平成28年度S2学期定期試験問題

試験科目　　　　　　出題教員名

上級デリバティブ (I)　　　　藤井優成

One must clearly state in the answer sheet when you make any additional assumptions or introduce new variables. One can use English and/or Japanese for answering the questions. One can assume the existence of the standard "idealistic" market conditions used in the lecture course. One can use the following functions for the standard normal distribution:

$$\phi(z) := \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right), \qquad N(x) := \int_{-\infty}^{x} \phi(z) dz .$$

Use the symbol $P(t,T)$ to denote the zero-coupon bond price with maturity T ($> t$) at time t. Unless otherwise stated, the *current time* should be taken as $t = 0$.

Problem 1

A forward rate agreement (FRA) is a binding contract between a lender and a borrower agreeing to let a certain fixed interest rate K act on a prefixed cash amount N over a certain future interval. For simplicity, let us put $N = 1$ throughout this problem.

(1.1) Consider a FRA for the period $[T_1, T_2]$ with $0 < T_1 < T_2$ in which the borrower borrows a unit amount of cash at T_1 and pays back the lender $(1 + K\delta)$ at T_2. Here, $\delta > 0$ is the day-count fraction for the period. The forward rate F is the fixed rate K that makes the present value of the FRA zero at $t = 0$. Give a replication strategy to derive the forward rate F, and give F in terms of the current zero-coupon bond prices.

(1.2) Consider a time partition $0 < T_1 < T_2 < T_3$ and the following two-period FRA in which the two parties (borrower and lender) agree on the followings;
(i) The borrower borrows a unit amount of cash at T_1.
(ii) The borrower pays the lender $K\delta$ at time T_2 and $(1 + K\delta)$ at time T_3.
Here, $\delta > 0$ is the day-count fraction common for the two periods, $[T_1, T_2]$ and $[T_2, T_3]$. Give a replication strategy and derive the forward rate F for the two-period FRA in terms of the current zero-coupon bond prices.[Hint: From a view point of the borrower, for example, start from a short sale of a zero-coupon bond with maturity T_1, and then divide its proceeds into two investments with different maturities.]

Problem 2

Consider a one-period binomial stock model. The current time is 0 and the terminal time is T (> 0). The stock is a risky asset whose current price is $S_0 = s$ and the price S_T at time T is either $s \times u$ with probability p_u or $s \times d$ with probability p_d without any dividend payments. Here, p_u, p_d are the probabilities under the empirical measure $\mathbb{P}$. There also exists a risk-free bond with the current price $B_0 = 1$. The bond price at T is given by $B_T = e^{rT}$. Here, all the parameters are constant satisfying $r, s, p_u, p_d > 0$, $0 < d < u$, and $p_u + p_d = 1$.

A portfolio strategy $h = (x, y)$ denotes the number of positions for the bond (x) and for the stock (y), respectively. Then the portfolio value of the strategy h is given by

$$V_t^h = xB_t + yS_t, \quad t \in \{0, T\} .$$

An arbitrage strategy is the strategy h satisfying

$$V_0^h = 0, \quad \mathbb{P}(V_T^h \geq 0) = 1, \quad \mathbb{P}(V_T^h > 0) > 0\,.$$

(2.1) Give q_u (the probability of the upward movement) and q_d (the probability of the downward movement) under the risk-neutral measure $\mathbb{Q}$ such that $\mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T] = s$ is satisfied. Furthermore give the conditions that make $\mathbb{Q}$ well-defined, i.e., $q_u > 0, q_d > 0$ and $q_u + q_d = 1$.

(2.2) Prove that the existence of the probability measure $\mathbb{Q}$ is necessary as well as sufficient the absence of arbitrage in the binomial model.

?

(2.3) Now, suppose that the stock pays the *known pre-fixed* dividend w (> 0) at time T per share regardless of its state (up or down). If this new binomial market is arbitrage-free, what is the current price of call option that pays $\max(S_T - sK, 0)$ at T with the strike K satisfying $d < K < u$?

(2.4) Under the same setup given in (2.3), give the risk-neutral probabilities for the up and dc movement (q_u, q_d) so that the well-posedness of the associated probability measure $\mathbb{Q}$ is necess as well as sufficient for the absence of arbitrage.

Problem 3

Let us consider the price dynamics of a non-dividend paying stock satisfying

$$dS(t) = rS(t)dt + \sigma S(t)dW_t^{\mathbb{Q}}, \quad S(0) = s$$

where s, r, σ are all strictly positive constants. r is the risk-free interest rate, and $W^{\mathbb{Q}}$ is a o dimensional Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. Let T and K positive constants.

(3.1) Derive the stochastic differential equation for $\ln(S(t))$, i.e., calculate $d\ln(S(t))$ by] formula.

(3.2) Calculate the next probability under the risk-neutral measure $\mathbb{Q}$:

$$\mathbb{Q}\Big(\{S(T) \geq K\}\Big)\,.$$

Based on the result of (3.2), one can immediately obtain the price C_D of a digital call opt which pays the unit amount of cash at T to the option holder if $S(T) \geq K$:

$$C_D = e^{-rT}\mathbb{E}^{\mathbb{Q}}\Big[\mathbf{1}_{S(T)\geq K}\Big] = e^{-rT}\mathbb{Q}\Big(\{S(T) \geq K\}\Big)\,.$$

(3.3) Give the Vega for the above digital call option and draw a schematic graph of the V by taking the current stock price as the horizontal axis. Moreover, based on the graph, exp an important risk-management issue for this product (in the *real* financial market) regarding change of the slope in the *implied volatility*.

Problem 1

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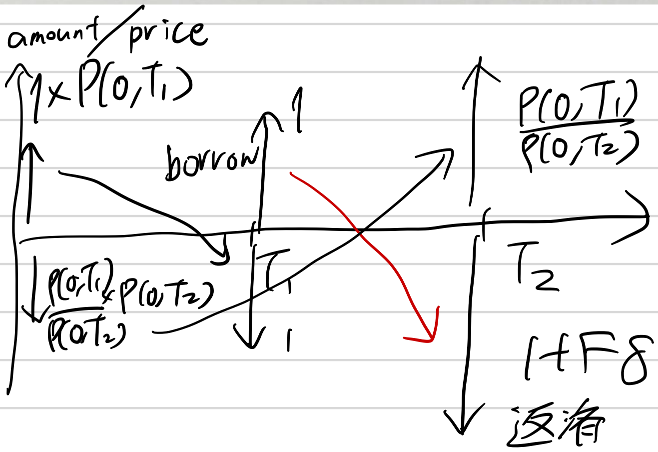

$(t=0)$

long ZCB mat. T_2 × $\dfrac{P(0,T_1)}{P(0,T_2)}$ 単位

short ZCB mat. T_1 × 1 単位

※ 短いZCBをshort, 長い方をlong

※ ZCB(T_1)の返済費用をFRAを借りて支払う

※ ZCB(T_2)の償還でFRAを返済

$$\text{net CF} = \frac{P(0,T_1)}{P(0,T_2)} - (1+F\delta) \quad \text{at } T_2$$

$$\therefore F = \frac{1}{\delta}\left(\frac{P(0,T_1)}{P(0,T_2)} - 1\right) \text{ //}$$

or

$$\frac{1}{P(0,T_2)} = \frac{1}{P(0,T_1)}(1+\delta F)$$

(1.2) Consider a time partition $0 < T_1 < T_2 < T_3$ and the following two-period FRA in which the two parties (borrower and lender) agree on the followings;

(i) The borrower borrows a unit amount of cash at T_1.

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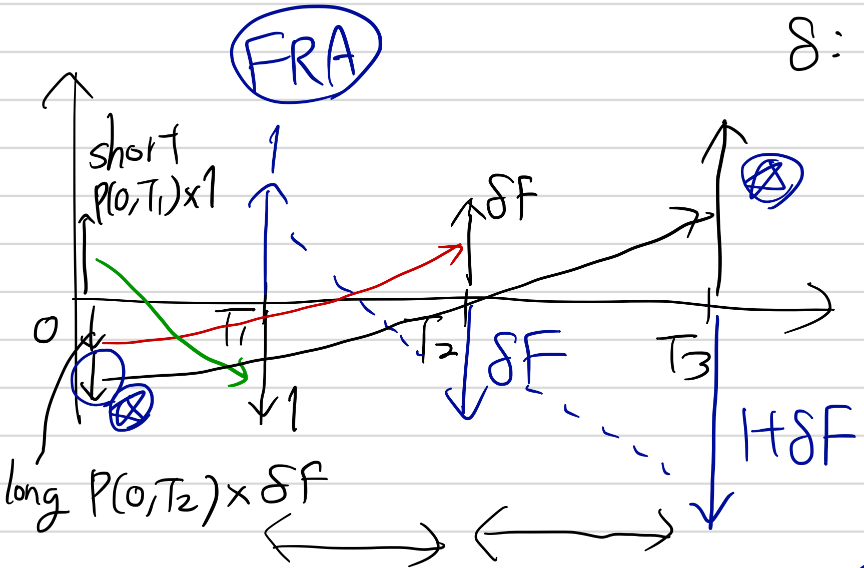
Here, $\delta > 0$ is the day-count fraction common for the two periods, $[T_1, T_2]$ and $[T_2, T_3]$. Give a replication strategy and derive the forward rate F for the two-period FRA in terms of the current zero-coupon bond prices.[Hint: From a view point of the borrower, for example, start from a short sale of a zero-coupon bond with maturity T_1, and then divide its proceeds into two investments with different maturities.]

δ: day-count fraction



$$P(0,T_3) \times \frac{P(0,T_1) - P(0,T_2)\delta F}{P(0,T_3)} \quad (単位)$$

strategy:

at $t=0$
- short ZCB mat T_1 × 1 単位
- long ZCB mat T_2 × δF 単位
- long ZCB mat T_3 × $\dfrac{P(0,T_1) - P(0,T_2)\delta F}{P(0,T_3)}$ 単位

net cash flow at T_3

$$= \frac{P(0,T_1) - P(0,T_2)\delta F}{P(0,T_3)} - (1+\delta F)$$

no-arbitrage ⇒

$$P(0,T_1) = P(0,T_2)\delta F + (1+\delta F)P(0,T_3)$$

$$F = \frac{P(0,T_1) - P(0,T_3)}{\delta\left(P(0,T_2) + P(0,T_3)\right)}$$

//

Problem 2

Consider a one-period binomial stock model. The current time is 0 and the terminal time is $T\ (>0)$. The stock is a risky asset whose current price is $S_0 = s$ and the price S_T at time T is either $s \times u$ with probability p_u or $s \times d$ with probability p_d without any dividend payments. Here, p_u, p_d are the probabilities under the empirical measure $\mathbb{P}$. There also exists a risk-free bond with the current price $B_0 = 1$. The bond price at T is given by $B_T = e^{rT}$. Here, all the parameters are constant satisfying $r, s, p_u, p_d > 0$, $0 < d < u$, and $p_u + p_d = 1$.

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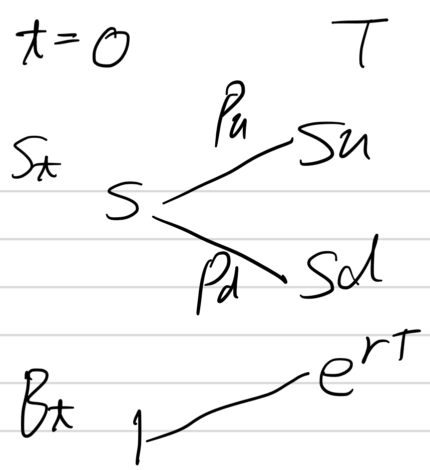

An arbitrage strategy is the strategy h satisfying

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$$e^{-rT}E^{Q}[S_T] = S_0$$

$$\begin{cases} e^{-rT}(q_u \cancel{s} u + q_d \cancel{s} d) = \cancel{s} \\ q_u + q_d = 1 \end{cases}$$

$$\Leftrightarrow q_u u + (1 - q_u) d = e^{rT}$$

$$q_u = \frac{e^{rT} - d}{u - d}, \quad q_d = \frac{u - e^{rT}}{u - d}$$ //

$$q_u > 0, \ q_d > 0$$

$$\Rightarrow \text{condition} : d < e^{rT} < u$$ //

(2.2) Prove that the existence of the probability measure $\mathbb{Q}$ is necessary as well as sufficient the absence of arbitrage in the binomial model.

existence of $\mathbb{Q}$ (EMM) $\iff$ no-arbitrage を示す.

① ($\Rightarrow$ sufficiency) $d<e^{rT}<u$

$d<e^{rT}<u$

Assume $V_0^h=0$ then $x+ys=0 \iff ys=-x$

Then, $V_T^h = xe^{rT}+ysZ = x(e^{rT}-Z)$

$$Z=\begin{cases} u & (p_u>0) \\ d & (p_d>0) \end{cases}$$

$\therefore P(V_T^h \geq 0) < 1 \Rightarrow$ no-arbitrage 証

② ($\Leftarrow$ necessarity)

no-arbitrage $\longrightarrow$ $d<e^{rT}<u$ の対偶を示す

②-1. $e^{rT} \leq d < u$ と仮定

Portfolio $h=(-s,1)$ $\quad V_0^h=-s+1\times s=0$

$V_T^h=-se^{rT}+sZ=s(-e^{rT}+Z)$

$P(V_T^h\geq 0)=1,\quad P(V_T^h>0)=P(Z=u)=p_u>0.$

$\Rightarrow$ arbitrage

②-2. $d<u\leq e^{rT}$ と仮定

Portfolio $h=(s,-1)$ $\quad V_0^h=s+(-1)\times s=0$

$V_T^h=se^{rT}-sZ=s(e^{rT}-Z)$

$P(V_T^h\geq 0)=1,\ P(V_T^h>0)=P(Z=d)=p_d>0$

$\Rightarrow$ arbitrage

証

(2.3) Now, suppose that the stock pays the *known pre-fixed* dividend w (> 0) at time T : share regardless of its state (up or down). If this new binomial market is arbitrage-free, w. is the current price of call option that pays $\max(S_T - sK, 0)$ at T with the strike K satisfy $d < K < u$?

at $t=0$

portfolio $\begin{cases} \text{risk-free Bond} : x \text{単位} \\ \text{Stock} : y \text{単位} \end{cases}$

call option $\max(S_T - sK, 0)$

$$V_0^h = x + ys$$

$$V_0^h \nearrow e^{rT}x + (su + w)y = s(u-K)$$

$$V_0^h \searrow e^{rT}x + (sd + w)y = 0$$

複製ポートフォリオ

$$s(u-d)y = s(u-K) \quad ※ d<K<u$$

$$\therefore \begin{cases} y = \dfrac{u-K}{u-d} \\ x = -e^{-rT}(sd+w)\dfrac{u-K}{u-d} \end{cases}$$

オプション価格 = 複製コスト = $V_0^h = x + ys$

$$= -e^{-rT}\frac{(sd+w)(u-K)}{u-d} + \frac{u-K}{u-d}s$$

$$= \frac{u-K}{u-d}\left(s - e^{-rT}(sd+w)\right)$$

$$= \frac{s(u-K)}{u-d}\left(1 - e^{-rT}\left(d + \frac{w}{s}\right)\right)$$ [ends with //]

(2.4) Under the same setup given in (2.3), give the risk-neutral probabilities for the up and dc movement (q_u, q_d) so that the well-posedness of the associated probability measure $\mathbb{Q}$ is necess as well as sufficient for the absence of arbitrage.

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[C_T]$$

$$\frac{u-K}{u-d}\left(s - e^{-rT}(sd+w)\right) = e^{-rT}\left(q_u' s(u-K) \cancel{+ q_d' \times 0}\right)$$

$\therefore$

$$\begin{cases} q_u' = \dfrac{e^{rT}}{(u-d)}\left(1-e^{-rT}\left(d+\dfrac{w}{s}\right)\right) = \dfrac{e^{rT}-\left(d+\frac{w}{s}\right)}{u-d} \\[2ex] q_d' = 1-q_u' = \dfrac{\left(u+\frac{w}{s}\right)-e^{rT}}{u-d} \end{cases}$$

$$q_u' > 0, \quad q_d' > 0$$

$$\Rightarrow d+\frac{w}{s} < e^{rT} < u+\frac{w}{s}$$

2.4(別解)

$$\begin{tikzcd}[row sep=small] & S\left(u+\frac{w}{S}\right) \\ S \arrow[ur, dash] \arrow[dr, dash] & \\ & S\left(d+\frac{w}{S}\right) \end{tikzcd}$$

$$S_0 + 0 = e^{-rT}\left[S_T + w\right]$$

$$Se^{rT} = q_u'(Su+w) + q_d'(Sd+w)$$

$$e^{rT} = q_u'\left(u+\frac{w}{S}\right) + (1-q_u')\left(d+\frac{w}{S}\right)$$

$$\begin{cases} q_u' = \dfrac{e^{rT}-\left(d+\frac{w}{S}\right)}{u-d} \\[2ex] q_d' = \dfrac{\left(u+\frac{w}{S}\right)-e^{rT}}{u-d} \end{cases}$$

Problem 3

Let us consider the price dynamics of a non-dividend paying stock satisfying

$$dS(t) = rS(t)dt + \sigma S(t)dW_t^{\mathbb{Q}}, \quad S(0) = s$$

where s, r, σ are all strictly positive constants. r is the risk-free interest rate, and $W^{\mathbb{Q}}$ is a o dimensional Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. Let T and K positive constants.

(3.1) Derive the stochastic differential equation for $\ln(S(t))$, i.e., calculate $d\ln(S(t))$ by I formula.

$$f(x) = \ln x \qquad \frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$$

$$d(\ln S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\frac{dS_t^2}{S_t^2}$$

$$= r\,dt + \sigma dW_t^{Q} - \frac{1}{2}\frac{1}{\cancel{S_t^2}}\left(\sigma^2 \cancel{S_t^2}\, dt\right)$$

$$= \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t^{Q}$$

(3.2) Calculate the next probability under the risk-neutral measure $\mathbb{Q}$:

$$\mathbb{Q}\Big(\{S(T) \geq K\}\Big).$$

Based on the result of (3.2), one can immediately obtain the price C_D of a digital call opt which pays the unit amount of cash at T to the option holder if $S(T) \geq K$:

$$C_D = e^{-rT}\mathbb{E}^{\mathbb{Q}}\Big[\mathbf{1}_{S(T)\geq K}\Big] = e^{-rT}\mathbb{Q}\Big(\{S(T) \geq K\}\Big).$$

$$\mathbb{Q}(\{S_T \geq K\})$$

$$\ln S_T - \ln S_0 = \left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T^{\mathbb{Q}}$$

$$S_T = S\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T + \sigma W_T^{\mathbb{Q}}\right\}$$

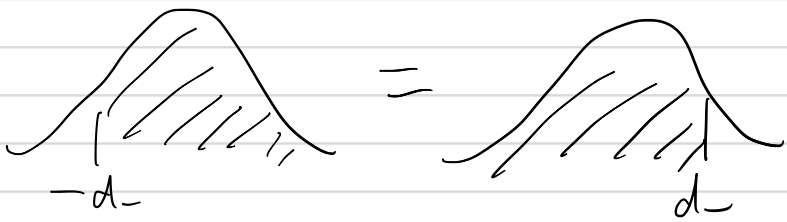
$$= S e^{\left(r-\frac{1}{2}\sigma^2\right)T} e^{\sigma\sqrt{T}Z} \qquad Z \sim N(0,1)$$

$$S_T \geq K \Leftrightarrow \ln S_T \geq \ln K$$

$$\ln S + \left(r-\frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z \geq \ln K$$

$$Z \geq -\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S}{K} + \left(r-\frac{1}{2}\sigma^2\right)T\right) := -d_-$$

$$\mathbb{Q}(\{S_T \geq K\}) = \mathbb{Q}(\{Z \geq -d_-\}) = \mathbb{Q}(\{Z \leq d_-\})$$



$$= \underline{N(d_-)} \quad //$$

$$C_D = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}\{S_T \geq K\}\right] = e^{-rT}\mathbb{Q}(\{S_T \geq K\}) = \underline{e^{-rT}N(d_-)} \quad //$$

$$\underline{d_- = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S}{K} + \left(r-\frac{1}{2}\sigma^2\right)T\right)} \quad //$$

$$\underline{= \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{F}{K} - \frac{\sigma^2}{2}T\right)} \quad // \qquad \left(F = Se^{rT}\right)$$

~~(3.3)~~ Give the Vega for the above digital call option and draw a schematic graph of the Vega by taking the current stock price as the horizontal axis. Moreover, based on the graph, express an important risk-management issue for this product (in the *real* financial market) regarding change of the slope in the *implied volatility*.

$$C_0 = e^{-rT} N(d_-)$$ digital call $\quad F = Se^{rT}$

$$Vega = \frac{\partial C_0}{\partial \sigma} = e^{-rT}\frac{\partial}{\partial\sigma}\int_{-\infty}^{d_-}\phi(z)\,dz = e^{-rT}\phi(d_-)\boxed{\frac{\partial d_-}{\partial \sigma}}$$

$$\boxed{\frac{\partial}{\partial\sigma}d_-} = \frac{\partial}{\partial\sigma}\left[\frac{1}{\sigma\sqrt T}\left(\ln\frac FK - \frac{\sigma^2}{2}T\right)\right] = -\frac{1}{\sigma^2\sqrt T}\ln\frac FK - \frac{\sqrt T}{2}$$

$$= -\frac{1}{\sigma^2\sqrt T}\left(\ln S - \ln K + \left(r+\frac{\sigma^2}{2}\right)T\right) = -\frac{1}{\sigma}d_+$$

$$Vega = \frac{e^{-rT}}{\sqrt{2\pi}}\exp\left\{-\frac{(\ln S - \ln K + (r - \frac{\sigma^2}{2})T)^2}{2\sigma^2 T}\right\}\cdot\left\{-\frac{1}{\sigma^2\sqrt T}\left(\ln S - \ln K + \left(r+\frac{\sigma^2}{2}\right)T\right)\right\}$$

$$= -\frac{e^{-rT}}{\sqrt{2\pi}}\frac{1}{\sigma}\frac{\ln S - A_+}{\sigma\sqrt T}\exp\left(-\frac{(\ln S - A_-)^2}{2\sigma^2 T}\right) \qquad d_\pm = \frac{\ln S - A_\pm}{\sigma\sqrt T}$$

$$= -\frac{e^{-rT}}{\sqrt{2\pi}\sigma}d_+\exp\left(-\frac{d_-^2}{2}\right) \qquad A_\pm = \ln K - \left(r \pm \frac{\sigma^2}{2}\right)T$$

$$A_+ < A_-$$

$$\frac{\partial}{\partial S}(Vega) = -\frac{e^{-rT}}{\sqrt{2\pi}\sigma}\left[\frac{1}{\sigma\sqrt T}\frac{1}{S}\times\exp\left(-\frac{(\ln S - A_-)^2}{2\sigma^2T}\right) + \frac{\ln S - A_+}{\sigma\sqrt T}\times\left(-\frac{\ln S - A_-}{\sigma^2 T}\right)\cdot\frac{1}{S}\exp\left(-\frac{(\ln S - A_-)^2}{2\sigma^2T}\right)\right]$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}\sigma}\frac{1}{\sigma\sqrt T}\frac{1}{S}\exp\left(-\frac{(\ln S - A_-)^2}{2\sigma^2T}\right)\left[-1 + \frac{(\ln S - A_+)(\ln S - A_-)}{\sigma^2 T}\right]$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}\sigma}\frac{1}{\sigma\sqrt T}\frac{1}{S}\exp\left(-\frac{d_-^2}{2}\right)\left[-1 + d_+\cdot d_-\right]$$

$$\frac{\partial}{\partial S}(\mathrm{Vega}) = \frac{e^{-rT}}{\sqrt{2\pi}\,\sigma}\frac{1}{\sigma\sqrt{T}}\frac{1}{S}\left[-1+\frac{(\ln S-A_+)(\ln S-A_-)}{\sigma^2 T}\right]\exp\left(-\frac{(\ln S-A_-)^2}{2\sigma^2 T}\right)$$

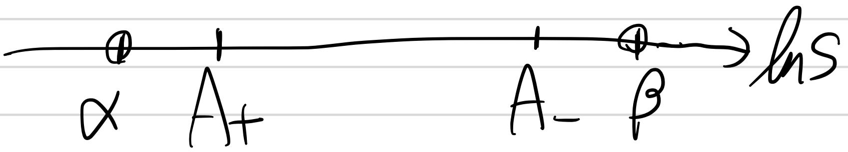
$$(\ln S-A_+)(\ln S-A_-)-\sigma^2 T=0$$

$$(\ln S)^2-(A_++A_-)\ln S+(A_+A_- -\sigma^2 T)=0$$

$$(\ln S)^2-2(\ln K-rT)\ln S$$
$$+\left\{(\ln K-rT)^2-\left(\frac{\sigma^2}{2}T\right)^2\right\}-\sigma^2 T=0$$

$$\ln S=(\ln K-rT)\pm\sqrt{(\ln K-rT)^2-(\ln K-rT)^2+\left(\frac{\sigma^2 T}{2}\right)^2+\sigma^2 T}$$

$$=(\ln K-rT)\pm\sqrt{\frac{\sigma^4T^2}{4}+\sigma^2 T}=\alpha,\beta$$
$$(\alpha<\beta)$$



$$\begin{cases}(\ln S-\alpha)(\ln S-\beta) > 0 & (\ln S>\beta)\\ \qquad\qquad " \qquad\qquad < 0 & (\alpha<\ln S<\beta)\\ \qquad\qquad " \qquad\qquad > 0 & (\ln S<\alpha)\end{cases}$$

$$A_\pm = \ln K - rT \mp \frac{\sigma^2}{2}T \qquad ※\ A_+ < A_-$$

$$\alpha, \beta = \ln K - rT \pm \sqrt{\frac{\sigma^4 T^2}{4} + \sigma^2 T} \qquad \alpha < \beta$$

S	0	$\cdots$	e^{α}	$\cdots$	e^{A_+}	$\cdots$	e^{A_-}	$\cdots$	e^{β}	$\cdots$
V'	/	$+$	0	$-$		$-$		$-$	0	$+$
V	/	$\nearrow$		$\searrow$	$\boxed{}$	$\searrow$	$\boxed{}$	$\searrow$		$\nearrow$

$$\mathrm{Vega}(S) = -\frac{e^{-rT}}{\sqrt{2\pi}} \frac{1}{\sigma} \frac{\ln S - A_+}{\sigma\sqrt{T}} \exp\left(-\frac{(\ln S - A_-)^2}{2\sigma^2 T}\right)$$

$$\left.\begin{array}{l} \displaystyle\lim_{S \to 0} \mathrm{Veg}(S) = \\ \displaystyle\lim_{S \to +\infty} \mathrm{Vega}(S) = \end{array}\right\} \text{意外とめんどくさい}$$

Vega(S)の形をみれば
Vega(S)=0 となるのは明らかに
$S = e^{A_+}$のみ

$$\mathrm{Vega}(e^{A_+}) = 0$$

$$\mathrm{Vega}(e^{A_-}) = -\frac{e^{-rT}}{\sqrt{2\pi}\sigma} \frac{A_- - A_+}{\sigma\sqrt{T}} = \frac{e^{-rT}}{\sqrt{2\pi}\sigma} \frac{\sigma^2 T}{\sigma\sqrt{T}} = -\sqrt{\frac{T}{2\pi}} e^{-rT}$$

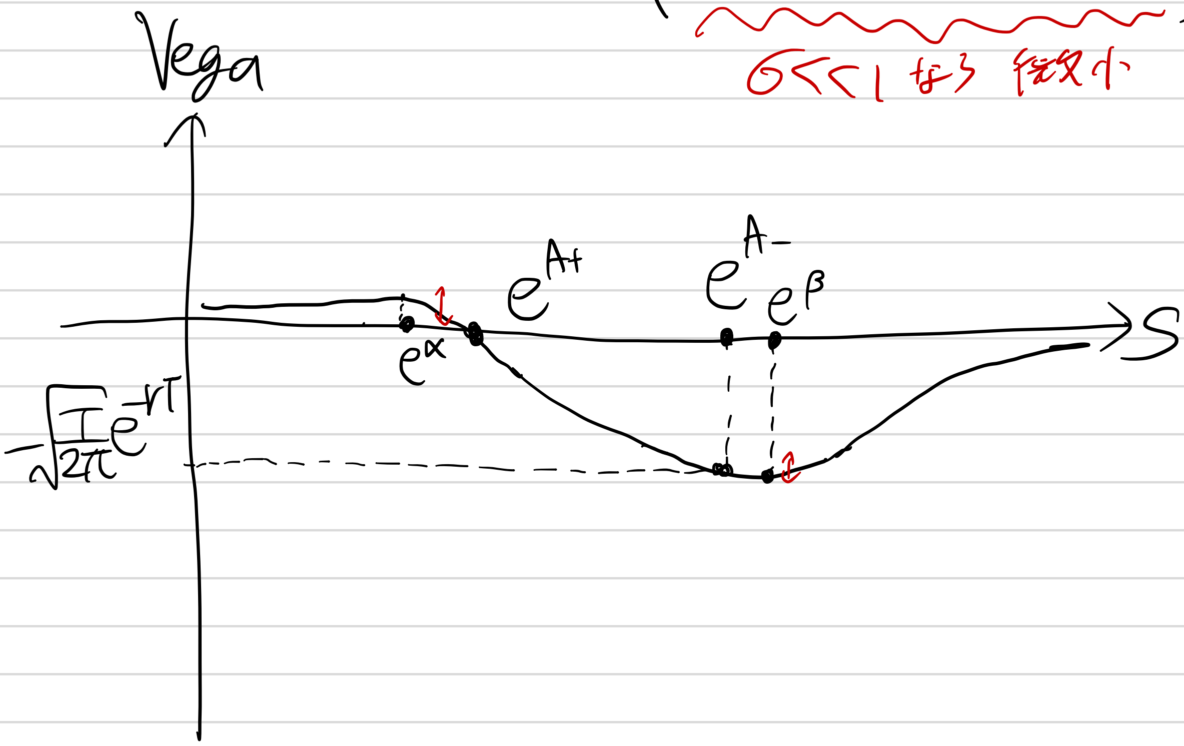
$$\begin{cases} \alpha - A_+ = -\left(\sqrt{\frac{\sigma^4 T^2}{4} + \sigma^2 T} - \frac{\sigma^2}{2}T\right) < 0 \\ \alpha - A_- = -\left(\sqrt{\frac{\sigma^4 T^2}{4} + \sigma^2 T} + \frac{\sigma^2}{2}T\right) < 0 \\ \beta - A_+ = \sqrt{\frac{\sigma^4 T^2 + \sigma^2 T}{4}} + \frac{\sigma^2}{2}T > 0 \\ \beta - A_- = \sqrt{\frac{\sigma^4 T^2 + \sigma^2 T}{4}} - \frac{\sigma^2}{2}T > 0 \end{cases}$$

if $\sigma \ll 1$

これらの差は全部微小

$$\mathrm{Vega}(e^{\alpha}) = \frac{e^{-rT}}{\sqrt{2\pi}\,\sigma}\left(\frac{\sqrt{\frac{\sigma^4T^2}{4}+\sigma^2T}-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}\,\sigma}\,\frac{1}{\sigma\sqrt{T}}\left(\underbrace{\sqrt{\left(1+\frac{\sigma^2T}{2}\right)^2-1}-\frac{\sigma^2T}{2}}_{\sigma \ll 1 \text{ なら 微小}}\right)$$



What's important risk-management issue?

...??