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AA 279 A – Space Mechanics
Lecture 14: Notes

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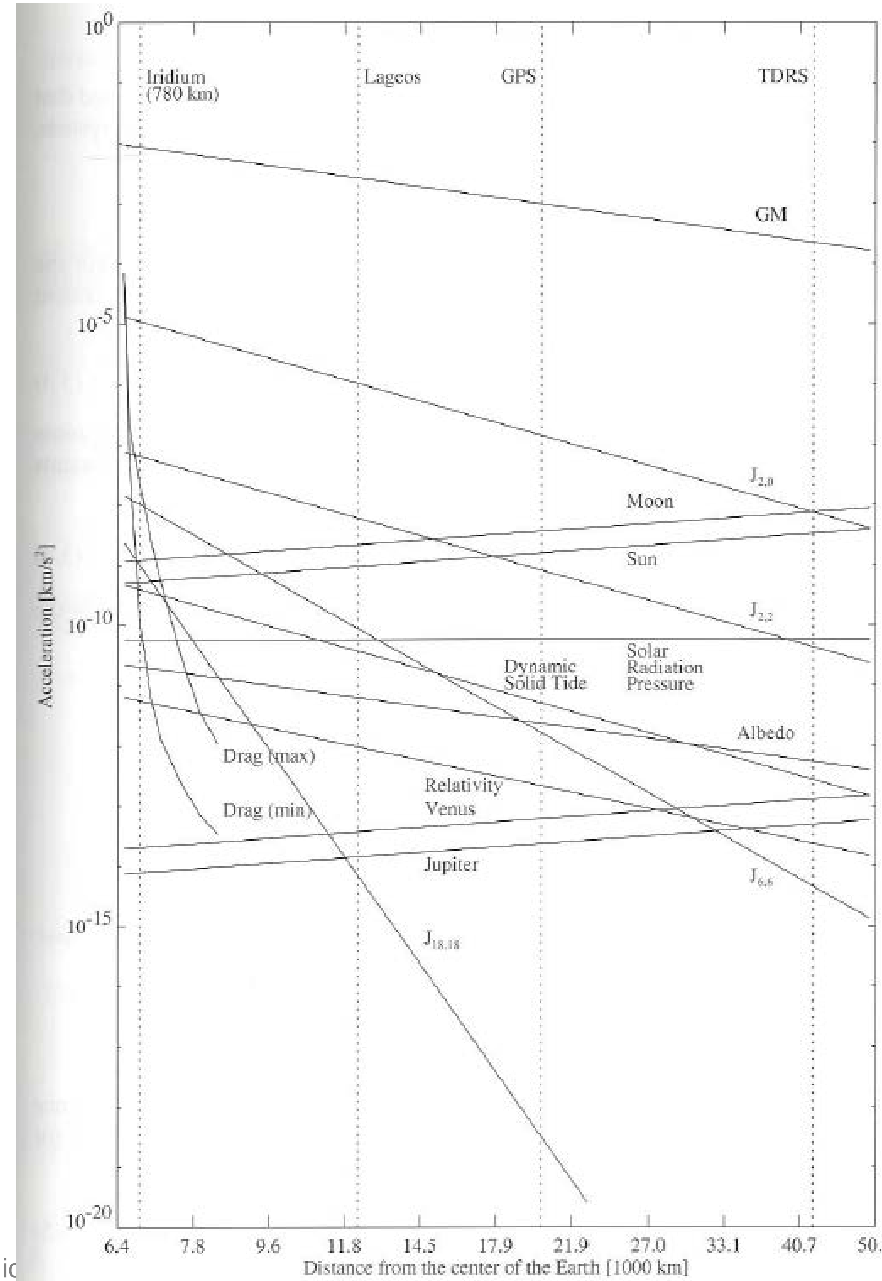
Dominant Orbit Perturbations (1)

- Earth's oblateness (J_2)
  - Equatorial bulge pulls the satellite tending to align orbit with equatorial plane
  - The orbit behaves like a gyroscope with precession about North pole
- Atmospheric drag
  - Caused by air molecules's impingement
  - Force opposes velocity and reduces mechanical energy
- Third body (Moon and Sun)
  - Caused by difference between force exerted on Earth and satellite
  - Gyroscopic precession is about normal to Moon's orbital plane or ecliptic plane
- Solar radiation pressure
  - Caused by photons' impingement
  - Force affects eccentricity vector mainly



Dominant Orbit Perturbations (2)

Contribution	LEO (1 rev)	LEO (1 day)	GEO (1 day)
Earth gravity; terms $>J_{2,0}$	600 m	5000 m	670 m
Earth gravity; terms $>J_{2,2}$	220 m	3000 m	2 m
Earth gravity; terms $>J_{4,4}$	150 m	1900 m	0 m
Earth gravity; terms $>J_{10,10}$	23 m	460 m	0 m
Third body solar gravity	3 m	34 m	3100 m
Third body lunar gravity	6 m	66 m	5100 m
Solar radiation pressure	1 m	14 m	415 m
Atmospheric drag	1 m	100 m	0 m

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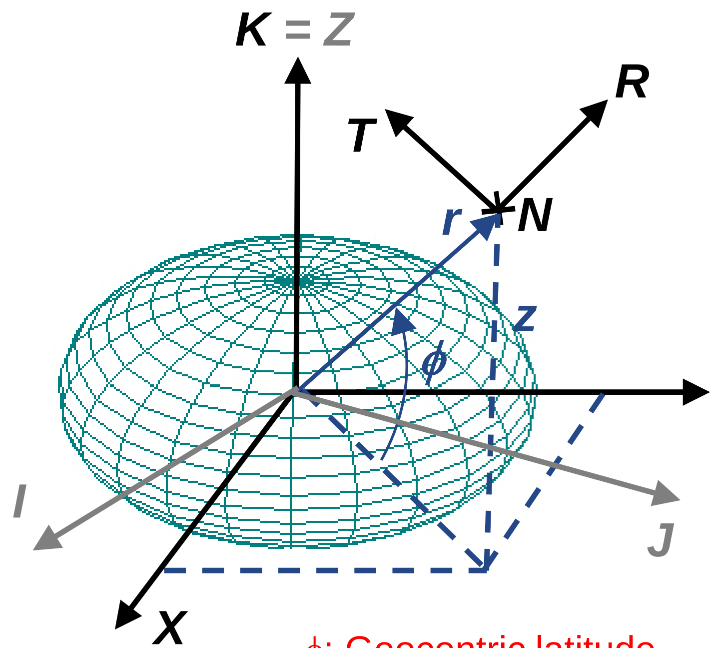
Oblate Earth Gravity Model (1)



- Earth is not a homogeneous sphere and its gravity field is not exactly

$$U(r) = \frac{\mu}{r} \Rightarrow \vec{\nabla} U = -\frac{\mu}{r^3}\vec{r}$$

- A more accurate model is an oblate spheroid with equatorial radius R_E

$$U(r,\phi) = \frac{\mu}{r}\left[1 - \underbrace{J_2\left(\frac{R_E}{r}\right)^2\left(\frac{3\sin^2\phi - 1}{2}\right)}_{\text{Perturbation from first term in spherical harmonic series expansion}}\right]$$

ϕ: Geocentric latitude
XYZ: ECEF
IJK: ECI
RTN: Orbital frame
$J_2 \approx 1.0826 \cdot 10^{-3}$

- Using $\sin\phi = z/r$, the perturbing force is derived as

$$\vec{f} = \vec{\nabla} U_{\text{Perturbing}} = -\frac{\mu J_2 R_E^2}{2}\left[6\left(\frac{z}{r^5}\right)\vec{K} + \left(\frac{3}{r^4} - 15\left(\frac{z^2}{r^6}\right)\right)\vec{R}\right]$$

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Oblate Earth Gravity Model (2)

- In order to use the Gauss Variational Equations we need ***f*** expressed in RTN
- Using the approach to convert from IJK to PQW coordinates we can show

$$\vec{K} = \sin i \sin u \vec{R} + \sin i \cos u \vec{T} + \cos i \vec{N} \qquad z = \vec{K} \cdot \vec{r}$$

- Substitution in ***f*** gives the desired result

$$\boxed{\begin{aligned} f_R &= -\frac{3\mu J_2 R_E^2}{2r^4}(1 - 3\sin^2 i \sin^2 u) \\ f_T &= -\frac{3\mu J_2 R_E^2}{2r^4}(\sin^2 i \sin u \cos u) \\ f_N &= -\frac{3\mu J_2 R_E^2}{2r^4}(\sin i \cos i \sin u) \end{aligned}}$$

These expressions for oblate gravity are ready to be fed into the GVE…

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Mean Earth's Oblateness Effects (1)

- The derivative of the RAAN can be expressed from GVE as follows

$$\frac{d\Omega}{d\nu} = \frac{d\Omega/dt}{d\nu/dt} = \frac{\dfrac{r\sin(\omega+\nu)}{na^2\sqrt{1-e^2}\sin i} f_N}{\underbrace{\dfrac{na^2}{r^2}\sqrt{1-e^2}}_{O(n)} + \underbrace{\dfrac{\sqrt{1-e^2}\cos\nu}{nae} f_R - \dfrac{\sqrt{1-e^2}}{nae}\dfrac{2+e\cos\nu}{1+e\cos\nu} f_T}_{O(J_2)\cdot O(n) \ll O(n)}} \approx$$

- neglecting small terms and after substitution of the oblate gravity

$$\approx -3J_2\left(\frac{R_E}{a(1-e^2)}\right)^2 \cos i \sin^2(u)(1+e\cos\nu)$$

- We calculate the average over one orbit to remove short-periodic J_2 effects

$$\left(\frac{d\Omega}{d\nu}\right)_{\mathrm{AVG}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\Omega}{d\nu}d\nu = \frac{-3J_2}{2\pi}\left(\frac{R_E}{a(1-e^2)}\right)^2 \cos i \int_0^{2\pi}\sin^2(u)(1+e\cos\nu)d\nu$$

Mean Earth's Oblateness Effects (2)

- Using the following integrals we can derive the secular effect of J_2 on the orbital elements

$$\int_0^{2\pi} \sin^2(u)dv = \pi \; ; \; \int_0^{2\pi} \sin^2(u)\cos v dv = 0$$

"Nodal rotation"

$$\left(\frac{d\Omega}{dt}\right)_{\mathrm{AVG}} = \left(\frac{d\Omega}{dv}\right)_{\mathrm{AVG}} n = -\frac{3}{2}nJ_2\left(\frac{R_E}{a(1-e^2)}\right)^2 \cos i$$

"Apsidal rotation"

$$\left(\frac{d\omega}{dt}\right)_{\mathrm{AVG}} = \frac{3}{4}nJ_2\left(\frac{R_E}{a(1-e^2)}\right)^2 (5\cos^2 i - 1)$$

Relatively small

$$\left(\frac{dM}{dt}\right)_{\mathrm{AVG}} = n + \frac{3}{4}nJ_2\left(\frac{R_E}{a(1-e^2)}\right)^2 \sqrt{1-e^2}(3\cos^2 i - 1)$$

Not affected by J_2 (secularly)

$$\left(\frac{di}{dt}\right)_{\mathrm{AVG}} = \left(\frac{de}{dt}\right)_{\mathrm{AVG}} = \left(\frac{da}{dt}\right)_{\mathrm{AVG}} = 0$$

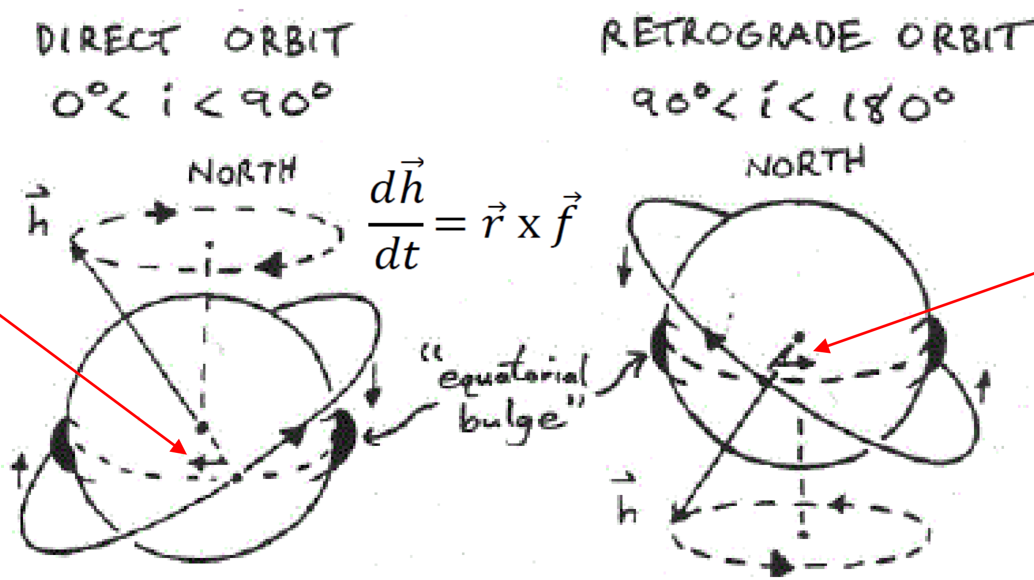
Selection of a, e, and i determines secular drifts !!

Nodal Rotation

$$\boxed{\left(\frac{d\Omega}{dt}\right)_{\text{AVG}} = \left(\frac{d\Omega}{d\nu}\right)_{\text{AVG}} n = -\frac{3}{2} n J_2 \left(\frac{R_E}{a(1-e^2)}\right)^2 \cos i}$$

- Orbits with $0° < i < 90°$ will have decreasing Ω: line of nodes rotates westward ("regresses") for an oblate central body
- In order to achieve repeat orbits (see TerraSAR-X) we need to adjust Ω through out-of-plane maneuvers (Δv_N) or alternatively adjust orbital period (a) through along-track maneuvers (Δv_T)

Line of nodes move "west" each orbit (regression)

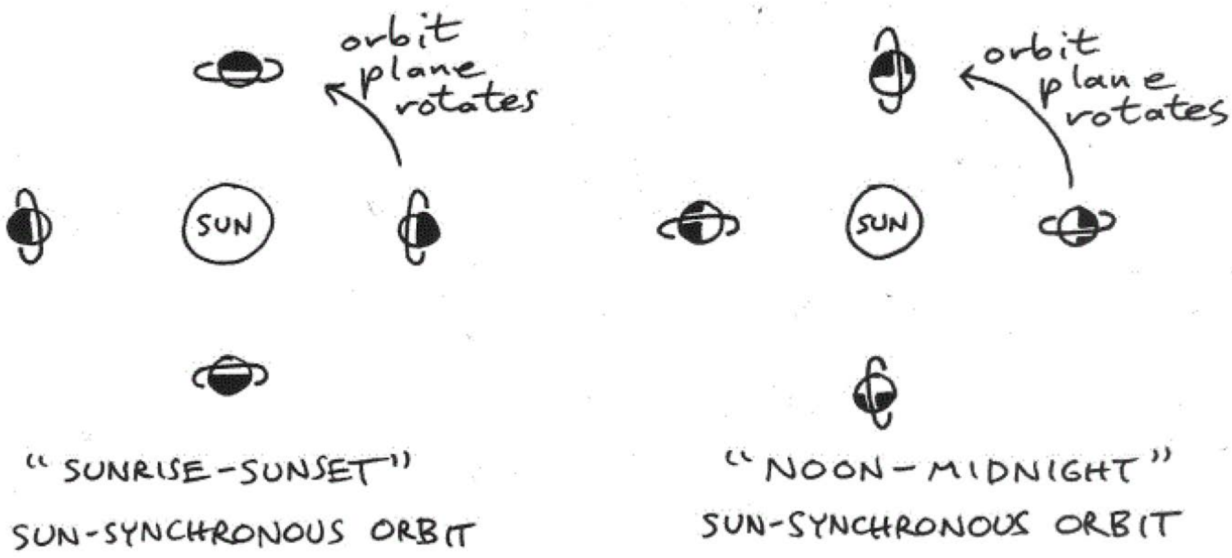


Line of nodes move "east" each orbit (advance)

Sun-Synchronous Orbits

$$\left(\frac{d\Omega}{dt}\right)_{\mathrm{AVG}} = \frac{360°}{365.25\ \mathrm{d}} = \frac{3}{2} n J_2 \left(\frac{R_E}{a(1-e^2)}\right)^2 \cos i$$

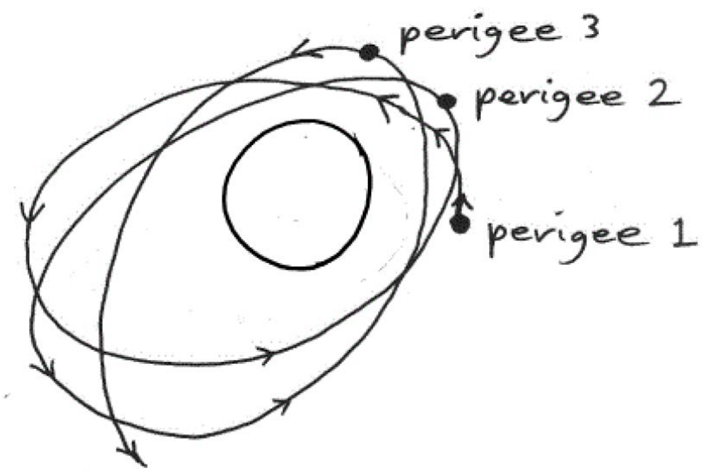
- Nodal regression or advance can be used to maintain a constant orientation of the orbital plane w.r.t. sun
- This allows observing locations on Earth at the same local time each orbit
- Note that this requires retrograde orbits (90° > i > 180°)



Apsidal Rotation

$$\boxed{\left(\frac{d\omega}{dt}\right)_{\mathrm{AVG}} = \frac{3}{4} n J_2 \left(\frac{R_E}{a(1-e^2)}\right)^2 (5\cos^2 i - 1)}$$

- Orbits with $e \neq 0$ will have rotation of line of apsides: changing ω or rotating eccentricity vector ***e*** (non-frozen orbit)
- The direction of the rotation depends on the inclination:
  - $i = 63.43°$ or $116.6°$ — Critical inclination at which J_2 effect is zeroed
  - $0° < i < 63.43°$ — Line of apsides advances (move east)
  - $63.43° < i < 116.6°$ — Line of apsides regresses (move west)



Gravity stronger than $1/r^2$ near perigee

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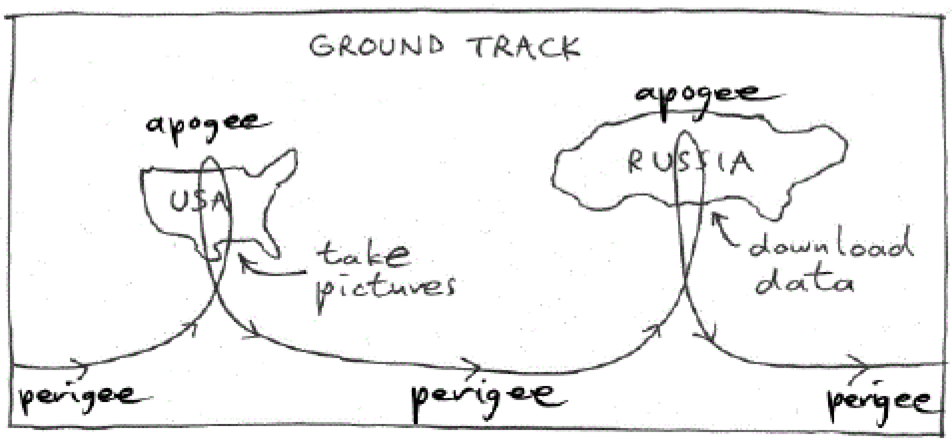
Molniya Orbits

$$\left(\frac{d\omega}{dt}\right)_{\mathrm{AVG}} = 0°/\mathrm{d} = \frac{3}{4} n J_2 \left(\frac{R_E}{a(1-e^2)}\right)^2 (5\cos^2 i - 1)$$

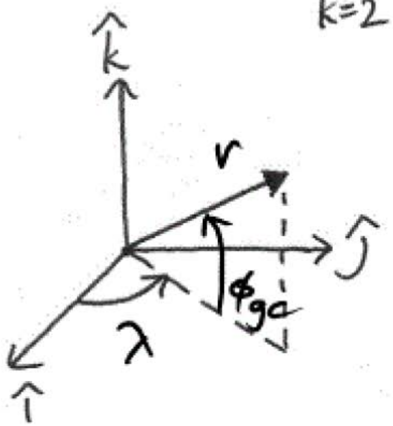
- Russia's high latitude makes geostationary satellites hard to see at their low elevation. Orbits are created which spend most of the time over Russia.
- This is done through proper choice of inclination and eccentricity vector. Critical inclination freezes the eccentricity vector which can be picked to keep perigee and apogee at desired spots.
- Communication and spy satellites can be placed in Molniya orbits with 12 hours periods so to have two apogees per day (e.g., over Russia and US)



Spherical Harmonic Earth Gravity Model (1)

- We have used a simple oblate earth gravity model so far
- Earth geoid is not symmetric about equator or about rotation axis
- A more general representation of the geopotential is

$$U_{earth}(r,\lambda,\phi_{gc}) = \frac{\mu}{r}\left[1 - \sum_{k=2}^{\infty}\left(\frac{R_\oplus}{r}\right)^k J_k P_k(\sin\phi_{gc}) + \sum_{k=2}^{\infty}\sum_{j=1}^{k}\left(\frac{R_\oplus}{r}\right)^k P_k^j(\sin\phi_{gc})\left\{C_k^j\cos j\lambda + S_k^j \sin j\lambda\right\}\right]$$



- effects of terms die off quickly as $k \to \infty$, $r \to \infty$
- J_k, C_k^j, S_k^j are coefficients on the terms

Spherical Harmonic Earth Gravity Model (2)

Associated Legendre functions P of degree k and order j

Magnitude and phase of spherical harmonics

$$U_{earth}(r,\lambda,\phi_{gd}) = \frac{\mu}{r}\left[1 - \sum_{k=2}^{\infty}\left(\frac{R_\oplus}{r}\right)^k J_k P_k(\sin\phi_{gc}) + \sum_{k=2}^{\infty}\sum_{j=1}^{k}\left(\frac{R_\oplus}{r}\right)^k P_k^j(\sin\phi_{gc})\left\{C_k^j\cos j\lambda + S_k^j \sin j\lambda\right\}\right]$$

- There are three different types of spherical harmonic:
  - Zonal (J)

    Order = zero, *Degree* = #zero crossings btw poles
  - Sectoral (C and S)

    Order = *Degree* = #cycles in longitude
  - Tesseral (C and S)

    Order = #cycles in longitude, *Degree–Order* = #zero crossings btw poles

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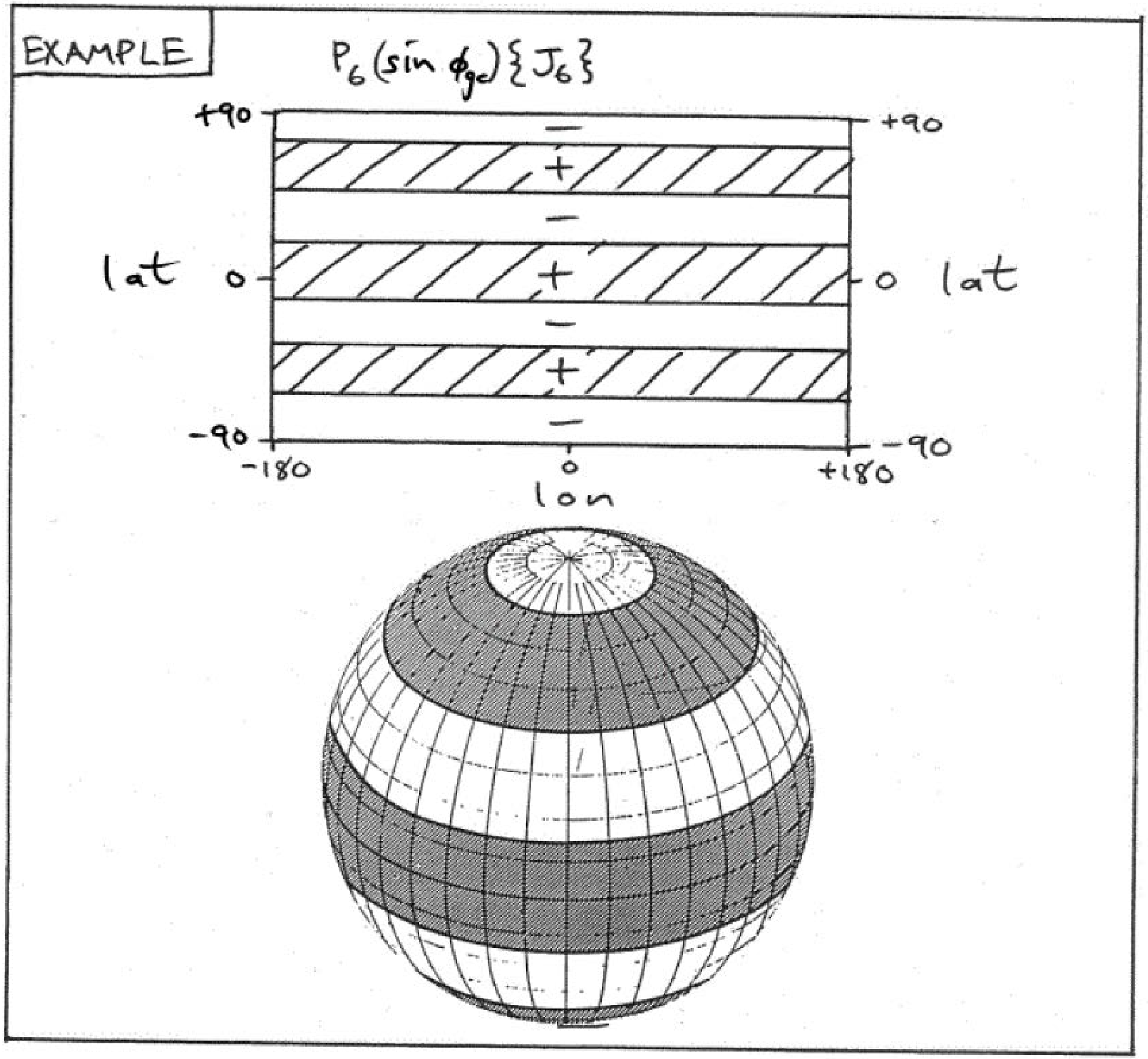
Spherical Harmonic Earth Gravity Model (3)



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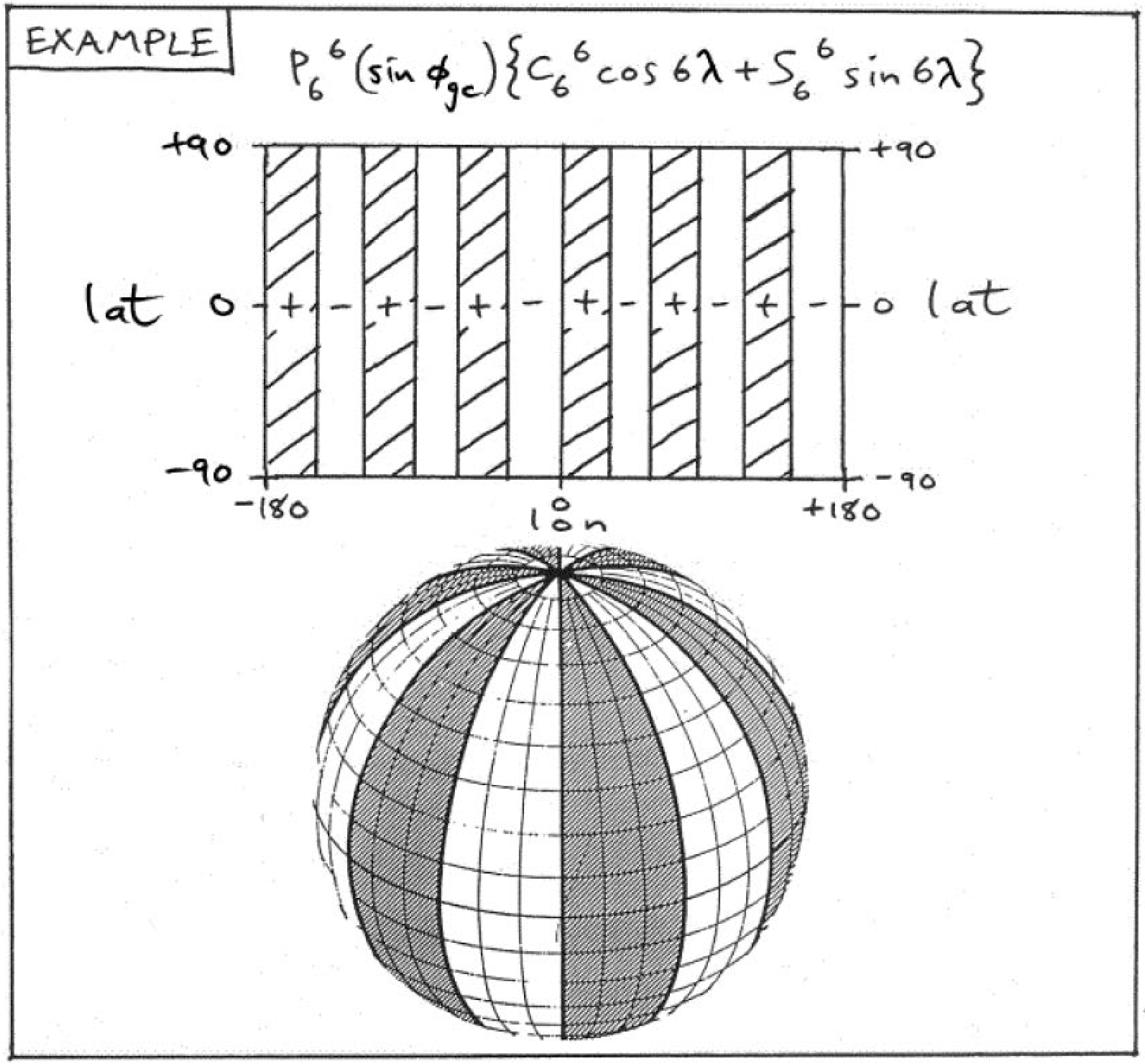
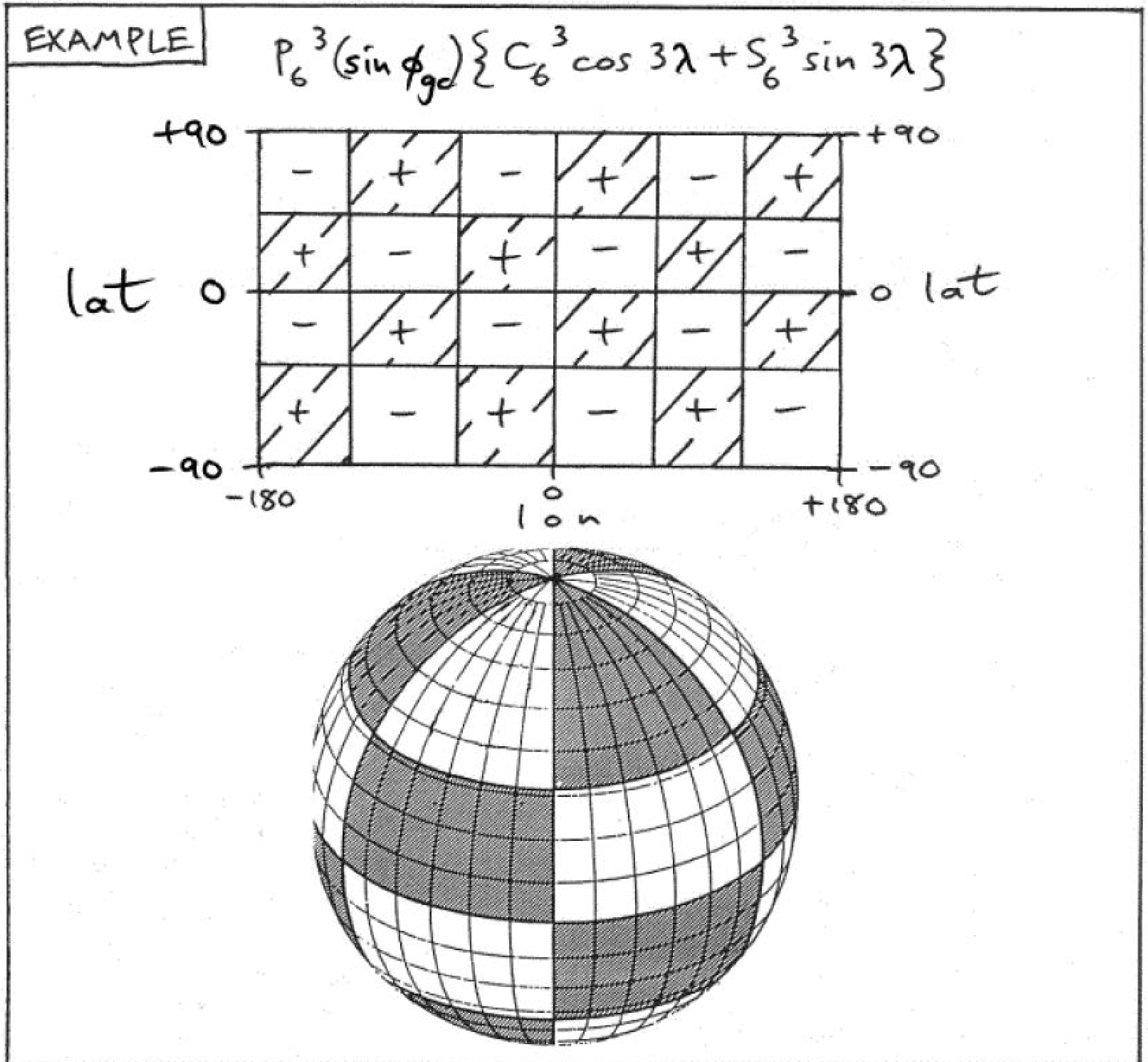
Spherical Harmonic Earth Gravity Model (4)



Spherical Harmonic Earth Gravity Model (5)



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Dominant Orbit Perturbations (3)

	Gravity		3rd Body	Atm Drag	Rad Press
	Zonal	Sect/Tess			
a	P	P	P	P S	P
e	P	P	P	P S	P
i	P	P	P	P S	P
Ω	P S	P	P S	P	P S
ω	P S	P	P S	P	P S
M_0	P S	P	P S	P	P S

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Atmospheric Drag

Orbit determination estimates force model parameters such as ballistic coefficient *B*

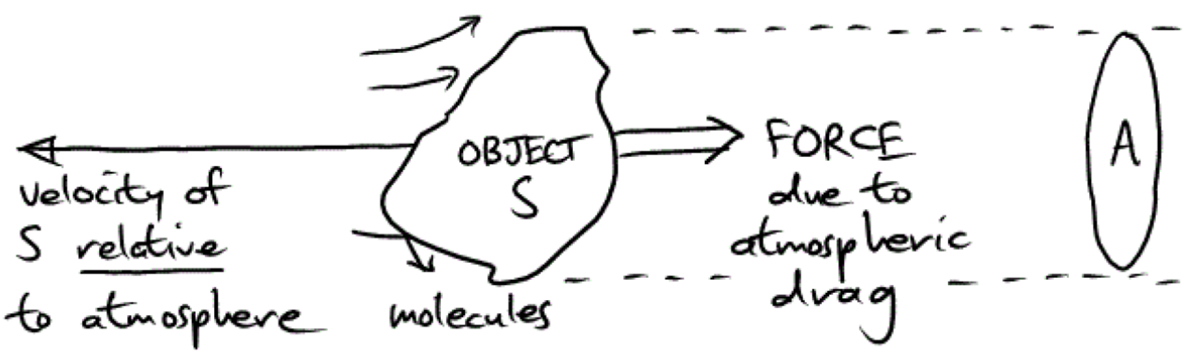
Density model required

- Molecules in atmosphere hitting object ‘S’ cause an aerodynamic force

$$\vec{f}_{\text{Drag}} = -\frac{1}{2} B \rho v_{rel}^2 \frac{\vec{v}_{rel}}{v_{rel}} \; ; \; B = \frac{C_D A}{M} \; ; \; \rho = \rho_0 e^{-\left(\frac{h-h_0}{H}\right)}$$

  - ρ = Atmospheric density (height, location, time, sun activity)
  - C_D = Drag coefficient (2.2 for flat plate, 2.0 for sphere)
  - $\boldsymbol{v}_{rel}$ = Velocity relative to atmosphere (moving w.r.t. Earth)
  - A = Cross-section area (size, attitude)
  - M = Mass of object ‘S’

Uncertainty



Atmospheric Drag Effects

- Effect on elliptical high orbit from GVE

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}\sin\nu}{na} f_R + \frac{\sqrt{1-e^2}}{na^2 e}\left(\frac{a^2(1-e^2)}{r} - r\right) f_T$$

$$\frac{d\omega}{dt} = -\frac{\sqrt{1-e^2}\cos\nu}{nae} f_R + \frac{\sqrt{1-e^2}}{nae}\frac{2+e\cos\nu}{1+e\cos\nu}\sin\nu f_T$$

- At perigee $r=r_p=a(1-e)$, $\nu = 0$, $f_R=0$ thus

$$\frac{de}{dt} = \frac{2\sqrt{1-e^2}}{na} f_{\mathrm{Drag}} < 0 \quad \Rightarrow \quad e \rightarrow 0$$

$$\frac{d\omega}{dt} = 0 \quad \Rightarrow \quad \omega = \text{const}$$

Decreasing eccentricity → Circular

Perigee constant

Apogee lowering

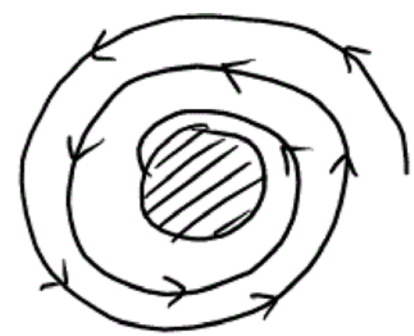
- Effect on circular low orbit from GVE

$$\frac{d\mathcal{E}}{dt} = \vec{f}\cdot\vec{v} < 0 \quad \Rightarrow \quad \mathcal{E} = -\frac{\mu}{2a}$$

DRAG PARADOX

$$\frac{da}{dt} = \frac{2}{n} f_T < 0 \quad \Rightarrow \quad v = \sqrt{\frac{\mu}{a}}$$

Spiral Decay → Splash!



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