The Nuclear Physics Weekly II

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August 15, 2021

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1 Scattering Theory : Complex Numbers review

1.1 Why Complex Numbers

- Complex numbers often come up as shortcuts for calculations in physics.
  Example : Wave equation $\Psi(x,t) = Re[Ae^{i(\omega t - kx)}]$
- Not just as shortcuts, Complex number is *fundamental* to Quantum Physics.

1.2 Scattering

- Scattering Amplitude M, a complex number, establishes the probability of a scattering process.
- M is a function of Center-of-momentum (frame in which total momentum of the system is zero) Energy , denoted as M.
- Probability of an interaction is proportional to $|M|^2$
- In a scattering amplitude-Energy plot, we observe a cusp at the *threshold energy.*
- Threshold energy is the minimum energy at which scattering can occur as below this energy level, probability of scattering is zero.

1.3 Basic Complex Number

- $z = a + i \cdot b$
- Often written in polar format as $z = r \cdot e^{i \cdot \alpha}$. Here, r is magnitude and α is the argument.

1.4 Further Reading

- Townsend, A Modern Approach to Quantum Mechanics
- Sakurai, Modern Quantum Mechanics
- Taylor, Scattering Theory
- International Summer School On Reaction Theory (2017 Lectures - Team 1,2)

2 More on Complex Numbers

2.1 Analytic functions

A complex analytic function, also known as *holomorphic function*, defined in some domain D of argand plane is such that it is single-valued and differentiable everywhere at all points in the domain.

Example: $f(z) = \sum_{n=0}^{N} a_n \cdot z^n$ where $a_n \in \mathbb{C}$

2.2 Visualising Complex functions

Plotting functions that are from $\mathbb{R}$ to $\mathbb{R}$ is a trivial task which can be done with 2D surface. But for complex functions where both domain and range in 2-dimensional, we need a 4D surface and hence impossible. But we can get (somewhat) around this problem by using a technique called *Domain Coloring.*

Domain coloring works by coloring the argand plane, say W, with help of a function that define color at each point in W. We then color each point of another argand plane, say Y, such that color of point $z \in Y$ corresponds to color of point $f(z)$ in W.

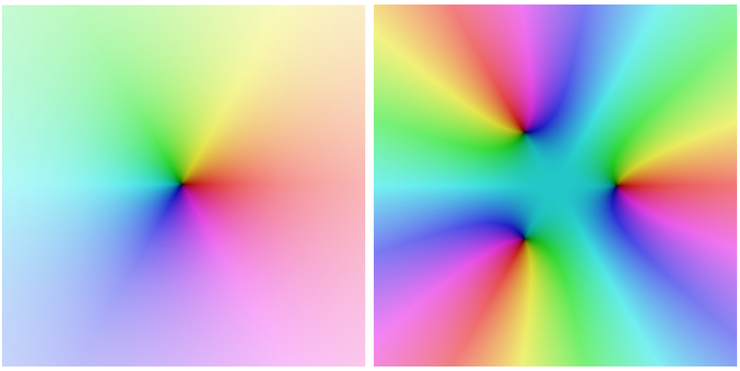

Figure 1: Color function (left) and Domain Coloring of $f(z) = z^3 - 1$ (right).

2.3 Pole singularity

A complex function $f(z)$ has pole singularity of order n at z_0 if $f(z) \cdot (z - z_0)^n$ is an holomorphic function.

Example: $f(z) = \frac{g(z)}{z - z_0}$ where $g(z)$ is holomorphic has pole singularity or order 1 at z_0.

Domain coloring can be helpful in inspecting singularities. $f(z) = \frac{z^2+z-1}{(z-i)(z+1)}$ is plotted in Figure 2 and we can clearly observe a peculiar character of the plot at the poles i.e. i and -1.

2.4 Cauchy's theorem

Cauchy's theorem is one of the most important theorem in complex analysis. It states that if $f(z)$ is analytic everywhere within a simply-connected region then for every simple closed path C lying in the region, we have,

$$\oint_C f(z) \cdot dz = 0$$

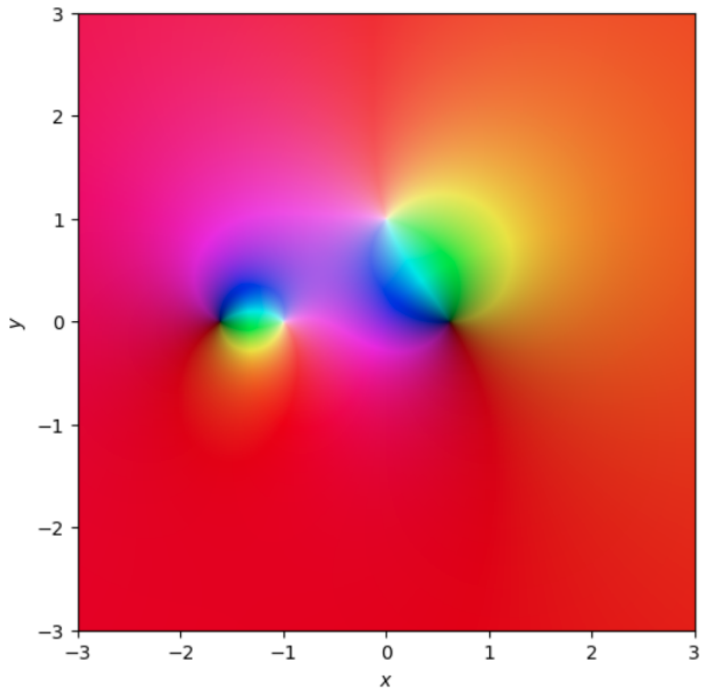

Figure 2: $f(z) = \frac{z^2+z-1}{(z-i)(z+1)}$

Example: z^2 is analytic everywhere. Hence for any simple contour C $\oint_C z^2 \cdot dz = 0$

2.5 Cauchy's Integral Formula

If $f(z)$ is analytic everywhere inside and on the boundary C fo a simply-connected region then for any point z_0 inside C, we have,

$$\oint_C \frac{f(z)}{z - z_0} \cdot dz = 2\pi i f(z_0)$$

Example: Evaluate $\oint_C \frac{z^2+z-1}{(z-i)(z+1)} \cdot dz$ where C is circular region of radius $r < \sqrt{2}$ around i.

The only pole of the integrand in C is at i. Rewrite the function as $\frac{\frac{z^2+z-1}{(z+1)}}{z-i}$. Thus, by Cauchy's integral formula the integral equates to $2 \cdot \pi \cdot \frac{i-2}{i+1}$.

2.6 Branch Cut

A branch cut in the curve along which an analytic multivalued function is discontinuous.

Example: Real and Imaginary parts of $\sqrt{z}$ are plotted separately in Figure 3 which shows a branch cut along the negative x-axis. The discontinuity is apparent in Figure 4.

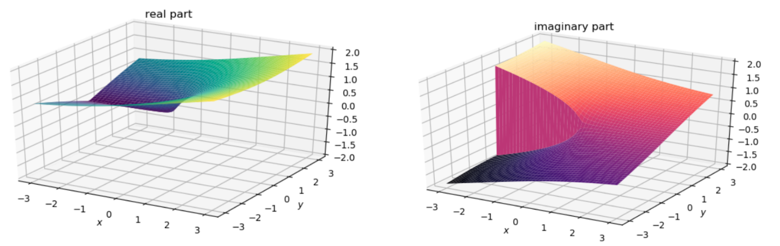

Figure 3: Real (left) and Imaginary(right) component of $f(z) = \sqrt{z}$.

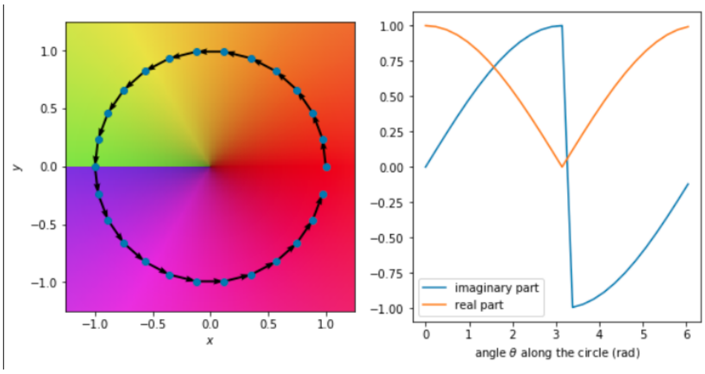

Figure 4: Discontinuity in $\sqrt{z}$

3 References

- Visualizing complex analytic functions using domain coloring - Hans Lundmark 2004
- Cauchy's Theorem - HELM 2008
- Winding Number and domain coloring - Grant Sanderson, 3blue1brown
- Weisstein, Eric W. "Branch Cut." From MathWorld–A Wolfram Web Resource