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AA 279 A – Space Mechanics
Lecture 2: Notes

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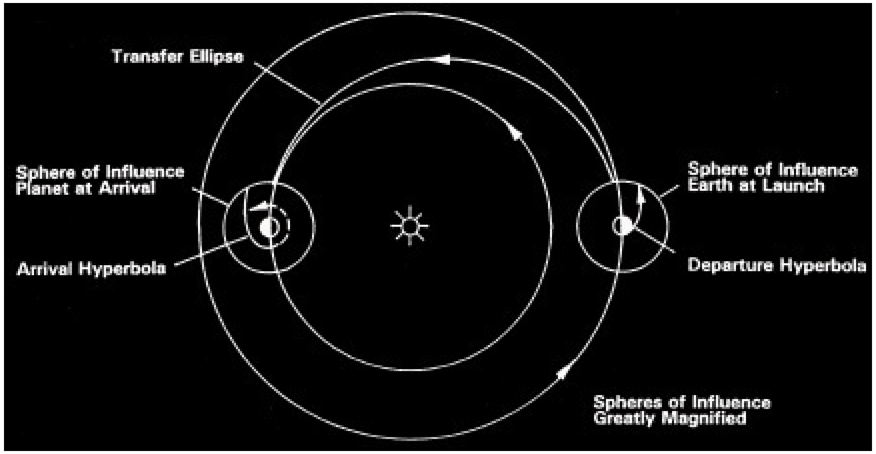

- Reading for next week
  - Bate 1.7-1.10, 4.1-4.3, 4.6.4
  - Montenbruck 2.2.1-2.2.2
  - Vallado 1.3.5-1.3.6, 2.2-2.2.3, 2.2.5, 2.3

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Gravitational Potential Energy (1)

Radially symmetric mass distribution

$$\vec{F}_{\mathrm{grav}} = -\vec{\nabla}\mathrm{V}(r)$$

- The gravity force field is conservative, i.e. its work done along a closed path is zero. This allows describing the force of gravity as the gradient of a scalar potential

$$\mathrm{V}(r) = \int_{r_{\mathrm{ref}}}^{r} -F_{\mathrm{grav}}dr =$$

$$= \int_{r_{\mathrm{ref}}}^{r} - \overbrace{-\frac{m\mu}{r^2}}^{\text{Newton IV.}} dr$$

- The gravitational potential represents the work required to raise a mass *m* from an arbitrary reference to a radius *r* in the gravity field

- In astrodynamics the arbitrary constant of the gravitational potential is chosen to be C = 0 ($r_{\mathrm{ref}} = \infty$) such that V < 0 and V($\infty$) = 0

$$\mathrm{V}(r) = \left[-\frac{m\mu}{r}\right]_{r_{\mathrm{ref}}}^{r} =$$

$$= -\frac{m\mu}{r} + \frac{m\mu}{r_{\mathrm{ref}}}$$

Arbitrary constant C

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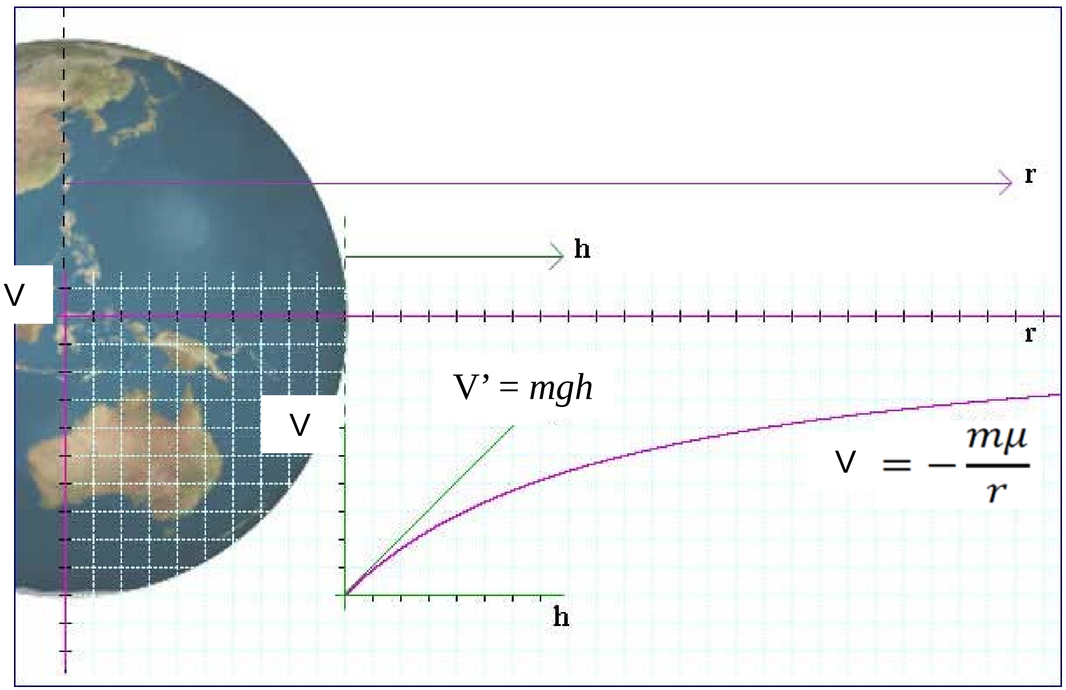
Gravitational Potential Energy (2)

- $V(r)$ is negative everywhere and increases with r
- $1/r$ potential gives $1/r^2$ field force
- If we "reset" reference so V = 0 at surface of central body, the slope of the potential becomes V' = *mgh*

$$V = -\frac{m\mu}{r}$$



Specific Mechanical Energy

Inertial velocity

- The specific mechanical energy is defined as kinetic plus potential energy per unit mass

$$\mathcal{E} = \frac{1}{m}\left(\frac{1}{2}mv^2 - \frac{m\mu}{r}\right)$$

V

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$$

Specific Mechanical Energy in Gravity Field

- Remember that potential energy concept is only valid for conservative force fields (e.g., gravity, spring, Coulomb)

- Since gravity is conservative, an object moving in this field does not lose or gain mechanical energy, hence $\mathcal{E}$ = constant

This provides a relationship between v and r magnitudes

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Specific Angular Momentum

Inertial velocity

- The specific angular momentum is defined as the linear angular momentum per unit mass (with respect to the central body)

$$\vec{h} = \frac{1}{m}(\vec{r} \times m\vec{v})$$

Momentum

$$\boxed{\vec{h} = \vec{r} \times \vec{v}}$$

Specific Angular Momentum

- The derivative with respect to time of the specific angular momentum equals

$$\frac{d\vec{h}}{dt} = \underbrace{\frac{d\vec{r}}{dt} \times \vec{v}}_{0} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{r} \times \vec{a}$$

- Since gravity pulls towards central body (point mass), the gravity force $\boldsymbol{F}_g$ is anti-parallel to the relative position vector $\boldsymbol{r}$, thus no torques are generated ($\boldsymbol{r} \times \boldsymbol{a} = 0$), hence $\boldsymbol{h}$ = constant

→ This provides a relationship between v and r directions

Two Connections for All Conic Sections

- Angular momentum and semi-parameter completely determine each other

$$h = \sqrt{\mu p} \; ; p = \frac{h^2}{\mu}$$

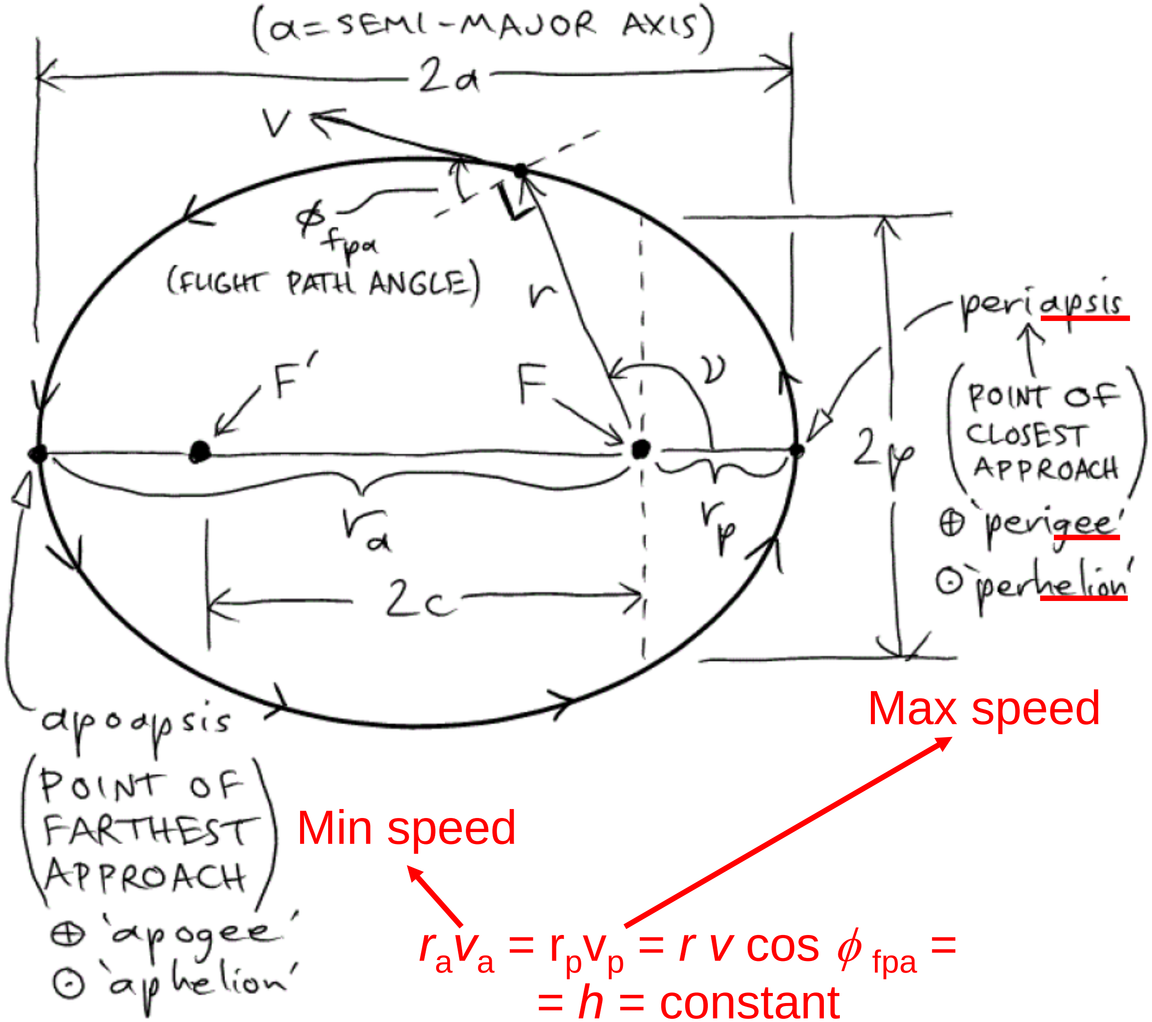
- Mechanical energy and semi-major axis completely determine each other, so that energy is solely determined by orbit "size"

$$\mathcal{E} = -\frac{\mu}{2a} \; ; a = -\frac{\mu}{2\mathcal{E}}$$

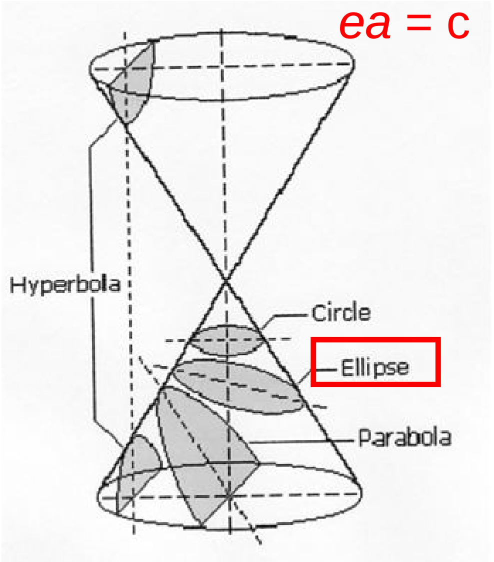
Semi-major axis of conic section

- Various combinations of only two quantities (e.g., *h*&$\mathcal{E}$, *p*&*a*, *h*&*a*, *p*&$\mathcal{E}$, etc.) can tell a lot about the orbit shape. Simple energy calculations can be used to derive orbit shape…

Conic Sections: Ellipse



$r_a v_a = r_p v_p = r\, v \cos \phi_{fpa} =$
$= h =$ constant

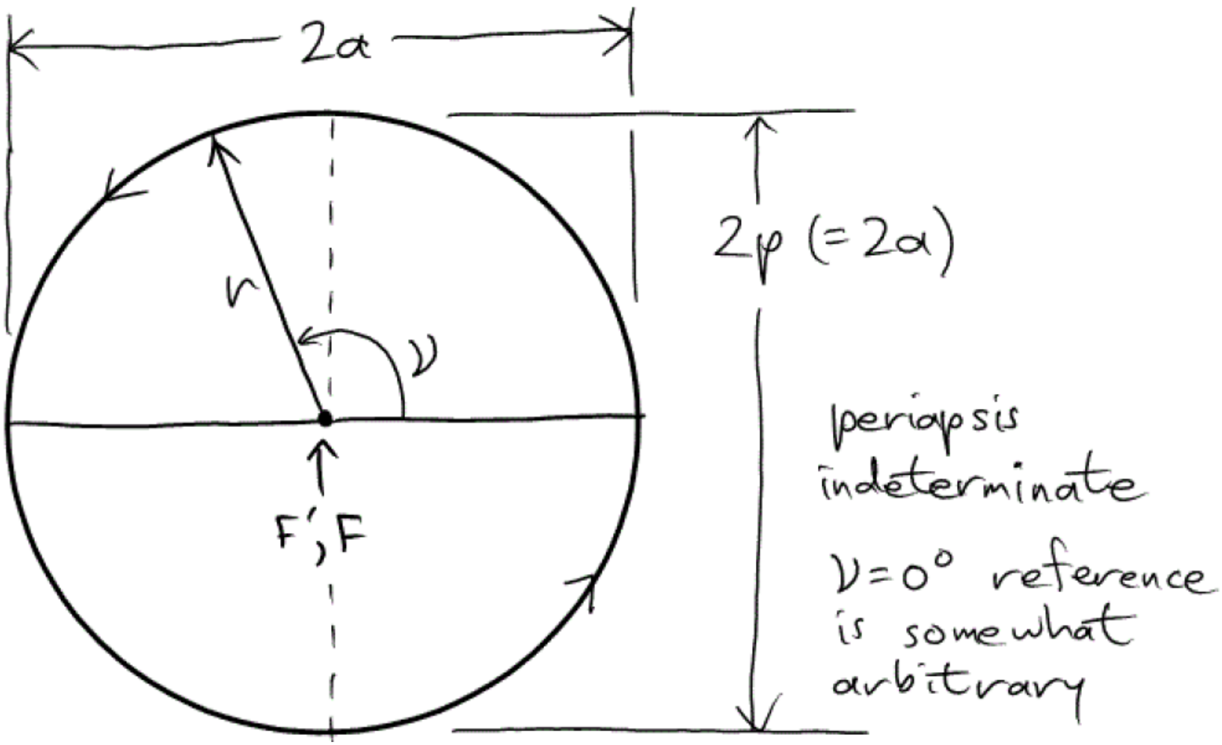


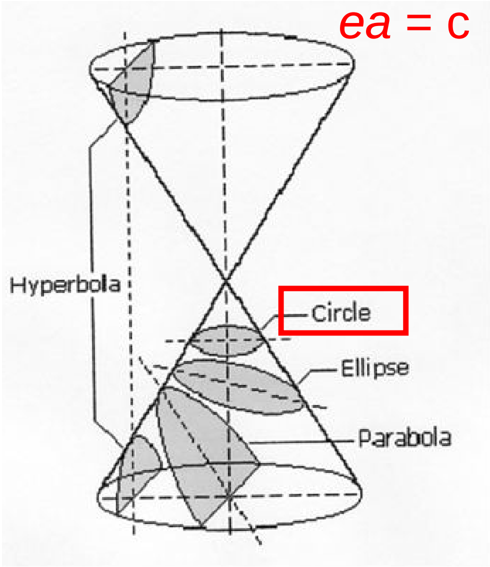
$$0 < e < 1$$

$$\mathcal{E} < 0$$

$$T = 2\pi \sqrt{\frac{a^3}{\mu}}$$

Conic Sections: Circle





$$e = 0$$

$$\mathcal{E} < 0$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Constant speed

$r_a v_a = r_p v_p = r\,v$
$= h =$ constant

$$v = \sqrt{\frac{\mu}{r}}$$

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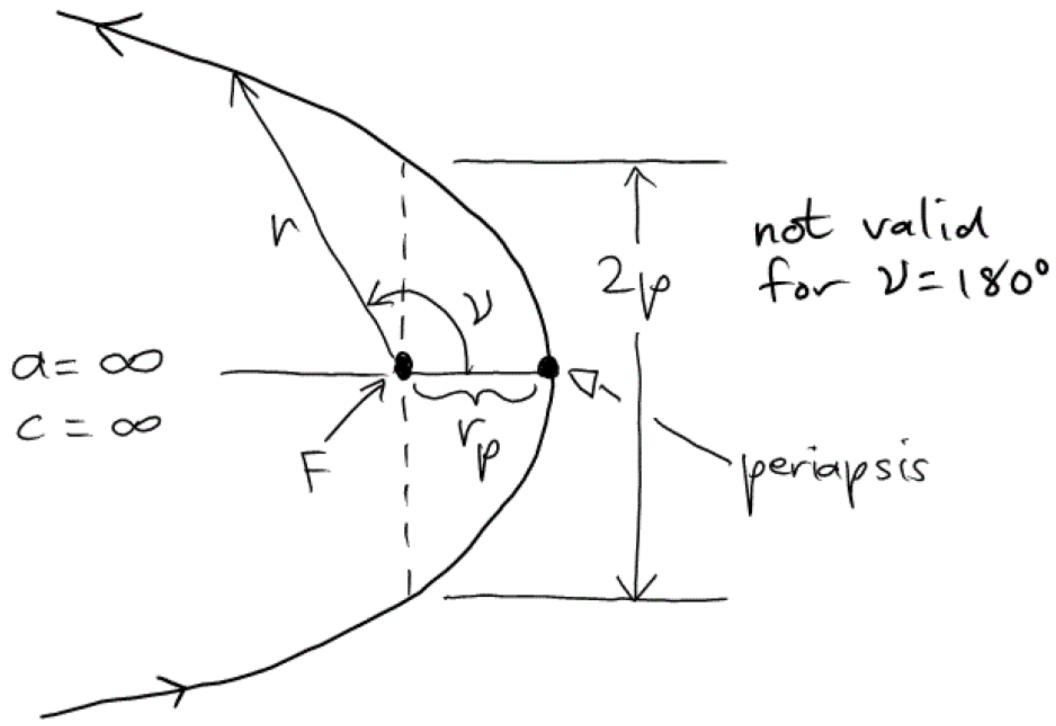
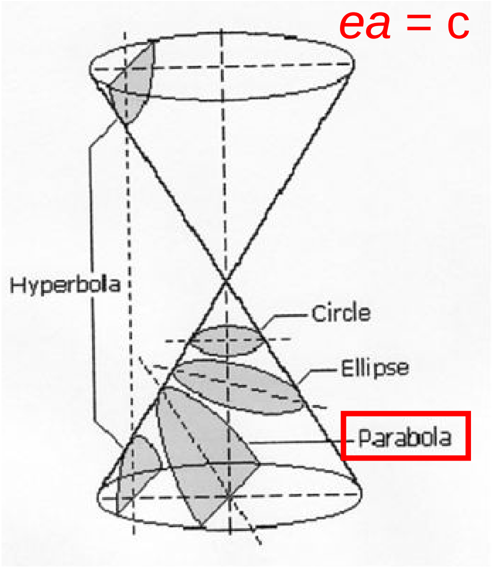
Conic Sections: Parabola





$\mathcal{E} = 0$ → Minimum energy to escape

$v = 0$ when $r = \infty$

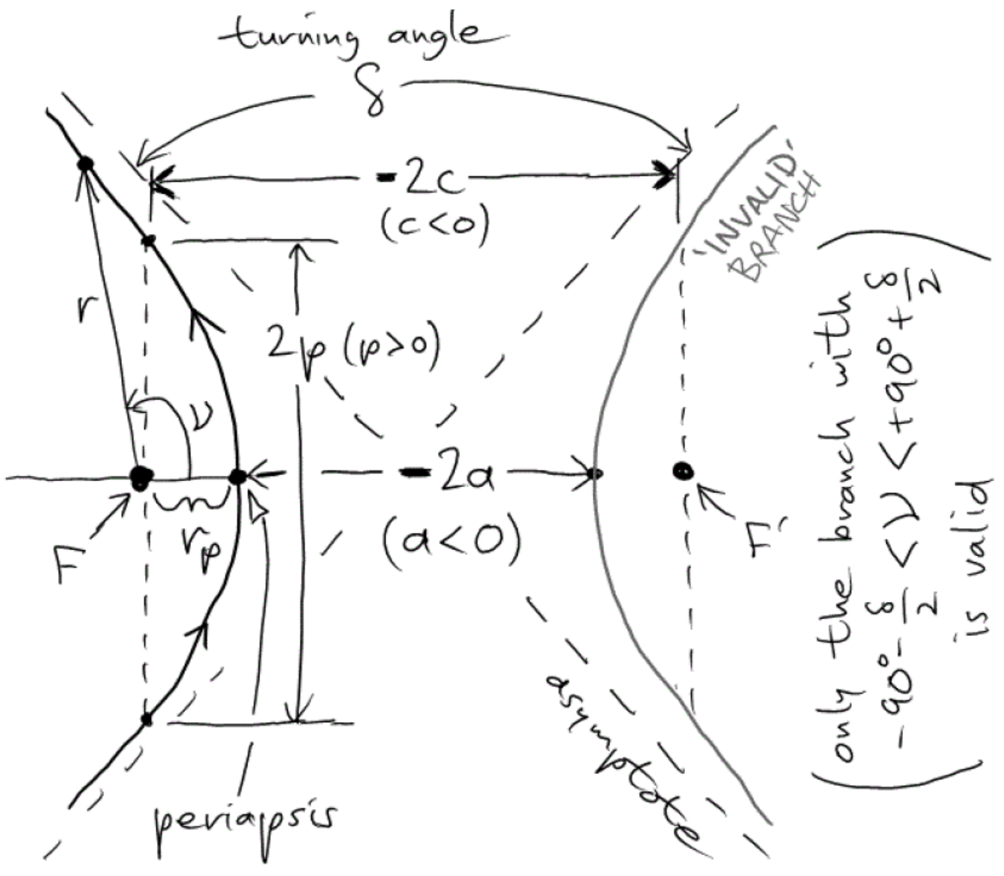
Escape speed is defined as speed on parabola

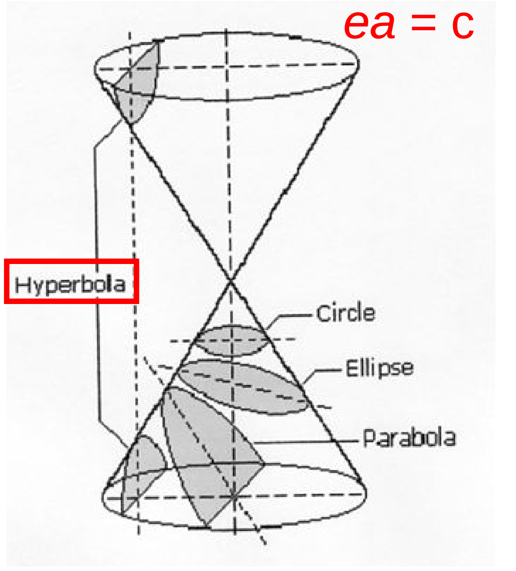
$$e = 1$$

$$\mathcal{E} = 0$$

T *Undefined*

Conic Sections: Hyperbola





$\mathcal{E} > 0$ → Enough energy to escape
$v \neq 0$ when $r = \infty$
Hyperbolic excess speed

$$e > 1$$

$$\mathcal{E} > 0$$

T *Undefined*

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Conic Sections: Conservation Properties

Specific Mechanical Energy in Gravity Field

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$$

$$\mathcal{E} = -\frac{\mu}{2a} \, ; a = -\frac{\mu}{2\mathcal{E}}$$

Velocity vs. distance

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

Vis-Viva Law

a = ∞

Escape velocity

$$v_0 = \sqrt{\frac{2\mu}{r_0}}$$

Regardless of velocity direction

Specific Angular Momentum

$$\vec{h} = \vec{r} \text{ x } \vec{v}$$

$$h = rv\cos\phi_{\text{fpa}}$$

$$h = \sqrt{\mu p} \, ; p = \frac{h^2}{\mu}$$

$$\frac{p}{a} = -\frac{2h^2\mathcal{E}}{\mu^2}$$

$$e = \sqrt{1 - \frac{p}{a}}$$

Eccentricity vs. energy and momentum

$$e = \sqrt{1 + \frac{2h^2\mathcal{E}}{\mu^2}}$$

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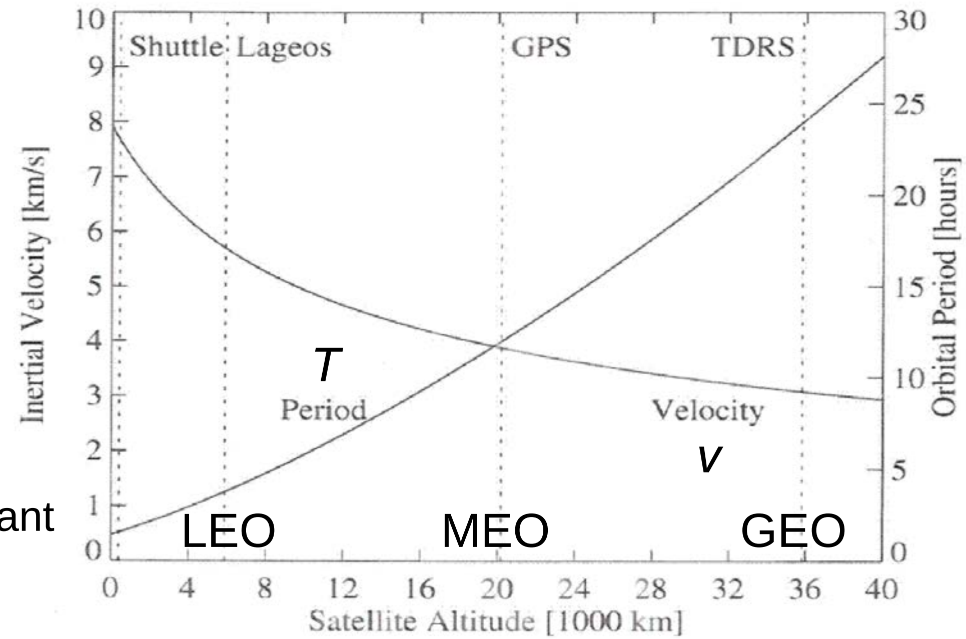
Summary for Conics

- Two-body problem yields conic section solutions with focus at central body
- We have explained terminology and properties of conic sections
- ***h*** remains constant
  - Orbital motion lies in plane fixed in inertial space
  - Flight path angle ϕ continually changes to keep $h = rv\cos\phi =$ constant
- $\mathcal{E}$ remains constant
  - Kinetic energy (speed) and potential energy (distance) are exchanged
- We have introduced many formulas and information with no proofs…
- and we cannot describe anything as a function of time…

Satellite in circular Earth orbit



$$v = \sqrt{\frac{\mu}{r}}$$

First cosmic speed (circular)

$$v_0 = \sqrt{\frac{2\mu}{r_0}}$$

Second cosmic speed (parabola, escape)

Canonical Units (Different from Astronomical Units)

- Astronomers have introduced a normalized system of length/time units to
  - Scale calculations to natural dimensions of the problem
  - Cope with inaccurate known physical constants (e.g., distance and mass)
- For a given central body CB, define
  - 1 Distance Unit as radius of body $$1\ \mathrm{DU_{CB}} = R_{\mathrm{CB}}$$
  - 1 Time Unit as time to travel 1 radian at 1 DU radius circular orbit $$1\ \mathrm{TU_{CB}} = \sqrt{\frac{R_{\mathrm{CB}}^3}{\mu_{\mathrm{CB}}}}$$
  - As a consequence by definition $$\begin{cases} v_{\mathrm{circ}}(R_{\mathrm{CB}}) = 1 \left[\dfrac{\mathrm{DU_{CB}}}{\mathrm{TU_{CB}}}\right] \\ \mu_{\mathrm{CB}} = 1 \left[\dfrac{\mathrm{DU_{CB}^3}}{\mathrm{TU_{CB}^2}}\right] \end{cases}$$
  - DUs and TUs are units of measure (nondimensional) strictly linked to the central body

Backup