APPENDIX D:

ANALYTICAL SOLUTION OF THE TWO BODY PROBLEM (KEPLERIAN MOTION)

An attractive general solution of the two-body problem, whose equation of motion is

$$\ddot{\mathbf{r}} = -\mu \mathbf{r}/r^3 \tag{D.1}$$

(with $\mu = G(m_1 + m_2)$, the gravitational mass constant) can be obtained in the elegant form

$$\mathbf{r}(t) = f(t)\mathbf{r}_o + g(t)\dot{\mathbf{r}}_o \tag{D.2}$$

$$\dot{\mathbf{r}}(t) = \dot{f}(t)\mathbf{r}_o + \dot{g}(t)\dot{\mathbf{r}}_o \tag{D.3}$$

These equations reflect the following truth: since the motion is planar, coefficients f and g <u>must exist</u> which map the initial conditions $(\mathbf{r}_o, \dot{\mathbf{r}}_o)$ onto the time varying position and velocity vectors $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$. Herrick (ref. D.1), develops functional forms for f and g and several alternative solutions in detail. We summarize equations in order of solution for computation of $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$, given $\mathbf{r}_o$, $\dot{\mathbf{r}}_o$ at time t_o, for the most common case of an elliptic orbit:

<u>Initial Calculations</u>

$$\Delta t = t - t_o$$

$$r_o = |\mathbf{r}_o| = +[\mathbf{r}_o \cdot \mathbf{r}_o]^{1/2}$$

$$V_o = |\dot{\mathbf{r}}_o| = +[\dot{\mathbf{r}}_o \cdot \dot{\mathbf{r}}_o]^{1/2}$$

$$D_o = \frac{1}{\sqrt{\mu}} \mathbf{r}_o \cdot \dot{\mathbf{r}}_o$$

$$\frac{1}{a} = \left(\frac{2}{r_o} - \frac{V_o^2}{\mu}\right)$$

a is the orbit semi major axis.

$$c_1 = e \cos E_o = 1 - r_o/a$$

$$c_2 = e \sin E_o = D_o/\sqrt{a}$$

e is the orbit eccentricity

$$e = +\,[c_1^2 + c_2^2]^{1/2}$$

E_0 is the initial eccentric anomaly

$$\tan E_0 = c_2/c_1$$

Kepler's Equation

First solve the following transcendental equation (a variant of Kepler's equation) for $\hat{E} \equiv E - E_0$ using Newton's method:

$$\sqrt{\mu}\,\Delta t = a^{3/2}\hat{E} - c_2 \sin \hat{E} + c_1(1 - \cos \hat{E})$$

This usually requires three or four corrections, initiating with the estimate

$$\hat{E} \cong \sqrt{\mu}\,\Delta t\, a^{-3/2}.$$

Solution for $\mathbf{r}(t)$, $\dot{\mathbf{r}}(t)$

The position and velocity at time are obtained as follows:

$$f = 1 - a(1 - \cos \hat{E})/r_0$$

$$g = \Delta t - a^{3/2}(\hat{E} - \sin \hat{E})/\sqrt{\mu}$$

$$\mathbf{r}(t) = f\mathbf{r}_0 + g\dot{\mathbf{r}}_0$$

$$r = |\mathbf{r}(t)| = [\mathbf{r} \cdot \mathbf{r}]^{1/2}$$

$$\dot{f} = -(\sqrt{\mu a}/r_0)\sin \hat{E}/r$$

$$\dot{g} = 1 - a(1 - \cos \hat{E})/r$$

$$\dot{\mathbf{r}}(t) = \dot{f}\mathbf{r}_0 + \dot{g}\dot{\mathbf{r}}_0$$

The above solution summary is a modification of a similar solution developed in detail by Herrick in reference D.1.

REFERENCE

D.1 S. Herrick, **Astrodynamics**, Vol. I, Van Nostrand Reinhold, Co., London, 1971, pp. 210-212.