constant, and we can express Eq. (A.82) in the body frame as

$$\boxed{\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mathbf{N}} \tag{A.83}$$

In the special case of the body axes being along the principal axes of inertia, the inertia matrix $\mathbf{I}$ is diagonal and Eq. (A.83) becomes

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_z\omega_y &= N_x \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= N_y \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_y\omega_x &= N_z \end{aligned} \tag{A.84}$$

From Eqs. (A.84) we see that a body cannot spin with constant angular velocity $\boldsymbol{\omega}$, except about a principal axis, unless external torques are applied. If $\dot{\boldsymbol{\omega}} = \mathbf{0}$, Eq. (A.83) becomes $\boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mathbf{N}$, and the left hand side is zero only if $\mathbf{I}\boldsymbol{\omega}$ is parallel to $\boldsymbol{\omega}$, that is, if $\boldsymbol{\omega}$ is along a principal axis of the body.

A.2.3 Nonlinear kinematics

For the kinematics we seek the differential equations of the motion of the body frame $\mathbf{F}_a$ with respect to the reference frame $\mathbf{F}_{\mathrm{lo}}$, relating the Euler (3,2,1) angles with the angular velocity vector $\boldsymbol{\omega}_{\mathrm{alo}}$.

The $\boldsymbol{\omega}_{\mathrm{alo}}$ between the frames is the sum of the individual rotation rates, referred to and added in the final frame. Using the individual rotation matrices from the Euler (3,2,1) rotation in Eq. (A.76) we can write

$$\boldsymbol{\omega}_{\mathrm{alo}} = \begin{bmatrix} \dot{\theta}_x \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_1(\theta_x) \begin{bmatrix} 0 \\ \dot{\theta}_y \\ 0 \end{bmatrix} + \mathbf{R}_1(\theta_x)\mathbf{R}_2(\theta_y) \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{bmatrix} \tag{A.85}$$

Multiplying the matrices $\mathbf{R}_1(\theta_x)$ and $\mathbf{R}_2(\theta_y)$ from section A.2.1 and collecting terms, Eq. (A.85) becomes

$$\boldsymbol{\omega}_{\mathrm{alo}} = \begin{bmatrix} 1 & 0 & -\sin(\theta_y) \\ 0 & \cos(\theta_x) & \sin(\theta_x)\cos(\theta_y) \\ 0 & -\sin(\theta_x) & \cos(\theta_x)\cos(\theta_y) \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \tag{A.86}$$

We need the inverse relationship of Eq. (A.86), and it shall be noted that the matrix is not a DCM and is not orthonormal, so we need to find the inverse matrix. The determinant becomes $\cos(\theta_y)$ and the inverse can be written as follows in the body frame:

$$\begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \frac{1}{\cos(\theta_y)} \begin{bmatrix} \cos(\theta_y) & 0 & 0 \\ \sin(\theta_x)\sin(\theta_y) & \cos(\theta_x)\cos(\theta_y) & \sin(\theta_x) \\ \cos(\theta_x)\sin(\theta_y) & -\sin(\theta_x)\cos(\theta_y) & \cos(\theta_x) \end{bmatrix} \boldsymbol{\omega}_{\mathrm{alo}} \tag{A.87}$$

A.2.4 Linear kinematics and dynamics attitude model

We will now develop the main steps taken in arriving at a combined linear model for dynamics and kinematics for the attitude motion of a spacecraft from the models in Eqs. (A.83) and (A.87), respectively.

The linearisation will be a general Taylor series expansion around a working point as both the kinematics and the dynamics models are functions of two variables. For a general function $f(x, u)$ we obtain to first order

$$f(x,u) = f(x_0,u_0) + \left.\frac{\partial f(x,u)}{\partial x}\right|_{x_0,u_0} (x - x_0) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{x_0,u_0} (u - u_0) \tag{A.88}$$

where the subscript 0 denotes the operating point.

We can write Eq. (A.83) for the dynamics as $\mathbf{I}\dot{\boldsymbol{\omega}} = f(\boldsymbol{\omega}, \mathbf{N})$, where the operating point for the torque is $\mathbf{N}_0 = \mathbf{0}$ and the operating point for the angular rate is the angular rate of the orbital frame $\boldsymbol{\omega}_0 = [0, -\omega_0, 0]^{\mathrm{T}}$.

We can write Eq. (A.87) for the kinematics as $\boldsymbol{\theta} = g(\boldsymbol{\theta}, \boldsymbol{\omega}_{\mathrm{alo}})$, where the operating point for the attitude angles is $\boldsymbol{\theta}_0 = \mathbf{0}$ and the operating point for the angular rate to the body frame is $\boldsymbol{\omega}_{\mathrm{alo}_0} = \mathbf{0}$.

We will not go through all the trivial derivations of the partial derivatives. We proceed to the combined linear model, defining the state vector as $\mathbf{x} = [\theta_x, \theta_y, \theta_z, \omega_{\mathrm{alo}_x}, \omega_{\mathrm{alo}_y}, \omega_{\mathrm{alo}_z}]^{\mathrm{T}}$ yielding

$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{N}} \tag{A.89}$$

where the system matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \left[\begin{array}{ccccc|c} 0 & 0 & \omega_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\omega_0 & 0 & 0 & 0 & 0 & 1 \\ \multicolumn{3}{c}{\mathbf{0}_{3\times3}} & \multicolumn{3}{c}{\omega_0 \mathbf{I}^{-1} \begin{bmatrix} I_{31} & 2I_{32} & I_{33} - I_{22} \\ -I_{32} & 0 & I_{12} \\ I_{22} - I_{11} & -2I_{12} & -I_{13} \end{bmatrix}_{3\times3}} \end{array}\right] \tag{A.90}$$

and the input matrix $\mathbf{B}$ becomes

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{3\times3} \\ \mathbf{I}^{-1}_{3\times3} \end{bmatrix} \tag{A.91}$$

In the case where the inertia matrix $\mathbf{I}$ is diagonal, meaning that the body axes coincide with the principal axes, the linear attitude dynamics can be simplified to

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & \omega_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\omega_0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \omega_0 \frac{I_{33}-I_{22}}{I_{11}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_0 \frac{I_{22}-I_{11}}{I_{33}} & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{I_{11}} & 0 & 0 \\ 0 & \frac{1}{I_{22}} & 0 \\ 0 & 0 & \frac{1}{I_{33}} \end{bmatrix} \mathbf{N} \tag{A.92}$$