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AA 279 A – Space Mechanics
Lecture 13: Notes

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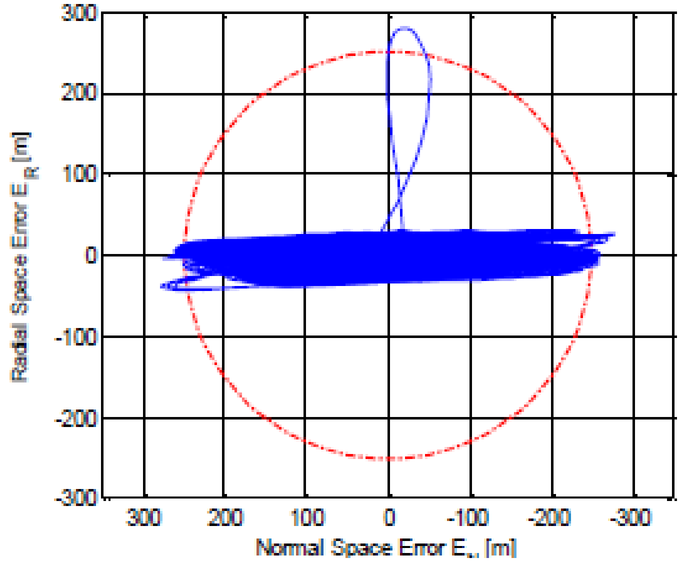
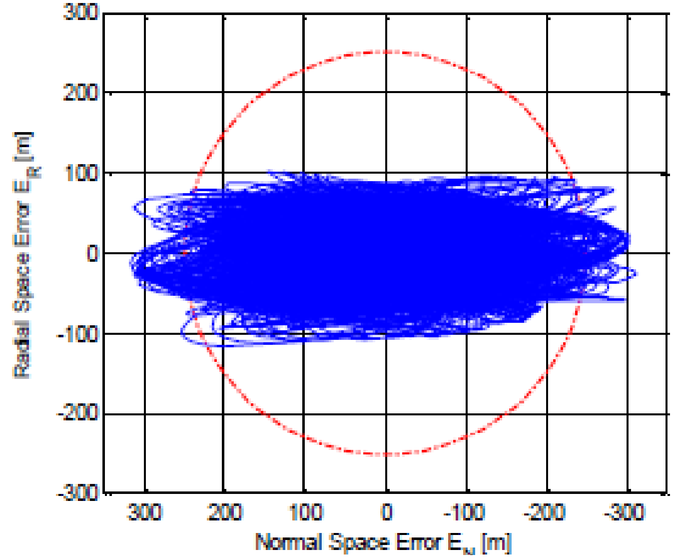
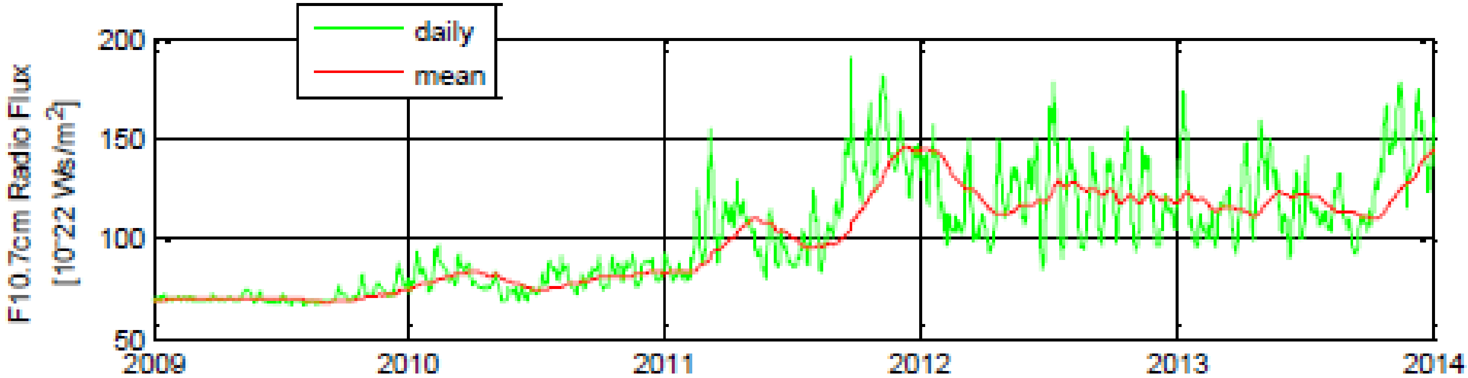
Precise Orbit Control (within 250 m tube)

Q4-2009





Q1-2014



Solar flux activity

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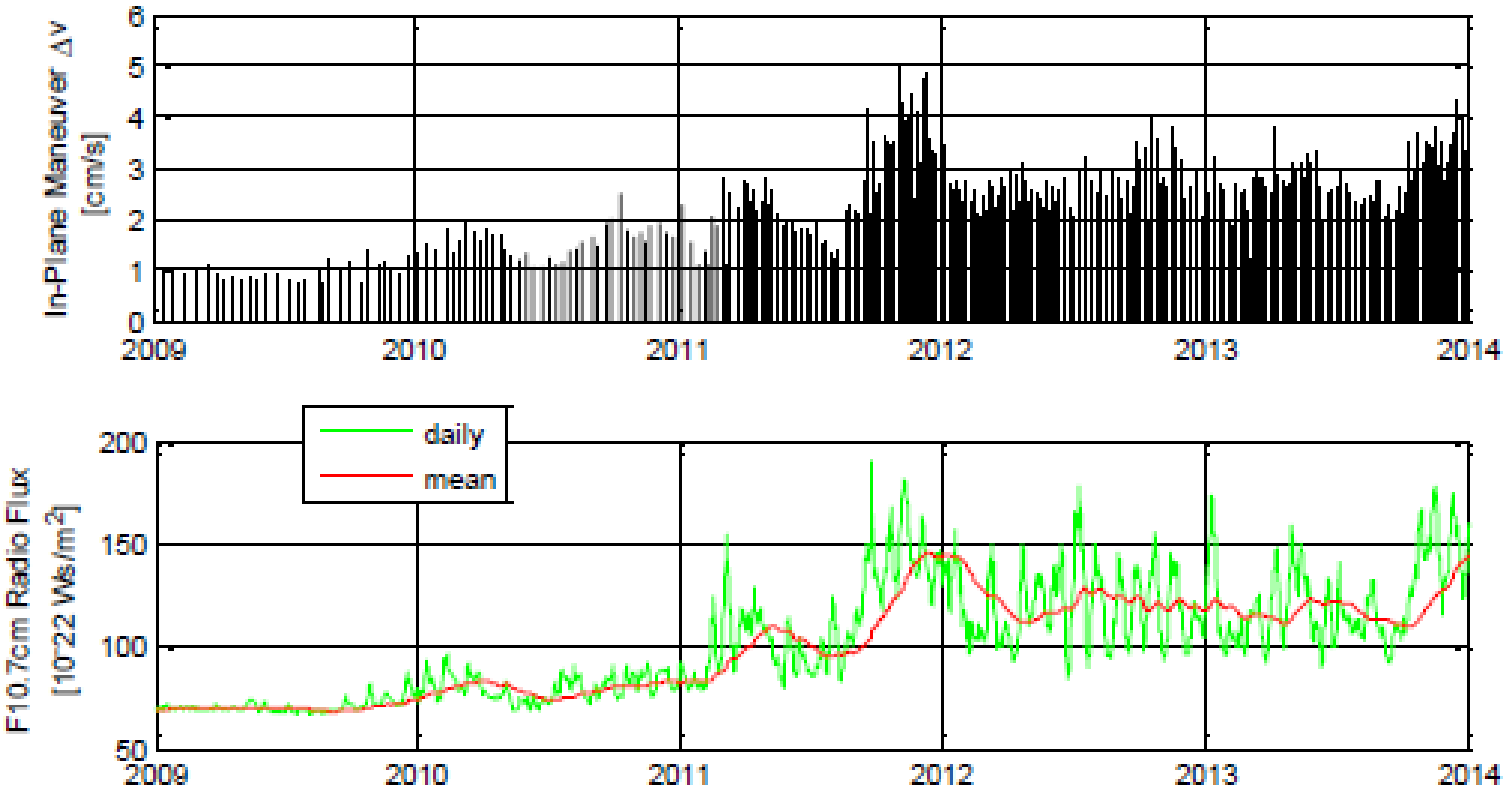
Precise Orbit Control (within 250 m tube)



Maneuver activity

Solar flux activity

Orbit Perturbations (Problem Description)

- Up until now, the only forces at work have been $1/r^2$ central gravity forces. We need to change that.

Perturbations

$$\vec{a} = \frac{d^2}{dt^2}\vec{r} = -\frac{\mu}{r^3}\vec{r} + \vec{f}$$

- The fundamental orbit differential equation can be modified to account for perturbations, i.e. forces ***f*** other than $1/r^2$ central gravity

- How do perturbing forces ***f*** affect the orbit? Typically we need to take the following forces
  - Non $1/r^2$ gravity
  - 3^{rd} body gravity
  - Atmospheric drag
  - Solar radiation pressure

Conservative

Non-conservative

Orbit Perturbations (Strategies)

- *Numerical integration* of equations of motion is simple and can yield accurate results, but
  - requires small time steps where things change fast, thus leading to long computer runs
  - does not provide insight into orbit's behavior, especially if position and velocity are used

State-space form

$$\frac{d}{dt}\vec{r} = \vec{v}$$

$$\frac{d}{dt}\vec{v} = -\frac{\mu}{r^3}\vec{r} + \vec{f}$$

Fast variables

- *General perturbation* techniques allow us to examine the effect of ***f*** on the orbital elements through an analytical approximation that captures the general character of the motion, but
  - Complicated series expansion of ***f*** are required which need to be truncated
  - This trade-off speeds up computation but decreases accuracy

Slow variables

$$\vec{f}$$

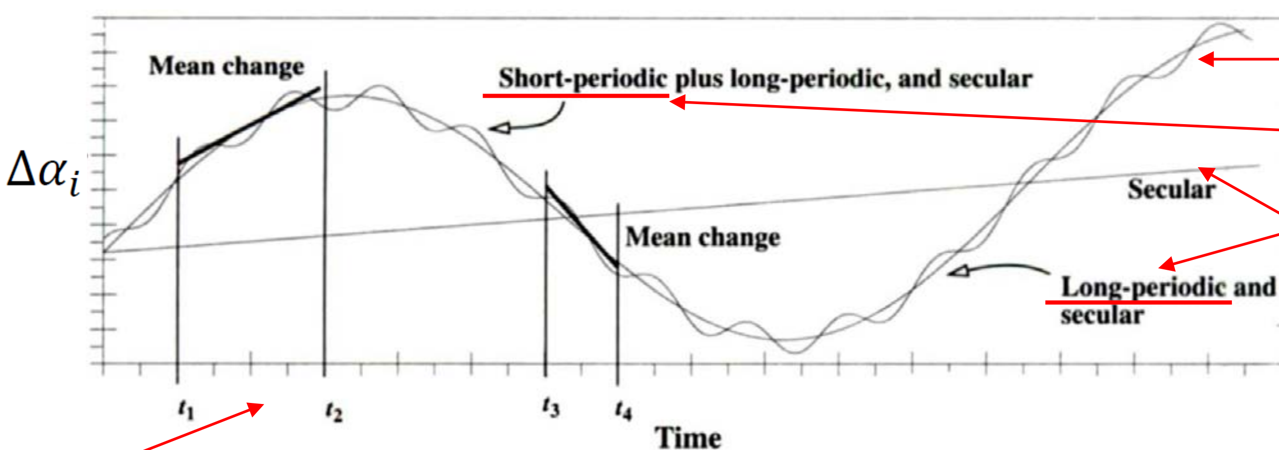
?

$$\dot{\vec{\alpha}} = [\dot{a} \quad \dot{e} \quad \dot{i} \quad \dot{\Omega} \quad \dot{\omega} \quad \dot{M}]^t$$

Orbit Perturbations (Terminology)

- In the absence of perturbations only $M = M_0 + n(t\text{-}t_0)$ is changing over time and repeats at orbital period T
- The effects of perturbations ***f*** on the orbital elements $\Delta\boldsymbol{\alpha}$ can be categorized according to their time properties: Secular, Long- and Short-Periodic

Example $$\Delta\alpha_i = c_0 + \dot{c}_1(t - t_0) + K_1\cos(2\omega) + K_2\sin(2\nu + \omega) + K_3\cos(2\nu)$$



Repeat after time interval

$\leq$ *T*

$>>$ *T*

Unbounded growth

- *Mean* orbital elements represent an average of the *osculating* orbital elements (instantaneous) over some selected time or anomaly range

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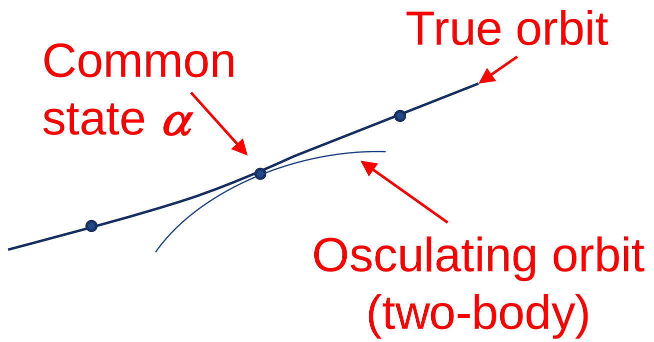
Gauss Variational Equations (Idea)



- General perturbation methods rely on a process called variation of parameters (VOP). Lagrange and Gauss used VOP to analyze perturbations

$$\frac{d\vec{\alpha}}{dt} = \vec{g}(\vec{\alpha},t) = \frac{\partial\vec{\alpha}}{\partial t} + \frac{\partial\vec{\alpha}}{\partial\vec{r}}\cdot\frac{d\vec{r}}{dt} + \frac{\partial\vec{\alpha}}{\partial\vec{v}}\cdot\frac{d\vec{v}}{dt} =$$

$$= \underbrace{\frac{\partial\vec{\alpha}}{\partial t} + \frac{\partial\vec{\alpha}}{\partial\vec{r}}\cdot\frac{d\vec{r}}{dt} - \frac{\partial\vec{\alpha}}{\partial\vec{v}}\cdot\frac{\mu}{r^3}\vec{r}}_{\text{Vanish for osculating orbit}} + \frac{\partial\vec{\alpha}}{\partial\vec{v}}\cdot\vec{f} =$$

- The VOP's concept is that we can use the solution of the unperturbed system to represent the solution of the perturbed system, provided that we can generalize the constants in the solution to be time-varying parameters

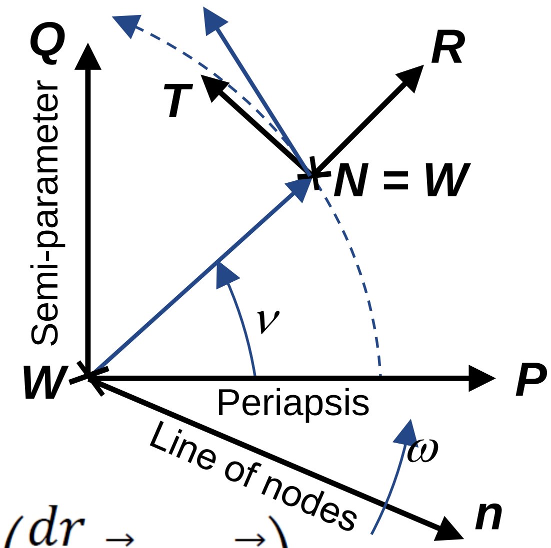
- Gauss variational equations (GVE) express the rates of change of the orbit elements explicitly in terms of the disturbing forces

$$= \frac{\partial\vec{\alpha}}{\partial\vec{v}}\cdot\vec{f}$$

Only the partial derivatives of orbit elements w.r.t. velocity

Gauss Variational Equations (Derivation)

- GVE are derived in the RTN frame, which is similar to perifocal coordinates PQW, rotated by ν



- Position vector: $\vec{r} = r\vec{R}$
- Velocity vector: $\vec{v} = \dot{r}\vec{R} + r\dot{\nu}\vec{T} = \dot{\nu}\left(\frac{dr}{d\nu}\vec{R} + r\vec{T}\right)$
- Angular velocity (w.r.t ECI): $\vec{\omega}_{\mathrm{IJK}\rightarrow\mathrm{RTN}} = \dot{\Omega}\vec{K} + \dot{i}\vec{n} + (\dot{\omega} + \dot{\nu})\vec{N}$

Argument of latitude

New terms due to ***f***

- Perturbing force (total): $\vec{f} = f_R\vec{R} + f_T\vec{T} + f_N\vec{N}$

GVE: Semi-major Axis

- The specific mechanical energy is not constant due to *nonconservative* forces

$$\frac{d\mathcal{E}}{dt} = \vec{f} \cdot \vec{v} = \dot{\nu}\left(\frac{dr}{d\nu} f_R + r f_T\right)$$

- This affects the semi-major axis according to the chain rule

$$\frac{da}{dt} = \frac{da}{d\mathcal{E}}\frac{d\mathcal{E}}{dt} = -\frac{2a^2}{\mu}\dot{\nu}\left(\frac{dr}{d\nu} f_R + r f_T\right)$$

- From angular momentum and trajectory equation (two-body)

$$h = r r\dot{\nu} = \sqrt{\mu p} \Rightarrow \dot{\nu} = \frac{na^2}{r^2}\sqrt{1-e^2} \qquad r = \frac{p}{1+e\cos\nu} \Rightarrow \frac{dr}{d\nu} = \frac{re\sin\nu}{1+e\cos\nu}$$

- The rate of change of semi-major axis is given by

$$\boxed{\frac{da}{dt} = \frac{2e\sin\nu}{n\sqrt{1-e^2}} f_R + \frac{2a\sqrt{1-e^2}}{nr} f_T}$$

Affected by IN-PLANE perturbations ONLY

GVE: Eccentricity

- The specific angular momentum is not constant due to *non-radial* forces

$$\frac{d\vec{h}}{dt} = \frac{d}{dt}(\vec{r} \text{ x } \vec{v}) = \vec{r} \text{ x } \vec{f} = -r f_N \vec{T} + r f_T \vec{N} = h\dot{\beta}\vec{T} + \dot{h}\vec{N}$$

Total change normal to position vector

Angular change in angular momentum

- We can differentiate the expression of the eccentricity as a function of *h* and *a*

$$e = \sqrt{1 - \frac{h^2}{\mu a}} \Rightarrow \frac{de}{dt} = -\frac{h}{\mu a e}\dot{h} + \frac{h^2}{2\mu a^2 e}\dot{a}$$

- After substitution the rate of change of the eccentricity is given by

$$\boxed{\frac{de}{dt} = \frac{\sqrt{1-e^2}\sin\nu}{na} f_R + \frac{\sqrt{1-e^2}}{na^2 e}\left(\frac{a^2(1-e^2)}{r} - r\right) f_T}$$

Affected by IN-PLANE perturbations ONLY

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GVE: Inclination

- The inclination is the angle between ***K*** and ***h***

$$\cos i = \frac{\vec{h}\cdot\vec{K}}{h} \Rightarrow -\sin i\,\frac{di}{dt} = \frac{1}{h^2}\left[h\frac{d}{dt}(\vec{h}\cdot\vec{K}) - \dot{h}(\vec{h}\cdot\vec{K})\right]$$

- Considering that ***K*** is constant

$$\frac{di}{dt} = \frac{1}{-h^2\sin i}\left[h\left(\dot{\vec{h}}\cdot\vec{K}\right) - \dot{h}(h\cos i)\right]$$

- Knowing the expression for the time derivatives of *h* and ***h***

$$\frac{di}{dt} = \frac{1}{h\sin i}\left[r f_T \cos i - \left(-r f_N \vec{T} + r f_T \vec{N}\right)\cdot\vec{K}\right] \quad \vec{T}\cdot\vec{K} = \sin i\cos(\omega+\nu)\,;\ \vec{N}\cdot\vec{K} = \cos i$$

- After substitution the rate of change of the inclination is given by

$$\boxed{\frac{di}{dt} = \frac{r\cos(\omega+\nu)}{na^2\sqrt{1-e^2}}f_N}$$

Affected by OUT-OF-PLANE perturbations ONLY

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GVE: RAAN

- The right ascension of the ascending node is the angle between ***I*** and ***n***

$$\cos\Omega = \vec{n}\cdot\vec{I} = \frac{\left(\vec{K}\times\vec{h}\right)\cdot\vec{I}}{\left\|\vec{K}\times\vec{h}\right\|} \Rightarrow -\sin\Omega\frac{d\Omega}{dt} = \frac{1}{\left\|\vec{K}\times\vec{h}\right\|^{2}}\left[\left\|\vec{K}\times\vec{h}\right\|\frac{d}{dt}\left(\vec{K}\times\vec{h}\cdot\vec{I}\right) - \dot{\left\|\vec{K}\times\vec{h}\right\|}\left(\vec{K}\times\vec{h}\cdot\vec{I}\right)\right]$$

- The quantities between parenthesis can be expressed as

$$\left\|\vec{K}\times\vec{h}\right\| = h\sin i\ ;\ \frac{d}{dt}\left(\vec{K}\times\vec{h}\cdot\vec{I}\right) = \vec{K}\times\dot{\vec{h}}\cdot\vec{I} = \vec{K}\times\left(-rf_N\vec{T} + rf_T\vec{N}\right)\cdot\vec{I}$$

- Where the products between unit vectors are given by

$$\vec{K}\times\vec{T}\cdot\vec{I} = -\vec{J}\times\vec{T} = \sin\Omega\sin(\omega+\nu) - \cos\Omega\cos(\omega+\nu)\cos i\ ;\ \vec{K}\times\vec{N}\cdot\vec{I} = -\vec{J}\times\vec{N} = \cos\Omega\sin i$$

- After substitution the rate of change of RAAN is given by

$$\boxed{\frac{d\Omega}{dt} = \frac{r\sin(\omega+\nu)}{na^{2}\sqrt{1-e^{2}}\sin i}f_N}$$

Affected by OUT-OF-PLANE perturbations ONLY

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GVE: Argument of Periapsis

d***r***/dt caused by perturbations vanish due to osculation

- The argument of latitude ($\omega + \nu$) is the angle between ***r*** and ***n***

$$\cos(\omega+\nu) = \frac{\vec{r}\cdot\vec{K}\times\vec{h}}{\left\|\vec{r}\cdot\vec{K}\times\vec{h}\right\|} \Rightarrow r\cos(\omega+\nu) = \frac{\vec{r}\cdot\vec{K}\times\vec{h}}{\left\|\vec{K}\times\vec{h}\right\|}$$

- Differentiating w.r.t. time the terms which perturbations affect (e, ν, h) yields

$$-r\sin(\omega+\nu)\left(\frac{d\omega}{dt}+\frac{d\nu}{dt}\right) = \frac{1}{\left\|\vec{K}\times\vec{h}\right\|^2}\left[\left\|\vec{K}\times\vec{h}\right\|\left(\vec{r}\cdot\vec{K}\times\dot{\vec{h}}\right) - \dot{\left\|\vec{K}\times\vec{h}\right\|}\left(\vec{r}\cdot\vec{K}\times\vec{h}\right)\right]$$

- The rate of change of the true anomaly caused by perturbations only is

$$r = \frac{h^2/\mu}{1+e\cos\nu} \Rightarrow \frac{d\nu}{dt} = \frac{\sqrt{1-e^2}\cos\nu}{nae}f_R - \frac{\sqrt{1-e^2}}{nae}\frac{2+e\cos\nu}{1+e\cos\nu}f_T$$

- After substitution and many simplifications the rate of change of ω is

$$\boxed{\frac{d\omega}{dt} = -\frac{\sqrt{1-e^2}\cos\nu}{nae}f_R + \frac{\sqrt{1-e^2}}{nae}\frac{2+e\cos\nu}{1+e\cos\nu}\sin\nu f_T - \frac{r\cot i\sin(\omega+\nu)}{na^2\sqrt{1-e^2}}f_N}$$

Affected by perturbations in ALL directions

GVE: Mean Anomaly

- The mean anomaly is defined through the Kepler's equation

$$M = M_0 + n(t - t_0) = E - e\sin E$$

- Which can be differentiated w.r.t. time to find effect of pertubations

$$\frac{dM}{dt} = \frac{dE}{dt} - \frac{de}{dt}\sin E - \frac{dE}{dt} e\cos E$$

- The missing time derivatives are given by the relationship between *E* and ν

$$E \leftrightarrow \nu \;\Rightarrow \frac{dE}{dt}\,, \cos E\,, \sin E$$

- Substitution leads to the rate of change of the mean anomaly (here total)

$$\boxed{\frac{dM}{dt} = n - \frac{1}{na}\left(\frac{2r}{a} - \frac{1-e^2}{e}\cos\nu\right) f_R - \frac{1-e^2}{nae}\left(1 + \frac{r}{a(1-e^2)}\right)\sin\nu f_T}$$

Affected by IN-PLANE perturbations ONLY (as true anomaly)

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GVE: Circular Orbits

- We have presented the 6 VOP relationships which provide instantaneous effects of forces on osculating orbital elements
- The equations are valid for $0<e<1$ (i.e., ellipses only), since $(1-e^2)^{0.5}$ and e appear as denominators
- Equatorial orbits cause problems as well, since $\sin i$ appear as denominator in the rate of change of i and ω
- For (near-)circular orbits, the GVE can be expressed in terms of the argument of latitude u and eccentricity vector $\boldsymbol{e}$ avoiding most singularities:

$$\frac{da}{dt} = \frac{2}{n} f_T \qquad \frac{di}{dt} = \frac{\cos u}{na} f_N$$

$$\frac{de_x}{dt} = \frac{1}{na}(\sin u \, f_R + 2\cos u \, f_T) \qquad \frac{d\Omega}{dt} = \frac{\sin u}{na \sin i} f_N$$

$$\frac{de_y}{dt} = \frac{1}{na}(-\cos u \, f_R + 2\sin u \, f_T) \qquad \frac{du}{dt} = \underbrace{n + \frac{1}{na}(-2\, f_R - \sin u \cot i \, f_N)}_{\text{Total}}$$

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GVE: Impulsive Maneuvers

- For impulsive maneuvers (instantaneous change of velocity), we can integrate the GVE over the impulse

$$\Delta\vec{v} = \begin{pmatrix} \Delta v_R \\ \Delta v_T \\ \Delta v_N \end{pmatrix} = \int_{t_M^-}^{t_M^+} \vec{f}\, dt \approx \vec{f}\Delta t$$

- To derive the instantaneous change of orbital elements caused by a maneuver at $u_M = u(t_M)$, e.g. for (near-)circular orbits

$$\Delta a = a^+ - a^- = \frac{2}{n}\Delta v_T \qquad \Delta i = \frac{\cos u_M}{na}\Delta v_N$$

$$\Delta e_x = \frac{1}{na}(\sin u_M\, \Delta v_R + 2\cos u_M\, \Delta v_T) \qquad \Delta\Omega = \frac{\sin u_M}{na\sin i}\Delta v_N$$

$$\Delta e_y = \frac{1}{na}(-\cos u_M\, \Delta v_R + 2\sin u_M\, \Delta v_T) \qquad \Delta u = \frac{1}{na}(-2\Delta v_R - \sin u_M \cot i\, \Delta v_N)$$

Knowing the desired Δ correction of orbital elements
we can compute maneuver size and location !

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Backup