CHAPTER 6

ELEMENTS OF OPTIMAL CONTROL THEORY

6.1 THE VARIATIONAL PROBLEM OF LAGRANGE

Modern optimal control has its roots in the calculus of variations, a subject placed upon solid foundations during the 1800's by the monumental works of Lagrange, Hamilton, and Jacobi. Variational calculus was motivated directly by the apparent existence of minimum principles and other variational laws (e.g., Hamilton's Principle) in analytical dynamics. In this chapter, we develop the fundamental concepts of calculus of variations and optimal control in a fashion that encompasses a very large class of spacecraft attitude control problems.

6.1.1 Statement of the Problem

A fundamental class of variational problems seeks an optimum space-time path x(t) which minimizes (or maximizes) the *performance index* functional

$$J = J[x(t),\ t_o,\ t_f] = \int_{t_o}^{t_f} F[x(t),\ \dot{x}(t),\ t]dt \tag{6.1}$$

with

$$x(t) = [x_1(t)\ x_2(t)\ \ldots\ x_n(t)]^T$$

Without loss of generality, we assume our task is to minimize Eq. 6.1. It is evident that a simple change of sign converts a maximization problem to a minimization problem.

To obtain the most fundamental classical results, we restrict initial attention to F and x of class C_2 (smooth, continuous functions having two continuous derivatives with respect to all arguments). Let $x(t)$, t_o, t_f represent the unknown path, and start and stop times, for which J of Eq. 6.1 has a local minimum value. Let an arbitrary neighboring, generally sub-optimal, path be denoted by $\tilde{x}(t)$, with neighboring terminal times $\tilde{t}_o$, $\tilde{t}_f$. We restrict the *varied path* $\tilde{x}(t)$ to be of class C_2 and to be near $x(t)$ in the sense that the *path variation*

$$\delta x(t) \equiv \tilde{x}(t) - x(t) \tag{6.2}$$

is of differential size for $\tilde{t}_0 \le t \le \tilde{t}_f$. We can consider $\tilde{x}(t)$, $\dot{\tilde{x}}(t)$ to be *generated* by small, arbitrary variations $\delta x(t)$ (of class C_2) as

$$\tilde{x}(t) = x(t) + \delta x(t) \tag{6.3}$$

$$\dot{\tilde{x}}(t) = \dot{x}(t) + \delta\dot{x}(t) \tag{6.4}$$

Clearly $\delta\dot{x}(t) = \frac{d}{dt}[\delta x(t)] = \frac{d}{dt}[\tilde{x}(t) - x(t)]$, since both $x(t)$ and $\delta x(t)$ are continuous.

Along the varied path $\tilde{x}(t)$ initiating at time $\tilde{t}_0 = t_0 + \delta t_0$ and terminating at $\tilde{t}_f = t_f + \delta t_f$, the performance index of Eq. 6.1 has the neighboring value

$$\tilde{J} = J[\tilde{x}(t),\tilde{t}_0,\tilde{t}_f] = \int_{\tilde{t}_0}^{\tilde{t}_f} F[x(t) + \delta x(t),\dot{x}(t) + \delta\dot{x}(t),t]dt \tag{6.5}$$

We define, for the case of finite $\delta x(t)$, the *finite variation* of J by differencing Eqs. 6.5 and 6.1 as

$$\Delta J = \tilde{J} - J = \int_{\tilde{t}_0}^{\tilde{t}_f} F(x + \delta x,\dot{x} + \delta\dot{x},t)dt - \int_{t_0}^{t_f} F(x,\dot{x},t)dt \tag{6.6}$$

Restricting attention to infinitesimal variations $\delta x(t)$, δt_0, δt_f, we define the differential *first variation* δJ as the linear part of ΔJ; we find δJ by expanding the first integral of Eq. 6.6 in a Taylor series in δx, $\delta\dot{x}$, δt_0, δt_f to be

$$\delta J = \int_{t_0}^{t_f} \left([\frac{\partial F}{\partial x}]\delta x + [\frac{\partial F}{\partial \dot{x}}]\delta\dot{x}\right)dt$$

$$+ F[x(t_f),\dot{x}(t_f),t_f]\delta t_f - F[x(t_0),\dot{x}(t_0),t_0]\delta t_0 \tag{6.7}$$

To compact Eq. 6.7 and subsequent discussion, we have introduced the $1 \times n$ *row gradient* $[\frac{\partial F}{\partial x}] \equiv [\frac{\partial F}{\partial x_1} \; \cdots \; \frac{\partial F}{\partial x_n}]$ so that for example

$$[\frac{\partial F}{\partial x}]\delta x \equiv [\frac{\partial F}{\partial x_1} \cdots \frac{\partial F}{\partial x_n}] \begin{Bmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{Bmatrix} \equiv \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \delta x_i$$

We will also make use of *column gradients*, which we denote by { } as

$$\{\frac{\partial F}{\partial x}\} = \begin{Bmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{Bmatrix} \equiv [\frac{\partial F}{\partial x}]^T$$

In preparation for making arguments on the arbitrariness of $\delta x(t), \delta t_o, \delta t_f$, we seek to eliminate the $\delta \dot{x}(t)$ term in Eq. 6.7. This is accomplished using the integration by parts

$$\int_{t_o}^{t_f} [\frac{\partial F}{\partial \dot{x}}]\delta \dot{x} dt \equiv [\frac{\partial F}{\partial \dot{x}}]\delta x(t)\Big|_{t_o}^{t_f} - \int_{t_o}^{t_f} \frac{d}{dt} [\frac{\partial F}{\partial \dot{x}}]\delta x(t)dt \tag{6.8}$$

Using Eq. 6.8 to replace the second term in the integrand of Eq. 6.7 yields

$$\delta J = \int_{t_o}^{t_f} \left[[\frac{\partial F}{\partial x}] - \frac{d}{dt} [\frac{\partial F}{\partial \dot{x}}] \right] \delta x(t)dt + [\frac{\partial F}{\partial x}]\delta x \Big|_{t_o}^{t_f}$$
$$+ F[x(t_f), \dot{x}(t_f), \delta t_f]\delta t_f - F[x(t_o), \dot{x}(t_o), t_o]\delta t_o = 0 \tag{6.9}$$

Equation 6.9 is set to zero as a *necessary condition* for J to have a minimum; that is, we require δJ to vanish for all admissible variations $\delta x(t)$, δt_o, δt_f. As a result, the trajectories $x(t)$ and terminal times t_o, t_f satisfying Eq. 6.9 yield *a stationary* value for $J[x(t), t_o, t_f]$. Since $\delta x(t)$ can assume an infinity of functional values, irrespective of the boundary conditions, we see that the integrand of the first term of Eq. 6.9 must vanish identically. Furthermore, since the boundary variations are generally independent of $\delta x(t)$, the boundary terms must also vanish independently. Thus Eq. 6.9 leads immediately to the *Euler - Lagrange necessary conditions*:

Euler-Lagrange Equations

$$[\frac{\partial F}{\partial \mathbf{x}}] - \frac{d}{dt}[\frac{\partial F}{\partial \dot{\mathbf{x}}}] = 0$$

or

$$\left\{\frac{\partial F}{\partial x_i}\right\} - \frac{d}{dt}\left\{\frac{\partial F}{\partial \dot{x}_i}\right\} = 0 \quad , \quad i = 1,2,\ldots,n \tag{6.10a}$$

Transversality Conditions

$$[\frac{\partial F}{\partial \mathbf{x}}|_{t_f}]\delta \mathbf{x}(t_f) \equiv \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}|_{t_f} \delta x_i(t_f) = 0 \tag{6.10b}$$

$$[\frac{\partial F}{\partial \mathbf{x}}|_{t_o}]\delta \mathbf{x}(t_o) \equiv \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}|_{t_o} \delta x_i(t_o) = 0 \tag{6.10c}$$

$$F[\mathbf{x}(t_f),\dot{\mathbf{x}}(t_f),t_f]\delta t_f = 0 \tag{6.10d}$$

$$F[\mathbf{x}(t_o),\dot{\mathbf{x}}(t_o),t_o]\delta t_o = 0 \tag{6.10e}$$

For example, if the initial and final times are fixed constants, and if the initial and final states are fully prescribed as

$$\mathbf{x}(t_o) = \mathbf{x}_o \quad , \quad \mathbf{x}(t_f) = \mathbf{x}_f \tag{6.11}$$

then the admissible path variatons $\delta\mathbf{x}(t)$ must vanish at t_o and t_f, and δt_o, δt_f must vanish as well. Thus for the *fixed time and fixed end point problem*, we find that the transversality conditions of Eqs. 6.10b - 6.10e are trivially satisfied and the necessary conditions reduce to the Euler-Lagrange equations of Eq. 6.10a subject to the 2n fixed boundary conditions of Eq. 6.11.

For more general boundary condition specifications, the transversality conditions provide *replacement* or *"natural" boundary conditions* for terminal variables not constrained to prescribed values. In the simplest such case, a single variable may be totally "free". For example, if the final time t_f is not constrained (and unknown), we must admit δt_f as non-zero and arbitrary. As a result, it is apparent by inspection of the transversality condition of Eq. 6.10d that the unknown "free" final time is implicitly determined from the generally nonlinear *stopping condition*

$$F[x(t_f),\ \dot{x}(t_f),\ t_f] = 0$$

Analogous transversality-derived boundary conditions are easily determined if other terminal conditions are free. For example, suppose that t_o is fixed; $x_r(t_o)$, $r = 1,\ldots,n$ are specified; t_f is fixed; $x_i(t_f)$, $i = 1,\ldots,p$ are free; and $x_j(t_f)$, $j = p + 1,\ldots,n$ are specified. As a result, the appropriate substitutions in Eqs. 6.10b and 6.10e are

$$\delta t_o = \delta t_f = 0;$$

$$\delta x_r(t_o) = 0,\ r = 1,\ldots,n;$$

$$\delta x_i(t_f) \text{ arbitrary},\ i = 1,\ldots,p;$$

$$\delta x_j(t_f) = 0 \quad ,\quad j = p + 1,\ldots,n$$

Upon introducing the variations above into Eqs. 6.10b and 6.10e, the governing transversality condition follows as

$$\sum_{i=1}^{p} \frac{\partial F}{\partial x_i}\Big|_{t_f}\ \delta x_i(t_f) = 0$$

Since $\delta x_i(t_f)$ is arbitrary for $i = 1,\ldots,p$, we conclude that the natural (replacement) boundary conditions are provided by

$$\frac{\partial F}{\partial x_i}\Big|_{t_f} = 0 \quad ,\ i = 1,\ldots,p$$

Consequently, the 2n initial and final states for the extermal trajectories are prescribed by the terminal constraints

$$x(t_o) = [x_i(t_o)\ \cdots\ x_p(t_o)\ \vdots\ x_{p+1}(t_o)\ \cdots\ x_n(t_o)]^T$$

$$\psi[x(t_f)] \equiv [\frac{\partial F}{\partial x_1}\Big|_{t_f}\ \cdots\ \frac{\partial F}{\partial x_p}\Big|_{t_f}\ \vdots\ (x_{p+1}(t_f) - C_{p+1})\cdots(x_n(t_f) - C_n)]^T = 0$$

with $\{C_{p+1}\ C_{p+2}\ \cdots\ C_n\}$ denoting the specified values for the boundary conditions on $x_{p+i}(t_f)$.

We will subsequently consider the more general case that the terminal states and times are functionally constrained to lie on a *generally* *nonlinear constraint manifold* of the form

$$\psi_j[x_1(t_f), \ldots, x_n(t_f), t_f] = 0 \quad , \quad j = 1,2,\ldots,n \qquad (6.12)$$

where the ψ_j are a set of independent functions of class C_2.

Notice, in any event, that typically n boundary conditions (i.e., specified boundary conditions and transversality replacement boundary conditions) will be available at time t_0, while the remaining conditions are associated with time t_f. Thus, the terminal boundary conditions on Eq. 6.10a are *split*, and as a result we have a *Two-Point Boundary-Value Problem* (TPBVP). Equation 6.10a generally provides n second-order nonlinear, stiff differential equations which can usually be solved for the second derivatives in the functional form

$$\ddot{x}_i = g_i(x_i, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, t) \quad , \quad i = 1,2,\ldots,n \qquad (6.10a)$$

Typically, numerical methods are required to solve Eq. 6.10a, even if we have an *initial-value problem* in which the $x_i(t_0)$ and $\dot{x}_i(t_0)$ are fully prescribed. Nonlinear TPBVPs are inherently more difficult to solve than nonlinear initial-value problems. In general, iterative numerical methods must be employed in some fashion to solve TPBVPs; where convergence is usually difficult to guarantee a priori.

Given a solution x(t), of the Euler-Lagrange Eq. 6.10a satisfying the appropriate terminal boundary conditions (Eqs. 6.10b through 6.10e, and/or 6.11), we have a *stationary trajectory*. If this stationary trajectory in fact minimizes (or maximizes) J, we have a local *extremal trajectory*. Analogous to maxima-minima theory in ordinary calculus, a curvature test is required to establish sufficiency for a local minimum (or maximum). Functional curvature of $J[x(t) + \delta x(t)]$ is tested using the *second variation* (ref. 1). Since formal sufficiency tests and the second variation play a relatively restricted role in practical applications, we elect not to treat these concepts here. Fortunately, a resourceful analyst can often achieve a high degree of confidence that a candidate trajectory is at least a local minimum, even if a formal sufficiency test proves intractable. The above developments can be usefully extended; if in lieu of the performance index of Eq. 6.1 we have

$$J[x(t)] = \int_{t_o}^{t_f} F[x(t),\ \dot{x}(t),\ \ddot{x}(t),\ \ldots,\ \frac{d^m}{dt^m}(x),\ t]dt \qquad (6.13)$$

Then, for F and x(t) smooth and 2m times, differentiable, it is shown in Reference 2 that the Euler-Lagrange Eq. 6.10a generalizes to

$$[\frac{\partial F}{\partial x}] - \frac{d}{dt}[\frac{\partial F}{\partial \dot{x}}] + \frac{d^2}{dt^2}[\frac{\partial F}{\partial \ddot{x}}] + \ldots + (-1)^m \frac{d^m}{dt^m}[\frac{\partial F}{\partial(\frac{d^m}{dt^m}(x))}] = 0 \qquad (6.14)$$

For the case of fixed end points, the boundary conditions of Eq. 6.11 generalize to

$$\begin{Bmatrix} x(t_o) \\ \dot{x}(t_o) \\ \vdots \\ \frac{d^m}{dt^m}(x)|_{t_o} \end{Bmatrix} = \begin{Bmatrix} x_o \\ \dot{x}_o \\ \vdots \\ x_o^{(m)} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} x(t_f) \\ \dot{x}(t_f) \\ \vdots \\ \frac{d^m}{dt^m}(x)|_{t_f} \end{Bmatrix} = \begin{Bmatrix} x_f \\ \dot{x}_f \\ \vdots \\ x_f^{(m)} \end{Bmatrix} \qquad (6.15)$$

For free or partially free end point problems, the transversality conditions of Eqs. 6.10b through 6.10e can be generalized in a straightforward fashion; hence we omit this developomont for brevity (see Sokolnikoff and Redheffer (ref. 2), for example).

6.1.2 Applications of the Calculus of Variations in Analytical Dynamics

We now digress briefly to point out several important connections with classical mechanics. The first obervation we make is that we must simply make the identifications

$$F = L;\ \{x_1,\ \ldots,\ x_n\} \equiv \{q_1,\ \ldots,\ q_n\} \qquad (6.16)$$

to see that Eq. 6.10a is identical to Lagrange's Eq. 3.64, for the case of $\overline{Q}_j$ = 0 (i.e., for conservative, holonomically constrained dynamical systems). From this comparison, we can infer that *conservative holonomic dynamical systems behave in such a fashion that the integral of the Lagrangian* $L[q_1(t),\ldots,q_n(t), \dot{q}_1(t),\ldots, \dot{q}_n(t),t]$ *has a stationary value along a "true" dynamical motion.* This beautiful truth is a narrow statement of a general result in variational mechanics known as *Hamilton's Principle* (see Section 3.2.6 and ref. 3). As is developed in Meirovitch (ref. 4), for example, the extended *Hamilton's Principle* formulated for fixed end points and a general non-conservative system has the form (c.f. Eq. 3.106)

$$\int_{t_o}^{t_f} (\delta T + \delta W)dt = 0 \tag{6.17}$$

where

δT is the variation in kinetic energy

δW is the virtual work of all external forces

$(\delta W = \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{R}_i$ and $\delta T = \sum_{i=1}^{N} m_i \underline{\dot{R}}_i \cdot \delta \underline{\dot{R}}_i$, for an N particle system)

Equation 6.17 can be directly applied (by expressing δT and δW as a function of n generalized coordinates $q_1, \ldots, q_n$ and their variations) to obtain the differential equations of motion and the corresponding boundary conditions (ref. 4). If terminal variations are admitted, to reflect the obvious truth that an infinity of boundary conditions are possible, Hamilton's Principle in Eq. 6.17 generalizes (Section 3.2.6) to include boundary terms as

$$\int_{t_o}^{t_f} (\delta T + \delta W)dt - \left[\sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \delta q_i \right]_{t_o}^{t_f} = 0 \tag{6.18}$$

This generalized form was called Hamilton's Law of Varying Action by Bailey (ref. 6); it has been shown by Rajan and Junkins (refs. 5, 7) to provide a novel starting point for constructing exact or approximate solutions for the $q_i(t)$ *without* first deriving differential equations. For conservative systems, the virtual work is derivable from a potential energy function $V(q_1, \ldots, q_n, t)$ as

$$\delta W = -\delta V \tag{6.19}$$

Using the usual definition of the Lagrangian

$$L = T - V \tag{6.20}$$

We see that Hamilton's Principle of Eq. 6.17 has the following form for conservative systems

$$\int_{t_o}^{t_f} \delta L dt = 0 \tag{6.21}$$

For the case of holonomic constraints, Pars (ref. 3) shows that Eq. 6.21 can be written in the most popular form

$$\delta S = 0 \tag{6.22}$$

where S is *Hamilton's principal function* or the "action integral"

$$S = \int_{t_o}^{t_f} L[q_1(t), \ldots, q_n(t), \dot{q}_1(t), \ldots, \dot{q}_n(t), t]dt \tag{6.23}$$

Clearly Lagrange's Eq. 3.64, with $\overline{Q} = 0$, follows immediately by comparing Eqs. 6.23, 6.1 and using Eq. 6.10a. Hamilton's principal function of Eq. 6.23 has many elegant properties (ref. 3); for example, the principal function plays a central role in canonical transformation theory, and satisfies an important partial differential equation, the *Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial t} + H(q_1, q_2 \ldots, q_n, p_1, p_2, \ldots, p_n t) = 0 \tag{6.24}$$

where

$$H \equiv \sum_{j=1}^{n} p_j \dot{q}_j - L, \text{ the dynamical } \textit{Hamiltonian} \text{ function} \tag{6.25}$$

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial S}{\partial q_j}, \quad j = 1,2,\ldots,n, \text{ the } \textit{conjugate momenta} \tag{6.26}$$

The most striking significance of finding a general analytical solution

$$S = S[q_1(t_o), \ldots, q_n(t_o), q_1(t), \ldots, q_n(t), t_o, t] \tag{6.27}$$

of the Hamilton-Jacobi Eq. 6.24 is that it reduces finding the solution of all possible two-point-boundary-value-problems to taking the partial derivatives of S as in Eq. 6.26. Unfortunately, only a small fraction of problems submit to analytical solutions for S, since the Hamilton-Jacobi equation is invariably nonlinear and is typically more difficult to solve that the Euler-Lagrange equations. Pars discusses the most significant of the known analytical solutions of Eq. 6.24 for the problems of classical mechanics (ref. 5). As is shown in Section 6.7, the *Hamilton-Jacobi-Bellman-Equation* is quite analogous to the Hamilton-Jacobi Equation of Eq. 6.24 and plays a similar role in optimal control theory.

6.2 FUNCTIONAL OPTIMIZATION WITH DIFFERENTIAL EQUATION CONSTRAINTS

We now turn our attention to development of the fundamental results needed for optimal control of nonlinear dynamical systems. Suppose we have a system whose behavior is described by solving ordinary differential equations. It is usually possible to arrange the system of differential equations in the standard first-order form

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_m, t) \quad , \quad i = 1,2,\ldots,n$$

or, compactly, as the column vector differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{6.28}$$

The $u_i(t)$ are m *control functions* of class C_2 which are to be chosen to maneuver the system described by Eq. 6.28 from the prescribed initial state

$$\mathbf{x}(t_o) = \mathbf{x}_o \quad , \quad t_o \text{ fixed,} \tag{6.29}$$

to a generally unspecified final time t_f and final state $\mathbf{x}(t_f)$ satisfying a nonlinear *manifold* system of n algebraic equations of the form

$$\boldsymbol{\psi}[\mathbf{x}(t_f), t_f] = \mathbf{0}, \tag{6.30}$$

in such a fashion that a performance index of the form

$$J = \phi[\mathbf{x}(t_f), t_f] + \int_{t_o}^{t_f} F[\mathbf{x}(t), \mathbf{u}(t), t]dt \tag{6.31}$$

is minimized. Introducing two n-vectors of *Lagrange Multipliers* $\lambda(t)$ and $\boldsymbol{\alpha}$, we form the *augmented functional* (Refs. 1,8) as

$$J = \phi + \boldsymbol{\alpha}^T\psi + \int_{t_o}^{t_f} [F + \lambda^T(\mathbf{f} - \dot{\mathbf{x}})]dt \tag{6.32}$$

Considering the neighboring trajectory associated with the variations $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$, $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}$, $\tilde{t}_f = t_f + \delta t_f$, we find from the linear part of $\Delta J = \tilde{J} - J$, that the first variation of J is

$$\begin{aligned}\delta J = &\int_{t_o}^{t_f} [\frac{\partial H}{\partial \mathbf{x}} + \dot{\lambda}^T]\delta\mathbf{x}(t)dt \\ &+ \int_{t_o}^{t_f} [\mathbf{f}^T - \dot{\mathbf{x}}^T]\delta\lambda(t)dt \\ &+ \int_{t_o}^{t_f} [\frac{\partial H}{\partial \mathbf{u}}]\delta\mathbf{u}(t)dt \\ &+ [H + \frac{\partial \Phi}{\partial t}]_{t_f} \delta t_f \\ &+ [\frac{\partial \Phi}{\partial \mathbf{x}} - \lambda^T]_{t_f} \delta\mathbf{x}(t_f) = 0\end{aligned} \tag{6.33}$$

with the auxiliary definition of the *control Hamiltonian*

$$H \equiv F[\mathbf{x}(t), \mathbf{u}(t),t] + \lambda^T(t)\ \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t),t], \tag{6.34}$$

and the augmented terminal function

$$\Phi \equiv \phi[\mathbf{x}(t_f), t_f] + \boldsymbol{\alpha}^T\psi[\mathbf{x}(t_f), t_f] \tag{6.35}$$

It follows, by inspection of the variational statement of Eq. 6.33 that the following necessary conditions hold:

$$\dot{\mathbf{x}} = [\frac{\partial H}{\partial \lambda}]^T \equiv \mathbf{f} \tag{6.36}$$

$$\dot{\lambda} = -[\frac{\partial H}{\partial \mathbf{x}}]^T \equiv -[\frac{\partial F}{\partial \mathbf{x}}]^T - [\frac{\partial \mathbf{f}^T}{\partial \mathbf{x}}]^T\lambda \tag{6.37}$$

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} \tag{6.38}$$

$$[H + \frac{\partial \Phi}{\partial t}]_{t_f} \; \delta t_f = 0 \tag{6.39}$$

$$[\frac{\partial \Phi}{\partial \mathbf{x}} - \lambda^T]_{t_f} \; \delta \mathbf{x}(t_f) = 0 \tag{6.40}$$

and, of course, the boundary conditions of Eqs. 6.29 and 6.30. If the final time is fixed, $\delta t_f = 0$ and Eq. 6.39 becomes trivially satisfied. If none of the $\mathbf{x}(t_f)$ are directly specified and the final time is free, conditions of Eqs. 6.39 and 6.40 provide the transversality conditions

$$[H + \frac{\partial \phi}{\partial t} + \alpha^T \frac{\partial \psi}{\partial t}]_{t_f} = 0 \tag{6.41}$$

and

$$\lambda(t_f) = [\frac{\partial \phi}{\partial \mathbf{x}}|_{t_f} + \alpha^T \frac{\partial \psi}{\partial \mathbf{x}}|_{t_f}]^T \tag{6.42}$$

Equation 6.41 is the "stopping condition" used to implicitly determine the optimal final time. Notice Eq. 6.42 determines a *final* boundary condition on the co-state $\lambda(t_f)$ (which must be considered simultaneously with Eq. 6.30 to determine α, whereas, Eq. 6.29 provides an *initial* condition on the state $\mathbf{x}(t_0)$. Thus the boundary conditions on Eqs. 6.36 and 6.37 are *split* and we generally have a two-point boundary-value problem.

The *algebraic* equation provided by Eq. 6.38 is usually simple enough to solve for $\mathbf{u}(t)$ as a function of $\mathbf{x}(t)$, $\lambda(t)$ and thereby eliminate $\mathbf{u}(t)$ from Eqs. 6.36 and 6.37 if desired.

6.3 PONTRYAGIN'S OPTIMAL CONTROL NECESSARY CONDITIONS

In many control applications, the above formulation suffers a serious shortcoming; the requirement (limitation!) that the admissible controls $\mathbf{u}(t)$ be smooth functions with two continuous derivatives immediately precludes on/off controls and the (often necessary) imposition of inequality bounds on $\mathbf{u}(t)$'s magnitude or it's derivatives. During the past two decades, several important generalizations of optimal control formulations have made it possible to routinely solve problems with inequality constraints on both the control and state variables.

If we allow admissible controls which are bounded and only piecewise continuous (in lieu of restricting them to belong to class C_2), the necessary conditions generalize in such a way that the only change from the conditions in Eqs. 6.36 through 6.40 is the replacement of $\{\frac{\partial H}{\partial \mathbf{u}}\} = 0$ by **Pontryagin's Principle: The optimal control u(t) is determined at each instant to render the Hamiltonian a minimum over all admissible control functions.** For example, Pontryagin's Principle (refs. 8, 9, 10), requires for controls of class C_2, that

$$\frac{\partial H}{\partial \mathbf{u}} = 0 \tag{6.43}$$

and

$$[\frac{\partial^2 H}{\partial \mathbf{u}^2}] \equiv \begin{bmatrix} \frac{\partial^2 H}{\partial u_2^2} & \cdots & \frac{\partial^2 H}{\partial u_1 \partial u_m} \\ \vdots & & \vdots \\ \frac{\partial^2 H}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 H}{\partial u_m^2} \end{bmatrix} \tag{6.44}$$

must be positive definite.

Thus Pontryagin's Principle is consistent with the developments of Section 6.2, since it yields Eq. 6.38; the additional necessary condition that $\frac{\partial^2 H}{\partial \mathbf{u}^2}$ be positive definite is also obtained. The most significant utility of Pontryagin's Principle, however, lies in finding optimal controls when the admissible controls *do not* belong to class C_2.

For example, suppose we have an optimal maneuver problem of form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + \mathbf{u} \ , \ \mathbf{x}(t_o) = \mathbf{x}_o, \ \mathbf{x}(t_f) = \mathbf{x}_f \tag{6.45a}$$

$$J = \frac{1}{2}\int_{t_o}^{t_f} (\mathbf{x}^T Q \mathbf{x}) dt \tag{6.45b}$$

$$H = \frac{1}{2}\,\mathbf{x}^T Q\mathbf{x} + \lambda^T[\mathbf{f} + \mathbf{u}] \tag{6.45c}$$

and the admissible control vectors must satisfy the constraints

$$|u_j(t)| \le U_{max_j}\ , \ j = 1,2,\ldots,m \tag{6.46}$$

The necessary conditions for the optimal maneuvers follow as

$$\dot{\mathbf{x}} = \left\{\frac{\partial H}{\partial \lambda}\right\}^T = \mathbf{f} + \mathbf{u} \tag{6.47a}$$

$$\dot{\lambda} = -\left\{\frac{\partial H}{\partial \mathbf{x}}\right\}^T = -\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]^T \lambda - Q\mathbf{x} \tag{6.47b}$$

and Pontryagin's Principle requires the Hamiltonian of Eq. 6.45c to be minimized with respect to $\mathbf{u}(t)$ over all admissible controls satisfying Eq. 6.46. Since H contains $\mathbf{u}$ linearly, we know that the extreme of H with respect to $\mathbf{u}$ must lie on the boundary of the region defined by Eqs. 6.46, we find that the $\lambda_i(t)$ are *switching functions* for each element $u_i(t)$ of the control vector $\mathbf{u}$:

$$\mathbf{u}(t) = \left\{\begin{matrix} S_1\ U_{max_1} \\ S_2\ U_{max_2} \\ \vdots \\ S_m\ U_{max_m} \end{matrix}\right\},\ S_i = -\ \mathrm{sign}\,[\lambda_i(t)], \tag{6.48}$$

except for the unusual event that one or more elements of $\lambda(t)$ vanishes identically for a finite time interval. This latter class of problems is known as *singular* optimal control problems (ref. 13); while the singular optimal control problem is of significant theoretical and some practical interest, we elect not to treat this subject formally here.

6.4 A SMOOTH CONTROL EXAMPLE: A SINGLE-AXIS ROTATIONAL MANEUVER

Consider the case of a rigid body constrained to rotate about a fixed axis, where the equation of motion is given by

$$\ddot{\phi} = \frac{1}{I} L(t) = u(t) \tag{6.49}$$

Suppose we seek a u(t) of Class C_2 which maneuvers the body from the prescribed initial conditions

$$\begin{aligned} \phi(0) &= \phi_0 \\ \dot{\phi}(0) &= \dot{\phi}_0 \end{aligned} \tag{6.50a}$$

to the desired final conditions

$$\begin{aligned} \phi(t_f) &= \phi_f \\ \dot{\phi}(t_f) &= \dot{\phi}_f \end{aligned} \tag{6.50b}$$

In such a fashion that the performance index

$$J = \frac{1}{2} \int_0^T u^2(t)dt \tag{6.51}$$

is minimized. For the moment, we restrict attention to the case that $t_0 = 0$ and $t_f = T$ are fixed. Two methods are considered to derive the optimal maneuver. First we note, that direct substitution of Eq. 6.49 into Eq. 6.51 yields a performance index of the form of Eq. 6.13 with

$$F(\phi,\dot{\phi},\ddot{\phi},t) = \frac{1}{2} \ddot{\phi}^2 \tag{6.52}$$

The generalized Euler-Lagrange equation then follows immediately from Eq. 6.14 as

$$\frac{d^4\phi}{dt^4} = 0 \tag{6.53}$$

which is trivially integrated to obtain the cubic polynomial

$$\phi(t) = a_1 + a_2 t + a_3 t^2 + a_4 t^3 \tag{6.54}$$

as the extremal trajectory.

The four integration constants can be determined as a function of the boundary conditions and the maneuver time T, by simply enforcing the boundary conditions of Eq. 6.50 on Eq. 6.54 and its time derivative. The solution of the resulting four algebraic equations gives

$$a_1 = \phi_0 \tag{6.55a}$$

$$a_2 = \dot{\phi}_o \quad (6.55b)$$

$$a_3 = 3(\phi_f - \phi_o)/T^2 - (2\dot{\phi}_o + \dot{\phi}_f)/T \quad (6.55c)$$

$$a_4 = -2(\phi_f - \phi_o)/T^3 + (\dot{\phi}_o + \dot{\phi}_f)/T^2 \quad (6.55d)$$

Further, it is obvious from Eqs. 6.49 and 6.54 that the optimal control is the linear function of time

$$u(t) = 2a_3 + 6a_4 t \quad (6.56)$$

For example, selecting the numerical values

$$\phi(0) = 0 \quad , \quad \dot{\phi}(0) = 0$$

$$\phi(1) = \pi/2 \quad , \quad \dot{\phi}(1) = 0$$

Leads to

$$u = \ddot{\phi} = 3\pi(1 - 2t)$$

and

$$\dot{\phi} = 3\pi(t - t^2)$$

$$\phi = 3\pi(t^2/2 - t^3/3)$$

the "rest-to-rest" maneuver and control history are shown in Figure 6.1.

Notice, since we admitted only controls of class C_2, we were able to use the generalized Euler-Lagrange Eq. 6.13 in lieu of the Pontryagin-form necessary conditions of Section 6.2. The primary distinction lies in how the constraining differential equation of Eq. 6.49 is enforced in minimizing Eq. 6.51. In the above, the constraint is enforced by simply substituting Eq. 6.49 into the integrand of Eq. 6.51. In the approach of Section 6.2, we enforce the differential equation constraints by using the Lagrange multiplier rule. To illustrate the equivalence in the present transparent example, we re-solve for the optimal maneuver using the approach and notations of Section 6.2.

Before we proceed, it is first necessary to convert Eq. 6.49 to the first order form of Eq. 6.28. This is accomplished by introducing the state variables

$$x_1 \equiv \phi, \; x_2 \equiv \dot{\phi} \quad (6.57)$$

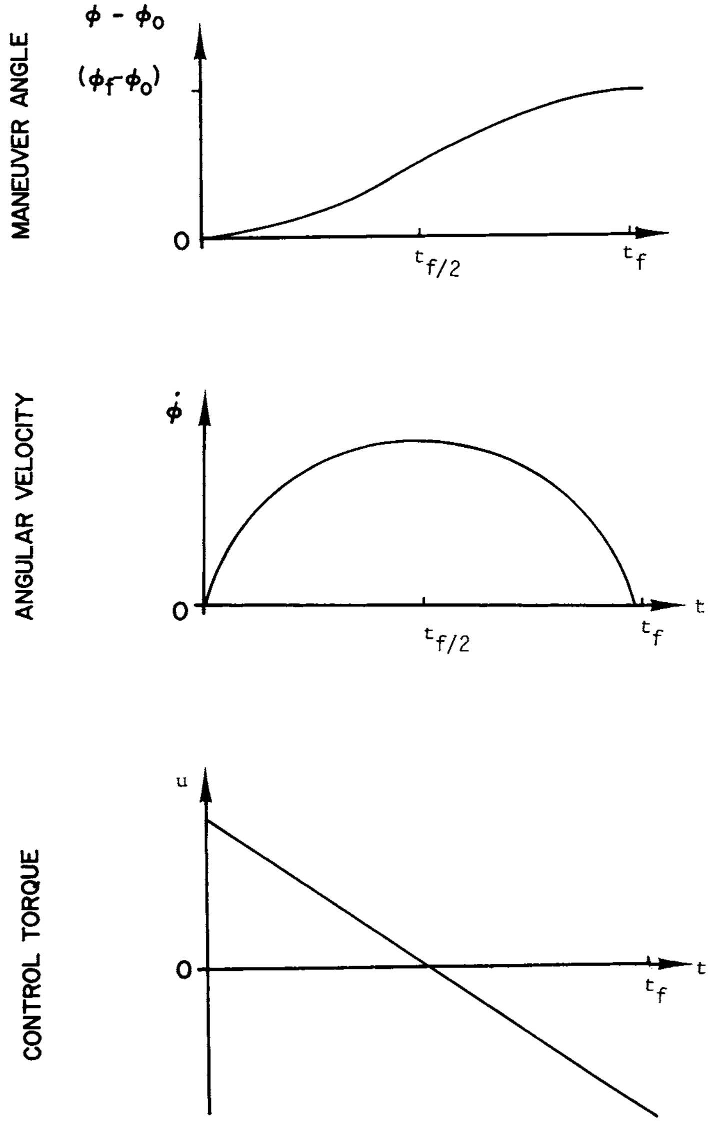


Figure 6.1 Optimal Rest-to-Rest Maneuvers for $\ddot{\phi} = u$,

$$J = \int_{t_0}^{t_f} \frac{1}{2} u^2 \, dt$$

Then the desired equivalent first-order equations follow as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u \tag{6.58}$$

To minimize Eq. 6.51 subject to Eq. 6.58, with the boundary conditions of Eq. 6.50, we first introduce the Hamiltonian of Eq. 6.34 as

$$H = \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 u \tag{6.59}$$

The necessary conditions for the optimal maneuver then follow from Eqs. 6.37 and 6.38 as Eq. 6.58 and

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \tag{6.60a}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \tag{6.60b}$$

$$\frac{\partial H}{\partial u} = 0 = u + \lambda_2 \tag{6.61}$$

The solution of Eq. 6.60 follows as

$$\lambda_1 = b_1 = \text{constant} \tag{6.62a}$$
$$\lambda_2 = -b_1 t + b_2 \tag{6.62b}$$

and Eq. 6.61 provides the optimal control

$$u = -\lambda_2 = -b_2 + b_1 t \tag{6.63}$$

Having u(t), Eq. 6.58 is solved to obtain $x_i(t)$ as

$$x_2 \equiv \dot{\phi} = b_3 - b_2 t + b_1 t^2/2$$
$$x_1 \equiv \phi = b_4 + b_3 t - b_2 t^2/2 + b_1 t^3/6 \tag{6.64}$$

Equation 6.64 is identical to the previous solution of Eq. 6.54 with the obvious relationship between integration constants b_i and a_i. For the case of one constraint, one state variable, and controls of Class C_2, it appears that the multiplier rule slightly increased the algebra. For the cases in which constraints can be eliminated by direct substitution and for controls of Class C_2, this pattern is typical. However, such ideal circumstances represent the minority of applications. Implicit, nonlinear constraints, nonlinear differential equations, and discontinuous controls abound in modern applications; for these cases, the introduction of Lagrange multipliers and the use of Pontryagin-form necessary conditions have been found to be advantageous.

6.4.1 Free Time and Free Final Angle

For the case that the final time T is free, we have from Eq. 6.39 the stopping condition $H(T) = 0$, which, considering Eqs. 6.59 and 6.62, 6.63, 6.64 leads to

$$H(T) = -\frac{2}{T^4}[aT^2 + bT + c] = 0 \tag{6.65}$$

with

$$c = 9(\phi_f - \phi_0)^2 \text{ , } b = -6(\phi_f - \phi_0)(\dot{\phi}_0 + \dot{\phi}_f) \text{ , and } a = \dot{\phi}_0^2 + \dot{\phi}_0\dot{\phi}_f + \dot{\phi}_f^2.$$

Thus, there are three final times for which $H(T) = 0$:

$$T_1^* = \infty \quad , \quad T_{2,3}^* = \frac{3(\phi_f - \phi_0)[\dot{\phi}_0 + \dot{\phi}_f \pm \sqrt{\dot{\phi}_0\dot{\phi}_f}]}{\dot{\phi}_0^2 + \dot{\phi}_0\dot{\phi}_f + \dot{\phi}_f^2} \tag{6.66}$$

$T_1^* = \infty$ corresponds to the global optimal free final time whereas T_2^* and T_3^*, when real, are local maxima or minima of J, at finite times; these have some significance in practical applications. It is obvious by inspection of Eq. 6.66 that, for the rest-to-rest case ($\dot{\phi}_0 = \dot{\phi}_f = 0$) the only zero of $H(T)$ is T $= \infty$; thus the optimum "rest-to-rest" maneuvers, to minimize Eq. 6.51, are carried out very slowly. Furthermore, consider the special class of maneuvers for which $\dot{\phi}_0 = 0$, from Eq. 6.66, we see that the discriminant $\sqrt{\dot{\phi}_0\dot{\phi}_f}$ vanishes and we have the double root

$$T^* = T_2^* = T_3^* = \frac{3(\phi_f - \phi_0)(\dot{\phi}_0 + \dot{\phi}_f)}{\dot{\phi}_0^2 + \dot{\phi}_f^2} = 3(\phi_f - \phi_0)/\dot{\phi}_f \tag{6.67}$$

at which J(T) has an inflection (Figure 6.2)

For $\phi_f = \pi/2$, $\phi_0 = 0$, $\dot{\phi}_f = 1$, we show in Figure 6.3 trajectories for the following three cases

Case 1, $T = T^* = 3(\phi_f - \phi_0)/\dot{\phi}_f = 3\pi/2 = 4.7124$

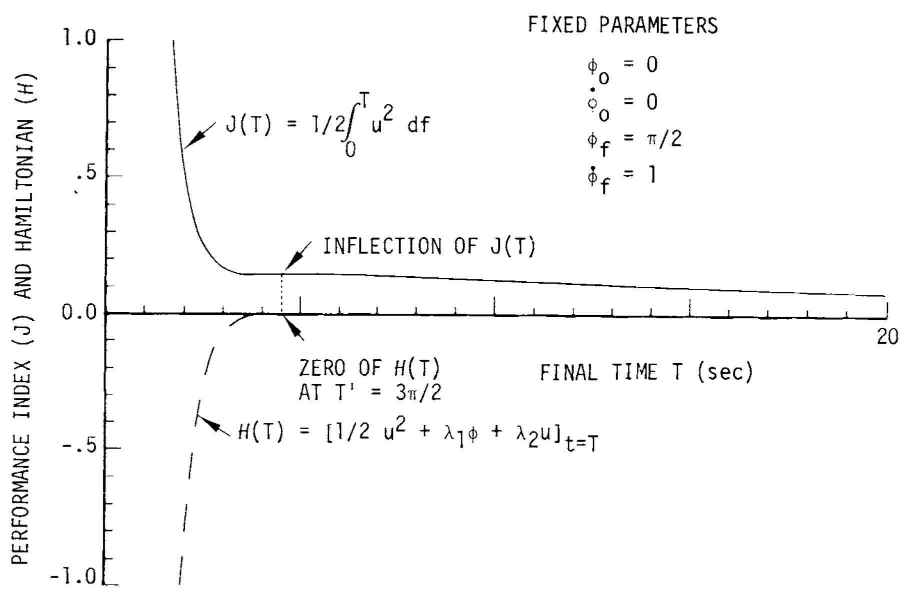


Figure 6.2 Spinup Maneuver: Effect of Final Time Variation Upon the Performance Functional

Case 2, $T = T^* - 1 = 3.7124$ $(T < T^*)$

Case 3, $T = T^* + 1 = 5.7124$ $(T > T^*)$

From Figure 6.3, it is evident that fixing the final time greater than T^* has the undesirable consequence that ϕ initially counter rotates (e.g., Case 3). The performance, as measured by J of Eq. 6.51 is actually slightly less for Case 3 than for Case 1; this example illustrates that counter intuitive and undesirable results sometimes stem from "optimal" control developments.

If *both* initial and final rates ($\dot{\phi}_0$ and $\dot{\phi}_f$) are zero, the inflection of J disappears; it is evident from Eq. 6.66 that the only zero of H(T) occurs as $T \to \infty$; the global minimum of J is zero and is approached as the maneuver time approaches infinity. The optimal control, angular velocity, and angle of rotation profiles (for this rest-to-rest class of maneuvers) are all completely analogous to the maneuver shown in Figure 6.1.

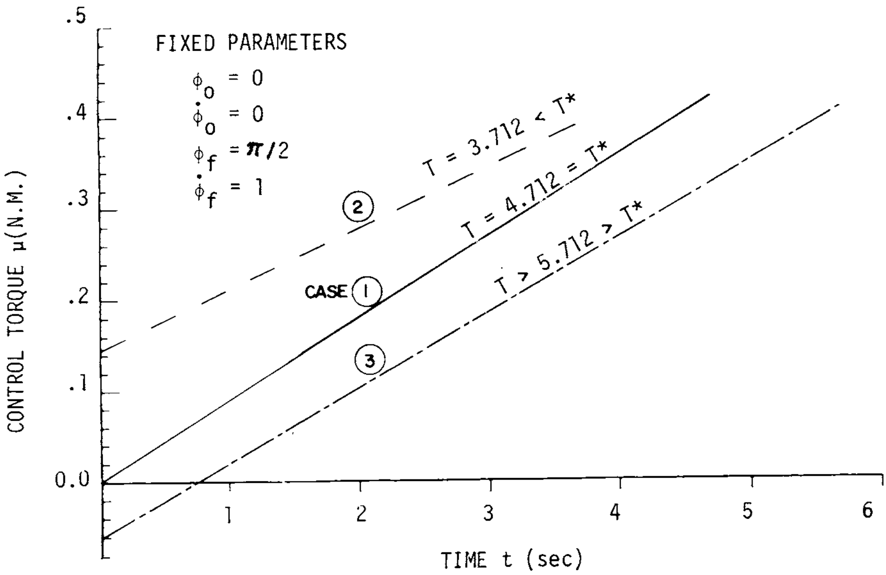


Figure 6.3a Spinup Maneuver: Effect of Final Time Variation Upon Maneuver Profile

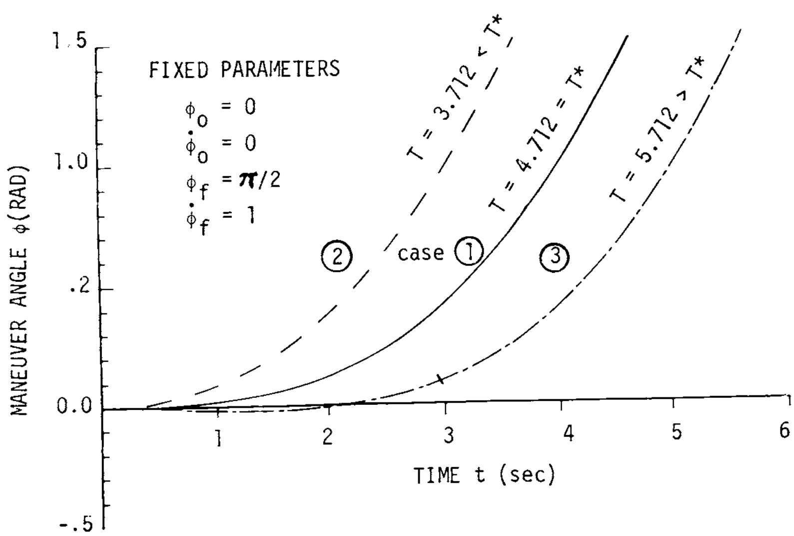


Figure 6.3b Spinup Maneuver: Effect of Final Time Variation Upon Torque History

Another interesting class of maneuvers is the case for which $\dot{\phi}_0 = \dot{\phi}_f \neq 0$. The performance index and Hamiltonian for such a case are shown as a function of maneuver time in Figure 6.4. Notice that $H(T)$ has the three zeros predicted by Eq. 6.66. The two local finite zeros correspond to a local minimum and a local maximum of J(T). Notice that the local minimum $J(T_1^*)$ is *zero*; this corresponds to a zero control *"coast arc"* at constant angular velocity. More generally, if $\dot{\phi}_0 \neq \dot{\phi}_f$, the minimum at T_1^* will be nonzero; but generally T_1^* *is the optimum finite* maneuver time. Furthermore, the region $T_1^* < T < T_3^*$ may be characterized by the "counter rotation" type of maneuver (T_3 = the instant corresponding to the right-most inflection point of $H(T)$). Figures 6.5a and 6.5b show the optimal control and optimal trajectories for five different final time specifications, including the two "free" final times ($T = T_1^*$, Case A2 and $T = T_2^*$, Case A4). Again it is evident that fixing final time arbitrarily is ill-advised since the behavior of the resulting "optimal" maneuver may be decidedly "non-optimal"!

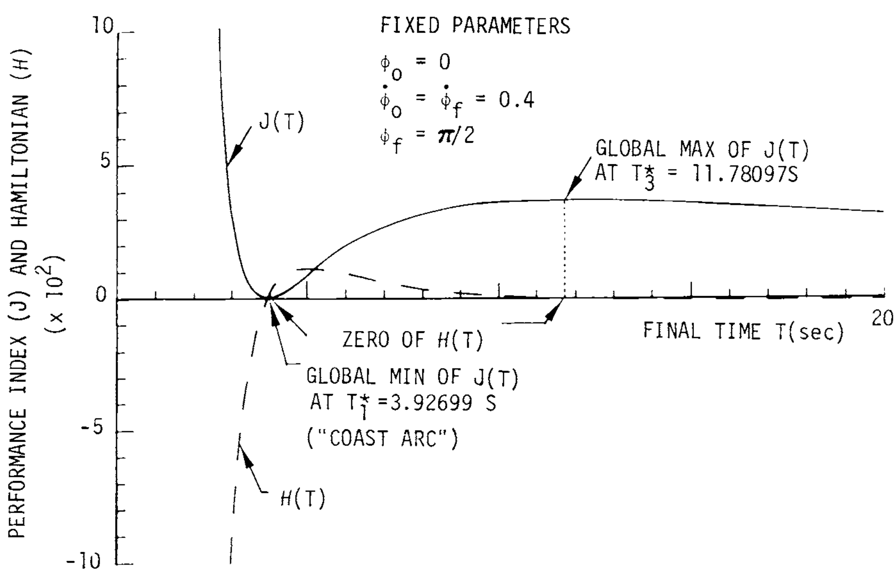


Figure 6.4 **Equal Final Angular Rate Maneuvers: Effect of Final Time Variation Upon Performance**

We are also concerned about artificially fixing other state variables which may, if left "free", lead to a more attractive maneuver. For maneuvering of spin-stabilized vehicles, for example, the exact phase of the spin axis is usually immaterial; an artificial specification of this "missing boundary condition" may prove unwise, as is evident below. Suppose $\phi(T)$ is "free", we infer from Eq. 6.42 that the $\phi(T) = \phi_f$ boundary condition is replaced by the transversality condition

$$\lambda_2(\phi_0,\phi_f,\dot{\phi}_0,\dot{\phi}_f,T) = 0 \tag{6.68}$$

In Figures 6.6a and 6.6b, we display three "fixed ϕ_f" solutions for the optimal control and corresponding trajectories. Case C2 corresponds to $\phi_f^\star$ = 2.3562 which is the "free ϕ_f"; i.e., the only root of Eq. 6.68 using Eqs. 6.62 and 6.67 for ϕ_f. The analytical form of this solution is

$$\phi_f^\star = \phi_0 + \frac{1}{2}(\dot{\phi}_0 + \dot{\phi}_f)T \tag{6.69}$$

Again, we see prescribing ϕ_f to an arbitrary value may have significant undesirable consequences. For the general case if either (or both) of $\dot{\phi}_0$ and $\dot{\phi}_f$ are non-zero, it can be shown that letting both ϕ_f and T be "free" leads to $T = \infty$, $\phi_f^\star = \infty$. Thus, if both final time and angle are free, the maneuver should be performed "as slowly as possible".

6.5 DISCONTINUOUS BOUNDED CONTROL: MINIMUM TIME BANG-BANG MANEUVERS

We consider the same single axis rotation problem of the above discussion (Eq. 6.49 re-arranged to the state space form of Eq. 6.58). We locate the origin of our state space at the desired final state; thus the terminal boundary conditions are

$$\begin{aligned} x(0) &= \phi_0 \ , \quad x(T) = 0 \\ \dot{x}(0) &= \dot{\phi}_0 \ , \quad \dot{x}(T) = 0 \end{aligned} \tag{6.70}$$

We adopt the system of Eq. 6.70 and consider only piecewise continuous controls satisfying

$$|u(t)| \le 1 \tag{6.71}$$

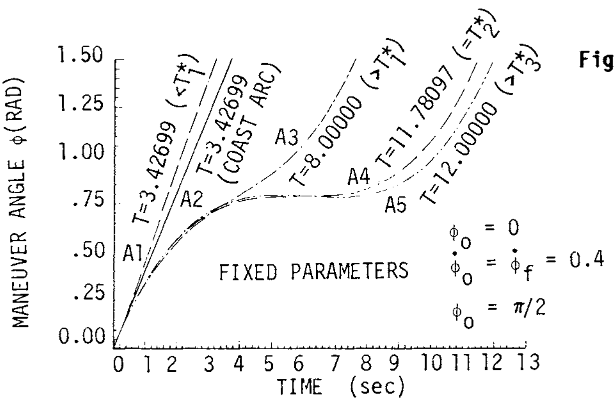


Fig. 6.5a Optimal Maneuver Angle History: Effect of Final Time Variation

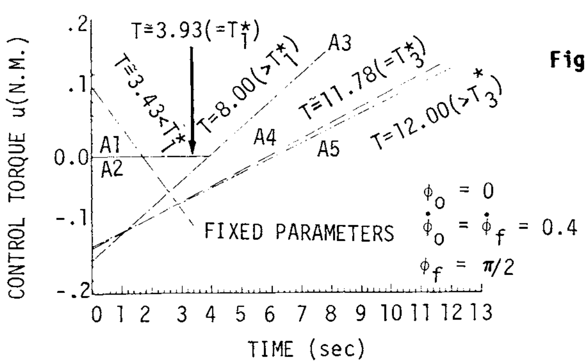


Fig. 6.5b Optimal Control for Equal Final Rate Maneuvers: Effect of Final Time Variation

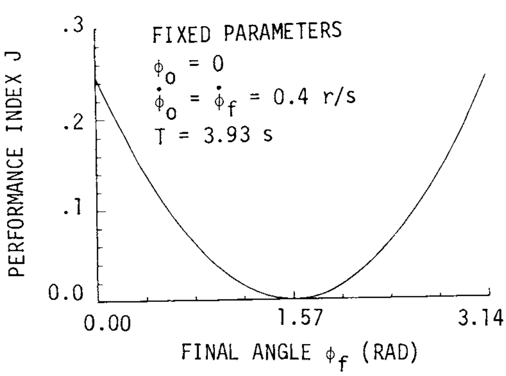


Fig. 6.5c Optimal Trajectories for Equal Final Rate Maneuvers: Effect of Final Angle Variation on Performance

Figure 6.5 Effect of Variations in Final Time and Final Angle Variations Upon the Optimal Maneuver

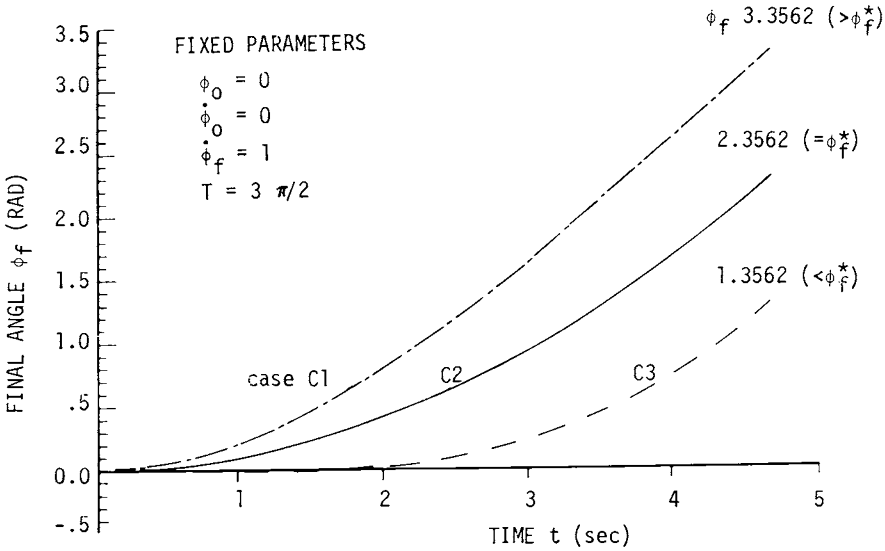


Figure 6.6a Spinup Maneuver: Effect of Final Angle Variation on Trajectory Profiles

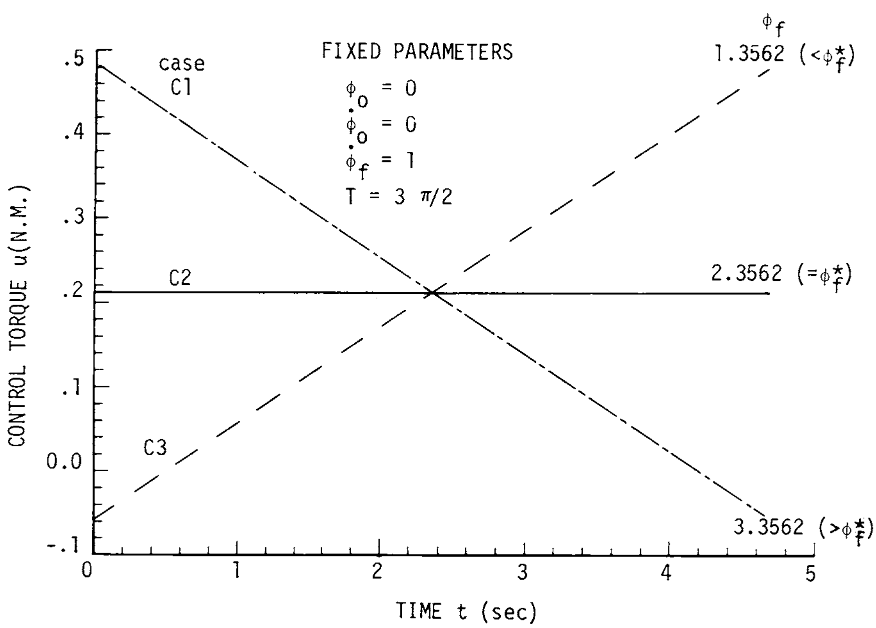


Figure 6.6b Spinup Maneuver: Effect of Final Angle Variation Upon the Optimal Control

Figure 6.6 Optimal Spinup Maneuver: Effect of Final Angle Variation

We seek to minimzie the maneuver "time to go"

$$K\ T = K \int_0^T dt \tag{6.72}$$

where K is a positive scale factor whose arbitrary value will be chosen to accomplish a useful normalization of the co-state variables.

The Hamiltonian functional of Eq. 6.34, for the present case, is

$$H = K + \lambda_1 x_2 + \lambda_2 u \tag{6.73}$$

The necessary conditions of Eqs. 6.36 through 6.38 yield

$$\dot{x}_1 = \frac{\partial H}{\partial \lambda_1} = x_2 \tag{6.74a}$$

$$\dot{x}_2 = \frac{\partial H}{\partial \lambda_2} = u \tag{6.74b}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \quad \rightarrow \lambda_1 = C_1 = \text{constant} \tag{6.74c}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \rightarrow \lambda_2 = C_2 - C_1 t \tag{6.74d}$$

Since Eq. 6.73 is a linear function of u, the optimal control follows from Pontryagin's Principle as

$$\min_u \{H(u)\} \rightarrow u = -\text{sgn}\ \lambda_2 \tag{6.74e}$$

It is evident from Eqs. 6.74d and 6.74e that only one sign change (at most) can occur in u(t), since λ_2 is a linear function of time. It is also evident that, if $\lambda_1(t)$ and $\lambda_2(t)$ are solutions of Eqs. 6.74c and 6.74d, $\alpha\lambda_1(t)$ and $\alpha\lambda_2(t)$ are also solutions (for α = an arbitrary positive constant); since all equations are linear the $\lambda_i(t)$, the optimal control u(t) from Eq. 6.74e is not affected by the choice of α. It is clear that assigning positive values to α generates an infinite family of co-state solutions which generate the same control history.

Since final time is unspecified, we have the formal stopping condition $H(T) = 0$ from Eq. 6.39. Imposing $H(T) = 0$ on Eq. 6.69, we have

$$H(T) = K + \lambda_1(T)x_2(T) + \lambda_2(T)u(T) = 0 \tag{6.75}$$

from which we deduce that the α-scaling on λ_i dictates a specific K value:

$$K = -[\lambda_1(T)x_2(T) + \lambda_2(T)u(T)] \tag{6.76}$$

Since an infinity of linearly scaled co-states generate the same control, we take advantage of this truth to scale initial conditions on the λ's so that the initial co-states lie on the unit circle

$$\lambda_1^2(0) + \lambda_2^2(0) = 1 \tag{6.77}$$

or, alternatively, we can define the complete family of trajectories by introducing an initial phase γ such that

$$C_1 = \lambda_1(0) = \cos\gamma$$
$$C_2 = \lambda_2(0) = \sin\gamma \tag{6.78}$$

where $0 \le \gamma < 360°$. Thus, the presence of the arbitrary scale factor allows us to reduce the number of unknown co-state boundary conditions from 2 to 1 and thereby *reduce the TPBVP problem to a bounded one variable search.*

From Eq. 6.78, it follows that the optimal control of Eq. 6.74e is given by

$$u = -\mathrm{sgn}[\sin\gamma - t\cos\gamma] \tag{6.79}$$

and, it is evident that the switch times (t_s) are related to the γ-values as follows

$$t_s = \tan\gamma \tag{6.80}$$

Since each γ choice generates a minimal time trajectory "going somewhere"; we seek the particular initial γ which generates the trajectory from the initial conditions ($x_1(0) = \phi_0$, $x_2(0) = \dot{\phi}_0$) to the origin ($x_1(T) = x_2(T) = 0$). We could proceed to iterate solutions of Eq. 6.74 for various γ choices to find the γ value (and corresponding final time, T) resulting in a trajectory $[x_1(t), x_2(t)]$ which passes through the origin. In Section 8.2, we consider a nonlinear bounded control problem in which a similar one-parameter numerical search is required. However by using a geometric approach and thereby avoiding the necessity of numerical iterative technqiues, we can gain important insight and *construct an analytical solution in the present example.*

Since $u = \pm 1$ from Eqs. 6.74e or 6.79, we can easily construct the global family of $x_1(t)$, $x_2(t)$ phase plane trajectories from integration of Eqs. 6.74a and 6.74b. For the $u = +1$ trajectories, we obtain

$$x_2 = c_1 + t \quad , \quad c_1 = x_2(0) = \dot{\phi}_0$$
$$x_1 = c_2 + c_1 t + t^2/2 \quad , \quad c_2 = x_1(0) = \phi_0$$

or, eliminating time, we obtain the equation for the phase plane trajectories as

$$x_1 = \frac{1}{2} x_2^2 + c_2 - c_1^2/2 \tag{6.81}$$

This family of positive control trajectories are the parabolas shown in Figure 6.7. Note that the *only positive control trajectory to the origin* is $x_1 = \frac{1}{2} x_2^2$.

For the $u = -1$ trajectories, we obtain

$$x_2 = b_1 - t \quad , \quad b_1 = x_2(0) = \dot{\phi}_0$$
$$x_1 = b_2 + b_1 t - t^2/2 \quad , \quad b_2 = x_1(0) = \phi_0$$

or

$$x_1 = -\frac{1}{2} x_2^2 + b_2 + \frac{b_1^2}{2}$$

The family of *negative torque* trajectories are the parabolas shown in Figure 6.8. Note that the *only negative control trajectory to the origin* is $x_1 = -\frac{1}{2} x_2^2$.

Thus we conclude by analysis of Figures 6.7 and 6.8 that either the curves $x_1 = -\frac{1}{2} x_2^2$ (second quadrant) or $x_1 = +\frac{1}{2} x_2^2$ (fourth quadrant) must be the final arc of each extremal satisfying the terminal boundary conditions. Since we already know, from Eq. 6.79, that at most one control switch occurs, it is evident that the control (in the second quadrant) switches from positive to negative when the positive torque trajectories intersect the *switching curve* $x_1 = -\frac{1}{2} x_2^2$, whereas the control (in the fourth quadrant) switches from

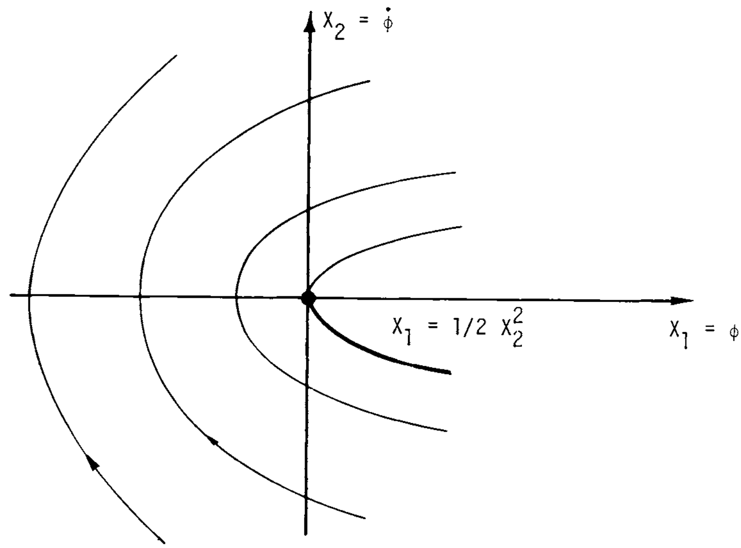


Figure 6.7 Positive Torque Trajectories

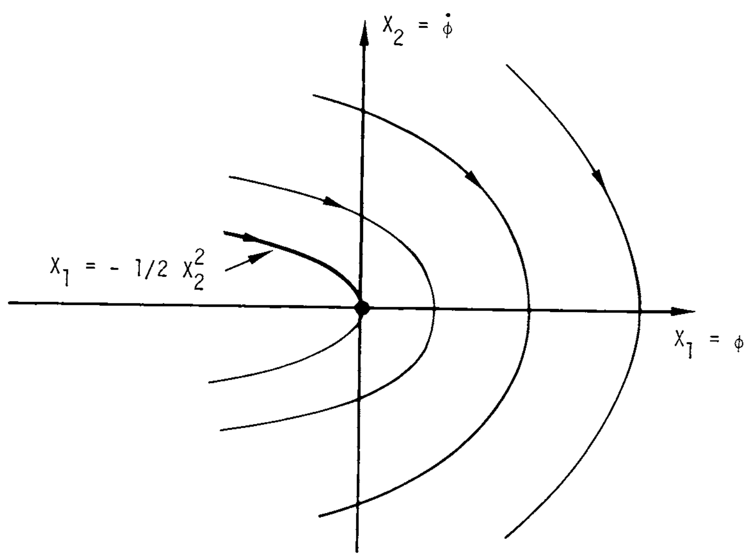


Figure 6.8 Negative Torque Trajectories

negative to positive when the initially negative torque trajectories intersect the *switching curve* $x_1 = + \frac{1}{2} x_2^2$.

The global portrait of time optimal "bang-bang" trajectories are shown in Figure 6.9. This simplest bang-bang control problem arises in many fields and represents the simplest special case of many multi-dimensional, nonlinear problems.

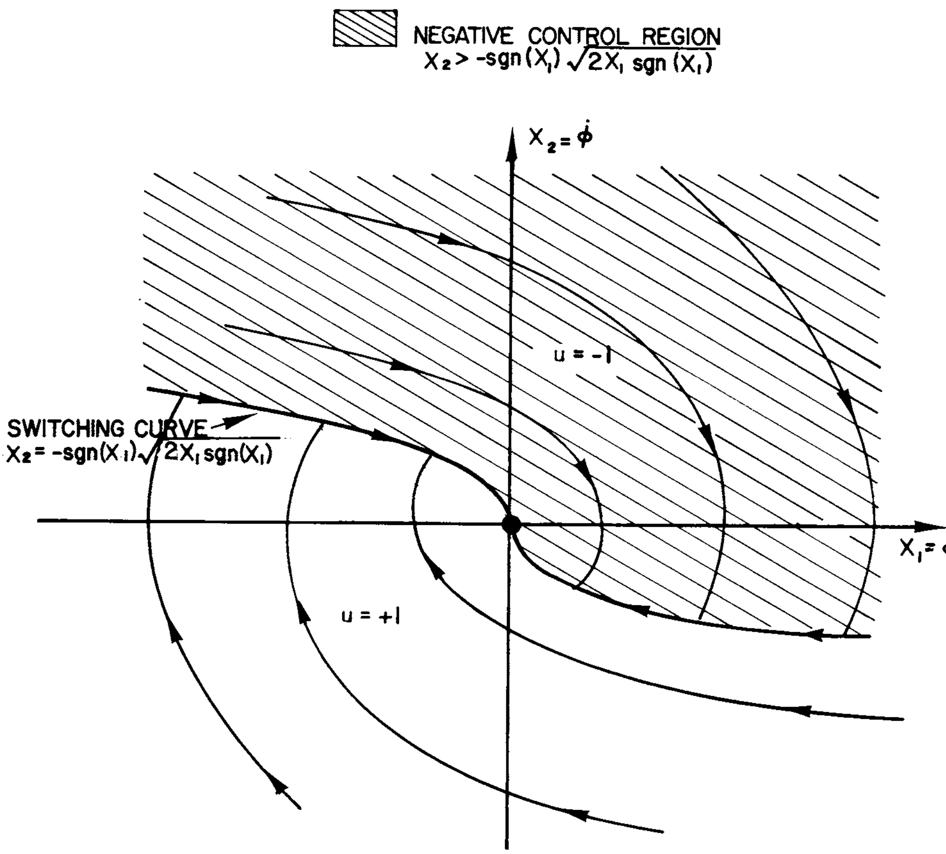


Figure 6.9 Switching Curves for Bang-Bang Minimum Time Attitude Maneuvers

6.6 DERIVATIVE PENALTY PERFORMANCE INDICES

We again consider the second order dynamical system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u \tag{6.58}$$

In Section 6.4, we minimized $J = \int_0^T \frac{1}{2} u^2 dt$ over all continuous functions. As a result, we found the optimal control $u(0) = -\lambda_2(t) = -C_2+C_1 t$ to be a linear function of time. One (often undesirable) consequence is that initial and final controls require a *jump discontinuity*) to begin or return to zero. Upon generalizing the model to include flexural degrees of freedom, the terminal jump discontinuities in the control profile become highly unattractive (see Chapters 9 and 10 for flexible body results). Jump discontinuities can excite the higher mode flexural degrees of freedom because of the high frequency content of the control torque history. In addition, the resulting control profiles are relatively sensitive to modeling errors and may therefore prove difficult to implement.

In order to develop the control-rate penalty techniques we now consider minimizing the performance index

$$J = \frac{1}{2} \int_0^{t_f} [w^2 u^2(t) + \dot{u}^2(t)] dt \tag{6.82}$$

where $u(t)$ is assumed to have two continuous time derivatives. This class of performance indices was first studied by Anderson and Moore in ref. 17. In Eq. 6.82, w is a real positive weight which permits a trade-off between penalizing amplitude (u^2) versus smoothness ($\dot{u}^2$) of the control. We can easily convert Eq. 6.82 into standard form for Pontryagin's Principle by simply introducing a new "state variable" $x_3 \equiv u$ and defining the new control variable $U \equiv \dot{u}$. Thus we seek to minimize

$$J = \frac{1}{2} \int_0^{t_f} [w^2 x_3^2(t) + U^2(t)] dt \tag{6.83}$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= U \end{aligned} \tag{6.84}$$

The Hamiltonian functional is

$$H = \frac{1}{2}\,(w^2\,x_3^2 + U^2) + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3\,U \tag{6.85}$$

and Pontryagin's necessary conditions follow as

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \\ \dot{\lambda}_3 &= -\frac{\partial H}{\partial x_3} = -\lambda_2 - w^2 x_3 \end{aligned} \tag{6.86}$$

Pontryagin's principle requires that H be minimized over admissible U. If U(t) is taken from the set of smooth unbounded functions with two continuous derivatives, we minimize H by requiring

$$\frac{\partial H}{\partial U} = 0 = U + \lambda_3 \rightarrow U(t) = -\lambda_3(t) \tag{6.87}$$

In terms of the original state and control variables $x_1 = \phi$, $x_2 = \dot{\phi}$, and u(t) (i.e., replacing x_3 by u and U by $\dot{u}$), Eqs. 6.84, 6.86 can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{6.88}$$

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \end{aligned} \tag{6.89}$$

$$\ddot{u} - w^2 u = \lambda_2 \tag{6.90}$$

We observe that minimizing H with respect to u yields a conventional second order Euler-Lagrange *differential* equation (due to the presence of the $\ddot{u}$ term) in lieu of the usual *algebraic* equation ($\frac{\partial H}{\partial u} = 0$), which determines u(t). Equations 6.88 through 6.90 constitute a sixth order system of linear differential equations. In addition to the four *state* boundary conditions

$$x_1(0) = \phi(0) = \phi_0$$
$$x_2(0) = \dot\phi(0) = \dot\phi_0$$
$$x_1(T) = \phi(T) = \phi_f \qquad (6.91)$$
$$x_2(T) = \dot\phi(T) = \dot\phi_f$$

We are free to prescribe two *control* boundary conditions for Eq. 6.90. Since we seek to eliminate the terminal control discontinuities, we require that the control be zero initially and vanish upon completion of the maneuver; thus, we also impose the following control boundary conditions

$$u(0) = 0$$
$$u(T) = 0 \qquad (6.92)$$

The analytical solution of Eqs. 6.88 through 6.90, subject to the boundary conditions of Eqs. 6.91 and 6.92, is given by

$$\phi(t) = x_1(t) = w^{-2}(k_1e^{wT} + k_2e^{-wT}) + w^{-2}(k_3t^3/6 - k_4t^2/2)$$
$$+ k_5t + k_6 \qquad (6.93)$$
$$u(t) = k_1e^{wT} + k_2e^{-\omega T} + \omega^{-2}(k_3t - k_4) \qquad (6.94)$$

where the six constants are determined to satisfy the six boundary conditions of Eq. 6.91; after considerable algebra we have

$$\Delta = [e^{wT}(wT - 2) + wT + 2][(w^2T^2 - 6\ wT + 12)e^{wT} - w^2T^2 - 6\ wT - 12]$$
$$(6.95a)$$

$$k_1 = -2w[\{[(2\dot\phi_f+\dot\phi_0)T^2 + 3(\phi_0-\phi_f)T]w^2 + 6(\phi_f-\phi_0-\dot\phi_fT)2 + 6(\dot\phi_f-\dot\phi_0)\}e^{wT}$$
$$+ [(\dot\phi_f + 2\dot\phi_0)T^2 + 3(\phi_0 - \phi_f)T]w^2$$
$$+ 6[\dot\phi_0T - \phi_f + \phi_0)w + 6(\dot\phi_0 - \dot\phi_f)]/\Delta \qquad (6.95b)$$

$$k_2 = 2we^{wT}[\{[(2\dot\phi_0 + \dot\phi_f)T^2 + 3(\phi_0 - \phi_f)T]w^2$$
$$+ 6(-\dot\phi_0T + \phi_f - \phi_0)w + 6(\dot\phi_0 - \dot\phi_f)\}e^{wT}$$
$$+ [(\dot\phi_0 + 2\dot\phi_f)T^2 + 3(\phi_0 - \phi_f)T]w^2$$
$$+ 6(\dot\phi_fT - \phi_f + \phi_0)w - 6(\dot\phi_0 - \dot\phi_f)]/\Delta \qquad (6.95c)$$

$$k_3 = \frac{6w^4[(\dot{\phi}_f + \dot{\phi}_0)T - 2(\phi_f - \phi_0)](e^{wT} - 1)}{T[(T^2w^2 - 6\ wT + 12)e^{2T} - T^2w^2 - 6Tw - 12]} \tag{6.95d}$$

$$\begin{aligned} k_4 = {} & 2w^3[\{[(\dot{\phi}_f + 2\dot{\phi}_0)T^2 + 3(\phi_0 - \phi_f)T]w^2 \\ & + 6\ w(-\dot{\phi}_0T + \phi_f - \phi_0) - 6(\dot{\phi}_f - \dot{\phi}_0)\}e^{2wT} \\ & + 12(\phi_0 - \phi_f + \dot{\phi}T)we^{wT} \\ & - w^2[(\dot{\phi}_f + 2\dot{\phi}_0)T^2 + 3(\phi_0 - \phi_f)T] \\ & + 6w(\phi_f - \phi_0 - \dot{\phi}_0T) + 6(\dot{\phi}_f - \dot{\phi}_0)]/\Delta \end{aligned} \tag{6.95e}$$

$$\begin{aligned} k_5 = {} & [\{\dot{\phi}_0T^3w^2 + [(2\dot{\phi}_f - 4\dot{\phi}_0)T^2 + 6(\phi_0 - \phi_f)T]w^2 \\ & + 12w(\phi_f - \phi_0 + \dot{\phi}_0T) - 12(\dot{\phi}_f + \dot{\phi}_0)\}e^{2wT} \\ & + \{[(8\dot{\phi}_f - 4\dot{\phi}_0)T^2 + 12(\phi_0 - \phi_f)T]w^2 + 24(\dot{\phi}_f + \dot{\phi}_0)\}e^{wT} \\ & - w^3T^2\dot{\phi}_0 + [(2\dot{\phi}_f - 4\dot{\phi}_0)T^2 + 6(\phi_0 - \phi_f)T]w^2 \\ & + 12w(\phi_0 - \phi_f - \dot{\phi}_0T) - 12(\dot{\phi}_0 + \dot{\phi}_f)]/\Delta \end{aligned} \tag{6.95f}$$

$$\begin{aligned} k_6 = {} & [\{w^4T^3\phi_0 - 8w^3T^2\phi_0 + [(-2\phi_f - 4\dot{\phi}_0)T^2 + (6\phi_f + 18\phi_0)T]w^2 \\ & + 12w(\dot{\phi}_0T - \phi_f - \phi_0) + 12(\dot{\phi}_f - \dot{\phi}_0)\}e^{2wT} \\ & + \{24w(\phi_0 + \phi_f - \dot{\phi}_fT) - 8\ w^3T^2\phi_0\}e^{2T} \\ & - w^4T^3\phi_0 - 8w^3T^2\phi_0 + \{(2\dot{\phi}_f + 4\dot{\phi}_0)T^2 - 6T(\phi_f + 3\phi_0)\}w^2 \\ & + 12w(\dot{\phi}_0T - \phi_f - \phi_0) - 12(\dot{\phi}_f - \dot{\phi}_0)]/\Delta \end{aligned} \tag{6.95g}$$

For particular missions, the structure of the resulting control profile can be studied parametrically as w assumes a range of values. Of particular interest is the special case solution where $w = 0$. As a result of setting $w = 0$ in Eq. 6.93), the optimal control solution for $\phi(t)$ simplifies to a fifth-order polynominal in time, as follows:

$$\phi(t) = \phi_0 + \dot{\phi}_0t + \frac{1}{6}\left(-\frac{b_2}{2}T + \frac{b_1}{6}T^2\right)t^3 + \frac{b_2}{24}t^4 - \frac{b_1}{120}t^5 \tag{6.96a}$$

$$\dot{\phi}(t) = \dot{\phi}_0 + \frac{1}{2}\left(-\frac{b_2}{2}T + \frac{b_1}{6}T^2\right)t^2 + \frac{1}{6}b_2t^3 - \frac{1}{24}b_1t^4 \tag{6.96b}$$

$$u(t) = \left(-\frac{1}{2}b_2T + \frac{b_1}{6}T^2\right)t + \frac{1}{2}b_2t^2 - \frac{1}{6}b_1t^3 \tag{6.96c}$$

with

$$b_1 = -\frac{120}{T^5}\left[6(\phi_f - \phi_0 - \dot{\phi}_0 T) - 3T(\dot{\phi}_f - \dot{\phi}_0)\right] \tag{6.96d}$$

$$b_2 = -\frac{24}{T^4}\left[15(\phi_f - \phi_0 - \dot{\phi}_0 T) - 7T(\dot{\phi}_f - \dot{\phi}_0)\right] \tag{6.96e}$$

In Figures 6.10 through 6.14, we study parametrically the behavior of the above optimal trajectories. We note, comparing Figures 6.10, 6.11, and 6.4 that the addition of the penalty on $\dot{u}^2$ does not change the qualitative behavior of the Hamiltonian and performance index functions. However, we see in Figure 6.12a that in lieu of the initial and final jump discontinuities evident in Figure 6.1, the control u(t) begins and ends at zero. We note, in passing, that we could easily impose penalties on $\ddot{u}^2$, and higher derivatives, and in so doing, allow correspondingly higher order terminal boundary condition osculations or other terminal constraints on u(t). Notice in Figure 6.12b the fact that the variations in w have a much more pronounced effect on u(t) than $\phi(t)$. Qualitatively, it is clear that controls which begin and end at zero are more attractive for many maneuvers, especially those involving large flexible spacecraft, because the high frequency content of the control torque is greatly reduced.

In Figure 6.13, we consider effects of varying final maneuver angle for this class of maneuvers. In particular, the condition $\lambda_2(\phi_0,\phi_f,\dot{\phi}_0,\dot{\phi}_f,T) = 0$ gives the optimal "free" final angle $\phi_f^\star$. As before (Figure 6.6), we find a coast (zero control) optimal trajectory associated with this free final angle. As is evident in Figure 6.13, the performance index has a well-defined minimum about $J(\phi_f^\star) = 0$. Figures 6.14a and 6.14b show the trajectories corresponding to $\phi_f > \phi_f^\star$, $\phi_f = \phi_f^\star$, $\phi < \phi_f^\star$.

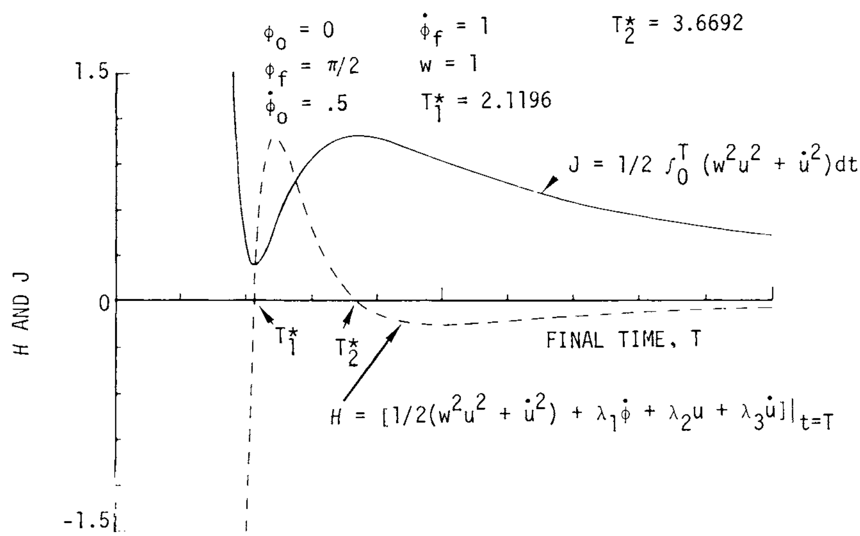
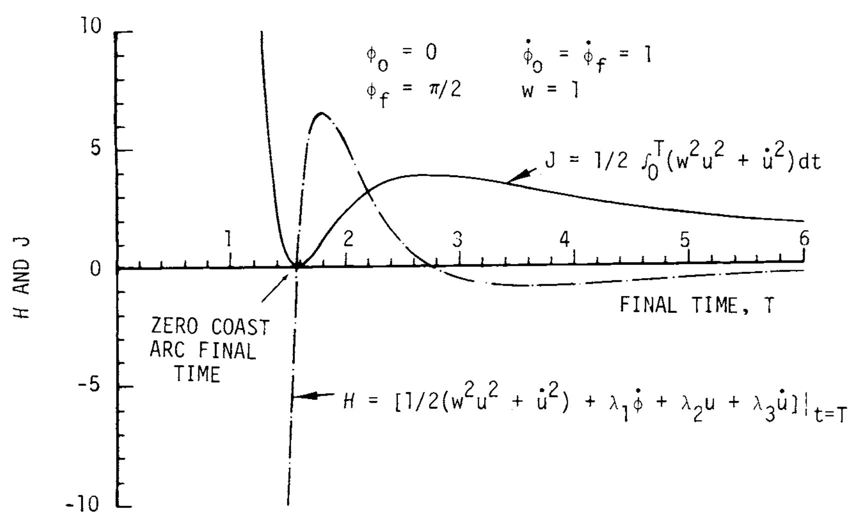


Figure 6.10 **The Performance Index and Hamiltonian as a Function of Final Time for** $J = \frac{1}{2}\int_0^T (w^2u^2 + \dot{u}^2)dt$



Figures 6.11 **The Existence of a Zero Control Coast Arc for Equal Initial and Final Angular Velocity**

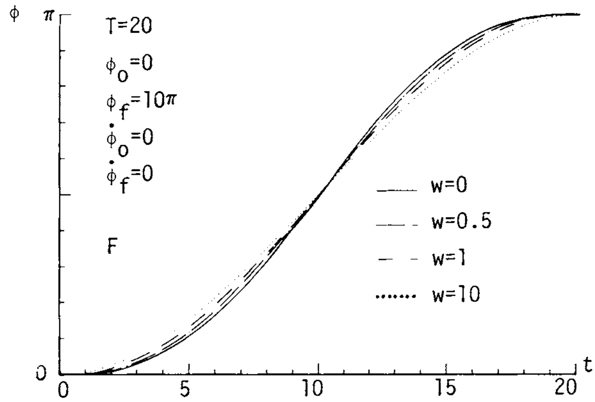


Figure 6.12a Maneuver Angle Versus Time for Four Weights (w) in

$$J = \frac{1}{2} \int_0^T (w^2u^2+\dot{u}^2)dt$$

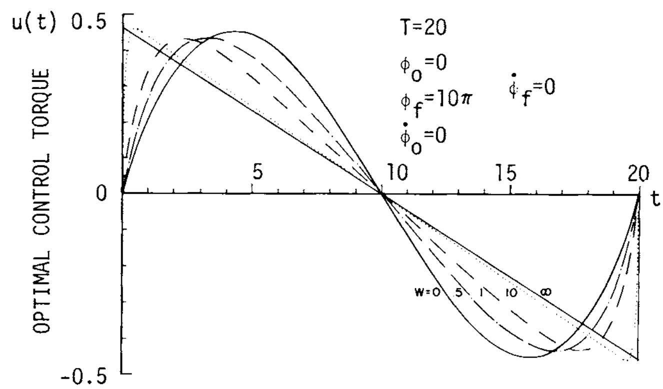


Figure 6.12b Optimal Control Torques Versus Time for Four Weights in

$$J = \frac{1}{2} \int_0^T (w^2u^2+\dot{u}^2)dt$$

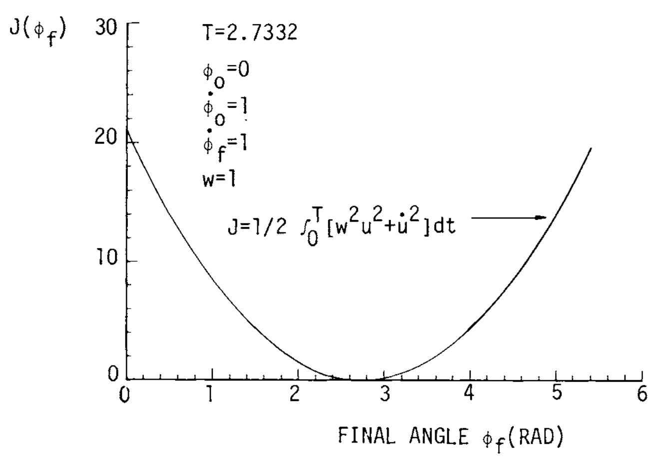


Figure 6.12c Performance

$$J = \frac{1}{2} \int_0^T (w^2u^2+\dot{u}^2)dt$$

Versus Final Maneuver Angle

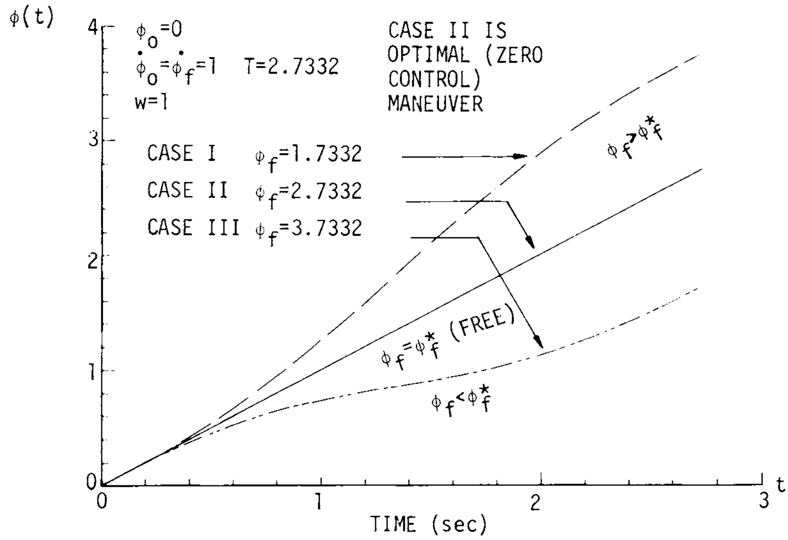


Figure 6.14a Effect of Final Angle Specification Versus Optimal Control

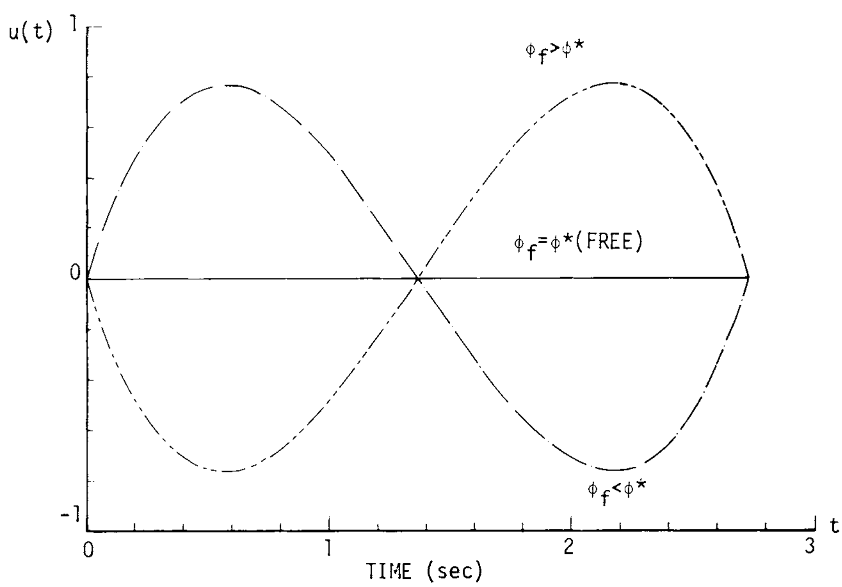


Figure 6.14b Effect of Final Angle Specification Upon the Optimal Maneuver Angle History

6.7 OPTIMAL FEEDBACK CONTROL

6.7.1 Motivation for Feedback Control

The formulations of the foregoing developments naturally lead to *open loop* optimal controls which are designed to calculate an optimal trajectory from a prescribed initial state to a prescribed final state. Such controls can be precomputed, under the assumption of a perfectly modeled system and perfectly known initial conditions. However, upon application of open loop controls to a real system, even small modeling errors and initial state errors, result in usually unacceptable divergence of the actual system's behavior from the optimal trajectory. In many cases *perturbation feedback controls* need to be superimposed (a 'la "guidance". in rocket flight path control) to continually correct for model errors and other disturbances. In some cases, we will see that it is possible to formulate optimal controls so that they can be calculated directly in a *terminal controller* feedback form

$$u = f[x(t) - x(t_f),\ t_f - t] \tag{6.97}$$

in which the optimal control is a function of instantaneous displacement from the desired final state and the "time to go" $\tau = t_f - t$. Such controls are of enormous practical impact, since we are in essence, continuously reinitializing the control calculations with current best estimates of $x(t)$ which can be updated continuously based upon measurements (and thereby counteract the accumulation of ever-present errors due to an erroneous model and other disturbances).

For motivation, we first consider a linear scalar problem (ref. 12). The system is described by the differential equation

$$\dot{x} = -\frac{1}{2} x + u \quad , \quad x(t_o) = x_o \tag{6.98}$$

We seek $u(t)$ to minimize

$$J = \frac{1}{2} s\, x^2(t_f) + \int_{t_o}^{t_f} [x^2 + \frac{1}{2} u^2]dt \quad , \quad s \text{ is a positive constant} \tag{6.99}$$

The Hamiltonian functional of Eq. 6.34 is

$$H = x^2 + \frac{1}{2} u^2 + \lambda \left(- \frac{1}{2} x + u\right) \tag{6.100}$$

The necessary conditions of Eqs. 6.36 through 6.40 yield, in addition to Eq. 6.98, the following equations

$$\dot{\lambda} = \frac{1}{2} \lambda - 2x \quad , \quad \lambda(t_f) = s\, x\,(t_f) \tag{6.101}$$

$$u = -\lambda \tag{6.102}$$

Since we seek a feedback control, it is reasonable to make the independent variable "time to go"

$$\tau = t_f - t \tag{6.103}$$

and to seek $\lambda(t) = -u(t)$ as a function of x and τ. As the simplest type of (proportional) feedback, we seek an optimal *time varying gain* $p(\tau)$ such that

$$\lambda(t) = -u = f(\tau,x) = p(\tau)x \quad , \quad p(0) = s \tag{6.104}$$

Making the variable change (Eq. 6.103) and substituting Eq. 6.104, Eqs. 6.98, 6.101, and 6.102, we have from Eq. 6.98

$$\frac{dx}{d\tau} = \frac{1}{2} x + p(\tau)x \tag{6.105a}$$

and from Eq. 6.101

$$\frac{dp}{d\tau} x + p \frac{dx}{d\tau} = - \frac{1}{2} p\, x + 2\, x \tag{6.105b}$$

and substitution of Eq. 6.105a into Eq. 6.105b yields

$$\left(\frac{dp}{d\tau} + p^2 + p - 2\right)x(\tau) = 0 \tag{6.106}$$

Since $x(\tau) \neq 0$, we find that the optimal feedback gain $p(\tau)$ is the solution of the scalar *Riccati equation*

$$\frac{dp}{d\tau} = - p - p^2 + 2 \tag{6.107}$$

While this equation is nonlinear, it does have an analytical solution. To introduce an important transformation, let us consider a more general version of the Riccati equation

$$\frac{dp}{d\tau} = a\, p + b\, p^2 + c \tag{6.108}$$

We introduce a new function $\alpha(t)$ defined implicitely by

$$p = \frac{-\frac{d\alpha}{d\tau}}{b\alpha} \tag{6.109}$$

It can be verified by direct substitution that the transformation of Eq. 6.109 maps Eq. 6.108 into the linear second-order differential equation

$$\frac{d^2\alpha}{d\tau^2} + g\,\frac{d\alpha}{d\tau} + e\,\alpha = 0 \tag{6.110}$$

with the constant coefficients determined by $g = -a$, $e = bc$. From Eq. 6.107, we have $a = b = -1$, $c = 2$, so that Eq. 6.110 can be written as

$$\frac{d^2\alpha}{d\tau^2} + \frac{d\alpha}{d\tau} - 2\alpha = 0 \tag{6.111}$$

with the general solution

$$\alpha(\tau) = c_1 e^{-2\tau} + c_2 e^{\tau} \tag{6.112}$$

Substitution of Eq. 6.112 into Eq. 6.109 with $b = -1$ leads to

$$p(\tau) = \frac{-2c_1 e^{-2\tau} + c_2 e^{\tau}}{c_1 e^{-2\tau} + c_2 e^{\tau}} \quad , \quad p(0) = s \tag{6.113}$$

In view of the fact that $p(\tau)$ has only one boundary condition $p(0) = s$, and by inspection of Eq. 6.113, it is evident that only the ratio $\beta = c_2/c_1$ of integration constants is of practical consequence; it follows that the analytical solution for the feedback gain is then

$$p(\tau) = \frac{\beta - 2\,e^{-3\tau}}{\beta + e^{-3\tau}} \tag{6.114}$$

with

$$\beta = -(s + 2)/(s - 1) \tag{6.115}$$

We note in passing that the apparent singularity at $s = 1$ in Eq. 6.115 can easily be eliminated by using $\delta = \beta^{-1} = c_1/c_2$ as the free constant and rewriting Eq. 6.114 as

$$p(\tau) = \frac{1 - 2\delta\, e^{-3\tau}}{1 + \delta\, e^{-3\tau}} \quad , \quad \delta = - \frac{s - 1}{s + 2} \tag{6.116}$$

It is obvious from Eqs. 6.114 through 6.116 that the following limits hold

$$\lim_{\substack{s \to \infty \\ \tau \to \infty}} (p(\tau)) = 1 \tag{6.117a}$$

$$\lim_{s \to \infty} (\beta) = \lim_{s \to \infty} (\delta) = -1 \tag{6.117b}$$

Since $\tau \equiv t_f - t$, for finite t, it is also clear that $\lim\limits_{t_f \to \infty} \{p(\tau)\} = 1$. The infinite time limit of $p(\tau)$ is known widely as the *steady state gain.* Note that the steady state gain can also be determined without solving the differential equation of Eq. 6.106; rather, we can set $\frac{dp}{d\tau} = 0$ and solve the *algebraic Riccati equation*

$$\frac{dp}{d\tau} = 0 = - p - p^2 + 2 \tag{6.118}$$

for the two roots p = +1, -2. It is easy to discard the negative root by the obvious truth that p = -2 leads to instability of Eq. 6.105a, whereas p = +1 results in a damped exponential decay, as desired. The p = -2 root is the limit of $p(\tau)$ for $\tau \to -\infty$, since we are concerned only with positive $\tau = t_f - t$, this limit is not of concern; thus $p(\tau) \geq 0$ for all $\tau > 0$, as desired. Figures 6.15a through 6.15c display the behavior of the feedback gains $p(\tau)$ versus time $t = t_f - \tau$ for several choices of final time t_f and weight factors s. Notice, in all cases, for $t_f \to \infty$, the solutions approach the steady state gain p = 1 as predicted by Eq. 6.116 and the limit of Eq. 6.117a.

Another transformation which linearizes the Riccati equation is the following, let

$$p(\tau) = p_s + 1/z(\tau) \tag{6.119}$$

where p_s is the desired root of the algebraic Riccati equation ($ap + bp^2 + c = 0$) and $z(\tau)$ is a to-be-determined function. Differentiation of Eq. 6.119 gives

Figure 6.15a
s = 0

Figure 6.15b
s = 10

Figure 6.15c
s = ∞

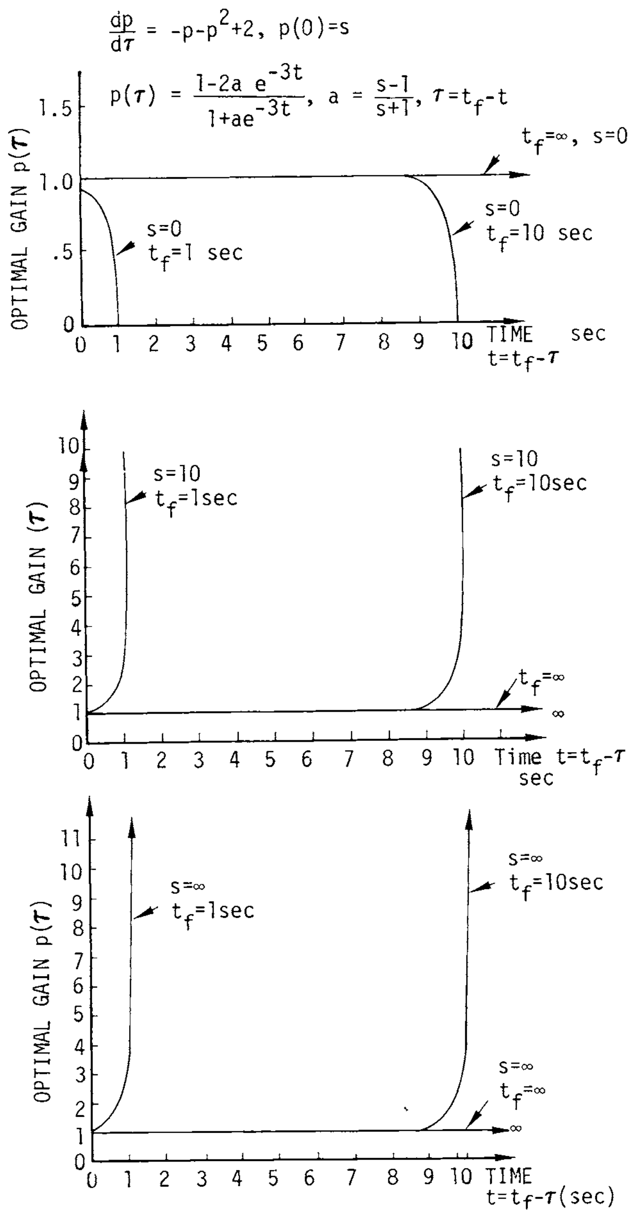


Figure 6.15 Solutions of the Scalar Riccati Equation

$$\frac{dp}{d\tau} = -\frac{1}{z^2}\frac{dz}{d\tau} \tag{6.120}$$

Substitution of Eqs. 6.119 and 6.120 into Eq. 6.108 gives

$$-\frac{1}{z^2}\frac{dz}{d\tau} = a(p_s + \frac{1}{z}) + b(p_s^2 + 2p_s\frac{1}{z} + \frac{1}{z^2}) + c$$

or

$$-\frac{1}{z^2}\frac{dz}{d\tau} = (ap_s + bp_s^2 + c) + (a + 2bp_s)\frac{1}{z} + b\frac{1}{z^2} \tag{6.121}$$

The first term on the right hand side is zero, since p_s is a root of $ap + bp^2 + c = 0$; multiplication of Eq. 6.121 by $-z^2$, remarkably, gives the *linear* differential equation for $z(\tau)$

$$\frac{dz}{d\tau} = -(a + 2bp_s)z - b \tag{6.122}$$

The solution of Eq. 5.122 is easily found to be

$$z = \frac{-b}{a+2bp_s} + Ke^{-(a+2bp_s)\tau} \tag{6.123}$$

where K is determined to satisfy the boundary conditions. Specifically, we require $z(0) = 1/[p(0) - p_s]$, so from Eq. 6.122, we find

$$z(\tau) = \left[e^{-(a+2bp_s)\tau} - 1\right]\left(\frac{b}{a+2bp_s}\right) + \frac{1}{s-p_s}e^{-(a+2bp_s)\tau} \tag{6.124}$$

This scalar solution is fully generalized to solve the matrix Riccati equation (Eq. 6.144) in Appendix A. Furthermore, in Chapter 11, the matrix generalization of Eq. 6.119 is found to be instrumental in obtaining an analytical solution for the time-varying gains associated with optimal tracking problems, as well as obtaining analytical solutions for the associated closed loop state trajectories and associated sensitivity partial derivatives.

In Figure 6.15, notice the exponentially explosive behavior of p near $t = t_f$ ($\tau = 0$). This rapid approach of a vertical asymptote is not troublesome when analytical solutions are possible, but such divergent, unstable gain growth is one of the sources of numerical difficulties in computation of optimal controls generally, and solutions of multi-dimensional Riccati equations in particular. In the context of particular applications, we shall discuss some methods available for resolving these difficulties.

6.7.2 The Hamilton-Jacobi-Bellman Equation and Bellman's Principle of Optimality

Bellman (ref. 14) first stated an important truth known widely as the *Principle of Optimality.* We shall use the performance index of Eq. 6.31. If we initiate a trajectory at an arbitrary start point (x,t) the cost-to-go for an arbitrary control u(t) is given by

$$J = \phi[x(t_f), t_f] + \int_t^{t_f} F[x(\tau), u(\tau), t]dt \tag{6.125}$$

We are concerned only with trajectories which satisfy the differential equation

$$\dot{x} = f(x, u, t) \tag{6.126}$$

and satisfy terminal constraints

$$\psi[x(t_f), t_f] = 0 \tag{6.127}$$

In Section 6.2, and 6.3, we developed the necessary conditions for minimizing Eq. 6.125 subject to $x(\tau)$ being a trajectory of Eq. (6.126) satisfying prescribed boundary conditions. The *Principle of Optimality* is concerned with the instantaneous time to go $t_f - t$ rather than the fixed $t_f - t_0$ interval. The principle of optimality states that J must be a minimum over every subinterval of time Δt, satisfying $t_f \geq t + \Delta t \geq t_0$, along an optimal trajectory. Having stated this principle, it seems so obviously true that we do not concern ourself with a formal proof. Clearly, if an optimal control had been employed everywhere *except* during the interval from t to $t + \Delta t$ the only way to minimize J of Eq. 6.125 is to choose u(t) to minimize J over the interval Δt in question.

The optimal control is implicitly defined by the requirement that it yield the minimum cost-to-go which we denote by

$$J^*[\mathbf{x},t] = \min_{\mathbf{u}(t)} \{\phi[\mathbf{x}(t_f),t_f] + \int_t^{t_f} F[\mathbf{x},\mathbf{u},\tau]dt\} \tag{6.128}$$

with the boundary condition requirement that

$$J^*(\mathbf{x},t) = \phi(\mathbf{x},t) \tag{6.129}$$

for all $(\mathbf{x},t)$ satisfying the terminal constraint

$$\boldsymbol{\psi}(\mathbf{x},t) = 0 \tag{6.130}$$

Notice $J = J(\mathbf{x},\mathbf{u},t)$ in Eq. 6.125, along a non-optimal trajectory, but $J^* = J^*(\mathbf{x},t)$ upon carrying out the minimization (6.125) over all admissible controls $\mathbf{u}$.

In order to develop an important partial differential equation, we now investigate Eq. 6.128 locally. Suppose optimal control is used everywhere on the interval (t,t_f) *except* during the initial Δt where a non-optimal $\mathbf{u}$ is employed. For Δt sufficiently small, the system will be displaced from $\{x,t\}$ to a neighboring point $\{x + \mathbf{f}(x, u, t)\Delta t, t + \Delta t\}$. Now, suppose from these perturbed initial conditions an optimal control is employed, it is apparent from Eqs. 6.128 through 6.130 that the perturbed cost-to-go is

$$\tilde{J}^*(x,t) = J^*[x + f(x, u, t)\Delta t, t + \Delta t] + F[x, \mathbf{u}, t]\Delta t \tag{6.131}$$

Since $\mathbf{u}$ over interval Δt is generally nonoptimal, it is clear that

$$\tilde{J}^*(x,t) \geq J^*(x,t) \tag{6.132}$$

and, the equality holds only if we choose $\mathbf{u}(t)$ to minimize Eq. 6.131. Thus

$$J^*(x,t) = \min_{\mathbf{u}} \{J^*[x + f(x,\mathbf{u},t)\Delta t, t + \Delta t] + F(\mathbf{x},\mathbf{u},t)\Delta t\} \tag{6.133}$$

Upon expanding in Taylor's series and taking the limit as $\Delta t \to 0$ (ref. 10), Eq. 6.133 leads immediately to the partial differential equation

$$\frac{\partial J^*(x,t)}{\partial t} + \min_{\mathbf{u}} [F(\mathbf{x}, \mathbf{u}, t) + \{\frac{\partial J^*}{\partial \mathbf{x}} (\mathbf{x},\mathbf{u},t)\}^T \mathbf{f}(\mathbf{x},\mathbf{u},t)] \tag{6.134}$$

Comparison of Eq. 6.134 with Eq. 6.34 reveals that Eq. 6.134 can be written as the **Hamiltonian-Jacobi-Bellman (HJB) equation**

$$\frac{\partial J^*(x,t)}{\partial t} + \min_{u}(H[x, \frac{\partial J^*}{\partial \mathbf{x}}, \mathbf{u},t]) = 0 \tag{6.135}$$

with the co-state

$$\lambda(t) = \{\frac{\partial J^*}{\partial x}\} = \text{function } (x,t) \tag{6.136}$$

The significance of finding a globally valid analytical solution of the HJB Equation for

$$J^* = J^*(x,\ t) \tag{6.137}$$

is that solution for the multiplier vector $\lambda(t)$ is reduced to taking the gradient of J^*. This immediately allows determination of the corresponding optimal control from Pontryagin's Principle, *in feedback form.* Unfortunately obtaining such global analytical solutions of the HJB equation can be accomplished only for special cases. The most important special case for which the HJB equation is solvable is the *linear quadratic regulator* for which we seek to minimize

$$J = \frac{1}{2}\, x^T(t_f)S_f x(t_f) + \frac{1}{2}\int_{t_o}^{t_f} [x^TQx + \mathbf{u}^TRu]dt \tag{6.138}$$

(where S_f, Q, R are symmetric, non-negative weight matrices), subject to the constraint

$$\dot{x} = A(t)x + B(t)\mathbf{u} \tag{6.139}$$

The HJB Equation of Eq. 6.135 becomes for this case

$$\frac{\partial J^*}{\partial t} + \min_{\mathbf{u}}\ [\{\frac{\partial J^*}{\partial x}\}^T(Ax + B\mathbf{u}) + \frac{1}{2}\,(x^TQx + \mathbf{u}^TRu)] = 0 \tag{6.140}$$

with the terminal boundary condition $J^*(x,t_f) = x^T(t_f)S_f x(t_f)$. Carrying out the minimization over $\mathbf{u}$ of Eq. 6.140 yields

$$\mathbf{u} = -\,R^{-1}\,B^T\,\{\frac{\partial J^*}{\partial x}\} \tag{6.141}$$

and thus the HJB of Eq. 6.140 becomes

$$\frac{\partial J^*}{\partial t} + \frac{1}{2}\,\{\frac{\partial J^*}{\partial x}\}^T Ax + \frac{1}{2}\,x^TA^T\{\frac{\partial J^*}{\partial x}\} + \frac{1}{2}\,x^TQx - \frac{1}{2}\,\{\frac{\partial J^*}{\partial x}\}^T BR^{-1}B^T\{\frac{\partial J^*}{\partial x}\} = 0 \tag{6.142}$$

It can be verified by direct substitution that the general solution of the HJB Eq. 6.142 is the quadratic form

$$J^*(x,t) = \frac{1}{2}\,x^TP(t)x \quad , \quad \{\frac{\partial J^*}{\partial x}\} = P(t)x \quad , \quad \frac{\partial J^*}{\partial t} = \frac{1}{2}\,x^T\dot{P}x \tag{6.143}$$

where P(t) is a symmetric positive matrix satisfying the *matrix Riccati* *equation*

$$\dot{P} = -PA - A^TP + PBR^{-1}B^TP - Q \quad (6.144)$$

with the terminal boundary condition

$$P(t_f) = S_f \quad (6.145)$$

Since we have Eqs. 6.136 and 6.141, the optimal control is thus obtained globally in the *time-varying linear feedback* form

$$u(t) = -K(t)x(t) \quad (6.146)$$

where the *optimal gain matrix* is

$$K(t) = R^{-1}(t)B^T(t)P(t) \quad (6.147)$$

Equations 6.144 and 6.146 are clearly the multi-dimensional generalizations of Eqs. 6.108 and 6.104, respectively, which we obtained for the scalar example in Section 6.7.1.

For the case of A, B, constant, and $t_f \to \infty$ in Eq. 6.138, it can be shown (for a controllable system), that P(t) approaches the constant positive semi-definite solution of the algebraic Riccati equation

$$PA + A^TP - PBR^{-1}B^TP + Q = 0 \quad (6.148)$$

and thus Eqs. 6.146 and 6.147 provide a constant gain feedback control.

In the context of the specific applications of subsequent chapters, we consider numerical and practical issues associated with determination of optimal feedback controls, for both linear and nonlinear systems. Equation 6.144 for the case of constant A, B, R, Q is solved in Appendix A.

6.7.3 Tuning Optimal Quadratic Regulators Vis-A-Vis Closed Loop Eigenvalue Placement

Consider the case of a linear constant coefficient system

$$\dot{x} = Ax + Bu \quad (6.149)$$

with the quadratic measure

$$J = \frac{1}{2}\int_0^\infty (x^TQx + u^TRu)dt \quad (6.150)$$

so the optimal feedback control is given by

$$\mathbf{u} = -R^{-1}B^TP\mathbf{x} \tag{6.151}$$

Thus the *closed loop system* is given by

$$\dot{\mathbf{x}} = \bar{A}\mathbf{x} \tag{6.152}$$

with

$$\bar{A} = A - BR^{-1}B^TP \tag{6.153}$$

Since $\bar{A}$ is constant, the stability properties associated with $e^{\lambda t}\mathbf{r}$ solutions of the closed-loop system are determined by the eigenvalues $(\lambda_1,\ldots,\lambda_n)$ of $\bar{A}$. Since $\bar{A}$ is a general matrix (see Appendix A), we must consider the *right* and *left* eigenvalue problems

$$\text{right:}\quad \lambda_i\mathbf{r}_i = \bar{A}\mathbf{r}_i \quad ; \quad \text{left:}\quad \lambda_i\boldsymbol{\ell}_i = \bar{A}^T\boldsymbol{\ell}_i \tag{6.154}$$

for $i = 1,2,\ldots,n$, where we choose the conventional normalizations:

$$[\ell]^T[r] = [I] \quad , \quad [\ell]^T\bar{A}[r] = \mathrm{Diag}(\lambda_1,\ldots,\lambda_n) \tag{6.155}$$

where

$$[\ell] = [\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_n] \quad , \quad [r] = [\mathbf{r}_1,\ldots,\mathbf{r}_n] \tag{6.156}$$

Clearly the eigenvalues' locations are affected by the particular weight matrices Q and R selected in Eq. 6.150, both explicitely, and implicitly, through solution of the algebraic Riccati Eq. 6.148 for P. In a more general context, $\bar{A} = \bar{A}(\mathbf{p})$, where $\mathbf{p}$ is a parameter vector which could include not only a parameterization of Q, R, but also the location of actuators [i.e., $B = B(\mathbf{p})$] and system model parameters [i.e., $A = A(\mathbf{p})$]. It is natural to ask questions vis-a-vis what choice on $\mathbf{p}$ leads to a "good" placement of the system closed loop eigenvalues. These questions are considered in Refs. 18-21; they are closely related to the literature on *pole placement.*

To address a few basic issues, consider $\bar{A}(\mathbf{p})$, $\lambda_i(\mathbf{p})$, $\mathbf{r}_i(\mathbf{p})$, $\boldsymbol{\ell}_i(\mathbf{p})$ to be continuous functions of $\mathbf{p}$; except near repeated eigenvalues, this continuity assumption can be shown to be justified. Differentiation of Eq. (6.154) gives

$$\frac{\partial\lambda_i}{\partial p_\ell}\mathbf{r}_i + \lambda_i\frac{\partial\mathbf{r}_i}{\partial p_\ell} = \frac{\partial\bar{A}}{\partial p_\ell}\mathbf{r}_i + \bar{A}\frac{\partial\mathbf{r}_i}{\partial p_\ell} \tag{6.157}$$

where p_ℓ is a typical element of **p**. Premultiplication of Eq. 6.157 by $\boldsymbol{\ell}_i^T$ and rearrangement leads to

$$\frac{\partial \lambda_i}{\partial p_\ell} \boldsymbol{\ell}_i^T \mathbf{r}_i = [\boldsymbol{\ell}_i^T \bar{A} - \lambda_i \boldsymbol{\ell}_i^T] \frac{\partial \mathbf{r}_i}{\partial p_\ell} + \boldsymbol{\ell}_i^T \frac{\partial \bar{A}}{\partial p_\ell} \mathbf{r}_i \tag{6.158}$$

Recognizing the bracketed coefficient as the left eigenvalue problem of Eq. 6.154, and making use of the normalization $\boldsymbol{\ell}_i^T \mathbf{r}_i = 1$, then Eq. 6.158 provides an analytical solution for eigenvalue sensitivity

$$\frac{\partial \lambda_i}{\partial p_\ell} = \boldsymbol{\ell}_i^T \frac{\partial \bar{A}}{\partial p_\ell} \mathbf{r}_i \quad , \quad i = 1,2,\ldots,n \tag{6.159}$$

If $\frac{\partial \bar{A}}{\partial p_\ell}$ can be conveniently calculated, then we can make use of conventional parameter optimization algorithms to solve a constrained optimization problem stated as functions of **p**, $\lambda_i(\mathbf{p})$, $\frac{\partial \lambda_i}{\partial \mathbf{p}}$, $\mathbf{r}_i(\mathbf{p})$, etc. References 18-20 provide several significant examples of this *eigenvalue optimization approach* to "tuning" of optimal quadratic regulators.

For the special case that $\mathbf{p} = \{q_{11}, q_{12}, \ldots, r_{11}, r_{12} \ldots\}$ where q_{ij} and r_{ij} are parameterizations of weight matrices $Q(q_{ij})$, $R(r_{ij})$, we need to give special attention to $\frac{\partial \bar{A}}{\partial q_{ij}}$ and $\frac{\partial \bar{A}}{\partial r_{ij}}$. Notice from Eq. 6.153 that

$$\frac{\partial \bar{A}}{\partial q_{ij}} = -BR^{-1}B \frac{\partial P}{\partial q_{ij}} \tag{6.160}$$

and

$$\frac{\partial \bar{A}}{\partial r_{ij}} = -B \frac{\partial}{\partial r_{ij}} [R^{-1}]BP - BR^{-1}B \frac{\partial P}{\partial r_{ij}} \tag{6.161}$$

where

$$\frac{\partial}{\partial r_{ij}} [R^{-1}] = -R^{-1} \frac{\partial R}{\partial r_{ij}} R^{-1} \tag{6.162}$$

The partial derivatives $\frac{\partial P}{\partial q_{ij}}$, $\frac{\partial P}{\partial r_{ij}}$ follow from differentiation of Eq. 6.148 as solutions of the following *algebraic Lyapunov equations*

$$\frac{\partial P}{\partial q_{ij}} \bar{A} + \bar{A}^T \frac{\partial P}{\partial q_{ij}} = - \frac{\partial Q}{\partial q_{ij}} \tag{6.163}$$

$$\frac{\partial P}{\partial r_{ij}} \bar{A} + \bar{A}^T \frac{\partial P}{\partial r_{ij}} = PB \frac{\partial}{\partial r_{ij}} [R^{-1}]B^T P \tag{6.164}$$

Finally, a preferred parameterization of Q and R (which allows easy enforcement of weight matrix positivity constraints) is the Cholesky decomposition

$$Q = [Q^{1/2}][Q^{1/2}] \quad , \quad R = [R^{1/2}][R^{1/2}] \tag{6.165}$$

where the Cholesky (square root) factors are the lower triangular matrices

$$Q^{1/2} = \begin{bmatrix} q_{11}^2 & 0 & \cdots & 0 \\ q_{12} & q_{22}^2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & \cdots & q_{nn}^2 \end{bmatrix} \tag{6.166}$$

$$R^{1/2} = \begin{bmatrix} r_{11}^2 & 0 & \cdots & 0 \\ r_{12} & r_{22}^2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_{1m} & r_{2m} & \cdots & r_{mm}^2 \end{bmatrix} \tag{6.167}$$

The products of Eqs. 6.165 are quaranteed to yield positive definite Q and R matrices, if all real $q_{ii} > 0$, $r_{ii} > 0$, and positive semi-definite Q and R if $q_{ii} \geq 0$, $r_{ii} \geq 0$, irregardless of the (real) values assigned to the off-diagonal q_{ij}, r_{ij}. Thus by iterating on real values q_{ij} and r_{ij} in Eqs. 6.166 and 6.167, we can easily quarantee positivity of Q and R; and the required partial derivatives $\partial Q/\partial q_{ij}$, $\partial R/\partial r_{ij}$ in Eqs. 6.163 and 6.164 are easily obtained from differentiation of Eqs. 6.165-6.167.

Reference 21 documents an example wherein Q and R are iterated successfully to achieve a rather general constrained eigenvalue placement optimization for a system of order 42. Reference 19 documents a related iteration process for synthesis of a direct output feedback controller for the same system. The Reference 19 developments also consider a more generalized optimization problem wherein sensors and actuator positions are optimized simultaneously with the design of the control gains to achieve improved placement of the closed loop eigenvalues.

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