AA 279 C – SPACECRAFT ADCS: LECTURE 3

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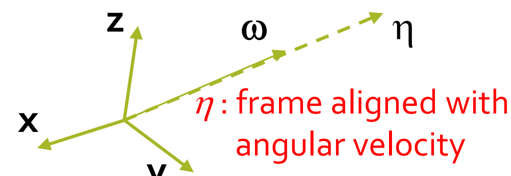
  - Wertz 16.3, 12.1

Ellipsoids of Rotational Motion

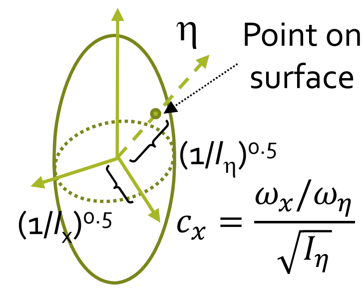


- Inertia ellipsoid (property of rigid body)

$$2T = \omega_y^2 I_y + \omega_z^2 I_z + \omega_x^2 I_x = \omega_\eta^2 I_\eta \Rightarrow$$

$$\frac{(\omega_y/\omega_\eta)^2}{I_\eta/I_y} + \frac{(\omega_z/\omega_\eta)^2}{I_\eta/I_z} + \frac{(\omega_x/\omega_\eta)^2}{I_\eta/I_x} = \frac{c_y^2}{b^2} + \frac{c_z^2}{c^2} + \frac{c_x^2}{a^2} = 1$$

Given ***I***, ellipsoid prescribes I_η for each η

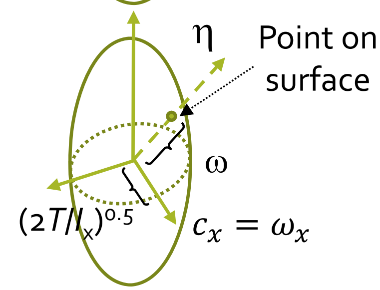


- Energy ellipsoid (depends on initial conditions)

$$2T = \omega_y^2 I_y + \omega_z^2 I_z + \omega_x^2 I_x \Rightarrow$$

$$\frac{(\omega_y)^2}{2T/I_y} + \frac{(\omega_z)^2}{2T/I_z} + \frac{(\omega_x)^2}{2T/I_x} = \frac{c_y^2}{b^2} + \frac{c_z^2}{c^2} + \frac{c_x^2}{a^2} = 1$$

Given *T*, ellipsoid prescribes ω for each η

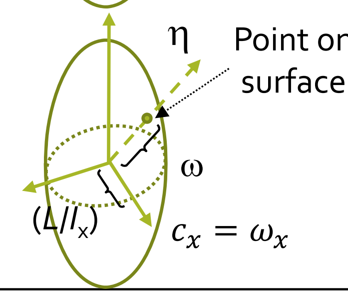


- Momentum ellipsoid (depends on initial conditions)

$$L^2 = \omega_y^2 I_y^2 + \omega_z^2 I_z^2 + \omega_x^2 I_x^2 \Rightarrow$$

$$\frac{(\omega_y)^2}{L^2/I_y^2} + \frac{(\omega_z)^2}{L^2/I_z^2} + \frac{(\omega_x)^2}{L^2/I_x^2} = \frac{c_y^2}{b^2} + \frac{c_z^2}{c^2} + \frac{c_x^2}{a^2} = 1$$

Given *L*, ellipsoid prescribes ω for each η



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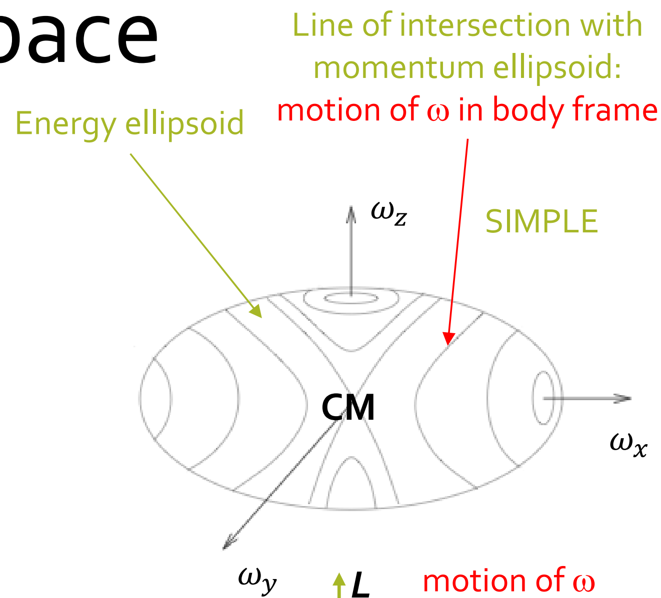
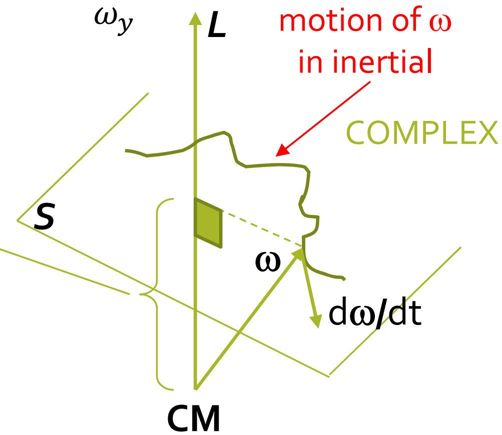
Ellipsoids in the Same Space

- Ellipsoids in principal axes
  - Energy ellipsoid gives all angular velocities compatible with the energetic content
  - Momentum ellipsoid gives all angular velocities compatible with the magnitude of the angular momentum
  - Motion is only possible along two close lines which are the intersection between the ellipsoids
- Ellipsoids in inertial space

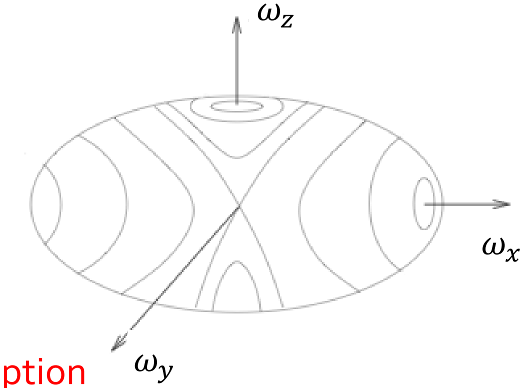
$$\left.\begin{aligned}\vec{L} &= \text{const}\\ 2T = \vec{\omega}\cdot\vec{L} &= \text{const}\end{aligned}\right\} \Rightarrow \vec{\omega}\cdot\vec{L}/L = \text{const} \Rightarrow \dot{\vec{\omega}}\cdot\vec{L} = 0$$

Since the time derivative of ***ω*** is tangent to the energy ellipsoid and perpendicular to ***h***, the energy ellipsoid rolls on the plane ***S***





Angular Velocity in Principal Axes: Polhode



- *Polhode* is the intersection between the energy and momentum ellipsoid and describes the evolution of the angular velocity with time

$$\frac{(\omega_y)^2}{2T/I_y}+\frac{(\omega_z)^2}{2T/I_z}+\frac{(\omega_x)^2}{2T/I_x}=\frac{(\omega_y)^2}{L^2/I_y^2}+\frac{(\omega_z)^2}{L^2/I_z^2}+\frac{(\omega_x)^2}{L^2/I_x^2}$$

$$\underbrace{(I_x-L^2/2T)}_{<0}I_x\omega_x^2+\underbrace{\left(I_y-L^2/2T\right)}_{><0}I_y\omega_y^2+\underbrace{(I_z-L^2/2T)}_{>0}I_z\omega_z^2=0$$

Assumption (always possible) $I_x < I_y < I_z$

- Thus the polhode is real only if $\boxed{I_x < \frac{L^2}{2T} < I_z}$ and its shape is given by

$$(I_y \overset{>0}{-} I_x)I_y\omega_y^2+(I_z \overset{>0}{-} I_x)I_z\omega_z^2=L^2 \overset{>0}{-} 2TI_x \Rightarrow \text{Ellipse in yz plane (seen from x)}$$

$$(I_x \overset{<0}{-} I_y)I_x\omega_x^2+(I_z \overset{>0}{-} I_y)I_z\omega_z^2=L^2 \overset{><0}{-} 2TI_y \Rightarrow \text{Hyperbola in xz plane (seen from y)}$$

$$(I_x \overset{<0}{-} I_z)I_x\omega_x^2+\left(I_y \overset{<0}{-} I_z\right)I_y\omega_y^2=L^2 \overset{<0}{-} 2TI_z \Rightarrow \text{Ellipse in xy plane (seen from z)}$$

- Rotation about axes of minimum and maximum inertia is STABLE, whereas rotation about intermedia inertia is UNSTABLE (large variation)

Equations of Attitude Motion

- Obtained by equality of time derivative of angular momentum vector ***L*** and total torque ***M*** exerted on spacecraft
- Time derivative taken in inertial frame

$$\dot{\vec{L}} = \vec{M}$$

- Time derivative taken in (and vectors expressed in) body frame (rotating)

$$\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{M} \Rightarrow \begin{cases} \dot{L}_x + \omega_y L_z - \omega_z L_y = M_x \\ \dot{L}_y + \omega_z L_x - \omega_x L_z = M_y \\ \dot{L}_z + \omega_x L_y - \omega_y L_x = M_z \end{cases} \Longrightarrow$$

Non-linear, coupled, no general analytical solution

- If body frame is aligned with principal axes (no products of inertia)

$$\begin{cases} I_x \dot{\omega}_x + (I_z - I_y)\omega_y \omega_z = M_x \\ I_y \dot{\omega}_y + (I_x - I_z)\omega_z \omega_x = M_y \\ I_z \dot{\omega}_z + (I_y - I_x)\omega_x \omega_y = M_z \end{cases}$$

Simpler, partially de-coupled, analytical solution for some cases

($M_i \rightarrow \omega_i$ if zero initial conditions)

Solutions of Equations of Attitude Motion

$$\boxed{\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z = M_x \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x = M_y \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y = M_z \end{cases}}$$

- In principal axes, if the initial angular velocity vector is zero, one disturbance component (e.g., M_x) acts only around its angular velocity component (e.g., ω_x)
- Possible solving methods $\omega_x(\dot{\omega}_x)$ $I_y = I_x$
  - Analytical for simple cases (phase plane, axial symmetry)
  - Partial integration (as a function of parameters)
  - Numerical integration (approximate)
- Typical cases
  - Periodic motion
    - $I_x > I_y > I_z \Rightarrow \omega_x \sim \omega_{xo}$, ω_y and ω_z exchange sign, but no sinusoids
    - $I_x \sim I_y > I_z \Rightarrow \omega_z \sim \omega_{zo}$, ω_x and ω_y exchange sign, but quasi-sinusoids
  - Harmonic motion (ellipsoids with circular section)
    - $I_x = I_y > I_z \Rightarrow \omega_z = \omega_{zo}$, ω_x and ω_y as above, but harmonic motion

Torque-Free Axial Symmetric Satellite (1)

$$\boxed{\begin{cases} I_x\dot{\omega}_x + (I_z - I_x)\omega_y\omega_z = 0 \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x = 0 \\ I_z\dot{\omega}_z = 0 \end{cases}}$$

- Assumption of axial symmetry is not so stringent
- External torques are first removed, and later introduced numerically

$$\boxed{\omega_z = \text{const}} = \overline{\omega}_z \Rightarrow \begin{cases} \dot{\omega}_x + \lambda\omega_y = 0 \\ \dot{\omega}_y - \lambda\omega_x = 0 \end{cases}, \lambda = \frac{(I_z - I_x)}{I_x}\overline{\omega}_z$$

Dimensions of angular velocity

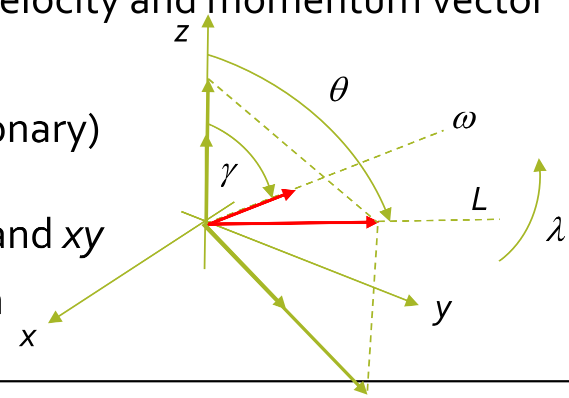
- We can multiply the first equation by ω_x and the second equation by ω_y

$$\frac{d}{dt}\left(\omega_x^2 + \omega_y^2\right) = 0 \Rightarrow \boxed{\left\|\vec{\omega}_{xy}\right\| = \text{const}}$$

- This implies constant magnitude of angular velocity and momentum vector

$$\omega_{xy}, \omega_z, \omega, L_{xy}, L_z, L = \text{const}$$

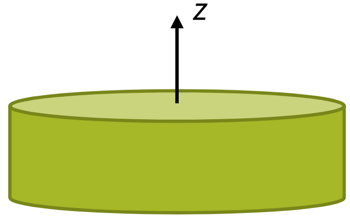
- A plane exists which rotates about *z* (or stationary) and contains angular velocity vector, angular momentum vector and their projection on *z* and *xy*
- These vectors describe circles and cones with semi-aperture θ and γ



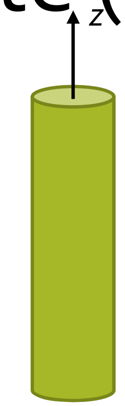
Torque-Free Axial Symmetric Satellite (2)

DYNAMICS

- $I_z > I_x \Rightarrow \lambda > 0,\ \theta < \gamma$



$I_z < I_x \Rightarrow \lambda < 0,\ \theta > \gamma$



- Analytical solution

$$\begin{cases} \dot{\omega}_x + \lambda\omega_y = 0 \\ \dot{\omega}_y - \lambda\omega_x = 0 \end{cases} \Rightarrow \begin{cases} \ddot{\omega}_x + \lambda^2\omega_x = 0 \\ \dot{\omega}_y = \lambda\omega_x \end{cases} \Rightarrow \begin{cases} \omega_x = c_1 e^{i\lambda t} + c_2 e^{-i\lambda t} \\ \dot{\omega}_y = \lambda\omega_x \end{cases}$$

- Initial conditions

$$\begin{cases} \omega_{x0} = c_1 + c_2 \\ \dot{\omega}_{x0} = i\lambda(c_1 - c_2) \end{cases} \Rightarrow \begin{cases} c_1 = \frac{\omega_{x0}}{2} - i\,\frac{\dot{\omega}_{x0}}{2\lambda} \\ c_2 = \frac{\omega_{x0}}{2} + i\,\frac{\omega_{x0}}{2\lambda} \end{cases} \Rightarrow \omega_{xy} = (\omega_{x0} + i\omega_{y0})e^{i\lambda t}$$

Angular acceleration is perpendicular to angular velocity

This vector rotates with constant angular velocity λ about z

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Torque-Free Axial Symmetric Satellite (3)

KINEMATICS

- Since the angular momentum vector is conserved in inertial space, the attitude motion is completely different relative to inertial axes
- To describe the attitude w.r.t. inertial frame, we introduce three angles which provide the orientation of the body axes relative to the inertial frame

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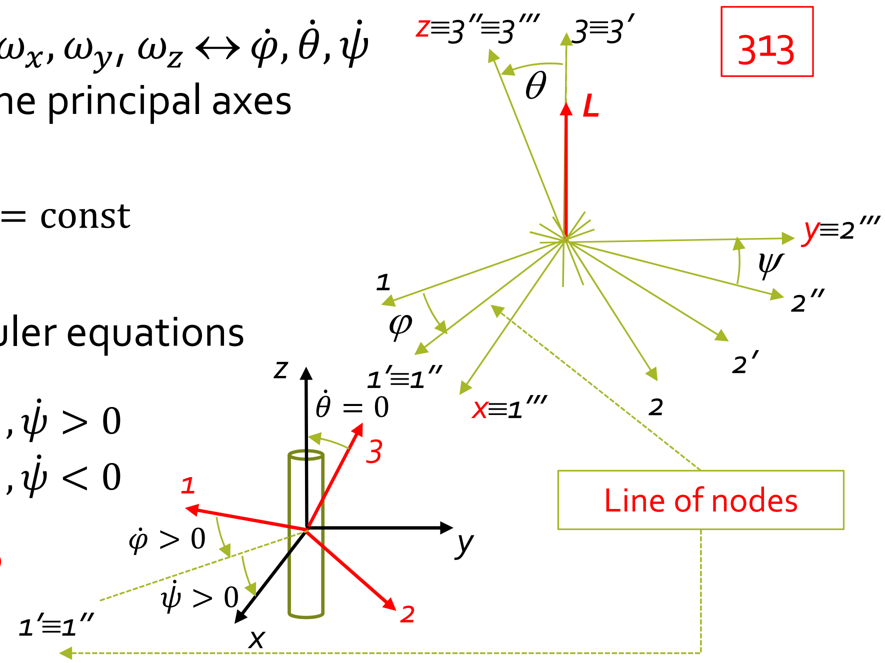
- First we seek the relationship $\omega_x, \omega_y, \omega_z \leftrightarrow \dot{\varphi}, \dot{\theta}, \dot{\psi}$ by projecting the latter onto the principal axes

$$\begin{cases} \omega_x = \dot{\varphi}\sin\theta\sin\psi \\ \omega_y = \dot{\varphi}\sin\theta\cos\psi \\ \omega_z = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases} \quad \text{with} \qquad \theta, \dot{\varphi}, \dot{\psi} = \text{const}$$

- Then we substitute into the Euler equations

$$\begin{cases} \dot{\psi} = \dot{\varphi}\overset{>0}{\cos\theta}\,\frac{I_x - I_z}{I_z} \\ \dot{\varphi} = \frac{I_z \omega_z}{I_x \cos\theta} > 0 \end{cases} \Rightarrow \begin{cases} I_x > I_z \,; \dot{\varphi} > 0 \,, \dot{\psi} > 0 \\ I_x < I_z \,; \dot{\varphi} > 0 \,, \dot{\psi} < 0 \end{cases}$$

We have chosen **3** // ***L***, thus $\omega_z > 0$



Backup