CHAPTER 3

BASIC PRINCIPLES OF DYNAMICS

We develop here the fundamental concepts which underlie the differential equations governing translational, rotational, and flexural motion of finite continuous bodies.

From one very basic point of view, we could begin with the fundamental truth that a rigid body possesses six degrees of freedom; choosing arbitrarily six coordinates (which uniquely position and orient the body in inertial space), then derive the differential equations governing these particular six coordinates. From the viewpoint of generality (since we are concerned with multiple body and flexible systems) and with the objective of illuminating the positive and negative consequences of particular coordinate selections, however, we choose an alternate approach. Using vector and vector/dyadic notations, we develop "momentum equations of motion" which hold for all valid coordinate choices; these equations, augmented by certain kinematical results (from Chapter 2) can be specialized for an infinite variety of particular coordinate specifications and system model assumptions.

We will also develop equations of motion via the generalized approaches due to Lagrange, Hamilton, et al.; which we will find, from one viewpoint, to be less general, in the sense that these methods constrain us to select particular coordinates at the onset. However, advantages associated with handling of internal constraint forces and algebraic advantages when we seek to include many internal degrees of freedom (associated with moving parts and flexibility), will be seen to strongly justify the generalized approaches. In addition to the algebraic order brought to formulating high dimensional equations of motion, the generalized approaches are very useful in developing approximate perturbation methods for analytically tractable cases. It is also significant that the generalized approach has been found more attractive for

computer automation of the algebra implicit in deriving the equations of motion (e.g., using the MACSYMA (ref. 1) computer automated algebraic manipulator).

3.1 NEWTONIAN/EULERIAN EQUATIONS OF MOTION

3.1.1 Translational Motion

We begin by considering the arbitrary (obviously variable density) body B depicted in Figure 3.1.

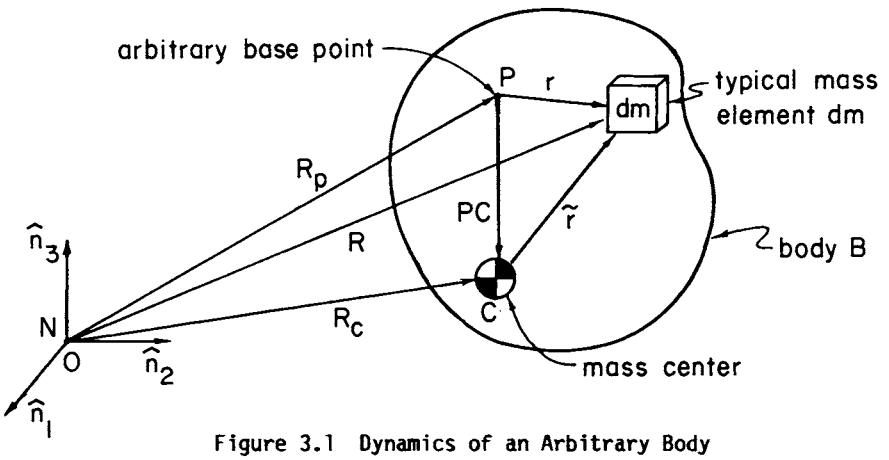

Figure 3.1 Dynamics of an Arbitrary Body

We assume the typical differential mass element, dm, to be a particle and apply Newton's laws to dm's motion. Newton's second law requires

$$dm\ddot{R} = dF + df \tag{3.1}$$

where

$(\dot{}) = \frac{d}{dt}(\ )_N$, the time derivative as seen in the inertial frame N

dF = the differential *external* force applied to dm (i.e., the surface and/or field force *not* due to interaction with other mass elements of B).

df = the differential *internal* forces applied to dm (i.e., any force arising from interaction with other elements of B).

The *mass center* or *centroid* of B is defined as that point C for which the

first mass moment is zero; with reference to Figure 3.1, the centroid is thus determined by the condition

$$\iiint_B (\mathbf{R} - \mathbf{R}_C)dm \equiv \iiint_B \tilde{\mathbf{r}}dm = 0 \tag{3.2}$$

or, since $\mathbf{R}_C$ locates a specific point fixed relative to B, it can be taken from under the volume integral to yield

$$\mathbf{R}_C = \frac{1}{m}\iiint_B \mathbf{R}dm \tag{3.3}*$$

where $m \equiv \iiint_B dm$ is B's total mass. Observe that the inertial acceleration of C follows from differentiation of Eq. 3.3 as

$$\ddot{\mathbf{R}}_C = \frac{1}{m}\iiint_B \ddot{\mathbf{R}}dm \tag{3.4}$$

where we have ignored the possible time variation of the boundary of the control volume over which the space integration is performed. Upon substitution of Eq. 3.1, Eq. 3.4 reduces to the *super particle theorem* or, more commonly "the translational equations of motion"

$$m\ddot{\mathbf{R}}_C = \mathbf{F} \tag{3.5}$$

where

$\mathbf{F} \equiv \iiint_B d\mathbf{F}$, the vector sum of all *external* forces acting on B

$\iiint_B d\mathbf{f} = 0$, for internal forces occurring in equal and opposite pairs.

3.1.2 Angular Momentum of Continuous Systems

Central to the Newtonian/Eulerian formulations of rotational motion is the time-variation of the system's angular momentum. For a finite continuous (not necessarily rigid) system B, we adopt the definition

$$\mathbf{H}^{B/P} \equiv \iiint_B \mathbf{r} \times \frac{d}{dt}(\mathbf{r})_F dm \tag{3.6}$$

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*It is significant to note that a similar result holds for *any* origin. For example, if we express $\tilde{\mathbf{r}} = \mathbf{R}-\mathbf{R}_C \equiv \mathbf{r} - \mathbf{PC}$ in Eq. 3.2, we find $\mathbf{PC} = 1/m \iiint_B \mathbf{r}dm$.

which is the system B's angular momentum about an arbitrary point P relative to reference frame F.

For most practical applications, we refer momenta to the inertial frame (although P may or may not be fixed in N or B), in which case we have

$$ {}^{N}H^{B/P} \equiv \iiint_B r \times \dot{r}\,dm \tag{3.7} $$

Please note that some authors use different base points for the position and velocity vectors in Eq. 3.7 which may cause confusion when comparing equations from several sources. For example, some authors (e.g., ref. 10) choose the angular momentum definition

$$ \mathbf{h} = \iiint_B \mathbf{r} \times \dot{\mathbf{R}}\,dm \tag{3.8} $$

where $\dot{\mathbf{R}} \equiv \frac{d}{dt}(\mathbf{R})_N$ is the velocity of dm relative to the inertial origin as seen from N rather than $\dot{\mathbf{r}} \equiv \frac{d}{dt}(\mathbf{r})_N$ which is the velocity of dm relative to the arbitrary base point P, as seen from N. Note that ${}^{N}H^{B/P}$ can be written as

$$ {}^{N}H^{B/P} = \iiint_B \mathbf{r} \times (\dot{\mathbf{R}} - \dot{\mathbf{R}}_P)\,dm = \mathbf{h} + \dot{\mathbf{R}}_P \times \iiint_B \mathbf{r}\,dm \tag{3.9} $$

Thus, we conclude that the two momentum definitions Eqs. 3.7 and 3.8 are identical if (i) P is fixed in N, and/or (ii) P is the centroid of B($\iiint \mathbf{r}\,dm = 0$), or (iii) $\dot{\mathbf{R}}_P$ is always parallel to $\iiint_B \mathbf{r}\,dm$. As will be evident, the selection of a particular definition for angular momentum is not a decision of overwhelming significance; it <u>is</u> important, however that a specific definition be adopted and that subsequent results be understood using a consistent underlying definition.

The definition Eq. 3.7 applies to an arbitrary, generally flexible body or system of bodies. The time variation of angular momentum, as seen in the inertial frame, follows immediately by differentiation of the defining Eq. 3.7 as

$$ {}^{N}\dot{H}^{B/P} \equiv \frac{d}{dt}\left[\iiint_B \mathbf{r} \times \dot{\mathbf{r}}\,dm\right]_N $$

$$ = \iiint_B \dot{\mathbf{r}} \times \dot{\mathbf{r}}\,dm + \iiint_B \mathbf{r} \times \ddot{\mathbf{r}}\,dm $$

$$= 0 + \iiint_B \mathbf{r} \times (\ddot{\mathbf{R}} - \ddot{\mathbf{R}}_P)dm$$

or

$$N\hat{H}^{B/P} = \iiint_B \mathbf{r} \times \ddot{\mathbf{R}}dm + \ddot{\mathbf{R}}_P \times \iiint_B \mathbf{r}dm$$

Since $\ddot{\mathbf{R}}dm \equiv d\mathbf{F} + d\mathbf{f}$, from Eq. 3.1, we have the fundamental truth ("Euler's Equations")

$$N\hat{H}^{B/P} = \mathbf{L}_P + \ddot{\mathbf{R}}_P \times \iiint_B \mathbf{r}dm \qquad (3.10)$$

where we have introduced the definition $\mathbf{L}_P = \iiint_B \mathbf{r} \times d\mathbf{F}$, the net *external* torque on B about point P and made use of $\mathbf{0} = \iiint_B \mathbf{r} \times d\mathbf{f}$, the net *internal* torque on B, for internal forces occurring in equal and opposite pairs.

Notice the second term of Eq. 3.10 vanishes if one or more of the following conditions is satisfied

(i) $\ddot{\mathbf{R}}_P = \mathbf{0}$ (P moves with constant velocity in N)

(ii) $\iiint_B \mathbf{r}dm = \mathbf{0}$ (P is the centroid C of B)

(iii) $\ddot{\mathbf{R}}_P$ is parallel to $\iiint_B \mathbf{r}dm$

Thus, we usually write Eqs. 3.10 as simply

$$\hat{H} = \mathbf{L} \qquad (3.10)$$

with an implicit understanding that the base point is the mass center and without the explicit reference frames designations. The latter may become useful, however, if multi-body configurations are analyzed. Equations 3.5 and 3.10, except for certain geometric/kinematic results (Chapter 2), provide a complete basis for determining six second order (or 12 first order) differential equations governing general translational and rotational motion of a rigid spacecraft. They can also play a central role for non-rigid spacecraft since these equations apply to an arbitrary, generally deformable, continuous body or collection of bodies. However they are *incomplete* for the multi-body and/or flexible system case; the internal and flexural degrees of freedom

require additional equations of motion; Eqs. 3.5 and 3.10 will, however, lead to correct *coupling* to these additional differential equations.

3.1.3 Translational Equations of Motion for Rigid Spacecraft

The above developments apply to dynamics of rigid satellites; the translational equations of motion follow immediately from Eq. 3.5. The two-body-problem idealization is commonly adopted as the zeroth approximation for translational (orbital) spacecraft motion. Referring to Figure 3.2, we consider two finite bodies; Body 1 (mass m_1) is taken as the "primary" (e.g., the earth), Body 2 (mass m_2) is taken as the satellite. If we assume both the primary and the satellite to be particles (or, equivalently, to be rigid bodies possessing spherically symmetric mass distributions), the net gravitational force acting on m_2 due to the presence of m_1 obeys Newton's law of universal gravitation

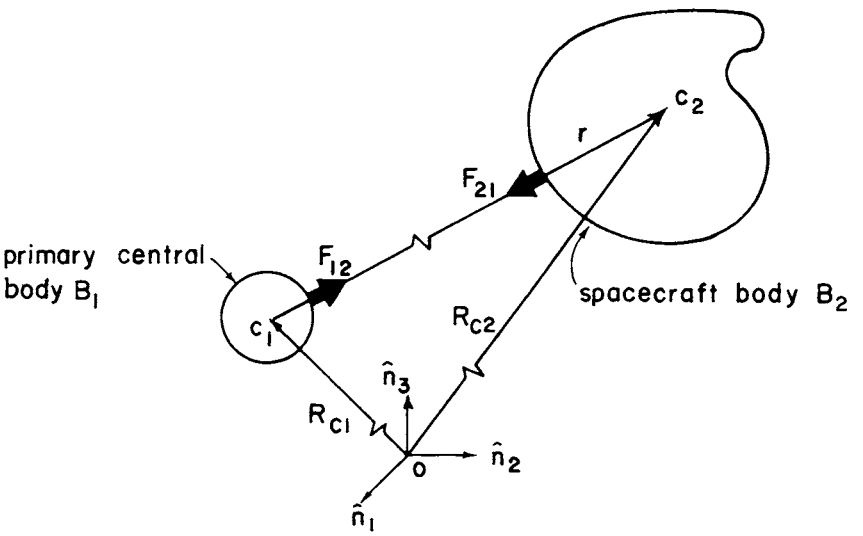

Figure 3.2 The Two Body Problem

$$F_{21} = -F_{12} = -\frac{Gm_1m_2}{r^3}\,r \tag{3.11}$$

where

G is the universal gravitation constant

$r = R_{C2} - R_{C1}$ is the spacecraft's mass center location relative to the earth's mass center.

In practice, the departure of the earth's gravity field from Eq. 3.11 is significant, (Eq. 3.11 provides only about four accurate digits for the gravitational force at 300 km above the earth's surface, due to departures of the earth from spherical symmetry), and additional effects (atmospheric drag, radiation pressure, lunar-solar attraction, thrust, ..., etc.) must be considered (refs. 2,3,4).

The primary's and the satellite's translational equations of motion (relative to an unspecified inertial frame) follow from application of Eq. 3.5 as

$$m_1\ddot{R}_{C1} = \frac{Gm_1m_2}{r^3}\,r + P_1 \tag{3.12a}$$

$$m_2\ddot{R}_{C2} = -\frac{Gm_1m_2}{r^3}\,r + P_2 \tag{3.12b}$$

where P_1 and P_2 are the vector sums of all perturbation forces other than the inverse square gravitational interaction. The inertial acceleration of m_2 relative to m_1 follows from differentiation of $r \equiv R_{C2} - R_{C1}$ as

$$\ddot{r} \equiv \frac{d^2}{dt^2}\,(R_{C2} - R_{C1})_N = \ddot{R}_{C2} - \ddot{R}_{C1} \tag{3.13}$$

Substitution of Eq. 3.12 into Eq. 3.13 immediately yields the 2-body equation of relative motion

$$\ddot{r} = -\frac{\mu}{r^3}\,r + p \tag{3.14}$$

where

$$\mu = G(m_1 + m_2) \cong 398601.2 \text{ km}^3/\text{sec}^2 \qquad (3.15)^*$$

$\mathbf{p} = \mathbf{P}_2/m_2 - \mathbf{P}_1/m_1$ is the relative "perturbative acceleration" (3.16)

For $\mathbf{p} = \mathbf{0}$, we have the two body problem whose non-trivial (but analytical) solution has played a most central historical role in the development of analytical mechanics, and, indeed, many facets of mathematics. There are several beautiful variants of this analytical solution which provide the solution of the initial value problem

$$\begin{Bmatrix} \mathbf{r}(t) \\ \dot{\mathbf{r}}(t) \end{Bmatrix} = \text{function } [(t - t_0) , \mathbf{r}(t_0) , \dot{\mathbf{r}}(t_0)]$$

One particularly attractive solution is summarized in Appendix D. Several other most elegant and important solutions can be derived (ref. 2,3,4). For example for $\mathbf{p} \neq \mathbf{0}$, we generally require numerical methods (ref. 2,3) to obtain $\mathbf{r}(t)$, $\dot{\mathbf{r}}(t)$; but for certain $\mathbf{p}$, approximate analytical perturbation solutions have been obtained (ref. 2,4). For $\mathbf{p}=\mathbf{0}$, or for $\mathbf{p}$ which do not depend significantly upon the orientation of the satellite B_2 (Fig. 3.2), the translational (orbital) differential equations are often found to be *uncoupled* from the rotational (attitude) differential equations, to a high degree of approximation. The reverse (uncoupling of the rotational equations from the solution of the translational motion) is less often an acceptable approximation (due primarily to gravity gradient and aerodynamic torques). As a consequence, one can usually solve for the translational motion first, without consideration of rotational motion, then use the resulting ephemerides (position and velocity history) to permit calculation of position and velocity

*Note that while G and m_1 are each known with about four digit accuracy, $\mu \cong Gm_1$ is known with 7+ digit accuracy. This is because μ appears in the equations of motion (Eq. 3.14) and can be recovered accurately, along with initial conditions and other model parameters, from best fits of the solution of Eq. 3.14 to satellite tracking data.

dependent torques during solution of the differential equations which govern the rotational motion.

3.1.4 Rotational Equations of Motion for Rigid Spacecraft

The general Eulerian equations of rotational motion of Section 3.1.2 are specialized here for the case of a rigid spacecraft. We begin by specializing the angular momentum definition of Eq. 3.7 for the case of a rigid body. Note

$$\dot{\mathbf{r}} \equiv \frac{d}{dt}(\mathbf{r})_N = \frac{d}{dt}(\mathbf{r})_B + \boldsymbol{\omega} \times \mathbf{r}$$

where $\boldsymbol{\omega}$ is the angular velocity of B relative to N. We restrict P to be a point fixed in B so that $\mathbf{r}$ is a constant vector; thus $\frac{d}{dt}(\mathbf{r})_B = 0$ and we have

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} \tag{3.17}$$

Substitution of Eq. 3.17 into Eq. 3.7 yields the integral definition of angular momentum for the case of a rigid body

$$\mathbf{H} \equiv {}^{N}\mathbf{H}^{B/P} = \iiint_B \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})dm \tag{3.18}$$

Referring to Figure 3.3, we introduce an orthogonal set of B-fixed unit vectors

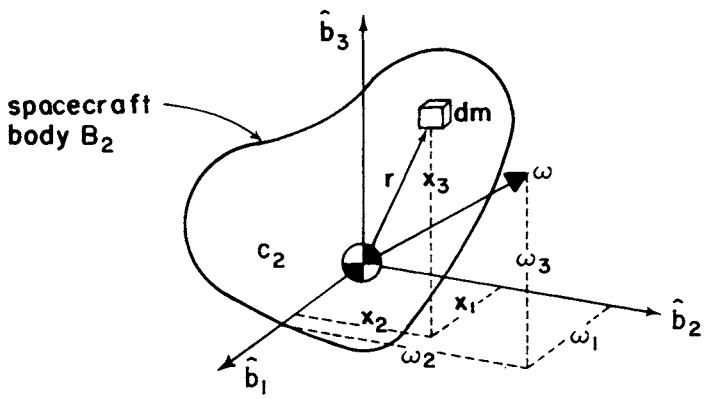

Figure 3.3 Spacecraft Centroidal Axes

$\{\hat{\mathbf{b}}\}$ with an origin at C_2 (the centroid of body 2, the spacecraft); defining components of $\mathbf{r}$, $\boldsymbol{\omega}$ and $\mathbf{H}$ along $\{\hat{\mathbf{b}}\}$ as

$$\mathbf{r} = \sum_{i=1}^{3} x_i \hat{\mathbf{b}}_i \quad , \quad \boldsymbol{\omega} = \sum_{i=1}^{3} \omega_i \hat{\mathbf{b}}_i \quad , \quad \mathbf{H} = \sum_{i=1}^{3} H_i \hat{\mathbf{b}}_i . \tag{3.19}$$

Substitution of Eq. 3.19 into Eq. 3.18, and carrying out the vector algebra and integrations, leads to the three component equations

$$\begin{aligned} H_1 &= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ H_2 &= I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ H_3 &= I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned} \tag{3.20}$$

or, in matrix form

$$\{H\} = [I]\{\omega\} \tag{3.21}$$

where the symmetric moment of inertia matrix is

$$[I] = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} = \iiint_B \begin{bmatrix} (x_2^2+x_3^2) & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & (x_1^2+x_3^2) & -x_2x_3 \\ -x_3x_1 & -x_3x_2 & (x_1^2+x_2^2) \end{bmatrix} dm, \tag{3.22}$$

$\{\omega\}^T = \{\omega_1\ \omega_2\ \omega_3\}$ and $\{H\}^T = \{H_1\ H_2\ H_3\}$. Alternatively, and without selecting any particular basis vectors, we can write $\mathbf{H}$ in vector/dyadic form. Notice that Eq. 3.18 can be manipulated as

$$\mathbf{H} = -\iiint_B \mathbf{r} \times (\mathbf{r} \times \boldsymbol{\omega}) dm$$

Using the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \equiv \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$, $\mathbf{H}$ can be further manipulated into the following forms

$$\begin{aligned} \mathbf{H} &= \iiint_B [\mathbf{r}\cdot\mathbf{r})\boldsymbol{\omega} - \mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega})]dm \\ &= \{\iiint_B [(\mathbf{r}\cdot\mathbf{r})\ \underline{\Delta} - \mathbf{r}\,\mathbf{r}]dm\} \cdot \boldsymbol{\omega} \end{aligned}$$

or

$$\mathbf{H} = \underline{\mathbf{I}} \cdot \boldsymbol{\omega} \tag{3.23}$$

where $\underline{\mathbf{I}} \equiv \iiint_B [(r^2)\underline{\Delta} - \mathbf{r}\,\mathbf{r}]dm$ is the *inertia dyadic* and $\underline{\Delta}$ is the *unit*

dyadic. In the $\{\hat{b}\}$ basis vectors, note Eq. 3.19 and the particular componentiations

$$\underline{I} = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\hat{b}_i\hat{b}_j \quad , \quad \underline{\Delta} = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\hat{b}_i\hat{b}_j = \sum_{i=1}^{3} \hat{b}_i\hat{b}_i$$

so that Eq. 3.23 can be shown to be identical to Eq. 3.20 for this special case. The inertia matrix and important practical issues associated with the inertia transformations for general coordinate axes rotation and translation are treated in References 6,9. In particular, we note that it is *always* possible to determine principal body fixed axes for which the inertia matrix is diagonal; thus the principal body fixed axes' components of angular momentum are denoted

$$H_1 = I_1\omega_1 \ , \ H_2 = I_2\omega_2 \ , \ H_3 = I_3\omega_3 \tag{3.24}$$

where I_1, I_2, I_3 are the eigenvalues (principal inertias) of the generally non-diagonal [I] for *any* (or all other) choices of body-fixed centroidal axes.

We note that the case of non-distinct eigenvalues (principal inertias), results in the corresponding eigenvectors (principal axes) not being unique. This is of no particular practical difficulty, however. For example, *any* choice of right-handed centroidal axes qualify as principal axes for the case of spherical symmetry. For the case of rotational symmetry, the one distinct eigenvector (along the symmetry axis) and *any* pair of mutually orthogonal vectors in the plane (whose normal is the distinct eigenvector) qualify as principal axes. Thus the lack of uniqueness is overcome by an arbitrary selection from the infinite set of physically admissible possibilities.

We restrict attention for this section to the case for which moments are taken about the centroid C_2, Eq. 3.10 is then simply

$$\dot{H} = L_C \tag{3.25}$$

and since

$$\dot{H} \equiv \frac{d}{dt}(H)_N$$

$$= \frac{d}{dt} (\underline{I} \cdot \omega)_N$$

$$\dot{H} = \frac{d}{dt} (\underline{I} \cdot \omega)_B + \omega \times (\underline{I} \cdot \omega)$$

or

$$\dot{H} = \frac{d}{dt} (\underline{I})_B \cdot \omega + \underline{I} \cdot \dot{\omega} + \omega \times (\underline{I} \cdot \omega) \tag{3.26}$$

For the case of a rigid body, $\underline{I}$ is a constant in B thus Eqs. 3.25 and 3.26 provide a general vector/dyadic set of *Eulerian Rotational Equations of Motion* as

$$\underline{I} \cdot \dot{\omega} + \omega \times (\underline{I} \cdot \omega) = L_C \tag{3.27}$$

This equation is valid with components taken in any body-fixed centroidal coordinate system; it is also valid for any choice of orientation coordinates. It is convenient to write the matrix equivalent of Eq. 3.27 as

$$[I]\{\dot{\omega}\} + [\tilde{\omega}][I]\{\omega\} = \{L_C\} \tag{3.28}$$

where

$\{L_C\}^T = \{L_1 \; L_2 \; L_3\}$, the $\{\hat{b}\}$ components of the external torque, and

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{3.29}$$

and $[I]$, $\{\omega\}$ are given by Eq. 3.22. Equation 3.28 is often solved for the angular acceleration as

$$\{\dot{\omega}\} = -[I]^{-1}[\tilde{\omega}][I]\{\omega\} + [I]^{-1}\{L_C\} \tag{3.30}$$

and, together with Eq. 3.14 and equations for modeling $\mathbf{p}$ and L_C, provide the central results for dynamics of *rigid* bodies in general and spacecraft in particular.

For the special choice of principal body fixed axes $[I] = \text{Diag.}(I_1, I_2, I_3)$, Eq. 3.30 yields the elegant and most commonly encountered version of Euler's equations

$$\dot{\omega}_1 = -(\frac{I_3-I_2}{I_1})\omega_2\omega_3 + \frac{L_1}{I_1}$$

$$\dot{\omega}_2 = -(\frac{I_1-I_3}{I_2})\omega_3\omega_1 + \frac{L_2}{I_2} \qquad (3.31)$$

$$\dot{\omega}_3 = -(\frac{I_2-I_1}{I_3})\omega_1\omega_2 + \frac{L_3}{I_3}$$

Notice the structure of the system of differential equations formed by Eqs. 3.14 and 3.31. Translation/rotational coupling (if present) occurs only through the forcing terms **p** and L_C. Of course, description of the complete "attitude state" dynamics requires an auxiliary set of kinematical differential equations relating the time derivatives of the chosen attitude coordinates to angular velocity components $(\omega_1,\omega_2,\omega_3)$. The most common choice is a set of three Euler angles. Table 2.1 summarizes the twelve sets of three kinematical differential equations corresponding to the twelve classical Euler angle sets. Simultaneous solution of the three (or more) kinematical differential equations with Eqs. 3.31 (and if necessary, Eqs. 3.14 due to coupling) provides the attitude as a function of time. For a general body, subject to general torques and initial conditions, the solution process requires numerical methods. However, for various special cases, the solution can be obtained analytically.

We now discuss briefly the particular case of torque-free motion (L_i=0); this case is of particular significance since it plays a "zeroth order approximation" role in rotational dynamics analogous to that of the two-body problem (**p**=**0**) in translational spacecraft dynamics. The zero torque, "pure spin special cases" are apparent by inspection; note if any two ω_i are zero, all three $\dot{\omega}_i$ are zero; thus we conclude "pure spin" initial conditions result in permanent pure spin motions. The stability of near-pure-spin motions are analyzed in Chapter 4 where it is found that (i) motions near the intermediate

axis of inertia are unstable, (ii) motions near pure spin about the largest (least) axis of inertia are stable (unstable) in the presence of energy dissipation, and (iii) in the absence of energy dissipation, motion near pure spin about the largest and least axes of inertia are oscillatory with ω_i periodic and the orientation history "quasi periodic". This can be shown considering Jacobi's classical 3-1-3 Euler angle solution, using the special inertial frame for which $\mathbf{H} = H\hat{n}_3$, where two of the three angles are periodic; the third angle oscillates about a linear function of time, the oscillations being determined by an incomplete elliptic integral of the third kind (ref. 6).

The torque-free rotational dynamics of an asymmetric body ("Poinsot's Problem") has historically given rise to many elegant motion analogies, but has not been adopted widely as a zeroth order approximation in solutions for spacecraft rotational motion. This situation (i.e., lack of wide adoption) is most likely due to the obscurity of Jacobi's classical solutions (being in terms of elliptic functions, elliptic integrals and/or theta functions), and the often poor degree of approximation of this solution in practical applications (i.e., the usual presence of significant departures from the torque-free and/or rigid body assumptions).

In references 5 and 6, it is established that the general time history of torque-free rotational motion of an arbitrary rigid body can be expressed using a set of Euler (quaternion) parameters $\{\beta(t)\}$, in terms of a 4 x 4 state transition $[\Phi(t)]$ which maps the initial Euler parameters $\{\beta(0)\}$ according to

$$\{\beta(t)\} = [\Phi(t)]\ \{\beta(0)\}$$

In reference 6 it is established that the state transition matrix can be decomposed as three "sequential" rotations (which capture the fundamental motions of precession, spin and nutation) as

$$[\Phi(t)] = [R(\beta''')][R(\beta'')][R(\beta')]$$

$$[R(\beta)] = \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & \beta_3 & -\beta_2 \\ \beta_2 & -\beta_3 & \beta_0 & \beta_1 \\ \beta_3 & \beta_3 & -\beta_1 & \beta_0 \end{bmatrix}$$

$$\{\beta'\} = \begin{Bmatrix} \cos \frac{1}{2} p\tau \\ \cos \delta \sin \frac{1}{2} p\tau \\ 0 \\ -\sin \delta \sin \frac{1}{2} p\tau \end{Bmatrix}, \quad \{\beta''\} = \begin{Bmatrix} \cos \frac{1}{2} \nu \\ -\sin \frac{1}{2} \nu \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\beta'''\} = \begin{Bmatrix} c_1 c_2 c_3 - \cos \delta s_1 c_2 s_3 + \sin \delta s_1 s_2 s_3 \\ c_1 c_2 s_3 + \cos \delta s_1 c_2 c_3 - \sin \delta s_1 s_2 c_3 \\ -c_1 c_2 c_3 - \cos \delta s_1 s_2 s_3 - \sin \delta s_1 c_2 s_3 \\ c_1 s_2 s_3 - \cos \delta s_1 s_2 c_3 - \sin \delta s_1 c_2 c_3 \end{Bmatrix}$$

$c_i \equiv \cos \frac{1}{2} \mu_i$ and $s_i \equiv \sin \frac{1}{2} \mu_i$ in $\{\beta'''\}$

p = constant precession rate

δ = mean (constant) nutation angle

ν = spin angle (a linear function of nondimensional time τ).

μ_i = three angles which can be expressed in terms of Jacobian elliptic functions and elliptic integrals.

The Euler parameters $\{\beta'\}$ describe the mean precessional motion (of the axis of largest or least inertia) about the angular momentum vector which is inertially fixed. The Euler parameters $\{\beta''\}$ describe the mean spin motion (rotation about the axis of largest or least inertia). The Euler parameters $\{\beta'''\}$ describe the "wobble" or nutation) due to the asymmetry of the body.

Calculation of $\{\beta'\}$ and $\{\beta''\}$ involves circular functions, whereas calculation of $\{\beta'''\}$ involves Jacobian elliptic functions and elliptic

integrals of the first and third kinds. This particular decomposition is due to H. S. Morton (ref. 6). It is conceptually elegant since $[R(\beta''')]$ is typically a small perturbation of an identity matrix (it becomes exactly an identity matrix if the body is axially symmetric). The above form of the solution holds for $\frac{H^2}{2I_1} \leq T \leq \frac{H^2}{2I_2}$; i.e., motion "nearer" spin about $\hat{b}_1$ than $\hat{b}_3$. Morton gives the details of both solution branches. In Chapter 4 we will give some detailed consideration to the classical solutions of Euler's equations for the single rigid body case and several multiple body cases.

3.2 GENERALIZED METHODS IN DYNAMICS

3.2.1 Kinetic Energy

With reference to Figure 3.1, we define the kinetic energy as

$$T \equiv \frac{1}{2} \iiint_B \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} dm \tag{3.32}$$

Using the mass center as a base point, we note $\mathbf{R} = \mathbf{R}_c + \tilde{\mathbf{r}}$ and expand Eq. 3.32 to the form

$$T = \frac{1}{2} m \dot{\mathbf{R}}_c \cdot \dot{\mathbf{R}}_c + \frac{1}{2} \iiint_B \dot{\tilde{\mathbf{r}}} \cdot \dot{\tilde{\mathbf{r}}} dm \tag{3.33}$$

where we made use of

$$m \equiv \iiint_B dm$$

$$\iiint_B \dot{\tilde{\mathbf{r}}} dm = \frac{d}{dt} \left[\iiint_B \tilde{\mathbf{r}} dm \right] = 0.$$

The first term of Eq. 3.33 is recognized as the translational kinetic energy of a particle of mass m concentrated at $\mathbf{R}_c$. The second term is recognized as the kinetic energy associated with the rotation and deformation of B.

We can specialize the general result of Eq. 3.33 for the case of a rigid body by making use of

$$\dot{\tilde{\mathbf{r}}} = \frac{d}{dt} (\tilde{\mathbf{r}})_B + \boldsymbol{\omega} \times \tilde{\mathbf{r}} = \boldsymbol{\omega} \times \tilde{\mathbf{r}}$$

so that the rotational energy (T_R)

$$T_R \equiv \frac{1}{2} \iiint_B \dot{\tilde{\mathbf{r}}} \cdot \dot{\tilde{\mathbf{r}}} dm \tag{3.34}$$

becomes

$$T_R = \frac{1}{2} \iiint (\omega \times \tilde{r}) \cdot (\omega \times \tilde{r}) dm \tag{3.35}$$

Making use of the vector identity

$$(a \times b) \cdot c \equiv a \cdot (b \times c) \tag{3.36}$$

then we can write Eq. 3.35 as

$$T_R = \frac{1}{2} \omega \cdot \int_B \tilde{r} \times (\omega \times \tilde{r}) dm \tag{3.37}$$

Referring to Eq. 3.18, we recognize the integral expression of Eq. 3.37 as being precisely the angular momentum H of B about its mass center. Thus

$$T_R = \frac{1}{2} \omega \cdot H = \frac{1}{2} \omega \cdot \underline{I} \cdot \omega \tag{3.38}$$

We can write Eq. 3.38 in matrix form as

$$T_R = \frac{1}{2} \{\omega\}^T [I] \{\omega\} \tag{3.39}$$

and for principal body-fixed axes, the general quadratic form of Eq. 3.39 reduces to

$$T_R = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2). \tag{3.40}$$

It is interesting to consider the time rate of change of the total energy in the form

$$T = \frac{1}{2} m \dot{R} \cdot \dot{R} + \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \tag{3.41}$$

Upon differentiation, we obtain

$$\frac{d}{dt} (T) = m \dot{R} \cdot \ddot{R} + I_1 \omega_1 \dot{\omega}_1 + I_2 \omega_2 \dot{\omega}_2 + I_3 \omega_3 \dot{\omega}_3. \tag{3.42}$$

Substitution of the translational equation of motion, (Eq. 3.5) and the Eulerian rotational equations of motion (Eq. 3.30), reduces Eq. 3.42 to

$$\frac{dT}{dt} = F \cdot \dot{R} + L \cdot \omega \tag{3.43}$$

which yields the unsurprising truth that the kinetic energy T is constant if F and L are zero. Integration of Eq. 3.43 between arbitrary limits yields the *work energy equation*

$$T - T_o = \int_{t_o}^{t} F \cdot \dot{R} dt + \int_{t_o}^{t} L \cdot \omega dt \tag{3.44}$$

where the first integral is the *translational work* done by the resultant

external force F(t), the second integral is the *rotational work* done by torque L(t).

3.2.2 D'Alembert's Principle

To make the subsequent discussion more transparent, we consider the general system as being a large collection of particles. In a careful limiting process, of course, a system so represented approaches the continuum case. Given N particles, Newton's second law applied to each requires

$$F_i - M_i\ddot{R}_i = 0 \quad , \quad i = 1,2,\ldots,N \tag{3.45}$$

where F_i is the force acting on the ith particle of mass M_i.

We introduce the concept of *virtual displacements* δR_i which are simply instantaneous variations in each R_i. These variations are introduced as analytical devices and are not necessarily displacements along a physical motion. Since Eq. 3.45 is true for all i, it is also true that

$$(F_i - M_i\ddot{R}_i) \cdot \delta R_i = 0 \quad , \quad i = 1,2,\ldots,N$$

and, summing over all particles, it is also true that

$$\sum_{i=1}^{N} (F_i - M_i\ddot{R}_i) \cdot \delta R_i = 0 \tag{3.46}$$

where we identify $\sum_{i=1}^{N} F_i \cdot \delta R_i \equiv \delta W$ as the *virtual work*. Equation 3.46 is the most fundamental version of a family of results referred to as *D'Alembert's Principle*. Suppose n independent *generalized* coordinates $(q_1,q_2,\ldots,q_n)$ are introduced which uniquely position all N particles. We can almost always construct explicit equations of the form

$$R_i = R_i(q_1,\ldots,q_n,t) \quad , \quad i = 1,2,\ldots,N \tag{3.47}$$

In the simplest case of constrained motion, we may have equality *(holonomic)* constraints of the form

$$\psi_j(q_1,q_2,\ldots,q_n,t) = 0 \quad , \quad j = 1,2,\ldots,m \tag{3.48}$$

If such constraints are present, and if they are sufficiently simple, we assume in this paragraph that m q's have been eliminated, by using the constraint

equations to solve for m q's as functions of the remaining n = 3N - m. We now consider the variations δR_i *to be generated*, through Eq. 3.47, from independent variations δq_j in the n coordinates q_j, specifically, we require

$$\delta R_i = \sum_{j=1}^{n} \frac{\partial R_i}{\partial q_j} \delta q_j .$$

Thus we can write Eq. 3.46 in the form

$$\sum_{j=1}^{n} \{ \sum_{i=1}^{N} (F_i - M_i \ddot{R}_i) \cdot \frac{\partial R_i}{\partial q_j} \} \delta q_j = 0. \tag{3.49}$$

Note the *virtual work* (δW) can now be written as

$$\delta W = \sum_{j=1}^{n} Q_j \delta q_j = \sum_{i=1}^{N} F_i \cdot \delta R_i \tag{3.50}$$

where

$$Q_j \equiv \sum_{i=1}^{N} F_i \cdot \frac{\partial R_i}{\partial q_j}$$

are the *generalized forces*. For the case that we can, in fact, eliminate all constraints and construct independent sets of generalized coordinates, Eq. 3.49 provides a most useful result (the *generalized D'Alembert's Principle*). By arguing that Eq. 3.49 holds for arbitrary, independent variations ($\delta q_1, \ldots, \delta q_n$), one concludes that each { } term must vanish. Thus we have the system's equations of motion in the D'Alembert form:

$$\sum_{i=1}^{N} (F_i - M_i \ddot{R}_i) \cdot \frac{\partial R_i}{\partial q_j} = 0 \quad , \quad j = 1,2,\ldots,n \tag{3.51}$$

3.2.3 Lagrange's Equations for Holonomic Systems

We restrict attention to the holonomic class of constraints of the form of Eq. 3.52. We further assume that these functions are simple enough to allow m of the q's to be solved as a function of the remaining q's; these remaining q's are now considered independent.

Let the forces acting on each particle be considered as being the sum of two parts

$$\mathbf{F}_i = \mathbf{f}_i + \mathbf{f}_{ci} \quad , \quad i = 1,2,\ldots,N \tag{3.52}$$

where $\mathbf{f}_{ci}$ is the *normal constraint* force acting on M_i. The normal constraint forces have the property that they are perpendicular to the surface on which M_i is constrained to move. Thus for $\mathbf{f}_{ci}$ nonzero, these forces do zero virtual work:

$$\mathbf{f}_{ci} \cdot \delta\mathbf{R}_i = 0 \quad , \quad i = 1,2,\ldots,N \tag{3.53}$$

We restrict the virtual displacements $\delta\mathbf{R}_i$ to be *consistent* with the constraints (in that both $\mathbf{R}_i$ and $\mathbf{R}_i + \delta\mathbf{R}_i$ satisfy all constraint equations). As a consequence of Eqs. 3.53 and $\delta\mathbf{R}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{R}_i}{\partial q_j} \delta q_j$ for an arbitrary (but consistent) set of δq_j, we conclude that

$$\mathbf{f}_{ci} \cdot \frac{\partial \mathbf{R}_i}{\partial q_j} \equiv 0 \quad , \quad \begin{cases} i = 1,2,\ldots,N \\ j = 1,2,\ldots,n \end{cases} \tag{3.54}$$

Thus the D'Alembert equations of motion, Eq. 3.51, takes on a simpler form

$$\sum_{i=1}^{N} (\mathbf{f}_i - M_i \ddot{\mathbf{R}}_i) \cdot \frac{\partial \mathbf{R}_i}{\partial q_j} = 0 \quad , \quad j = 1,2,\ldots,n \tag{3.55}$$

which is considerably easier to use, since one can ignore all "virtually nonworking" constraint forces [those which do no virtual work; i.e., satisfy Eqs. 3.53, 3.54].

Returning to Eqs. 3.49-3.51 the *generalized forces* simplify as

$$Q_j \equiv \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{R}_i}{\partial q_j} \equiv \sum_{i=1}^{N} \mathbf{f}_i \cdot \frac{\partial \mathbf{R}_i}{\partial q_j} \quad , \quad j = 1,2,\ldots,n \tag{3.56}$$

so that Eqs. 3.51 and 3.55 both assume the form

$$\sum_{i=1}^{N} M_i \ddot{\mathbf{R}}_i \cdot \frac{\partial \mathbf{R}_i}{\partial q_j} = Q_j \quad , \quad j = 1,2,\ldots,n \tag{3.57}$$

It is useful to recognize from

$$\mathbf{R}_i = \mathbf{R}_i (q_1,\ldots,q_n,t) \quad , \quad i = 1,2,\ldots,N$$

that

$$\dot{\mathbf{R}}_i = \frac{\partial \mathbf{R}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \mathbf{R}_i}{\partial q_j} \dot{q}_j \quad , \quad i = 1,2,\ldots,N \tag{3.58}$$

so that the "cancellation of dots" identity

$$\frac{\partial \dot{\mathbf{R}}_i}{\partial \dot{q}_j} \equiv \frac{\partial \mathbf{R}_i}{\partial q_j} \tag{3.59}$$

holds for all i and j.

Using the identity (Eq. 3.59), Eq. 3.57 can be written as

$$\sum_{i=1}^{N} M_i \ddot{\mathbf{R}}_i \cdot \frac{\partial \dot{\mathbf{R}}_i}{\partial \dot{q}_j} = Q_j \quad , \quad j = 1,2,\ldots,n \tag{3.60}$$

Making use of the definition of kinetic energy of a system of particles

$$T = \frac{1}{2} \sum_{i=1}^{N} M_i \dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i \tag{3.61}$$

we note that the left-hand side of Eq. 3.60 can be manipulated as

$$\frac{d}{dt} \left\{ \sum_{i=1}^{N} M_i \dot{\mathbf{R}}_i \cdot \frac{\partial \dot{\mathbf{R}}_i}{\partial \dot{q}_j} \right\} - \sum_{i=1}^{N} M_i \dot{\mathbf{R}}_i \cdot \frac{d}{dt} \frac{\partial \dot{\mathbf{R}}_i}{\partial \dot{q}_i} = Q_j$$

or

$$\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} M_i \dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i \right) \right\} - \sum_{i=1}^{N} M_i \dot{\mathbf{R}}_i \cdot \frac{d}{dt} \frac{\partial \mathbf{R}_i}{\partial q_j} = Q_j$$

or making use of Eq. 3.61 and reversing the differentiation in the last term on the left hand side, we obtain

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \sum_{i=1}^{N} M_i \dot{\mathbf{R}}_i \cdot \frac{\partial \dot{\mathbf{R}}_i}{\partial q_j} = Q_j$$

Finally, recognizing the last term on the left hand side as $\frac{\partial}{\partial q_j} \left[\frac{1}{2} M_i \dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i \right]$, we arrive at a fundamental version of *Lagrange's equations*

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \quad , \quad j = 1,2,\ldots,n \tag{3.62}$$

These second-order differential equations hold for a large class of systems of particles and rigid bodies, under arbitrary external loading conditions.

For the case of *conservative* external forces which satisfy

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{3.63}$$

for $V = V(q_1,\ldots,q_n,t)$ a *potential energy function* of position (but not velocity) coordinates, then Eq. 3.62 reduces to the most famous form of Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \bar{Q}_j \quad , \quad j = 1,2,\ldots,n \tag{3.64}$$

where

$$L \equiv T - V = L(q_1,\ldots,q_n,\dot{q}_1,\ldots\dot{q}_n,t), \text{ the system's } \textit{Lagrangian} \text{ function.} \tag{3.65}$$

We left a residual generalized force $\bar{Q}_j$ on the right-hand-side of Eq. 3.64 to allow for any remaining forces not absorbed into the potential energy function via Eq. 3.63. Clearly the Lagrangian L is a scalar function which contains all of the "dynamical information", exclusive of the non-conservative forces, which are accounted for in $\bar{Q}_j$.

Application of Lagrange's Equations 3.62 or 3.64 to a system of particles is straightforward. However additional insights are useful when the system involves one or more finite bodies as well as one or more internal degrees of freedom.

First of all, note that a rigid body has six degrees of freedom. In the context of the many particle idealization, one can arrive at this conclusion by the following logic. Let us use three coordinates to locate any chosen reference particle in the body. Any second particle must maintain a fixed distance from the first, so only two additional coordinates are required to uniquely position the second particle. Any third particle must maintain a fixed distance from each of the first two particles, so only one additional coordinate is required to uniquely position the third particle. Any fourth particle is uniquely positioned by the constraint that it must maintain a fixed distance with respect to each of the first three; no additional coordinates are

required. Thus a total of 3 + 2 + 1 = 6 coordinates are required to uniquely position every element of mass in a rigid body. It is only necessary to express the potential and kinetic energy as a function of the chosen set of six independent coordinates; then application of Lagrange's equations (for the conservative holonomic case) is straightforward.

Secondly, the generalized force definition of Eq. 3.56 is an inconvenient starting point if the system under consideration is a continuum. For the particular case of a rigid body, we now develop more convenient generalized force expressions, starting with the definition of Eq. 3.56, and making use of Eq. 3.59, we write

$$Q_j = \sum_{i=1}^{N} F_i \cdot \frac{\partial \dot{R}_i}{\partial \dot{q}_j}$$

Selecting some reference point P in the body, we write $\dot{R}_j = \dot{R}_p + \dot{r}_j$ so that

$$Q_j = \sum_{i=1}^{N} F_i \cdot \frac{\partial \dot{R}_p}{\partial \dot{q}_j} + \sum_{i=1}^{N} F_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \tag{3.66}$$

For the case of a rigid body, $\dot{r}_i = \omega \times r_i = - r_i \times \omega$ and thus

$$Q_j = F \cdot \frac{\partial \dot{R}_p}{\partial \dot{q}_j} - \sum_{i=1}^{N} F_i \cdot r_i \times \frac{\partial \omega}{\partial \dot{q}_j}$$

or

$$Q_j = F \cdot \frac{\partial \dot{R}_p}{\partial \dot{q}_j} + L_p \cdot \frac{\partial \omega}{\partial \dot{q}_j} \quad , \quad j = 1,2,\ldots,n \tag{3.67}$$

where we made use of $F = \sum_{i=1}^{N} F_i$, $L_p = \sum_{i=1}^{N} r_i \times F_i$ and the identity $a \cdot b \times c = a \times b \cdot c$.

Eq. 3.67 is easy to apply when we need the generalized forces for a rigid body. Of course, one can often write the Q_j down by inspection (e.g., by calculating the virtual work $\delta W = \sum_{j=1}^{n} Q_j \delta q_j$ done by the "virtually working" forces when the coordinates undergo a virtual displacement). In Chapter 5, Eq. 3.67 is generalized to include flexible body effects.

3.2.4 Lagrange's Equations for Rigid Bodies and Multiple Body Systems

As a fundamental application of the above developments, let us consider the case of an asymmetric ($I_1 \neq I_2 \neq I_3$) rigid body with arbitrary external forces and moments. The kinetic energy is given by Eq. 3.41 and the potential energy is zero. Adopting as coordinates ($q_1 = x$, $q_2 = y$, $q_3 = z$, $q_4 = \phi$, $q_5 = \theta$, $q_6 = \psi$) we have

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{3.68}$$

where

(x,y,z) are rectangular coordinate displacements from an inertial origin and

(ϕ,θ,ψ) are 3-1-3 Euler angles.

The kinetic energy is implicitly a function of $(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi})$ through the (Table 2.1) kinematic relationships

$$\begin{aligned} \omega_1 &= \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \omega_2 &= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \omega_3 &= \dot{\phi} \cos\theta + \dot{\psi} \end{aligned} \tag{3.69}$$

Thus, the equations of motion follow from Eq. 3.64 by simply carrying out the differentiations and algebra implicit in the six Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_x \quad , \quad x \to y,z \tag{3.70a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = Q_\phi \quad , \quad \phi \to \theta,\psi \tag{3.70b}$$

Prior to engaging in this development, let us record the results already obtained from the Newton/Euler developments of Sections 3.1 and 3.2.

Equation 3.5 yields the translational equations of motion

$$\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y \\ m\ddot{z} &= F_z \end{aligned} \tag{3.71a}$$

and Eq. 3.30 provides the rotational equations of motion

$$\begin{aligned} I_1\dot{\omega}_1 &= -(I_3 - I_2)\omega_2\omega_3 + L_1 \\ I_2\dot{\omega}_2 &= -(I_1 - I_3)\omega_1\omega_3 + L_2 \\ I_3\dot{\omega}_3 &= -(I_2 - I_1)\omega_1\omega_2 + L_3 \end{aligned} \tag{3.71b}$$

which for the 3-1-3 Euler angles, must be augmented by the kinematic equations (Table 2.1)

$$\begin{aligned} \dot{\phi} &= (\sin\psi/\sin\theta)\omega_1 + (\cos\psi/\sin\theta)\omega_2 \\ \dot{\theta} &= (\cos\psi)\omega_1 + (-\sin\psi)\omega_2 \\ \dot{\psi} &= (-\sin\psi\cos\theta/\sin\theta)\omega_1 + (-\cos\psi\cos\theta/\sin\theta)\omega_2 + \omega_3 \end{aligned} \tag{3.72}$$

The Lagrangian developments lead naturally to second-order equations, so we anticipate that some manipulation may be required to establish the equivalence between Eqs. 3.71b, 3.72 and the resulting Lagrangian equations for (ϕ,θ,ψ).

First we note $\dot{R} = \dot{x}\hat{n}_1 + \dot{y}\hat{n}_2 + \dot{z}\hat{n}_3$, $\omega = \sum_{i=1}^{3} \omega_i \hat{b}_i$ so that

$$\frac{\partial \dot{R}}{\partial \dot{x}} = \hat{n}_1 \ , \ \frac{\partial \dot{R}}{\partial \dot{y}} = \hat{n}_2 \ , \ \frac{\partial \dot{R}}{\partial \dot{z}} = \hat{n}_3 \tag{3.73}$$

and

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial \dot{\phi}} &= \sum_{i=1}^{3} \frac{\partial \omega_i}{\partial \dot{\phi}} \hat{b}_i = (\sin\theta \sin\psi)\hat{b}_1 + (\sin\theta \cos\psi)\hat{b}_2 + \cos\theta \hat{b}_3 \\ \frac{\partial \boldsymbol{\omega}}{\partial \dot{\theta}} &= \sum_{i=1}^{3} \frac{\partial \omega_i}{\partial \dot{\theta}} \hat{b}_i = (\cos\psi \hat{b}_1 - \sin\psi \hat{b}_2) \\ \frac{\partial \boldsymbol{\omega}}{\partial \dot{\psi}} &= \sum_{i=1}^{3} \frac{\partial \omega_i}{\partial \dot{\psi}} \hat{b}_i = \hat{b}_3 \end{aligned} \tag{3.74}$$

Thus the six generalized forces are thus determined from Eq. 3.67 to be

$$\begin{aligned} Q_x &= F_x \\ Q_y &= F_y \\ Q_z &= F_z \end{aligned} \tag{3.75}$$

$$\begin{aligned} Q_\phi &= L_1 \sin\theta \sin\psi + L_2 \sin\theta \cos\psi + L_3 \cos\theta \\ Q_\theta &= L_1 \cos\psi - L_2 \sin\psi \\ Q_\psi &= L_3 \end{aligned} \tag{3.76}$$

Considering the translational motion, substitution of Eq. 3.68 and Eq. 3.75 into Eq. 3.70a immediately verifies Eq. 3.71a.

Considering now the rotational motion, substitution of Eqs. 3.68, 3.69, and 3.76 into 3.70b leads to the three intermediate equations

$$\begin{aligned} & I_1\dot{\omega}_1 (\sin\theta \sin\psi) + I_2\dot{\omega}_2 (\sin\theta \cos\psi) + I_3\dot{\omega}_3 (\cos\theta) \\ & \quad + I_1\omega_1 (\dot{\theta} \cos\theta \sin\psi + \dot{\psi} \sin\theta \cos\psi) \\ & \quad + I_2\omega_2 (\dot{\theta} \cos\theta \cos\psi - \dot{\psi} \sin\theta \cos\psi) \\ & \quad + I_3\omega_3 (-\dot{\theta} \cos\theta) \\ & \quad = L_1 \sin\theta \sin\psi + L_2 \sin\theta \cos\psi + L_3 \cos\theta \end{aligned} \tag{3.77a}$$

$$\begin{aligned} & I_1\dot{\omega}_1 (\cos\psi) + I_2\dot{\omega}_2 (-\sin\psi) \\ & \quad + I_1\omega_1 (-\dot{\psi} \sin\psi - \dot{\phi} \cos\theta \sin\psi) \\ & \quad + I_2\omega_2 (-\dot{\psi} \cos\psi - \dot{\phi} \cos\theta \cos\psi) \\ & \quad + I_3\omega_3 (\dot{\phi} \sin\theta) \\ & \quad = L_1 \cos\psi - L_2 \sin\psi \end{aligned} \tag{3.77b}$$

$$\begin{aligned} & I_3\dot{\omega}_3 - I_1\omega_1(\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi) \\ & \quad + I_2\omega_2(\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi) = L_3 \end{aligned} \tag{3.77c}$$

Considerable algebra is required to verify Eq. 3.71b from Eq. 3.77 or vice-versa. The verification is accomplished by using Eq. 3.72 to eliminate $(\dot{\phi},\dot{\theta},\dot{\psi})$ from Eqs. 3.77 in favor of $(\omega_1,\omega_2,\omega_3)$; a considerable number of cancellations and $\sin^2 + \cos^2 = 1$ and other simplifications ultimately eliminates all functions of (ϕ,θ,ψ) and results in verifying Eq. 3.71b. Note, however that the third equation of 3.71b is identical to Eq. 3.77c; upon recognizing from Eq. 3.69 that the two parenthetic terms of Eq. 3.77c are identical to ω_2 and ω_1, respectively. The fact that the third Euler equation is much easier to obtain than the first two (by Lagrangian mechanics, using the 3-1-3 Euler angles) led Goldstein (ref. 9) to use an ingenious device to bypass all of the algebra implicit in reducing Eqs. 3.77a and 3.77b to the first two of Euler's Equations 3.71b. The device makes use of the fact that labeling of

the $(\hat{b}_1,\hat{b}_2,\hat{b}_3)$ axes *is arbitrary*, so long as we use a cyclic permutation of 1-2-3. Thus having established the third Euler equation of Eq. 3.71b, the other two follow from cyclic permutation of the indices. This device, while clever, probably simply obscures a fundamental truth: The Eulerian equations of motion, and their generalizations, are more easily obtained by the Eulerian methods of Section 3.1.4 than via the Lagrangian methods of the present discussion, and more importantly: *The Eulerian equations are of fundamental significance in general rotational dynamics, and do not depend upon any particular choice of Euler angles.*

On the other hand, the "natural" end product of applying the Lagrangian approach to rotational dynamics of a rigid body is a set of three second-order equations for (ϕ,θ,ψ). These equations are obtained from Eqs. 3.77 by making use of Eq. 3.69 and their derivatives to eliminate ω's in favor of $(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi},\ddot{\phi},\ddot{\theta},\ddot{\psi})$; the result (see Broucke, ref. 13) can be collected in matrix form as

$$[A]\begin{Bmatrix}\ddot{\phi}\\ \ddot{\theta}\\ \ddot{\psi}\end{Bmatrix}+[B]\begin{Bmatrix}\dot{\phi}^2\\ \dot{\theta}^2\\ \dot{\psi}^2\\ \dot{\phi}\dot{\theta}\\ \dot{\phi}\dot{\psi}\\ \dot{\theta}\dot{\psi}\end{Bmatrix}=[C]\begin{Bmatrix}L_1\\ L_2\\ L_3\end{Bmatrix} \tag{3.78}$$

where the coefficient matrices A, B, C are lenghty transcendental functions of (θ,ψ). It is significant to note that, if we in fact seek three second-order equations of the form Eq. 3.78 rather than the six first-order equations 3.71b and 3.72c, there appears to be a "conservation of misery principle" present; both the Eulerian and Lagrangian approaches lead to comparable levels of algebraic development. Equation 3.78, for example, eventually follows from differentiation of Eq. 3.72c with substitutions from Eqs. 3.71b and 3.69.

In summary, the justification for Lagrangian methods lies elsewhere, there is no advantage to be gained over Newtonian/Eulerian approaches if one is concerned with a single rigid body. The justification of Lagrangian methods, in fact, lies in formulation of the equations of motion for many degree of freedom systems *having many virtually nonworking internal forces* which can be ignored in the Lagrangian approach, whereas they must first be included, then elminated in the Newtonian/Eulerian approach (in which Newtonian/Eulerian equations are derived for each internal degree of freedom).

As a very simple illustration of the above remarks, consider the system depicted in Figure 3.4.

The rigid body B, is constrained to rotate about a fixed axis through point O. An external torque $L(t)\hat{b}_3$ is present; the mass m is constrained to oscillate in a slot fixed in B.

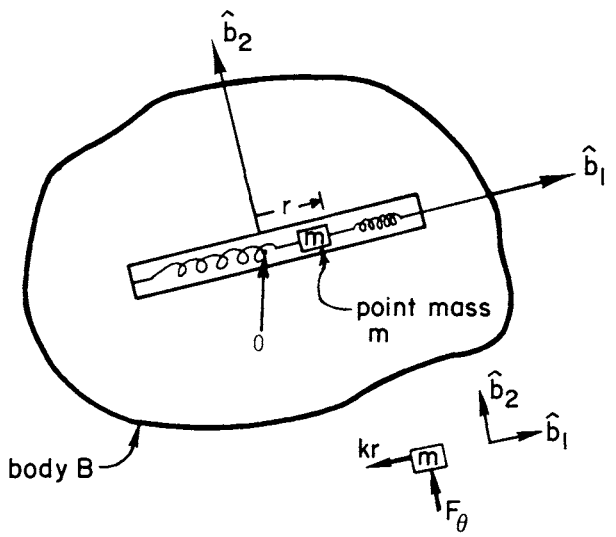

Figure 3.4 Two Coupled Bodies

The *external* degree of freedom is described by $\theta(t)$, the rigid rotation of B. The *internal* degree of freedom is described by $r(t)$, the position of m relative to the equilibrium point O on the axis of rotation.

The external degree of freedom's governing equation of motion can be obtained several ways. One very convenient way is to apply Euler's equation to the entire system, thus

$$L_{system} = \dot{H}_{system} \tag{3.79}$$

in which

$$L_{system} = L\,\hat{b}_3 \tag{3.80}$$

and

$$H_{system} = [I_B + mr^2]\omega\hat{b}_3 \tag{3.81}$$

I_B = moment of inertia of B about $\hat{b}_3$

Thus Eq. 3.79 yields the differential equation for $\omega(t)$

$$[I_B + mr^2]\dot{\omega} + 2mr\dot{r}\omega = L(t) \tag{3.82}$$

and we have the obvious kinematic relationship between $\theta(t)$ and $\omega(t)$

$$\dot{\theta} = \omega \tag{3.83}$$

Thus we efficiently arrive at one of the two equations of motion, but we must apply an alternative viewpoint to get the differential equation governing internal degrees of freedom. In particular, perhaps the most obvious and elementary procedure is to apply Newton/Euler principles to the subsystems (namely B and m). The rotational equation for B's motion follows from

$$L_B = \dot{H}_B \tag{3.84}$$

with

$$L_B = (L - rF_\theta)\hat{b}_3$$

$$H_B = I_B\omega\hat{b}_3 \tag{3.85}$$

Thus B's motion is governed by

$$I_B\dot{\omega} = L - rF_\theta \tag{3.86}$$

Mass m's motion is governed by Newton's second law

$$\mathbf{F} = m\ddot{\mathbf{r}} \tag{3.87}$$

in which

$$\mathbf{F} = -kr\hat{\mathbf{b}}_1 + F_\theta\hat{\mathbf{b}}_2 \tag{3.88}$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\omega^2)\hat{\mathbf{b}}_1 + (r\dot{\omega} + 2\dot{r}\omega)\hat{\mathbf{b}}_2 \tag{3.89}$$

Thus, Eq. 3.87 yields an equation for the constraint force

$$F_\theta = m(r\dot{\omega} + 2\dot{r}\omega) \tag{3.90}$$

and the *internal* equation of motion for r(t)

$$\ddot{r} + (\frac{k}{m} - \omega^2)r = 0. \tag{3.91}$$

Substitution of Eqs. 3.90 into Eq. 3.86 immediately verifies the *external* equation of motion, Eq. 3.82. Thus, applying Newton's laws and Euler's equations to the subsystems led to a complete set of differential equations for both the internal and external ("rigid body") degrees of freedom. But notice, it was necessary to introduce (and then eliminate!) the constraint reaction F_θ. The analogous simultaneous algebraic elimination of many constraint forces is a messy and significant disadvantage of approaching complicated dynamical systems using the Newton/Euler principles applied to substructures. However, applying Newton/Euler principles to the entire system is a most efficient path to establishing the fundamental equations governing the *external* (or rigid body) degrees of freedom. For the *internal* degrees of freedom, we believe the *generalized* class of approaches to be more efficient, owing to the *implicit* elimination of virtually nonworking constraint forces. In the present example, the system Lagrangian is

$$L = T - V = \frac{1}{2} I_B\omega^2 + \frac{1}{2} m(\dot{r}^2 + r^2\omega^2) - \frac{1}{2} kr^2 \tag{3.92}$$

and the internal equation of motion follows immediately from Eq. 3.64 as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{3.93}$$

as Eq. 3.91. It is also true, in the present single axis rotation example, that the θ equation follows near-trivially from Lagrange's equations. However when large angle, three-dimensional rotations are involved, we have already established that Lagrange's equations are a round-about means to derive Euler's Eq. 3.31 and their generalizations for systems of nonrigid bodies. Thus, we offer as an attractive operational rule-of-thumb: *Derive external equations of motion (for the "rigid body degrees of freedom") via Newton/Euler methods; derive internal equations of motion via Lagrange's equations.*

Several modifications (refs. 10 through 13) of the Langrangian formulation have been developed so that differential equations involving $(q_i, \omega_i, \dot{\omega}_i)$ are obtained in lieu of $(q_i, \dot{q}_i, \ddot{q}_i)$ where the $\omega_i = f_i(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$ are "more general" generalized velocity coordinates ("quasi-coordinates"), such as the orthogonal components of angular velocity (in Euler's Eq. 3.31). These quasi-coordinate approaches offer a unifying alternative of using a Lagrange-type formulation to universally derive equations for the external as well as the internal degrees of freedom. However the associated increase in abstraction is a disadvantage which leads us to prefer the position stated in the foregoing paragraph. A significant number of investigators, however, prefer the quasi-coordinate approach; we therefore include a summary of a quasi-coordinate formulation of multi-body flexible spacecraft equations of motion in Chapter 5.

3.2.5 Lagrange's Equations for Non-Holonomic Constraints

In the above developments, it is assumed that all constraints are holonomic and are eliminated initially to reduce the number of coordinates to the minimum. For the significantly more general case of nonholonomic constraints in the *Pfaffian* form

$$0 = \phi_j(q_1,q_2,\ldots,q_n,t) + \sum_{i=1}^{n} \psi_{ji}(q_1,\ldots,q_n,t)\dot{q}_i \quad , \quad j = 1,2,\ldots,m \tag{3.94}$$

or

$$0 = \phi_j dt + \sum_{i=1}^{n} \psi_{ji} dq_i \quad , \quad j = 1,2,\ldots,m \tag{3.95}$$

The developments of Section 3.2.3. can be generalized (ref. 9), so that Lagrange's Eq. 3.64 becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \bar{Q}_j + \sum_{i=1}^{m} \lambda_i \psi_{ij} \quad , \quad j = 1,2,\ldots,n \tag{3.96}$$

where the m algebraic Eqs. 3.94 together with the n differential equations determine the m unknown *Lagrange Multipliers* $(\lambda_1,\ldots,\lambda_m)$ and the n q_i's. The final term in Eq. 3.96 is clearly the generalized constraint force. In some circumstances, we may use Eq. 3.96 even though Eq. 3.95 is integrable to Eq. 3.48; this will be useful if we are concerned with the constraint forces. Observe, Eq. 3.96 can often be solved for acceleration as

$$\ddot{\mathbf{q}} = f(\mathbf{q},\dot{\mathbf{q}},\bar{Q}) + M^{-1}\psi^T\lambda \tag{3.96a}$$

$$\psi\dot{\mathbf{q}} + \boldsymbol{\phi} = 0 \tag{3.95a}$$

where M is the generally time-varying mass matrix. We can often eliminate explicit dependence upon the Lagrange multipliers; there are three basic steps in the procedure. First, we differentiate the matrix form of the constraint equation (3.95a) with respect to time, yielding

$$\psi\ddot{\mathbf{q}} + \dot{\psi}\dot{\mathbf{q}} + \dot{\boldsymbol{\phi}} = 0$$

Second, introducing the constrained accelerations from Eq. 3.96a leads to

$$\psi\{\mathbf{f} + M^{-1}\psi^T\lambda\} + \dot{\psi}\dot{\mathbf{q}} + \dot{\boldsymbol{\phi}} = 0$$

which yields the following linear algebraic equation to solve for λ:

$$[\psi M^{-1}\psi^T]\lambda = -\{\psi\mathbf{f} + \dot{\psi}\dot{\mathbf{q}} + \dot{\boldsymbol{\phi}}\}$$

Third, we introduce the numerical values (or algebraic expressions) obtained for λ into the constrained acceleration equation, which can now be

integrated. We observe, however, that the solution for λ must be computed on each derivative evaluation in the numerical integration process.

3.2.6 Hamilton's Principles

The above pluralization of Hamilton's Principle is not accidental, we find several variations of this variational concept in recent textbooks and the literature. We will confine our attention to the most fundamental issues here, but in doing so, we point out the main distinctions between various formulations and viewpoints.

We begin with D'Alembert's Principle, which, from Eq. 3.46 we write as

$$\delta W = \sum_{i=1}^{N} M_i \ddot{R}_i \cdot \delta R_i \tag{3.97}$$

Equation 3.97 holds with the same generality as Newton's second law of Eq. 3.45 and is therefore a very general starting point for the results of this section. In the present discussion, we consider the $\delta R_i(t)$ to be smooth, twice differentiable "path variations" which are otherwise arbitrary displacements from a "true dynamical path"; $R_i(t)$ to a varied path $\tilde{R}_i(t) = R_i(t) + \delta R_i(t)$; the true path $R_i(t)$ is assumed to satisfy Lagrange's equations of motion (or, equivalently, Newton's laws) while $\tilde{R}_i(t)$ is generally <u>not</u> a true path. Since the $\delta R_i(t)$ are differentiable, then it follows from

$$\delta R_i(t) = \tilde{R}_i(t) - R_i(t)$$

that

$$\dot{\delta R}_i(t) = \frac{d}{dt}(\delta R_i(t)) = \dot{\tilde{R}}_i(t) - \dot{R}_i(t) \tag{3.98}$$

Note that the variation of the kinetic energy between $\tilde{R}_i(t)$ and $R_i(t)$ is

$$\delta T = \delta\left[\frac{1}{2}\sum_{i=1}^{N} M_i \dot{R}_i \cdot \dot{R}_i\right] = \sum_{i=1}^{N} M_i \dot{R}_i \cdot \dot{\delta R}_i$$

Also note the identity

$$\sum_{i=1}^{N} \frac{d}{dt}(M_i \dot{R}_i \cdot \delta R_i) = \sum_{i=1}^{N} M_i \ddot{R}_i \cdot \delta R_i + \delta T \tag{3.99}$$

Use of Eq. 3.99 in Eq. 3.97 leads immediately to the variational statement of D'Alembert's principle:

$$\delta T + \delta W = \sum_{i=1}^{N} \frac{d}{dt} (M_i \dot{R}_i \cdot \delta R_i) \tag{3.100}$$

Introducing $R_i = R_i(q_1, q_2, \ldots, q_n, t)$, the right hand side of Eq. 3.100 can be manipulated as follows

$$\sum_{i=1}^{N} \frac{d}{dt} (M_i \dot{R}_i \cdot \delta R_i) = \frac{d}{dt} \left(\sum_{j=1}^{n} \sum_{i=1}^{N} (M_i \dot{R}_i \cdot \frac{\partial R_i}{\partial q_j} \delta q_j) \right) \tag{3.101}$$

and making use of the identity of Eq. 3.59 and substitution of Eq. 3.101 into Eq. 3.100, we obtain the modified variational statement of D'Alembert's Principle

$$\delta T + \delta W = \frac{d}{dt} \left[\sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j} \delta q_j \right] \tag{3.102}$$

Since we have a perfect differential on the right hand side, this variational statement can be integrated between arbitrary instants in time to obtain the following result:

$$\int_{t_o}^{t_1} (\delta T + \delta W) dt = \left[\sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j} \delta q_j \right]_{t_o}^{t_1} \tag{3.103}$$

This is the most general of Hamilton's Principles and is known in the recent literature [refs. 14-18] as *Hamilton's principle of varying action.* In References [14,15,18] it is established that Eq. 3.103 can be used as a basis for direct numerical approximation of the solution $q_j(t)$ of either initial or two-point boundary value problems. These methods expand $q_j(t)$ in terms of prescribed linearly independent basis functions $\{\phi_1(t), \ldots, \phi_M(t)\}$ of time as

$$q_j(t) \cong \sum_{\ell=1}^{M} a_{\ell j} \phi_\ell(t) \tag{3.104}$$

and by making use of relations such as

$$\delta(q_j^2) = 2q_j\delta q_j = 2[\sum_{\ell=1}^{M} a_{\ell j}\phi_\ell(t)][\sum_{m=1}^{M} \delta a_{mj}\phi_m(t)]$$

$$\delta(\dot{q}_j^2) = 2\dot{q}_j\delta\dot{q}_j = 2[\sum_{\ell=1}^{M} a_{\ell j}\dot{\phi}_\ell(t)][\sum_{m=1}^{M} \delta a_{mj}\dot{\phi}_m(t)]$$

the integral of Eq. 3.103 can often be performed term-by-term analytically (or, if not, numerically) and one is led to equations of the form

$$\sum_{k=1}^{M}\sum_{j=1}^{n} \text{function}_{kj}\,(a_{1j},a_{2j},\ldots,a_{Mj},\phi_1(t_1),\ldots,\phi_M(t_1))\delta a_{kj} = 0$$

Upon imposing boundary condition constraints upon the $\delta a_{\ell j}$'s, and observing that the coefficients of all free $\delta a_{\ell j}$'s must vanish independently, one can usually manipulate each of these coefficient equations to the form

$$\sum_{\ell=1}^{M} [\text{function}_\ell(a_{1j},\ldots,a_{Mj})]\phi_\ell(t_1) = 0 \qquad (3.105)$$

Since the $\phi_\ell(t)$ are linearly independent, each of the algebraic coefficient functions of $\phi_\ell(t)$ must also vanish independently, provided these algebraic equations can be solved for the $(a_{1j},\ldots,a_{Mj})$, we have a direct method for construction of approximate solutions from Hamilton's law of varying.

Recently, a more analytical approach was taken by Rajan and Junkins [refs. 16-18], wherein direct methods for solving Eq. 3.103 for $q_j(t)$ were developed *without* the necessity of approximations like Eq. 3.104. Specifically, the integral variational analogies to the following methods (traditionally used for solving differential equations): (i) Laplace and Fourier transform methods, and (ii) asymptotic perturbation methods (e.g., the method of multiple time scales, the method of Lindstedt-Poincare, and the method of averaging, etc., for systems which do not admit closed analytical solutions) have been developed [refs. 16-18] and illustrated for several examples. In summary, Eq. 3.103 provides a general starting point for solving for the motion of a dynamical

system -- it is not mandatory that one first "work backwards" to get Lagrange's equations!

If we restrict the discussion to the fixed end point, two-point boundary-value-problem; we can then impose that $R_i(t_o)$ and $R_i(t_1)$ are fixed and that the admissible $\delta R_i(t_o)$, $\delta R_i(t_1)$, $\delta q_j(t_o)$ and $\delta q_j(t_1)$ must all vanish. We are then led to the immediate conclusion that the boundary term on the right-hand side of Eq. 3.103 vanishes and we obtain the *Generalized Hamilton's Principle*

$$\int_{t_o}^{t_1} (\delta T + \delta W)dt = 0 \tag{3.106}$$

This principle can be used to derive the Lagrangian equations of motion of a system, (see developments in Chapter 6) but it cannot generally be used to *directly* solve for motion as in the above discussion unless one possesses the unusual insight required, for example, to constrain the δa's so that all final variations of $\delta R_j(t_1)$ vanish, i.e., the generally non-zero contribution of the boundary terms of Eq. 3.103 are important. One standard practice is to use Eq. 3.106 (which holds for the <u>two-point boundary-value problem</u>), to derive Lagrange's equations and then promptly use these Lagrange's equations to solve an <u>initial-value problem</u>! Of course the same differential equations govern both initial-value and (the infinity of possible) two-point boundary-value problems, so there is nothing wrong with this practice. But perhaps, this inconsistency does undermine the position of those who argue that the fixed end point version of Hamilton's principle is the most fundamental (or, in some instances, the *only valid*) version! We wish to point out the following truths:

(1) Eq. 3.103 is more general and more powerful than Eq. 3.106 and provides a starting point of several paths for solving for motion which do not necessarily require first deriving Lagrange's equations; it obviously

includes Eq. 3.106 for the special case of fixed end points.

(2) There are an infinity of physically admissible choices for the boundary conditions for the "fixed" end conditions for the two-point boundary problems - if we admit that boundary condition variations can (and do) generally exist, then we obtain immediately Eq. 3.106 on physical grounds.

(3) The use of Eq. 3.106 as a general principle is certainly not invalid, but it offers only a minor advantage over Lagrange's equations themselves in that, for distributed systems, the spatial boundary conditions are obtained *automatically* (in addition to rederivation of Lagrange's equations [ref. 10] for the special case at hand, upon carrying through the first variation for the particular Lagrangian).

Finally, if we restrict attention to conservative holonomic systems, then Eq. 3.106 is specialized [ref. 12] to the most famous form of Hamilton's Principle

$$\delta \int_{t_o}^{t_1} L dt = 0 \tag{3.107}$$

where $L = T - V$ is the system Lagrangian. As is evident in Chapter 6, a necessary condition for the first variation of Eq. 3.107 to vanish is the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{3.108}$$

Which is, not surprisingly, Lagrange's equations of motion for a conservative holonomic system. Note the statement of Eq. 3.107 is only a necessary condition that the integral of the Lagrangian be an extreme along a "true" motion connecting fixed end points.

An occasional misconception is that Hamilton's principle is a minimum principle; the integral of the Lagrangian is not generally minimized, as can be readily estabished by the near-trivial counter example:

True Motion

$$\ddot{q} = 0 \quad , \quad t_o = 0 \quad , \quad t_1 = t_1$$

$$\dot{q}(t) = 1/t_1 = \text{constant}$$

$$q(t) = t/t_1 \quad ; \quad q(0) = 0 \quad , \quad q(t_1) = 1$$

Potential and Kinetic Energy

$$L = T = \frac{1}{2}\dot{q}^2 = \frac{1}{2t_1^2} \quad , \quad V = 0$$

Varied Motion

$$\tilde{q}(t) = q(t) + \delta q(t)$$

$$\dot{\tilde{q}}(t) = \dot{q}(t) + \delta\dot{q}(t)$$

$$\delta q(t) = \varepsilon t(t_1 - t) \quad , \text{ a specific choice.}$$

so

$$\tilde{q}(t) = \frac{t}{t_1} + \varepsilon t(t_1 - t)$$

$$\dot{\tilde{q}}(t) = \frac{1}{t_1} + \varepsilon t_1 - 2\varepsilon t$$

$$\tilde{L} = \tilde{T} = \frac{1}{2}\dot{\tilde{q}}^2 = \frac{1}{2}\left[\frac{1}{t_1^2} + 2\varepsilon - 4\varepsilon\frac{t}{t_1} + \varepsilon^2(t_1^2 - 4t_1 t + 4t^2)\right]$$

Action Integrals

$$S = \int_{t_o}^{t_1} L\, dt = \int_0^{t_1} \frac{1}{2}\dot{q}^2 dt = \frac{1}{2t_1}$$

$$\tilde{S} = \int_{t_o}^{t_1} \tilde{L}\, dt = \int_0^{t_1} \frac{1}{2}\dot{\tilde{q}}^2\, dt = \frac{1}{2t_1} + \varepsilon^2 t_1^3/6$$

From the above it follows that the first variation of the action integral is

$$\delta S = \tilde{S} - S = \varepsilon^2 t_1^3/6$$

and it is evident that

(i) $\delta S \to 0$ as $\varepsilon \to 0$, so the first variation is zero for differential path variations.

(ii) S is minimized if $t_1 > 0$

(iii) S is maximized if $t_1 < 0$

(iv) S has an inflection at $t_1 = 0$

This is a typical result; for any specific choice of a dynamical system, the system boundary conditions, and for a specific family of admitted variations, S is usually minimized only for a certain restricted sub-set of choices on the initial and final times. In the above special case, the branching at $t_1 = 0$ is not of general significance, is easy to construct examples for which the integral is maximized for one finite positive range of t_1 values and minimized for another. Thus Hamilton's Principle is not generally a minimum principle.

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