AA 279 C – SPACECRAFT ADCS: LECTURE 7

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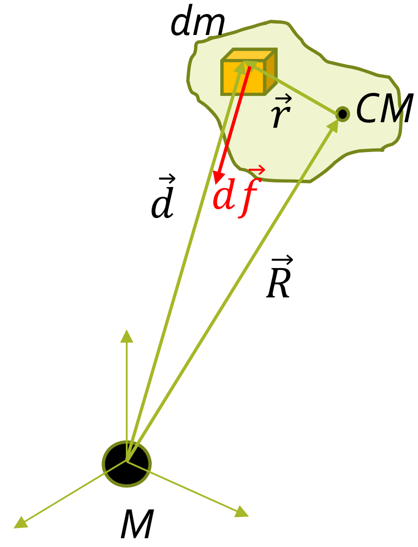
Gravity Gradient Modeling (1)



- This environmental torque is intimately related to the attitude dynamics and can be incorporated in the Euler equations
- Rigorously speaking, once modeled in the dynamics, it is not a disturbance anymore, but a property of the system we can exploit
- Each satellite's mass element is subject to a different gravity force depending on its distance from the Earth's center

$$\begin{cases} d\vec{f} = -GM\frac{\vec{d}}{d^3}dm \\ \vec{d} = \vec{R} + \vec{r} \\ d\vec{M} = \vec{r}\,\mathrm{x}\,d\vec{f} \end{cases} \Rightarrow \vec{M} = \int_m d\vec{M} = -GM\int_m \vec{r}\,\mathrm{x}\,\frac{\vec{R}+\vec{r}}{|\vec{R}+\vec{r}|^3}dm$$

Non-zero torque if points are at different distances from *M*

Torque proportional to ***r***

Gravity Gradient Modeling (2)

- Approximation: ***r*** is an infinitesimal of the first order and is treated as a small perturbation of ***R***

$$\left|\vec{R}+\vec{r}\right|^{-3} = R^{-3}\left|1+\frac{\vec{r}}{\vec{R}}\right|^{-3} \sim R^{-3}\left(1-3\frac{\vec{r}}{\vec{R}}\right) = R^{-3}\left(1-3\frac{\vec{r}\cdot\vec{R}}{R^2}\right) \Rightarrow$$

$$\vec{M} = -\frac{3GM}{R^5}\int_m (\vec{r} \times \vec{R})\vec{r}\cdot\vec{R}dm$$

- Expressing the integral in the orbital frame (physics)

$$\begin{cases} \vec{r} = r\hat{\vec{R}} + t\hat{\vec{T}} + n\hat{\vec{N}} \\ \vec{R} = R\hat{\vec{R}} \end{cases} \Rightarrow \vec{M} = \frac{3GM}{R^3}\int_m \left(n\hat{\vec{N}} - t\hat{\vec{T}}\right) r dm = \frac{3GM}{R^3}\begin{bmatrix} 0 \\ I_{rn} \\ -I_{rt} \end{bmatrix}$$

No torque in radial direction

Change with time because RTN are not body axes

RTN=XYZ implies no torque

- Expressing the integral in principal axes (Euler equations)

$$\begin{cases} \vec{r} = x\hat{\vec{X}} + y\hat{\vec{Y}} + z\hat{\vec{Z}} \\ \vec{R} = R(c_x\hat{\vec{X}} + c_y\hat{\vec{Y}} + c_z\hat{\vec{Z}}) \end{cases} \Rightarrow \vec{M} = \frac{3GM}{R^3}\int_m \begin{bmatrix} (y^2-z^2)c_y c_z \\ (z^2-x^2)c_x c_z \\ (x^2-y^2)c_x c_y \end{bmatrix} dm = \frac{3GM}{R^3}\begin{bmatrix} (I_z-I_y)c_y c_z \\ (I_x-I_z)c_z c_x \\ (I_y-I_x)c_x c_y \end{bmatrix}$$

Larger for larger differences btw. moments of inertia

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Attitude Dynamics with Gravity Gradient

Note similarity

1. Euler equations

$$\begin{cases} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z = 3n^2(I_z - I_y)c_y c_z \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x = 3n^2(I_x - I_z)c_z c_x \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y = 3n^2(I_y - I_x)c_x c_y \end{cases}$$

To compare results

2. Equilibrium (as for single-spin)

$$\begin{cases} \bar{\omega}_x = 0 \\ \bar{\omega}_y = 0 \\ \bar{\omega}_z \neq 0 \end{cases} \Rightarrow \boxed{\begin{cases} c_y = c_z = 0 \\ c_x = 1 \\ \bar{\omega}_z = n \end{cases}}$$

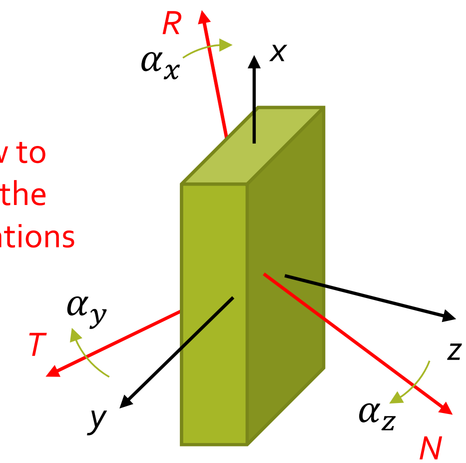
RTN=XYZ

Mean motion

3. Linearization through perturbation of equilibrium

$$\begin{cases} \vec{\omega}_{xyz} = \vec{A}\vec{\omega}_{RTN} = \begin{bmatrix} \dot{\alpha}_x - \alpha_y n \\ \dot{\alpha}_y + \alpha_x n \\ \dot{\alpha}_z + n \end{bmatrix} \\ \vec{c}_{xyz} = \vec{A}\vec{c}_{RTN} = \vec{A}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -\alpha_z \\ \alpha_y \end{bmatrix} \end{cases}$$

These transformations allow to write the Euler equations in the variables which represent rotations about the RTN axes



Stability with Gravity Gradient (1)

3. Substitution of kinematics in the Euler equations provides

$$\begin{cases} I_x(\ddot{\alpha}_x - \dot{\alpha}_y n) + n(I_z - I_y)(\dot{\alpha}_y + \alpha_x n) = 0 \\ I_y(\ddot{\alpha}_y + \dot{\alpha}_x n) + n(I_x - I_z)(\dot{\alpha}_x - \alpha_y n) = 3n^2(I_x - I_z)\alpha_y \\ I_z\ddot{\alpha}_z = -3n^2(I_y - I_x)\alpha_z \end{cases}$$

3rd equation is still decoupled from other 2, but now it is an harmonic oscillator

- The 3rd equation represents pitch motion (rotation about N), which is stable if

$$I_z\ddot{\alpha}_z + 3n^2(I_y - I_x)\alpha_z = 0 \Rightarrow \ddot{\alpha}_z + k_N\alpha_z = 0 \Rightarrow \boxed{k_N = \frac{(I_y - I_x)}{I_z} > 0}$$

Inertia tangent to trajectory must be larger than in radial

- The first 2 equations represent roll and yaw motion (rotation about T and R)

$$\begin{cases} \ddot{\alpha}_x + n(k_R - 1)\dot{\alpha}_y + n^2\alpha_x k_R = 0 \\ \ddot{\alpha}_y - n(k_T - 1)\dot{\alpha}_x + 4n^2\alpha_y k_T = 0 \end{cases} \Rightarrow \begin{bmatrix} s^2 + n^2 k_R & sn(k_R - 1) \\ -sn(k_T - 1) & s^2 + 4n^2 k_T \end{bmatrix} \begin{bmatrix} \alpha_x(s) \\ \alpha_y(s) \end{bmatrix} = 0$$

Laplace transform

- The new stability condition becomes $\boxed{\mathrm{Re}(s_i) \leq 0\ \forall i, \text{ where } s_i \text{ satisfies } \det A = 0}$

Stability with Gravity Gradient (2)

4. From the determinant

If s_1 is a root, then $-s_1$ is also a root, thus stability requires s_1 to be pure imaginary and s_1^2 to be real negative

$$\det A = 0 \Rightarrow s^4 + s^2 n^2 (1 + 3k_T + k_T k_R) + 4n^4 k_T k_R = 0 \quad \Rightarrow$$
$$s''^2 + s''(1 + 3k_T + k_T k_R) + 4k_T k_R = 0\,;\ s'' = s^2/n^2 \quad \Rightarrow$$
$$2s''_{1,2} = -1 - 3k_T - k_T k_R \pm \sqrt{(1 + 3k_T + k_T k_R)^2 - 16k_T k_R} < 0 \quad \Rightarrow$$
$$\boxed{\begin{cases} 1 + 3k_T + k_R k_T > 4\sqrt{k_R k_T} \\ k_R k_T > 0 \end{cases}}$$

Inertia constraints for stability of roll and yaw motion

- The stability conditions can be put all together considering that

$$k_N = \frac{(I_y - I_x)}{I_z} > 0 \Rightarrow \frac{I_x}{I_y} < 1$$

$$k_T = \frac{(I_z - I_x)}{I_y}\,;\ k_R = \frac{(I_z - I_y)}{I_x}$$

$$\begin{cases} k_T > k_R & \text{Stable pitch} \\ k_R k_T > 0 & \\ 1 + 3k_T + k_R k_T > 4\sqrt{k_R k_T} & \end{cases}$$

Stable roll and yaw

$$k_T I_y + I_x = k_R I_x + I_y \Rightarrow \underbrace{(k_T - 1)}_{<0} = \overbrace{\frac{I_x}{I_y}}^{<1} \underbrace{(k_R - 1)}_{<0} \Rightarrow k_T > k_R$$

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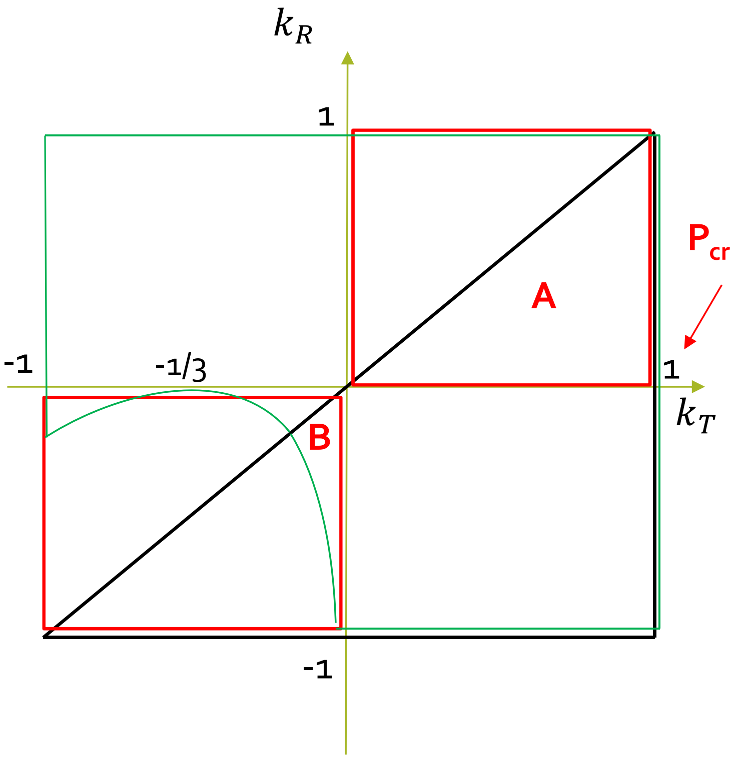
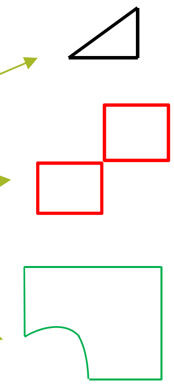
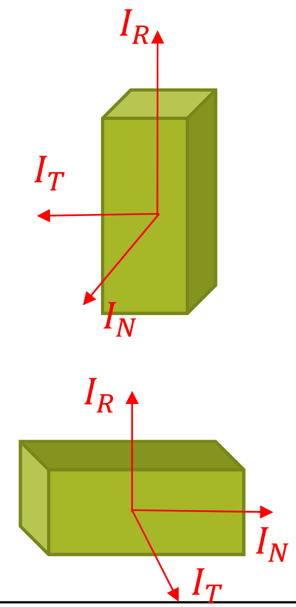
Stability with Gravity Gradient (3)



$$\begin{cases} k_T > k_R \\ k_R k_T > 0 \\ 1 + 3k_T + k_R k_T > 4\sqrt{k_R k_T} \end{cases}$$





Three axis passive stability through gravity gradient is possible in 2 configurations (rotation about min and max inertia)

A: *Sure*

$$\boxed{\begin{array}{l} k_N > 0 \Rightarrow I_y > I_x \Rightarrow I_T > I_R \\ k_R k_T > 0 \Rightarrow I_z = I_{max} \\ I_N > I_T > I_R \end{array}}$$

B: *Maybe*

$$\boxed{\begin{array}{l} k_N > 0 \Rightarrow I_y > I_x \Rightarrow I_T > I_R \\ k_R k_T > 0 \Rightarrow I_z = I_{min} \\ I_T > I_R > I_N \end{array}}$$

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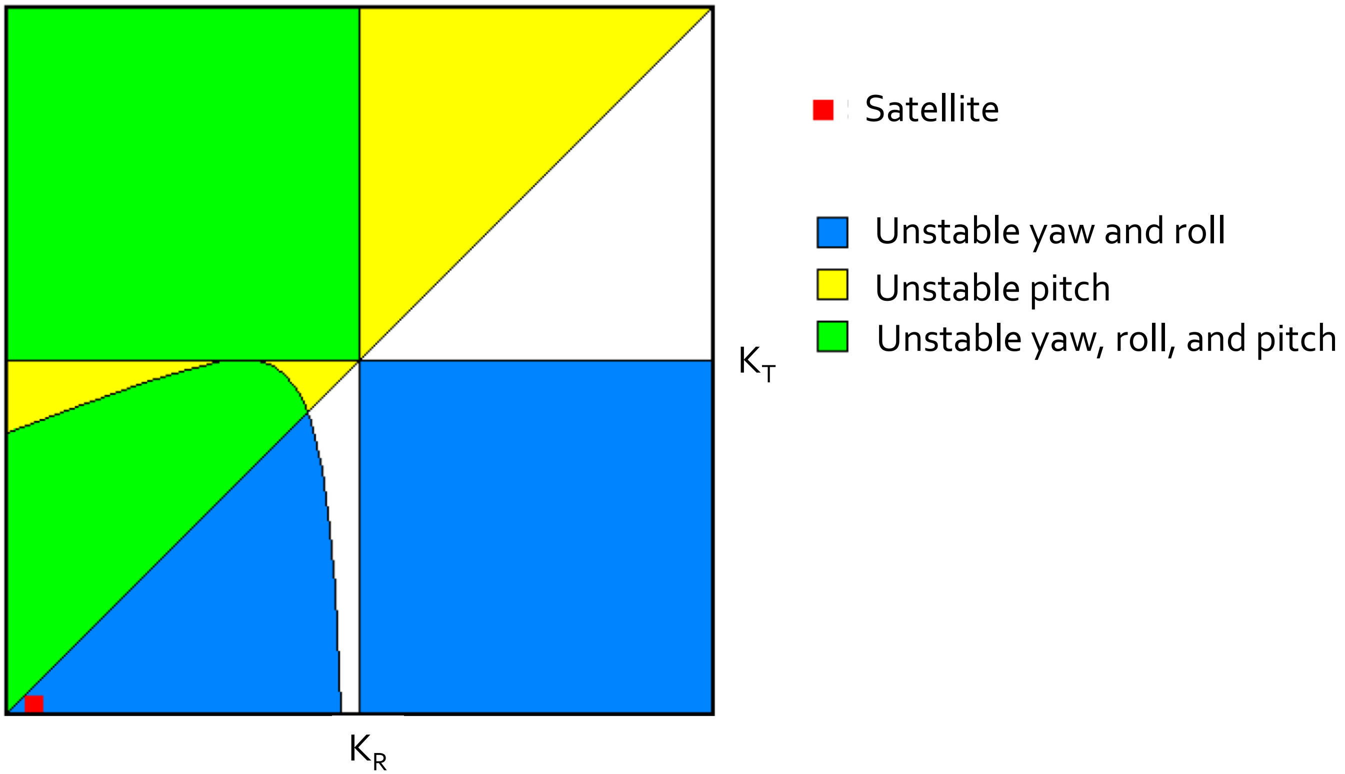
Stability with Gravity Gradient (4)



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Dual-Spin Subject to Gravity Gradient

- The most general satellite can be described by combining all previous models

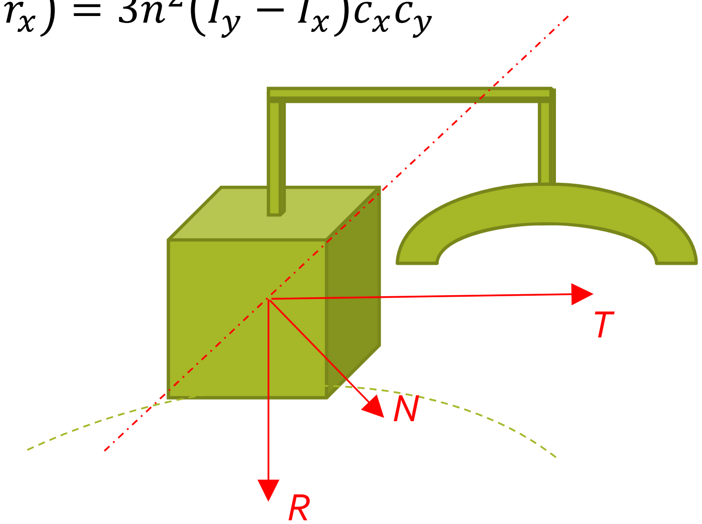
$$\begin{cases} I_x\dot{\omega}_x + I_r\dot{\omega}_r r_x + (I_z - I_y)\omega_y\omega_z + I_r\omega_r(\omega_y r_z - \omega_z r_y) = 3n^2(I_z - I_y)c_y c_z \\ I_y\dot{\omega}_y + I_r\dot{\omega}_r r_y + (I_x - I_z)\omega_z\omega_x + I_r\omega_r(\omega_z r_x - \omega_x r_z) = 3n^2(I_x - I_z)c_z c_x \\ I_z\dot{\omega}_z + I_r\dot{\omega}_r r_z + (I_y - I_x)\omega_x\omega_y + I_r\omega_r(\omega_x r_y - \omega_y r_x) = 3n^2(I_y - I_x)c_x c_y \\ I_r\dot{\omega}_r = M_r \end{cases}$$

- The trimming problem consists of finding the equilibrium of the Euler equations, since

$$\vec{\omega} = f(\vec{\alpha}, \dot{\vec{\alpha}}),\ \vec{c} = g(\vec{\alpha}),\ \|\hat{\vec{r}}\| = 1,\ \dot{\vec{\alpha}} = 0$$

Kinematics RTN→XYZ Equilibrium

- We can find 3 equations in 6 unknowns given by $\vec{\alpha}$, r_i, r_j, and ω_r which can be solved by fixing $\vec{\alpha}$ and then finding rotor parameters



Principal axes of inertia are rotated w.r.t. RTN

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Passive Damping of Modes of Oscillations

- The actual motion of a dual spin satellite subject to gravity gradient (generic satellite) is the sum of multiple harmonic motions at different frequencies
  - **Nutation**: conic motion (e.g. of pitch axis) about the spin axis
    - ω_x and ω_y (perturbations) are undesired and need to be removed
  - **Precession**: pendulum motion (e.g. of roll and yaw axis) about the spin axis
    - ω_z (perturbation) is undesired and need to be removed
- These perturbations can be removed through a damper which brings the satellite back to its nominal configuration
- This is a passive device which makes use of dissipation, thus the target attitude must be a configuration of stable equilibrium
- Dissipation can be created inside the satellite through relative motion caused by the motion of the satellite itself

Backup