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AA 279 A – Space Mechanics
Lecture 4: Notes

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- Relationship between eccentric and true anomaly
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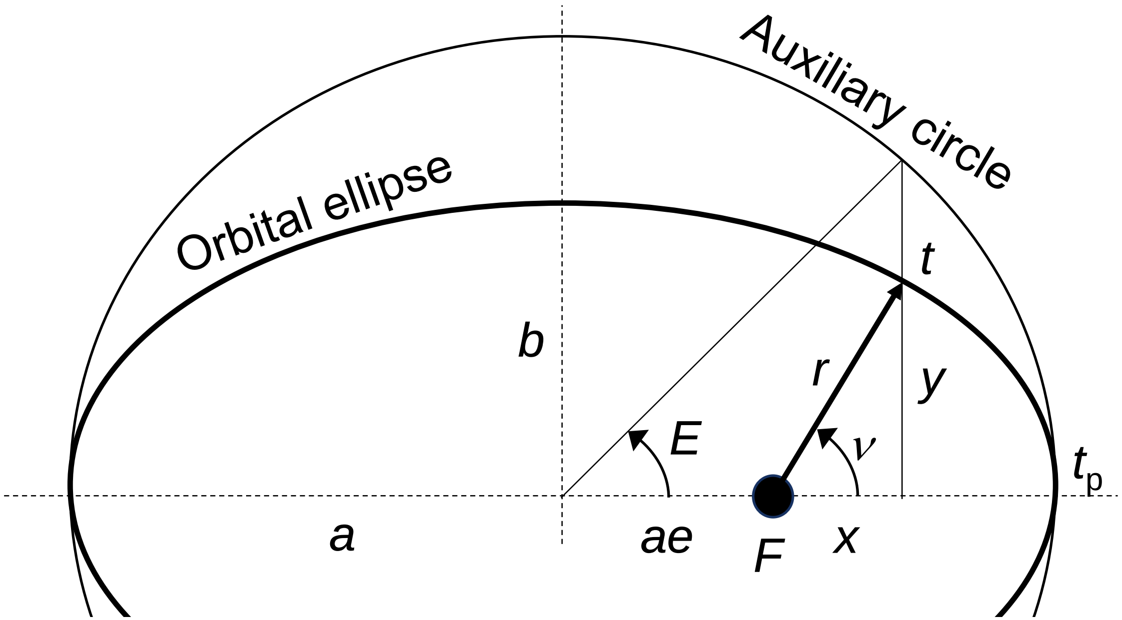
- Orbital elements

- Reading for next week (lectures 5 and 6)
  - Bate 2.2-2.6, 2.8, 2.9, 2.15, 3.2
  - Montenbruck 2.2.3-2.2.5, 2.3, 5
  - Vallado 2.4, 2.6-2.8, 3.2-3.5.2, 11.1.2-11.2.2

Kepler's Equation

$$M = n(t - t_p)$$
$$M = M_0 + n(t - t_0)$$
Mean Anomaly

$$E - e\sin E = M$$
Kepler's Equation [rad]



- Mean, *M*, Eccentric, *E*, and True, ν, Anomalies always in same semi-plane
- $M = E = \nu$ at 0, π, 2π, etc. [rad]
- *M* represents a mean motion on auxiliary circle and cannot be shown above
- The Kepler's equation is only valid for ellipses

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Time Dependence of Motion

$M\,(t)$ = known by its definition

+

$E\,(t)$ = known through Kepler's Equation

+

$\nu\,(E)$ = known by definition of E

t = input

Non linear equation which can only be solved iteratively

Explicit formulas to be derived

$\nu\,(t) = \nu\,(E(M(t)))$

+

$r(\nu)$ = known through Trajectory Equation

$r(t)$ = ?

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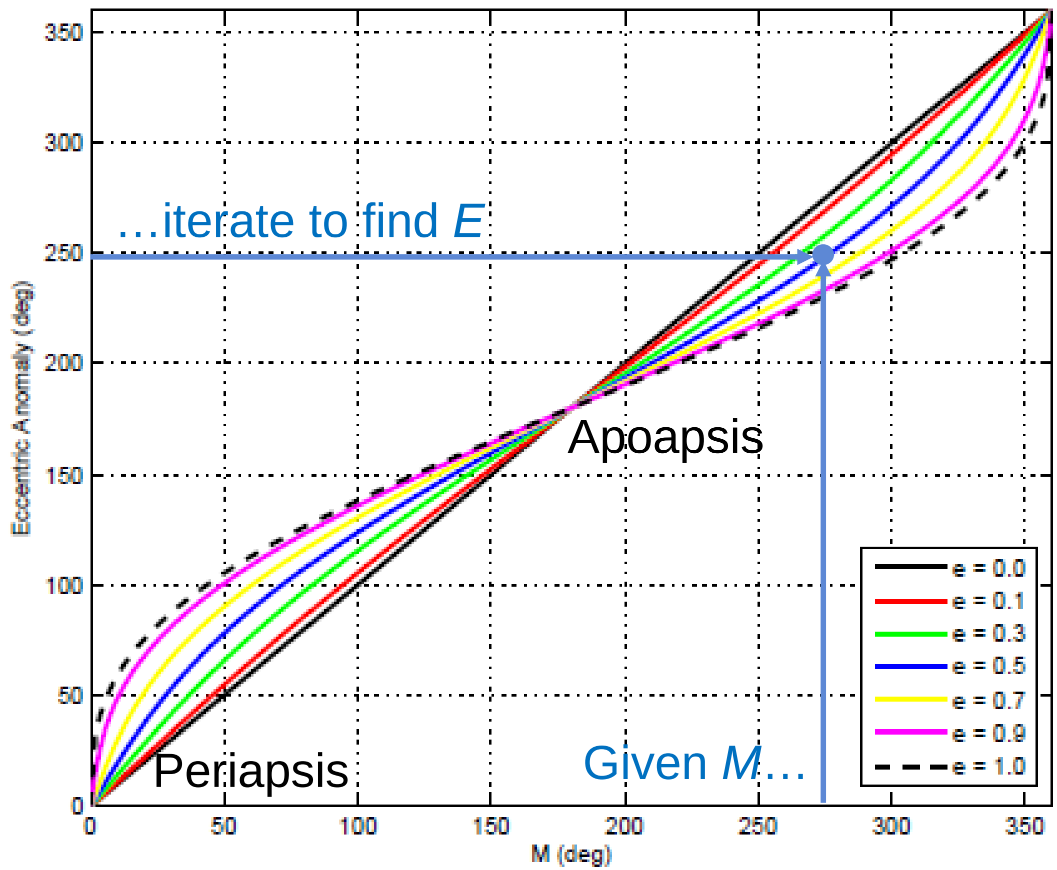
Solving Kepler's Equation (1)

- Given *E* and *e*, *M* is found by substitution
- The larger *e*, the larger the maximum difference between *M* and *E*
- Given *M* and *e*, *E* can be found numerically through Newton-Raphson method
- Finding roots of *f*(*E*)

$$E - e\sin E = M$$



$$f(E) = E - e\sin E - M = 0$$

$$f(E) = f(\tilde{E} + \delta) \approx f(\tilde{E}) + f'(\tilde{E})\delta + \cancel{\frac{1}{2!} f''(\tilde{E})\delta^2} + \cdots$$

Neglect second order terms and solve for δ

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Solving Kepler's Equation (2)

- Solving for δ provides the successive correction $\delta = E_{i+1} - E_i$ towards an improved solution

$$\delta = -\frac{f(\tilde{E})}{f'(\tilde{E})}$$

Correction

$$E_{i+1} = E_i + \delta_i = E_i - \frac{E_i - e\sin E_i - M_i}{1 - e\cos E_i}$$

Iterative algorithm

- The iterative algorithm can be stopped when δ_i drops below some tolerance value (e.g.,$1{\cdot}10^{-8}$)

- This method converges well and the number of iterations depends on two factors: e and E_0. The best initial guess is either M ($\pm$ e) or π (safe)

$$E_k = e\sin E_{k-1} + M$$

Alternative selection of initial guess, k gives order

Relationship between Anomalies

$$\frac{y_{Ellipse}}{y_{Circle}} = \frac{b}{a} = \sqrt{1-e^2}$$

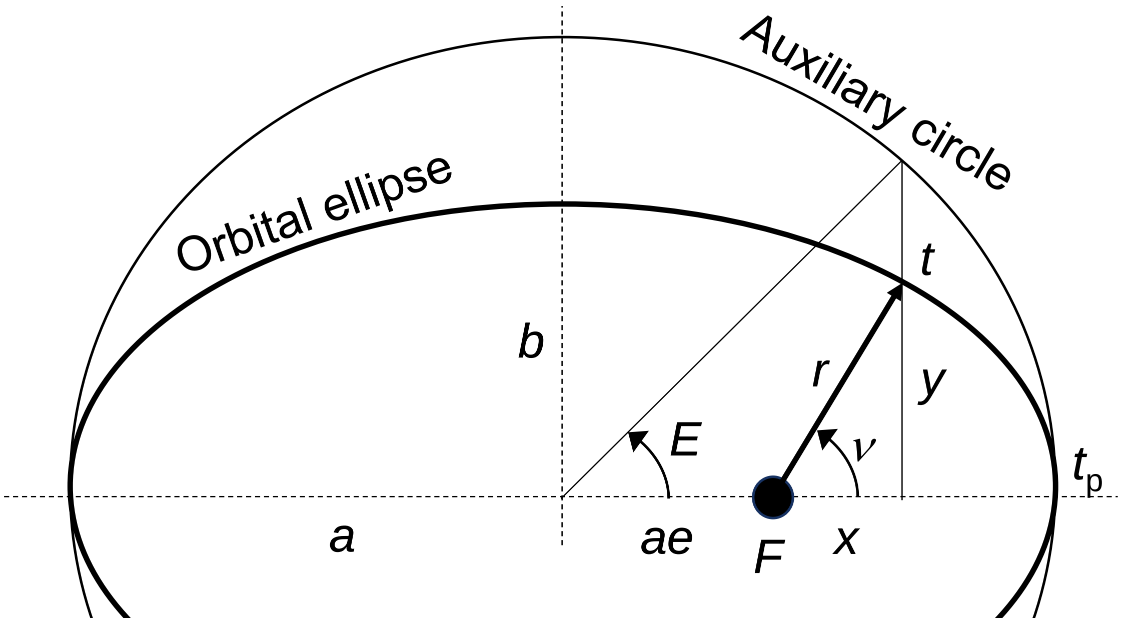
Scaling factor (only for ellipses)

$$x = r\cos\nu = a(\cos E - e)$$

$$y = r\sin\nu = a\sqrt{1-e^2}\sin E$$

$$r = a(1 - e\cos E)$$

Definition of Eccentric Anomaly



$$\left.\begin{aligned} \cos E &= \frac{e+\cos\upsilon}{1+e\cos\upsilon}\ ;\ \cos\upsilon = \frac{\cos E - e}{1-e\cos E} \\ \sin E &= \frac{\sin\nu\sqrt{1-e^2}}{1+e\cos\nu}\ ;\ \sin\upsilon = \frac{\sin E\sqrt{1-e^2}}{1-e\cos E} \end{aligned}\right\}\ \tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\upsilon}{2}\ ;\ \tan\frac{\upsilon}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

Relating Eccentric and True Anomaly

Kepler's Problem (1)

- Calculate **time of flight** (t-t_0) between two points ν and ν_0 on orbit where $0 \leq \nu < 2\pi$, $0 \leq \nu_0 < 2\pi$, and $k = 0,1,\ldots,N$ is the number of times object passes though periapsis between t and t_0

1) Find E from ν using Eccentric/True anomaly relationships
2) Find E_0 from ν_0 using Eccentric/True anomaly relationships
3) Find (t-t_0) using

Always ensure $0 \leq E, E_0 < 2\pi$

k is given !

$$t - t_0 = \frac{M - M_0}{n} = \frac{1}{n}[2\pi k + (E - esinE) - (E_0 - esinE_0)]$$

Generalized form of Kepler's equation

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Kepler's Problem (2)

- Predict **position** ν at desired time t given position ν_0 at time t_0. Note that the unknowns are $0 \leq \nu < 2\pi$ and $k = 0,1,\ldots,N$ (number of times object passes though periapsis between t and t_0), whereas $0 \leq \nu_0 < 2\pi$ is given.

1) Find E_0 from ν_0 using Eccentric/True anomaly relationships
2) Calculate M_0 from E_0 using Kepler's equation
3) Find M and k from

$$\boxed{M + 2\pi k = M_0 + n(t - t_0)}$$

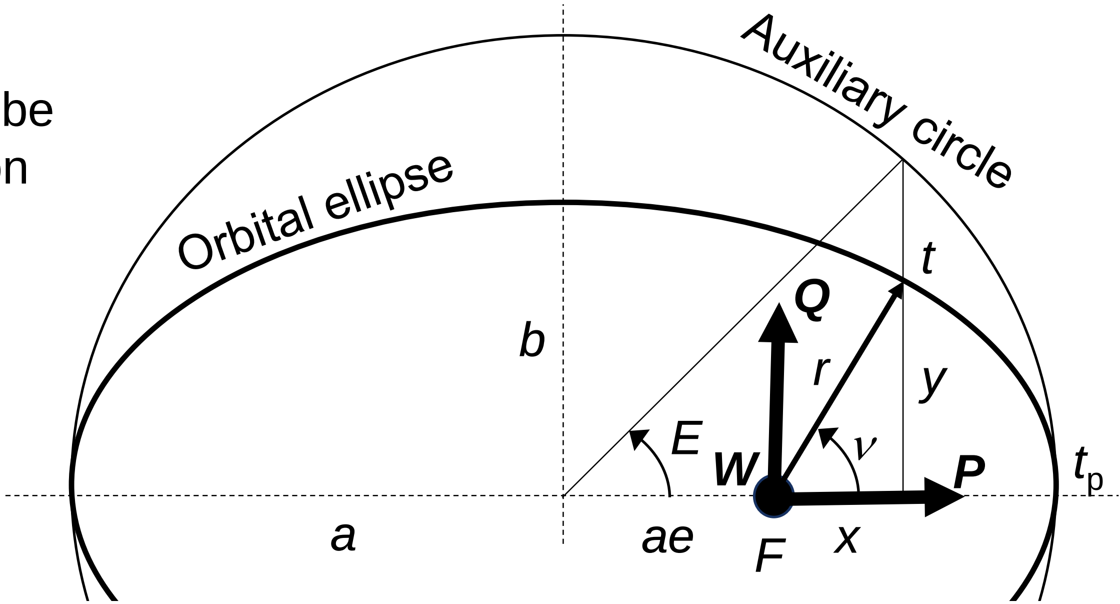
Always ensure
$0 \leq E_0, M_0, E, \nu < 2\pi$

where k is an integer chosen to cause $0 \leq M < 2\pi$

4) Find E via numerical solution of Kepler's equation
5) Find ν from E using Eccentric/True anomaly relationships

The Orbit in Space (1)

- The prediction method based on the Kepler's equation can be used to simulate orbital motion versus time
- We are only describing 3 Degrees of Freedom though
  - 2 DOFs for "size" and "shape" (e.g., *a* and *e*)
  - 1 DOF for "timing" or "phasing" (e.g., ν_0)



- We can use other combinations of parameters such as *a*&*p* for size&shape or M_0 or t_p for timing
- We are working in perifocal coordinates defined by ***P*** = ***B***/*B*, ***Q*** (ν=90°) and ***W*** = ***h***/*h* unit vectors

$$\vec{r} = x\vec{P} + y\vec{Q} =$$
$$= r\cos\upsilon\vec{P} + r\sin\upsilon\vec{Q} =$$
$$= a(\cos E - e)\vec{P} + a\sqrt{1-e^2}\sin E\vec{Q}$$

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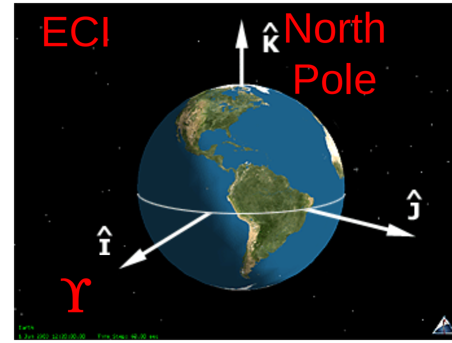
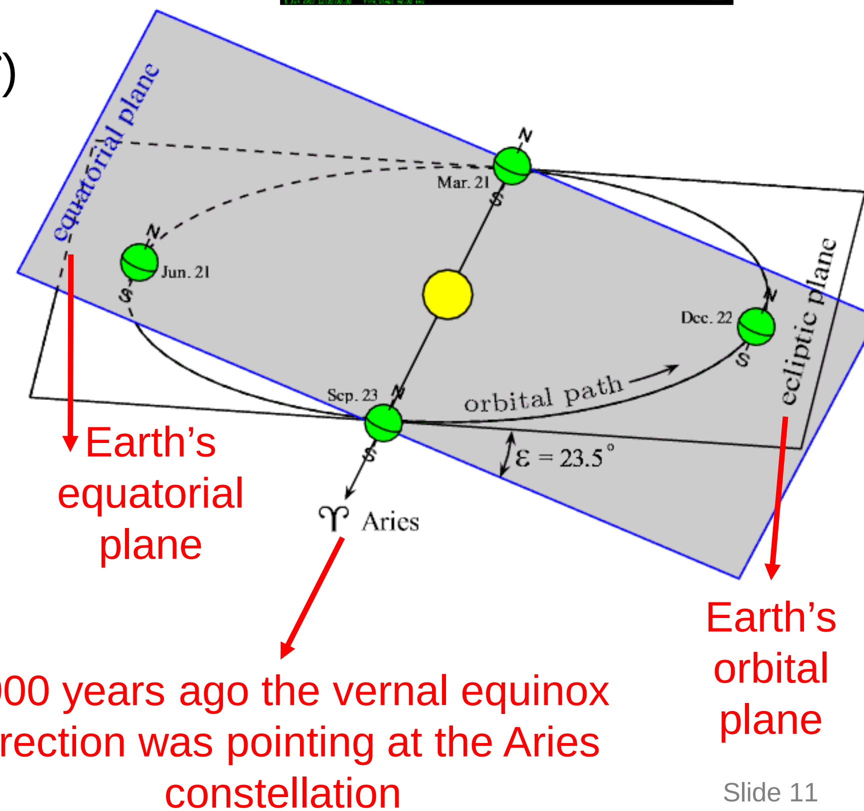
The Orbit in Space (2)

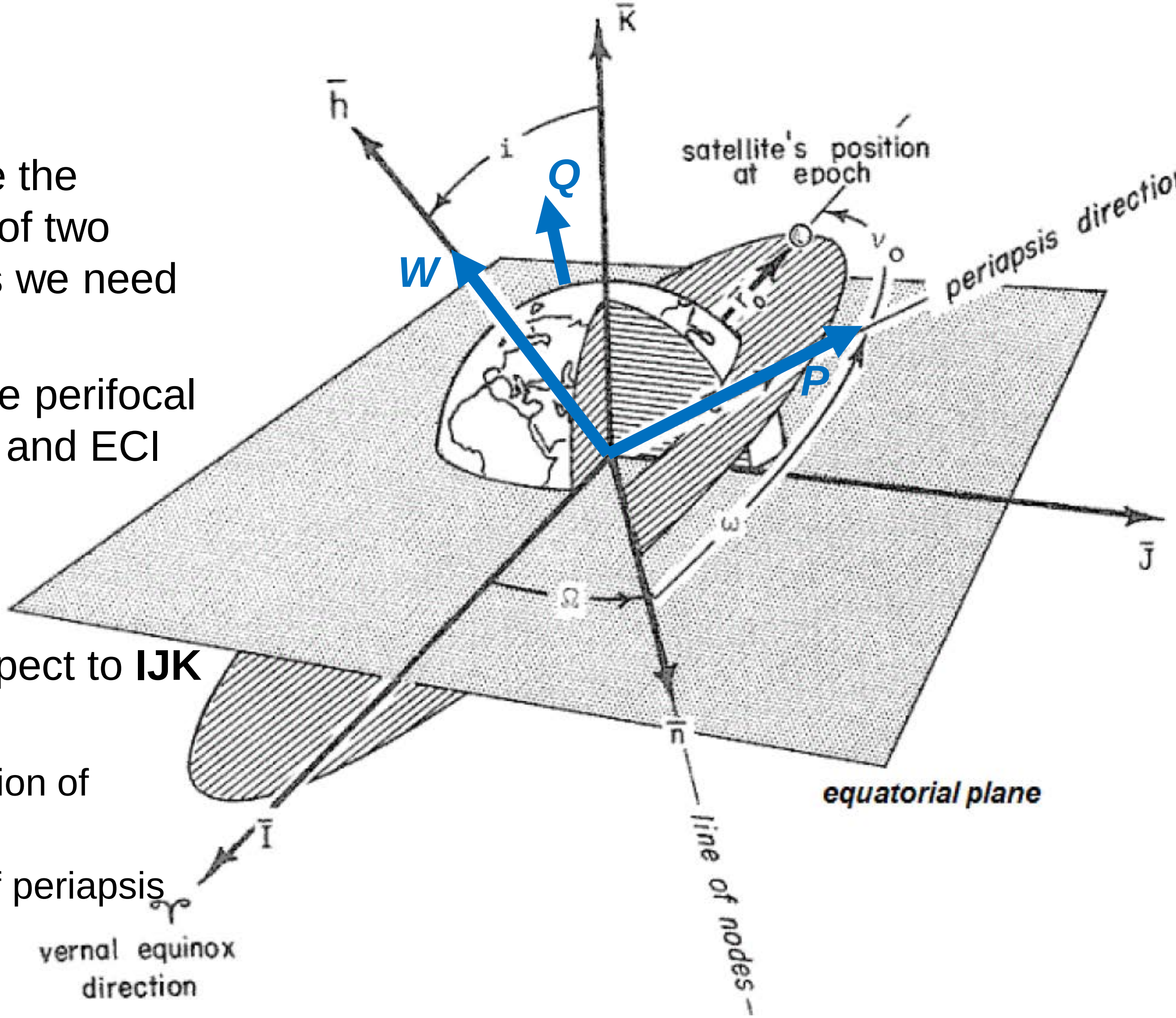
- The most common coordinate system for Earth-bound satellite orbits is the geocentric equatorial coordinate system
- This is an Earth-Centered Inertial (ECI) reference system with axis
  - ***I*** aligned with the *vernal equinox* (Υ)
  - ***J*** completing the right-handed triad
  - ***K*** pointing to the north pole
- The *vernal equinox* describes the direction of the Sun as seen from the Earth at the beginning of spring time or, equivalently, the intersection of the equatorial plane (***I***-***J***) with the Earth's orbital plane



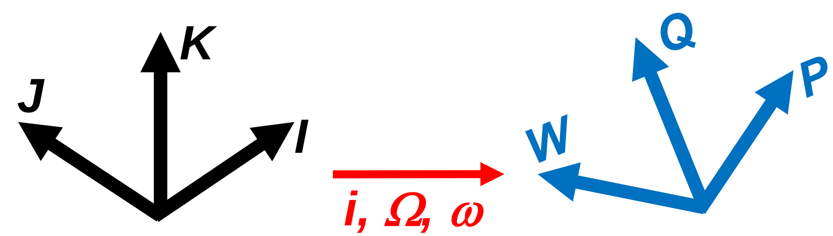


The Orbit in Space (3)

- In order to describe the relative orientation of two coordinate systems we need three angles
- In our case we have perifocal coordinates, **PQW**, and ECI coordinates, **IJK**
- The missing three DOFs are the orbit orientation with respect to **IJK**
  - *i* = inclination
  - Ω = right ascension of ascending node
  - ω = argument of periapsis



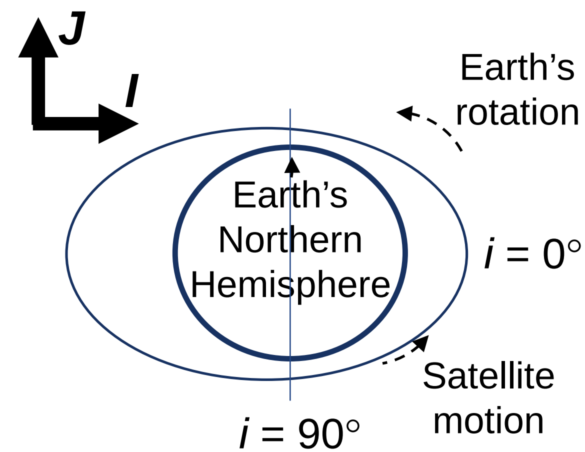
The Orbit in Space (4)



- ***i* = inclination**

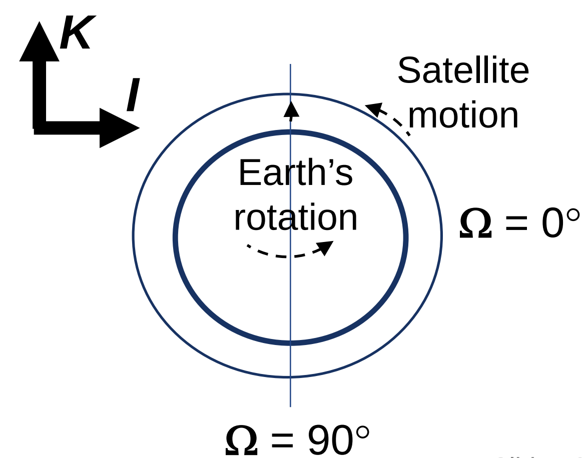
  Angle of intersection between orbital plane and equator

  i > 90° implies that satellite motion is retrograde, i.e. its direction of revolution around the Earth being opposite to that of the Earth's rotation



- ***Ω* = right ascension of ascending node**

  Angle between vernal equinox and the point on the orbit at which the satellite crosses the equator from south to north

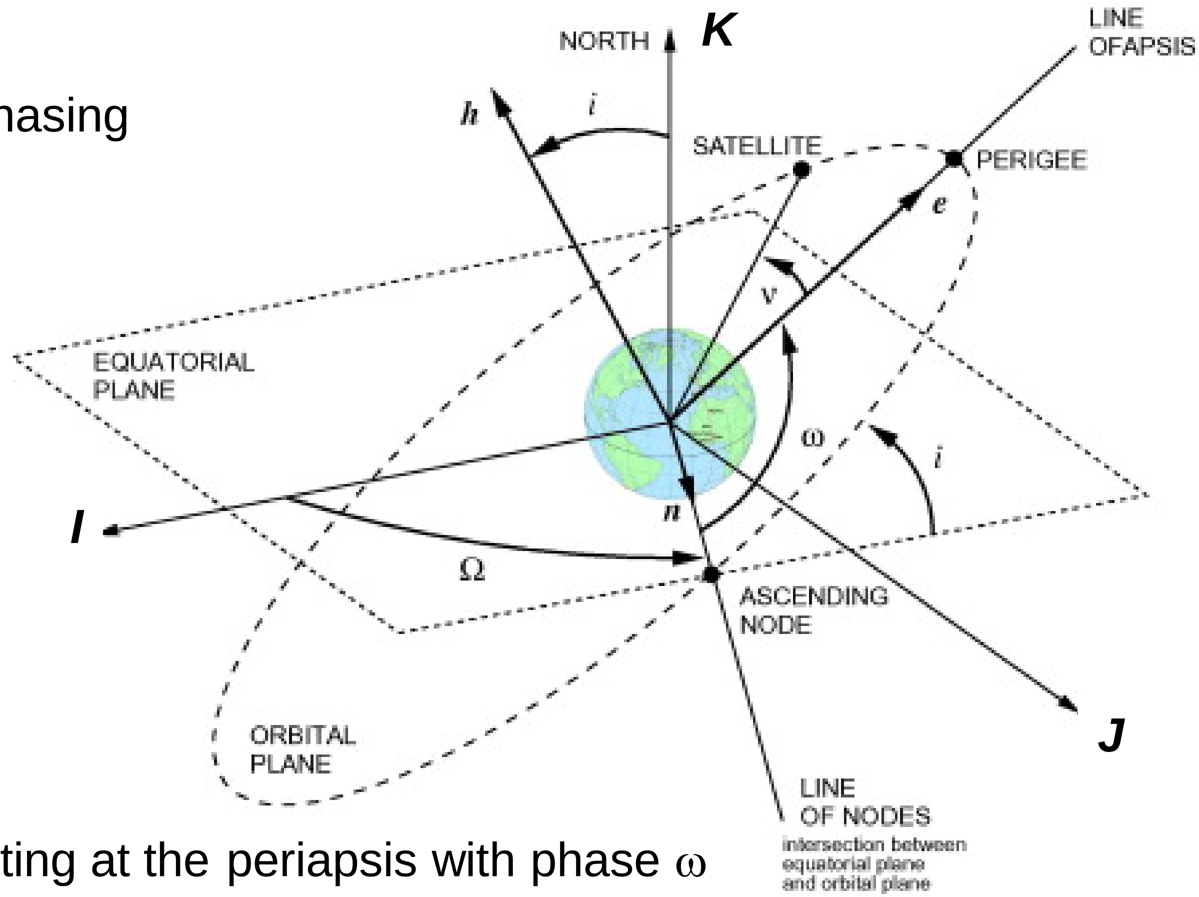


- ***ω* = argument of periapsis**

  Angle between the direction of the ascending node and the direction of the periapsis

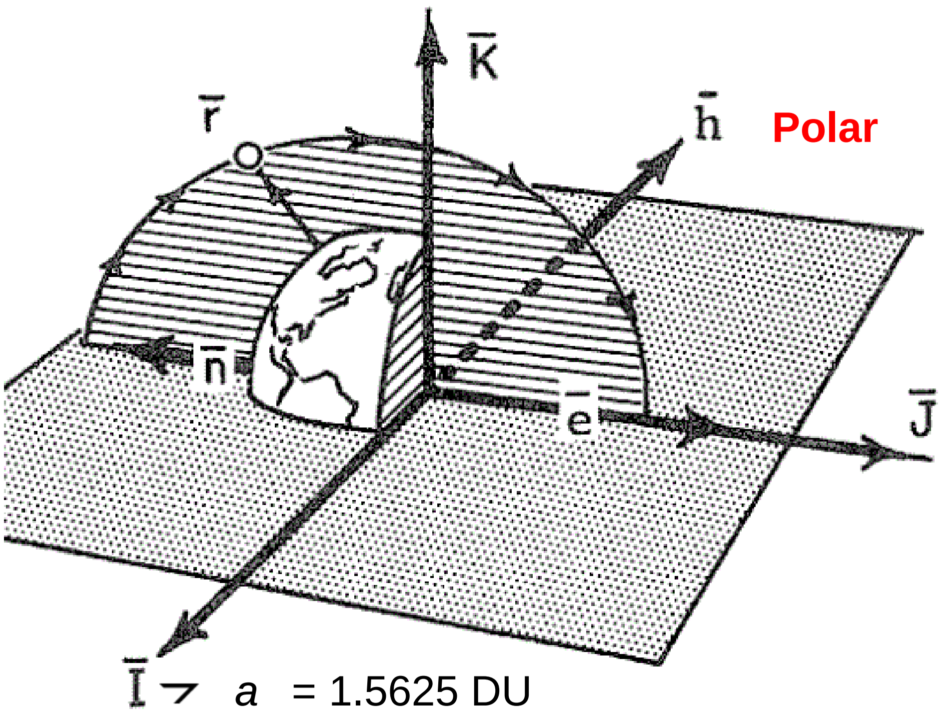
Keplerian or Classical Orbital Elements

- In-plane scale, shape, phasing
  - *a*
  - *e*
  - ν_0
- In-plane orientation
  - ω
- Out-of-plane orientation
  - *i*
  - Ω
- For a total of 6 DOFs

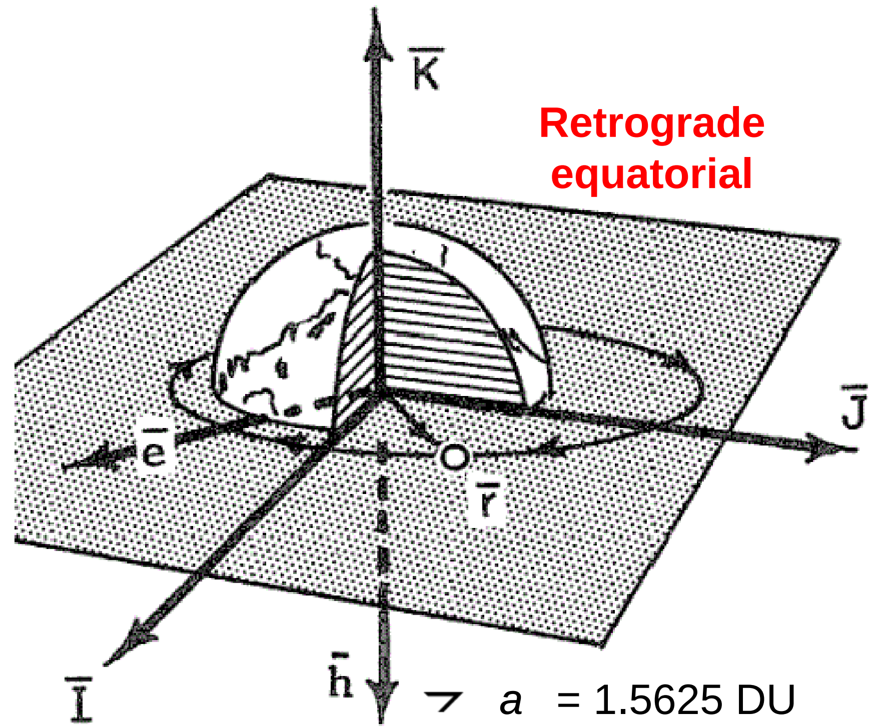


- Note that the vector pointing at the periapsis with phase ω and magnitude *e* is called eccentricity vector ***e*** (not a classical orbital element, $\boldsymbol{e} = \boldsymbol{B}/\mu$ from trajectory equation)

Orbital Elements Examples



- a = 1.5625 DU
- e = 0.2
- ν_0 = 225°
- ω = 180°
- i = 90°
- Ω = 270°



- a = 1.5625 DU
- e = 0.2
- ν_0 = 270°
- ω = undefined
- i = 180°
- Ω = undefined

Backup