Table 1 Conditions of numerical experiments

1) Orbit	near synchronous orbit
2) Perturbation forces considered	gravitational force of the Moon gravitational force of the Sun perturbation forces due to the Earth's nonsphericity up to orders 15 solar radiation pressure
3) Epoch of used orbital elements	77.01.12.
4) Latitude of the satellite	60°E
5) Calculation of Eq. (7)	double precision
6) Calculation of Eq. (8)	single precision except for the calculation of $F_0(X_0+X_1)-F_0(X_0)$

Table 2 Deviations of satellite's position from DODS' values after one day integration from epoch

Δt	
2 h	about 20 m
1 h	about 3 m
30 min	less than 1 m
10 min	less than 1 m

from

$$V_{1,i+1}=V_{1,i}+\frac{\Delta t}{6}\{(f_{1,i}+f_{2,i})+4(f_{1,i+1/2}+f_{2,i+1/2})+(f_{1,i+1}+f_{2,i+1})\} \tag{13}$$

Thus, X_1 and V_1 at the next time step are obtained.

From the foregoing equations we notice that the predictor-corrector formulas chosen are very simple, requiring no starter at the beginning of integration. These facts enable the orbit generation programs to be compact. Next we introduce some approximations. As the values of $X^c_{1,i+1/2}-X^p_{1,i+1/2}$ and $X^c_{1,i+1}-X^p_{1,i+1}$ are small, we approximate $f^c_{2,i+1/2}$ by $f^p_{2,i+1/2}$ and $f^c_{2,i+1}$ by $f^p_{2,i+1}$. By introducing these approximations, we can reduce the frequency of calculating the perturbation forces to as little as twice per step, which contributes to the reduction of computation time. Successive corrections of the solution are continued by the correctors, Eqs. (11) and (12) applied only to the f_1 part. The error in the solution occurring from this approximation proves to be small as seen from the numerical experiment in the next section.

III. Results and Conclusions

According to the method described in the previous section, the results of numerical experiments are presented for ephemeris generation of satellites. Conditions for these experiments are shown in Table 1. The results are compared with those generated by the Adams-Cowell method of orders 12 using NASA's DODS (Table 2). In Table 2, deviations of the satellite's position from DODS's values are shown after one day of integration from the epoch. From these results, we can see that the choice of $\Delta t=1$ h can be sufficient for the ephemeris generation, and the errors become negligibly small even if approximations noted in the last section are used. In our method, these approximations are very effective to reduce computational time. The fact that numerical integrations are all performed by a single type of algorithm in the computer is also effective in reducing computation time.

When a long-range ephemeris is generated, some additional calculations are required. At the time when X_1 and V_1 become large, we calculate X and V from Eqs. (3) and convert them to osculating orbital elements. Calculations are continued using these new elements setting X_1 and V_1 to zero.

When we apply the integration method described in this Note to orbit determination, Eqs. (8) can also be applicable to the calculation of ephemeris partials. In this case, Eqs. (8) are integrated with appropriate initial conditions. Thus, partials of finite-difference formulas are calculated easily.

The integration method described in this Note has proved to be useful. Using this method, orbit determination programs can be very compact and high-speed computations can be realized without reduction of accuracy or neglecting perturbation forces. According to our experience, it is possible to realize accurate orbit determination using minicomputers.

References

[1]Tsukamoto, K., Kajikawa, M., Uchikado, S., Yokoyama, M., and Tanaka, T., "Main Transmit and Receive Station of BSE Program," *Proceedings of the Twelfth International Symposium on Space Technology and Science,* May, 1977.

[2]Roche, W., et. al., "Definitive Orbit Determination Operating System Description Edition II," Goddard Space Flight Center, July, 1971.

[3]Wagner, W.E., and Velez, C.E., "Goddard Trajectory Determination Subsystem Mathematical Specifications," Goddard Space Flight Center, March, 1972.

Quaternion from Rotation Matrix

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Introduction

FOR the purposes of this Note, a quaternion will be thought of as a four-parameter representation of a coordinate transformation matrix. The quaternion (q) may be written in many different forms, three of which are

$$q\equiv\begin{bmatrix} s \\ v \end{bmatrix}\equiv\begin{bmatrix} \cos\dfrac{\alpha}{2} \\ -u\sin\dfrac{\alpha}{2} \end{bmatrix}\equiv\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{1}$$

The first representation, with a scalar (s) component and a vector (v) component, is perhaps the most traditional. The second representation is in the same format, but is expressed in terms of the parameters α and u which lend themselves to a geometrical interpretation of the quaternion. The unit vector u is the eigenvector of the rotation matrix, and α is the angle about u that one coordinate frame is rotated with respect to the other. This interpretation is included only to emphasize the relevance of the quaternion extraction algorithm to the important problem of finding the eigenvector of a rotation matrix.

The third representation of the quaternion shown in Eq. (1) will be the one used throughout the remainder of this Note. This choice best reflects the unifying philosophy of the new approach, which is that each of the components should be treated the same way. Other authors perhaps have been misled by the special nature of the scalar component. Grubin[1] presents a very simple algorithm; however, it degrades for large rotation angles and, indeed, has a zero-over-zero singularity at exactly 180 deg. Klumpp,[2] on the other hand,

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places less emphasis on the scalar term and does eliminate the singularity problem, but only at the expense of four square roots and some cumbersome logic to determine the component signs. The procedure developed here will be compact, singularity-free, and require only one square root.

Analysis

The quaternion to rotation matrix conversion is purely algebraic. In component form, the elements of the matrix are simply expressed as a function of the quaternion components by the well-known equation[1,2]

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

This equation plus the unitary condition

$$q_0^2+q_1^2+q_2^2+q_3^2=1 \tag{3}$$

is the starting point for the solution of the inverse problem: given the matrix elements, find the quaternion components. It should be pointed out that there will always be two solutions to this problem, Eqs. (2) and (3); if a quaternion satisfies them so will its negative. This ambiguity, though not mathematically significant, is usually resolved by choosing the solution for which q_0 is nonnegative. This choice corresponds to a rotation angle α less than 180 deg, as opposed to a rotation of 360 deg $-\alpha$ in the opposite sense.

Before proceeding, a few definitions are made which will prove useful, namely

$$p \equiv 2q \tag{4}$$

$$M_{00} \equiv T \tag{5}$$

$$T \equiv M_{11}+M_{22}+M_{33} \tag{6}$$

Note that T is the trace of the matrix, and $\boldsymbol{p}$ is the quaternion $\boldsymbol{q}$ scaled up by a factor of two, purely as a matter of convenience. With the aid of these definitions, and Eq. (3), the diagonal terms of Eq. (2) may be manipulated to yield a symmetric set of equations for the $\boldsymbol{p}$ component magnitudes, namely

$$p_0^2=1+2M_{00}-T \tag{7a}$$

$$p_1^2=1+2M_{11}-T \tag{7b}$$

$$p_2^2=1+2M_{22}-T \tag{7c}$$

$$p_3^2=1+2M_{33}-T \tag{7d}$$

The off-diagonal terms of Eq. (2) yield all combinations of $\boldsymbol{p}$ component products, namely

$$p_0p_1=M_{32}-M_{23} \qquad p_2p_3=M_{32}+M_{23} \tag{8a,b}$$

$$p_0p_2=M_{13}-M_{31} \qquad p_3p_1=M_{13}+M_{31} \tag{9a,b}$$

$$p_0p_3=M_{21}-M_{12} \qquad p_1p_2=M_{21}+M_{12} \tag{10a,b}$$

Clearly, Eqs. (7-10) depict ten equations in only four unknowns (the four components of $\boldsymbol{p}$) and, as a result, there are many possible ways to solve them.

Recommended Solution

In words, the best strategy for solving the preceding equations is to first select the component of $\boldsymbol{p}$ from Eqs. (7) with the largest magnitude. Then, the other components follow directly by division of the appropriate three equations among Eqs. (8-10) (those equations which involve the maximum component). It should be noted that the first step in this outline requires only one square root, and the second step insures the proper signs for the other components.

The search for the component of maximum magnitude is greatly facilitated by the form of Eqs. (7). Clearly, if i $(0 \le i \le 3)$ is the index, such that

$$M_{ii}=\max(M_{00},M_{11},M_{22},M_{33}) \tag{11}$$

then, it follows that

$$|p_i|=\sqrt{1+2M_{ii}-T} \tag{12a}$$

$$|p_i|=\max(|p_0|,\ |p_1|,\ |p_2|\ ,\ |p_3|) \tag{12b}$$

Furthermore, from Eqs. (3) and (4), it may be shown that this component must lie in the range $(1 \le |p_i| \le 2)$. A value of unity is possible only if all the components of $\boldsymbol{p}$ have unity magnitude, which corresponds to the situation where the diagonal terms of the matrix are all zero. It should be noted that choosing the maximum $|p_i|$ extracts the most possible accuracy from the diagonal and off-diagonal equations.

As seen from Eq. (12a), the sign ascribed to p_i could be either plus or minus, which is as it should be since there are two possible solutions. Thus, whichever sign is chosen, the off-diagonal equations (8a) or (8b), (9a) or (9b), and (10a) or (10b) then determine the appropriate signs for the other components of $\boldsymbol{p}$. In the end, of course, the components of $\boldsymbol{p}$ must be scaled down by a factor of two to obtain $\boldsymbol{q}$, and, if desired, all the signs may be changed if it happens that p_0 emerges negative.

Concluding Remarks

It must be emphasized that nowhere in this paper has any of the components of $\boldsymbol{p}$ been treated differently from any of the others. Indeed, if the quaternion represents a truly random orientation then it is also true that each of the components is equally likely to be the largest. This situation may or may not occur in practice, and a computer implementation should take advantage of any a priori knowledge. However, such opportunism does not detract from the philosophy of treating each of the components the same way. It is this concept that forms the basis for the unified, compact, and singularity-free approach to the quaternion extraction problem presented here.

An optimal computer implementation of the recommended solution is dependent on the computer language and the environment in which it will be used, and thus cannot be given here. The portion of the algorithm which has not been described in detail is how to choose the appropriate three off-diagonal equations from among Eqs. (8a,b), (9a,b), and (10a,b). However, many relationships and symmetries may be found among the subscripts of these six equations which should facilitate this process.

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References

[1] Grubin, C., "Derivation of the Quaternion Scheme via the Euler Axis and Angle," *Journal of Spacecraft and Rockets,* Vol. 7, Oct. 1970, pp. 1261-1263.

[2] Klumpp, A.R., "Singularity-Free Extraction of a Quaternion from a Direction-Cosine Matrix," *Journal of Spacecraft and Rockets,* Vol. 13, Dec. 1976, pp. 754-755.