AA 279 C – SPACECRAFT ADCS: LECTURE 5

Prof. Simone D'Amico

Stanford's Space Rendezvous Laboratory (SLAB)

Table of Contents

- Reading
  - Wertz 17, 18.3

Kinematic Equations of Motion

- In order to fully describe the attitude motion, we need to express the variations of the attitude parameters as a function of the angular velocities in body frame
- Each attitude parameterization has its specific set of kinematic equations of motion
- Given an arbitrary time history of $\boldsymbol{\omega}(t)$, the kinematic equations are differential equations which can be integrated to yield the instantaneous attitude
- The kinematic differential equations are **linear** only for the direction cosine matrix and quaternions
- All other attitude representations contain some degree of nonlinearity and singularity

Kinematics for Direction Cosine Matrix

- The variation over Δt of the direction cosine matrix can be seen as a rotation about the Euler axis ***e*** by an angle φ which tends to zero for Δt → o

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{A}'\vec{A}(t) - \vec{A}(t)}{\Delta t}$$

- The infinitesimal rotation is given by

$$\vec{A}' = \cos\varphi \vec{I} + (1-\cos\varphi)\vec{e}\vec{e}^t - \sin\varphi[\vec{e}\text{x}] = \vec{I} - \varphi[\vec{e}\text{x}] = \vec{I} - \omega\Delta t[\vec{e}\text{x}]$$
$$= \vec{I} - \Delta t[\vec{\omega}\text{x}]$$

- Substitution provides

$$\boxed{\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{-\Delta t[\vec{\omega}\text{x}]\vec{A}(t)}{\Delta t} = -[\vec{\omega}\text{x}]\vec{A}(t)}$$

- Knowing initial angular velocity and direction cosine matrix, we can always integrate over time to predict the spacecraft attitude in the future
- Heavy redundancy is a disadvantage of this approach

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Kinematics for Quaternions

- The variation over Δt of the quaternion vector can be seen as a composition of quaternion rotations by an angle φ which tends to zero for $\Delta t \rightarrow$ 0

$$\frac{d\vec{q}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{q}(t+\Delta t) - \vec{q}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{B}'\vec{q}(t) - \vec{q}(t)}{\Delta t}$$

- The infinitesimal quaternion rotation is given by

$$\vec{B}' \to \vec{I} + \frac{1}{2}\vec{\Omega}\Delta t \;\; ; \vec{\Omega} = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}$$

- Substitution provides

$$\boxed{\frac{d\vec{q}}{dt} = \lim_{\Delta t \to 0} \frac{\left(\vec{I} + \frac{1}{2}\vec{\Omega}\Delta t\right)\vec{q}(t) - \vec{q}(t)}{\Delta t} = \frac{1}{2}\vec{\Omega}\vec{q}(t)}$$

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Kinematics for Gibbs Vector

- The variation over Δt of the Gibbs vector can be seen as a composition of Gibbs rotations by an angle φ which tends to zero for $\Delta t \rightarrow 0$

$$\frac{d\vec{g}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{g}(t+\Delta t) - \vec{g}(t)}{\Delta t} = \cdots$$

- Directly from its definition, the infinitesimal Gibbs rotation is given by

$$\vec{g}' \rightarrow \frac{1}{2}\vec{\omega}\Delta t$$

- Substitution provides

$$\boxed{\frac{d\vec{g}}{dt} = \frac{1}{2}[\vec{\omega} - \vec{\omega} \times \vec{g} + (\vec{\omega} \cdot \vec{g})\vec{g}]}$$

- Although non-linear, the differential equation contains no trigonometric functions and only quadratic nonlinearity
- It is singular for odd multiples of 180°

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Kinematics for Euler Angles

- Let's consider prior example of 313 symmetric Euler angle sequence
- We seek relationship between time derivatives of Euler angles and angular velocity in body frame
- First we construct the angular velocity in inertial frame from the composition of 313 Euler angles

$$\vec{\omega} = \dot{\varphi}\vec{3} + \dot{\theta}\vec{1}' + \dot{\psi}\vec{z}$$

- Next we project the vector onto body axes

$$\begin{cases} \omega_x = \vec{\omega} \cdot \vec{\hat{x}} = \dot{\varphi}\vec{3} \cdot \vec{\hat{x}} + \dot{\theta}\vec{1}' \cdot \vec{\hat{x}} = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_y = \vec{\omega} \cdot \vec{\hat{y}} = \dot{\varphi}\vec{3} \cdot \vec{\hat{y}} + \dot{\theta}\vec{1}' \cdot \vec{\hat{y}} = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_z = \vec{\omega} \cdot \vec{\hat{z}} = \dot{\varphi}\vec{3} \cdot \vec{\hat{z}} + \dot{\theta}\vec{1}' \cdot \vec{\hat{z}} + \dot{\psi} = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases}$$

- Finally we solve for the time derivative of the angles

$$\boxed{\begin{cases} \dot{\varphi} = (\omega_x\sin\psi + \omega_y\cos\psi)/\sin\theta \\ \dot{\theta} = \omega_x\cos\psi - \omega_y\sin\psi \\ \dot{\psi} = \omega_z - (\omega_x\sin\psi + \omega_y\cos\psi)\cot\theta \end{cases}}$$

Note trigonometric functions and singularity for division by zero

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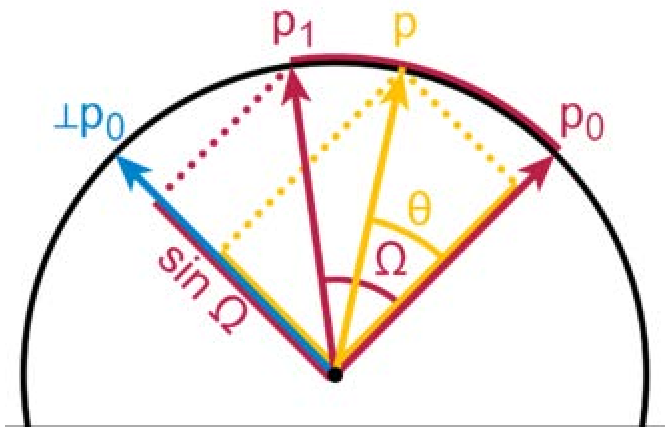
Notes on Quaternions and Interpolation

- As the norm of the quaternion is unitary, a quaternion must always be normalized after each manipulation to remove numerical errors
- Interpolating Euler angles provides ill specified attitudes because, in general, there is no unique path across any two orientations
- Interpolating Quaternions ensures a unique path under all circumstances
- Since the norm of the quaternion is unitary, all quaternions lie on a 4D sphere
- Spherical Linear Interpolation follows the shortest great arc on the unit sphere

0<=h<=1

$$\cos(\Omega) = q_0 \bullet q_1$$

$$Slerp(q_0, q_1, h) = \frac{q_0 \sin((1-h)\Omega) + q_1 \sin(h\Omega)}{\sin(\Omega)}$$



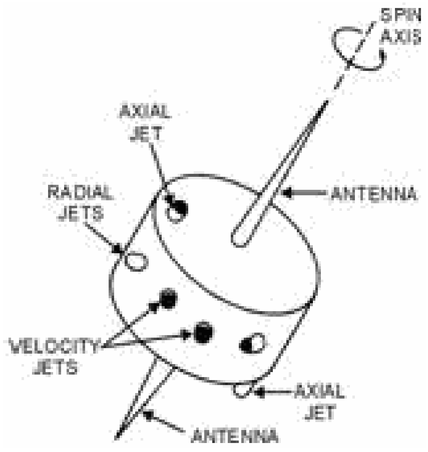
- For small steps, this gives Linear Interpolation

$$Lerp(q_0, q_1, h) = q_0(1-h) + q_1 h$$

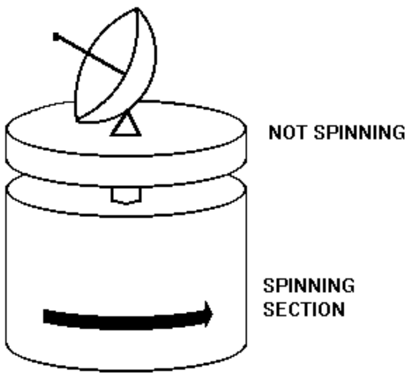
- An efficient implementation computes a quaternion describing the differential rotation, then computes its phase angle and scales it by *h*, to compute a new quaternion corresponding to the differential rotation to interpolate

Single- vs Dual-Spin Satellites

- Single-Spin
  - Entire spacecraft spins about axis of max inertia
  - Nutation damper dumps nutation angle
  - Examples: COBE, WMAP, PLANCK (1-10 rpm)
  - Disadvantages
    - Poor maneuverability due to high angular momentum
    - Constraints due to spinning (sensors, solar panels)



- Dual-Spin (rotor or flywheel)
  - Platform doesn't rotate or slowly, while rotor or flywheel spin
  - Nutation damper dumps nutation angle
  - Examples: Geosynchronous satellites
  - Disadvantages
    - Poor maneuverability due to high angular momentum
    - Constraints due to spinning part (less than single-spin)



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Attitude Stability: Methodology

- First we study attitude stability in the absence of external torques
- The procedure applies the following standard steps
  1. Express Euler equations in principal axes without torques
  2. Find equilibrium point(s) defined by constant angular velocity
  3. Linearize Euler equations about equilibrium
  4. Analysis response after small perturbation is applied
  5. Identify stable and unstable equilibrium point(s)
- This procedure is applied to two typical satellite configurations
  - Single-spin satellite
  - Dual-spin satellite (i.e., with flywheel or rotor)

Single-Spin, Velocity (1)

1. Euler equations

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y = 0 \end{cases}$$

2. Equilibrium

$$\begin{cases} \dot{\omega}_x = 0 \\ \dot{\omega}_y = 0 \\ \dot{\omega}_z = 0 \end{cases} \Rightarrow \begin{cases} (I_z - I_y)\bar{\omega}_y\bar{\omega}_z = 0 \\ (I_x - I_z)\bar{\omega}_z\bar{\omega}_x = 0 \\ (I_y - I_x)\bar{\omega}_x\bar{\omega}_y = 0 \end{cases} \Rightarrow \boxed{\begin{cases} \bar{\omega}_x = 0 \\ \bar{\omega}_y = 0 \\ \bar{\omega}_z \neq 0 \end{cases}}$$

Single spin

3. Linearization through perturbation of equilibrium

II order perturbations

$$\begin{cases} \omega_x = \bar{\omega}_x + \Delta\omega_x = \omega_x \\ \omega_y = \bar{\omega}_y + \Delta\omega_y = \omega_y \\ \omega_z = \bar{\omega}_z + \Delta\omega_z = \bar{\omega}_z + \omega_z \end{cases} \Rightarrow \begin{cases} I_x\dot{\omega}_x + (I_z - I_y)(\omega_y\bar{\omega}_z + \cancel{\omega_y\omega_z}) = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)(\omega_x\bar{\omega}_z + \cancel{\omega_x\omega_z}) = 0 \\ I_z\dot{\omega}_z + (I_y - I_x)\cancel{\omega_x\omega_y} = 0 \end{cases} \Rightarrow$$

Notation changed !

Now represents variation w.r.t. equilibrium

$$\Rightarrow \boxed{\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)(\omega_y\bar{\omega}_z) = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)(\omega_x\bar{\omega}_z) = 0 \\ I_z\dot{\omega}_z = 0 \end{cases}}$$

Single-Spin, Velocity (2)

Decoupling: 3rd equation independent from other 2

4. Linear system

Eigenvalues

$$\dot{\vec{x}} = \vec{A}\vec{x} \;\; ; \; \vec{A} = \begin{bmatrix} 0 & \frac{I_y - I_z}{I_x}\bar{\omega}_z & 0 \\ \frac{I_z - I_x}{I_y}\bar{\omega}_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \det(\lambda\vec{I} - \vec{A}) = 0 \Rightarrow \begin{cases} \lambda_{1,2} = \pm\sqrt{\frac{\bar{\omega}_z^2}{I_x I_y}(I_y - I_z)(I_z - I_x)} \\ \lambda_3 = 0 \end{cases}$$

5. Stability (mathematical, not yet physical)

$\mathrm{Re}(\lambda_i) < 0\,, \forall i \Rightarrow$ The attitude motion is stable (periodic, not asymptotic) if the rotation axis is a principal axis of maximum or minimum inertia

$$(I_y - I_z)(I_z - I_x) < 0$$

Rotation about axis of intermediate inertia is unstable and the solution grows exponentially in the vicinity of the equilibrium point

$$(I_y - I_z)(I_z - I_x) > 0$$

Backup