AA 279 C – SPACECRAFT ADCS: LECTURE 15

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Generalized minimum time maneuver (1)

- If we are able to express our attitude dynamics through a linear system in state-space form

$$\dot{\vec{x}} = \vec{A}\vec{x} + \vec{B}\vec{u}$$

- A general optimum attitude control problem can be expressed through a functional that we want to minimize

$$J = \int_{t0}^{tf} G(\vec{x},\vec{u},t)dt \ ; \ G(\vec{x},\vec{u},t) = \begin{cases} |u| & \text{Minimum fuel} \\ 1 & \text{Minimum time} \end{cases}$$

- The minimum problem is constrained by the dynamics, thus we build the Hamiltonian

Lagrange multipliers or co-states

$$H = G(\vec{x},\vec{u},t) - \vec{p}^{t}\left[\vec{A}\vec{x} + \vec{B}\vec{u}\right]$$

- Let's try to reproduce the results from the previous lecture (minimum time about one axis), starting from the dynamics

$$\begin{Bmatrix} \ddot{\alpha} \\ \dot{\alpha} \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\alpha} \\ \alpha \end{Bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Generalized minimum time maneuver (2)

- Then building the Hamiltonian and using the Pontryagin's maximum (or minimum) principle

$$H = 1 - p_1 u - p_2 \dot{\alpha} \;\; ; \;\; \dot{\vec{p}} = -\vec{\nabla}_x H \;\rightarrow \begin{cases} \dot{p}_1 = p_2 \\ \dot{p}_2 = 0 \end{cases} \rightarrow \begin{cases} p_1 = c_1 t + c_2 \\ \quad p_2 = c_1 \end{cases}$$

- It is clear that minimizing H is equivalent to maximizing $-H$

$$\min_u(H) = \max_u(-H) = \max_u(-1 + p_1 u + p_2 \dot{\alpha})$$

- Thus the control input must be maximum and must have the same sign of p_1

$$u = u_{max}\operatorname{sign}(p_1) = u_{max}\operatorname{sign}(c_1 t + c_2)$$

- The change of sign happens at most 1 time, and the optimum solution corresponds to the maximum torque with only 1 commutation, which is the same result obtained previously (switching curve: parabola in phase space)
- This approach can be generalized to three-axis attitude control using an Euler angle, Euler axis attitude parameterization

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Generalized minimum effort maneuver

- If our goal is to minimize the effort (e.g., propellant consumption) the procedure can be repeated identically

$$-H = -|u| + p_1 u + p_2 \dot{\alpha} \ ; \ \dot{\vec{p}} = -\vec{\nabla}_x H \ \rightarrow \begin{cases} \dot{p}_1 = p_2 \\ \dot{p}_2 = 0 \end{cases} \rightarrow \begin{cases} p_1 = c_1 t + c_2 \\ p_2 = c_1 \end{cases}$$

- As before, we seek

$$\min_u(H) = \max_u(-H) = \max_u(-|u| + p_1 u + p_2 \dot{\alpha})$$

- If we apply a positive torque u>0, the minimum effort is given by

$$\begin{cases} p_1 > 1 \\ p_1 < 1 \end{cases} \Rightarrow \begin{cases} u > 0 \\ u = 0 \end{cases}$$

- If we apply a negative torque u<0, the minimum is

$$\begin{cases} p_1 < -1 \\ p_1 > -1 \end{cases} \Rightarrow \begin{cases} u < 0 \\ u = 0 \end{cases}$$

Final result for control law

$$u = \begin{cases} |p_1| > 1 \ \rightarrow u = u_{max}\mathrm{sign}(p_1) \\ |p_1| < 1 \ \rightarrow u = 0 \end{cases}$$

Tethers

- Space tethers are long cables with masses at both ends which are designed to act as a rigid body
- Tethers have been conceived for several purposes but rarely flown, they can be used to perform attitude maneuvers or as space elevators
- Main techniques under development/study
  - Electrodynamic tethers
    - Conducting cable can generate forces through interaction with a planetary magnetic field (Lorentz force)
  - Momentum exchange tethers
    - Capture arriving spacecraft and release it to a different orbit
  - Tethered formation-flying
    - Accurately maintain a set distance

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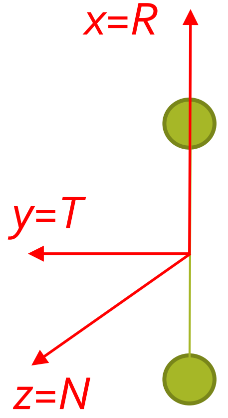
Tethers Dynamics (from Euler equations)

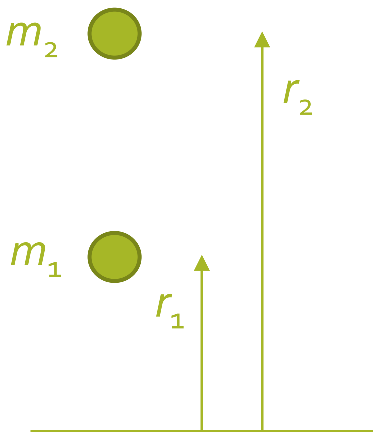
- The tension forces which are induced by the tether with masses at the ends are such that we can treat the tether as a rigid body
  - $I_x \rightarrow 0$ (Depends on masses)
  - $I_y = I_z$
- Since the x dimension is much larger than the others, the gravity gradient torque becomes relevant, and the linearized Euler equations become

$$\begin{cases} \ddot{\alpha}_y = -4n^2\alpha_y & \text{roll} \\ \ddot{\alpha}_z = -3n^2\alpha_z & \text{pitch} \end{cases}$$



- The yaw equation degenerates and is not considered, whereas roll and pitch are characterized by the usual eigenvalues with frequencies
  - $2n$ in the orbital plane (roll)
  - $\sqrt{3}n$ out of the orbital plane (pitch)

Tethers Dynamics (from Force diagram, 1)



- The two masses orbit in a synchronous manner such that the cable is always aligned with the radial direction, at the equilibrium

$$F_{g1} + F_{g2} + F_{c1} + F_{c2} = 0$$

gravity $\quad F_{g1,2} = -\frac{G m_T m_{1,2}}{r_{1,2}^2}\ ;\ F_{c1,2} = m_{1,2} n^2 r_{1,2}$ $\quad$ centrifugal

- The degrees of freedom are $r_2 - r_1$, n
- Given the masses and the distance from the Earth, the only unknown is the angular velocity n which guarantees the equilibrium of the system

$$-G m_T \left(\frac{m_1}{r_1^2} + \frac{m_2}{r_2^2}\right) + n^2 (m_1 r_1 + m_2 r_2) = 0 \Rightarrow n^2 = \frac{G m_T \left(\frac{m_1}{r_1^2} + \frac{m_2}{r_2^2}\right)}{m_1 r_1 + m_2 r_2}$$

- For each tether’s length, a unique angular velocity n provides zero total force (equilibrium), n corresponds to the mean motion of an intermediate circular orbit between r_2 and r_1

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Tethers Dynamics (from Force diagram, 2)

- The distance from Earth corresponding to the circular orbit with mean motion *n* is given by

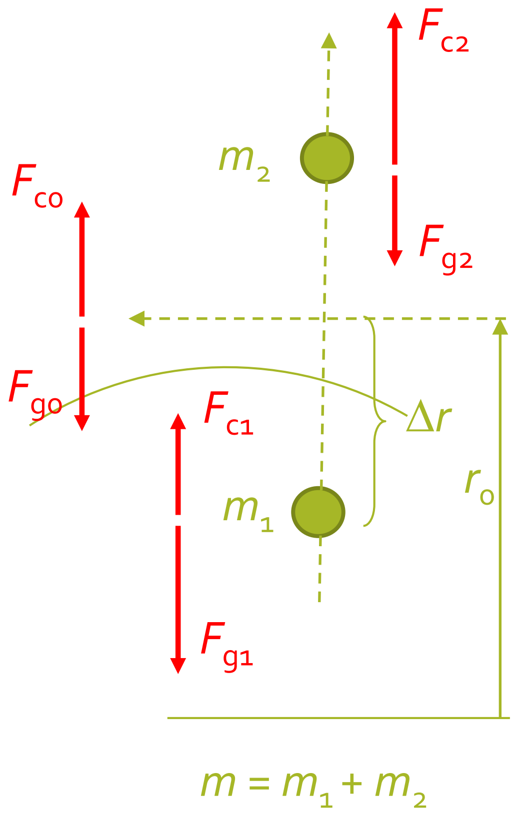
$$n^2 r_0 = \frac{G m_T}{r_0^2} \Rightarrow r_0^3 = \frac{G m_T}{n^2} \Rightarrow r_0^3 = \frac{m_1 r_1 + m_2 r_2}{\left(\frac{m_1}{r_1^2} + \frac{m_2}{r_2^2}\right)}$$

- The point at r_o is a zero-g point, where forces are equal, whereas at r_1 and r_2 prevail gravity and centrifugal forces respectively
- The system behaves as a rigid body because it is pulled by the masses, the generated tension can be computed through the linearization about r_o

$$T = \Delta F = \Delta F_g + \Delta F_c \sim \left(\frac{\partial F_g}{\partial r} + \frac{\partial F_c}{\partial r}\right) \Delta r = 3 m n^2 \Delta r$$

- Considering typical values

$$\begin{cases} m \sim 1000 \text{ kg} \\ \Delta r \sim 1 \text{ km} \\ n \sim 0.001\ 1/s \end{cases} \Rightarrow T \sim 3 \text{ N}\ ; \begin{cases} m \sim 1000 \text{ kg} \\ \Delta r \sim 100 \text{ km} \\ n \sim 0.001\ 1/s \end{cases} \Rightarrow T \sim 300 \text{ N}$$



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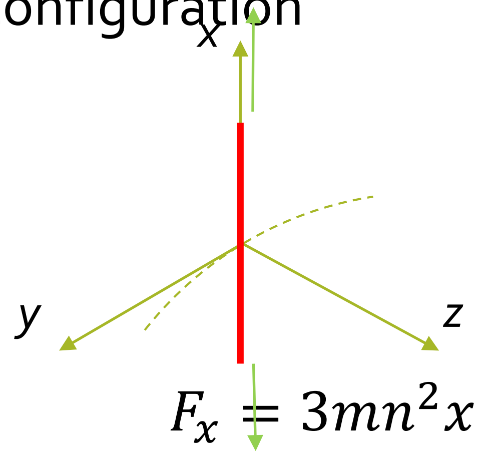
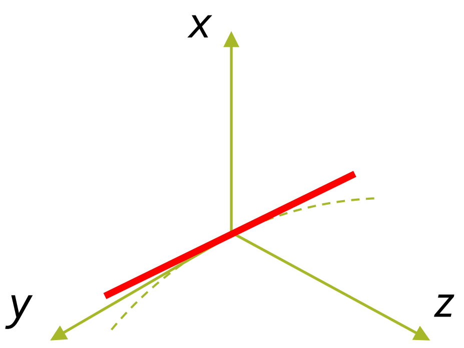
Tethers Dynamics (Linear Equations, 1)

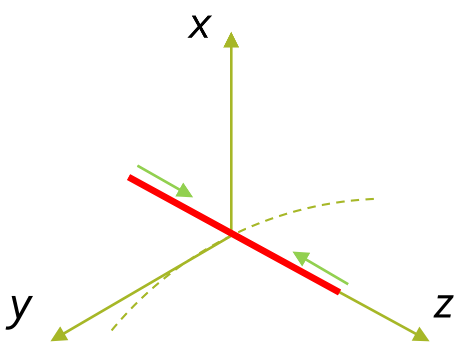
- We can generalize the approach and use the superposition of effects for linear systems to derive the equations of motion for a generic configuration



$$F_x = 3mn^2x$$

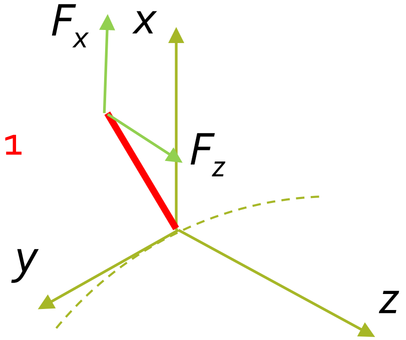


$$F_y = 0$$



$$F_z = -mn^2z$$

Generic position of 1 mass



$$\begin{cases} M_x = F_z y = -mn^2 zy \\ M_y = -F_z x + F_x z = 4mn^2 zx \\ M_z = -F_x y = -3mn^2 xy \end{cases}$$

Small perturbation of equilibrium through small angles

$x \rightarrow l$

$y \rightarrow l\, \alpha_z$

$z \rightarrow -l\, \alpha_y$

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Tethers Dynamics (Linear Equations, 2)

- We have computed the external torques acting on the rigid body for a small departure from the equilibrium (alignment with radial direction)
- Reconsidering the Euler equations and computing the inertia tensor

$$\begin{cases}(m_1 l_1^2 + m_2 l_2^2)\ddot{\alpha}_y = -4n^2(m_1 l_1^2 + m_2 l_2^2)\alpha_y \\ (m_1 l_1^2 + m_2 l_2^2)\ddot{\alpha}_z = -3n^2(m_1 l_1^2 + m_2 l_2^2)\alpha_z\end{cases} ; l = l_1 + l_2$$

- These equations are identical to the Euler equations with gravity gradient for a rigid body
- When the cable changes its length, the non-zero velocity taken in the rotating frame causes a Coriolis force

$$\dot{l} \neq 0 \ \rightarrow \ \vec{F}_{Coriolis} = -2m\vec{\omega} \times \vec{v}$$

Angular velocity of tether reference system

Relative velocity

- And a Coriolis torque

$$\vec{F}_{Coriolis} \neq 0 \ \rightarrow \ \vec{M}_{Coriolis} = -2m\vec{r} \times \vec{\omega} \times \vec{v} = -2m\begin{bmatrix} 0 \\ l\dot{l}\dot{\alpha}_y \\ l\dot{l}(n+\dot{\alpha}_z) \end{bmatrix} ; \ \vec{r} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} ; \ \vec{v} = \begin{pmatrix} \dot{l} \\ 0 \\ 0 \end{pmatrix} ; \ \vec{\omega} = \begin{pmatrix} 0 \\ \dot{\alpha}_y \\ n+\dot{\alpha}_z \end{pmatrix}$$

Tethers Dynamics (Linear Equations, 3)

- If one mass is much larger than the other, then its forces have negligible momentum arm because they are close to the zero-g point

$$l_2 = 0\ ;\ m_2 \gg m_1$$

- After inclusion of the Coriolis torque, the Euler equations simplify to

$$\begin{cases} \ddot{\alpha}_y + 2\dfrac{\dot{l}_1}{l_1}\dot{\alpha}_y + 4n^2\alpha_y = 0 \\ \ddot{\alpha}_z + 2\dfrac{\dot{l}_1}{l_1}\dot{\alpha}_z + 3n^2\alpha_z = -2\dfrac{\dot{l}_1}{l_1}n \end{cases}$$

Second order linear differential equations (non-homogeneous about cross-track)

- The roll equation represents an harmonic oscillator with damping given by the variation of length of the cable

Small mass gets closer to large mass $\quad \dot{l}_1 > 0\ ;\ l_1 < 0 \quad$ Forced oscillations due to negative damping: UNSTABLE

Small mass gets farer from large mass $\quad \dot{l}_1 < 0\ ;\ l_1 < 0 \quad$ Damped oscillations due to positive damping: STABLE

- It is only possible to achieve roll passive stability if the length of the cable is increased with the payload (small mass) below the carrier (large mass)

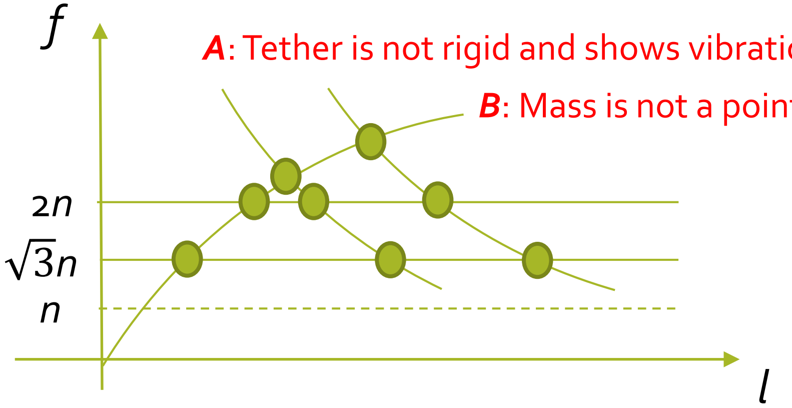
Tethers Dynamics (Linear Equations, 4)

- The forced unstable oscillations are dangerous if we want to recover a mass using a tether
- We need active control, for example by keeping the pitch angle constant during the complete maneuver, from the pitch equation we can impose

$$\frac{\dot{l}_1}{l_1} = -\frac{3}{2} n \alpha_z$$

The longer the tether, the larger the velocity needed to keep constant pitch

- The characteristic oscillations are at frequencies $2n$ and $\sqrt{3}n$, is there a danger of resonance with other modes? Are there other modes at the same frequency?



All these configurations are to be avoided

Resonance with Tethers

- The tether configuration represents the critical point in the gravity gradient stability chart and is subject to resonance due to two main phenomena
  - ***A***: *Tether is not rigid and shows vibration modes*
    - Longitudinal and transversal movements have frequencies which go with the inverse square of the tether length

$$f \propto \frac{1}{\sqrt{l}}$$

  - ***B***: *Small mass is not a point and is subject to oscillations about attach point*
    - This phenomenon can be modeled as a composite pendulum with frequency which is proportional to the square of the tether length

$$f = \frac{1}{2\pi}\sqrt{\frac{Td}{I}} \propto \sqrt{l}$$

Diameter of small mass

Cable's tension depends on length

Inertia of small mass

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Backup