AA 279 C – SPACECRAFT ADCS: LECTURE 2

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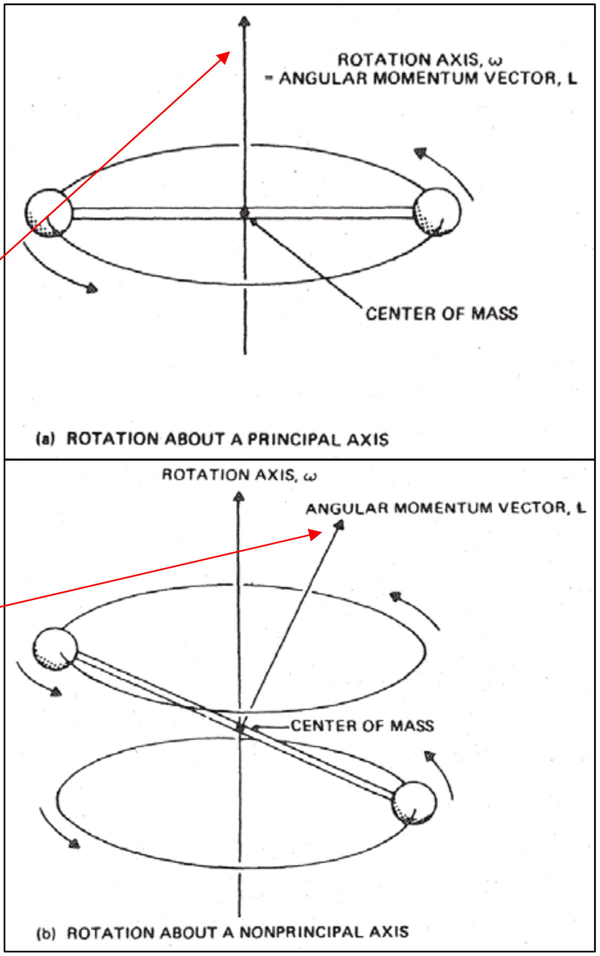
Rigid Body Dynamics: Sets of Axes

- Geometrical or body axes (***x,y,z***)
  - Defined relative to structure
  - Gives orientation of ADCS hardware
- Angular momentum axis (***L***)
  - Parallel to angular momentum vector
  - Pass through center of mass
- Instantaneous rotation axis ($\boldsymbol{\omega}$)
  - Axis about which spacecraft rotates
  - Each point has = angular velocity
- Principal axis (***P***)
  - In general axes ***L*** and $\boldsymbol{\omega}$ are not the same
  - ***L*** = const $\Rightarrow$ $\boldsymbol{\omega}$ must rotate
  - ***L*** and $\boldsymbol{\omega}$ are the same if $\boldsymbol{\omega} /\!/ \boldsymbol{P}$
  - ***P*** is a property of the rigid body



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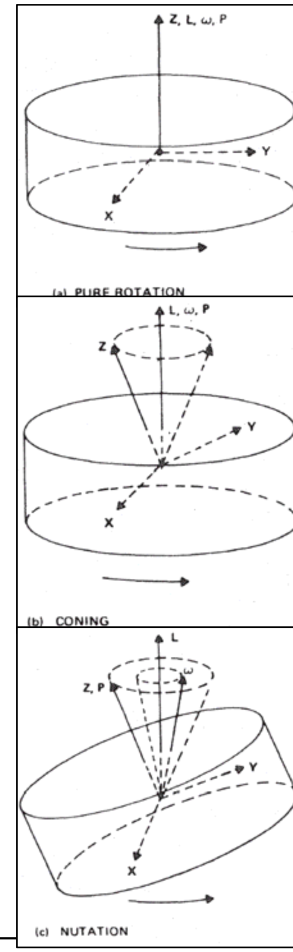
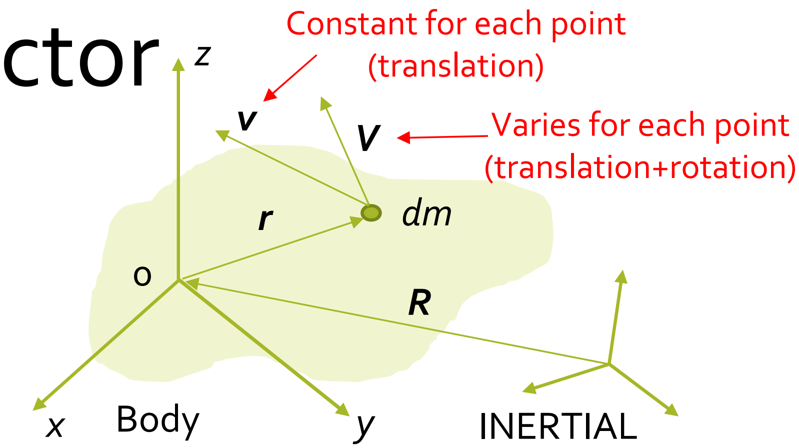
Rigid Body Dynamics: Types of Attitude

- Pure rotation (limit case)
  - ***ω // P // z // L***
- Coning (physical motion same as pure)
  - ***z*** not// ***P***
  - ***z*** rotates about ***L*** in inertial space
  - Due to coordinate system misalignment
  - Can be corrected if ***P*** is known in body frame
- Nutation (different than classical mechanics)
  - ***ω*** *not//* ***P***
  - Both ***P*** and ***ω*** rotate about ***L*** = *const*
  - Neither ***L*** nor ***ω*** are fixed in body frame
  - Angle between ***P*** and ***L*** is the nutation angle
  - Nutation and coning can occur together such that all axes are different !



(a) PURE ROTATION
(b) CONING
(c) NUTATION

Angular Momentum Vector



- Angular momentum vector relative to o
  - Inertial or absolute velocity

$$\vec{V} = \vec{v} + \vec{\omega} \times \vec{r}$$

  - For a point mass

$$\overrightarrow{L_0} = \vec{r} \times (dm)\vec{V}$$

  - For a body

$$\overrightarrow{L_0} = \int \vec{r} \times (dm)\vec{V} = -\vec{v} \times \int \vec{r}dm + \int \vec{r} \times \vec{\omega} \times \vec{r}dm$$

- Simplification: angular momentum vector relative to center of mass (o = CM)

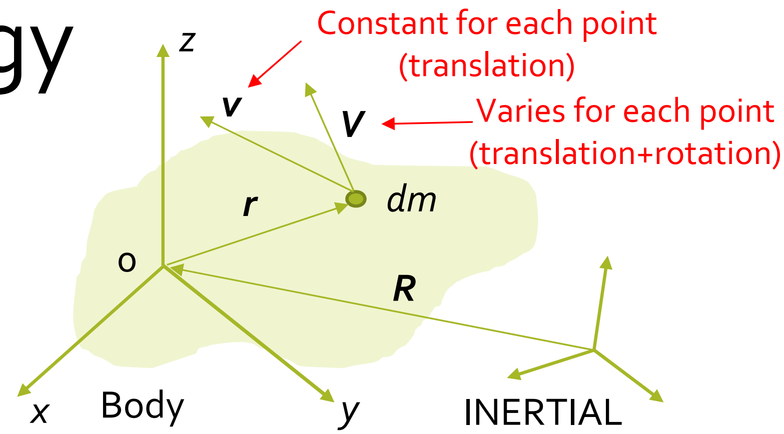
$$-\vec{v} \times \int \vec{r}dm = 0 \qquad \vec{r} \times \vec{\omega} \times \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} \omega_y z - \omega_z y \\ \omega_z x - \omega_x z \\ \omega_x y - \omega_y x \end{pmatrix} = \begin{bmatrix} y(\omega_x y - \omega_y x) \\ z(\omega_y z - \omega_z y) \\ x(\omega_z x - \omega_x z) \end{bmatrix} - \begin{bmatrix} z(\omega_z x - \omega_x z) \\ x(\omega_x y - \omega_y x) \\ y(\omega_y z - \omega_z y) \end{bmatrix}$$

$$\vec{L} = \begin{bmatrix} \int(y^2+z^2)dm & -\int(xy)dm & -\int(xz)dm \\ -\int(xy)dm & \int(x^2+z^2)dm & -\int(yz)dm \\ -\int(yz)dm & -\int(yz)dm & \int(x^2+y^2)dm \end{bmatrix} \vec{\omega} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix} \vec{\omega} = \vec{I}\,\vec{\omega}$$

Inertia matrix is symmetric, does not depend on motion, but varies with choice of body frame

L and ***w*** are expressed in body frame "xyz" and varies with time in body frame

Rotational Kinetic Energy



- Involves rotational motion only
  - Inertial or absolute velocity

    $$\vec{V} = \vec{v} + \vec{\omega} \times \vec{r}$$

  - For a point mass

    $$2T = V^2 dm$$

  - For a body

Translation only

Constant

$$2T = \int \vec{V} \cdot \vec{V} dm = \int v^2 dm + \int (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) dm + 2 \int \vec{v} \cdot (\vec{\omega} \times \vec{r}) dm$$

- Simplification: kinetic energy relative to center of mass (o = CM)

$$\int \vec{r} dm = 0 \quad (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) = \begin{pmatrix} \omega_y z - \omega_z y \\ \omega_z x - \omega_x z \\ \omega_x y - \omega_y x \end{pmatrix} \cdot \begin{pmatrix} \omega_y z - \omega_z y \\ \omega_z x - \omega_x z \\ \omega_x y - \omega_y x \end{pmatrix}$$

$$2T = \omega_y^2 I_y + \omega_z^2 I_z + \omega_x^2 I_x - 2\left(\omega_y \omega_z I_{zy} + \omega_x \omega_z I_{xz} + \omega_x \omega_y I_{xy}\right) = \vec{\omega} \cdot I \vec{\omega} = \vec{\omega} \cdot \vec{L}$$

For torque-free motion the kinetic energy is constant

Thus, ***L*** and ***w*** are constrained in the way they can change

Principal Axes: Further Simplification

- We can always find a convenient set of axes (principal) such that the inertia matrix is diagonal (zero inertia products)
- We use the fact that the rotational kinetic energy is invariant to changes of the body reference frame
- If we rotate the body frame, the components of the angular velocity and inertia matrix change, but not *T*

$$2T = \vec{\omega} \cdot I\vec{\omega} = A\vec{\omega}' \cdot IA\vec{\omega}' = \vec{\omega}' \cdot I'\vec{\omega}'$$

Rotation matrix   Invariance

- Changing from vector to matrix algebra

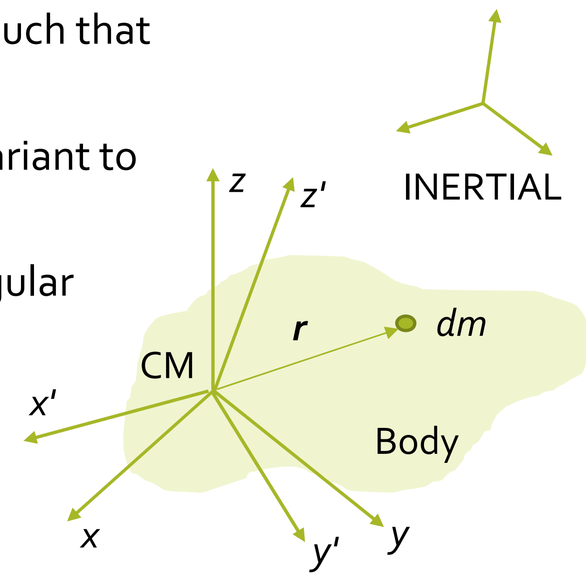
$$2\mathrm{T} = A\vec{\omega}' \cdot IA\vec{\omega}' = \vec{\omega}' A^t IA\vec{\omega}' \Rightarrow A^t IA = I'$$

- We seek a direction cosine matrix *A* so that *I′* is diagonal

$$IA = AI'$$

Appendix C.6 (Wertz)

- If *I* is not diagonal with the given body frame, its *eigenvalues* provide the principal moments of inertia, while its *eigenvectors* provide the rotation matrix



Principal Axes

$$\vec{L} = I_{x'}\omega_{x'}\vec{x}' + I_{y'}\omega_{y'}\vec{y}' + I_{z'}\omega_{z'}\vec{z}'$$
$$2T = I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2$$

$$I = \begin{bmatrix} I_{x'} & 0 & 0 \\ 0 & I_{y'} & 0 \\ 0 & 0 & I_{z'} \end{bmatrix}$$

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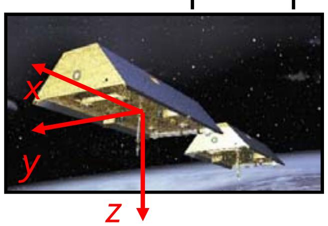
Properties of Inertia Matrix

- The inertia matrix, ***I***, is a property of the *rigid* body because the body frame (***x***,***y***,***z***) and body do not change with time or motion
- ***I*** is constant for a rigid body, but its elements change with the choice of body frame
- ***I*** is fixed by satellite mass distribution and body frame, its elements cannot be picked randomly for a real system
  - $I_i \geq 0, \forall i$ e.g. $I_x = \int (y^2 + z^2) dm \geq 0$
  - $I_i + I_j \geq I_k, i \neq j \neq k$ e.g. $I_x + I_y = \int (y^2 + z^2 + x^2 + z^2) dm = I_z + \int 2z^2 dm$
  - $I_i - I_j \leq I_k, i \neq j \neq k$ e.g. $I_x - I_y = \int (y^2 + z^2 - x^2 - z^2) dm = I_z - \int 2x^2 dm$
  - $I_i \geq 2I_{jk}, i \neq j \neq k$ e.g. $I_x = \int (y^2 + z^2) dm = \int (y - z)^2 dm + \int 2yzdm$

= holds for wire

- For real objects ***r*** is not a straight line, det(***I***) > 0, thus ***I***$^{-1}$ exists ($\neq$ ***I***^{t} though)
- The mass is symmetrically distributed about principal axes, and any axis of rotational symmetry is a principal axis, thus $I_{ij} = 0, i \neq j$
- Rotation about principal axis: other moments of inertia are multiplied by zero



$$I = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix} = \overbrace{\begin{bmatrix} 87 & -0.8 & 0.1 \\ -0.8 & 486 & 0 \\ 0.1 & 0 & 538 \end{bmatrix}}^{\text{GRACE (Jan, 2002)}} \text{ kg/m}^2$$

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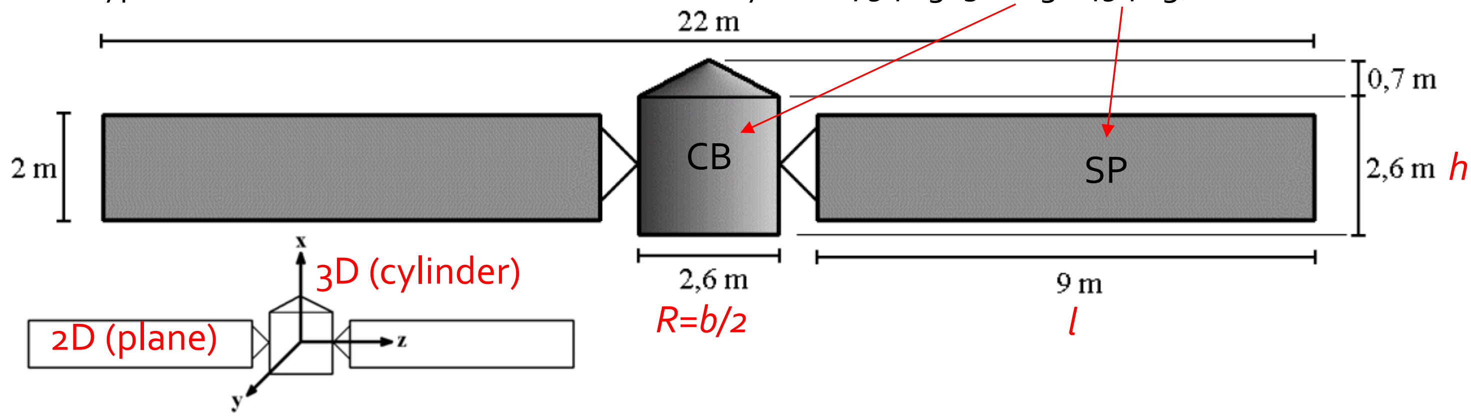
Mass and Moments of Inertia: Example

- Hypothesis from data available on internet: dry mass 794 kg ($\overset{2m_{\mathrm{SP}}}{300\text{ kg}}$+ $\overset{m_{\mathrm{CB}}}{494\text{ kg}}$)



- Convenient choice of body axes as axes of symmetry (principal axes)

- Computation of moments of inertia of each part about parallel axes though CoM

$$
\begin{aligned}
I_{x\mathrm{SP}} &= m_{\mathrm{SP}} l^2/12 = 1012.5\ \mathrm{kg\ m^2} \\
I_{z\mathrm{SP}} &= m_{\mathrm{SP}} h^2/12 = 1062.5\ \mathrm{kg\ m^2} \\
I_{y\mathrm{SP}} &= I_{x\mathrm{SP}} + I_{z\mathrm{SP}} = 50\ \mathrm{kg\ m^2}
\end{aligned}
$$

$$
\begin{aligned}
I_{x\mathrm{CB}} &= m_{\mathrm{CB}} R^2/2 = 417.43\ \mathrm{kg\ m^2} \\
I_{y\mathrm{CB}} &= m_{\mathrm{CB}} (R^2/2 + b^2/12) = 487.0\ \mathrm{kg\ m^2} \\
I_{z\mathrm{CB}} &= I_{y\mathrm{CB}} = 487.0\ \mathrm{kg\ m^2}
\end{aligned}
$$

$$
\begin{aligned}
I_x &= 2\left(I_{x\mathrm{SP}} + m_{\mathrm{SP}} z_{gSP}^2\right) + I_{x\mathrm{CB}} = 15117\ \mathrm{kg\ m^2} \\
I_y &= 2\left(I_{y\mathrm{SP}} + m_{\mathrm{SP}} z_{gSP}^2\right) + I_{y\mathrm{CB}} = 15287\ \mathrm{kg\ m^2} \\
I_z &= 2 I_{z\mathrm{SP}} + I_{z\mathrm{CB}} = 587\ \mathrm{kg\ m^2}
\end{aligned}
$$

These numbers meet the properties of an inertia matrix

Backup

Analogy between translation and rotational motion

Table 3: *The analogies between translational and rotational motion.*

Translational motion		*Rotational motion*	
Displacement	$d\mathbf{r}$	Angular displacement	$d\boldsymbol{\phi}$
Velocity	$\mathbf{v} = d\mathbf{r}/dt$	Angular velocity	$\boldsymbol{\omega} = d\boldsymbol{\phi}/dt$
Acceleration	$\mathbf{a} = d\mathbf{v}/dt$	Angular acceleration	$\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$
Mass	M	Moment of inertia	$I = \int \rho \lvert\hat{\boldsymbol{\omega}} \times \mathbf{r}\rvert^2 \, dV$
Force	$\mathbf{f} = M\,\mathbf{a}$	Torque	$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{f} = I\,\boldsymbol{\alpha}$
Work	$W = \int \mathbf{f} \cdot d\mathbf{r}$	Work	$W = \int \boldsymbol{\tau} \cdot d\boldsymbol{\phi}$
Power	$P = \mathbf{f} \cdot \mathbf{v}$	Power	$P = \boldsymbol{\tau} \cdot \boldsymbol{\omega}$
Kinetic energy	$K = M\,v^2/2$	Kinetic energy	$K = I\,\omega^2/2$