APPENDIX B:

CLOSED-FORM SOLUTION FOR AN INTEGRAL CONTAINING MATRIX EXPONENTIALS

As defined by Eq. 11.42 we must often compute a number of matrix integrals of the form

$$\Psi(A_1,A_2,A_3,t_f,t_o) = \int_{t_o}^{t_f} e^{A_1\tau} A_2 e^{A_3\tau} d\tau \tag{B.1}$$

where the constant matrices A_1, A_2, and A_3 are specified. To simplify the evaluation of the integral in Eq. B.1 we make the following change of variables

$\tau = \sigma + t_o$

leading to

$$\Psi(A_1,A_2,A_3,t_f,t_o) = e^{A_1 t_o}\Psi(A_1,A_2,A_3,t_f-t_o,0)e^{A_3 t_o} \tag{B.2}$$

The integral involving the matrix exponentials in Eq. B.2 can be easily evaluated by forming the constant matrix

$$\Pi = \begin{array}{cc} \begin{bmatrix} -A_1 & A_2 \\ 0 & A_3 \end{bmatrix} & \begin{matrix} \}n_1 \\ \}n_2 \end{matrix} \\ \begin{matrix} n_1 & \quad n_2 \end{matrix} & \end{array} \tag{B.3}$$

and computing the matrix exponential (ref. B.1)

$$e^{\Pi(t_f-t_o)} = \begin{array}{cc} \begin{bmatrix} F_1(t_f-t_o) & G_1(t_f-t_o) \\ 0 & F_2(t_f-t_o) \end{bmatrix} & \begin{matrix} \}n_1 \\ \}n_2 \end{matrix} \\ \begin{matrix} n_1 & \qquad\qquad n_2 \end{matrix} & \end{array} \tag{B.4}$$

where we have the "Van-Loan Identities":

$$F_1(t_f-t_o) = e^{-A_1(t_f-t_o)} \quad , \quad F_2(t_f-t_o) = e^{A_3(t_f-t_o)} \tag{B.5}$$

$$G_1(t_f-t_o) = e^{-A_1(t_f-t_o)}\Psi(A_1,A_2,A_3,t_f-t_o,0) \tag{B.6}$$

Thus the solution for the integral of Eq. B.2 follows as

$$\Psi(A_1,A_2,A_3,t_f-t_o,0) = e^{A_1(t_f-t_o)}G_1(t_f-t_o) \tag{B.7}$$

and the solution for Eq. B.1 can be shown to be

$$\Psi(A_1,A_2,A_3,t_f,t_o) = e^{A_1t_f}G_1(t_f - t_o)e^{A_3t_o} \tag{B.8}$$

In order to evaluate Eq. B.8 at discrete times, the semigroup properties of exponential matrices are now exploited, yielding the following recursion relationships for the matrix partitions of $e^{\Pi(t_f-t_o)}$:

$$F_j(t + \Delta t) = F_j(\Delta t)F_j(t) \quad , \quad F_j(t_o) = I \quad , \quad j = 1,2 \tag{B.9}$$

$$G_1(t + \Delta t) = F_1(\Delta t)G_1(t) + G_1(\Delta t)F_2(t) \quad , \quad G_1(t_o) = 0 \tag{B.10}$$

where $\Delta t = (t_f - t_o)/m$ and m is the total number of discrete time steps. As a result, the integral in Eq. B.1 can be written as

$$\Psi(A_1,A_2,A_3,t+\Delta,t_o) = e^{A_1t_o}F_3(t + \Delta)G_1(t + \Delta)e^{A_3t_o} \tag{B.11}$$

where $F_3(t) = e^{A_1t}$ is obtained in a separate calculation and $F_3(t + \Delta t)$ is recursively generated by using a formula similar to Eq. B.9. A further simplification results when Eq. B.11 is post-multiplied by a vector

$$\begin{aligned}\Psi(A_1,A_2,A_3,t+\Delta,t_o)\sigma &= e^{A_1t_o}F_3(t + \Delta)G_1(t + \Delta)e^{A_3t_o}\sigma \\ &= e^{A_1t_o}F_3(t + \Delta)\nu_1(t + \Delta)\end{aligned} \tag{B.12}$$

where the recursion relationship for $\nu_1(t)$ is given by

$$\nu_1(t + \Delta) = F_1(\Delta t)\nu_1(t) + G_1(\Delta t)\nu_2(t) \quad , \quad \nu_1(0) = \mathbf{0} \tag{B.13}$$

$$\nu_2(t + \Delta) = F_2(\Delta t)\nu_2(t) \quad , \quad \nu_2(0) = e^{A_3t_o}\sigma \tag{B.14}$$

References

B.1 Van Loan, C. F., "Computing Integrals Involving Matrix Exponentials," IEEE Transactions on Automatic Control, Vol. Ac-23, No. 3, June 1978, pp. 395-404.