AA 279 C – SPACECRAFT ADCS: LECTURE 4

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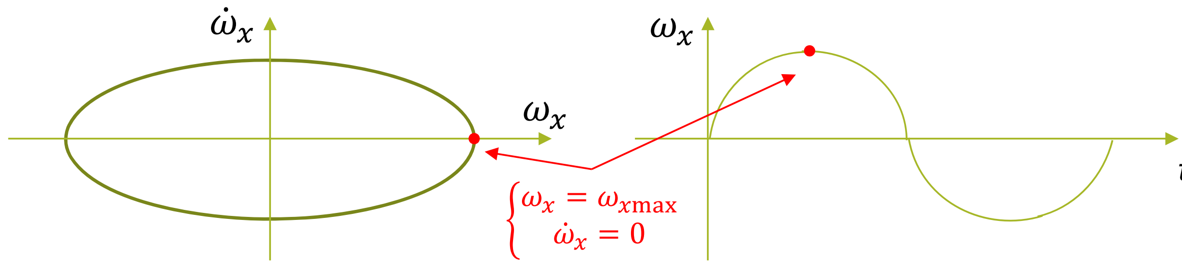
Torque-Free General Satellite (1) (Non Symmetric)

- The polhode represents an analytical solution of the periodic trajectory in the angular velocity vector space $\omega_x, \omega_y, \omega_z$
- It is possible to derive an analytical solution in the so-called phase planes which have coordinates $(\omega_x, \dot{\omega}_x)$, $(\omega_y, \dot{\omega}_y)$ $(\omega_z, \dot{\omega}_z)$ respectively



- The solution is found by comparing Euler equations and conservation laws

$$\times 3 \begin{cases} I_i\dot{\omega}_i + (I_j - I_k)\omega_j\omega_k = 0 \\ h^2 = \sum (I_i\omega_i)^2 \\ 2T = \sum I_i(\omega_i)^2 \end{cases} \Rightarrow \begin{cases} I_i\ddot{\omega}_i + (I_j - I_k)(\dot{\omega}_j\omega_k + \dot{\omega}_k\omega_j) & \times 3 \\ h^2 - 2TI_i = (\omega_k)^2(I_k^2 - I_iI_k) + (\omega_j)^2(I_j^2 - I_iI_j) & \times 3 \end{cases} \Rightarrow$$

$$\begin{cases} I_i\ddot{\omega}_i + (I_j - I_k)\left(\frac{I_i - I_k}{I_j}(\omega_j)^2\omega_i + \frac{I_j - I_i}{I_k}(\omega_k)^2\omega_i\right) = 0 & \times 3 \\ 2T(I_j - I_k) = (\omega_i)^2(I_jI_i - I_kI_i) + (\omega_jI_j)^2 - (\omega_kI_k)^2 + \left((\omega_k)^2 - (\omega_j)^2\right)I_kI_j & \times 3 \end{cases}$$

Torque-Free General Satellite (2) (Non Symmetric)

Coefficients known from mass distribution

Coefficients known from initial conditions

$$\text{x3} \quad \begin{cases} I_i \ddot{\omega}_i + (I_j - I_k)\omega_i A_i \left((\omega_j)^2 + (\omega_k)^2 \right) = 0 \\ \left((\omega_j)^2 + (\omega_k)^2 \right) = f_i(\omega_i^2, B_i) \end{cases} \quad \text{x3} \quad \Rightarrow \ddot{\omega}_i + \omega_i P_i + \omega_i^3 Q_i = 0 \Rightarrow$$

$$\text{x3} \quad \boxed{\dot{\omega}_i^2 + \omega_i^2 \left(P_i + \frac{1}{2} Q_i \omega_i^2 \right) = K_i}$$

Similar to a conic section in the phase plane, shape depends mainly on *P* and *Q*

- Although mathematically the conic section could be an hyperbola (open line) for $P_i + \frac{1}{2} Q_i \omega_i^2 < 0$, this is not physically possible because the polhode prescribes max and min bounds for the angular velocity
- In reality the argument $P_i + \frac{1}{2} Q_i \omega_i^2 > 0$ and the angular velocity is kept between minimum and maximum values constrained by *P* and *Q*

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Attitude Parameterization

- We seek the coordinate transformation necessary to overlap a triad fixed to the satellite ***x,y,z*** with an inertial reference triad ***1,2,3***
- Five possible representations are available

Representation	Singularities	Trigonometric functions	Redundant parameters	Notes
Direction cosine matrix	NO	NO	6	3x3 Matrix
Euler axis and angle	YES	YES	1	Physical interpretation
Quaternions	NO	NO	1	4x1 Vector
Gibbs vector	YES	NO	0	Manipulation of quaternions
Euler angles	YES	YES	0	12 possible combinations

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Direction Cosine Matrix (Mother of all transformations)

- Elements of the direction cosine matrix are the angles between the unit vectors of the two right-handed triads

$$\vec{v}_{xyz} = \vec{A}\vec{v}_{123} \;\; ; \;\; \vec{v}_{123} = \vec{A}^t\vec{v}_{xyz} \;\; ; \;\; \vec{A} = \begin{bmatrix} \vec{x}\cdot\vec{1} & \vec{x}\cdot\vec{2} & \vec{x}\cdot\vec{3} \\ \vec{y}\cdot\vec{1} & \vec{y}\cdot\vec{2} & \vec{y}\cdot\vec{3} \\ \vec{z}\cdot\vec{1} & \vec{z}\cdot\vec{2} & \vec{z}\cdot\vec{3} \end{bmatrix}$$

- Since the unit vectors refer to right-handed triads the direction cosine matrix is orthogonal, which provides the following 6 constraint equations

$$\vec{A}^t = \vec{A}^{-1} \;\; ; \;\; \vec{A}^t\vec{A} = I$$

- The rotation preserves the magnitude of vectors and the angles between vectors
  - Number of parameters: 9
  - Number of independent parameters: 3
- Terminology
  - Direct transformation: computing ***A*** from arbitrary parameterization
  - Inverse transformation: computing arbitrary parameterization from ***A***

Euler Axis and Angle (Commanding slew maneuvers)

- Coordinate transformation is done though a single rotation about an axis ***e*** of an angle φ for a total of 4 parameters
- The unit vector ***e*** is the eigenvector associated to the unitary eigenvalue of the direction cosine matrix

$$\vec{e} \quad = \vec{A}\vec{e} \quad ; \quad \vec{e}_{123} = \vec{e}_{xyz}$$

Trigonometric functions

- Direct and inverse transformations

The trace of ***A*** is invariant for a rotation φ about an arbitrary axis

$$\vec{A} = \cos\varphi \vec{I} + (1 - \cos\varphi)\vec{e}\vec{e}^t - \sin\varphi[\vec{e}\mathrm{x}]$$

$$\begin{cases} e_1 = (A_{23} - A_{32})/2\sin\varphi \\ e_2 = (A_{31} - A_{13})/2\sin\varphi \\ e_3 = (A_{12} - A_{21})/2\sin\varphi \end{cases} ; \quad \varphi = \arccos\left[\frac{1}{2}\left(\mathrm{tr}\left(\vec{A}\right) - 1\right)\right]$$

- Singularity for sinφ = 0, ambiguity of sign ($\pm\varphi$), subsequent rotations cannot be combined directly $\vec{e}'', \vec{\varphi}'' \neq f\left(\vec{e} \quad , \vec{\varphi} \quad , \vec{e}', \vec{\varphi}'\right)$
  - Number of parameters: 4
  - Number of independent parameters: 3

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Quaternions (On-board navigation)

- Also called Euler symmetric parameters, defined by a 4x1 unit vector

$$\begin{cases} q_1 = e_1 \sin(\varphi/2) \\ q_2 = e_2 \sin(\varphi/2) \\ q_3 = e_3 \sin(\varphi/2) \\ q_4 = \cos(\varphi/2) \end{cases} ; \ \vec{q}_t = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \vec{q} \\ q_4 \end{Bmatrix}$$

- Direct and inverse transformations

$$\vec{A} = (q_4^2 - \vec{q}^{\,2})\vec{I} + 2\vec{q}\vec{q}^{\,t} - 2q_4[\vec{q}\mathrm{x}]$$

No trigonometric functions

$$\begin{cases} q_1 = (A_{23} - A_{32})/4q_4 \\ q_2 = (A_{31} - A_{13})/4q_4 \\ q_3 = (A_{12} - A_{21})/4q_4 \end{cases} ; \quad q_4 = \pm\frac{1}{2}(1 + A_{11} + A_{22} + A_{33})^{1/2}$$

One of 4 possible ways to compute $\boldsymbol{q}_t$, we typically choose one that maximize q_4

- Changing sign of $\boldsymbol{q}_t$ does not change rotation, changing sign of only $\boldsymbol{q}$ provides transpose, subsequent rotations can be combined directly and efficiently
  - Number of parameters: 4
  - Number of independent parameters: 3

$$\vec{A}(\vec{q}_t{}'') = \vec{A}(\vec{q}_t{}')\vec{A}(\vec{q}_t)$$ 27 products

$$(\vec{q}_t{}'') = (\vec{q}_t{}')^{\circ}(\vec{q}_t)$$ 16 products

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Gibbs Vector
(Not widely used because of singularity)

- Non-normalized Euler symmetric parameters, defined by a 3x1 vector

$$\begin{cases} g_1 = q_1/q_4 = e_1 \tan(\varphi/2) \\ g_2 = q_2/q_4 = e_2 \tan(\varphi/2) \\ g_3 = q_3/q_4 = e_3 \tan(\varphi/2) \end{cases}$$

No trigonometric functions

No sign ambiguity

- Direct and inverse transformations

$$\vec{A} = \{(1 - \vec{g}^2)\vec{I} + 2\vec{g}\vec{g}^t - 2[\vec{g}\mathrm{x}]\}/(1 + \vec{g}^2)$$

$$\begin{cases} q_1 = (A_{23} - A_{32})/(1 + A_{11} + A_{22} + A_{33}) \\ q_2 = (A_{31} - A_{13})/(1 + A_{11} + A_{22} + A_{33}) \\ q_3 = (A_{12} - A_{21})/(1 + A_{11} + A_{22} + A_{33}) \end{cases}$$

- The inverse transformation is singular for φ odd multiple of 180°, subsequent rotations can be combined directly through a complex product law
  - Number of parameters: 3
  - Number of independent parameters: 3

$$(\vec{g}'') = (\vec{g} + \vec{g}' - \vec{g}' \mathrm{x}\, \vec{g})/(1 - \vec{g} \cdot \vec{g}')$$

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Euler Angles
(Useful for small angles approximation)

- Represent 3 consecutive rotations necessary to overlap triads ***x,y,z*** and ***1,2,3***
- A total of 12 possible combinations are possible, respectively with 2 or 3 different indexes (313,323,212,232,121,131,123,132,213,231,321,312)

yaw, roll, pitch

- Direct and inverse transformations (313 example)

$$\vec{A}_{313}(\theta,\varphi,\psi)=\vec{A}_3(\psi)\vec{A}_1(\theta)\vec{A}_3(\varphi)=\begin{bmatrix} c\psi c\varphi - c\theta s\psi s\varphi & c\psi s\varphi + c\theta s\psi c\varphi & s\theta s\psi \\ -s\psi c\varphi - c\theta c\psi s\varphi & -s\psi s\varphi + c\theta c\psi c\varphi & s\theta c\psi \\ s\theta s\varphi & -s\theta c\varphi & c\theta \end{bmatrix}$$

$$\begin{cases} \theta = \arccos(A_{33}) \\ \varphi = -\arctan(A_{31}/A_{32}) \\ \psi = \arctan(A_{13}/A_{23}) \end{cases}$$

Each Euler angle sequence has a singularity

- The inverse transformation is singular for sinθ= 0, for small angles the rotation matrix for different indexes become simple (not dependent on order of rotations)
  - Number of parameters: 3
  - Number of independent parameters: 3

$$\vec{A}_{312}(\varphi,\theta,\psi)\sim\begin{bmatrix} 1 & \varphi & -\psi \\ -\varphi & 1 & \theta \\ \psi & -\theta & 1 \end{bmatrix}$$

Backup

Example of Direction Cosine Matrix

- Let the body frames of two spacecraft *B* and *F* be defined relative to the inertial reference frame *N* by the unit vectors

$$\hat{\boldsymbol{b}}_1 = (0,1,0)^T \qquad \hat{\boldsymbol{b}}_2 = (1,0,0)^T \qquad \hat{\boldsymbol{b}}_3 = (0,0,-1)^T$$

$$\hat{\boldsymbol{f}}_1 = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right)^T \qquad \hat{\boldsymbol{f}}_2 = (0,0,1)^T \qquad \hat{\boldsymbol{f}}_3 = \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}, 0\right)^T$$

- The matrix [***BN***] maps vectors written in the *N* frame into vectors written in the *B* frame, while [***FB***] maps vectors in the *B* frame into vectors in the *F* frame

$$[FB]_{ij} = \cos\alpha_{ij} = \hat{\boldsymbol{f}}_i \cdot \hat{\boldsymbol{b}}_j \qquad [BN]_{ij} = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j \qquad [FN]_{ij} = \hat{\boldsymbol{f}}_i \cdot \hat{\boldsymbol{n}}_j$$

- It is not necessary to find the angles between set of vectors, instead the inner products provide

$$[FB] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \qquad [BN] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad [FN] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

Example of Euler Angles

- Let the body frames of two spacecraft *B* and *F* be defined relative to the inertial reference frame *N* by the asymmetric (321) Euler angles

$$\boldsymbol{\theta}_{\mathcal{B}} = (30, -45, 60)^T \text{ and } \boldsymbol{\theta}_{\mathcal{F}} = (10, 25, -15)^T \text{ degrees}$$

- We seek the relative orientation of *B* relative to *F*, i.e. [***BF***] which maps vectors in the *F* frame into vectors in the *B* frame, in terms of (321) Euler angles. We need to use the direct transformations first

$$[BN] = \begin{bmatrix} 0.612372 & 0.353553 & 0.707107 \\ -0.78033 & 0.126826 & 0.612372 \\ 0.126826 & -0.926777 & 0.353553 \end{bmatrix} \qquad [FN] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

$$[BF] = [BN][FN]^T = \begin{bmatrix} 0.303372 & -0.0049418 & 0.952859 \\ -0.935315 & 0.1895340 & 0.298769 \\ -0.182075 & -0.9818620 & 0.052877 \end{bmatrix}$$

- And the inverse transformation next

$$\psi = \tan^{-1}\left(\frac{-0.0049418}{0.303372}\right) = 0.933242\ deg \qquad \theta = -\sin^{-1}(0.952859) = -1.26252\ deg$$

$$\phi = \tan^{-1}\left(\frac{0.298769}{0.052877}\right) = \boxed{-57.6097\ deg}$$

B differs from *F* by a rotation of -57.6° about first axis

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Example of Euler Axis and Angle

- Let the body frame B relative to N be given by the asymmetric (321) Euler angles

$$\boldsymbol{\theta}_{\mathcal{B}} = (10, 25, -15)^T \text{ degrees} \qquad [BN] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

- The Euler rotation angle is found through the trace operator

$$\Phi = \cos^{-1}\left(\frac{1}{2}(0.892539 + 0.932257 + 0.875426 - 1)\right) = 31.7762^\circ$$

- The corresponding Euler rotation axis is given by

$$\hat{e} = \frac{1}{2\sin(31.7762^\circ)} \begin{pmatrix} -0.23457 - 0.325773 \\ 0.357073 - (-0.422618) \\ 0.157379 - (-0.275451) \end{pmatrix} = \begin{pmatrix} -0.532035 \\ 0.740302 \\ 0.410964 \end{pmatrix}$$

- A second Euler angle exists, $\Phi' = 31.7762^\circ - 360^\circ = -328.2238^\circ$, which provides two perfectly equivalent sets of Euler Axis and Angles

$$(\hat{\boldsymbol{e}}, \Phi) \text{ or } (\hat{\boldsymbol{e}}, \Phi')$$

Example of Quaternion (1)

- Let's use Stanley's method to find the Quaternion $\boldsymbol{\beta}$ of the direction cosine matrix

$$[BN] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

- The method uses the following equations

$$\beta_4^2 = \frac{1}{4}(1 + Trace[C])$$
$$\beta_1^2 = \frac{1}{4}(1 + 2C_{11} - Trace[C])$$
$$\beta_2^2 = \frac{1}{4}(1 + 2C_{22} - Trace[C])$$
$$\beta_3^2 = \frac{1}{4}(1 + 2C_{33} - Trace[C])$$

Select maximum component and compute others by division

$$\Rightarrow$$

$$\beta_4\beta_1 = (C_{23} - C_{32})/4$$
$$\beta_4\beta_2 = (C_{31} - C_{13})/4$$
$$\beta_4\beta_3 = (C_{12} - C_{21})/4$$
$$\beta_2\beta_3 = (C_{23} + C_{32})/4$$
$$\beta_3\beta_1 = (C_{31} + C_{13})/4$$
$$\beta_1\beta_2 = (C_{12} + C_{21})/4$$

- Substitution provides

$$\beta_4^2 = 0.925055 \quad \beta_1^2 = 0.021214$$
$$\beta_2^2 = 0.041073 \quad \beta_3^2 = 0.012657$$

$$\Rightarrow$$

$$\boldsymbol{\beta} = (\underset{4,}{0.961798}, \underbrace{\underset{1,}{-0.14565}, \underset{2,}{0.202665}, \underset{3}{0.112505}}_{\text{Vector part}})^T$$

Scalar part

Example of Quaternion (2)

- Let's combined successive rotations described by the Quaternions $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}''$ into a total rotation described by $\boldsymbol{\beta}$

$$[FB] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \qquad [BN] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad [FN] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

$$\boldsymbol{\beta}'' = \left(\frac{1}{2}\sqrt{\frac{\sqrt{3}}{2}+1}, -\frac{1}{2}\sqrt{\frac{\sqrt{3}}{2}+1}, \frac{-\sqrt{2}}{4\sqrt{2+\sqrt{3}}}, \frac{\sqrt{2}}{4\sqrt{2+\sqrt{3}}}\right)^T \quad \boldsymbol{\beta}' = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T$$

$\beta = ?$

- The quaternion product can be used to directly compute the composite

$$\begin{pmatrix} \beta_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta''_4 & -\beta''_1 & -\beta''_2 & -\beta''_3 \\ \beta''_1 & \beta''_4 & \beta''_3 & -\beta''_2 \\ \beta''_2 & -\beta''_3 & \beta''_4 & \beta''_1 \\ \beta''_3 & \beta''_2 & -\beta''_1 & \beta''_4 \end{bmatrix} \begin{pmatrix} \beta'_4 \\ \beta'_1 \\ \beta'_2 \\ \beta'_3 \end{pmatrix} = \frac{1}{2\sqrt{2}}\left(\sqrt{3}, \sqrt{3}, 1, 1\right)^T$$