CHAPTER 11

COMPUTATIONAL METHODS FOR CLOSED - LOOP CONTROL PROBLEMS

11.1 INTRODUCTION

In this chapter we present a summary of recently developed techniques for generating closed-form solutions for the feedback gains and state trajectory equations of a broad class of finite time linear quadratic optimization problems. The solutions for the feedback gains are useful for both simulation studies and generating commands to actuators. On the other hand, the state trajectory solutions are of value for sensitivity studies during the control design process; wherein, for example, the performance index weighting matrices can be adjusted in order to enhance the closed-loop system performance. The great utility of these solutions is that the associated computational issues are reduced from a problem of numerical integral calculus to a problem of numerical linear algebra. Throughout the discussion the plant dynamics equations are assumed to be linear and time-invariant. The success of these new methods is based on the use of the formal representation of the solution of the differential matrix Riccati (DMR) equation presented in Appendix A, where the solution of the DMR equation is expressed as the sum of a steady-state constant plus a variable transient term. The key step in the solution process for all of the control synthesis problems to be discussed is the use of the transient part of the Riccati solution, to generate simplifying coordinate transformations for the dependent variables. For example, in many problems the governing equations possess time-varying coefficients, however, by a proper choice of transformation variables the coefficients for the transformed equations become time-invariant. As a result, the transformed equations are easily solved by the well-known methods of classical linear systems theory.

Moreover, many of the resulting solutions can be cast in recursive form for efficient computer implementations, as shown in the spacecraft closed-loop control papers of References 7, 11, 12, 15, 16, and 18.

The following three classes of linear-quadratic optimal feedback problems are considered. First, the standard linear regulator problem is presented in Section 11.3. Second, the optimal terminal control problem is presented in Section 11.4. Third, the dual tracking and disturbance-accommodating control problem is presented in Section 11.5. Section 11.6 presents a short integral table for the Riccati-like matrix equations presented in Sections 11.3-5. Conclusions and recommendations are presented in Section 11.7.

11.2 THE OPTIMAL CONTROL PROBLEM

In this section we define the performance indicies to be minimized for the feedback control problems considered in Sections 11.3-5.

Type I: Linear Regulator Problem

$$J = \frac{1}{2}\|\mathbf{y}(t_f)\|^2_{S'} + \frac{1}{2}\int_{t_o}^{t_f} \{\|\mathbf{y}(t)\|^2_{Q'} + \|\mathbf{u}(t)\|^2_R\}dt \tag{11.1}$$

Subject to: $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$, given $\mathbf{x}(t_o)$ (11.2)

$$\mathbf{y}(t) = F\mathbf{x}(t) \tag{11.3}$$

Type II: Terminal Control Problem

$$J = \frac{1}{2}\int_{t_o}^{t_f} \{\|\mathbf{y}(t)\|^2_{Q'} + \|\mathbf{u}(t)\|^2_R\}dt \tag{11.4}$$

Subject to: $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$, given $\mathbf{x}(t_o)$ (11.5)

$$\mathbf{y}(t) = F\mathbf{x}(t) \tag{11.6}$$

$$x_i(t_f) = x_{if} \quad , \quad i = 1,\ldots,q;\ q \le n \tag{11.7}$$

Type III: Tracking/Disturbance-Accommodating Control Problem

$$J = \frac{1}{2}\|\mathbf{y}^*(t_f) - \mathbf{y}(t_f)\|^2_{S'} + \frac{1}{2}\int_{t_o}^{t_f} \{\|\mathbf{y}^*(t) - \mathbf{y}(t)\|^2_{Q'} + \|\mathbf{u}(t)\|^2_R\}dt \tag{11.8}$$

Subject to: $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + \mathbf{d}(t)$, given $\mathbf{x}(t_0)$ (11.9)

$$\mathbf{y}(t) = F\mathbf{x}(t) \tag{11.10}$$

where in Eqs. 11.1-.10 $\|\mathbf{p}\|_W^2 = \mathbf{p}^T W\mathbf{p}$, $\mathbf{y}(t)$ is the output state, A is the system dynamics matrix, B is the control influence matrix, F is the measurement influence matrix, $S' = (S')^T = F^T SF \geq 0$ is the terminal state weight matrix, $Q' = (Q')^T = F^T QF \geq 0$ is the state weight matrix, $R = R^T > 0$ is the control weight matrix, $\mathbf{x}(t)$ is the n-dimensional state vector, $\mathbf{u}(t)$ is the m-dimensional control vector, and $\mathbf{d}(t)$ is the n-dimensional disturbance state vector. Moreover, it is assumed that the systems listed above are controllable and observable, and that the controllers have perfect measurements for the states.

11.3 LINEAR REGULATOR PROBLEM: NECESSARY CONDITIONS AND SOLUTIONS

The necessary conditions for the linear regulator problem are obtained by defining the following Hamiltonian for Eqs. 11.1-.3 (see Chapter 6)

$$H(\mathbf{x}(t),\mathbf{u}(t),\lambda(t),t) = \frac{1}{2}\|\mathbf{y}(t)\|_{Q'}^2 + \frac{1}{2}\|\mathbf{u}(t)\|_R^2 + \lambda^T(t)[A\mathbf{x}(t) + B\mathbf{u}(t)] \tag{11.11}$$

and the terminal penalty function

$$\Theta(\mathbf{x}(t_f),t_f) = \|\mathbf{y}(t)\|_{S'}^2 \tag{11.12}$$

where $\lambda(t)$ is the Lagrange multiplier vector.

Application of Pontryagin's principle yields the necessary conditions

$$\partial H/\partial\lambda(t) = \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{11.13}$$

$$\partial H/\partial\mathbf{x}(t) = -\dot{\lambda}(t) = F^T QF\mathbf{x}(t) + A^T\lambda(t) \tag{11.14}$$

$$\partial H/\partial\mathbf{u}(t) = 0 \rightarrow \mathbf{u}(t) = -R^{-1}B^T\lambda(t) \tag{11.15}$$

$$\partial\Theta/\partial\mathbf{x}(t_f) = \lambda(t_f) = F^T SF\mathbf{x}(t_f) \tag{11.16}$$

Introducing Eq. 11.15 into Eq. 11.13 yields the modified state equation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) - BR^{-1}B^T\lambda(t) \tag{11.17}$$

The closed-loop control is determined by assuming that

$$\lambda(t) = P(t)\mathbf{x}(t) \tag{11.18}$$

where P(t) is a time-varying matrix function which is to be determined.

Substituting Eq. 11.18 into Eqs. 11.14 and 11.17, leads to the matrix differential Riccati equation:

$$\dot{P}(t) + P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + F^T QF = 0 \quad ; \quad P(t_f) = F^T SF \tag{11.19}$$

where the linear state feedback control is given by

$$\mathbf{u}(t) = -R^{-1}B^T P(t)\mathbf{x}(t) \tag{11.20}$$

As shown in Appendix A, the solution for Eq. 11.19 can be written as

$$P(t) = P_{ss} + Z^{-1}(t) \tag{11.21}$$

where P_{ss} is the solution for the algebraic Riccati equation of Eq. A.112 and $Z(t)$ satisfies the differential Lyapunov equation of Eq. A.116. The solution for $Z(t)$ can be shown to be (refs. 1,2,3)

$$Z(t) = Z_{ss} + e^{\bar{A}(t-t_f)}[Z(t_f) - Z_{ss}]e^{\bar{A}^T(t-t_f)} \tag{11.22}$$

where $\bar{A} = A - BR^{-1}B^T P_{ss}$ is the system stability matrix, $Z(t_f) = (F^T SF - P_{ss})^{-1}$, and Z_{ss} satisfies the algebraic Lyapunov equation (refs. 3,4-6)

$$\bar{A}Z_{ss} + Z_{ss}\bar{A}^T - BR^{-1}B^T = 0 \tag{11.23}$$

Introducing Eq. 11.21 into Eq. 11.20 produces the time-varying control

$$\mathbf{u}(t) = -R^{-1}B^T[P_{ss} + Z^{-1}(t)]\mathbf{x}(t) \tag{11.24}$$

where $Z(t)$ is given by Eq. 11.22. Clearly, if this solution is to be of practical value then it is necessary that efficient techniques be available for generating $Z(t)$ and its inverse. To this end, the solution for $Z(t)$ in Eq. 11.24 can be recursively generated from the following difference equation by exploiting the semi-group properties of the matrix exponentials appearing in Eq. 11.22:

$$\begin{aligned} Z(t + \Delta t) &= Z_{ss} + e^{\bar{A}\Delta t}[Z(t) - Z_{ss}]e^{\bar{A}^T\Delta t} \\ &= C + e^{\bar{A}\Delta t}Z(t)e^{\bar{A}^T\Delta t} \end{aligned}$$

where $C = Z_{ss} - e^{\bar{A}\Delta t}Z_{ss}e^{\bar{A}^T\Delta t}$

$$Z(t_o) = Z_{ss} + e^{\bar{A}(t_o - t_f)}[Z(t_f) - Z_{ss}]e^{\bar{A}^T(t_o - t_f)}$$

$$\Delta t = (t_f - t_o)/N$$

and the solution for $\mathbf{u}(t)$ can be obtained using the method outlined in Eqs. A.151-154. Alternatively, we can establish the following direct recursion for $Z^{-1}(t + \Delta t)$:

$$\begin{aligned} Z^{-1}(t + \Delta t) &= [C + e^{\bar{A}\Delta t}Z(t)e^{\bar{A}^T\Delta t}]^{-1} \\ &= e^{-\bar{A}^T\Delta t}[Z(t) + D]^{-1}e^{-\bar{A}\Delta t} \\ &= e^{-\bar{A}^T\Delta t}[I + Z^{-1}(t)D]^{-1}Z^{-1}(t)e^{-\bar{A}\Delta t} \end{aligned}$$

where $D = e^{-\bar{A}\Delta t}Z_{ss}e^{-\bar{A}^T\Delta t} - Z_{ss}$. The relative computational merits of the recursive solutions listed above remains a subject for future investigations.

11.3.1 State Trajectory Equation

In order to determine the closed-loop system response under the assumptions of perfect plant knowledge and perfect state estimation, we introduce Eq. 11.24 into Eq. 11.13, leading to

$$\dot{\mathbf{x}}(t) = [\bar{A} - BR^{-1}B^TZ^{-1}(t)]\mathbf{x}(t) \quad ; \quad \mathbf{x}_o = \mathbf{x}(t_o) \tag{11.25}$$

where the equation above is nonautonomous.

To simplify Eq. 11.25, we introduce the following coordinate transformation for the dependent variable $\mathbf{x}(t)$ (ref. 7):

$$\mathbf{x}(t) = Z(t)\mathbf{r}(t) \tag{11.26}$$

where $Z(t)$ is given by Eq. 11.22 and $\mathbf{r}(t)$ is a vector function which is to be determined.

Upon differentiating Eq. 11.26 we find

$$\dot{\mathbf{x}}(t) = \dot{Z}(t)\mathbf{r}(t) + Z(t)\dot{\mathbf{r}}(t) \tag{11.27}$$

or

$$\dot{\mathbf{x}}(t) = [\bar{A}Z(t) + Z(t)\bar{A}^T - BR^{-1}B^T]\mathbf{r}(t) + Z(t)\dot{\mathbf{r}}(t) \tag{11.28}$$

where $\dot{Z}(t)$ in Eq. 11.27 has been replaced by the right hand side of Eq. A.116.

The differential equation for $\mathbf{r}(t)$ is obtained by introducing Eqs. 11.26 and 11.28 into Eq. 11.25, leading to

$$Z(t)[\dot{\mathbf{r}}(t) + \bar{A}^T\mathbf{r}(t)] = 0 \tag{11.29}$$

from which it follows that the linear constant-coefficient vector differential equation for $\mathbf{r}(t)$ is given by

$$\dot{\mathbf{r}}(t) = -\bar{A}^T\mathbf{r}(t) \quad ; \quad \mathbf{r}_o = Z^{-1}(t_o)\mathbf{x}_o \tag{11.30}$$

where the solution for $\mathbf{r}(t)$ follows as

$$\mathbf{r}(t) = e^{-\bar{A}^T(t-t_o)}\mathbf{r}_o \tag{11.31}$$

Substituting Eq. 11.31 into Eq. 11.26 produces the desired solution for the state trajectories as

$$\mathbf{x}(t) = \Phi(t,t_o)\mathbf{x}_o \tag{11.32}$$

where $\Phi(t,t_o) = Z(t)e^{-\bar{A}^T(t-t_o)}Z^{-1}(t_o)$ is the system state transition matrix, and $\Phi(t,t_o)$ satisfies the nonautonomous matrix differential equation:

$$\dot{\Phi}(t,t_o) = [A - BR^{-1}B^TP(t)]\Phi(t,t_o) \quad ; \quad \Phi(t_o,t_o) = I$$

It is of interest to observe that $\Phi(t,t_o)$ satisfies the following semi-group and inversion properties:

$$\begin{aligned}\Phi(t_2,t_o) &= \Phi(t_2,t_1)\Phi(t_1,t_o)\\ &= [Z(t_2)e^{-\bar{A}^T(t_2-t_1)}Z^{-1}(t_1)][Z(t_1)e^{-\bar{A}^T(t_1-t_o)}Z^{-1}(t_o)]\\ &= Z(t_2)e^{-\bar{A}^T(t_2-t_o)}Z^{-1}(t_o)\end{aligned}$$

and

$$\begin{aligned}\Phi(t_o,t) &= \Phi^{-1}(t,t_o)\\ &= [Z(t)e^{-\bar{A}^T(t-t_o)}Z^{-1}(t_o)]^{-1}\\ &= [Z^{-1}(t_o)]^{-1}e^{\bar{A}^T(t-t_o)}Z^{-1}(t)\end{aligned}$$

$$= Z(t_o)e^{-\bar{A}^T(t_o-t)}Z^{-1}(t)$$

11.3.2 Recursion Relationship for Evaluating the State and Control at Discrete Time Steps

If the solution for $x(t)$ is required at the discrete times $t_k = t_o + k\Delta t$ $(k = 1,\ldots,N)$ for $\Delta t = (t_f - t_o)/N$, then Eq. 11.32 can be written as (refs. 7 and 18)

$$x(t) = \{Z_{ss} + e^{\bar{A}(t-t_f)}[Z(t_f) - Z_{ss}]e^{\bar{A}^T(t-t_f)}\}e^{-\bar{A}^T(t-t_o)}Z^{-1}(t_o)x_o$$

$$= Z_{ss}e^{-\bar{A}^T(t-t_o)}Z^{-1}(t_o)x_o + e^{\bar{A}(t-t_f)}[Z(t_f) - Z_{ss}]e^{\bar{A}^T(t_o-t_f)}Z^{-1}(t_o)x_o \tag{11.33}$$

leading to the linear recursion

$$x(t_k) = Z_{ss}a_k + b_k \quad , \quad k = 1,\ldots,N \tag{11.34}$$

where

$$a_o = Z^{-1}(t_o)x_o$$

$$b_o = e^{\bar{A}(t_o-t_f)}[Z(t_f) - Z_{ss}]e^{\bar{A}^T(t_o-t_f)}Z^{-1}(t_o)x_o$$

$$a_k = e^{-\bar{A}^T\Delta t}a_{k-1}$$

$$b_k = e^{\bar{A}\Delta t}b_{k-1}$$

To compute the control, $u(t)$, at discrete times we observe that Eq. 11.20 can be written as

$$u(t) = -R^{-1}B^TP(t)x(t)$$

$$= -R^{-1}B^T[P_{ss} + Z^{-1}(t)]\Phi(t,t_o)x_o$$

where the closed-form solutions for $P(t)$ and $x(t)$ have been introduced from Eqs. 11.21 and 11.32, respectively. Simplifying the equation above leads to

$$u(t) = -R^{-1}B^T[P_{ss}Z(t)e^{-\bar{A}^Tt} + e^{-\bar{A}^Tt}]e^{\bar{A}^Tt_o}Z^{-1}(t_o)x_o$$

where $Z(t)$ is defined by Eq. 11.22. The linear difference equation for $u(t)$ at the discrete times $t_k = t_o + k\Delta t$ $(k = 1,\ldots,N)$ for $\Delta t = (t_f - t_o)/N$ can be shown to be:

$$u(t_k) = E\mathbf{a}_k + F\mathbf{b}_k \quad , \quad k = 0,\ldots,N$$

where

$$E = -R^{-1}B^T[P_{ss}Z_{ss} + I]$$

$$F = -R^{-1}B^TP_{ss}$$

and $\mathbf{a}_k$ and $\mathbf{b}_k$ are defined by Eq. 11.34.

11.3.3 Residual State Trajectory Equation

An issue of great practical concern in the design of control algorithms for flexible spacecraft is the extent to which the applied control excites the response of previously truncated (i.e. high frequency plant dynamics) degrees of freedom. In order to assess the magnitude of the so-called control spillover effects, the applied control is used to force the response of a selected number of degrees of freedom which were not included in the plant model used for the control system design. However, as in Section 11.3.1, it is assumed that we have perfect knowledge of the plant and that we experience no state estimation errors. To this end we assume that the time-invariant residual state trajectory equation is given by (ref. 18)

$$\dot{x}_r(t) = A_rx_r(t) + B_r\mathbf{u}(t) \quad , \quad x_{ro} = x_r(t_o) \text{ given} \tag{11.35}$$

where $\mathbf{u}(t)$ is defined by Eq. 11.24, A_r is the residual plant dynamics equation, B_r is the residual control influence matrix, and $x_r(t)$ is the n_r-dimensional residual state vector. Introducing Eq. 11.24 into Eq. 11.35, leads to

$$\dot{x}_r(t) = A_rx_r(t) - B_rR^{-1}B^T[P_{ss} + Z^{-1}(t)]x(t) \tag{11.36}$$

or

$$\dot{x}_r(t) = A_rx_r(t) - B_rR^{-1}B^T[P_{ss} + Z^{-1}(t)]\Phi(t,t_o)x_o \tag{11.37}$$

where Eq. 11.32 has been substituted into Eq. 11.36. The solution for $x_r(t)$ can be shown to be

$$\mathbf{x}_r(t) = e^{A_r(t-t_o)}\mathbf{x}_{ro} - I_1(t,t_o) - I_2(t,t_o) \tag{11.38}$$

where

$$I_1(t,t_o) = \int_{t_o}^{t} e^{A_r(t-v)} B_r R^{-1} B^T P_{ss} Z(v) e^{-\bar{A}^T(v-t_o)} Z^{-1}(t_o)\mathbf{x}_o dv \tag{11.39}$$

$$I_2(t,t_o) = \int_{t_o}^{t} e^{A_r(t-v)} B_r R^{-1} B^T e^{-\bar{A}^T(v-t_o)} Z^{-1}(t_o)\mathbf{x}_o dv \tag{11.40}$$

Introducing $Z(t)$ from Eq. 11.22 into Eq. 11.39, leads to

$$I_1(t,t) = \mathbf{J}_1(t,t_o) + \mathbf{J}_2(t,t_o) \tag{11.41}$$

where

$$\mathbf{J}_1(t,t_o) = \Psi(-A_r,C_1,-\bar{A}^T,t,t_o)\mathbf{b}_1 \tag{11.42}$$

$$\mathbf{J}_2(t,t_o) = \Psi(-A_r,C_2,\bar{A},t,t_o)\mathbf{b}_2 \tag{11.43}$$

and

$$\Psi(A_1,A_2,A_3,t,t_o) = \int_{t_o}^{t} e^{A_1 v} A_2 e^{A_3 v} dv$$

$$C_1 = B_r R^{-1} B^T P_{ss} Z_{ss} \quad , \quad C_2 = B_r R^{-1} B^T$$

$$\mathbf{b}_1 = e^{\bar{A}^T t_o} Z^{-1}(t_o)\mathbf{x}_o \quad , \quad \mathbf{b}_2 = e^{-\bar{A}t_f}[Z(t_f) - Z_{ss}]e^{\bar{A}^T(t_o-t_f)} Z^{-1}(t_o)\mathbf{x}_o$$

A_1 is $n_1 \times n_1$, A_2 is $n_1 \times n_2$, A_3 is $n_2 \times n_2$, and $\Psi(\cdot)$ is $n_1 \times n_2$. The matrix exponential solution for $\Psi(\cdot)$ and its recursive formulas are presented in Appendix B. Subject to the constants defined in Eqs. 11.42 and 11.43, Eq. 11.40 can be written as

$$I_2(t,t_o) = e^{A_r t}\Psi(-A_r,C_2,-\bar{A}^T,t,t_o)\mathbf{b}_1 \tag{11.44}$$

Observing the argument list of Eqs. 11.42 and 11.44, the solution for $\mathbf{x}_r(t)$ in Eq. 11.38 can be simplified as follows:

$$\mathbf{x}_r(t) = e^{A_r(t-t_o)}\mathbf{x}_{ro} - e^{A_r t}\{\Psi(-A_r,C_1 + C_2,-\bar{A}^T,t,t_o)\mathbf{b}_1 - \Psi(-A_r,C_2,\bar{A},t,t_o)\mathbf{b}_2\} \tag{11.45}$$

It should be recognized, however, that the calculations above must be modified if the derivative penalty method of Chapters 6 and 10 is used for frequency-shaping of the control.

11.3.4 State Trajectory Sensitivity Calculations

In order to optimize the closed-loop system performance, the weighting matrix parameters for S, Q, and R in Eq. 11.1 must usually be adjusted. One way to approach this is by defining an alternative measure of the system performance which is to be minimized. For example, if we assume that the terminal values of the state are to be small, we can seek to minimize the norm of the terminal state vector

$$\sigma = \mathbf{x}_f^T W \mathbf{x}_f \tag{11.46}$$

where $\mathbf{x}_f$ is defined by Eq. 11.32 and $W = W^T > 0$ is a weight matrix. Let γ be one of the free parameters. Taking the partial derivative of Eq. 11.46 with respect to γ we find

$$\frac{\partial \sigma}{\partial \gamma} = 2\left(\frac{\partial \mathbf{x}_f}{\partial \gamma}\right)^T W \mathbf{x}_f \tag{11.47}$$

from which the parameter corrections are obtained from a gradient search algorithm as (e.g., the steepest descent method, conjugate gradient methods, etc.):

$$\Delta \mathbf{p} = -\alpha \frac{\partial \sigma}{\partial \mathbf{p}} / \left\| \frac{\partial \sigma}{\partial \mathbf{p}} \right\| \tag{11.48}$$

where $\mathbf{p}^T = [\gamma, \beta, \sigma, \ldots]$ is the vector of free variables for the optimization process, α is a scalar step size constant, and $\|\cdot\|$ denotes the Euclidian norm. More general numerical optimization criteria are considered in ref. 8.

In order to efficiently evaluate $\partial \mathbf{x}_f / \partial \gamma$ in Eq. 11.47, we compute:

$$\frac{\partial \mathbf{x}_f}{\partial \gamma} = \frac{\partial}{\partial \gamma} [\Phi(t_f, t_o)] \mathbf{x}_o \tag{11.49}$$

where the analytic solutions for the required partial derivatives follow as:

State Transition Matrix Partials

$$\frac{\partial}{\partial\gamma}[\Phi(t_f,t_o)] = \frac{\partial Z(t_f)}{\partial\gamma} e^{-\bar{A}^T(t_f-t_o)} Z^{-1}(t_o) + Z(t_f)\frac{\partial e^{-\bar{A}^T(t_f-t_o)}}{\partial\gamma} Z^{-1}(t_o)$$

$$+ Z(t_f)e^{-\bar{A}^T(t_f-t_o)}\frac{\partial Z^{-1}(t_o)}{\partial\gamma} \tag{11.50}$$

Terminal State Lyapunov Partials

$$\frac{\partial Z(t_f)}{\partial\gamma} = \frac{\partial}{\partial\gamma}(F^TSF - P_{ss})^{-1}$$

$$= -(F^TSF - P_{ss})^{-1}[F^T \frac{\partial S}{\partial\gamma} F - \frac{\partial P_{ss}}{\partial\gamma}](F^TSF - P_{ss})^{-1} \tag{11.51}$$

where $\frac{\partial S}{\partial\gamma}$ is problem dependent and $\frac{\partial P_{ss}}{\partial\gamma}$ is defined by Eq. (11.54).

Matrix Exponential Partials

$$\frac{\partial e^{-\bar{A}^T(t_f-t_o)}}{\partial\gamma} = e^{-\bar{A}^T(t_f-t_o)}\,\Psi(\bar{A}^T,-\frac{\partial\bar{A}^T}{\partial\gamma},-\bar{A}^T,t_f-t_o,0) \tag{11.52}$$

where $\Psi(\cdot)$ is defined following Eq. 11.43, the derivation for the matrix exponential derivative is given in Appendix C, and $\frac{\partial\bar{A}^T}{\partial\gamma}$ is defined by Eq. 11.55.

Inverse Initial State Lyapunov Partials

$$\frac{\partial Z^{-1}(t_o)}{\partial\gamma} = -Z^{-1}(t_o)\frac{\partial Z(t_o)}{\partial\gamma}Z^{-1}(t_o)$$

$$= -Z^{-1}(t_o)\{\frac{\partial Z_{ss}}{\partial\gamma} + \frac{\partial e^{-\bar{A}(t_f-t_o)}}{\partial\gamma}[Z(t_f) - Z_{ss}]e^{-\bar{A}^T(t_f-t_o)}$$

$$+ e^{-\bar{A}(t_f-t_o)}[\frac{\partial Z(t_f)}{\partial\gamma} - \frac{\partial Z_{ss}}{\partial\gamma}]e^{-\bar{A}^T(t_f-t_o)}$$

$$+ e^{-\bar{A}(t_f-t_o)}[Z(t_f) - Z_{ss}]\frac{\partial e^{-\bar{A}^T(t_o-t_f)}}{\partial\gamma}\}Z^{-1}(t_o) \tag{11.53}$$

where

$$Z(t_o) = Z_{ss} + e^{-\bar{A}(t_f-t_o)}[Z(t_f) - Z_{ss}]e^{-\bar{A}^T(t_f-t_o)}$$

$\frac{\partial Z_{ss}}{\partial \gamma}$ is defined by Eq. 11.56, $\frac{\partial e^{-\bar{A}(t_f-t_o)}}{\partial \gamma}$ and $\frac{\partial e^{-\bar{A}^T(t_f-t_o)}}{\partial \gamma}$ are defined by Eq. 11.52, and $\frac{\partial Z(t_f)}{\partial \gamma}$ is defined by Eq. 11.51.

Steady-State Riccati Matrix Partials

$$\bar{A}^T \frac{\partial P_{ss}}{\partial \gamma} + \frac{\partial P_{ss}}{\partial \gamma} \bar{A} = - \frac{\partial A^T}{\partial \gamma} P_{ss} - P_{ss} \frac{\partial A}{\partial \gamma} - F^T \frac{\partial Q}{\partial \gamma} F + P_{ss} \Xi P_{ss} \quad (11.54)$$

where Ξ is defined following Eq. 11.56.

Steady-State System Stability Matrix Partials

$$\frac{\partial \bar{A}}{\partial \gamma} = \frac{\partial A}{\partial \gamma} - \Xi P_{ss} - BR^{-1}B^T \frac{\partial P_{ss}}{\partial \gamma} \quad (11.55)$$

where Ξ is defined following Eq. 11.56.

Steady-State Lyapunov Matrix Partials

$$\bar{A} \frac{\partial Z_{ss}}{\partial \gamma} + \frac{\partial Z_{ss}}{\partial \gamma} \bar{A}^T = \Xi - \frac{\partial \bar{A}}{\partial \gamma} Z_{ss} - Z_{ss} \frac{\partial \bar{A}^T}{\partial \gamma} \quad (11.56)$$

where in Eqs. 11.54-.56 the term Ξ is given by

$$\Xi = \frac{\partial}{\partial \gamma} [BR^{-1}B^T] = \frac{\partial B}{\partial \gamma} R^{-1}B^T - BR^{-1} \frac{\partial R}{\partial \gamma} R^{-1}B^T + BR^{-1} \frac{\partial B^T}{\partial \gamma}$$

The remaining partial derivatives for $\frac{\partial A}{\partial \gamma}$, $\frac{\partial B}{\partial \gamma}$, $\frac{\partial S}{\partial \gamma}$, $\frac{\partial Q}{\partial \gamma}$, and $\frac{\partial R}{\partial \gamma}$ are problem dependent. Furthermore, the partials for $\frac{\partial A}{\partial \gamma}$ only exist if the performance index of Eq. 11.1 and the state equation of Eq. 11.2 has been transformed to eliminate a cross-coupling integral penalty term of the form (ref. 8):

$$2y^T(t)Nu(t) \quad (11.57)$$

where A and Q are then given by

$$A = \tilde{A} - BR^{-1}N^T \quad (11.58)$$

$$Q = \tilde{Q} - NR^{-1}N^T \tag{11.59}$$

and $\tilde{A}$ and $\tilde{Q}$ are the original plant dynamics and state penalty matrices. For the special case of $N \neq 0$ the required partials for A and Q follow as:

A and Q Partials When $N \neq 0$

$$\frac{\partial A}{\partial \gamma} = -\frac{\partial B}{\partial \gamma} R^{-1}N^T + BR^{-1}\frac{\partial R}{\partial \gamma} R^{-1}N^T - BR^{-1}\frac{\partial N^T}{\partial \gamma} \tag{11.60}$$

$$\frac{\partial Q}{\partial \gamma} = \frac{\partial \tilde{Q}}{\partial \gamma} - \frac{\partial N}{\partial \gamma} R^{-1}N^T + NR^{-1}\frac{\partial R}{\partial \gamma} R^{-1}N^T - NR^{-1}\frac{\partial N^T}{\partial \gamma} \tag{11.61}$$

Of course a more general criterion than that presented in Eq. 11.46 can be imposed in the optimization process; indeed either time or frequency domain performance objectives can be minimized. Nevertheless, the basic partial derivative structure presented in Eqs. 11.49-61 will remain valid if the state trajectory equation is used to define the system performance. In addition it should be recognized in Eqs. 11.54 and 11.56 that transform methods exist which exploit the presence of the $\bar{A}$ matrix on the right and left hand side of the algebraic Lyapunov equations of Eqs. 11.54 and 11.56; thus greatly reducing the computational costs when there are many independent variables for γ in the optimization parameter vector (Ref. 8). In an analogous way the costs associated with computing the matrix exponential partials of Eq. 11.52 can be greatly reduced (an example is presented in Appendix C, Section C.2.2).

11.4 TERMINAL CONTROL PROBLEM: NECESSARY CONDITIONS AND SOLUTIONS

In order to obtain the necessary conditions for the terminal control problem defined by Eqs. 11.4-.7, we first append the terminal constraints of Eq. 11.7 to the performance index, leading to

$$J = \nu^T\psi(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \{\|\mathbf{y}(t)\|^2_{Q'} + \|\mathbf{u}(t)\|^2_R\}dt \tag{11.62}$$

where ν is a q-dimensional vector of constant Lagrange multipliers, the terminal constraint vector $\psi(t_f)$ is given by

$$\psi(t_f) = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_q]^T = MFx_D$$

M is a q x n selection operator, and

$$(x_D)_i = \begin{cases} \bar{x}_i & , \text{ for } i \le q \\ 0 & , \text{ for } i > q \end{cases} \quad (i = 1, \ldots, n)$$

or more generally,

$$(x_D)_i = \begin{cases} \bar{x}_i & , \text{ if the ith x-value is constrained} \\ 0 & , \text{ if the ith x-value is not constrained} \end{cases} \quad (i=1,\ldots,n)$$

where $(\cdot)_i$ denotes the ith element of the vector quantity $(\cdot)$.

The necessary conditions for performance index of Eq. 11.62 are obtained by defining the following Hamiltonian for Eqs. 11.5-.7 (see Chapter 6)

$$H(\mathbf{x}(t),\mathbf{u}(t),\lambda(t),t) = \frac{1}{2}\|\mathbf{y}(t)\|^2_{Q'} + \frac{1}{2}\|\mathbf{u}(t)\|^2_R + \lambda(t)^T[A\mathbf{x}(t) + B\mathbf{u}(t)] \tag{11.63}$$

and the terminal penalty function is

$$\Theta(\mathbf{x}(t_f),t_f) = \nu^T\psi(t_f) \tag{11.64}$$

where $\lambda(t)$ is the Lagrange multiplier for the state.

Application of Pontryagin's principle yields the necessary conditions

$$\partial H/\partial\lambda(t) = \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{11.65}$$

$$\partial H/\partial\mathbf{x}(t) = -\dot{\lambda}(t) = F^TQF\mathbf{x}(t) + A^T\lambda(t) \tag{11.66}$$

$$\partial H/\partial\mathbf{u}(t) = \mathbf{0} \rightarrow \mathbf{u}(t) = -R^{-1}B^T\lambda(t) \tag{11.67}$$

$$\partial\Theta/\partial\mathbf{x}(t_f) = \lambda(t_f) = \frac{\partial\psi^T}{\partial\mathbf{x}(t_f)}\,\nu \tag{11.68}$$

Introducing Eq. 11.67 into Eq. 11.65 yields the modified state equation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) - BR^{-1}B^T\lambda(t) \tag{11.69}$$

The closed-loop control is determined by assuming that

$$\lambda(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{S}(t)\nu(t) \tag{11.70}$$

$$\psi(t_f) = \mathbf{S}^T(t)\mathbf{x}(t) + G(t)\nu \tag{11.71}$$

where $P(t)$ is $n \times n$, $S(t)$ is $n \times q$, and $G(t)$ is $q \times q$ are sweep variables (refs. 9 and 10) which must be determined.

Substituting Eq. 11.70 into Eqs. 11.66 and 11.69 yields the following necessary conditions

$$\dot{P}(t) + P(t)A + A^TP(t) - P(t)BR^{-1}B^TP(t) + F^TQF = 0 \quad ; \quad P(t_f) = 0 \tag{11.72}$$

$$\dot{S}(t) + [A^T - P(t)BR^{-1}B^T]S(t) = 0 \quad ; \quad S(t_f) = F^TM^T \tag{11.73}$$

for an arbitrary state vector $x(t)$ and constant Lagrange multiplier ν.

To find the necessary conditions for $G(t)$, we differentiate Eq. 11.71 with respect to time, while treating ψ and ν as constants, leading to

$$0 = \dot{S}^T(t)x(t) + S^T(t)\dot{x}(t) + \dot{G}(t)\nu$$

Substituting $\dot{x}(t)$ from Eq. 11.69, and $\lambda(t)$ from Eq. 11.70, into the equation above, we find:

$$\dot{G}(t) - S^T(t)BR^{-1}B^TS(t) = 0 \quad ; \quad G(t_f) = 0 \tag{11.74}$$

The solution for ν is obtained from Eq. 11.71 at the initial time as

$$\nu = G^{-1}(t_0)[\psi - S^T(t_0)x(t_0)] \tag{11.75}$$

Assuming that Eq. 11.75 is true for all values of time in order to obtain a continuous time feedback law, the optimal control is obtained by introducing Eq. 11.75 into Eq. 11.70 and the result into Eq. 11.67, yielding

$$u(t) = -R^{-1}B^T[\{P(t) - S(t)G^{-1}(t)S^T(t)\}x(t) + S(t)G^{-1}(t)\psi] \tag{11.76}$$

This feedback law shows explicit dependence on the terminal constraints, $\psi(t_f) = MFx_0$, of Eq. 11.62. It is of interest to note that if $\psi(t_f) = 0$, then Eq. 11.76 reduces to the regulator case of Eq. 11.20.

11.4.1 Exponential Solutions for the Riccati-Like Matrix Equations

As shown in Appendix A, the solution for Eq. 11.72 is given by

$$P(t) = P_{ss} + Z^{-1}(t) \tag{11.77}$$

where P_{ss} is the solution for the algebraic Riccati equation of Eq. A.112 and $Z(t)$ satisfies the differential Lyapunov equation of Eq. A.116. The solution for $Z(t)$ can be shown to be (refs. 1,2,3)

$$Z(t) = Z_{ss} + e^{\bar{A}(t-t_f)}[Z(t_f) - Z_{ss}]e^{\bar{A}^T(t-t_f)} \tag{11.78}$$

where $\bar{A} = A - BR^{-1}B^TP_{ss}$ is the system stability matrix, $Z(t_f) = -P_{ss}^{-1}$, and Z_{ss} satisfies the algebraic Lyapunov equation (refs. 3,4-6)

$$\bar{A}Z_{ss} + Z_{ss}\bar{A}^T - BR^{-1}B^T = 0 \tag{11.79}$$

The solution for the rectangular matrix $S(t)$ in Eq. 11.73 follows on assuming the product form solution (refs. 11 and 18):

$$S(t) = Z^{-1}(t)V(t) \tag{11.80}$$

where $Z(t)$ is defined by Eq. 11.78 and $V(t)$ is a matrix function which is to be determined. Substituting Eqs. 11.77 and 11.80 into Eq. 11.73, while recalling Eq. A.116 and the following definition for the time derivative of the inverse of $Z(t)$:

$$\frac{d}{dt}[Z^{-1}(t)] = -Z^{-1}(t)\dot{Z}(t)Z^{-1}(t)$$

leads to the following linear time-invariant matrix differential equation for $V(t)$:

$$\dot{V}(t) - \bar{A}V(t) = 0 \quad ; \quad V(t_f) = Z(t_f)S(t_f) \tag{11.81}$$

The solution for $V(t)$ follows as

$$V(t) = e^{\bar{A}(t-t_f)}Z(t_f)S(t_f) \tag{11.82}$$

and from Eq. (11.80) the solution for $S(t)$ is given by

$$S(t) = \tau(t,t_f)S(t_f) \tag{11.83}$$

where $\tau(t,t_f) = Z^{-1}(t)e^{\bar{A}(t-t_f)}Z(t_f)$ is the system state transition matrix, and $\tau(t,t_f)$ satisfies the following nonantonomous matrix differential equation:

$$\dot{\tau}(t,t_f) = -[A - BR^{-1}B^TP(t)]^T\tau(t,t_f) \quad ; \quad \tau(t_f,t_f) = I \tag{11.84}$$

where $P(t)$ is defined by Eq. 11.77.

It is of interest to note that $\tau(t,t_f)$ is the formal adjoint of $\Phi(t,t_o)$ in Eq. 11.32, where the following product is constant for all time

$$\tau^T(t,t_f)\Phi(t,t_o) = [Z(t_f)e^{\bar{A}^T(t-t_f)}Z^{-1}(t)][Z(t)e^{-\bar{A}^T(t-t_o)}Z^{-1}(t_o)]$$

$$= Z(t_f)e^{-\bar{A}^T(t_f-t_o)}Z^{-1}(t_o) \qquad (11.85)$$

Moreover, $\tau(t,t_f)$ satisfies the following semi-group and inversion properties:

$$\tau(t_2,t_f) = \tau(t_2,t_1)\tau(t_1,t_f) = [Z^{-1}(t_2)e^{\bar{A}(t_2-t_1)}Z(t_1)][Z^{-1}(t_1)e^{\bar{A}(t_1-t_f)}Z(t_f)]$$

$$= Z^{-1}(t_2)e^{\bar{A}(t_2-t_f)}Z(t_f)$$

and

$$\tau(t_f,t) = \tau^{-1}(t,t_f) = [Z^{-1}(t)e^{\bar{A}(t-t_f)}Z(t_f)]^{-1}$$

$$= Z^{-1}(t_f)e^{-\bar{A}(t-t_f)}[Z^{-1}(t)]^{-1} = Z^{-1}(t_f)e^{\bar{A}(t_f-t)}Z(t)$$

These results are analogous to the state transition matrix properties shown for $\Phi(t,t_o)$ following Eq. 11.32.

The solution for the matrix G(t) in Eq. 11.74 can be shown to be (ref. 11 and 18):

$$G(t) = S^T(t)Z(t)S(t) - S^T(t_f)Z(t_f)S(t_f) \qquad (11.86)$$

or

$$G(t) = S^T(t_f)[\tau^T(t,t_f)Z(t)\tau(t,t_f) - Z(t_f)]S(t_f)$$

which can be easily verified by straightforward differentiation.

The optimal control is then computed by introducing Eqs. 11.77, 11.83, and 11.86 into Eq. 11.76 for each value of time required in the control application.

11.4.2 Change of Variables for the Riccati-Like Matrix Equations

Alternatively, as shown by Juang, Turner, and Chun (ref. 12), the time-varying matrix quantities in Eq. 11.76 can be considerably simplified by defining the modified matrix variables

$$Z_m^{-1}(t) = Z^{-1}(t) - S(t)G^{-1}(t)S^T(t) \tag{11.87}$$

and

$$S_m(t) = S(t)G^{-1}(t) \tag{11.88}$$

where the differential equations for $Z_m(t)$ and $S_m(t)$ can be shown to be (ref. 12)

$$\dot{Z}_m(t) - \bar{A}Z_m(t) - Z_m(t)\bar{A}^T + BR^{-1}B^T = 0; \; Z_m(t_o) = [Z^{-1}(t_o) - S(t_o)G^{-1}(t_o)S^T(t_o)]^{-1} \tag{11.89}$$

and

$$\dot{S}_m(t) + [\bar{A}^T - Z_m^{-1}(t)BR^{-1}B^T]S_m(t) = 0 \quad ; \quad S_m(t_o) = S(t_o)G^{-1}(t_o) \tag{11.90}$$

It is remarkable that Eq. 11.89 is identical to Eq. A.116 and Eq. 11.90 is identical to Eq. 11.73 after P(t) is introduced from Eq. 11.77; thus greatly simplifying the solution process.

In order to more fully investigate the relationship between $Z_m(t)$ and $Z(t)$ we now apply the matrix inversion lemma (ref. 13)

$$[P^{-1} + H^TQH]^{-1} = P - PH^T[HPH^T + Q^{-1}]^{-1}HP \tag{11.91}$$

to the following inverse of Eq. 11.87:

$$Z_m(t) = [Z^{-1}(t) - S(t)G^{-1}(t)S^T(t)]^{-1} \tag{11.92}$$

Comparing Eqs. 11.91 and 11.92 we make the identifications

$$P = Z(t) \quad , \quad H = S^T(t) \quad , \quad Q = -G^{-1}(t)$$

leading to

$$Z_m(t) = Z(t) - Z(t)S(t)[S^T(t)Z(t)S(t) - G(t)]^{-1}S^T(t)Z(t) \tag{11.93}$$

Recalling the solutions for S(t) and G(t) given by Eqs. 11.80 and 11.86 respectively, it follows that Eq. 11.93 can be written as

$$Z_m(t) = Z(t) - V(t)[S^T(t_f)Z(t_f)S(t_f)]V^T(t)$$

or

$$Z_m(t) = Z(t) + V(t)WV^T(t) \tag{11.94}$$

where $W = MFP_{ss}^{-1}F^TM^T$, M is defined following Eq. 11.62 and $Z_m(t_f)$ is singular because $G^{-1}(t_f)$ is singular. On comparing Eqs. 11.87 and 11.94, it is quite striking to observe that no inverses of time-varying matrices appear in Eq. 11.94. It should be recalled, however, that the simplifying application of the matrix inversion lemma is only possible because of the availability of the closed-form solutions for S(t) and G(t).

To further simplify the expression for $Z_m(t)$ we introduce the solutions for Z(t) in Eq. 11.78 and V(t) in Eq. 11.82, into Eq. 11.94, leading to:

$$Z_m(t) = Z_{ss} + e^{\bar{A}(t-t_f)}[Z_m(t_f) - Z_{ss}]e^{\bar{A}^T(t-t_f)} \tag{11.95}$$

where $Z_m(t_f) = Z(t_f) + Z(t_f)S(t_f)WS^T(t_f)Z(t_f)$. Comparing Eqs. 11.78 and 11.95 it is clear that Z(t) and $Z_m(t)$ differ only in the value of the terminal boundary condition.

The solution for $S_m(t)$ in Eq. 11.88 can be written as (i.e., by exploiting the known solution for S(t) in Eq. 11.73)

$$S_m(t) = \tau_m(t,t_o)S_m(t_o) \tag{11.96}$$

where $\tau_m(t,t_o) = Z_m^{-1}(t)e^{\bar{A}(t-t_o)}Z_m(t_o)$ is the system state transition matrix, $S_m(t_o) = S(t_o)G^{-1}(t_o)$, and $\tau_m(t,t_o)$ satisfies the following nonautonomous matrix differential equation:

$$\dot{\tau}_m(t,t_o) = -[\bar{A} - BR^{-1}B^TZ_m^{-1}(t)]^T\tau_m(t,t_o) \quad ; \quad \tau_m(t_o,t_o) = I$$

The solution for $S_m(t)$ is referenced to the initial time, t_o, since $G^{-1}(t_f)$ is singular.

Substituting Eqs. 11.87 and 11.88 into the control of Eq. 11.76, yields

$$u(t) = -R^{-1}B^T[\{P_{ss} + Z_m^{-1}(t)\}x(t) + S_m(t)\nu] \tag{11.97}$$

where the solutions for $Z_m(t)$ and $S_m(t)$ are given by Eqs. 11.95 and 11.96, respectively. Equation 11.97 is the desired final form for the optimal control, and in the next section the solution for u(t) is used for solving the state trajectory differential equation in closed-form.

11.4.3 State Trajectory Equation

In order to determine the closed-loop system response under the assumptions of perfect plant knowledge and perfect state estimation, we introduce Eq. 11.97 into Eq. 11.13, leading to

$$\dot{x}(t) = [\bar{A} - BR^{-1}B^TZ_m^{-1}(t)]x(t) - BR^{-1}B^TS_m(t)\psi \quad ; \quad \text{given } x(t_o) \qquad (11.98)$$

The solution for Eq. 11.98 is obtained by a two-stage process. First, we recognize that the homogeneous part of Eq. 11.98 is identical to Eq. 11.25 with $Z(t)$ replaced by $Z_m(t)$; accordingly, the homogeneous solution for $x_h(t)$ in Eq. 11.98 is obtained from Eq. 11.32 by replacing $Z(t)$ with $Z_m(t)$. Second, the particular solution is obtained from a *variation of parameters* approach.

The solution for the homogeneous part of Eq. 11.98 can be shown to be (see Section 11.3.1)

$$x_h(t) = \Phi_m(t,t_o)x(t_o) \qquad (11.99)$$

where $\Phi_m(t,t_o) = Z_m(t)e^{-\bar{A}^T(t-t_o)}Z_m^{-1}(t_o)$ is the system state transition matrix, and $\Phi_m(t,t_o)$ satisfies the nonautonomous matrix differential equation:

$$\dot{\Phi}_m(t,t_o) = [\bar{A} - BR^{-1}B^TZ_m^{-1}(t)]\Phi_m(t,t_o) \quad ; \quad \Phi_m(t_o,t_o) = I \qquad (11.100)$$

The *variation of parameters* solution for Eq. 11.98 is assumed to be (refs. 11 and 18)

$$x(t) = \Phi_m(t,t_o)g(t) \quad ; \quad g(t_o) = x(t_o) \qquad (11.101)$$

where $g(t)$ is to be determined. Introducing Eq. 11.101 into 11.98 produces the following differential equation for $g(t)$:

$$\dot{g}(t) = -\Phi_m^{-1}(t,t_o)BR^{-1}B^TS_m(t)\psi \qquad (11.102)$$

or

$$\dot{g}(t) = -\Phi_m^{-1}(t,t_o)BR^{-1}B^T\tau_m(t,t_o)S_m(t_o)\psi \qquad (11.103)$$

where the solution for $S_m(t)$ in Eq. 11.96 has been introduced in Eq. 11.102.

The solution for $g(t)$ can be shown to be

$$g(t) = -\Phi_m^{-1}(t,t_o)Z_m(t)\tau_m(t,t_o)S_m(t_o)\psi + x(t_o) + Z_m(t_o)S_m(t_o)\psi \quad (11.104)$$

which can be verified by straightforward differentiation, where the differential equation for $\Phi_m^{-1}(t,t_o)$ follows as (refs. 11,12, and 18):

$$d[\Phi_m^{-1}(t,t_o)]/dt = -\Phi_m^{-1}(t,t_o)[\bar{A} - BR^{-1}B^T Z_m^{-1}(t)] \quad ; \quad \Phi_m^{-1}(t_o,t_o) = I$$

and the differential equations for $Z_m(t)$ and $\tau_m(t,t_o)$ are given by Eqs. 11.89 and 11.96, respectively. Substituting Eq. 11.104 into Eq. 11.101, yields:

$$x(t) = \Phi_m(t,t_o)\{x(t_o) + Z_m(t_o)S_m(t_o)\psi\} - Z_m(t)\tau_m(t,t_o)S_m(t_o)\psi$$

which simplifies to

$$x(t) = \Phi_m(t,t_o)\{x(t_o) + Z_m(t_o)S_m(t_o)\psi\} - e^{\bar{A}(t-t_o)} Z_m(t_o)S_m(t_o)\psi \quad (11.105)$$

where the solution for $\tau_m(t,t_o)$ has been introduced from Eq. 11.96. The solution above shows the explicit presence of the terminal constraint vector ψ.

11.4.4 Recursion Relationship for Evaluating the State and Control at Discrete Time Steps

If the solution for $x(t)$ is required at the discrete times $t_k = t_o + k\Delta t$ $(k = 1,\ldots,N)$ for $\Delta t = (t_f - t_o)/N$, then Eq. 11.105 can be written as (refs. 18 and 19)

$$\begin{aligned} x(t) &= \{Z_{ss} + e^{\bar{A}(t-t_f)}[Z_m(t_f) - Z_{ss}]e^{\bar{A}^T(t-t_f)}\}e^{-\bar{A}^T(t-t_o)} Z_m^{-1}(t_o)\{x_o \\ &\quad + Z_m(t_o)S_m(t_o)\psi\} - e^{\bar{A}(t-t_o)} Z_m(t_o)S_m(t_o)\psi \\ &= Z_{ss}e^{\bar{A}^T(t-t_o)} Z_m^{-1}(t_o)\{x_o + Z_m(t_o)S_m(t_o)\psi\} \\ &\quad + e^{\bar{A}t}\{e^{-\bar{A}t_f}[Z_m(t_f) - Z_{ss}]e^{\bar{A}^T(t_o-t_f)} Z_m^{-1}(t_o)\{x_o \\ &\quad + Z_m(t_o)S_m(t_o)\psi\} - e^{-\bar{A}t_o} Z_m(t_o)S_m(t_o)\psi\} \end{aligned} \quad (11.106)$$

leading to the linear difference equation

$$x(t_k) = Z_{ss}\mathbf{a}_k + \mathbf{b}_k \quad (k = 1,\ldots,N) \quad (11.107)$$

where

$$\mathbf{a}_o = Z_m^{-1}(t_o)\{x_o + Z_m(t_o)S_m(t_o)\psi\}$$

$$\mathbf{b}_o = e^{\bar{A}(t_o-t_f)}[Z_m(t_f) - Z_{ss}]e^{\bar{A}^T(t_o-t_f)} Z_m^{-1}(t_o)\{x_o$$

$$+ Z_m(t_o)S_m(t_o)\psi\} - Z_m(t_o)S_m(t_o)\psi$$

$$\mathbf{a}_k = e^{-\bar{A}^T\Delta t}\mathbf{a}_{k-1}$$

$$\mathbf{b}_k = e^{\bar{A}\Delta t}\mathbf{b}_{k-1}$$

Comparing Eqs. 11.34 and 11.107 it is clear that the structure of the linear difference equations are identical, and only the initial conditions change.

To compute the control, $\mathbf{u}(t)$, at the discrete times we observe that Eq. 11.97 can be written as

$$\mathbf{u}(t) = -R^{-1}B^T[\{P_{ss} + Z_m^{-1}(t)\}\mathbf{x}(t) + S_m(t)\psi]$$

or

$$= -R^{-1}B^T[\{P_{ss} + Z_m^{-1}(t)\}\{\Phi_m(t,t_o)\{x(t_o) + Z_m(t_o)S_m(t_o)\psi\}$$
$$- e^{\bar{A}(t-t_o)}Z_m(t_o)S_m(t_o)\psi\} + S_m(t)\psi]$$

where the closed-form solution for $\mathbf{x}(t)$ has been introduced from Eq. 11.105. Simplifying the equation above leads to

$$\mathbf{u}(t) = -R^{-1}B^T[P_{ss}Z_m(t)e^{-\bar{A}^T t} + e^{-\bar{A}^T t}]e^{\bar{A}^T t_o}Z_m^{-1}(t_o)\{x_o + Z_m(t_o)S_m(t_o)\psi\}$$
$$+ R^{-1}B^T[P_{ss} + Z_m^{-1}(t)]e^{\bar{A}(t-t_o)}Z_m(t_o)S_m(t_o)\psi$$
$$- R^{-1}B^TS_m(t)\psi$$

where $\Phi_m(t,t_o)$ has been introduced from Eq. 11.99. Recalling the solution for $S_m(t)$, given by Eq. 11.96, it follows that the equation above reduces to

$$\mathbf{u}(t) = -R^{-1}B^T[P_{ss}Z_m(t)e^{-\bar{A}^T t} + e^{-\bar{A}^T t}]e^{-\bar{A}^T t_o}Z_m^{-1}(t_o)\{x_o + Z_m(t_o)S_m(t_o)\psi\}$$
$$+ R^{-1}B^TP_{ss}e^{\bar{A}(t-t_o)}Z_m(t_o)S_m(t_o)\psi$$

where $Z_m(t)$ is defined by Eq. 11.95. After some simple algebra, the linear difference equation for $\mathbf{u}(t)$ at the discrete times $t_k = t_o + k\Delta t$ $(k = 0,\ldots,N)$

for $\Delta t = (t_f - t_o)/N$ can be shown to be (refs. 18 and 19):

$$\mathbf{u}(t_k) = E\mathbf{a}_k + F\mathbf{b}_k \quad , \quad k = 1,\ldots,N$$

where

$$E = -R^{-1}B^T[P_{ss}Z_{ss} + I]$$

$$F = -R^{-1}B^TP_{ss}$$

and $\mathbf{a}_k$ and $\mathbf{b}_k$ are defined by Eq. 11.107.

11.4.5 State Trajectory Sensitivity Calculations

In order to compute the sensitivity of the terminal state, we set $t = t_f$ in Eq. 11.105 and let γ be one of the free parameters. The results of this section are analogous to those presented in Section 11.3.4 for the linear regulator. From Eq. 11.105 the terminal state sensitivity follows as:

$$\begin{aligned}\frac{\partial \mathbf{x}(t_f)}{\partial \gamma} = {} & \frac{\partial}{\partial \gamma}[\Phi_m(t_f,t_o)]\{\mathbf{x}_o + Z_m(t_o)S_m(t_o)\psi\} + \Phi_m(t_f,t_o)\frac{\partial Z_m(t_o)}{\partial \gamma}S_m(t_o)\psi \\ & + \Phi_m(t_f,t_o)Z_m(t_o)\frac{\partial S_m(t_o)}{\partial \gamma}\psi - \frac{\partial}{\partial \gamma}[e^{\bar{A}(t_f-t_o)}]Z_m(t_o)S_m(t_o)\psi \\ & - e^{\bar{A}(t_f-t_o)}\frac{\partial Z_m(t_o)}{\partial \gamma}S_m(t_o)\psi - e^{\bar{A}(t_f-t_o)}Z_m(t_o)\frac{\partial S_m(t_o)}{\partial \gamma}\psi\end{aligned} \tag{11.108}$$

The partial derivatives for $\frac{\partial}{\partial \gamma}[\Phi_m(t_f,t_o)]$ are obtained from Eqs. 11.50 through 11.61 by replacing $Z(t)$ by $Z_m(t)$. The remaining partials are presented in order of solution.

<u>Matrix Exponential Partials</u>

$$\begin{aligned}\frac{\partial}{\partial \gamma}[e^{\bar{A}(t_f-t_o)}] &= e^{\bar{A}(t_f-t_o)}\int_0^{(t_f-t_o)} e^{-\bar{A}v}\frac{\partial \bar{A}}{\partial \gamma}e^{\bar{A}v}dv \\ &= e^{\bar{A}(t_f-t_o)}\Psi(-\bar{A},\frac{\partial \bar{A}}{\partial \gamma},\bar{A},t_f-t_o,0)\end{aligned} \tag{11.109}$$

where $\frac{\partial \bar{A}}{\partial \gamma}$ is defined by Eq. 11.55, $\Psi(\cdot)$ is defined by Eq. B.1, and the derivation for the matrix exponential partial derivative is given in Appendix 11.C.

Initial State Lyapunov Partials

$$\frac{\partial Z_m(t_o)}{\partial\gamma} = \frac{\partial}{\partial\gamma}\{Z_{ss} + e^{-\bar{A}(t_f-t_o)}[Z_m(t_f) - Z_{ss}]e^{-\bar{A}^T(t_f-t_o)}\}$$

$$= \frac{\partial Z_{ss}}{\partial\gamma} + \frac{\partial e^{-\bar{A}(t_f-t_o)}}{\partial\gamma}[Z_m(t_f) - Z_{ss}]e^{-\bar{A}^T(t_f-t_o)}$$

$$+ e^{-\bar{A}(t_f-t_o)}[\frac{\partial Z_m(t_f)}{\partial\gamma} - \frac{\partial Z_{ss}}{\partial\gamma}]e^{-\bar{A}^T(t_f-t_o)}$$

$$+ e^{-\bar{A}(t_f-t_o)}[Z_m(t_f) - Z_{ss}]\frac{\partial e^{-\bar{A}^T(t_f-t_o)}}{\partial\gamma} \qquad (11.110)$$

where $\frac{\partial Z_{ss}}{\partial\gamma}$ is defined by Eq. 11.56, $\frac{\partial e^{-\bar{A}^T(t_f-t_o)}}{\partial\gamma}$ is defined by Eq. 11.52, and $\frac{\partial e^{-\bar{A}(t_f-t_o)}}{\partial\gamma}$ is defined by the transpose of Eq. 11.52.

Terminal State Lyapunov Partials

$$\frac{\partial Z_m(t_f)}{\partial\gamma} = \frac{\partial}{\partial\gamma}\{Z(t_f) + Z(t_f)S(t_f)WS^T(t_f)Z(t_f)\} \qquad (11.111)$$

$$= \frac{\partial Z(t_f)}{\partial\gamma} + \frac{\partial Z(t_f)}{\partial\gamma}S(t_f)WS^T(t_f)Z(t_f)$$

$$+ Z(t_f)S(t_f)\frac{\partial W}{\partial\gamma}S^T(t_f)Z(t_f)$$

$$+ Z(t_f)S(t_f)WS^T(t_f)\frac{\partial Z(t_f)}{\partial\gamma} \qquad (11.112)$$

where $\frac{\partial Z(t_f)}{\partial\gamma} = P_{ss}^{-1}\frac{\partial P_{ss}}{\partial\gamma}P_{ss}^{-1}$, $\frac{\partial P_{ss}}{\partial\gamma}$ is defined by Eq. 11.54, and W is defined by Eq. 11.94.

W Matrix Partials

$$\frac{\partial W}{\partial \gamma} = -MFP_{ss}^{-1} \frac{\partial P_{ss}}{\partial \gamma} P_{ss}^{-1}F^TM^T \tag{11.113}$$

where $\frac{\partial P_{ss}}{\partial \gamma}$ is defined by Eq. 11.54.

Partials for $S_m(t_o)$

$$\frac{\partial}{\partial \gamma}[S_m(t_o)] = \frac{\partial}{\partial \gamma}[S(t_o)G^{-1}(t_o)] = \frac{\partial}{\partial \gamma}[S(t_o)]G^{-1}(t_o)$$
$$- S(t_o)G^{-1}(t_o) \frac{\partial}{\partial \gamma}[G(t_o)]G^{-1}(t_o)$$

where

$$\frac{\partial}{\partial \gamma}[S(t_o)] = \frac{\partial}{\partial \gamma}[\tau(t_o,t_f)]S(t_f) = -Z^{-1}(t_o) \frac{\partial Z(t_o)}{\partial \gamma} Z^{-1}(t_o)e^{\bar{A}(t_o-t_f)}Z(t_f)S(t_f)$$
$$+ Z^{-1}(t_o) \frac{\partial}{\partial \gamma} [e^{\bar{A}(t_o-t_f)}]Z(t_f)S(t_f)$$
$$+ Z^{-1}(t_o)e^{\bar{A}(t_o-t_f)} \frac{\partial Z(t_f)}{\partial \gamma} S(t_f)$$

and

$$\frac{\partial}{\partial \gamma}[G(t_o)] = \frac{\partial}{\partial \gamma}[S^T(t_o)Z(t_o)S(t_o) - S^T(t_f)Z(t_f)S(t_f)]$$
$$= \frac{\partial}{\partial \gamma}[S^T(t_o)]Z(t_o)S(t_o) + S^T(t_o) \frac{\partial Z(t_o)}{\partial \gamma} S(t_o)$$
$$+ S^T(t_o)Z(t_o) \frac{\partial}{\partial \gamma} [S(t_o)] - S^T(t_f) \frac{\partial Z(t_f)}{\partial \gamma} S(t_f)$$

where $\frac{\partial Z(t_f)}{\partial \gamma}$ is defined by Eq. 11.112 and $\frac{\partial Z(t_o)}{\partial \gamma}$ is obtained from Eq. 11.110 by replacing $Z_m(t)$ is with $Z(t)$.

11.5 DISTURBANCE-ACCOMMODATING TRACKING PROBLEM: NECESSARY CONDITIONS AND SOLUTIONS

In order to obtain the necessary conditions for the tracking/disturbance-accommodating control problem defined by Eqs. 11.8-.10, we require a state variable model for the deterministic disturbance which is to be suppressed. To this end we model the disturbance state as

$$d(t) = \Lambda e^{Mt}\beta \tag{11.114}$$

where Λ is the $n \times n_d$ disturbance state influence matrix, e^{Mt} is the $n_d \times n_d$ matrix exponential for the disturbance dynamics, and β is the n_d-dimensional initial condition vector for the disturbance. One general model for defining Eq. 11.114 consists of representing the disturbance dynamics in terms of Fouier series (Refs. 14,15), where the rows of Λ consist of the Fourier series coefficients for the individual disturbances, M is a constant block diagonal matrix:

$$M = \text{Block Diag}\left[0, \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}\right]$$

The dynamic reference state for the tracking prccess is assumed to be given by

$$y^*(t) = Fx^*(t) = FHe^{\Omega t}s_o \qquad (11.115)$$

where H is a $n \times n_t$ tracking state influence matrix, $e^{\Omega t}$ is the $n_t \times n_t$ matrix exponential for the tracking state dynamics, and s_o is the n_t-dimensional tracking state initial condition vector. For the special case that the reference output state is obtained from the solution of a disturbance free open-loop control problem, then $H = [I,0]$ is $n \times 2n$, Ω is the $2n \times 2n$ Hamiltonian matrix (see Chapters 9 and 10)

$$\Omega = \begin{bmatrix} A^* & -B^*(R^*)^{-1}(B^*)^T \\ -Q^* & -(A^*)^T \end{bmatrix}$$

where A^*, B^*, Q^*, and R^* denote reference state matrices for the state, control, state weight, and control weight respectively, and s_o is given by

$$s_o = [x^*(t_o), \lambda^*(t_o)]^T$$

where $\lambda^*(t)$ is the reference output open-loop co-state vector.

In order to obtain the necessary conditions for the disturbance-accommodating tracking problem defined by Eqs. 11.8-.10, the following

Hamiltonian defined (see Chapter 6)

$$\mathcal{H}(\mathbf{x}(t),\lambda(t),\mathbf{u}(t),t) = \frac{1}{2}\,\|\mathbf{y}^*(t) - \mathbf{y}(t)\|^2_{Q'} + \frac{1}{2}\,\|\mathbf{u}(t)\|^2_R$$

$$+ \lambda^T(t)[A\mathbf{x}(t) + B\mathbf{u}(t) + \mathbf{d}(t)] \qquad (11.116)$$

and the terminal penalty function

$$\Theta(\mathbf{x}(t_f),t_f) = \frac{1}{2}\,\|\mathbf{y}^*(t_f) - \mathbf{y}(t_f)\|^2_{S'} \qquad (11.117)$$

where $\lambda(t)$ is the Lagrange multiplier for the state.

Application of Pontryagin's principle yields the following necessary conditions

$$\partial\mathcal{H}/\partial\lambda(t) = \dot{x}(t) = Ax(t) + Bu(t) + \mathbf{d}(t) \qquad (11.118)$$

$$\partial\mathcal{H}/\partial x(t) = -\,\dot{\lambda}(t) = F^TQF\mathbf{x}(t) + A^T\lambda(t) \qquad (11.119)$$

$$\partial\mathcal{H}/\partial u(t) = 0 \rightarrow u(t) = -R^{-1}B^T\lambda(t) \qquad (11.120)$$

$$\partial\Theta/\partial\mathbf{x}(t_f) = \lambda(t_f) = -F^TSF[\mathbf{x}^*(t_f) - x(t_f)] \qquad (11.121)$$

Introducing Eq. 11.120 into Eq. 11.119 yields the modified state equation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) - BR^{-1}B^T\lambda(t) + \mathbf{d}(t) \qquad (11.122)$$

The closed-loop control is determined by assuming that

$$\lambda(t) = P(t)x(t) - \xi(t) \qquad (11.123)$$

where $P(t)$ is $n \times n$ and $\xi(t)$ is $n \times 1$ are matrices and vectors which must be determined.

Substituting Eq. 11.123 into Eqs. 11.119 and 11.122 yields the following necessary conditions

$$\dot{P}(t) + P(t)A + A^TP(t) - P(t)BR^{-1}B^TP(t) + F^TQF = 0 \quad ; \quad P(t_f) = F^TQF \qquad (11.124)$$

$$\dot{\xi}(t) + [A - BR^{-1}B^TP(t)]^T\xi(t) + F^TQFx^*(t) - P(t)d(t) = 0 \quad ; \quad \xi(t_f) = F^TQFx^*(t_f) \tag{11.125}$$

for an arbitrary state vector $x(t)$, where the linear state and prefilter feedback control is given by

$$u(t) = -R^{-1}B^T[P(t)x(t) - \xi(t)] \tag{11.126}$$

It should be observed that Eqs. 11.19 and 11.124 are identical in both form and boundary condition; consequently, the the solution for $P(t)$ in Eq. 11.124 is given by Eqs. 11.21-.23.

11.5.1 Closed-Form Solution for the Prefilter Equation

The solution for $\xi(t)$ in Eq. 11.125 follows on assuming the product form solution (Refs. 15,16)

$$\xi(t) = Z^{-1}(t)r(t) \tag{11.127}$$

where $Z(t)$ is defined by Eq. 11.22 and $\mathbf{r}(t)$ is a vector function which is to be determined. The linear constant-coefficient vector differential equation for $\mathbf{r}(t)$ can be shown to be

$$\dot{\mathbf{r}}(t) - \bar{A}\mathbf{r}(t) = Z(t)P_{ss}\mathbf{d}(t) + \mathbf{d}(t) - Z(t)F^TQF\mathbf{x}^*(t), \ \mathbf{r}(t_f) = Z(t_f)F^TQF\mathbf{x}^*(t_f) \tag{11.128}$$

The solution for $\mathbf{r}(t)$ follows as

$$\mathbf{r}(t) = e^{\bar{A}t}[\mathbf{r}(t_o) - \gamma(t)] \tag{11.129}$$

$$\gamma(t) = \int_{t_o}^{t} e^{\bar{A}v}[Z(v)F^TQC\mathbf{x}^*(v) - \mathbf{d}(v) - Z(v)P_{ss}\mathbf{d}(v)]dv \tag{11.130}$$

where the initial condition for $\mathbf{r}(t_o)$ is given by

$$\mathbf{r}(t_o) = e^{-\bar{A}t_f}\mathbf{r}(t_f) + \gamma(t_f) \tag{11.131}$$

The solution for the integral expression in Eq. 11.130 follows on introducing $Z(t)$ from Eq. 11.22, $\mathbf{x}^*(t)$ from Eq. 11.115, and $\mathbf{d}(t)$ from Eq. 11.114 into Eq. 11.130, leading to

$$\begin{aligned}\gamma(t) &= \Psi(-\bar{A},D_1,\Omega,t,t_o)\mathbf{s}_o + D_2\Psi(\bar{A}^T,D_3,\Omega,t,t_o)\mathbf{s}_o \\ &\quad - \Psi(-\bar{A},D_4,M,t,t_o)\beta - D_2\Psi(\bar{A}^T,D_5,M,t,t_o)\beta \qquad (11.132)\end{aligned}$$

where

$$D_1 = Z_{ss}F^TQFH \quad , \quad D_2 = e^{-\bar{A}t_f}[Z(t_f) - Z_{ss}]e^{-\bar{A}^Tt_f}$$

$$D_3 = F^TQFH \quad , \quad D_4 = Z_{ss}P_{ss}\Lambda + \Lambda \quad , \quad D_5 = P_{ss}\Lambda$$

and $\Psi(\cdot)$ is defined following Eq. 11.43.

Introducing Eq. 11.129 into Eq. 11.127 yields the following solution for $\xi(t)$:

$$\xi(t) = T(t,t_f)\xi(t_f) - Z^{-1}(t)e^{\bar{A}t}\{\gamma(t) - \gamma(t_f)\} \qquad (11.133)$$

where $T(t,t_f)$ is the state transition matrix defined by Eqs. 11.83-.85. Substituting Eqs. 11.21 and 11.133 into Eq. 11.126, the feedback control becomes

$$\begin{aligned}\mathbf{u}(t) &= -R^{-1}B^T[\{P_{ss} + Z^{-1}(t)\}\mathbf{x}(t) - T(t,t_f)\xi(t_f) \\ &\quad + Z^{-1}(t)e^{\bar{A}t}\{\gamma(t) - \gamma(t_f)\}] \qquad (11.134)\end{aligned}$$

We further observe that the tracking/disturbance-accommodation formulation above reduces to the special case of a tracking solution by setting $\beta = 0$ in Eq. 11.132, or to the special case of a disturbance-accommodating solution by setting $\mathbf{s}_o = 0$ in Eq. 11.132.

11.5.2 State Trajectory Equation

In order to determine the closed-loop system response under the assumptions of perfect plant knowledge and perfect state estimation, we introduce Eq. 11.126 into Eq. 11.118, leading to

$$\dot{\mathbf{x}}(t) = [\bar{A} - BR^{-1}B^T Z^{-1}(t)]\mathbf{x}(t) + BR^{-1}B^T \xi(t) + \mathbf{d}(t) \qquad (11.135)$$

The solution for $\mathbf{x}(t)$ in Eq. 11.135 is obtained by carrying out a two-stage process. First, the homogeneous solution is obtained by using the approach presented in Section 11.3.1. Second, using a *variation of parameters* approach the complete solution for Eq. 11.135 is obtained.

As shown in Section 11.3.1 the homogeneous solution for Eq. 11.135 is given by

$$\mathbf{x}_h(t) = \Phi(t,t_o)\mathbf{x}(t_o) \qquad (11.136)$$

where $\Phi(t,t_o) = Z(t)e^{\bar{A}^T(t-t_o)}Z^{-1}(t_o)$ is the system state transition matrix, $\Phi(t,t_o)$ satisfies the nonautonomous matrix differential equation:

$$\dot{\Phi}(t,t_o) = [\bar{A} - BR^{-1}B^T Z^{-1}(t)]\Phi(t,t_o) \quad ; \quad \Phi(t_o,t_o) = I \qquad (11.137)$$

and $\Phi(t,t_o)$ satisfies the semi-group and inversion properties presented following Eq. 11.32.

The *variation of parameters* solution for Eq. 11.135 is assumed to be

$$\mathbf{x}(t) = \Phi(t,t_o)\mathbf{g}(t) \quad ; \quad \mathbf{g}(t_o) = \mathbf{x}(t_o) \qquad (11.138)$$

where $\mathbf{g}(t)$ is to be determined. Introducing Eq. 11.138 into Eq. 11.135 produces the following differential equation for $\mathbf{g}(t)$:

$$\dot{\mathbf{g}}(t) = \Phi^{-1}(t,t_o)[BR^{-1}B^T\xi(t) + \mathbf{d}(t)] \qquad (11.139)$$

where the solution for $\xi(t)$ is defined by Eq. 11.127 and $\mathbf{d}(t)$ is defined by Eq. 11.114.

The solution for $\mathbf{g}(t)$ can be shown to be (Refs. 15 and 18)

$$\mathbf{g}(t) = \mathbf{g}(t_o) + \Phi^{-1}(t,t_o)\mathbf{r}(t) + \mathbf{r}_1(t) + \mathbf{r}_2(t) - \mathbf{r}(t_o) \tag{11.140}$$

where $\mathbf{r}(t)$ is defined by Eqs. 11.129 and 11.132

$$\mathbf{r}_1(t) = \int_{t_o}^{t} \Phi^{-1}(v,t_o)Z(v)F^TQFx^*(v)dv \tag{11.141}$$

$$\mathbf{r}_2(t) = -\int_{t_o}^{t} \Phi^{-1}(v,t_o)Z(v)P_{ss}\mathbf{d}(v)dv \tag{11.142}$$

and $\mathbf{r}(t_o)$ is defined by Eq. 11.131. The solution for $\mathbf{g}(t)$ in Eq. 11.140 can be verified by straightforward differentiation, where the differential equation for $\Phi^{-1}(t,t_o)$ follows as

$$\frac{d}{dt}[\Phi^{-1}(t,t_o)] = -\Phi^{-1}(t,t_o)[\bar{A} - BR^{-1}B^TZ^{-1}(t)] \quad ; \quad \Phi^{-1}(t_o,t_o) = I$$

and

$$\Phi^{-1}(t,t_o) = Z(t_o)e^{-\bar{A}^T(t_o-t)}Z^{-1}(t) \tag{11.143}$$

Substituting Eq. 11.140 into Eq. 11.138 yields the following solution for the state trajectory equation:

$$x(t) = \Phi(t,t_o)[x(t_o) - r(t_o) + r_1(t) + r_2(t)] + r(t) \tag{11.144}$$

The solutions for $r_1(t)$ and $r_2(t)$ are obtained by introducing Eq. 11.143 into Eqs. 11.141 and 11.142, leading to

$$\mathbf{r}_1(t) = Z(t_o)\Psi(\bar{A}^T,D_3,\Omega,t,t_o)s_o \tag{11.145}$$

$$\mathbf{r}_2(t) = Z(t_o)\Psi(\bar{A}^T,D_5,M,t,t_o)\beta \tag{11.146}$$

where $\Psi(\cdot)$ is defined following Eq. 11.43. Furthermore, we observe that the integrals defined in Eqs. 11.145 and 11.146 have been previously defined as part of the solution for Eq. 11.132; thus no additional computational effort is required in order to produce $r_1(t)$ and $r_2(t)$.

11.5.3 Recursion Relationship for Evaluating the State and Control at Discrete Time Steps

If the solution for $x(t)$ is required at the discrete times $t_k = t_0 + k\Delta t$ $(k = 1,\ldots,N)$, for $\Delta t = (t_f - t_0)/N$, then Eq. 11.144 can be written as

$$\begin{aligned}
x(t) &= Z(t)e^{-\bar{A}^T(t-t_0)}Z^{-1}(t_0)\{x_0 - r_0\} \\
&+ Z(t)e^{\bar{A}^T t_0}\{e^{-\bar{A}^T t}\Psi(\bar{A}^T,D_3,\Omega,t,t_0)s_0 \\
&+ e^{-\bar{A}^T t}\Psi(\bar{A}^T,D_5,M,t,t_0)\beta\} + e^{\bar{A}t}r_0 \\
&- e^{\bar{A}t}\Psi(-\bar{A},D_1,\Omega,t,t_0)s_0 + e^{\bar{A}t}\Psi(-\bar{A},D_4,M,t,t_0)\beta \\
&- e^{\bar{A}t}D_2\Psi(\bar{A}^T,D_3,\Omega,t,t_0)s_0 + e^{\bar{A}t}D_2\Psi(\bar{A}^T,D_5,M,t,t_0)\beta \qquad (11.147)
\end{aligned}$$

In order to simplify the recursion relationships for the $\Psi(\cdot)$ integrals above, we assume that $t_0 \equiv 0$; however, the generalization for $t_0 \neq 0$ is straightforward, though more cumbersome to present. A key step in the development of the recursion relationships is to view matrix functions of the form $e^{(\cdot)}C\Psi(\cdot)$, where C is constant matrix, as operators which map "initial condition" vectors, σ, to a future time. As a result, we can arrange the calculations in two ways: first, we can hold σ fixed and propagate $e^{(\cdot)}C\Psi(\cdot)$; and second, we can hold $e^{(\cdot)}C\Psi(\cdot)$ fixed and propagate σ. The latter option turns out to be the most efficient computationally. The recursion relationships are obtained by exploiting the semi-group properties of the matrix exponential solution required for computing $\Psi(\cdot)$. The basic solution algorithm is presented in Appendix 11.B. The recursive solutions for the terms appearing Eq. 11.147 are presented below. The order of presentation of terms corresponds to the order of appearance in Eq. 11.147.

Recursion Relationship for $Z(t)e^{-\bar{A}^T(t-t_o)}Z^{-1}(t_o)\{\mathbf{x}_o - \mathbf{r}_o\}$

$$Z(t+\Delta)e^{-\bar{A}^T(t+\Delta)}Z^{-1}(0)\{\mathbf{x}_o - \mathbf{r}_o\} = Z_{ss}\mathbf{a}(t+\Delta) + \mathbf{b}(t+\Delta) \tag{11.148}$$

where

$$\mathbf{a}(t_o) = Z^{-1}(0)\{\mathbf{x}_o - \mathbf{r}_o\}$$

$$\mathbf{b}(t_o) = e^{-\bar{A}t_f}[Z(t_f) - Z_{ss}]e^{-\bar{A}^T t_f}Z^{-1}(0)\{\mathbf{x}_o - \mathbf{r}_o\}$$

$$\mathbf{a}(t+\Delta) = e^{-\bar{A}^T\Delta t}\mathbf{a}(t)$$

$$\mathbf{b}(t+\Delta) = e^{\bar{A}\Delta t}\mathbf{b}(t)$$

Recursion Relationship for $Z(t)$

$$Z(t+\Delta) = C + e^{\bar{A}\Delta t}Z(t)e^{\bar{A}^T\Delta t} \tag{11.149}$$

where

$$Z(0) = Z_{ss} + e^{-\bar{A}t_f}[Z(t_f) - Z_{ss}]e^{-\bar{A}^T t_f}$$

$$C = Z_{ss} - e^{\bar{A}\Delta t}Z_{ss}e^{\bar{A}^T\Delta t}$$

Recursion Relationship for $e^{-\bar{A}^T t}\Psi(\bar{A}^T, D_3, \Omega, t, 0)\mathbf{s}_o$

From Eqs. C.1 and C.11-.14, we obtain

$$e^{-\bar{A}^T(t+\Delta)}\Psi(\bar{A}^T, D_3, \Omega, t+\Delta, 0)\mathbf{s}_o = \nu_1^1(t+\Delta) \tag{11.150}$$

where

$$\nu_1^1(t+\Delta) = F_1^1(\Delta t)\nu_1^1(t) + G_1^1(\Delta t)\nu_2^1(t) \quad , \quad \nu_1^1(0) = \mathbf{0}$$

$$\nu_2^1(t+\Delta) = F_2^1(\Delta t)\nu_2^1(t) \quad , \quad \nu_2^1(0) = \mathbf{s}_o$$

$$C_1 = \begin{bmatrix} -\bar{A}^T & D_3 \\ 0 & \Omega \end{bmatrix}$$

$$e^{C_1 \Delta t} = \begin{bmatrix} F_1^1(\Delta t) & G_1^1(\Delta t) \\ 0 & F_2^1(\Delta t) \end{bmatrix}$$

Recursion Relaionship for $e^{-\bar{A}^T t}\Psi(\bar{A}^T, D_5, M, t, 0)\beta$

From Eqs. C.1 and C.11-.14, we obtain

$$e^{-\bar{A}^T(t+\Delta)}\Psi(\bar{A}^T, D_5, M, t+\Delta, 0)\beta = \nu_1^2(t + \Delta) \tag{11.151}$$

where

$$\nu_1^2(t + \Delta) = F_1^2(\Delta t)\nu_1^2(t) + G_1^2(\Delta t)\nu_2^2(t) \quad , \quad \nu_1^2(0) = \mathbf{0}$$

$$\nu_2^2(t + \Delta) = F_2^2(\Delta t)\nu_2^2(t) \quad , \quad \nu_2^2(0) = \beta$$

$$C_2 = \begin{bmatrix} -\bar{A}^T & D_5 \\ 0 & M \end{bmatrix}$$

$$e^{C_2 \Delta t} = \begin{bmatrix} F_1^2(\Delta t) & G_1^2(\Delta t) \\ 0 & F_2^2(\Delta t) \end{bmatrix}$$

Recursion Relationship for $e^{\bar{A}t}\mathbf{r}(0)$

$$\mathbf{p}(t + \Delta) = e^{\bar{A}\Delta t}\mathbf{p}(t) \quad , \quad \mathbf{p}(0) = \mathbf{r}(0) \tag{11.152}$$

Recursion Relationship for $e^{\bar{A}t}\Psi(-\bar{A}, D_1, \Omega, t, 0)\mathbf{s}_o$

From Eqs. C.1 and C.11-.14, we obtain

$$e^{\bar{A}(t+\Delta)}\Psi(-\bar{A}, D_1, \Omega, t+\Delta, 0)\mathbf{s}_o = \nu_1^3(t + \Delta t) \tag{11.153}$$

where

$$\nu_1^3(t + \Delta) = F_1^3(\Delta t)\nu_1^3(t) + G_1^3(\Delta t)\nu_2^3(t) \quad , \quad \nu_1^3(0) = 0$$

$$\nu_2^3(t + \Delta) = F_2^3(\Delta t)\nu_2^3(t) \quad , \quad \nu_2^3(0)s_o$$

$$C_3 = \begin{bmatrix} \bar{A} & D_1 \\ 0 & \Omega \end{bmatrix}$$

$$e^{C_3\Delta t} = \begin{bmatrix} F_1^3(\Delta t) & G_1^3(\Delta t) \\ 0 & F_2^3(\Delta t) \end{bmatrix}$$

<u>Recursion Relationship for $e^{\bar{A}t}\Psi(-\bar{A},D_4,M,t,0)\beta$</u>

From Eqs. C.1 and C.11-.14, we obtain

$$e^{\bar{A}(t+\Delta)}\Psi(-\bar{A},D_4,M,t+\Delta,0)\beta = \nu_1^4(t + \Delta) \tag{11.154}$$

where

$$\nu_1^4(t + \Delta) = F_1^4(\Delta t)\nu_1^4(t) + G_1^4(\Delta t)\nu_2^4(t) \quad , \quad \nu_1^4(0) = 0$$

$$\nu_2^4(t + \Delta) = F_2^4(\Delta t)\nu_2^4(t) \quad , \quad \nu_2^4(0) = \beta$$

$$C_4 = \begin{bmatrix} \bar{A} & D_4 \\ 0 & M \end{bmatrix}$$

$$e^{C_4\Delta t} = \begin{bmatrix} F_1^4(\Delta t) & G_1^4(\Delta t) \\ 0 & F_2^4(\Delta t) \end{bmatrix}$$

<u>Recursion Relationship for $e^{\bar{A}t}D_2\Psi(\bar{A}^T,D_3,\Omega,t,0)s_o$</u>

From Eqs. C.1 and C.11-.14, we obtain

$$e^{\bar{A}(t+\Delta)}D_2\Psi(\bar{A}^T,D_3,\Omega,t+\Delta,0)s_o = K(t + \Delta)\nu_1^1(t + \Delta) \tag{11.155}$$

where

$$K(t + \Delta) = F_1^3(\Delta t)K(t)[F_1^3(\Delta t)]^T \quad , \quad K(0) = D_2$$

$\nu_1^1(t + \Delta)$ is defined by Eq. 11.150 and $F_1^3(\Delta t)$ is defined by Eq. 11.151.

Recursion Relationship for $e^{\overline{A}t}D_2\Psi(\overline{A}^T,D_5,M,t,0)\boldsymbol{\beta}$

From Eqs. C.1 and C.11-.14, we obtain

$$e^{\overline{A}(t+\Delta)}D_2\Psi(\overline{A}^T,D_5,M,t+\Delta,0)\boldsymbol{\beta} = K(t + \Delta)\nu_1^2(t + \Delta) \tag{11.156}$$

where

$\nu_1^2(t + \Delta)$ is defined by Eq. 11.151 and $K(t + \Delta)$ is defined by Eq. 11.155.

Substituting Eqs. 11.148-.156 into Eq. 11.147 yields state trajectory equations for the following three classes of control problems.

Disturbance-Accommodating Tracking Formulation

$$\begin{aligned}\mathbf{x}(t) = Z_{ss}\mathbf{a}(t) + \mathbf{b}(t) + Z(t)\{\nu_1^1(t) + \nu_2^2(t)\} \\ + \mathbf{p}(t) - \nu_1^3(t) + \nu_1^4(t) + K(t)\{-\nu_1^1(t) + \nu_1^2(t)\}\end{aligned} \tag{11.157}$$

Tracking Formulation (i.e. $\boldsymbol{\beta} \equiv \mathbf{0}$)

$$\begin{aligned}\mathbf{x}(t) = Z_{ss}\mathbf{a}(t) + \mathbf{b}(t) + [Z(t) - K(t)]\nu_1^1(t) \\ + \mathbf{p}(t) - \nu_1^3(t)\end{aligned} \tag{11.158}$$

Disturbance-Accommodating Formulation (i.e. $\mathbf{s}_0=\mathbf{0}$ and $\mathbf{r}(t_f)=\mathbf{0}$)

$$\begin{aligned}\mathbf{x}(t) = Z_{ss}\mathbf{a}(t) + \mathbf{b}(t) + [Z(t) + K(t)]\nu_1^2(t) \\ + \mathbf{p}(t) + \nu_1^4(t)\end{aligned} \tag{11.159}$$

To compute the control, $\mathbf{u}(t)$, at discrete times we observe that Eq. 11.126 can be written as

$$u(t) = -R^{-1}B^T[\{P_{ss} + Z^{-1}(t)\}x(t) - \xi(t)]$$

$$= -R^{-1}B^T[\{P_{ss} + Z^{-1}(t)\}\{\Phi(t,0)[x_o - r(0) + r_1(t) + r_2(t)] + r(t)\}$$

$$- \xi(t)]$$

which simplifies to

$$u(t) = -R^{-1}B^T[P_{ss}\Phi(t,0)\{x_o - r(0) + r_1(t) + r_2(t)\} + P_{ss}r(t)$$

$$+ Z^{-1}(t)\Phi(t,0)\{x_o - r(0) + r_1(t) + r_2(t)\}] \qquad (11.160)$$

where $\xi(t) = Z^{-1}(t)r(t)$ as defined by Eq. 11.127.

Substituting $\Phi(t,0)$ from Eq. 11.32, $r(0)$ from Eq. 11.131, $r_1(t)$ from Eq. 11.141, and $r_2(t)$ from Eq. 11.142, we obtain

$$u(t) = -R^{-1}B^T[(P_{ss}Z(t) + I)e^{-\bar{A}^T t}Z^{-1}(0)\{x_o - r(0)$$

$$+ Z(0)\Psi(\bar{A}^T,D_3,\Omega,t,0)s_o + Z(0)\Psi(\bar{A}^T,D_5,M,t,0)\beta\}$$

$$+ P_{ss}\{e^{At}r(0) - e^{At}\Psi(-\bar{A},D_1,\Omega,t,0)s_o + e^{At}\Psi(-\bar{A},D_4,M,t,0)\beta$$

$$- e^{\bar{A}t}D_2\Psi(\bar{A}^T,D_3,\Omega,t,0)s_o + e^{\bar{A}t}D_2\Psi(\bar{A}^T,D_5,M,t,0)\beta\}] \qquad (11.161)$$

From Eqs. 11.148-.156 we obtain the recursive linear difference equations for generating the control time histories for the following classes of control problems:

Distrubance-Accommodating Tracking Formulation

$$u(t) = -R^{-1}B^T[\{P_{ss}Z(t) + I\}\{q(t) + \nu_1^1(t) + \nu_1^2(t)\}$$

$$+ P_{ss}\{p(t) - \nu_1^3(t) + \nu_1^4(t) + K(t)[-\nu_1^1(t) + \nu_1^2(t)]\}] \qquad (11.162)$$

where

$$q(t + \Delta) = e^{-\bar{A}^T\Delta t}q(t) \quad , \quad q(0) = Z^{-1}(0)\{x_o - r(0)\}$$

$Z(t)$ is defined by Eq. 11.149 and $K(t)$ is defined by Eq. 11.155.

Tracking Formulation (i.e., $\beta = 0$)

$$\mathbf{u}(t) = -R^{-1}B^T[\{P_{ss}Z(t) + I\}\{\mathbf{q}(t) + \nu_1^1(t)\} + P_{ss}\{\mathbf{p}(t) - \nu_1^3(t) - K(t)\nu_1^1(t)\}] \tag{11.163}$$

Disturbance-Accommodating Formulation (i.e. $s_0 = 0$ and $\mathbf{r}(t_f) = \mathbf{0}$)

$$\mathbf{u}(t) = -R^{-1}B^T[\{P_{ss}Z(t) + I\}\{\mathbf{q}(t) + \nu_1^2(t)\} + P_{ss}\{\mathbf{p}(t) + \nu_1^4(t) + K(t)\nu_1^2(t)\}] \tag{11.164}$$

11.5.3 Constant Reference State and Disturbance State Special Cases

Moreover, it should be observed that the special case of a constant dynamic reference state for $\mathbf{y}^*(t)$ in Eq. 11.115 is handled by imposing the constraint

$$\Omega = 0$$

Similarly, the special case of a constant disturbance state for $\mathbf{d}(t)$ in Eq. 11.114 is handled by imposing the constraint

$$M = 0$$

As a result, the structure of the closed-form solutions presented in Sections 11.5.1-.3 remains unchanged when either of the above mentioned special cases are considered.

11.5.5 Residual State Trajectory Equation

Assuming that the residual state trajectory equation is given by

$$\dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r\mathbf{u}(t) + \mathbf{d}_r(t) \quad , \quad \mathbf{x}_r(0) \text{ given} \tag{11.165}$$

where $\mathbf{u}(t)$ is defined by Eq. 11.126, A_r is the $n_r \times n_r$ residual system dynamics matrix, B_r is the $n_r \times n$ residual control influence matrix, $\mathbf{x}_r(t)$ is the n_r-dimensional residual state vector (for the previously truncated plant dynamics), and $\mathbf{d}_r(t)$ is the n_r-dimensional residual disturbance state vector given by

$$\mathbf{d}_r(t) = \Lambda_r e^{Mt}\beta$$

which is analogous to Eq. 11.114, where Λ_r is the $n_r \times n_d$ residual disturbance state influence matrix. By substituting Eq. 11.126 into Eq. 11.165 we obtain

$$\dot{\mathbf{x}}_r(t) = A_r \mathbf{x}_r(t) - B_r R^{-1} B^T [P_{ss}\mathbf{x}(t) + Z^{-1}(t)\{\mathbf{x}(t) - \mathbf{r}(t)\}] + \mathbf{d}_r(t)$$

where $\mathbf{x}(t)$ is defined by Eq. 11.144 and $\mathbf{r}(t)$ is defined by Eq. 11.129. Using standard techniques the solution for $\mathbf{x}_r(t)$ can be shown to be

$$x_r(t) = e^{A_r t} x_r(0) - e^{A_r t} \int_0^t e^{-A_r v} B_r R^{-1} B^T [P_{ss}\mathbf{x}(v)$$
$$+ Z^{-1}(v)\{x(v) - \mathbf{r}(v)\}]dv + \{e^{A_r t} \int_0^t e^{-A_r v} \Lambda_r e^{Mv} dv\}\beta \quad (11.166)$$

After considerable labor the integrals above can be expressed as

$$x_r(t) = e^{A_r t} x_r(0) + \sum_{i=1}^{11} I_i(t) \quad (11.167)$$

where

$$I_1(t) = -[\Gamma_1]_{12}\alpha \quad , \quad I_2(t) = -[\Gamma]_{13}s_o \quad , \quad I_3(t) = [\Gamma_2]_{13}\beta$$

$$I_4(t) = -[\Gamma_3]_{12}D_2\alpha \quad , \quad I_5(t) = -[\Gamma_3]_{12}r_o \quad , \quad I_6(t) = [\Gamma_3]_{13}s_o$$

$$I_7(t) = -[\Gamma_4]_{13}\beta \quad , \quad I_8(t) = -[\Gamma_5]_{12}\alpha \quad , \quad I_9(t) = -[\Gamma_5]_{13}s_o$$

$$I_{10}(t) = [\Gamma_6]_{13}\beta \quad , \quad I_{11}(t) = e^{A_r t}\Psi(-A_r, \Lambda_r, M, t, 0)\beta$$

and

$$\alpha = Z^{-1}(0)\{x(0) - r(0)\}$$

$$\Gamma_1 = \Gamma_1(A_4,D_6,-\bar{A}^T,D_3,\Omega,t) \quad , \quad \Gamma_2 = \Gamma_2(A_4,D_6,-\bar{A}^T,D_5,M,t)$$

$$\Gamma_3 = \Gamma_3(A_4,D_7,A,D_1,\Omega,t) \quad , \quad \Gamma_4 = \Gamma_4(A_4,D_7,A,D_4,M,t)$$

$$\Gamma_5 = \Gamma_5(A_4,D_8,-\bar{A}^T,D_3,\Omega,t) \quad , \quad \Gamma_6 = \Gamma_6(A_r,D_8,-\bar{A}^T,D_5,M,t)$$

where Γ is defined by Eq. C.3, $[\Gamma]_{12} = G_1(t)$ as defined by Eq. C.1, $[\Gamma]_{13} = H_1(t)$ as defined by Eq. C.2, and

$$D_6 = B_r R^{-1} B^T P_{ss} Z_{ss} \quad , \quad D_7 = B_r R^{-1} B^T P_{ss} \quad , \quad D_8 = B_r R^{-1} B^T$$

11.5.6 State Trajectory Sensitivity Calculations

In order to compute the sensitivity of the terminal state, we set $t = t_f$ and $t_0 = 0$ in Eq. 11.147 and let α be one of the free parameters. The results of this section are analogous to those presented in Sections 11.3.4 and 11.4.5 for the linear regulator problem and the terminal control problem. From Eq. 11.147 the terminal state sensitivity follows as (Ref. 18,19):

$$\begin{aligned}
\frac{\partial x(t_f)}{\partial \alpha} = {} & \frac{\partial}{\partial \alpha}[\Phi(t_f,0)]\{x_0 - r(0)\} - \Phi(t_f,0)\frac{\partial r(0)}{\partial \alpha} \\
& + [\frac{\partial Z(t_f)}{\partial \alpha}]\{e^{-\bar{A}^T t}\Psi(\bar{A}^T,D_3,\Omega,t,0)s_0 + e^{-\bar{A}^T t}\Psi(\bar{A}^T,D_5,M,t,0)\beta\} \\
& + Z(t_f)\{\frac{\partial}{\partial \alpha}[e^{-\bar{A}^T t}\Psi(\bar{A}^T,D_3,\Omega,t_f,0)]s_0 \\
& + \frac{\partial}{\partial \alpha}[e^{-\bar{A}^T t}\Psi(\bar{A}^T,D_5,M,t_f,0)]\beta\} + \frac{\partial}{\partial \alpha}[e^{\bar{A}t_f}]r(0) + e^{\bar{A}t_f}\frac{\partial r(0)}{\partial \alpha} \\
& - \frac{\partial}{\partial \alpha}[e^{\bar{A}t_f}\Psi(-\bar{A},D_1,\Omega,t_f,0)]s_0 + \frac{\partial}{\partial \alpha}[e^{\bar{A}t_f}\Psi(-\bar{A},D_4,M,t_f,0)]\beta \\
& - [\frac{\partial}{\partial \alpha}\{e^{\bar{A}t_f}\}D_2 + e^{\bar{A}t_f}\frac{\partial}{\partial \alpha}\{D_2\}]\{\Psi(\bar{A}^T,D_3,\Omega,t_f,0)s_0
\end{aligned}$$

$$- \Psi(\bar{A}^T, D_5, M, t_f, 0)\beta\} - e^{\bar{A}t_f} D_2 \{\frac{\partial}{\partial\alpha}[\Psi(\bar{A}^T, D_3, \Omega, t_f, 0)]s_0$$

$$- \frac{\partial}{\partial\alpha}[\Psi(\bar{A}^T, D_5, M, t_f, 0)]\beta\} \tag{11.168}$$

The partials appearing above are presented in the order in which they occur in Eq. 11.168. The partials for $\frac{\partial}{\partial\alpha}[\Phi(t_f,0)]$ are defined by Eqs. 11.50-.61.

<u>Initial $\mathbf{r}(t)$ Partials</u>

$$\frac{\partial \mathbf{r}(0)}{\partial\alpha} = \frac{\partial}{\partial\alpha}[e^{-\bar{A}t_f}]\mathbf{r}(t_f) + e^{-\bar{A}t_f}\frac{\partial \mathbf{r}(t_f)}{\partial\alpha} + \frac{\partial \gamma(t_f)}{\partial\alpha} \tag{11.169}$$

where $\frac{\partial \mathbf{r}(t_f)}{\partial\alpha}$ is defined by Eq. 11.170, $\frac{\partial}{\partial\alpha}[e^{-\bar{A}t_f}]$ is defined by Eq. 11.171, and $\frac{\partial\gamma(t_f)}{\partial\alpha}$ is defined by Eq. 11.190.

<u>Final $\mathbf{r}(t)$ Partials</u>

$$\frac{\partial \mathbf{r}(t_f)}{\partial\alpha} = \frac{\partial Z(t_f)}{\partial\alpha} F^T QF\mathbf{x}^*(t_f) + Z(t_f)F^T \frac{\partial Q}{\partial\alpha} F\mathbf{x}^*(t_f) \tag{11.170}$$

where $\frac{\partial Z(t_f)}{\partial\alpha}$ is defined by Eq. 11.51 and $\frac{\partial Q}{\partial\alpha}$ is defined by Eqs. 11.59 and 11.61.

<u>Matrix Exponential Partials for $e^{-\bar{A}t_f}$</u>

$$\frac{\partial}{\partial\alpha}[e^{-\bar{A}t_f}] = e^{-\bar{A}t_f}\int_0^{t_f} e^{\bar{A}v}(-\frac{\partial\bar{A}}{\partial\alpha})e^{-\bar{A}v}dv$$

$$= e^{-\bar{A}t_f}\Psi(\bar{A}, -\frac{\partial\bar{A}}{\partial\alpha}, -\bar{A}, t_f, 0) \tag{11.171}$$

where $\frac{\partial\bar{A}}{\partial\alpha}$ is defined by Eq. 11.55, $\Psi(\cdot)$ is defined by Eq. B.1, and the derivation for the matrix exponential partial derivative is given in Appendix C.

Matrix Exponential Partials for $e^{\bar{A}^T t_f}$

$$\frac{\partial}{\partial\alpha}[e^{\bar{A}^T t_f}] = e^{\bar{A}^T t_f}\int_0^{t_f} e^{-\bar{A}^T v}\,\frac{\partial\bar{A}^T}{\partial\alpha}\,e^{\bar{A}^T v}dv$$

$$= e^{\bar{A}^T t_f}\Psi(-\bar{A}^T,\ \frac{\partial\bar{A}^T}{\partial\alpha},\bar{A}^T,t_f,0) \qquad (11.172)$$

where $\frac{\partial\bar{A}}{\partial\alpha}$ is defined by Eq. 11.55, $\Psi(\cdot)$ is defined by Eq. B.1, and the derivation for the matrix exponential partial derivative is given in Appendix C.

Partials for $e^{-\bar{A}^T t_f}\Psi(\bar{A}^T,D_3,\Omega,t_f,0)$

From Eqs. C.33-.42 it follows that (Ref. 20)

$$\frac{\partial}{\partial\alpha}[e^{-\bar{A}^T t_f}\Psi(\bar{A}^T,D_3,\Omega,t_f,0)] = Ze^{\tilde{A}t_f}Y(0) \qquad (11.173)$$

where $Z = [I \quad 0 \quad 0 \quad 0]$ (selection operator)

$$\tilde{A} = \begin{bmatrix} -\bar{A}^T & D_3 & -\partial\bar{A}^T/\partial\alpha & \partial D_3/\partial\alpha \\ 0 & \Omega & 0 & \partial\Omega/\partial\alpha \\ 0 & 0 & -\bar{A}^T & D_3 \\ 0 & 0 & 0 & \Omega \end{bmatrix}$$

$Y(0)$ is defined by Eq. C.37, $\frac{\partial\bar{A}^T}{\partial\alpha}$ is defined by Eq. 11.55, $\frac{\partial D_3}{\partial\alpha}$ is defined by Eq. 11.187, and $\frac{\partial\Omega}{\partial\alpha}$ is defined by Eqs. 11.59 and 11.61.

Partials for $e^{-\bar{A}^T t}\Psi(\bar{A}^T,D_5,M,t_f,0)$

From Eqs. C.33-.42 it follows that (Ref. 20)

$$\frac{\partial}{\partial\alpha}[e^{-\bar{A}^T t}\Psi(\bar{A}^T,D_5,M,t_f,0)] = Ze^{\tilde{A}t_f}Y(0) \qquad (11.174)$$

where $Z = [I \quad 0 \quad 0 \quad 0]$ (selection operator)

$$\tilde{A} = \begin{bmatrix} -\bar{A}^T & D_5 & -\partial\bar{A}^T/\partial\alpha & \partial D_5/\partial\alpha \\ 0 & M & 0 & \partial M/\partial\alpha \\ 0 & 0 & -\bar{A}^T & D_5 \\ 0 & 0 & 0 & M \end{bmatrix}$$

$Y(0)$ is defined by Eq. C.37, $\frac{\partial\bar{A}^T}{\partial\alpha}$ is defined by Eq. 11.55, $\frac{\partial D_5}{\partial\alpha}$ is defined by Eq. 11.189, and $\frac{\partial M}{\partial\alpha}$ is problem dependent.

Partials for $e^{\bar{A}^T t_f}\Psi(-\bar{A},D_1,\Omega,t_f,0)$

From Eqs. C.33-.42 it follows that (Ref. 20)

$$\frac{\partial}{\partial\alpha}[e^{\bar{A}t_f}\Psi(-\bar{A},D_1,\Omega,t_f,0)] = Ze^{\tilde{A}t_f}Y(0) \qquad (11.175)$$

where $Z = [I \quad 0 \quad 0 \quad 0]$ (selection operator)

$$\tilde{A} = \begin{bmatrix} \bar{A} & D_1 & \partial\bar{A}/\partial\alpha & \partial D_1/\partial\alpha \\ 0 & \Omega & 0 & \partial\Omega/\partial\alpha \\ 0 & 0 & \bar{A} & D_1 \\ 0 & 0 & 0 & \Omega \end{bmatrix}$$

$X(0)$ is defined by Eq. C.37, $\frac{\partial\bar{A}}{\partial\alpha}$ is defined by Eq. 11.55, $\frac{\partial D_1}{\partial\alpha}$ is defined by Eq. 11.185, and $\frac{\partial\Omega}{\partial\alpha}$ is defined by Eqs. 11.59 and 11.61.

Partials for $e^{\bar{A}t_f}\Psi(-\bar{A},D_4,M,t_f,0)$

From Eqs. C.33-.42 it follows that (Ref. 20)

$$\frac{\partial}{\partial\alpha}[e^{\bar{A}t_f}\Psi(-\bar{A},D_4,M,t_f,0)] = Ze^{\tilde{A}t_f}X(0) \qquad (11.176)$$

where $Z = [I \quad 0 \quad 0 \quad 0]$ (selection operator)

$$\tilde{A} = \begin{bmatrix} \bar{A} & D_4 & \partial\bar{A}/\partial\alpha & \partial D_4/\partial\alpha \\ 0 & M & 0 & \partial M/\partial\alpha \\ 0 & 0 & \bar{A} & D_4 \\ 0 & 0 & 0 & M \end{bmatrix}$$

$Y(0)$ is defined by Eq. C.37, $\frac{\partial\bar{A}}{\partial\alpha}$ is defined by Eq. 11.55, $\frac{\partial D_4}{\partial\alpha}$ is defined by Eq. 11.188, and $\frac{\partial M}{\partial\alpha}$ is problem dependent.

Partials for $\Psi(\bar{A}^T,D_3,\Omega,t_f,0)$

From Eqs. C.55 and C.56 it follows that (Ref. 20)

$$\frac{\partial}{\partial\alpha}\left[\Psi(\bar{A}^T,D_3,\Omega,t_f,0)\right] = \frac{\partial F}{\partial\alpha}G + F\frac{\partial G}{\partial\alpha} \tag{11.177}$$

where F and $\frac{\partial F}{\partial\alpha}$ are obtained from Eqs. C.18-.24 as follows:

$$\begin{bmatrix} \frac{\partial F}{\partial\alpha} \\ F \end{bmatrix} = e^{\tilde{A}t_f}\begin{bmatrix} 0 \\ I \end{bmatrix} \tag{11.178}$$

$$\tilde{A} = \begin{bmatrix} \bar{A}^T & \partial\bar{A}^T/\partial\alpha \\ 0 & \bar{A}^T \end{bmatrix}$$

$\frac{\partial\bar{A}^T}{\partial\alpha}$ is defined by Eq. 11.55, and G and $\frac{\partial G}{\partial\alpha}$ are obtained from Eqs. C.33-.42 as follows:

$$\frac{\partial G}{\partial\alpha} = Z_1 e^{\tilde{B}t_f}Y(0) \tag{11.179}$$

$$G = Z_2 e^{\tilde{B}t_f}Y(0) \tag{11.180}$$

where

$Z_1 = [I \quad 0 \quad 0 \quad 0]$ (selection operator)

$Z_2 = [0 \quad 0 \quad I \quad 0]$ (selection operator)

$$\tilde{B} = \begin{bmatrix} -\bar{A}^T & D_3 & -\partial\bar{A}^T/\partial\alpha & \partial D_3/\partial\alpha \\ 0 & \Omega & 0 & \partial\Omega/\partial\alpha \\ 0 & 0 & -\bar{A}^T & D_3 \\ 0 & 0 & 0 & \Omega \end{bmatrix}$$

$Y(0)$ is defined by Eq. C.37, $\frac{\partial\bar{A}^T}{\partial\alpha}$ is defined by Eq. 11.55, $\frac{\partial D_3}{\partial\alpha}$ is defined by Eq. 11.187, and $\frac{\partial\Omega}{\partial\alpha}$ is defined by Eqs. 11.59 and 11.61.

Partials for $\Psi(\bar{A}^T,D_5,M,t_f,0)$

From Eqs. C.55 and C.56 it follows that (ref. 20)

$$\frac{\partial}{\partial\alpha}\left[\Psi(\bar{A}^T,D_5,M,t_f,0)\right] = \frac{\partial F}{\partial\alpha} G + F\frac{\partial G}{\partial\alpha} \tag{11.181}$$

where F and $\frac{\partial F}{\partial\alpha}$ are obtained from Eqs. C.18-.24 as follows:

$$\begin{bmatrix} \frac{\partial F}{\partial\alpha} \\ F \end{bmatrix} = e^{\tilde{A}t_f}\begin{bmatrix} 0 \\ I \end{bmatrix} \tag{11.182}$$

$$\tilde{A} = \begin{bmatrix} \bar{A}^T & \partial\bar{A}^T/\partial\alpha \\ 0 & \bar{A}^T \end{bmatrix}$$

$\frac{\partial\bar{A}^T}{\partial\alpha}$ is defined by Eq. 11.55, and G and $\frac{\partial G}{\partial\alpha}$ are obtained from Eqs. C.33-.42 as follows:

$$\frac{\partial G}{\partial\alpha} = Z_1 e^{\tilde{B}t_f} Y(0) \tag{11.183}$$

$$G = Z_2 e^{\tilde{B}t_f} Y(0) \tag{11.184}$$

where

$Z_1 = [I \quad 0 \quad 0 \quad 0]$ (selection operator)

$Z_2 = [0 \quad 0 \quad I \quad 0]$ (selection operator)

$$\tilde{B} = \begin{bmatrix} -\bar{A}^T & D_5 & -\partial\bar{A}^T/\partial\alpha & \partial D_5/\partial\alpha \\ 0 & M & 0 & \partial M/\partial\alpha \\ 0 & 0 & -\bar{A}^T & D_5 \\ 0 & 0 & 0 & M \end{bmatrix}$$

$Y(0)$ is defined by Eq. C.37, $\frac{\partial\bar{A}^T}{\partial\alpha}$ is defined by Eq. 11.55, $\frac{\partial D_5}{\partial\alpha}$ is defined by Eq. 11.189, and $\frac{\partial M}{\partial\alpha}$ is problem dependent.

Partials for D_i $(i = 1,\ldots,5)$

From Eq. 11.132 we have

$$\frac{\partial D_1}{\partial\alpha} = \frac{\partial Z_{ss}}{\partial\alpha} F^T QFH + Z_{ss} F^T \frac{\partial Q}{\partial\alpha} FH \tag{11.185}$$

$$\begin{aligned}\frac{\partial D_2}{\partial\alpha} &= \frac{\partial}{\partial\alpha}[e^{-\bar{A}t_f}][Z(t_f) - Z_{ss}]e^{-\bar{A}^T t_f} \\ &+ e^{-\bar{A}t_f}\left[\frac{\partial Z(t_f)}{\partial\alpha} - \frac{\partial Z_{ss}}{\partial\alpha}\right]e^{-\bar{A}^T t_f} + e^{-\bar{A}t_f}[Z(t_f) - Z_{ss}]\frac{\partial}{\partial\alpha}[e^{-\bar{A}^T t_f}]\end{aligned} \tag{11.186}$$

$$\frac{\partial D_3}{\partial\alpha} = F^T \frac{\partial Q}{\partial\alpha} FH \tag{11.187}$$

$$\frac{\partial D_4}{\partial\alpha} = \frac{\partial Z_{ss}}{\partial\alpha} P_{ss}\Lambda + Z_{ss}\frac{\partial P_{ss}}{\partial\alpha}\Lambda \tag{11.188}$$

$$\frac{\partial D_5}{\partial\alpha} = \frac{\partial P_{ss}}{\partial\alpha}\Lambda \tag{11.189}$$

where $\frac{\partial Z_{ss}}{\partial\alpha}$ is defined by Eq. 11.56, $\frac{\partial Z(t_f)}{\partial\alpha}$ is defined by Eq. 11.51, $\frac{\partial}{\partial\alpha}[e^{-At_f}]$ is defined by Eq. 11.171, $\frac{\partial P_{ss}}{\partial\alpha}$ is defined by 11.54, and $\frac{\partial Q}{\partial\alpha}$ is defined by Eqs. 11.59 and 11.61.

Partials for $\gamma(t_f)$

$$\frac{\partial \gamma(t_f)}{\partial \alpha} = \frac{\partial}{\partial \alpha}[\Psi(-\bar{A},D_1,\Omega,t_f,0)]s_0 + \frac{\partial D_2}{\partial \alpha} \Psi(\bar{A}^T,D_3,t_f,0)s_0$$

$$+ D_2 \frac{\partial}{\partial \alpha} [\Psi(\bar{A}^T,D_3,\Omega,t_f,0)]s_0 - \frac{\partial}{\partial \alpha} [\Psi(-\bar{A},D_4,M,t_f,0)]\beta$$

$$- \frac{\partial D_2}{\partial \alpha} \Psi(\bar{A}^T,D_5,M,t_f,0)\beta - D_2 \frac{\partial}{\partial \alpha} [\Psi(\bar{A},D_5,M,t_f,0)]\beta \quad (11.190)$$

where $\frac{\partial D_2}{\partial \alpha}$ is defined by Eq. 11.186, $\frac{\partial}{\partial \alpha}[\Psi(\bar{A}^T,D_3,\Omega,t_f,0)]$ is defined by Eqs. 11.177-.180, and $\frac{\partial}{\partial \alpha}[\Psi(\bar{A}^T,D_5,M,t_f,0)]$ is defined by Eq. 11.180. The partial derivatives for $\Psi((-\bar{A},D_1,\Omega,t_f,0)$ and $\Psi(-\bar{A},D_4,M,t_f,0)$ are presented below. Equation 11.190 is computed as part of Eq. 11.169.

Partials for $\Psi(-\bar{A},D_1,\Omega,t_f,0)$

From Eqs. C.55 and C.56 it follows that (ref. 20)

$$\frac{\partial}{\partial \alpha}[\Psi(-\bar{A},D_1,\Omega,t_f,0)] = \frac{\partial F}{\partial \alpha} G + F \frac{\partial G}{\partial \alpha} \quad (11.191)$$

where F and $\frac{\partial F}{\partial \alpha}$ are obtained from Eqs. C.18-.24 as follows:

$$\begin{bmatrix} \frac{\partial F}{\partial \alpha} \\ F \end{bmatrix} = e^{\tilde{A}t_f} \begin{bmatrix} 0 \\ I \end{bmatrix} \quad (11.192)$$

$$\tilde{A} = \begin{bmatrix} -\bar{A} & -\partial\bar{A}/\partial\alpha \\ 0 & -\bar{A} \end{bmatrix}$$

$\frac{\partial \bar{A}}{\partial \alpha}$ is defined by Eq. 11.55 and G and $\frac{\partial G}{\partial \alpha}$ are obtained from Eqs. C.33-.42 as follows:

$$\frac{\partial G}{\partial \alpha} = Z_1 e^{\tilde{B}t_f} Y(0) \quad (11.193)$$

$$G = Z_2 e^{\tilde{B}t_f} Y(0) \quad (11.194)$$

where

$Z_1 = [I \quad 0 \quad 0 \quad 0]$ (selection operator)

$Z_2 = [0 \quad 0 \quad I \quad 0]$ (selection operator)

$$\tilde{B} = \begin{bmatrix} \bar{A} & D_1 & \partial\bar{A}/\partial\alpha & \partial D_1/\partial\alpha \\ 0 & \Omega & 0 & \partial\Omega/\partial\alpha \\ 0 & 0 & \bar{A} & D_1 \\ 0 & 0 & 0 & \Omega \end{bmatrix}$$

$Y(0)$ is defined by Eq. C.37, $\frac{\partial\bar{A}}{\partial\alpha}$ is defined by Eq. 11.55, $\frac{\partial D_1}{\partial\alpha}$ is defined by Eq. 11.185, and $\frac{\partial\Omega}{\partial\alpha}$ is problem dependent.

<u>Partials for $\Psi(-\bar{A},D_4,M,t_f,0)$</u>

From Eqs. C.55 and C.56 it follows that (ref. 20)

$$\frac{\partial}{\partial\alpha}\left[\Psi(-\bar{A},D_4,M,t_f,0)\right] = \frac{\partial F}{\partial\alpha} G + F \frac{\partial G}{\partial\alpha} \tag{11.195}$$

where F and $\frac{\partial F}{\partial\alpha}$ are obtained from Eq. C.18-.24 as follows:

$$\begin{bmatrix} \frac{\partial F}{\partial\alpha} \\ F \end{bmatrix} = e^{\tilde{A}t_f} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{11.196}$$

$$\tilde{A} = \begin{bmatrix} -\bar{A} & -\partial\bar{A}/\partial\alpha \\ 0 & -A \end{bmatrix}$$

$\frac{\partial\bar{A}}{\partial\alpha}$ is defined by Eq. 11.55 and G and $\frac{\partial G}{\partial\alpha}$ are obtained from Eqs. C.33-.42 as follows:

$$\frac{\partial G}{\partial\alpha} = Z_1 e^{\tilde{B}t_f} Y(0) \tag{11.197}$$

$$G = Z_2 e^{\tilde{B}t_f} Y(0) \tag{11.198}$$

where

$Z_1 = [I \quad 0 \quad 0 \quad 0]$ (selection operator)

$Z_2 = [0 \quad 0 \quad I \quad 0]$ (selection operator)

$$\tilde{B} = \begin{bmatrix} -\bar{A} & D_4 & -\partial\bar{A}/\partial\alpha & \partial D_4/\partial\alpha \\ 0 & M & 0 & \partial M/\partial\alpha \\ 0 & 0 & -\bar{A} & D_4 \\ 0 & 0 & 0 & M \end{bmatrix}$$

$Y(0)$ is defined by Eq. C.37, $\frac{\partial\bar{A}}{\partial\alpha}$ is defined by Eq. 11.55, $\frac{\partial D_4}{\partial\alpha}$ is defined by Eq. 11.188, and $\frac{\partial M}{\partial\alpha}$ is problem dependent.

11.6 A Short Integral Table for Riccati-Like Matrix Differential Equations

As an aid to the reader, Table 11.1 presents a summary of the known solutions for Riccati matrix differential equations. The central result is that the state transition matrices of Eqs. T.3-.6 can be solved in closed-form. These equations arise in many applications as seen in Sections 11.3-.5 and their use provides a unifying framework for obtaining new results. For example Eq. T.3 is required in solving Eqs. 11.25, 11.98, and 11.135; Eq. T.5 is required in solving Eqs. 11.73, 11.96, and 11.125 and so on.

11.7 CONCLUSIONS AND RECOMMENDATIONS

The methods presented in Sections 11.3-11.5 clearly demonstrate that a broad class of finite time linear quadratic optimization problems can be readily handled by introducing special coordinate transformations. The key to the simplifying coordinate transformations is the use of the transient part of the differential matrix Riccati equation. Using this approach is has been possible to obtain matrix exponential solutions for the time-varying gains, state trajectory equations, residual state trajectory equations, and associated sensitivity partial derivatives. Future work can consider the coupled problems of the plant and estimation, model uncertainties, and real-time implementation approaches. Generalized perturbation approaches are also of interest, as well

Table 11.1
MATRIX EXPONENTIAL SOLUTIONS OF RICCATI LIKE DIFFERENTIAL EQUATIONS

Eq. #	D. Eq.	Typical B. C.	Solution	Comment
T.1	$\dot{P}=-PA-A^TP+PEP-Q$	$P(t_f)=S$	$P_{ss}+Z^{-1}(t)$*	Standard Riccati Eq./Well Known
T.2	$\dot{Z}=\bar{A}Z+Z\bar{A}^T-E$**	$Z(t_f)=(S-P_{ss})^{-1}$	$Z_{ss}+e^{\bar{A}(t-t_f)}[Z(t_f)-Z_{ss}]e^{\bar{A}^T(t-t_f)}$†	Lyapunov Eq./Well Known
T.3	$\dot{\phi}_1=F(t)\phi_1$††	$\phi_1(t_o)=I$	$Z(t)e^{-\bar{A}^T(t-t_o)}Z^{-1}(t_o)$	State Transition Matrix/New Result
T.4	$\dot{\phi}_2=-\phi_2F(t)$	$\phi_2(t_o)=I$	$Z(t_o)e^{-\bar{A}^T(t_o-t)}Z^{-1}(t)$	State Transition Matrix/New Result
T.5	$\dot{\phi}_3=-F^T(t)\phi_3$	$\phi_3(t_f)=I$	$Z^{-1}(t)e^{\bar{A}(t-t_f)}Z(t_f)$	State Transition Matrix/New Result
T.6	$\dot{\phi}_4=\phi_4F^T(t)$	$\phi_4(t_f)=I$	$Z^{-1}(t_f)e^{\bar{A}(t_f-t)}Z(t)$	State Transition Matrix/New Result
T.7	$\dot{H}=P^TEQ$	$H(t_f)=0$	$P^T(t)Z(t)Q(t)-P^T(t_f)Z(t_f)Q(t_f)$	New Result where $P=\phi_2^T\zeta$ or $\phi_3\zeta$, ζ is n_1xn_2; $Q=\phi_3\eta$ or $\phi_2^T\eta$, η is n_1xn_3

*$P_{ss}=P_{ss}^T>0$ is the solution for the algebraic Riccati equation.

**$\bar{A}=A-EP_{ss}$ is the system stability matrix.

†Z_{ss} is the solution for the algebraic Lyapunov equation.

††$F(t)=A-EP(t)=\bar{A}-EZ^{-1}(t)$

as the applications of these methods to three-dimensional maneuvers of flexible spacecraft.

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