AA 279 C – SPACECRAFT ADCS: LECTURE 10

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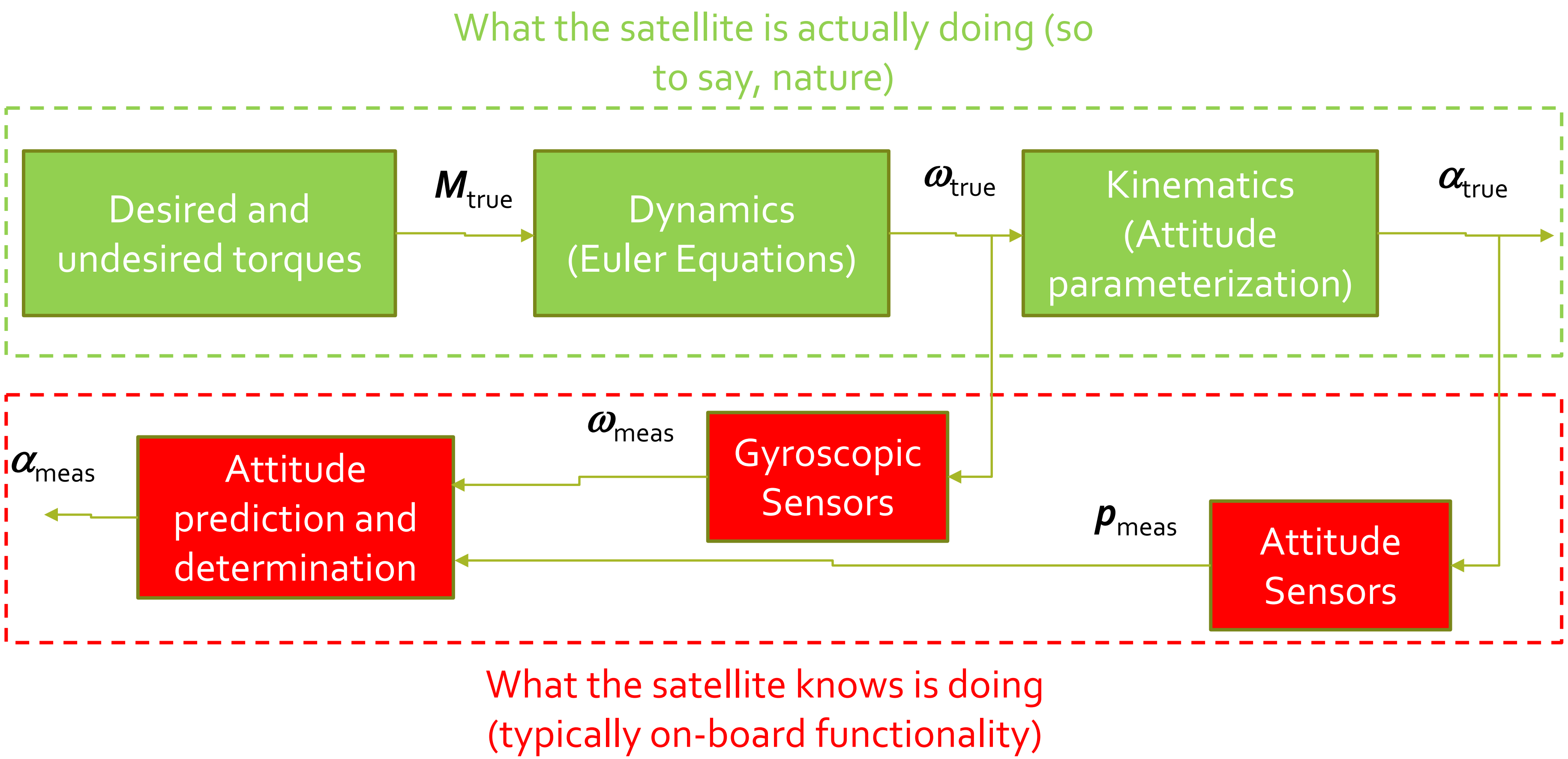
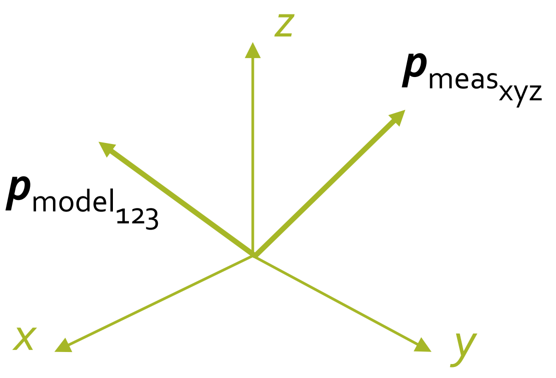


Attitude Determination Problem

- The goal is to determine the orientation of the principal frame (or body frame) relative to the inertial frame from
  1. Measured angular velocities (from satellite motion)
  2. Measured directions (from objects or fields in space)

1. Angular velocities can be used to compute attitude parameters through the integration of the kinematic equations
   - Error on initial conditions increases with time

2. The body frame can be rotated in such a way that the directions (unit vectors) measured in body frame (*xyz*) coincide with the known directions in inertial frame (*123*)
   - Since a rotation of *xyz* about one axis keeps the two unit vectors aligned, two measured directions are necessary for 3-axis attitude determination



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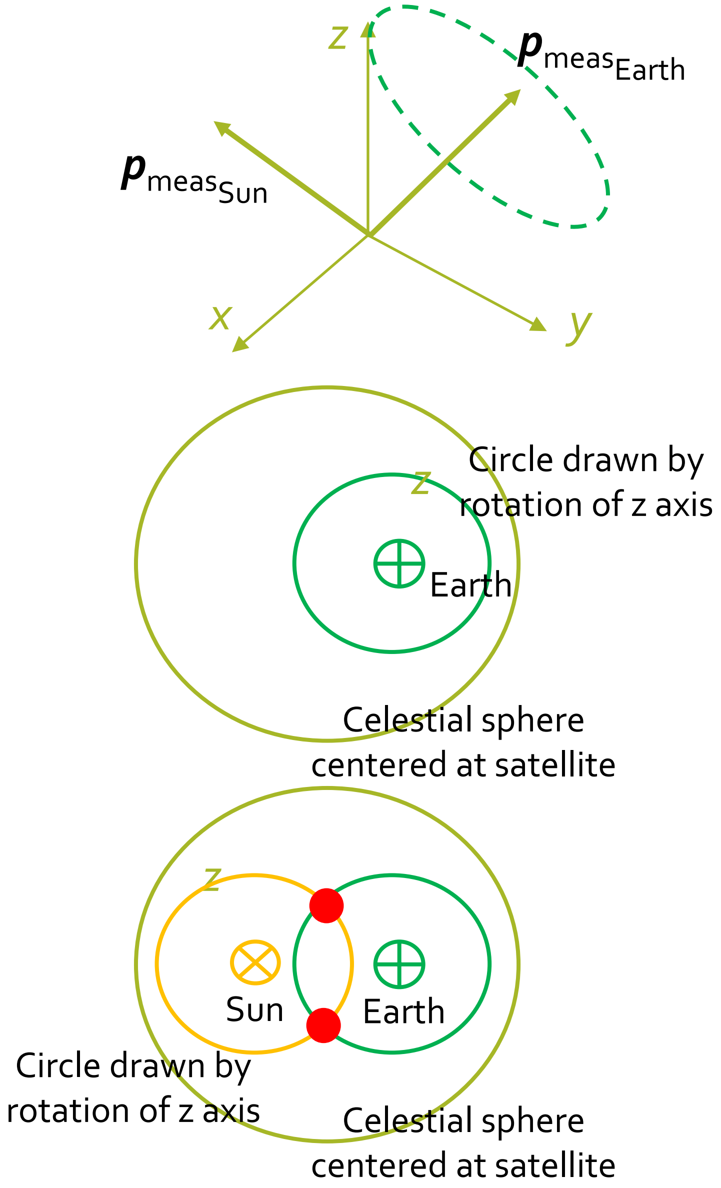
Geometric Attitude Determination

- One strategy is to use the minimum amount of information, i.e., from two measured directions
- If one unit vector is aligned with the inertial counterpart, it is possible to rotate the body frame about that unit vector to find all compatible configurations
- If the two celestial bodies are not aligned, the intersection between the circles provides the actual orientation of the body axis in inertial frame
- Only one of the two mathematical solutions will be the true one and is clearly distinguished from the fictitious one which changes quickly over time
- Sensor errors produce an annulus or rings rather than circles, and minimum errors are obtained if the intersection is nearly perpendicular rather than tangent



Deterministic Attitude Determination (1)

- More sensors can provide a set of simultaneous measurements of direction to objects in principal axes (*xyz*)
- Correspondingly we need directions to the same objects in the inertial frame from a model (*123*). Note that typically this implies some orbit and time knowledge.
- In the case of one measurement (one unit vector)

$$\vec{p}_{meas_{xyz}} = \vec{A}\vec{p}_{model_{123}} \Leftrightarrow \vec{m}_i = \vec{A}\vec{v}_i$$

Change of notation

- In the case of three independent measurements (three unit vectors)

$$\vec{M} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{m}_1 & \vec{m}_2 & \vec{m}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \vec{A}\begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \vec{A}\vec{V}$$

- The matrices ***V*** and ***M*** are not singular if the unit vectors are not aligned, thus the direction cosine matrix is given by

$$\vec{A} = \vec{M}\vec{V}^{-1} \text{ or } \vec{A} = \left(\vec{V}\vec{M}^{-1}\right)^t$$

Unnecessary redundancy

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Deterministic Attitude Determination (2)

- More efficiently, we can use only two unit vectors and build the corresponding triads

$$\vec{p}_m = \vec{m}_1 \,;\, \vec{q}_m = \frac{\vec{m}_1 \times \vec{m}_2}{\|\vec{m}_1 \times \vec{m}_2\|} \,;\, \vec{r}_m = \vec{p}_m \times \vec{q}_m$$

From measurements

Reference axis

$$\vec{p}_v = \vec{v}_1 \,;\, \vec{q}_v = \frac{\vec{v}_1 \times \vec{v}_2}{\|\vec{v}_1 \times \vec{v}_2\|} \,;\, \vec{r}_v = \vec{p}_v \times \vec{q}_v$$

From model

- Knowing the inertial correspondence based on the measurements

$$\vec{M} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{p}_m & \vec{q}_m & \vec{r}_m \\ \vdots & \vdots & \vdots \end{bmatrix} = \vec{A} \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{p}_v & \vec{q}_v & \vec{r}_v \\ \vdots & \vdots & \vdots \end{bmatrix} = \vec{A}\vec{V} \Rightarrow \vec{A} = \vec{M}\vec{V}^{-1}$$

Pseudoinverse is used for more than 3 measurements

- Errors are minimized if the two unit vectors are as perpendicular as possible
- Errors can be reduced by computing $\boldsymbol{p}_m$ from the most accurate among $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, or even by using fictitious measurements which distribute errors

$$\vec{\tilde{m}}_1 = \frac{\vec{m}_1 + \vec{m}_2}{2} \,;\, \vec{\tilde{m}}_2 = \frac{\vec{m}_1 - \vec{m}_2}{2}$$

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Statistical Attitude Determination (1)

- In contrast to algebraic methods, it is possible to devise attitude estimation methods which are optimal in a statistical sense
- The most popular statistical attitude determination method is the *q*-method which allows weighting the accuracy of the difference sensors involved
- Due to measurement errors, $\vec{m}_i - \vec{A}\vec{v}_i \neq 0$ in general. Thus we seek an optimal attitude matrix which minimizes the loss function

$$J(\vec{A}) = \sum_{i=1}^{n} w_i \left|\vec{m}_i - \vec{A}\vec{v}_i\right|^2$$

Number of measurements

Error of *i*th measurement

Weight of *i*th measurement

- While minimizing *J*(***A***), we want to give more importance to the more accurate measurements, thus w_i will be larger for a star tracker than a magnetometer

$$J(\vec{A}) = -2\sum_{i=1}^{n} w_i\, \vec{m}_i^t \vec{A}\vec{v}_i + \text{const} = -2\sum_{i=1}^{n} \vec{w}_i \vec{A}\vec{u}_i + \text{const} = -2\,\text{tr}\left(\vec{W}^t \vec{A}\vec{U}\right) + \text{const}$$

$$\vec{w}_i = \sqrt{w_i}\vec{m}_i \,;\, \vec{u}_i = \sqrt{w_i}\vec{v}_i \,;\, \vec{W} = [\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_n] \,;\, \vec{U} = [\vec{u}_1 \quad \vec{u}_2 \quad \cdots \quad \vec{u}_n]$$

Statistical Attitude Determination (2)

Parameterize ***A***
in terms of
quaternion ***q***

- Minimizing *J*(***A***) is equivalent to maximizing

$$J'(\vec{A}) = \mathrm{tr}\left(\vec{W}^t \vec{A}\vec{U}\right) = \mathrm{tr}\left(\vec{W}^t \vec{A}(\vec{q})\vec{U}\right) = \cdots = \vec{q}^t \vec{K}\vec{q}$$

Lots of algebra

$$\vec{K} = \begin{bmatrix} \vec{S} - \vec{I}\sigma & \vec{Z} \\ \vec{Z}^t & \sigma \end{bmatrix}; \vec{B} = \vec{W}\vec{V}^t;\ \vec{S} = \vec{B} + \vec{B}^t;\ \vec{Z} = (B_{23} - B_{32} \quad B_{31} - B_{13} \quad B_{12} - B_{21})^t; \sigma = tr(\vec{B})$$

- The extrema of *J'* subject to unit quaternion constraint can be found through the method of Lagrange multipliers as the unconstrained extrema of

$$J''(\vec{q}) = \vec{q}^t \vec{K}\vec{q} - \lambda\vec{q}^t\vec{q}$$

- Differentiation of *J''* w.r.t. ***q*** ^{t} provides $\vec{K}\vec{q} = \lambda\vec{q}$, thus the optimum quaternion is an eigenvector of ***K***. Substituting back in *J'* provides $J'(\vec{q}) = \lambda$
- Thus, *J* is minimum if ***q*** is the eigenvector corresponding to the maximum eigenvalue of *J'*
- The optimum attitude determination problem is reduced to an eigenvector eigenvalue problem
- The main disadvantages are the construction of weighted vector measurements, and the assumption of well known sensor properties

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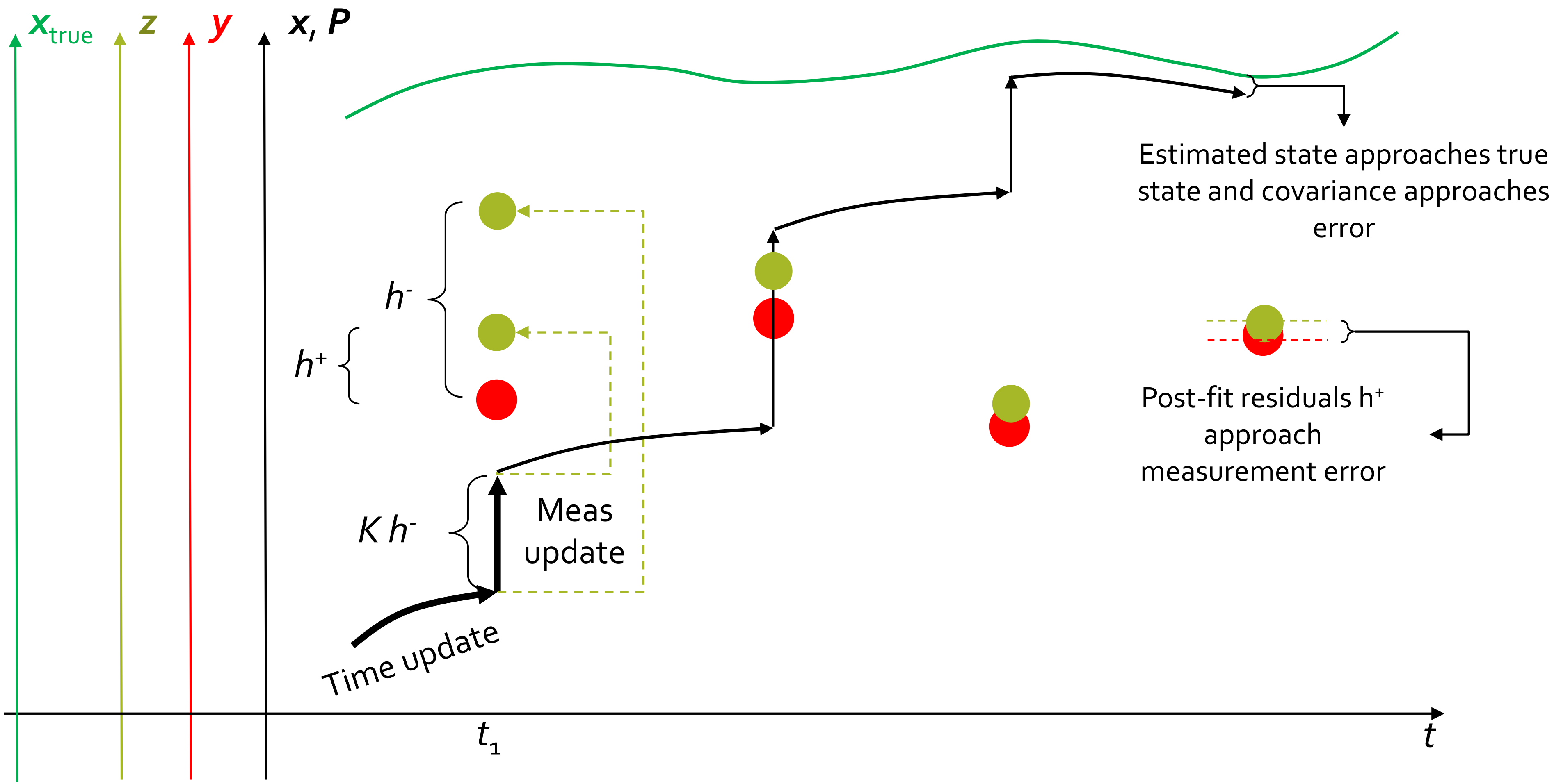
Deterministic vs state attitude estimation

- A method is required to deal with multiple solved-for parameters to obtain accurate estimates
- State estimation methods use the partial derivatives of the observables with respect to various solved-for parameters (state) to correct an a-priori estimate
- Two basic methods exist to do so
  - *Sequential estimation*: a new estimate of the state vector is obtained after each observation
  - *Batch estimation*: all observation are processed and combined to obtain a single update state vector
- The methods can be combined and show advantages and disadvantages. In general sequential estimation is more sensitive to individual data points and converges more quickly at the cost of stability
- Deterministic methods always provide a solution, require a very rough estimate (if at all), are easy to interpret, but are not able to account for uncertainties and include more state parameters

Ingredients for state attitude estimation

- State vector, ***x***
  - m-dimensional, all variables necessary for accurate attitude determination, e.g. sensor biases, misalignments, attitude parameters
  - State parameters may be constant during the processing interval or time-varying: $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t)$
- Observation vector, ***y***
  - n-dimensional, all sensor measurements, e.g. direct sensor readouts such as event times or processed observations
- Modeled observation vector, ***z***
  - n-dimensional, predicted values of the observation vector, ***y***, based on estimated values of the state, ***x***
  - The observation model is typically based on the hardware model of the sensor which is providing the measurements: $\vec{z} = \vec{g}(\vec{x}, t)$
- The way ***x***, ***y***, and ***z*** are used to obtain an estimate is similar across methods

Kalman Filter (Procedure)



Extended Kalman Filter (Updates)

- Time Update (first step, always possible)
  - Propagate state vector and error covariance forward in time

State transition matrix

Control input torque

$$\vec{x}_{k-} = \vec{\Phi}_{k-1}\vec{x}_{k-1} + \vec{u}_{k-1}$$

State error covariance matrix

Process noise matrix

$$\vec{P}_{k-} = \vec{\Phi}_{k-1}\vec{P}_{k-1}\vec{\Phi}^{t}_{k-1} + \vec{Q}_{k-1}$$

- Measurement Update (second step, measurement must be available)
  - Compute gain matrix from uncertainty and partial derivatives

Sensitivity matrix

$$\vec{K}_k = \vec{P}_{k-}\vec{H}^{t}_{k}\left(\vec{H}_k\ \vec{P}_{k-}\vec{H}^{t}_{k} + \vec{R}_k\right)^{-1}$$

Measurement error covariance

  - Update state and covariance estimates

$$\vec{x}_k = \vec{x}_{k-} + \vec{K}_k(\vec{y}_k - \vec{z}_k)$$

Joseph formulation

$$\vec{P}_k = \left(\vec{I} - \vec{K}_k\vec{H}_k\right)\vec{P}_{k-}\left(\vec{I} - \vec{K}_k\vec{H}_k\right)^{t} + \vec{K}_k\ \vec{R}_k\vec{K}^{t}_{k}$$

- Initialization of filter done through diagonal matrices for uncorrelated errors
  - ***P*** [*mxm*]: uncertainty of state parameters, $P_{ii} = (\sigma_{ii})^2$
  - ***Q*** [*mxm*]: uncertainty of dynamics model
  - ***R*** [*nxn*]: uncertainty of measurements

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