CHAPTER 2

GEOMETRY AND KINEMATICS OF ROTATIONAL MOTION

The geometry and kinematics of rotational motion is a rich and often fascinating branch of mechanics and applied mathematics. It has attracted the serious extended attention of Euler, Jacobi, Hamilton, Cayley, Klein and Gibbs; these pioneering scholars left permanent contributions. The most significant of their contributions are imbedded in the following developments.

There are several "macroscopic truths" which we state without proof prior to engaging in geometric/mathematical development of the subject matter. Four "truths" are as follows:

(i) A minimum of three coordinates (typically angles) are required to generally specify a relative orientation of two reference frames F_1 and F_2.

(ii) For any chosen set of three orientation coordinates, one or more geometrically singular orientations of F_2 relative to F_1 exists for which two of the three coordinates are undefined.

(iii) At or near a geometric singularity (introduced through the particular selection of orientation coordinates), the corresponding kinematic differential equations defining the coordinates' time derivatives are likewise singular.

(iv) The geometric singularities and associated difficulties (ii) and (iii) can often be avoided altogether through a regularization [4]; in lieu of any three parameter set, several redundant (four or more parameter) sets are available which are uniformly and universally determined. The most useful of the redundant nonsingular parameterizations are associated with **Euler's Principal Rotation Theorem**, as discussed in Section 2.4.

These truths are reasonably well known and will be evident in the developments below.

2.1 SPECIAL VECTOR KINEMATIC NOTATIONS

It has been our observation that a very large fraction, perhaps the majority, of errors (committed in formulating dynamical equations), are of kinematic origin. When faced with three or more reference frames with general relative translation and rotation, ample room exists for confusion, even in interpretation of linear velocities and accelerations. Therefore, particularly in the early stages of formulating the kinematic and dynamic equations of motion, we will usually choose to follow a pattern motivated by the work of Likins and Kane (refs. 1,2) and adopt very explicit kinematical notations.

For example, the symbol

$${}^{F}\mathbf{v}^{A/B}$$

should be read as "the velocity of point A with respect to point B as seen in reference frame F". Clearly, in terms of differentiation of the displacement vector **BA** (from B to A), ${}^{F}\mathbf{v}^{A/B}$ is symbolic for

$${}^{F}\mathbf{v}^{A/B} \equiv \frac{d}{dt}(\mathbf{BA})_F \equiv \lim_{\Delta t \to 0}\left[\frac{\mathbf{BA}(t+\Delta t) - \mathbf{BA}(t)}{\Delta t}\right]_F \qquad (2.1)$$

Vector derivatives taken in frames with F_1 and F_2 having relative rotation are related by the "transport theorem" (ref. 1) which, in this explicit notation is

$$\frac{d}{dt}(\mathbf{Q})_{F_1} = \frac{d}{dt}(\mathbf{Q})_{F_2} + \boldsymbol{\omega} \times \mathbf{Q} \qquad (2.2)$$

where

Q is an arbitrary vector,

F_1 and F_2 are reference frames with arbitrary relative motion (e.g., F_1 and F_2 could be conceived of as being imbedded in two generally translating and tumbling rigid bodies), and

$\boldsymbol{\omega} \equiv \boldsymbol{\omega}^{F_2/F_1}$ is the angular velocity of F_2 relative to F_1.

Observe that a vector is quite often differentiated in one frame and componentiated in another. For example, consider the position vector $\mathbf{r}$ = OP in Figure 2.1.

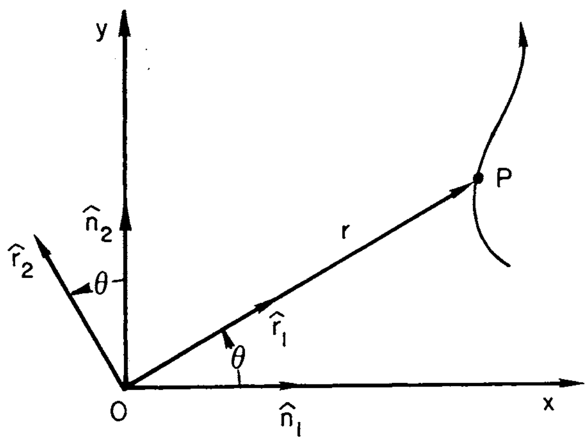

Figure 2.1 General Planar Motion

Clearly $\mathbf{r}$ has inertial (N) components

$$\mathbf{r} = x\,\hat{\mathbf{n}}_1 + y\,\hat{\mathbf{n}}_2 \tag{2.3}$$

and polar, rotating (R) components

$$\mathbf{r} = r\,\hat{\mathbf{r}}_1;\ \text{where } (\hat{\ }) \text{ denotes a unit vector.} \tag{2.4}$$

Observe that inertial velocity $\dot{\mathbf{r}} \equiv {}^N\mathbf{V}^{P/O} \equiv \frac{d}{dt}(\mathbf{r})_N$ can be written with either inertial rectangular components

$$\dot{\mathbf{r}} = \dot{x}\,\hat{\mathbf{n}}_1 + \dot{y}\,\hat{\mathbf{n}}_2 \tag{2.5)*}$$

*Here we have introduced a notational compaction we employ throughout this text, $(\dot{\ }) \equiv d/dt(\ )_N$; an overdot denotes time differentiation as seen from the inertial reference frame.

or rotating polar components

$$\dot{\mathbf{r}} = \frac{d}{dt}(r\ \hat{\mathbf{r}}_1)_N = \frac{d}{dt}(r\ \hat{\mathbf{r}}_1)_R + \boldsymbol{\omega} \times (r\ \hat{\mathbf{r}}_1)\ , \text{ where } \boldsymbol{\omega} = \boldsymbol{\omega}^{R/N} = \dot{\theta}\ \hat{\mathbf{n}}_3$$

and thus

$$\dot{\mathbf{r}} = \dot{r}\ \hat{\mathbf{r}}_1 + r\dot{\theta}\ \hat{\mathbf{r}}_2 \tag{2.6}$$

Likewise, the inertial acceleration $\ddot{\mathbf{r}} = \frac{d}{dt}(\dot{\mathbf{r}})_N \equiv {}^N\mathbf{a}^{P/O} \equiv \frac{d^2}{dt^2}(\mathbf{r})_N$ can be componentiated along either $\{\hat{\mathbf{n}}\}$ or $\{\hat{\mathbf{r}}\}$*. Note

$$\ddot{\mathbf{r}} = \ddot{x}\ \hat{\mathbf{n}}_1 + \ddot{y}\ \hat{\mathbf{n}}_2 \tag{2.7}$$

but we can also determine $\{\hat{\mathbf{r}}\}$ components of $\ddot{\mathbf{r}}$ as

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{d}{dt}(\dot{\mathbf{r}})_N \\ &= \frac{d}{dt}[\dot{r}\ \hat{\mathbf{r}}_1 + r\dot{\theta}\ \hat{\mathbf{r}}_2]_N \\ &= \frac{d}{dt}[\dot{r}\ \hat{\mathbf{r}}_1 + r\dot{\theta}\ \hat{\mathbf{r}}_2]_R + \boldsymbol{\omega} \times [\dot{r}\ \hat{\mathbf{r}}_1 + r\dot{\theta}\ \hat{\mathbf{r}}_2]\end{aligned}$$

which, after carrying out the implied operations yield the well-known truth

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{r}}_2 \tag{2.8}$$

The equivalence of Eq. 2.5 to Eq. 2.6 and Eq. 2.7 to Eq. 2.8 can be established by "brute force" kinematics (i.e., avoiding differentiation of rotating vectors) by substituting into Eqs. 2.5 and 2.7 the geometric relationships

$$x = r\cos\theta,\ y = r\sin\theta$$

$$\hat{\mathbf{n}}_1 = \cos\theta\hat{\mathbf{r}}_1 - \sin\theta\hat{\mathbf{r}}_2$$

$$\hat{\mathbf{n}}_2 = \sin\theta\hat{\mathbf{r}}_1 + \cos\theta\hat{\mathbf{r}}_2$$

and carrying out the ensuing differentiation and associated algebra (even this

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*We denote a column of right-handed unit vectors by $\{\ \}$; i.e.,

$$\{\mathbf{n}\} = \begin{Bmatrix}\hat{\mathbf{n}}_1 \\ \hat{\mathbf{n}}_2 \\ \hat{\mathbf{n}}_3\end{Bmatrix}$$

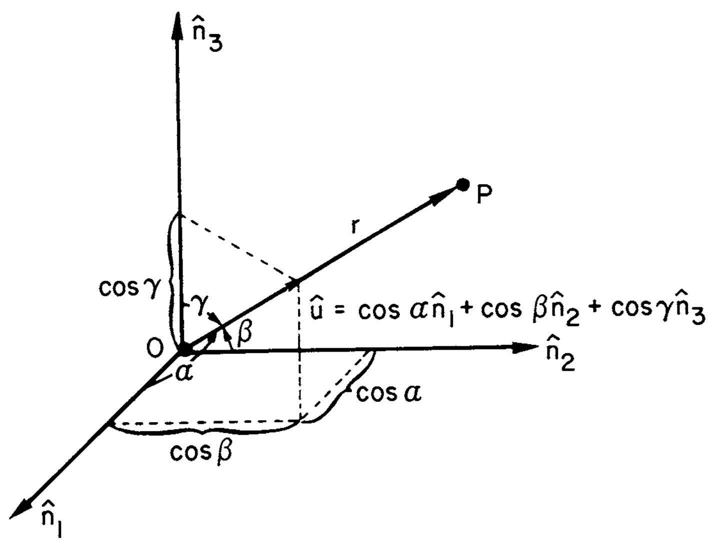

Figure 2.2 Direction Cosines

simple, classical example will help reinforce the utility of vector kinematics).

2.2 DIRECTION COSINES AND ORTHOGONAL PROJECTIONS

Referring to Figure 2.2, a general vector **r** can be written as

$$\mathbf{r} = r\hat{\mathbf{u}} = r[\cos\alpha\hat{\mathbf{n}}_1 + \cos\beta\hat{\mathbf{n}}_2 + \cos\gamma\hat{\mathbf{n}}_3] \tag{2.9}$$

where the components ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) of $\hat{\mathbf{u}}$ are the **"direction cosines"** defining the direction of $\hat{\mathbf{u}}$ and **r**. Consider two sets of three orthogonal unit vectors $\{\hat{\mathbf{b}}\}$ and $\{\hat{\mathbf{n}}\}$ having a general relative orientation. The components of $\hat{\mathbf{b}}_i$ on $\{\hat{\mathbf{n}}\}$ are denoted as

$$\hat{\mathbf{b}}_i = \sum_{j=1}^{3} C_{ij}\hat{\mathbf{n}}_j \quad , \quad i = 1,2,3 \tag{2.10}$$

Clearly, $C_{ij} \equiv$ cosine of the angle between $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{b}}_j$; the matrix equivalent of

Eq. 2.10 is

$$\{\hat{\mathbf{b}}\} = [C]\{\hat{\mathbf{n}}\} \tag{2.11}$$

where [C] is the 3×3 "direction cosine matrix". The direction cosine matrix plays a central role in spacecraft dynamics and control. A [C] matrix exists for any pair of orthogonal sets of three axes. There are several elegant properties of this class of matrices; we develop here the more important ones.

Since $\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$ and $\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3\}$ are orthogonal right-handed triads of unit vectors, we investigate the nine scalar products implicit in

$$\{\hat{\mathbf{b}}\}\cdot\{\hat{\mathbf{b}}\}^T \equiv [C]\{\hat{\mathbf{n}}\}\cdot\{\hat{\mathbf{n}}\}^T[C]^T \tag{2.12}$$

Due to orthogonality of $\hat{\mathbf{b}}_i$, we have the conditions

$$\{\hat{\mathbf{b}}\}\cdot\{\hat{\mathbf{b}}\}^T = \begin{bmatrix} \hat{\mathbf{b}}_1\cdot\hat{\mathbf{b}}_1 & \hat{\mathbf{b}}_1\cdot\hat{\mathbf{b}}_2 & \hat{\mathbf{b}}_1\cdot\hat{\mathbf{b}}_3 \\ \hat{\mathbf{b}}_2\cdot\hat{\mathbf{b}}_1 & \hat{\mathbf{b}}_2\cdot\hat{\mathbf{b}}_2 & \hat{\mathbf{b}}_2\cdot\hat{\mathbf{b}}_3 \\ \hat{\mathbf{b}}_3\cdot\hat{\mathbf{b}}_1 & \hat{\mathbf{b}}_3\cdot\hat{\mathbf{b}}_2 & \hat{\mathbf{b}}_3\cdot\hat{\mathbf{b}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and likewise, due to the orthongonality of $\{\hat{\mathbf{n}}\}$

$$\{\hat{\mathbf{n}}\}\cdot\{\hat{\mathbf{n}}\}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus we obtain from Eq. 2.12 the well-known and most important truth

$$[C][C]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.13a}$$

or

$$[C]^{-1} = [C]^T \tag{2.13b}$$

This important "inverse equals transpose" property is a necessary and sufficient condition characterizing *orthogonal* matrices.

Another important property deals with projecting components of an arbitrary vector **v**. Let the $\{\hat{\mathbf{b}}\}$ and $\{\hat{\mathbf{n}}\}$ components of **v** be denoted as

$$\mathbf{v} = v_{b1}\hat{\mathbf{b}}_1 + v_{b2}\hat{\mathbf{b}}_2 + v_{b3}\hat{\mathbf{b}}_3 = \{v_b\}^T\{\hat{\mathbf{b}}\} \tag{2.14a}$$

and

$$\mathbf{v} = v_{n1}\hat{\mathbf{n}}_1 + v_{n2}\hat{\mathbf{n}}_2 + v_{n3}\hat{\mathbf{n}}_3 = \{v_n\}^T\{\hat{\mathbf{n}}\} \tag{2.14b}$$

Substitution of Eq. 2.11 into Eq. 2.14a and equating to Eq. 2.14b yields the conclusion that

$$\{v_b\} = [C]\{v_n\} \tag{2.15a}$$

$$\{v_n\} = [C]^T\{v_b\} \tag{2.15b}$$

Thus we conclude that the orthogonal components of a general vector project precisely as do the respective unit vectors (i.e., given Eqs. 2.11 and 2.14, Eq. 2.15 necessarily hold for all **v**).

The direction cosine matrix has several other important properties, as established by Goldstein (ref. 3) and summarized here as:

(1) Det[C] = ± 1, + 1 if both $\{\hat{\mathbf{b}}\}$ and $\{\hat{\mathbf{n}}\}$ are right-handed (as well as orthogonal).

(2) [C] has only one real eigenvalue, it is ± 1; +1 for $\{\hat{\mathbf{b}}\}$ and $\{\hat{\mathbf{n}}\}$ right-handed.

(3) Successive rotations obeying

$$\{\hat{\mathbf{b}}''\} = [C'']\{\mathbf{b}'\}$$
$$\{\hat{\mathbf{b}}'\} = [C']\{\hat{\mathbf{b}}\}$$
$$\{\hat{\mathbf{b}}\} = [C]\{\hat{\mathbf{n}}\}$$

can be written in terms of a composite projection $\{\hat{\mathbf{b}}''\} = [C''']\{\hat{\mathbf{n}}\}$, with the associated direction cosine matrix

$$[C'''] = [C''][C'][C] \tag{2.16}$$

which is also orthogonal, so if [C], [C'], and [C"] obey (1) and (2), [C'''] does as well.

Since we are interested in spacecraft rotational dynamics, we are naturally concerned with the time behavior of [C]. Suppose the instantaneous angular velocity $\boldsymbol{\omega}$ of $\{\hat{\mathbf{b}}\}$ relative to $\{\hat{\mathbf{n}}\}$ is taken with $\{\hat{\mathbf{b}}\}$ components as

$$\boldsymbol{\omega} = \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \tag{2.17}$$

Since a general vector differentiates according to Eq. 2.2, we know

$$\frac{d}{dt}(\hat{\mathbf{b}}_i)_N = \boldsymbol{\omega} \times \hat{\mathbf{b}}_i \quad , \quad i = 1,2,3$$

or

$$\frac{d}{dt}\{\hat{\mathbf{b}}\}_N = \boldsymbol{\omega} \times \{\hat{\mathbf{b}}\} \tag{2.18}$$

Upon substitution of Eq. 2.17, Eq. 2.18 becomes

$$\frac{d}{dt}\{\hat{\mathbf{b}}\}_N = -\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\{\hat{\mathbf{b}}\} = -[\tilde{\omega}]\{\hat{\mathbf{b}}\} \tag{2.19}$$

with

$$[\tilde{\omega}] \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = -[\tilde{\omega}]^T \tag{2.20}$$

Since we also have Eq. 2.11, upon differentiation we also have

$$\frac{d}{dt}\{\hat{\mathbf{b}}\}_N = [\dot{C}]\{\hat{\mathbf{n}}\} \tag{2.21}$$

Substitution of Eq. 2.11 into Eq. 2.19 and equating the result to Eq. 2.21 we immediately establish the important and universally valid kinematic differential equation

$$[\dot{C}] = -[\tilde{\omega}][C] \tag{2.22}$$

The orthogonality condition, Eq. 2.13a, can be considered a set of nine scalar constraint equations (only six of which are independent). These constraints are implicit in the differential Eq. 2.22 as can be readily established. Equation 2.13 must in fact be an exact integral of Eq. 2.22; this truth is established as follows

$$\frac{d}{dt}([C][C]^T) = [\dot{C}][C]^T + [C][\dot{C}]^T$$

Substitution of Eq. 2.22 yields

$$\frac{d}{dt}([C][C]^T) = -[\tilde{\omega}][C][C]^T - [C][C]^T[\tilde{\omega}]^T$$

making use of Eqs. 2.13a and 2.20; i.e., at the arbitrary initial time, immediately yields

$$\frac{d}{dt}\left([C][C]^T\right) \equiv [0];$$

Thus, since $[C][C]^T$ is a constant of the solution of Eq. 2.22, if Eq. 2.13a is satisfied initially, the solutions of Eq. 2.22 satisfy Eq. 2.13a for all time.

Equation 2.22 provides a very convenient and universally applicable differential equation whose integration yields the instantaneous direction cosines for arbitrary $\omega(t)$. For the cases in which $\omega(t)$ can be integrated a priori or can be accurately measured (e.g., via rate gyro systems), Eq. 2.22 is a linear ordinary differential equation with a known time varying coefficient matrix. Since the orthogonality constraints Eq. 2.13 are implicit in Eq. 2.22, Eq. 2.13 can be used as necessary condition test for error analysis of numerical solutions. As will be evident, by comparison with subsequent developments, the nine scalar equations of Eq. 2.22 are redundant, since only three degrees of freedom exist. However, using the minimum number (3) of coordinates invariably replaces Eq. 2.22 by a nonlinear system of differential equations containing one or more division-by-zero singular points. It is possible to retain analogous universal, linear, and successive rotation advantages (of the direction cosines) by using the four Euler (Quaternion) parameters, which are only once-redundant. Upon considering the more popular Euler angle parameterization of [C] in Section 2.4, we will consider the Euler parameters in Sections 2.5 and 2.6.

2.3 ROTATIONS ABOUT A FIXED AXIS

Referring to Figure 2.3, we consider rotation of a rigid body B about a fixed axis (colinear with $\hat{\ell}$ through O); the $\hat{\ell}$ axis is fixed in both the body and in the inertial frame N.

Consider the displacement vector **r** from body fixed point O (somewhere on the axis of rotation) to an arbitrary body-fixed point P. Upon rotating the body through an angle ϕ, it is clear that P will be displaced along a circle (of radius $r \sin\theta$) to a position P'. Consider the projection of $\mathbf{r}$ and $\mathbf{r}'$, as seen in the plane of the circular arc swept out by P (see sketch below).

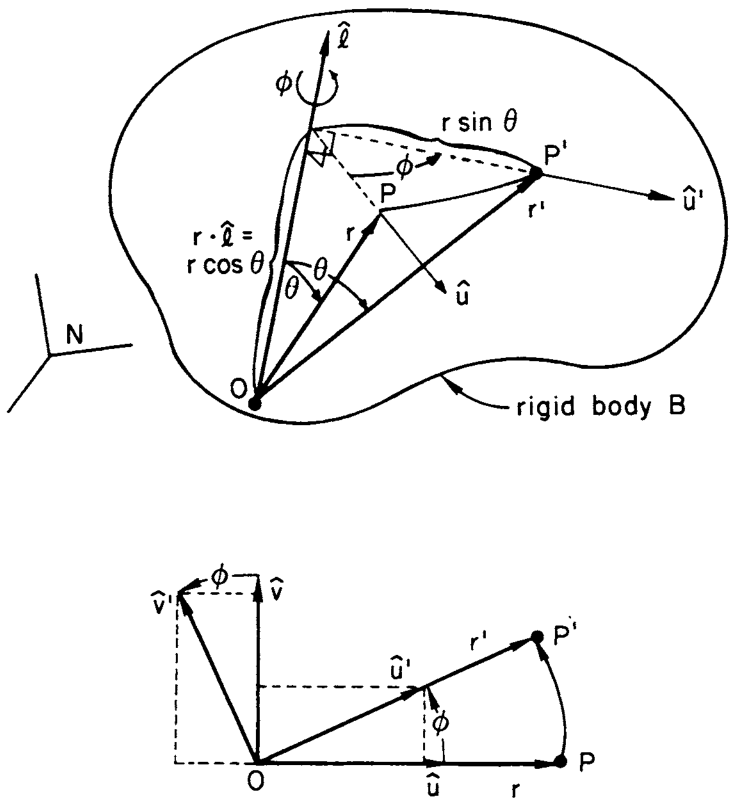

Figure 2.3 Rotation About a Fixed Axis

We see from Figure 2.3 that

$$\mathbf{r} = r\cos\theta\,\hat{\boldsymbol{\ell}} + r\sin\theta\,\hat{\mathbf{u}} \tag{2.23}$$

and

$$\mathbf{r}' = r\cos\theta\,\hat{\boldsymbol{\ell}} + r\sin\theta\,\hat{\mathbf{u}}' \tag{2.24}$$

We can see from the above sketch that

$$\hat{\mathbf{u}}' = \cos\phi\ \hat{\mathbf{u}} + \sin\phi\ \hat{\mathbf{v}} \tag{2.25}$$

It also follows from the above geometry

$$\hat{\mathbf{v}} = \frac{\hat{\boldsymbol{\ell}} \times \mathbf{r}}{|\hat{\boldsymbol{\ell}} \times \mathbf{r}|} = \left(\frac{1}{r\ \sin\theta}\right)\ \hat{\boldsymbol{\ell}} \times \mathbf{r} \tag{2.26}$$

$$\hat{\mathbf{u}} = \hat{\mathbf{v}} \times \hat{\boldsymbol{\ell}} = \left(\frac{1}{r\ \sin\theta}\right)(\hat{\boldsymbol{\ell}} \times \mathbf{r}) \times \hat{\boldsymbol{\ell}} \tag{2.27}$$

Making use of the vector triple product identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Eq. 2.27 reduces to

$$\hat{\mathbf{u}} = \left(\frac{1}{r\ \sin\theta}\right)[\mathbf{r} - (\hat{\boldsymbol{\ell}} \cdot \mathbf{r})\hat{\boldsymbol{\ell}}] \tag{2.28}$$

Substitution of Eqs. 2.28, 2.26 and 2.25 into Eq. 2.24 yields an important geometrical result

$$\mathbf{r}' = (1 - \cos\phi)(\hat{\boldsymbol{\ell}} \cdot \mathbf{r})\hat{\boldsymbol{\ell}} + \cos\phi\ \mathbf{r} + \sin\phi\ (\hat{\boldsymbol{\ell}} \times \mathbf{r}) \tag{2.29}$$

Equation 2.29 is the general vector equation for the space curve (which is a circle!) generated by an arbitrary point P fixed in a rigid body rotating about a fixed axis. This equation holds for arbitrarily large displacements. For the limiting case of infinitesimally small displacements, note the limits

$$\phi \to d\phi$$

$$\sin\phi \to d\phi$$

$$\cos\phi \to 1$$

$$\mathbf{r}' - \mathbf{r} \to d\mathbf{r}$$

Thus Eq. 2.29 immediately yields the familiar differential tangential displacement result

$$d\mathbf{r} = d\phi(\hat{\boldsymbol{\ell}} \times \mathbf{r}) = (r\ \sin\theta)d\phi\ \hat{\mathbf{v}} \tag{2.30}$$

Also, if ϕ is varying with time, consideration of $\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ and $\Delta\phi = \phi(t + \Delta t) - \phi(t)$ in Eq. 2.29 leads, in the limit as $\Delta t \to 0$, to the often used result:

$$\dot{\mathbf{r}} = \lim_{\Delta t \to 0}\left(\frac{\Delta\mathbf{r}(t)}{\Delta t}\right) = \dot{\phi}\hat{\boldsymbol{\ell}} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}\ ;\ \boldsymbol{\omega} = \dot{\phi}\hat{\boldsymbol{\ell}} \tag{2.31}$$

2.4 EULERIAN ANGLES

For many practical applications, the relative orientation of two orthogonal reference frames is defined in terms of three angles. In many physical systems (e.g., polar telescope mounts, a zenith-elevation mounted radar antenna, gyroscope gimbals, etc.) a particular set of two or three angles are "built into" the gimbal axes of the particular hardware. For example, see the gyro assembly of Figure 2.4. In such a case, the most obvious choice of angles to orient the rigid body are those implicit in the gimbal design.

For the case of a reference frame imbedded in an unconstrained body (e.g., a space vehicle), however, an infinity of orientation coordinates is possible;

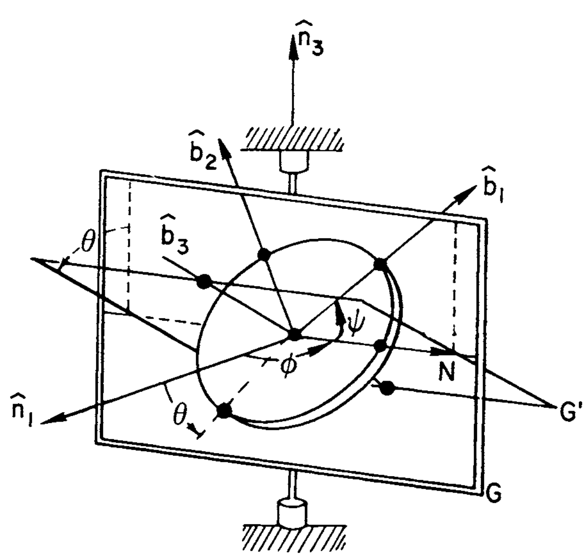

Figure 2.4 Two Gimbal Gyro with 3-1-3 Euler Angles

the particular choice of coordinates should be strongly influenced by the ease of motion visualization and perhaps more importantly, the absence of analytical or numerical singularities for a particular application or class of applications.

The most popular orientation coordinates in analytical dynamics generally and space vehicle dynamics in particular are a set of three Eulerian angles. The classical ("3-1-3") set of Euler angles are depicted in Figure 2.5. These were first used by astronomers to define the orientation of orbit planes of the planets relative to the earth's orbit plane (the eccliptic plane). In this context, $\theta_1 = \phi$ (the longitude of the ascending node), $\theta_2 = \theta$ (the inclination), $\theta_3 = \psi$ (the argument of perihelion). These "3-1-3" angles were first used in rigid body rotational dynamics by Euler during the early 1700's.

The successive rotational transformation property of direction cosines, as defined by Eq. 2.16, suggests using a set of three elementary rotations to parametrize (at any instant) the direction cosine matrix. If we restrict the three elementary rotations to be rigid right-handed rotations about fixed axes (as in Section 2.3), there still exists an infinity of three angle sets (owing to the infinity of available directions for fixing the three axes of rotation). If we restrict the axes of rotation to be colinear with one of the three orthogonal, right-handed body-fixed vectors $\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$, however, there are only twelve distinct cases. These are the classical Euler angles (although there is not universal conformity in adopting right-handed definitions for these angles). We introduce three indices α-β-γ to characterize these rotations

- α denotes the axis of the first rotation θ_1 about $\hat{\mathbf{b}}_\alpha = \hat{\mathbf{b}}'_\alpha$, which brings $\{\hat{\mathbf{b}}\}$ into position $\{\hat{\mathbf{b}}'\}$.
- β denotes the axis of the second rotation θ_2 about $\hat{\mathbf{b}}'_\beta = \hat{\mathbf{b}}''_\beta$, which brings $\{\hat{\mathbf{b}}'\}$ into $\{\hat{\mathbf{b}}''\}$

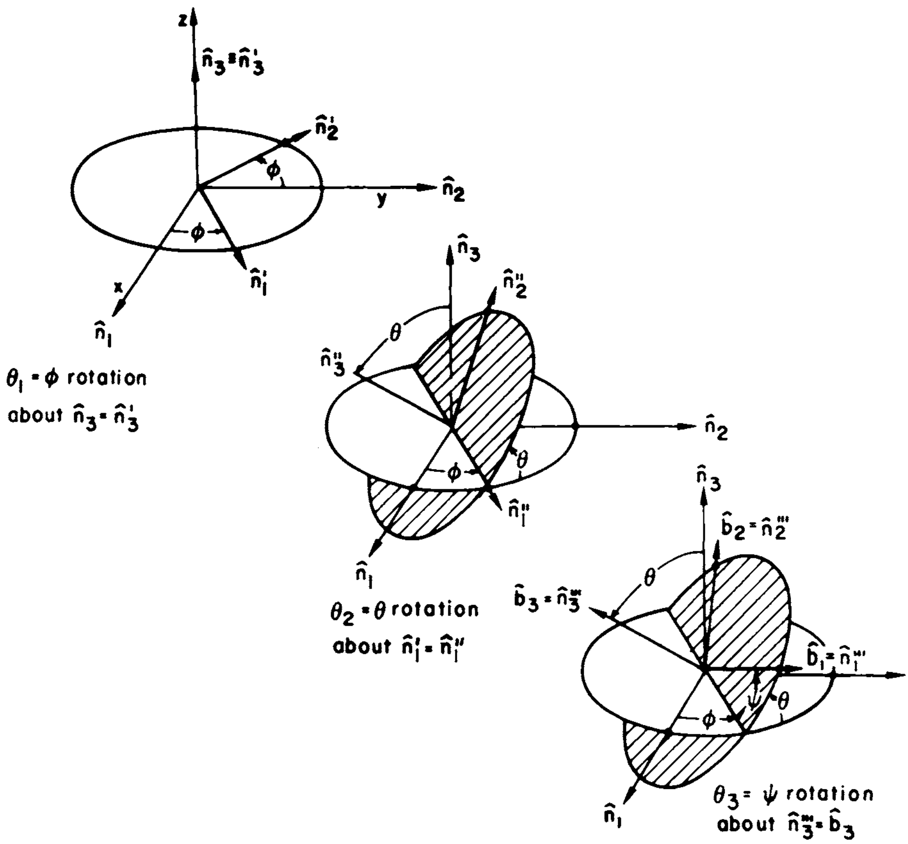

Figure 2.5 The 3-1-3 Euler Angles

γ denotes the axis of the third rotation θ_3 about $\hat{\mathbf{b}}'' _\gamma = \hat{\mathbf{b}}_\gamma'''$, which brings $\{\hat{\mathbf{b}}''\}$ into $\{\hat{\mathbf{b}}'''\}$.

For example, the "3-1-3" description of the Euler angles of Figure 2.5 is clearly consistent with this designation. In this particular case, we observe rotation θ_1 about $\hat{\mathbf{n}}_3 \equiv \hat{\mathbf{n}}_3'$ results in the orthogonal projection

$$\begin{Bmatrix} \hat{n}_1' \\ \hat{n}_2' \\ \hat{n}_3' \end{Bmatrix} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} \tag{2.32a*}$$

Rotation θ_2 about $\mathbf{n}_1' = \mathbf{n}_1''$ results in

$$\begin{Bmatrix} \hat{n}_1'' \\ \hat{n}_2'' \\ \hat{n}_3'' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_2 & s\theta_2 \\ 0 & -s\theta_2 & c\theta_2 \end{bmatrix} \begin{Bmatrix} \hat{n}_1' \\ \hat{n}_2' \\ \hat{n}_3' \end{Bmatrix} \tag{2.32b}$$

Rotation θ_3 about $\hat{\mathbf{n}}_3'' = \hat{\mathbf{n}}_3''' \equiv \hat{\mathbf{b}}_3$ yields

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{n}_1'' \\ \hat{n}_2'' \\ \hat{n}_3'' \end{Bmatrix} \tag{2.32c}$$

Substitution of Eq. 2.32a into Eq. 2.32b and the result into Eq. 2.32c, the direction cosine matrix has the 3-1-3 Euler angle parameterization

$$\{\hat{\mathbf{b}}\} = [C(\theta_1,\theta_2,\theta_3)]\{\hat{\mathbf{n}}\}$$

with

$$[C(\theta_1,\theta_2,\theta_3)] = \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_2 & s\theta_2 \\ 0 & -s\theta_2 & c\theta_2 \end{bmatrix} \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.33}$$

or, carrying out the implied matrix multiplications

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*Here we introduce the abbreviations c ≡ cos, s ≡ sin, which we will employ throughout this text to compact transcendental expressions.

$$[C(\theta_1,\theta_2,\theta_3)] = \begin{bmatrix} c\theta_3 c\theta_1 - s\theta_3 c\theta_2 s\theta_1 & c\theta_3 s\theta_1 + s\theta_3 c\theta_2 c\theta_1 & s\theta_3 s\theta_2 \\ -s\theta_3 c\theta_1 - c\theta_3 c\theta_2 s\theta_1 & -s\theta_3 s\theta_1 + c\theta_3 c\theta_2 c\theta_1 & c\theta_3 s\theta_2 \\ s\theta_2 s\theta_1 & -s\theta_2 c\theta_1 & c\theta_2 \end{bmatrix} \quad (2.34)$$

From Eq. 2.34, we see that the angles can be calculated, given the direction cosines, from the inverse transformations

$$\theta_1 = \tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right) \quad , \quad \theta_2 = \cos^{-1}(C_{33}) \quad , \quad \theta_3 = \tan^{-1}\left(\frac{C_{13}}{C_{23}}\right) \quad (2.35a,b,c)$$

In general, the direction cosines can be formed from any of the twelve sets of Euler angles via multiplication of three elementary rotation matrices; for a general α-β-γ rotation sequence, the direction cosine matrix has the form

$$[C(\theta_1,\theta_2,\theta_3)] = [M_\gamma(\theta_3)][M_\beta(\theta_2)][M_\alpha(\theta_1)] \quad (2.36)$$

where the three elementary rotation matrices are

$$[M_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \quad (2.37a)$$

$$[M_2(\theta)] = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \quad (2.37b)$$

$$[M_3(\theta)] = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.37c)$$

It is probably obvious, but we emphasize that the above discussion employs "sequential rotations" in the instantaneous geometric sense. By this we mean that the *instantaneous* position of $\{\hat{\mathbf{b}}\}$ relative to $\{\hat{\mathbf{n}}\}$ has direction cosines which can be calculated via Eq. 2.36. Clearly, we do not restrict our interpretation of Eq. 2.36 to the special case that the rotations are in fact α-β-γ sequential stop-start angular motions about fixed axes. An infinity of sequential motions *could have* led to any instantaneous values for [C], but the instantaneous [C] matrix, exclusive of certain singularities, can still be *described* as an instantaneous composite of three Eulerian rotations. This situation is quite analogous to the more familiar truth that one can choose rectangular or spherical coordinates to describe the same dynamical path of a particle. However, the issue is sometimes clouded by particular physical gimbaled devices which *do* execute specific Euler angle rotations, either sequentially or simultaneously.

In the rotational dynamics of Chapter 3 and subsequent developments, we will find that the differential equations for the three angles (or other parameters used to describe orientation) play a central role. Usually one encounters three or more kinematic equations of the functional form

$$\dot{\theta}_i = f_i(\theta_1,\theta_2,\theta_3,\omega_1,\omega_2,\omega_3,t,\ldots) \quad , \quad i = 1,2,3 \tag{2.38}$$

where

$$\boldsymbol{\omega} = \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \tag{2.39}$$

is the angular velocity of $\{\hat{\mathbf{b}}\}$ relative to $\{\hat{\mathbf{n}}\}$.

To illustrate the general process for establishing these differential equations for the α-β-γ Euler angles, we consider the 3-1-3 case in detail.

From Figure 2.5, it is apparent that the angular velocity can be written as

$$\boldsymbol{\omega} = \dot{\theta}_1\hat{\mathbf{n}}_3 + \dot{\theta}_2\hat{\mathbf{n}}''_1 + \dot{\theta}_3\hat{\mathbf{b}}_3 \tag{2.40}$$

From Eqs. 2.32c and 2.34, it follows that

$$\hat{\mathbf{n}}''_1 = \cos\theta_3\hat{\mathbf{b}}_1 - \sin\theta_3\hat{\mathbf{b}}_2 \tag{2.41a}$$

$$\hat{\mathbf{n}}_3 = \sin\theta_3\ \sin\theta_2\hat{\mathbf{b}}_1 + \cos\theta_3\ \sin\theta_2\hat{\mathbf{b}}_2 + \cos\theta_2\hat{\mathbf{b}}_3 \tag{2.42b}$$

which upon substituting into Eq. 2.40 and equating the result to Eq. 2.39 yields the kinematic equation

$$\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Bmatrix} = \begin{bmatrix} s\theta_3 s\theta_2 & c\theta_3 & 0\\ c\theta_3 s\theta_2 & -s\theta_3 & 0\\ c\theta_2 & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{Bmatrix}; \tag{2.43}$$

the inverse of Eq. 2.43 is the kinematic differential equation for 3-1-3 Euler angles

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \frac{1}{s\theta_2} \begin{bmatrix} s\theta_3 & c\theta_3 & 0 \\ c\theta_3 s\theta_2 & -s\theta_3 s\theta_2 & 0 \\ -s\theta_3 c\theta_2 & -c\theta_3 c\theta_2 & s\theta_2 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{2.44}$$

which has an obvious singularity for $\theta_2 \to 0, \pi$ (in which $\dot{\theta}_1$ and $\dot{\theta}_3$ are undefined, regardless of the behavior of the $\omega_i(t)$). This singularity is also evident in the inverse transformations of Eq. 2.35 for θ_1 and θ_3. Referring to Figure 2.5, the geometric interpretation of this singularity is that the $\theta_2 = 0, \pi$ conditions correspond to the vanishing of the line of nodes (the $(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2)$ and $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ planes are coincident). In general, when two of three Euler angles are measured in the same plane, their values are not uniquely determined and a singularity occurs. The kinematic relationships Eqs. 2.43 and 2.44 can be written compactly as

$$\{\omega\} = [B(\theta_2, \theta_3)]\{\dot{\theta}\} \tag{2.45a}$$

$$\{\dot{\theta}\} = [B(\theta_2, \theta_3)]^{-1}\{\omega\} \tag{2.45b}$$

The $[B]$ and $[B]^{-1}$ matrices are summarized in Table 2.1 for all 12 sets of α-β-γ Euler angles. Table 2.1 also summarizes the inverse transformations from direction cosines of Eq. 2.36 to the corresponding Euler angle parameterization of $[C]$.

In many applications, it is possible to select a judicious set of the Euler angles which avoids, for all practical purposes, the singularity at $\theta_2 = 0, \pm\pi$ or $\theta_2 = \pm\, \pi/2$ (see Table 2.1). In a significant subset of applications, it is desirable to linearize the kinematic relationships of Table 2.1. In this situation, it is extremely important that an Euler angle set be chosen so that the anticipated small motions are "far away" from the singularity (preferably 90° away). Regardless of the "smallness" of the physical angular motion, linearizations of the results in Table 2.1 are likely to be invalid near a

TABLE 2.1 EULER ANGLE GEOMETRIC AND KINEMATIC FORMULA SUMMARY

Direction Cosine Parameterization: $[C(\theta_1,\theta_2,\theta_3)] = [M_\gamma(\theta_3)][M_\beta(\theta_2)][M_\alpha(\theta_1)]$

Angular Velocity/Angular Rate Transformation: $\{\omega\} = [B(\theta_2,\theta_3)]\{\dot{\theta}\}$

Abbreviation : $c_i = \cos(\theta_i)$, $s_i = \sin(\theta_i)$, $c^{-1}(\ )$ = arc cosine( ),

$s^{-1}(\ )$ = arc sine ( ), $t^{-1}(\ )$ = arc tangent( )

α-β-γ ROTATION SEQUENCE	B	B^{-1}	ANGLES AS FUNCTIONS OF DIRECTION COSINES	SINGULAR AT θ_2 =
1-2-1	$\begin{bmatrix} c_2 & 0 & 1 \\ s_2s_3 & c_3 & 0 \\ s_2c_3 & -s_3 & 0 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} 0 & s_3 & c_3 \\ 0 & s_2c_3 & -s_2c_3 \\ s_2 & -c_2s_3 & -c_2c_3 \end{bmatrix}$	$\theta_1=T^{-1}(C_{12}/-C_{13})$ $\theta_2=c^{-1}(C_{11})$ $\theta_3=t^{-1}(C_{21}/C_{31})$	$0,\pm\pi$
1-2-3	$\begin{bmatrix} c_2c_3 & s_3 & 0 \\ -c_2s_3 & c_3 & 0 \\ s_2 & 0 & 1 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} c_3 & -s_3 & 0 \\ c_2s_3 & c_2c_3 & 0 \\ -s_2c_3 & s_2s_3 & c_2 \end{bmatrix}$	$\theta_1=t^{-1}(-C_{32}/C_{33})$ $\theta_2=s^{-1}(C_{31})$ $\theta_3=t^{-1}(-C_{21}/C_{11})$	$\pm\pi/2$
1-3-1	$\begin{bmatrix} c_2 & 0 & 1 \\ -s_2c_3 & s_3 & 0 \\ s_2s_3 & c_3 & 0 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} 0 & -c_3 & s_3 \\ 0 & s_2s_3 & s_2c_3 \\ s_2 & c_2c_3 & -c_2s_3 \end{bmatrix}$	$\theta_1=t^{-1}(C_{13}/C_{12})$ $\theta_2=c^{-1}(C_{11})$ $\theta_3=t^{-1}(C_{31}/-C_{21})$	$0,\pm\pi$
1-3-2	$\begin{bmatrix} c_2c_3 & -s_3 & 0 \\ -s_2 & 0 & 1 \\ c_2s_3 & c_3 & 0 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} c_3 & 0 & s_3 \\ -c_2s_3 & 0 & c_2c_3 \\ s_2c_3 & c_2 & s_2s_3 \end{bmatrix}$	$\theta_1=t^{-1}(C_{23}/C_{22})$ $\theta_2=s^{-1}(-C_{21})$ $\theta_3=t^{-1}(C_{31}/C_{11})$	$\pm\pi/2$
2-1-2	$\begin{bmatrix} s_2s_3 & c_3 & 0 \\ c_2 & 0 & 1 \\ -s_2c_3 & s_3 & 0 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} s_3 & 0 & -c_3 \\ s_2c_3 & 0 & s_2s_3 \\ -c_2s_3 & s_2 & c_2c_3 \end{bmatrix}$	$\theta_1=t^{-1}(C_{21}/C_{23})$ $\theta_2=c^{-1}(C_{22})$ $\theta_3=t^{-1}(C_{12}/-C_{32})$	$0,\pm\pi$

Table 2.1 Continued

α-β-γ ROTATION SEQUENCE	B	B^{-1}	ANGLES AS FUNCTIONS OF DIRECTION COSINES	SINGULAR AT θ_2 =
2-1-3	$\begin{bmatrix} c_2s_3 & c_3 & 0 \\ c_2c_3 & -s_3 & 0 \\ -s_2 & 0 & 1 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} s_3 & c_3 & 0 \\ c_2c_3 & -c_2s_3 & 0 \\ s_2s_3 & s_2c_3 & c_2 \end{bmatrix}$	$\theta_1=t^{-1}(C_{31}/C_{33})$ $\theta_2=s^{-1}(-C_{32})$ $\theta_3=t^{-1}(C_{12}/C_{22})$	$\pm\pi/2$
2-3-1	$\begin{bmatrix} s_2 & 0 & 1 \\ c_2c_3 & s_3 & 0 \\ -c_2s_3 & c_3 & 0 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} 0 & c_3 & -s_3 \\ 0 & c_2s_3 & c_2c_3 \\ c_2 & -s_2c_3 & s_2s_3 \end{bmatrix}$	$\theta_1=t^{-1}(-C_{13}/C_{11})$ $\theta_2=s^{-1}(C_{12})$ $\theta_3=t^{-1}(-C_{32}/C_{22})$	$\pm\pi/2$
2-3-2	$\begin{bmatrix} s_2c_3 & -s_3 & 0 \\ c_2 & 0 & 1 \\ s_2s_3 & c_3 & 0 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} c_3 & 0 & s_3 \\ -s_2s_3 & 0 & s_2c_3 \\ -c_2c_3 & s_2 & -c_2s_3 \end{bmatrix}$	$\theta_1=t^{-1}(C_{23}/-C_{21})$ $\theta_2=c^{-1}(C_{22})$ $\theta_3=t^{-1}(C_{32}/C_{12})$	$0,\pm\pi$
3-1-2	$\begin{bmatrix} -c_2s_3 & c_3 & 0 \\ s_2 & 0 & 1 \\ c_2c_3 & s_3 & 0 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} -s_3 & 0 & c_3 \\ c_2c_3 & 0 & c_2s_3 \\ s_2s_3 & c_2 & -s_2c_3 \end{bmatrix}$	$\theta_1=t^{-1}(-C_{21}/C_{22})$ $\theta_2=s^{-1}(C_{23})$ $\theta_3=t^{-1}(-C_{13}/C_{33})$	$\pm\pi/2$
3-1-3	$\begin{bmatrix} s_2s_3 & c_3 & 0 \\ s_2c_3 & -s_3 & 0 \\ c_2 & 0 & 1 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} s_3 & c_3 & 0 \\ c_3s_2 & -s_3s_2 & 0 \\ -s_3c_2 & -c_3c_2 & s_2 \end{bmatrix}$	$\theta_1=t^{-1}(C_{31}/-C_{32})$ $\theta_2=c^{-1}(C_{33})$ $\theta_3=t^{-1}(C_{13}/C_{23})$	$0,\pm\pi$
3-2-1	$\begin{bmatrix} -s_2 & 0 & 1 \\ c_2s_3 & c_3 & 0 \\ c_2c_3 & -s_3 & 0 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} 0 & s_3 & c_3 \\ 0 & c_2c_3 & -c_2s_3 \\ c_2 & s_2s_3 & s_2c_3 \end{bmatrix}$	$\theta_1=t^{-1}(C_{12}/C_{11})$ $\theta_2=s^{-1}(-C_{13})$ $\theta_3=t^{-1}(C_{23}/C_{33})$	$\pm\pi/2$
3-2-3	$\begin{bmatrix} -s_2c_3 & s_3 & 0 \\ s_2s_3 & c_3 & 0 \\ c_2 & 0 & 1 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} -c_3 & s_3 & 0 \\ s_2s_3 & s_2c_3 & 0 \\ c_2c_3 & -c_2s_3 & s_2 \end{bmatrix}$	$\theta_1=t^{-1}(C_{32}/C_{31})$ $\theta_2=c^{-1}(C_{33})$ $\theta_3=t^{-1}(C_{23}/-C_{13})$	$0,\pm\pi$

singularity. A very familiar device in space vehicle dynamics is to define a moving frame which is the "nominal" or desired motion, then introduce three small Euler angles to describe departure motion. A casual inspection of Table 2.1 reveals any of the six α-β-γ Euler angle sets which *have non-repeated indices* (i.e., 1-2-3, 3-2-1, 2-3-1, 3-1-2, 2-1-3, 1-3-2) are well-suited for describing small de arture motions. The most common ("yaw, pitch, roll") set of Euler angles for aircraft and spacecraft applications is the 3-2-1 sequence, taken as typically small displacements from a moving "local vertical" set of reference axes. So long as these angles remain small, of course, no analytical or numerical difficulty associated with the geometric singularities will occur.

It is a remarkable truth that the corresponding kinematic relationships for the four Euler (quaternion) parameters of Section 2.6 are *rigorously*, and *universally* linearly related to the angular velocity components, for all orientations. These orientation parameters are closely related to Euler's principal rotation theorem developed in the following section.

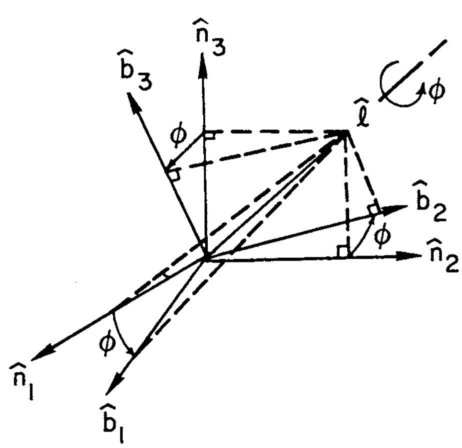

Figure 2.6 Euler's Principal Rotation

2.5 EULER'S PRINCIPAL ROTATION THEOREM

Euler (refs. 5, 6, 7) is generally credited with being responsible for the **Principal Rotation Theorem:**

> A rigid body can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rotation of the body through a *principal angle* (ϕ) about a *principal line* ($\hat{\mathbf{l}}$); the principal line being a judicious axis fixed in the body and fixed in space.

Letting (see Figure 2.6) the body fixed axes $\{\hat{\mathbf{b}}\}$ be "initially" coincident with fixed axes $\{\hat{\mathbf{n}}\}$, we can use Euler's principal rotation theorem to develop several elegant parameterizations of the direction cosine matrix [C] defining $\{\hat{\mathbf{b}}\}$'s instantaneous angular position in the sense

$$\{\hat{\mathbf{b}}\} = [C]\{\hat{\mathbf{n}}\}.$$

Let us denote the $\{\hat{\mathbf{b}}\}$ and $\{\hat{\mathbf{n}}\}$ components of $\hat{\mathbf{l}}$ as

$$\hat{\mathbf{l}} = \ell_{b1}\hat{\mathbf{b}}_1 + \ell_{b2}\hat{\mathbf{b}}_2 + \ell_{b3}\hat{\mathbf{b}}_3 \tag{2.46a}$$

and

$$\hat{\mathbf{l}} = \ell_{n1}\hat{\mathbf{n}}_1 + \ell_{n2}\hat{\mathbf{n}}_2 + \ell_{n3}\hat{\mathbf{n}}_3 \tag{2.46b}$$

As a direct consequence of the fact that ℓ_{bi} and ℓ_{ni} are constants (during a rotation ϕ about a fixed $\hat{\mathbf{l}}$) and $\hat{\mathbf{b}}_i$ are "initially" coincident with $\hat{\mathbf{n}}_i$, we see that $\ell_{ni} \equiv \ell_{bi} \equiv \ell_i$, for i = 1,2,3. According to Eq. 2.15, we have

$$\begin{Bmatrix} \ell_{b1} \\ \ell_{b2} \\ \ell_{b3} \end{Bmatrix} = [C] \begin{Bmatrix} \ell_{n1} \\ \ell_{n2} \\ \ell_{n3} \end{Bmatrix} \tag{2.47}$$

but since $\ell_{bi} = \ell_{ni} = \ell_i$, we have

$$\begin{Bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{Bmatrix} = [C] \begin{Bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{Bmatrix} \tag{2.48}$$

Careful inspection of Eq. 2.48 reveals the truth that $\hat{\mathbf{l}}$ exists if and only if

[C] has an eigenvalue of +1; in which case $\hat{\boldsymbol{\ell}}$ is the corresponding unit eigenvector of [C]. Goldstein (ref. 3) proves that all "proper" direction cosine matrices (those corresponding to displacement of right-handed axes imbedded in a rigid body) do in fact have an eigenvalue of +1; the eigenvalue and corresponding eigenvector are unique (to within a sign on $\hat{\boldsymbol{\ell}}$ and ϕ) except for the case of zero angular displacement. The lack of sign uniqueness, as will be evident, does not cause a practical difficulty.

Since Euler's theorem reduces the general angular displacement to a single rotation about a fixed line, we can make immediate use of the developments in Section 2.3 to parameterize the direction cosine matrix in terms of $\hat{\boldsymbol{\ell}}$ and ϕ. Specifically, if we take $\mathbf{r} = \hat{\mathbf{n}}_i$ and $\mathbf{r}' = \hat{\mathbf{n}}_i' \equiv \hat{\mathbf{b}}_i$ in the general Eq. 2.29, we obtain

$$\hat{\mathbf{b}}_i = (1 - \cos\phi)(\hat{\boldsymbol{\ell}} \cdot \hat{\mathbf{n}}_i)\hat{\boldsymbol{\ell}} + \cos\phi\hat{\mathbf{n}}_i + \sin\phi(\hat{\boldsymbol{\ell}} \times \hat{\mathbf{n}}_i) \ , \ i = 1,2,3 \qquad (2.49)$$

If we substitute $\hat{\boldsymbol{\ell}} = \ell_1\hat{\mathbf{n}}_1 + \ell_2\hat{\mathbf{n}}_2 + \ell_3\hat{\mathbf{n}}_3$ and carry out the implied algebra in Eq. 2.49, we immediately obtain Eq. 2.11 with

$$[C] = \begin{bmatrix} \ell_1^2(1-c\phi)+c\phi & \ell_1\ell_2(1-c\phi)+\ell_3 s\phi & \ell_1\ell_3(1-c\phi)-\ell_2 s\phi \\ \ell_2\ell_1(1-c\phi)-\ell_3 s\phi & \ell_2^2(1-c\phi)+c\phi & \ell_2\ell_3(1-c\phi)+\ell_1 s\phi \\ \ell_3\ell_1(1-c\phi)+\ell_2 s\phi & \ell_3\ell_2(1-c\phi)-\ell_1 s\phi & \ell_3^2(1-c\phi)+c\phi \end{bmatrix} \qquad (2.50)$$

Since $\ell_1^2 + \ell_2^2 + \ell_3^2 = 1$, we have $[C(\ell_1,\ell_2,\ell_3,\phi)]$, but only three degrees of freedom, as expected. Notice we can verify immediately (by summing the trace of Eq. 2.50) that

$$\cos\phi = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1) \qquad (2.51)$$

and, by differencing the symmetric elements, we see that

$$2\ell_3\sin\phi = C_{12} - C_{21}$$
$$2\ell_2\sin\phi = C_{31} - C_{13}$$
$$2\ell_1\sin\phi = C_{23} - C_{32}$$

Multiplying the above three equations by ℓ_3, ℓ_2, ℓ_1, respectively (and making use of $\sum_{i=1}^{3} \ell_i^2 = 1$), we have

$$\sin\phi = \frac{1}{2}\,[\ell_3(C_{12} - C_{21}) + \ell_2(C_{31} - C_{13}) + \ell_1(C_{23} - C_{32})] \qquad (2.52)$$

Thus, given the direction cosines [C], we can solve for $\hat{\boldsymbol{\ell}}$ from Eq. 2.48, normalized so that $\hat{\boldsymbol{\ell}}$ is a unit vector, and ϕ from Eqs. 2.51 and 2.52. Notice that the sign choice on ℓ_i correctly affects the quadrant of ϕ in Eq. 2.52. Clearly a positive rotation about $\pm\,\hat{\boldsymbol{\ell}}$ is equivalent to a negative rotation about $\mp\,\hat{\boldsymbol{\ell}}$.

2.6 EULER PARAMETERS

Following Euler, we define the four *Euler Parameters* in terms of the *Principal Line* ($\hat{\boldsymbol{\ell}}$) and *Principal Angle* (ϕ) as

$$\begin{aligned} \beta_0 &= \cos\frac{\phi}{2} \\ \beta_1 &= \ell_1 \sin\frac{\phi}{2} \\ \beta_2 &= \ell_2 \sin\frac{\phi}{2} \\ \beta_3 &= \ell_3 \sin\frac{\phi}{2} \end{aligned} \qquad (2.53)$$

It is obvious that the four β_i's satisfy the constraint equation

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1. \qquad (2.54)$$

If we make use of the half-angle identities

$$\begin{aligned} \sin\phi &= 2\sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ \cos\phi &= 2\cos^2\frac{\phi}{2} - 1 \end{aligned}$$

and Eq. 2.54, the direction cosine matrix of Eq. 2.50 can be parameterized as a function of the Euler Parameters:

$$[C(\beta)] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix} \qquad (2.55)$$

The inverse relationships (β's in terms of the elements of [C]) can be deduced from Eq. 2.55:

$$\begin{aligned} \beta_0 &= \pm\frac{1}{2}\,(C_{11} + C_{22} + C_{33} + 1)^{1/2} \\ \beta_1 &= (C_{23} - C_{32})/4\beta_0 \end{aligned}$$

$$\beta_2 = (C_{31} - C_{13})/4\beta_0$$

$$\beta_3 = (C_{12} - C_{21})/4\beta_0 \qquad (2.56)$$

The first of these equations has an apparent sign ambiguity. However, there is no loss of generality in adopting the positive sign, since it is evident that changing all four signs of the β_i's in Eq. 2.55 *does not* change the direction cosine matrix. This is another reflection of the equivalence of rotating by a positive angle ϕ about $\pm \hat{\ell}$ to a $-\phi$ rotation about $\mp \hat{\ell}$. It is evident that Eqs. 2.56 contain a 0/0 indeterminacy whenever β_0 goes through zero; a computationally superior algorithm has been developed by Stanley (ref. 8):

$$\beta_0^2 = \frac{1}{4}(1 + 2\, C_{00} - T)$$

$$\beta_1^2 = \frac{1}{4}(1 + 2\, C_{11} - T)$$

$$\beta_2^2 = \frac{1}{4}(1 + 2\, C_{22} - T) \qquad (2.57)$$

$$\beta_3^2 = \frac{1}{4}(1 + 2\, C_{33} - T)$$

where Stanley defines

$$T = \text{Trace } [C] = C_{11} + C_{22} + C_{33} \qquad (2.58a)$$

$$C_{00} = T \qquad (2.58b)$$

and selects (for division by) the β_i computed from Eq. 2.57 which has the largest absolute value assuming it to be positive; the other three β_j's can be obtained by dividing β_i into the appropriate three of the following six equations (obtained by differencing and summing the symmetric elements of [C]):

$$\beta_0\beta_1 = (C_{23} - C_{32})/4$$

$$\beta_0\beta_2 = (C_{31} - C_{13})/4$$

$$\beta_0\beta_3 = (C_{12} - C_{21})/4 \qquad (2.59)$$

$$\beta_2\beta_3 = (C_{23} + C_{32})/4$$

$$\beta_3\beta_1 = (C_{31} + C_{13})/4$$

$$\beta_1\beta_2 = (C_{12} + C_{21})/4$$

The Euler parameters have many elegant properties; we shall develop the more important relationships in the following discussion.

One important property is the simple fashion in which Euler parameters of sequential rotations parameterize an equivalent single rotation. To develop these results, let $\{\hat{\mathbf{b}}\}$, $\{\hat{\mathbf{b}}'\}$, and $\{\hat{\mathbf{b}}''\}$ be three arbitrary positions of a triad of unit vectors; the direction cosine matrix [C] defining the relative orientations of $\{\hat{\mathbf{b}}\}$, $\{\hat{\mathbf{b}}'\}$, and $\{\hat{\mathbf{b}}''\}$ can be parameterized in terms of three sets of Euler parameters as

$$\{\hat{\mathbf{b}}'\} = [C(\beta')]\{\hat{\mathbf{b}}\} \tag{2.60a}$$

$$\{\hat{\mathbf{b}}''\} = [C(\beta'')]\{\hat{\mathbf{b}}'\} \tag{2.60b}$$

$$\{\hat{\mathbf{b}}''\} = [C(\beta)]\{\hat{\mathbf{b}}\} \tag{2.60c}$$

We seek a relationship of the form

$$\beta_i = \text{function}\ (\beta_0',\beta_1',\beta_2',\beta_3';\beta_0'',\beta_1'',\beta_2'',\beta_3'') \quad , \quad i = 0,1,2,3 \tag{2.61}$$

relating the three sets of Euler parameters. From Eq. 2.60, it is obvious that

$$[C(\beta)] = [C(\beta'')][C(\beta')] \tag{2.62}$$

Direct substitution of Eq. 2.55 into Eq. 2.62 and equating the corresponding elements leads to the most elegant result

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \begin{bmatrix} \beta_0'' & -\beta_1'' & -\beta_2'' & -\beta_3'' \\ \beta_1'' & \beta_0'' & \beta_3'' & -\beta_2'' \\ \beta_2'' & -\beta_3'' & \beta_0'' & \beta_1'' \\ \beta_3'' & \beta_2'' & -\beta_1'' & \beta_0'' \end{bmatrix} \begin{Bmatrix} \beta_0' \\ \beta_1' \\ \beta_2' \\ \beta_3' \end{Bmatrix} \tag{2.63}$$

or, by *transmutation* of Eq. 2.63

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \begin{bmatrix} \beta_0' & -\beta_1' & -\beta_2' & -\beta_3' \\ \beta_1' & \beta_0' & -\beta_3' & \beta_2' \\ \beta_2' & \beta_3' & \beta_0' & -\beta_1' \\ \beta_3' & -\beta_2' & \beta_1' & \beta_0' \end{bmatrix} \begin{Bmatrix} \beta_0'' \\ \beta_1'' \\ \beta_2'' \\ \beta_3'' \end{Bmatrix} \tag{2.64}$$

It is obvious by inspection that the coefficient matrices in Eqs. 2.63 and 2.64 are orthogonal; thus, any set of β's can be solved universally as a simple, nonsingular, bilinear combination of the other two. The successive rotation properties of the Euler parameters are thus quite elegant compared to any alternative parameterizations of [C]. For example, the analogous transformations for 3-1-3 (ϕ,θ,ψ) Euler angles can be verified to be

$$\begin{aligned}\theta &= \cos^{-1}(C_{33})\\ \phi &= \tan^{-1}(C_{31}/-C_{32})\\ \psi &= \tan^{-1}(C_{13}/C_{23})\end{aligned} \tag{2.65}$$

where the C_{ij} are the lengthy transcendental functions obtained by carrying out the matrix multiplication

$$[C(\phi,\theta,\psi)] = [C(\phi'',\theta'',\psi'')][C(\phi',\theta',\psi')] \tag{2.66}$$

using Eq. 2.34 to parameterize the two matrices on the right side of Eq. 2.66. These nine equations can be inverted for (ϕ,θ,ψ) as in Eq. 2.65, where the C_{ij} are the functions of $(\phi',\theta',\psi',\phi'',\theta'',\psi'')$ obtained from Eq. 2.66. It is a trivial observation that Eqs. 2.63 and 2.64 are vastly more attractive that Eqs. 2.65 and 2.66!

In applications, we often require a transformation from a set of Euler angles to the corresponding Euler parameters, and the inverse transformation. The transformation from Euler parameters into Euler angles is entirely straightforward, we simply calculate the direction cosine matrix from Eq. 2.55, then employ the inverse trigonometric relationships in Table 2.1, corresponding to the particular Euler angle rotation sequence. The transformation from Euler angles to Euler parameters can proceed by calculating numerical values for the direction cosines from Eq. 2.36, then using Eqs. 2.56, 2.57, or 2.59 to calculate the β's. However it is possible to derive very compact analytical expressions that are more efficient and avoid the branching logic of the path through the direction cosines.

To develop the direct transformation from Euler angles to Euler parameters, we first note from Eq. 2.53 that the elementary Euler angle rotations have the corresponding Euler parameter values

Rotation about a "1" axis: $\beta_o = \cos\frac{1}{2}\alpha$

$\beta_1 = \sin\frac{1}{2}\alpha$

$\beta_2 = \beta_3 = 0$

Rotation about a "2" axis: $\beta_o = \cos\frac{1}{2}\beta$

$\beta_2 = \sin\frac{1}{2}\beta$

$\beta_1 = \beta_3 = 0$

Rotation about a "3" axis: $\beta_o = \cos\frac{1}{2}\gamma$

$\beta_3 = \sin\frac{1}{2}\gamma$

$\beta_1 = \beta_2 = 0$

If we let $c_i = \cos\frac{1}{2}\theta_i$ and $s_i = \sin\frac{1}{2}\theta_i$, for the 3-1-3 Euler angles $(\theta_1,\theta_2,\theta_3)$, we can write from Eq. 2.63 the equivalent Euler parameters (replacing the above three elementary rotations) as

$$\begin{Bmatrix} \beta_o \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \begin{bmatrix} c_3 & 0 & 0 & -s_3 \\ 0 & c_3 & s_3 & 0 \\ 0 & -s_3 & c_3 & 0 \\ s_3 & 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & c_2 & s_2 \\ 0 & 0 & -s_2 & c_2 \end{bmatrix} \begin{Bmatrix} c_1 \\ 0 \\ 0 \\ s_1 \end{Bmatrix}$$

Upon carrying out the matrix multiplications we obtain

$$\beta_o = \cos\frac{1}{2}\theta_3 \cos\frac{1}{2}\theta_2 \cos\frac{1}{2}\theta_1 - \sin\frac{1}{2}\theta_3 \cos\frac{1}{2}\theta_2 \sin\frac{1}{2}\theta_1$$

$$\beta_1 = \cos\frac{1}{2}\theta_3 \sin\frac{1}{2}\theta_2 \cos\frac{1}{2}\theta_1 + \sin\frac{1}{2}\theta_3 \sin\frac{1}{2}\theta_2 \sin\frac{1}{2}\theta_1$$

$$\beta_2 = -\sin\frac{1}{2}\theta_3 \sin\frac{1}{2}\theta_2 \cos\frac{1}{2}\theta_1 + \cos\frac{1}{2}\theta_3 \sin\frac{1}{2}\theta_2 \sin\frac{1}{2}\theta_1$$

$$\beta_3 = \sin\frac{1}{2}\theta_3 \cos\frac{1}{2}\theta_2 \cos\frac{1}{2}\theta_1 + \cos\frac{1}{2}\theta_3 \cos\frac{1}{2}\theta_2 \sin\frac{1}{2}\theta_1$$

Finally, using trigonometric identities to simplify to the final transformation

$$\begin{aligned}
\beta_0 &= \cos \tfrac{1}{2}\,\theta_2 \cos \tfrac{1}{2}\,(\theta_1 + \theta_3) \\
\beta_1 &= \sin \tfrac{1}{2}\,\theta_2 \cos \tfrac{1}{2}\,(\theta_1 - \theta_3) \\
\beta_2 &= \sin \tfrac{1}{2}\,\theta_2 \sin \tfrac{1}{2}\,(\theta_1 - \theta_3) \\
\beta_3 &= \cos \tfrac{1}{2}\,\theta_2 \sin \tfrac{1}{2}\,(\theta_1 + \theta_3)
\end{aligned} \qquad (2.67)$$

The above can be paralleled for the other 11 sets of Euler angle sequences; the transformations for all 12 sets of Euler angles are summarized in Table 2.2. For ease in computer programming, we can employ the most useful universal transformation

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = [R_{\alpha\beta\gamma}] \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2.68)^*$$

where

$$\begin{aligned}
[R_{\alpha\beta\gamma}] = {} & [\cos \tfrac{1}{2}\,\theta_3\, R_0 + \sin \tfrac{1}{2}\,\theta_3\, R_\gamma]\,\cdot \\
& [\cos \tfrac{1}{2}\,\theta_2\, R_0 + \sin \tfrac{1}{2}\,\theta_2\, R_\beta]\,\cdot \\
& [\cos \tfrac{1}{2}\,\theta_1\, R_0 + \sin \tfrac{1}{2}\,\theta_1\, R_\alpha]\,.
\end{aligned} \qquad (2.69)$$

$$R_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

*Eq. 2.68 was developed by H. S. Morton, Jr. of the University of Virginia, Dec. 1973 (ref. 4).

TABLE 2.2

TRANSFORMATION FROM THE TWELVE SETS OF EULER ANGLES TO EULER PARAMETERS*

α-β-γ	β_0	β_1	β_2	β_3
1-2-1	$c\,\frac{\theta_2}{2}\,c(\frac{\theta_1+\theta_3}{2})$	$c\,\frac{\theta_2}{2}\,s(\frac{\theta_1+\theta_3}{2})$	$s\,\frac{\theta_2}{2}\,c(\frac{\theta_1-\theta_3}{2})$	$s\,\frac{\theta_2}{2}\,s(\frac{\theta_1-\theta_3}{2})$
2-3-2	$c\,\frac{\theta_2}{2}\,c(\frac{\theta_1+\theta_3}{2})$	$s\,\frac{\theta_2}{2}\,s(\frac{\theta_1-\theta_3}{2})$	$c\,\frac{\theta_2}{2}\,s(\frac{\theta_1+\theta_3}{2})$	$s\,\frac{\theta_2}{2}\,c(\frac{\theta_1-\theta_3}{2})$
3-1-3	$c\,\frac{\theta_2}{2}\,c(\frac{\theta_1+\theta_3}{2})$	$s\,\frac{\theta_2}{2}\,c(\frac{\theta_1-\theta_3}{2})$	$s\,\frac{\theta_2}{2}\,s(\frac{\theta_1-\theta_3}{2})$	$c\,\frac{\theta_2}{2}\,s(\frac{\theta_1+\theta_3}{2})$
1-3-1	$c\,\frac{\theta_2}{2}\,c(\frac{\theta_3+\theta_1}{2})$	$c\,\frac{\theta_2}{2}\,s(\frac{\theta_3+\theta_1}{2})$	$s\,\frac{\theta_2}{2}\,s(\frac{\theta_3-\theta_1}{2})$	$s\,\frac{\theta_2}{2}\,c(\frac{\theta_3-\theta_1}{2})$
2-1-2	$c\,\frac{\theta_2}{2}\,c(\frac{\theta_3+\theta_1}{2})$	$s\,\frac{\theta_2}{2}\,c(\frac{\theta_3-\theta_1}{2})$	$c\,\frac{\theta_2}{2}\,s(\frac{\theta_3+\theta_1}{2})$	$s\,\frac{\theta_2}{2}\,s(\frac{\theta_3-\theta_1}{2})$
3-2-3	$c\,\frac{\theta_2}{2}\,c(\frac{\theta_3+\theta_1}{2})$	$s\,\frac{\theta_2}{2}\,s(\frac{\theta_3-\theta_1}{2})$	$s\,\frac{\theta_2}{2}\,c(\frac{\theta_3-\theta_1}{2})$	$c\,\frac{\theta_2}{2}\,s(\frac{\theta_3+\theta_1}{2})$
1-2-3	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$
	$-s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$+c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$+s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$
2-3-1	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$
	$-s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$+s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$+c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$
3-1-2	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$
	$-s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$+s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$+c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$
1-3-2	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$
	$+s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$+s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$
2-1-3	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$
	$+s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$+s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$
3-2-1	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$
	$+s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$+s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$-c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$

*c $\equiv$ cos, s $\equiv$ sin

Equation 2.68 captures all 12 transformations of Table 2.2 in a general form suitable for a universal computer algorithm.

The above properties are significant, attractive features of the Euler prameters; however, the most impressive property is the kinematical differential equation which we now develop. We seek equations of the form $\dot{\beta}_i = \text{function}_i\ (\beta_0, \beta_1, \beta_2, \beta_3, \omega_1, \omega_2, \omega_3)$. These can be obtained by differentiating Eq. 2.56. For example, consider the derivation of the equation for $\dot{\beta}_0$, by differentiating the first term of Eq. 2.56, we have

$$\dot{\beta}_0 = \frac{(\dot{C}_{11} + \dot{C}_{22} + \dot{C}_{33})}{8\beta_0} \tag{2.70}$$

From Eq. 2.22, substitute for the $\dot{C}_{ij}$'s

$$\begin{aligned} \dot{C}_{11} &= \omega_3 C_{21} - \omega_2 C_{31} \\ \dot{C}_{22} &= -\omega_3 C_{12} + \omega_1 C_{32} \\ \dot{C}_{33} &= \omega_2 C_{13} - \omega_1 C_{23} \end{aligned} \tag{2.71}$$

so that Eq. 2.70 becomes

$$\dot{\beta}_0 = \frac{(C_{32} - C_{23})\omega_1 + (C_{13} - C_{31})\omega_2 + (C_{21} - C_{12})\omega_3}{8\beta_0} \tag{2.72}$$

and eliminating the C_{ij}'s in favor of the β_i's using the first three terms of Eq. 2.59, we have the desired equation for $\dot{\beta}_0$.

$$\dot{\beta}_0 = -\frac{1}{2}(\beta_1\omega_1 + \beta_2\omega_2 + \beta_3\omega_3) \tag{2.73}$$

Similarly, we can derive equations for $\dot{\beta}_1$, $\dot{\beta}_2$, $\dot{\beta}_3$; the resulting four equations can be written in matrix form

$$\begin{Bmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} \tag{2.74}$$

or, by *transmutation* of Eq. 2.74

$$\begin{Bmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \qquad (2.75)$$

Eq. 2.74 or, alternatively, Eq. 2.75 are indeed useful results. Observe that the transformation matrix in Eq. 2.75 relating $\dot{\beta}_i$'s and ω_j's is orthogonal; therefore this relationship is universally nonsingular (whereas the corresponding kinematical equations for any three angle set is transcendental, nonlinear, and contains a 0/0 singularity). Further, note Eq. 2.74, for $\omega_i(t)$ measured or integrated a priori, is a *rigorously linear* differential equation

$$\{\dot{\beta}\} = [\omega(t)]\{\beta\} \qquad (2.76)$$

Since $[\omega(t)]$ is skew symmetric (compare Eqs. 2.74 and 2.76), we can show that $\sum_i \beta_i^2 = 1$ is a rigorous integral of the solution. To see this, let

$$\|\beta\| = \sum_{i=0}^{3} \beta_i^2 = \{\beta\}^T\{\beta\} \qquad (2.77)$$

then

$$\frac{d}{dt}\|\beta\| = \{\dot{\beta}\}^T\{\beta\} + \{\beta\}^T\{\dot{\beta}\} \qquad (2.78)$$

substitution of Eq. 2.76 into Eq. 2.78 yields

$$\frac{d}{dt}\|\beta\| = \{\beta\}^T[[\omega]^T + [\omega]]\{\beta\} \qquad (2.79)$$

and since $[\omega]^T = -[\omega]$, we see from Eq. 2.79 that $\frac{d}{dt}\|\beta\| = 0$. Since $\|\beta(t_0)\| = 1$ will be established by any valid choice of initial Euler parameters, we can see that any (accurate) solution of Eqs. 2.74, 2.75, or 2.76 will guarantee $\|\beta(t)\| = 1$. In fact, this is a standard (necessary) condition used to test numerical solutions of Eq. 2.75, to, for example, control step size.

The Euler parameters represent a fascinating example of *regularization* in mechanics. Through a judicious choice of coordinates, we are able to eliminate the singularities usually present. It is also a beautiful truth that all possible rotational motions correspond to a path on the surface of a four dimensional unit sphere. For certain cases, analytical solution for the Euler parameters are possible, see refs. 4 and 11.

2.7 OTHER ORIENTATION PARAMETERS

Aside from Euler angles, Euler parameters, and the direction cosines, there are an infinity of less commonly adopted possible descriptions of orientation. We summarize here the most prominent members of this large family of possibilities.

Rodriquez Parameters $(\lambda_1, \lambda_2, \lambda_3)$

These parameters (ref. 6) are intimately related to the principal rotation and, therefore, to the Euler parameters. In fact, the λ_i's are simply

$$\begin{aligned} \lambda_1 &= \beta_1/\beta_o = \ell_1 \tan \tfrac{1}{2}\phi \\ \lambda_2 &= \beta_2/\beta_o = \ell_2 \tan \tfrac{1}{2}\phi \\ \lambda_3 &= \beta_3/\beta_o = \ell_3 \tan \tfrac{1}{2}\phi \end{aligned} \tag{2.80}$$

Clearly, these parameters have an unbounded behavior near $\phi = (2n+1)\pi$ where n is integer and therefore are less attractive than the β's themselves for most purposes. The geometric and kinematic relationships governing the λ_i's can be immediately derived from the corresponding Euler parameter equations.

Cayley-Klein Parameters

These four complex parameters (ref. 7) are the combinations of the Euler parameters

$$\begin{aligned} \alpha &= \beta_o + i\beta_3 \quad , \quad i^2 = -1 \\ \beta &= -\beta_2 + i\beta_1 \\ \gamma &= \beta_2 + i\beta_1 \\ \delta &= \beta_o - i\beta_3 \end{aligned} \tag{2.81}$$

with the inverse relations being

$$\begin{aligned}
\beta_0 &= (\alpha + \delta)/2 \\
\beta_1 &= -i(\beta + \gamma)/2 \\
\beta_2 &= -(\beta - \gamma)/2 \\
\beta_3 &= -i(\alpha - \delta)/2
\end{aligned} \tag{2.82}$$

and the parameterization of the direction cosine matrix being

$$[C(\alpha,\beta,\gamma,\delta)] = \begin{bmatrix} (\alpha^2-\beta^2-\gamma^2+\delta^2)/2 & i(-\alpha^2+\beta^2-\gamma^2+\delta^2)/2 & (\beta\delta-\alpha\gamma) \\ i(\alpha^2+\beta^2-\gamma^2-\delta^2)/2 & (\alpha^2+\beta^2+\gamma^2+\delta^2)/2 & -i(\alpha\gamma+\beta\delta) \\ (\gamma\delta-\alpha\beta) & i(\alpha\beta+\gamma\delta) & (\alpha\delta+\beta\gamma) \end{bmatrix} \tag{2.83}$$

An alternate set of four complex Cayley-Klein-like parameters (having advantages for certain applications) is discussed in Ref. [11].

Quaternions

Hamilton (ref. 7) developed quaternion algebra due to the intimate connection with Euler Parameters and rotation of vectors. Quaternions are four dimensional entities having one real and three complex parts,

$$q = q_o + iq_1 + jq_2 + kq_3 \tag{2.84}$$

where

$$i\cdot i = -1,\ j\cdot j = -1,\ k\cdot k = -1$$

$$i\cdot j = k,\ j\cdot k = i,\ -k\cdot j = i,\ \text{etc.}$$

For a unit quaternion $\sum_{i=o}^{3} q_i^2 = 1$. From Ames and Murnahan's (ref. 10) development of quaternion algebra, it can be shown that the quaternion multiplication $q = q''q'$ yields a new quaternion (q) whose components relate to those of q'' and q' as

$$\begin{Bmatrix} q_o \\ q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{bmatrix} q_o'' & -q_1'' & -q_2'' & -q_3'' \\ q_1'' & q_o'' & -q_3'' & q_2'' \\ q_2'' & q_3'' & q_o'' & -q_1'' \\ q_3'' & -q_2'' & q_1'' & q_o'' \end{bmatrix} \begin{Bmatrix} q_o' \\ q_1' \\ q_2' \\ q_3' \end{Bmatrix} \tag{2.84}$$

Comparison of Eq. 2.84 with Eq. 2.63 allows us to identify

$$q_o = \beta_o$$

$$q_i = -\beta_i \quad , \quad i = 1,2,3 \qquad (2.85)$$

Or, alternatively, we can consider the Euler parameters to be the elements of the quaternion

$$q = \beta_o - i\beta_1 - j\beta_2 - k\beta_3. \qquad (2.86)$$

Historically, quaternions played a "stepping stone" role en route to modern vector algebra and quaternion algebra is not presently widely used. Unfortunately, Hamilton's association of Euler parameters with quaternions served to make them more obscure when quaternion algebra fell largely into a "museum-piece" status. As we have seen in the foregoing, the basic kinematic and geometric relationships governing the Euler parameters can be easily developed without relying upon interpreting them as elements of a quaternion.

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