Pseudo-Boolean Optimization

in memory of P.L. Hammer

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April 12, 2013

Outline

1. Pseudo-Boolean Optimization
   - Pseudo-Boolean Functions
   - Representations and Bounds
   - Persistencies and Autarkies
   - Graph Cut Models
   - Implication Networks
2. Results
   - Computational Results

Pseudo-Boolean Optimization (PBO)

Variables and Literals

- **Variables**: $x_1,\ x_2,\ \ldots,\ x_n \quad \in \{0,1\}; \quad V = \{1,2,\ldots,n\}$
- **Negations**: $\overline{x}_i \ = \ 1 - x_i \quad \in \{0,1\}$ for $i = 1,\ldots,n$

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$$f(x_1,\ldots,x_n) \ = \ \sum_{S \subseteq V} a_S \prod_{i \in S} x_i$$

Unconstrained Binary Optimization (PBO)

$$\min_{(x_1,\ldots,x_n) \in \{0,1\}^n} f(x_1,\ldots,x_n)$$

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 & = & {\color{orange}-\mathbf{5}} + \overline{x}_1 + \overline{x}_2 + \overline{x}_3 + x_1x_2 + x_1x_3 + x_2x_3 & \text{quadratic posiform} \\
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Roof Dual Bound: $C_2(f) \leq f$ (Hammer, Hansen and Simeone, 1984)

$\mathbf{C_2}(f)$ = largest C s.t. $f = C + \phi$ for some **quadratic posiform** ϕ.

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Complete Hierarchy of Bounds: (B, Crama and Hammer, 1990)

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Recent generalizations

Bisubmodular functions: (Kolmogorov, 2010; Kahl and Strandmark, 2011)

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$S = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ and $y = (\mathbf{1}, \mathbf{0}, \mathbf{1})$ is an autarky of the posiform

$$\phi = \mathbf{x_1 x_2} + 5\mathbf{\overline{x}_1 x_3} x_6 + 4\mathbf{\overline{x}_2 \overline{x}_3} x_7 + 4\mathbf{\overline{x}_1} x_4 + 5\mathbf{x_2} x_5 + 6x_4x_5$$

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Basic facts about persistencies and autarkies

Every persistency of a function f is an autarky for some posiform ϕ representing f.

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Every posiform ϕ has a unique maximal subset $S = S(\phi)$ for which it has an autarky $y \in \{0,1\}^S$.

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For a **quadratic** function f it is **"easy" to find** the unique maximal persistent set $S(f)$ provable by quadratic posiforms. (B. and Hammer, 1990)

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Network Model for Submodular QPBO (Hammer, 1965)

- A QPBF is submodular IFF all quadratic coefficients are nonpositive. (*Doit Yourself, anytime*)
- To a submodular QPBF f associate a network G_f as follows

$$=$$

- There is a one-to-one correspondence between values of f and $s-t$ cut values of G_f. (*Hammer, 1965*)
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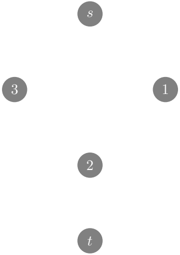

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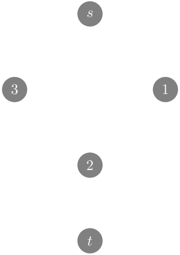

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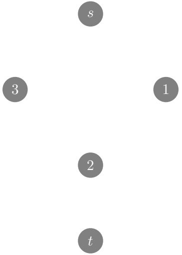

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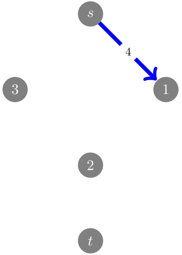

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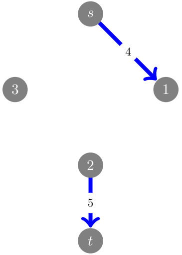

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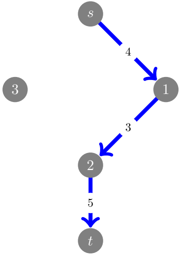

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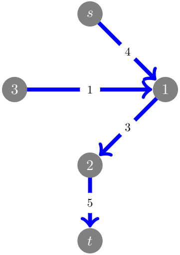

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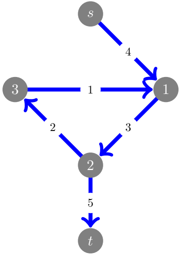

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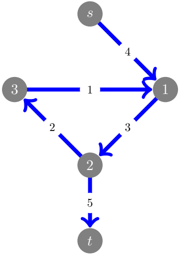
- There is a one-to-one correspondence between values of f and $s-t$ cut values of G_f. (*Hammer, 1965*)
- **Graph cuts** (325000) in computer vision: (Greig, Porteous, and Seheult, 1989), (Boykov, Veksler and Zabih, 1998)

Network Model for Submodular QPBO (Hammer, 1965)



- A QPBF is submodular IFF all quadratic coefficients are nonpositive. (*Doit Yourself, anytime*)
- To a submodular QPBF f associate a network G_f as follows

$$
\begin{aligned}
\mathbf{f} &= \mathbf{4 - x_1 + 7x_2 + x_3 - 3x_1x_2 - x_1x_3 - 2x_2x_3} \\
&= \mathbf{4\overline{x}_1 + 5x_2 + 3x_1\overline{x}_2 + \overline{x}_1x_3 + 2x_2\overline{x}_3} \\
&= \mathbf{4s\overline{x}_1 + 5x_2\overline{t} + 3x_1\overline{x}_2 + \overline{x}_1x_3 + 2x_2\overline{x}_3}
\end{aligned}
$$

- There is a one-to-one correspondence between values of f and $s-t$ cut values of G_f. (*Hammer, 1965*)
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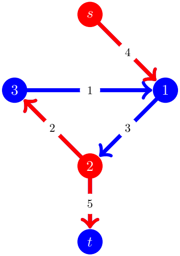

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- To a submodular QPBF f associate a network G_f as follows

$$\mathbf{f} = \mathbf{4 - x_1 + 7x_2 + x_3 - 3x_1x_2 - x_1x_3 - 2x_2x_3}$$

$$= \mathbf{4\overline{x}_1 + 5x_2 + 3x_1\overline{x}_2 + \overline{x}_1x_3 + 2x_2\overline{x}_3}$$

$$= \mathbf{4s\overline{x}_1 + 5x_2\overline{t} + 3x_1\overline{x}_2 + \overline{x}_1x_3 + 2x_2\overline{x}_3}$$

- There is a one-to-one correspondence between values of f and $s-t$ cut values of G_f. (*Hammer, 1965*)

$$\mathbf{f(0,1,0) \;=\; C(\{s,2\},\{1,3,t\}) \;=\; 11}$$

- **Graph cuts** (**325000**) in computer vision: (Greig, Porteous, and Seheult, 1989), (Boykov, Veksler and Zabih, 1998)

Network Model for Submodular QPBO (Hammer, 1965)

- A QPBF is submodular IFF all quadratic coefficients are nonpositive. (*Doit Yourself, anytime*)
- To a submodular QPBF f associate a network G_f as follows

$$\begin{aligned} \mathbf{f} &= \mathbf{4 - x_1 + 7x_2 + x_3 - 3x_1x_2 - x_1x_3 - 2x_2x_3} \\ &= \mathbf{4\overline{x}_1 + 5x_2 + 3x_1\overline{x}_2 + \overline{x}_1x_3 + 2x_2\overline{x}_3} \\ &= \mathbf{4s\overline{x}_1 + 5x_2\overline{t} + 3x_1\overline{x}_2 + \overline{x}_1x_3 + 2x_2\overline{x}_3} \end{aligned}$$

- There is a one-to-one correspondence between values of f and $s - t$ cut values of G_f. (*Hammer, 1965*)

$$\mathbf{f(0,1,0) = C(\{s,2\},\{1,3,t\}) = 11}$$

- **Graph cuts (325000)** in computer vision: (Greig, Porteous, and Seheult, 1989), (Boykov, Veksler and Zabih, 1998)

Outline

QPBF $\longrightarrow$ Posiform

$$f = \mathbf{10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3}$$

QPBF $\longrightarrow$ Posiform

$$
\begin{aligned}
f &= \mathbf{10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3} \\
&= 10 - 2x_1 - 6x_2 + 2x_1x_2 - {\color{blue}2x_1(1 - \overline{x}_3)} + 4x_2x_3
\end{aligned}
$$

QPBF $\longrightarrow$ Posiform

$$
\begin{aligned}
f &= \mathbf{10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3} \\
&= 10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1(1 - \overline{x}_3) + 4x_2x_3 \\
&= 10 - 4x_1 - 6x_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3
\end{aligned}
$$

QPBF $\longrightarrow$ Posiform

$$
\begin{aligned}
f &= \mathbf{10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3} \\
&= 10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1(1-\overline{x}_3) + 4x_2x_3 \\
&= 10 - 4x_1 - 6x_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\
&= 10 - 4(1-\overline{x}_1) - 6(1-\overline{x}_2) + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3
\end{aligned}
$$

QPBF $\longrightarrow$ Posiform

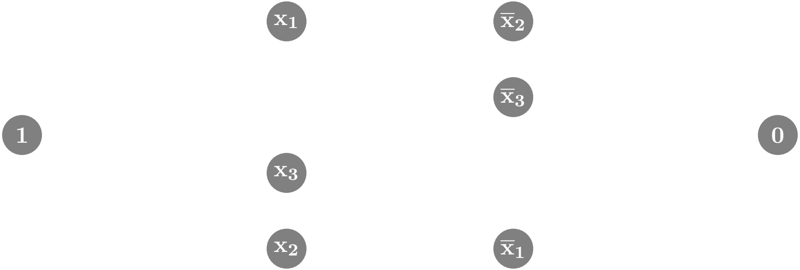
$$
\begin{aligned}
f &= \mathbf{10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3} \\
&= 10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1(1 - \overline{x}_3) + 4x_2x_3 \\
&= 10 - 4x_1 - 6x_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\
&= 10 - 4(1 - \overline{x}_1) - 6(1 - \overline{x}_2) + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\
&= \mathbf{4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3} \;\; = \;\; \phi
\end{aligned}
$$

QPBF $\longrightarrow$ Posiform

$$\begin{aligned} f &= \mathbf{10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3} \\ &= 10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1(1 - \overline{x}_3) + 4x_2x_3 \\ &= 10 - 4x_1 - 6x_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\ &= 10 - 4(1 - \overline{x}_1) - 6(1 - \overline{x}_2) + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\ &= \mathbf{4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3} \quad = \quad \phi \end{aligned}$$

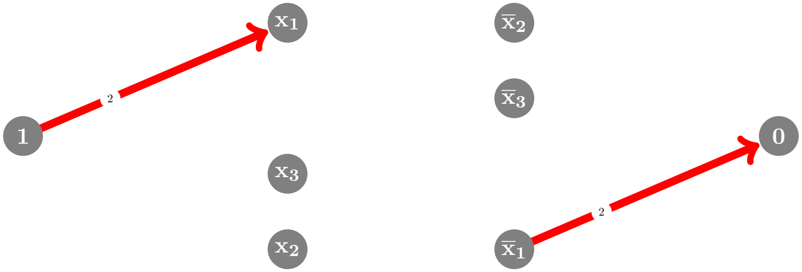
$$\mathbf{C}(\phi) = \mathbf{0} \qquad \text{and} \qquad \mathbf{S}(\phi) = \emptyset$$

Implication Networks



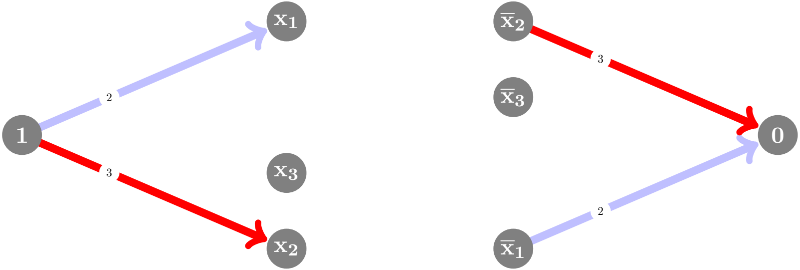
$$\mathbf{f} = 4\overline{\mathbf{x}}_1 + 6\overline{\mathbf{x}}_2 + 2\mathbf{x}_1\mathbf{x}_2 + 2\mathbf{x}_1\overline{\mathbf{x}}_3 + 4\mathbf{x}_2\mathbf{x}_3$$

Implication Networks



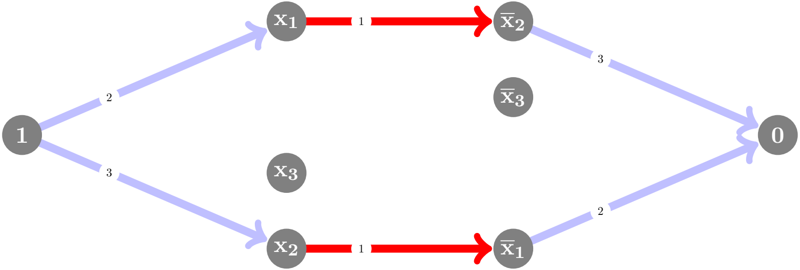
$$f = \color{red}{4\overline{x}_1} + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3$$

Implication Networks



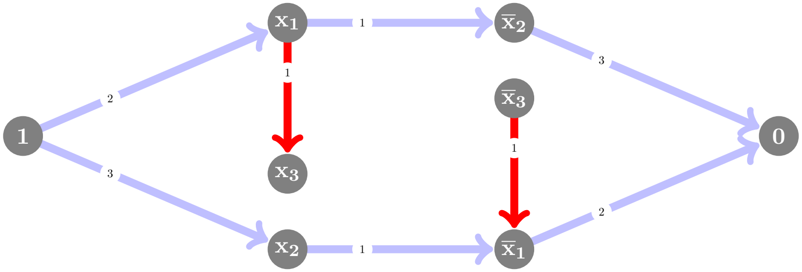
$$\mathbf{f} = 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3$$

Implication Networks



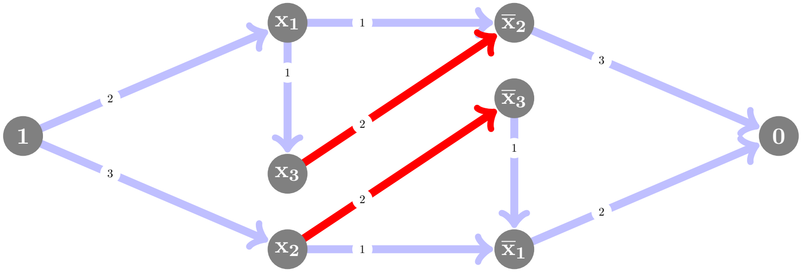
$$f = 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3$$

Implication Networks



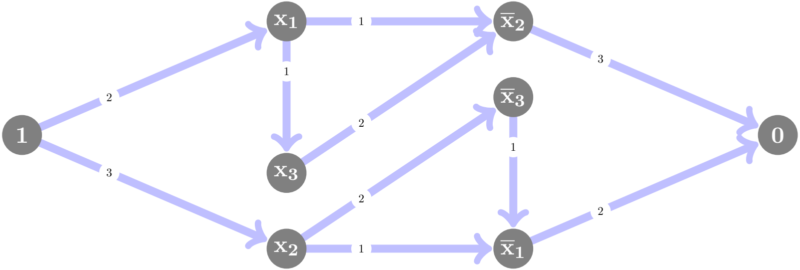
$$\mathbf{f} \quad = 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3$$

Implication Networks



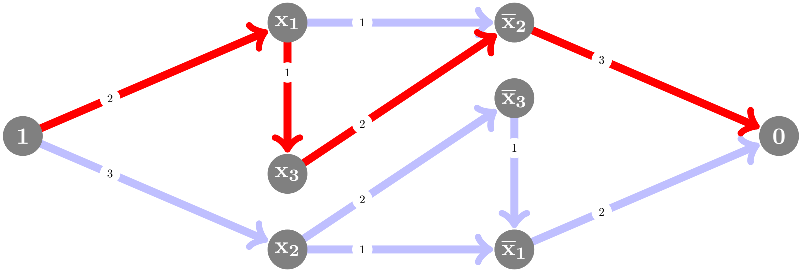
$$f \quad = 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3$$

Implication Networks



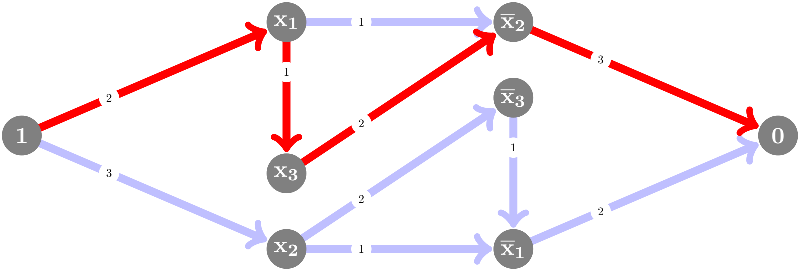
$$\mathbf{f} = 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3$$

Flows and Posiform Transformations



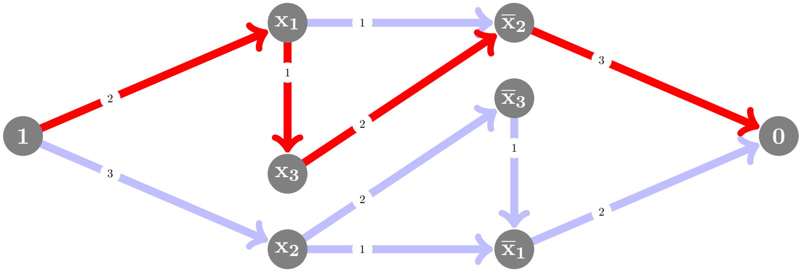
$$f \quad = 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3$$

Flows and Posiform Transformations



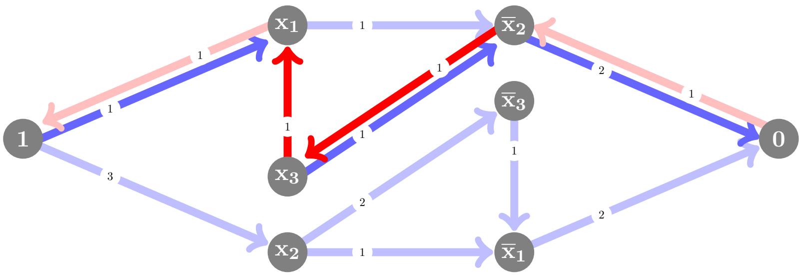
$$
\begin{aligned}
\mathbf{f} &= 4\bar{x}_1 + 6\bar{x}_2 + 2x_1x_2 + 2x_1\bar{x}_3 + 4x_2x_3 \\
&= 3\bar{x}_1 + 5\bar{x}_2 + 2x_1x_2 + x_1\bar{x}_3 + 3x_2x_3 \\
&\quad + \left(\bar{x}_1 + x_1\bar{x}_3 + x_3x_2 + \bar{x}_2\right)
\end{aligned}
$$

Flows and Posiform Transformations



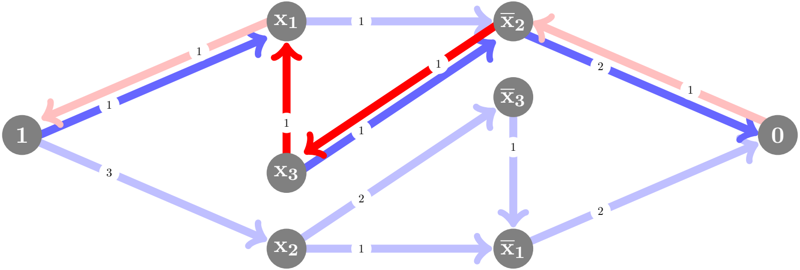
$$
\begin{aligned}
f &= 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\
&= 3\overline{x}_1 + 5\overline{x}_2 + 2x_1x_2 + x_1\overline{x}_3 + 3x_2x_3 \\
&\quad + (\overline{x}_1 + x_1\overline{x}_3 + x_3x_2 + \overline{x}_2) \\
&= 3\overline{x}_1 + 5\overline{x}_2 + 2x_1x_2 + x_1\overline{x}_3 + 3x_2x_3 \\
&\quad + (1 + \overline{x}_1x_3 + \overline{x}_3\overline{x}_2)
\end{aligned}
$$

Flows and Posiform Transformations



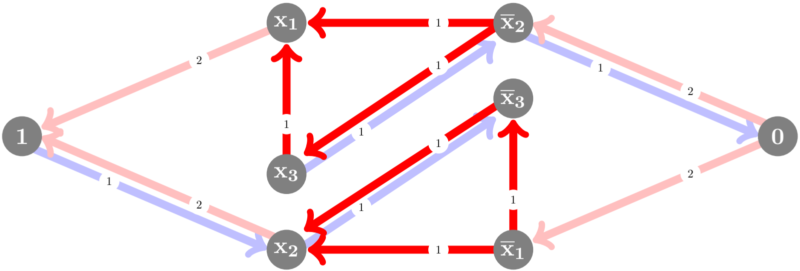
$$
\begin{aligned}
f &= 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\
&= 3\overline{x}_1 + 5\overline{x}_2 + 2x_1x_2 + x_1\overline{x}_3 + 3x_2x_3 \\
&\quad + (\overline{x}_1 + x_1\overline{x}_3 + x_3x_2 + \overline{x}_2) \\
&= 3\overline{x}_1 + 5\overline{x}_2 + 2x_1x_2 + x_1\overline{x}_3 + 3x_2x_3 \\
&\quad + (1 + \overline{x}_1x_3 + \overline{x}_3\overline{x}_2) \\
&= 1 + 3\overline{x}_1 + 5\overline{x}_2 + 2x_1x_2 + x_1\overline{x}_3 + 3x_2x_3 + \overline{x}_1x_3 + \overline{x}_3\overline{x}_2
\end{aligned}
$$

Flows and Posiform Transformations



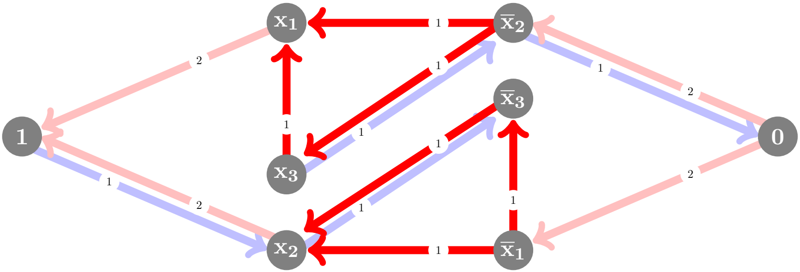
$$\begin{aligned} f &= 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\ &= 1 + 3\overline{x}_1 + 5\overline{x}_2 + 2x_1x_2 + x_1\overline{x}_3 + 3x_2x_3 + \overline{x}_1x_3 + \overline{x}_3\overline{x}_2 \end{aligned}$$

Flows and Posiform Transformations



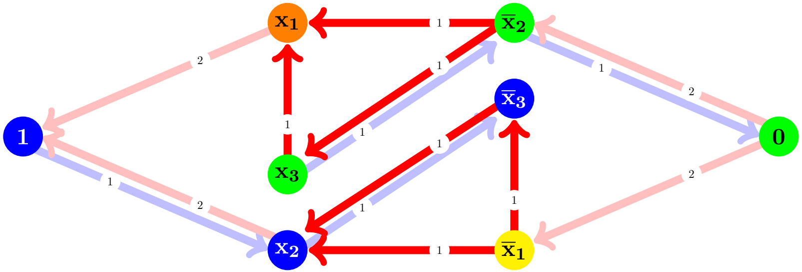
$$
\begin{aligned}
\mathbf{f} &= 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\
&= 1 + 3\overline{x}_1 + 5\overline{x}_2 + 2x_1x_2 + x_1\overline{x}_3 + 3x_2x_3 + \overline{x}_1x_3 + \overline{x}_3\overline{x}_2 \\
&= 4 + 2\overline{x}_2 + 2\overline{x}_1\overline{x}_2 + 2\overline{x}_1x_3 + 2x_2x_3 + 2\overline{x}_2\overline{x}_3
\end{aligned}
$$

Persistencies and Decompositions



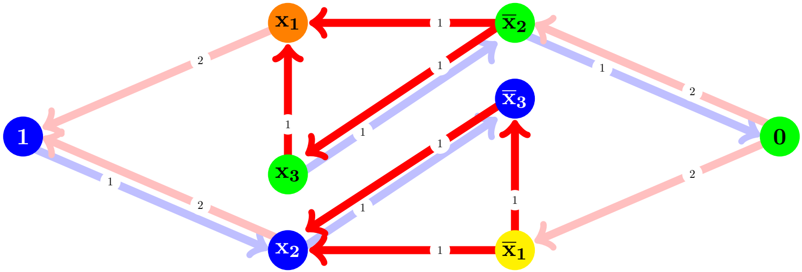
$$
\begin{aligned}
\mathbf{f} &= 10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3 \\
&= 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\
&= 4 + 2\overline{x}_2 + 2\overline{x}_1\overline{x}_2 + 2\overline{x}_1x_3 + 2x_2x_3 + 2\overline{x}_2\overline{x}_3
\end{aligned}
$$

Persistencies and Decompositions



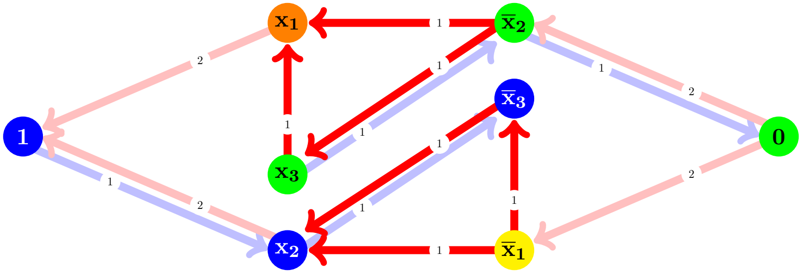
$$
\begin{aligned}
f &= 10 - 2x_1 - 6x_2 + 2x_1x_2 - 2x_1x_3 + 4x_2x_3 \\
&= 4\overline{x}_1 + 6\overline{x}_2 + 2x_1x_2 + 2x_1\overline{x}_3 + 4x_2x_3 \\
&= 4 + 2\overline{x}_2 + 2\overline{x}_1\overline{x}_2 + 2\overline{x}_1x_3 + 2x_2x_3 + 2\overline{x}_2\overline{x}_3
\end{aligned}
$$

Persistencies and Decompositions



- Strong persistency: $x_2 = 1$, $x_3 = 0$
- Weak persistency: $x_1 = 1$ (or $x_1 = 0$)
- This method computes the unique maximal set of autark variables justifiable by any quadratic posiform of the given quadratic PBF (B. and Hammer, 1990).

Persistencies and Decompositions



- Strong persistency: $x_2 = 1$, $x_3 = 0$
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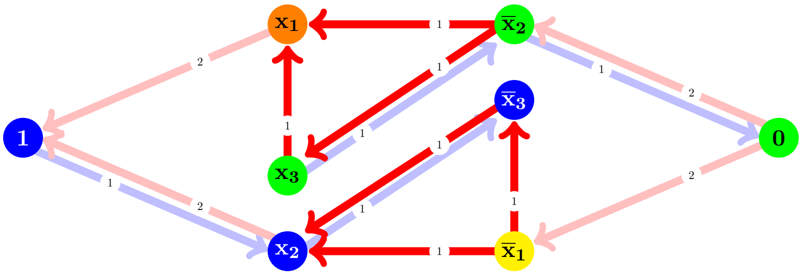
Persistencies and Decompositions



- Strong persistency: $x_2 = 1$, $x_3 = 0$
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- This method computes the **unique maximal set** of autark variables justifiable by any quadratic posiform of the given quadratic PBF (B. and Hammer, 1990).

Outline

Via Minimization in VLSI Design

		Percentage of Variables Fixed by					
Problem[1]	n	Persistency		Probing		ALL TOOLS	Time (sec)
		(strong)	(weak)	(forc)	(equal)		
via.c1y	829	93.6%	6.4%	0%	0%	**100%**	**0.03**
via.c2y	981	94.7%	5.3%	0%	0%	**100%**	**0.06**
via.c3y	1328	94.6%	5.4%	0%	0%	**100%**	**0.09**
via.c4y	1367	96.4%	3.6%	0%	0%	**100%**	**0.09**
via.c5y	1203	93.1%	6.9%	0%	0%	**100%**	**0.08**
via.c1n	828	57.4%	9.6%	32.4%	0.6%	**100%**	**0.49**
via.c2n	980	12.4%	4.4%	83.1%	0.1%	**100%**	**7.14**
via.c3n	1327	6.8%	5.7%	87.3%	0.2%	**100%**	**18.17**
via.c4n	1366	11.1%	1.3%	87.6%	0%	**100%**	**23.08**
via.c5n	1202	3.4%	1.4%	95.0%	0.2%	**100%**	**17.13**

[1] S. Homer and M. Peinado. Design and performance of parallel and distributed approximation algorithms for maxcut. Journal of Parallel and Distributed Computing 46 (1997) 48-61.

Vertex Cover in Planar Graphs

	Averages for 100 graphs in each of the 4 groups			
	Variables Fixed (%)		Time (sec)	
n	A. D. N.[2]	**QUBO**[3]	A. D. N.[2]	**QUBO**[3]
1000	68.4	**100**	4.06	**0.05**
2000	67.4	**100**	12.24	**0.16**
3000	65.5	**100**	30.90	**0.27**
4000	62.7	**100**	60.45	**0.53**

[2] Alber, Dorn, Niedermeier. Experimental evaluation of a tree decomposition based algorithm for vertex cover on planar graphs. Discrete Applied Mathematics 145 (2005) 219-231; 750 GHz, Linux PC, 720 MB

[3] Pentium 4, 2.8 GHz, Windows XP, 512 MB

Jumbo Vertex Cover in Planar Graphs

Vertices	Computing Times (min)[4]		
	Planar Density		
	10%	50%	90%
50,000	0.7	2.3	0.9
100,000	2.9	10.2	3.9
250,000	19.5	69.8	26.3
500,000	79.3	277.3	106.9

QUBO fixed all variables for all problems!

[4] Averages over 3 experiments on a Xeon 3.06 GHz, XP, 3.5 GB RAM.

One Dimensional Ising Models

σ	Number of Spins	Average Computing Time (s) Branch, Cut & Price[5]	BiqMaq[5]	**QUBO**[6]
2.5	100	699	68	**1**
	150	92 079	388	**3**
	200	N/A	993	**9**
	250	N/A	6 567	**14**
	300	N/A	34 572	**21**
3.0	100	256	59	**1**
	150	13 491	293	**2**
	200	61 271	1 034	**3**
	250	55 795	3 594	**4**
	300	55 528	8 496	**5**

[5] F. Rendl, G. Rinaldi, A. Wiegele. (2007). Solving max-cut to optimality by intersecting semidefinite and polyhedral relaxations.

[6] ALL problems were solved by QUBO.

Larger One Dimensional Ising Models

σ	n	Average of 3 Problems: Variables not fixed	Average of 3 Problems: QUBO Time (s)[7]
2.5	500	**5**	**13**
	750	**22**	**30**
	1000	**24**	**53**
	1250	**20**	**81**
	1500	**32**	**124**
3.0	500	**0**	**4**
	750	**0**	**12**
	1000	**0**	**23**
	1250	**0**	**37**
	1500	**0**	**59**

[7] Pentium M, 1.6 GHz 760 MB RAM

Open Ends

- How can we extend the above results for general PBF-s?
- How can we find autarkies for a given posiform?
- Quadratization via graph stability (Ebenegger, Hammer, de Werra, 1984): Can we map persistencies?
- Quadratization in higher dimension: given f, find quadratic g such that

$$f(x) = \min_{y \in \{0,1\}^m} g(x, y) \text{ for all } x \in \{0,1\}^n$$

  - Does it exists? How many new variables do we need? (Rosenberg, 1975), ...
  - Which quadratization provides us with the most persistencies?

Open Ends

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- How can we find autarkies for a given posiform?
- Quadratization via graph stability (Ebenegger, Hammer, de Werra, 1984): Can we map persistencies?
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$$f(x) = \min_{y \in \{0,1\}^m} g(x, y) \text{ for all } x \in \{0,1\}^n$$

  - Does it exist? How many new variables are needed? (Rosenberg, 1975)...
  - Which quadratization provides us with the most persistencies?

Open Ends

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