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AA 279 A – Space Mechanics
Lecture 5: Notes

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- Orbital elements
- Types of orbits
- Orbital elements to/from position and velocity
- Basic coordinate systems

- Reading for next week (lectures 7 and 8)
  - Bate n/a
  - Montenbruck 6.2.2, 9.1.1
  - Vallado 6.8

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Special Cases

- Equatorial non-circular orbit

  Longitude of periapsis

  $$\boxed{\Pi = \Omega + \omega}$$

  Since the line of nodes is undefined we can't locate periapsis w.r.t. ascending node. Instead locate periapsis w.r.t. vernal equinox (always defined)

- Circular non-equatorial orbit

  Argument of latitude

  $$\boxed{u = \omega + \nu}$$

  Since the line of apsis is undefined we can't locate satellite w.r.t. periapsis. Instead locate satellite w.r.t. ascending node (defined for inclined orbits)

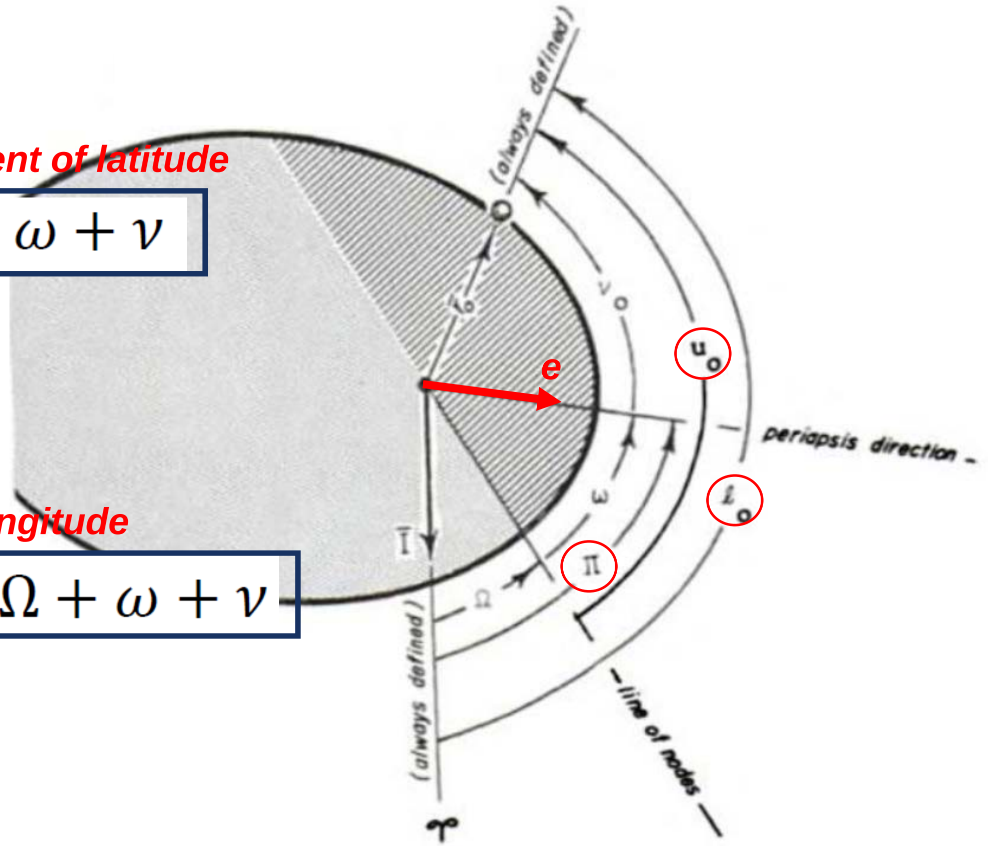
- Equatorial and circular orbit

  True longitude

  $$\boxed{l = \Omega + \omega + \nu}$$

  Since the lines of apsis and nodes are undefined we locate satellite w.r.t. vernal equinox (always defined)

Often *M* is used instead of ν in these definitions, thus *u* and *l* become "mean"



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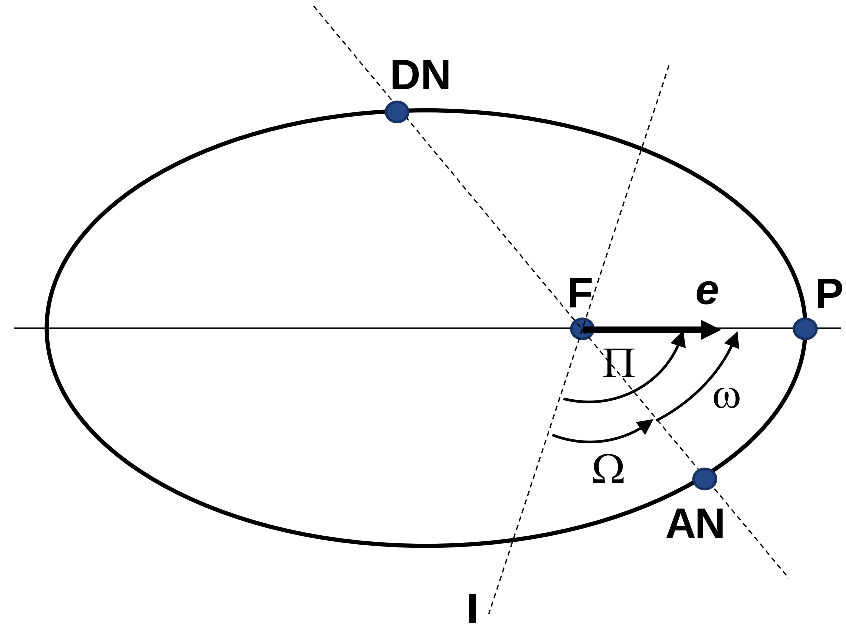
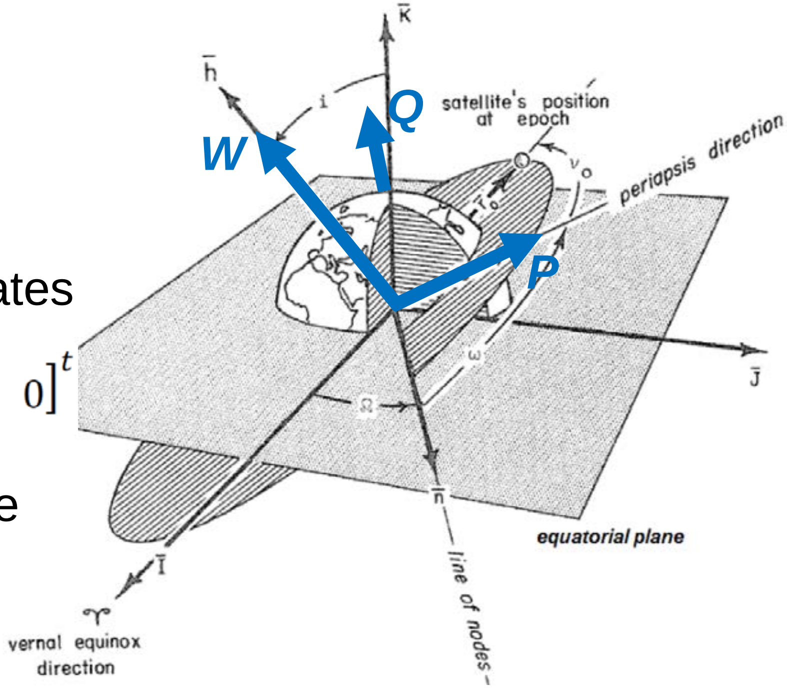
Eccentricity Vector

- Vector with magnitude *e* and direction pointing from focus towards periapsis
- Frequently used to handle orbits of small eccentricity (nearly circular) and perturbations to the 2-body problem
- It is null, (0,0), for "perfectly" circular orbits instead of (*e* = 0, ω = undefined)
- Mathematically it can be expressed in many different ways depending on coordinate basis or state representation



$$
\begin{cases}
\vec{e} = e\hat{\vec{P}} \\
\vec{e} = (e\cos\omega \quad e\sin\omega)^t \\
\vec{e} = (e\cos\Pi \quad e\sin\Pi)^t \\
\vec{e} = \dfrac{\vec{B}}{\mu} = \dfrac{\vec{v} \times \vec{h}}{\mu} - \dfrac{\vec{r}}{r}
\end{cases}
$$

From PQW to IJK Coordinates
(Perifocal to Earth-Centered Inertial)



- Recall satellite position in perifocal coordinates

$$\vec{r}_{\mathrm{PQW}} = (r\cos\nu \quad r\sin\nu \quad 0)^t = \left[a(\cos E - e) \quad a\sqrt{1-e^2}\sin E \quad 0\right]^t$$

- The satellite position in inertial space can be expressed through a sequence of three elementary rotations -ω, -i, -Ω as

$$\vec{r}_{\mathrm{IJK}} = \vec{R}_{\mathrm{PQW}\rightarrow\mathrm{IJK}}\vec{r}_{\mathrm{PQW}} = \vec{R}_{\mathrm{z}}(-\Omega)\vec{R}_{\mathrm{x}}(-i)\vec{R}_{\mathrm{z}}(-\omega)\vec{r}_{\mathrm{PQW}} \qquad \vec{R}_{\mathrm{x}}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}; \; \vec{R}_{\mathrm{z}}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Evaluating this expression (using u) we find $\boldsymbol{r}_{\mathrm{IJK}}$ and vectors ***P***, ***Q***, ***W***

$$\vec{r}_{\mathrm{IJK}} = r\begin{pmatrix} \cos u \cos\Omega - \sin u \cos i \sin\Omega \\ \cos u \sin\Omega + \sin u \cos i \cos\Omega \\ \sin u \sin i \end{pmatrix}$$

Notes:

1. **The unit vectors *P*, *Q*, *W* are the columns of the rotation matrix**

$$\vec{R}_{\mathrm{PQW}\rightarrow\mathrm{IJK}} = (\vec{P} \quad \vec{Q} \quad \vec{W})$$

2. **The rotation matrix is orthonormal**

$$\vec{R}^{-1}_{\mathrm{PQW}\rightarrow\mathrm{IJK}} = \vec{R}^{t}_{\mathrm{PQW}\rightarrow\mathrm{IJK}}$$

Position and Velocity from Orbital Elements

- The transformation from perifocal to ECI coordinates gives an algorithm to compute position

$$\vec{r}_{\mathrm{PQW}} = (r\cos\nu \quad r\sin\nu \quad 0)^t = \left[a(cosE - e) \quad a\sqrt{1-e^2}\sin E \quad 0\right]^t$$ **Scaling and shape**

$$\vec{r}_{\mathrm{IJK}} = \vec{R}_{\mathrm{PQW}\rightarrow\mathrm{IJK}}\vec{r}_{\mathrm{PQW}}$$ **Orientation**

- and velocity from orbital elements

$$\vec{v}_{\mathrm{PQW}} = \dot{\vec{r}}_{\mathrm{PQW}} = \left[-a\dot{E}\sin E \quad a\dot{E}\sqrt{1-e^2}\cos E \quad 0\right]^t =$$

$$= \frac{an}{(1-e\cos E)}\left[-\sin E \quad \sqrt{1-e^2}\cos E \quad 0\right]^t$$ **Scaling and shape**

$$\vec{v}_{\mathrm{IJK}} = \vec{R}_{\mathrm{PQW}\rightarrow\mathrm{IJK}}\vec{v}_{\mathrm{PQW}}$$ **Orientation**

- here used has been made of the Kepler's Equation $(1 - e\cos E)\dot{E} = n$

Orbital Elements from Position and Velocity (1)

- Equivalently there is exactly one set of orbital elements that corresponds to given values of position and velocity at time t_0

1) Find specific angular momentum

$$\vec{h} = \vec{r} \text{ x } \vec{v} = \begin{pmatrix} r_j v_k - r_k v_j \\ r_k v_i - r_i v_k \\ r_i v_j - r_j v_i \end{pmatrix}$$

2) From previous discussion, express ***W*** = ***h***/*h*

$$\begin{pmatrix} \sin i \sin\Omega \\ \sin i \cos\Omega \\ \cos i \end{pmatrix} = \begin{pmatrix} +h_i/h \\ -h_j/h \\ +h_k/h \end{pmatrix} = \begin{pmatrix} +W_i \\ -W_j \\ +W_k \end{pmatrix}$$

3) Calculate *i* and *Ω*

$$i = \arctan\left(\frac{\sqrt{W_i^2 + W_j^2}}{W_k}\right)$$

$$\Omega = \arctan\left(\frac{W_i}{-W_j}\right)$$

Note: for $\alpha = \arctan\left(\frac{y}{x}\right)$ make sure that -90°<α<+90° for *x*>0 and +90°<α<+270° for x<0

Orbital Elements from Position and Velocity (2)

4) Compute semi-latus rectum from *h*

$$p = \frac{h^2}{\mu}$$

5) Compute *a* and *n* from vis-viva law

$$a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1} \qquad n = \sqrt{\frac{\mu}{a^3}}$$

6) Compute eccentricity from *a* and *p*

$$e = \sqrt{1 - \frac{p}{a}}$$

7) Consider the following identity

$$\vec{r} \cdot \vec{v} = -a(\cos E - e) \cdot a \sin E \dot{E} + \\ + a\sqrt{1-e^2}\sin E \cdot a\sqrt{1-e^2}\cos E \dot{E} = \\ = \quad a^2 n e \sin E$$

8) Solve for *e*sin*E* and *e*cos*E*

$$E = \arctan\left[\frac{\vec{r} \cdot \vec{v}/(a^2 n)}{1 - r/a}\right]$$

9) Compute *M* and ν from *E* using Kepler's Equation

10) Compute *u* from rotation formula PQW to IJK

$$u = \arctan\left(\frac{r_k/\sin i}{r_i \cos\Omega + r_j \sin\Omega}\right)$$

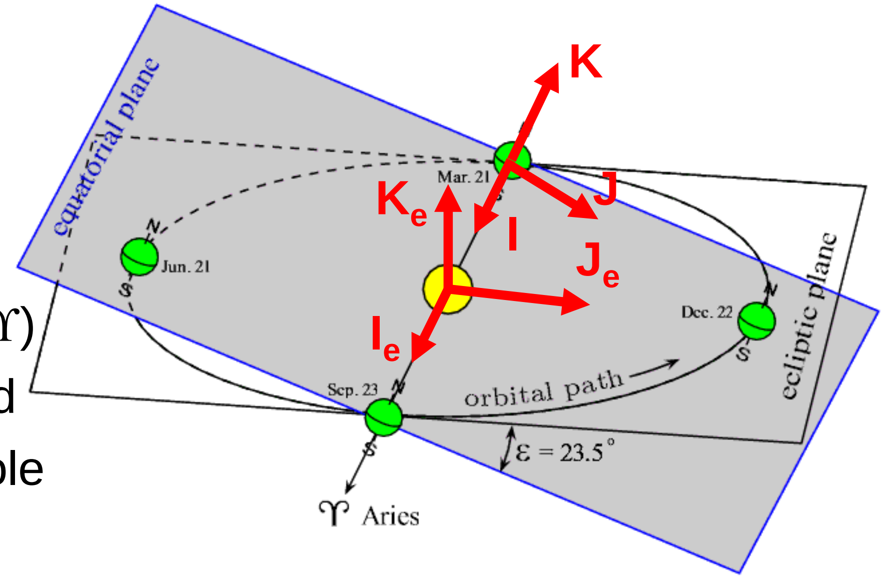
11) Compute argument of perigee

$$\omega = u - \nu$$

Note: $e = 0$, $i = 0°$ or $180°$ are to be handled separately

Heliocentric Coordinate System

- Common coordinate system for interplanetary missions or planets
- This is a Sun-Centered Inertial (SCI) reference system with axis
  - $\boldsymbol{I_e}$ aligned with the *vernal equinox* (Υ)
  - $\boldsymbol{J_e}$ completing the right-handed triad
  - $\boldsymbol{K_e}$ pointing to the ecliptic's north pole
- ***Ideally*** ECI and SCI differ by a rotation angle ε about the shared first axis ($\boldsymbol{I_e} = \boldsymbol{I}$)
- $\varepsilon \approx 23.5°$ is the angle between the Earth's orbital plane (ecliptic) and the Earth's equatorial plane, called *obliquity*



Function of time

$$\vec{r}_{\mathrm{IeJeKe}} = \vec{R}_{\mathrm{X}}(\varepsilon)\left(\vec{r}_{\mathrm{IJK}}^{\,\mathrm{Satellite}} - \vec{r}_{\mathrm{IJK}}^{\,\mathrm{Sun}}\right)$$

$$\vec{R}_{\mathrm{X}}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

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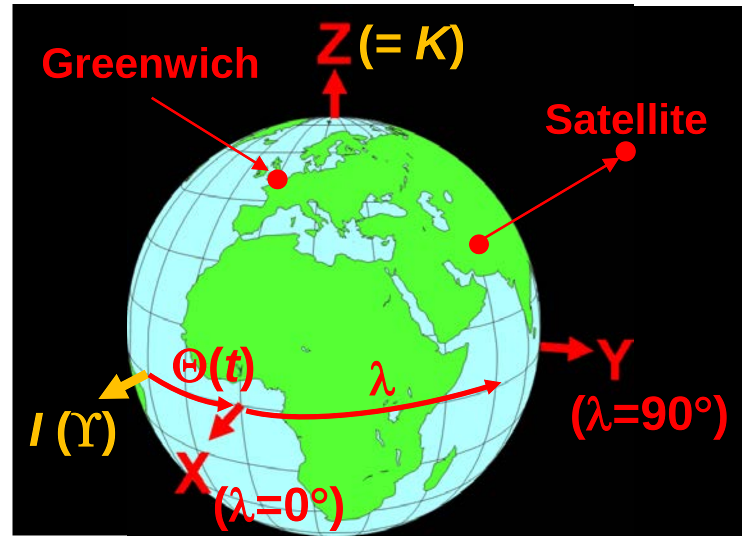
Earth-Centered Earth-Fixed Coordinate System

- The most common coordinate system for satellite measurements and navigation
- This is an Earth-Centered Earth-Fixed (ECEF) reference system with axis
  - ***X*** aligned with Greenwich meridian, i.e. zero geocentric longitude (λ=0°)
  - ***Y*** completing the right-handed triad, i.e. λ=90° East
  - ***Z*** pointing to the north pole
- ***Ideally*** ECI and ECEF differ by a rotation angle Θ about the shared third axis (***Z***=***K***)
- Θ measures the time between meridian crossings of a star for an observer on Earth, hence it is called *Greenwich sidereal time*

Ideal configuration ECI/ECEF



Function of time

$$\vec{r}_{\mathrm{XYZ}} = \vec{R}_{\mathrm{IJK}\to\mathrm{XYZ}}\vec{r}_{\mathrm{IJK}} = \vec{R}_{\mathrm{z}}(\Theta)\vec{r}_{\mathrm{IJK}}$$

$$\vec{R}_{\mathrm{z}}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Backup

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