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AA 279 A – Space Mechanics
Lecture 3: Notes

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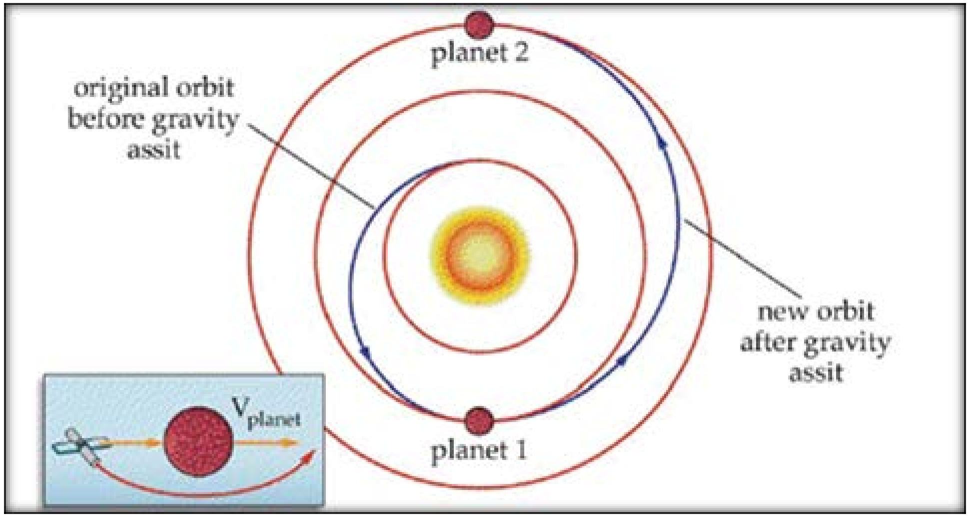
Table of Contents

- Finish general discussion of orbits
- Derivation of fundamental formulas
- Introduce timing relations
- Kepler's Equation



Patch conic approximation through multiple 2-body problems

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Escape Velocity

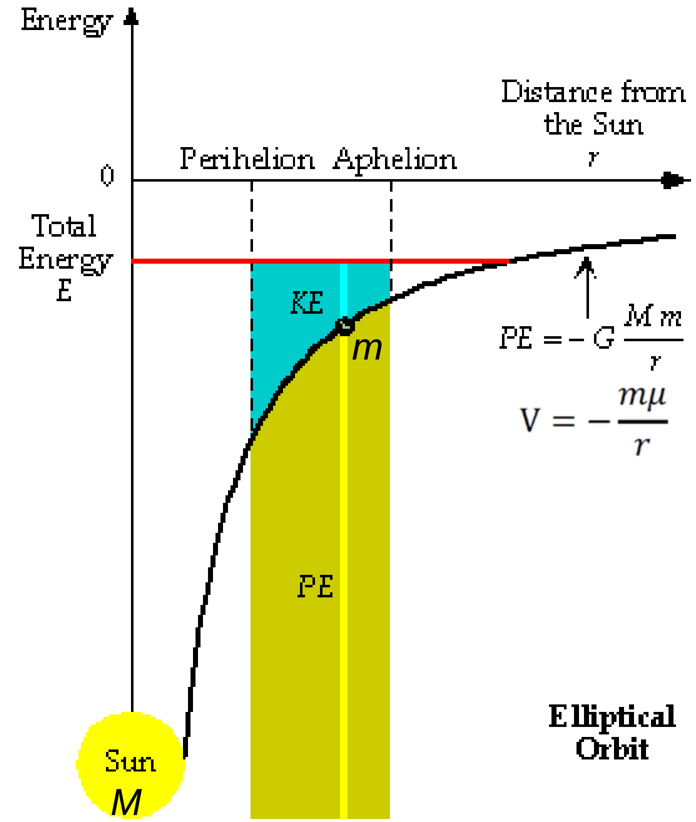
- Object moving at "escape velocity" will barely escape the central body's gravity well: *v tends to 0 as r tends to ∞*

$$\mathcal{E}(r_\infty, v_\infty) = \frac{v_\infty^2}{2} - \frac{\mu}{r_\infty} = 0$$

$$\mathcal{E}(r, v) = \frac{v^2}{2} - \frac{\mu}{r} = \mathcal{E}(r_\infty, v_\infty) = 0$$

$$v_0 = \sqrt{\frac{2\mu}{r_0}}$$

- The escape velocity is a function of r_0 and decreases as r_0 increases. Any object traveling with v_0 is on a parabolic path



Excess Hyperbolic Velocity

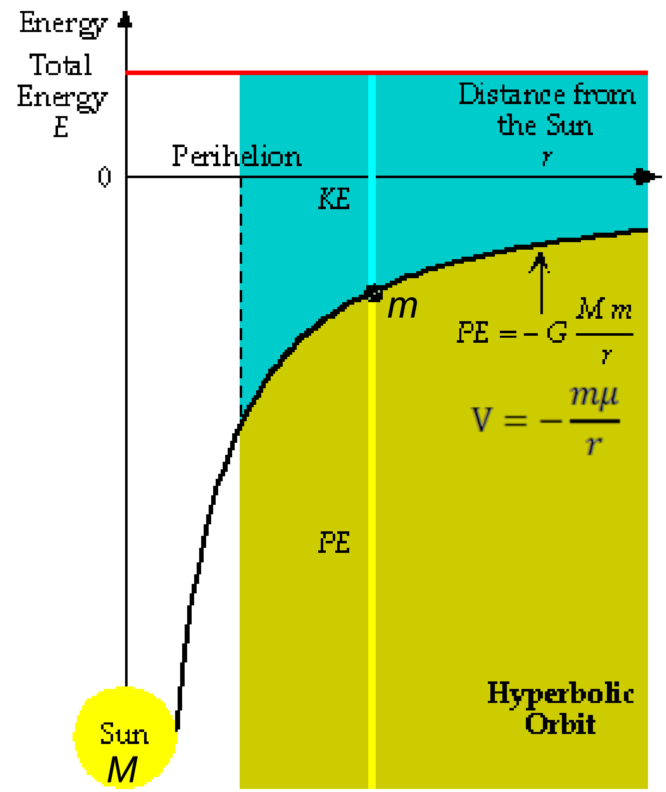
- If object at a given *r* is traveling any faster than v_0, then it is on a hyperbolic path

$$\mathcal{E}(r_\infty, v_\infty) = \frac{v_\infty^2}{2} - \frac{\mu}{r_\infty} = \frac{v_\infty^2}{2} > 0$$

$$\mathcal{E}(r, v) = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v^2}{2} - \frac{v_0^2}{2} = \mathcal{E}(r_\infty, v_\infty)$$

$$v_\infty^2 = v^2 - v_0^2$$

- It will eventually escape the central body and maintain an extra velocity v_∞ at $r = \infty$ which is called excess hyperbolic velocity



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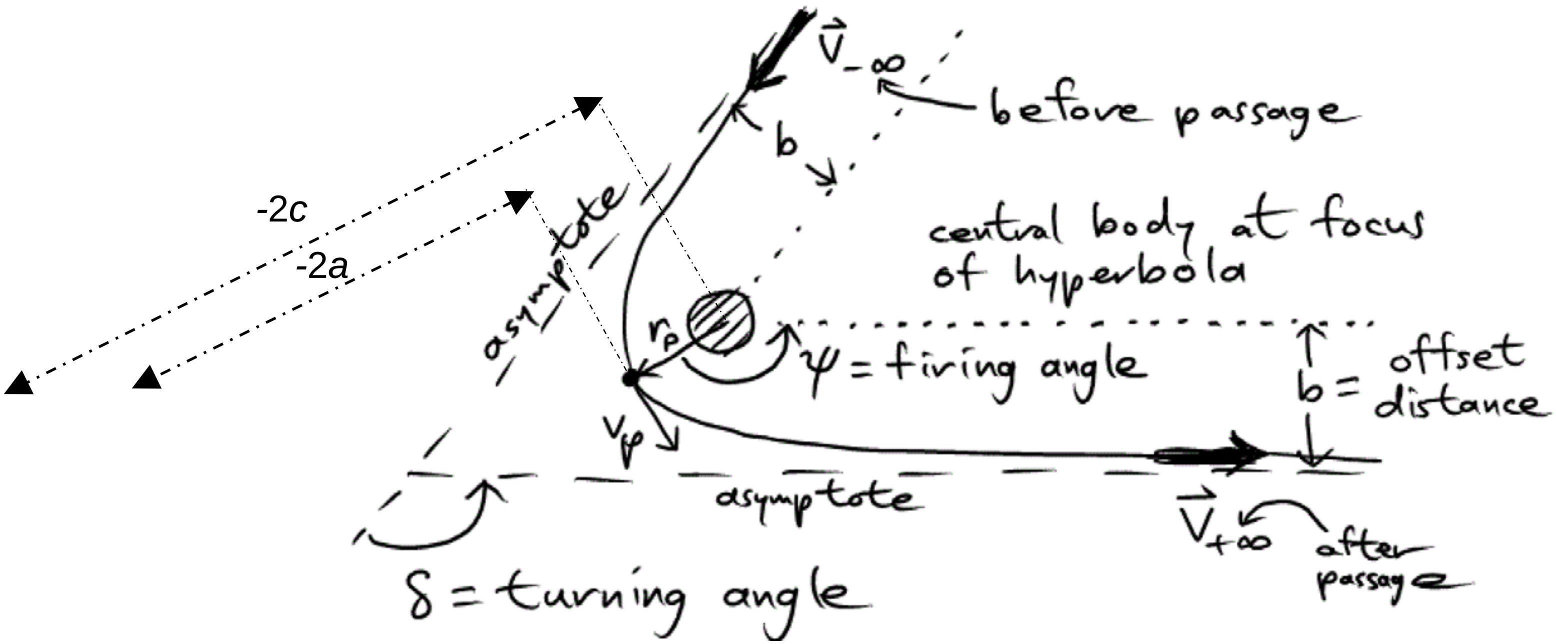
Hyperbolic Passage



$$\frac{1}{e} = \frac{a}{c} = \sin\left(\frac{\delta}{2}\right) \rightarrow \delta = 2\mathrm{asin}\left(\frac{1}{e}\right)\ ;\ \psi = 2\mathrm{acos}\left(-\frac{1}{e}\right)$$ Fire angle parameter

$$\mathcal{E} = \frac{v_\infty^2}{2} = \frac{v_p^2}{2} - \frac{\mu}{r_p} > 0$$

$$e = \sqrt{1 + \frac{2\mathcal{E}h^2}{\mu^2}} = \sqrt{1 + \frac{b^2 v_\infty^4}{\mu^2}} = 1 + \frac{r_p v_\infty^2}{\mu}$$

$$h = \text{constant} = b v_\infty = r_p v_p$$

$$b = \frac{\mu}{v_\infty^2}\sqrt{e^2 - 1}$$ Impact parameter

Some Key Derivations

- Based on the Fundamental Orbital Differential Equation (FODE)…

$$\frac{d^2}{dt^2}\vec{r} + \mu\frac{\vec{r}}{r^3} = 0$$

- We asserted without demonstration…

$$r(v) = \frac{p}{1 + e\cos v} \qquad \mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = \text{const} \qquad \mathcal{E} = -\frac{\mu}{2a}$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \qquad \vec{h} = \vec{r} \times \vec{v} = \text{const} \qquad h = \sqrt{\mu p}$$

- We also assumed Kepler's three laws and several properties of conic sections
- Today, we will derive some of these. Others are in your texts.

Derivation of Conservation Laws

- Try the scalar and vector products of ***r*** and ***v*** with FODE…

$$\vec{v}\cdot\left(\frac{d^2}{dt^2}\vec{r}+\mu\frac{\vec{r}}{r^3}\right)=0$$

[m/s] · [N/kg] = [J/s] / [kg]

$$\vec{v}\cdot\frac{d}{dt}\vec{v}\ +\ \left(\vec{r}\cdot\frac{d}{dt}\vec{r}\right)\frac{\mu}{r^3}=0$$

! $$v\dot{v}+r\dot{r}\frac{\mu}{r^3}=0$$

$$\frac{d}{dt}\left(\frac{v^2}{2}-\frac{\mu}{r}\right)=\frac{d\mathcal{E}}{dt}=0$$

Specific mechanical energy is conserved !

$$\vec{r}\times\left(\frac{d^2}{dt^2}\vec{r}+\mu\frac{\vec{r}}{r^3}\right)=0$$

[m] · [N/kg] = [Nm] / [kg]

$$\vec{r}\times\frac{d^2}{dt^2}\vec{r}\ +\ \mu\frac{\vec{r}\times\vec{r}}{r^3}=0$$

$$\frac{d}{dt}\left(\vec{r}\times\frac{d}{dt}\vec{r}\right)=\vec{v}\times\vec{v}+\vec{r}\times\frac{d^2}{dt^2}\vec{r}=0$$

$$\frac{d}{dt}(\vec{r}\times\vec{v})=\frac{d\vec{h}}{dt}=0$$

Linear angular momentum is conserved !

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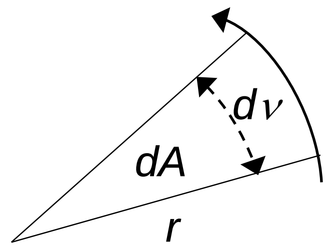
Derivation of Kepler's II and III Law

- Look at the swept area in *dt* and integrate over entire orbital period *T*



$$dA = \frac{1}{2} r (r d\nu)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\nu} \overset{!}{=} \frac{h}{2}$$

$$\frac{dA}{dt} = \text{const}$$

Kepler's II Law Verified !

$$\int_0^T dt = \int_0^{A_{Ellipse}} \frac{2}{h} dA = \frac{2}{h} \int_0^{A_{Ellipse}} dA = \frac{2}{h} A_{Ellipse}$$

$$A_{Ellipse} = \pi ab = \pi a \left(a\sqrt{1 - e^2} \right) = \pi a \left(\sqrt{aa(1 - e^2)} \right)$$

$$A_{Ellipse} = \pi a \sqrt{ap} = \pi a \sqrt{\frac{ah^2}{\mu}} = \pi \sqrt{\frac{a^3 h^2}{\mu}}$$

$$T = 2\pi \sqrt{\frac{a^3}{\mu}}$$

Kepler's III Law Verified !

$$\boxed{n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}}$$

Mean motion
[rad/s]

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Derivation of Trajectory Equation

- Try the vector product of ***h*** with FODE…and the scalar product with ***r*** after integration…

$$\vec{h} \times \left(\frac{d^2}{dt^2}\vec{r} + \mu \frac{\vec{r}}{r^3} \right) = 0$$

$$\frac{d}{dt}(\vec{h} \times \vec{v}) + \frac{\mu}{r^3}[\vec{v}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{r}\cdot\vec{v})] = 0$$

!

$$\frac{d}{dt}(\vec{h} \times \vec{v}) + \frac{\mu}{r}\vec{v} - \frac{\mu\dot{r}}{r^2}\vec{r} = 0$$

$$\frac{d}{dt}(\vec{h} \times \dot{\vec{r}}) + \mu\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = 0$$

!

$$\vec{h} \times \dot{\vec{r}} + \mu\frac{\vec{r}}{r} = \vec{B}$$

Integration constant

$$\vec{r}\cdot\vec{h} \times \dot{\vec{r}} + \mu\frac{\vec{r}\cdot\vec{r}}{r} = \vec{r}\cdot\vec{B}$$

Angle between ***B*** and ***r***

$$\vec{h}\cdot\vec{r} \times \dot{\vec{r}} + \mu r = rB\cos v$$

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos v} = \frac{p}{1 + e\cos v}$$

Orbital trajectories are conic sections !

Kepler's I Law Verified !

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Momentum, Energy and Orbit's Shape

- Compare the solution of FODE with the polar equation of a conic section…

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos\nu} = \frac{p}{1 + e\cos\nu}$$

$$h = \sqrt{\mu p}$$

Linear angular momentum is completely determined by p !

$$h = rv\cos\phi_{\text{fpa}} = r_p v_p$$

The flight path angle is zero at periapsis (and apoapsis)

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$$

$$\mathcal{E} = \frac{v_p^2}{2} - \frac{\mu}{r_p} = \frac{h^2}{2r_p^2} - \frac{\mu}{r_p}$$

From conic section

$$r_p = a(1-e)\,;\; h^2 = \mu p = \mu a(1-e^2)$$

$$\mathcal{E} = -\frac{\mu}{2a}$$

Specific mechanical energy is completely determined by a !

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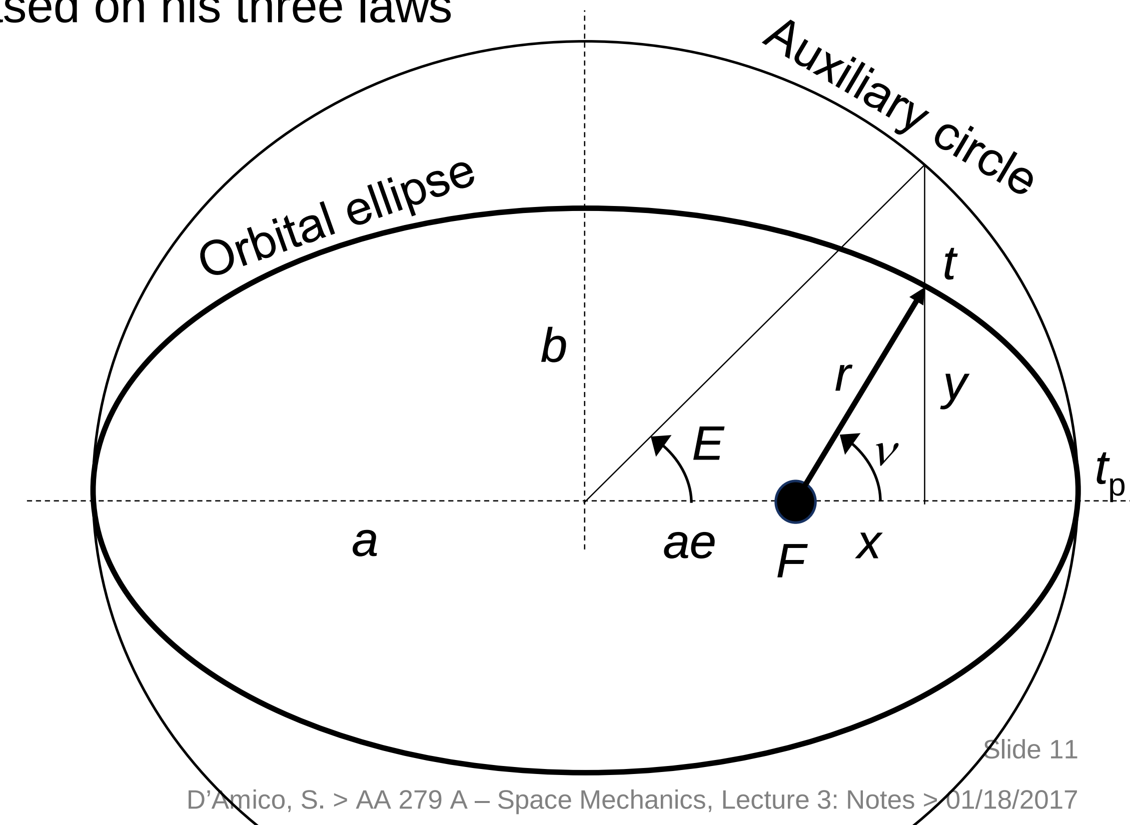
Time Dependence of Motion (1)

- From the law of gravity we have concluded that the orbital motion has the shape of a conic section $r(\nu)$, but no information on the time dependence of the motion $\nu(t)$ has yet been derived

- In order to find the orbital position at a specific time, Kepler has developed a geometrical construction based on his three laws

  - Ellipse geometry
  - $$\frac{dA}{dt} = \text{const}$$
  - $$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$



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Time Dependence of Motion (2)

$$\frac{y_{Ellipse}}{y_{Circle}} = \frac{b}{a} = \sqrt{1-e^2}$$

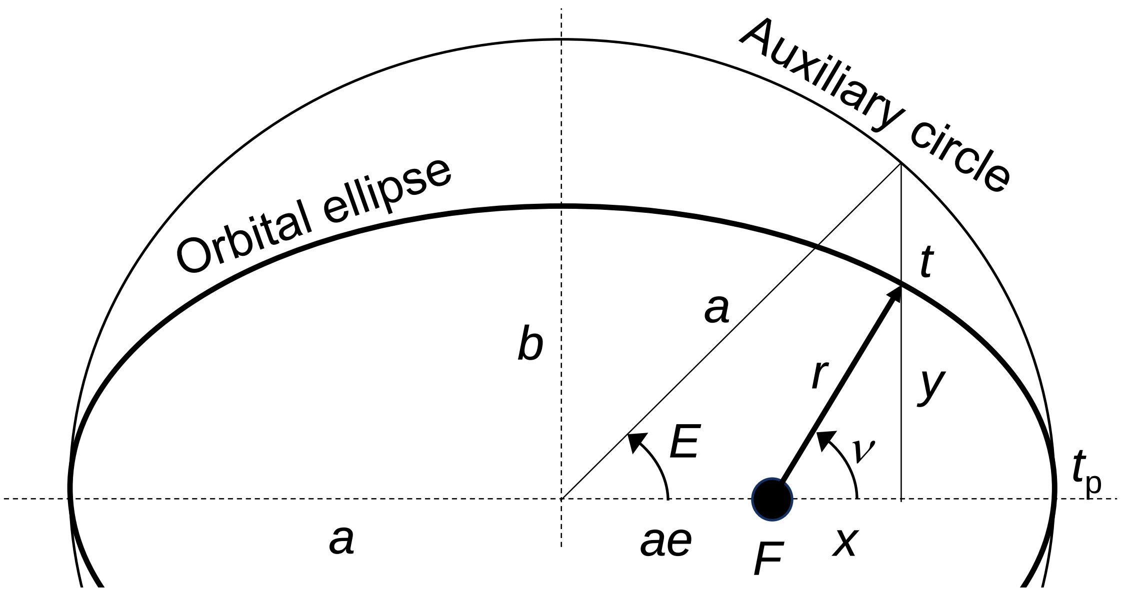
Scaling factor (only for ellipses)

$$x = r\cos\nu = a(\cos E - e)$$

$$y = r\sin\nu = a\sqrt{1-e^2}\sin E$$

$$r = a(1 - e\cos E)$$

Definition of Eccentric Anomaly



$$h = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)}$$

$$h = |\vec{r} \times \vec{v}| = x\dot{y} - y\dot{x} = a(\cos E - e)a\sqrt{1-e^2}\cos E\dot{E} + a\sqrt{1-e^2}\sin E a \sin E\dot{E} =$$

$$= a^2\sqrt{1-e^2}\dot{E}(1 - e\cos E)$$

Angular momentum
or areal velocity

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Time Dependence of Motion (3)

- We can compare the equivalent expressions for the angular momentum *h* and introduce the mean motion *n* to simplify notation

$$(1 - e\cos E)\dot{E} = n$$

- Integrating with respect to time e.g. from t_p to *t* yields the *Kepler's Equation*

$$E(t) - e\sin E(t) = n(t - t_p)$$

- Kepler defined the *mean anomaly, M,* which increases linearly with *t* and proportionally to *n* so to simplify notation

$$M = n(t - t_p)$$

$$M = M_0 + n(t - t_0)$$

Mean Anomaly

$$\boxed{E - e\sin E = M}$$

Kepler's Equation [rad]

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Time Dependence of Motion (4)

$M(t)$ = known by its definition

+

$E(t)$ = known through Kepler's Equation

+

$\nu(E)$ = known by definition of E

t = input

Non linear equation which can only be solved iteratively

$\nu(t) = \nu(E(M(t)))$

+

$r(\nu)$ = known through Trajectory Equation

$r(t)$ = ?

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Backup