試験科目 出題教員名

上級デリバティブ I
(債券分析) 藤井優成

Remark

One must clearly state in the answer sheet when you make any additional assumptions or define variables which are not stated in the problem. One can use either English or Japanese. One can assume the existence of the standard "idealistic" market conditions used in the lecture course. $W^{\mathbb{Q}}$ stands for a one-dimensional Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. One can use the following functions for the standard normal distribution:

$$\phi(z) := \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{z^2}{2}\Big), \qquad N(x) := \int_{-\infty}^{x} \phi(z) dz \ . \tag{0.1}$$

Problem 1

A forward rate agreement (FRA) is a binding contract between a lender and a borrower agreeing to let a certain fixed interest rate K act on a prefixed cash amount N over a certain future time interval $[T_M, T_N]$. The day-count fraction for the period is given by $\delta(T_M, T_N)$. The current time is $t = 0$ and assume $0 < T_M < T_N$. K is defined as a simple rate and there is no compounding in the time interval $[T_M, T_N]$.

(1.1): Explain the definition of the *forward rate* F for the above contract.

(1.2): Derive the forward rate F at $t = 0$ for the time interval $[T_M, T_N]$ based on a *replication strategy*, and express it by the risk-free zero-coupon bond prices and the day-count fraction. One must answer not only the result of F but also the details of the replication strategy.

Problem 2

Consider a one-period binomial stock model. The current time is 0 and the terminal time is T (> 0). There is a stock whose price at $t = 0$ is s. The stock price $S(T)$ at time T will be either $s \times u$ with probability p_u or $s \times d$ with probability p_d without any dividend payments. Here, p_u, p_d are the probabilities under the empirical measure. There also exists a risk-free bond whose price at $t = 0$ is 1 and will be e^{rT} at time T for sure. Here, all the parameters are constant satisfying $r, s, p_u, p_d > 0$, $0 < d < u$, and $p_u + p_d = 1$.

(2.1) Give q_u (the probability of the upward movement) and q_d (the probability of the downward movement) under the risk-neutral measure $\mathbb{Q}$ such that $\mathbb{E}^{\mathbb{Q}}[e^{-rT} S_T] = s$ is satisfied. Furthermore, give the conditions that makes $\mathbb{Q}$ well-defined, i.e., $q_u > 0, q_d > 0$ and $q_u + q_d = 1$.

(2.2) Explain why the existence of the probability measure $\mathbb{Q}$ is necessary as well as sufficient for the absence of arbitrage in this one-period binomial stock model.

Problem 3

Let us consider the price dynamics of a non-dividend paying stock satisfying

$$dS(t) = rS(t)dt + \sigma S(t) dW_t^{\mathbb{Q}}, \quad S(0) = s$$

where s, r, σ are all strictly positive constants. r is the risk-free interest rate.

(3.1) Derive the stochastic differential equation for $\ln(S(t))$, i.e., calculate $d\ln(S(t))$ by Itô-formula.

(3.2) Give a European call option price at time zero with the expiry T (> 0) and the strike price K (> 0). You do not have to give the derivation.

(3.3) Derive a European put option price at time zero with the expiry T (> 0) and the strike price K (> 0) by using the put-call parity and the result of (3.2).

Problem 4

Let the domestic currency be JPY (Japanese Yen) and assume its risk-free interest rate is zero. We denote NZD (New Zealand dollar) risk-free interest rate by r_f (> 0) and consider NZD as a foreign currency. We assume both interest rates are constant. The dynamics of NZD/JPY exchange rate $X(t)$ (i.e., the price of one NZD in terms of JPY at time t) is assumed to follow a geometric Brownian motion:

$$
\begin{aligned}
dX(t) &= (0 - r_f)X(t)dt + \sigma X(t) dW_t^{\mathbb{Q}} \\
X(0) &= x
\end{aligned}
$$

where σ is a positive constant and $\mathbb{Q}$ is the risk-neutral measure for JPY. Since the JPY interest rate is so low, a bank is offering a "seemingly attractive" structured deposit linked to NZD exchange rate for Japanese customers. It is designed as follows;

(i) At time 0, the customer deposit ¥ x (i.e., x units of JPY) to the bank.
(ii) At the maturity T, the bank pays either (a) or (b) based on the realization of $X(T)$;
 (a) $\left(xe^{RT}\right)$ in JPY (i.e. offering a domestic deposit with rate R) if $X(T) \geq x$
 (b) $\left(e^{r_f T}\right)$ in NZD (i.e. offering a foreign deposit with rate r_f) if $X(T) < x$.
Here, R is a constant and announced to the depositor (just) before the contract is made.

Now, answer the following questions:
(4.1) Calculate the probability $\mathbb{Q}\Big(X(T) \geq x\Big)$, i.e. the probability of the events $\{X(T) \geq x\}$ under the risk-neutral measure. (You know the distribution of $X(T)$ under $\mathbb{Q}$!)

(4.2) Prove the following R^* makes the above deposit "fair" i.e., the present value of the bank payment at time T is equal to ¥ x;

$$
R^* = \frac{1}{T}\ln\left(N\Big(\frac{\sigma\sqrt{T}}{2} - \frac{r_f}{\sigma\sqrt{T}}\Big)\Big/N\Big(-\frac{\sigma\sqrt{T}}{2} - \frac{r_f}{\sigma\sqrt{T}}\Big)\right)
$$

with $N(\cdot)$ defined in Eq.(0.1).

(4.3) One can see that R^* is positive since $N(\cdot)$ is a positive and increasing function. In fact, R^* can be very large for relatively short deposit period. What is an important risk that the depositors need to accept in return for a seemingly high return?

Problem 1

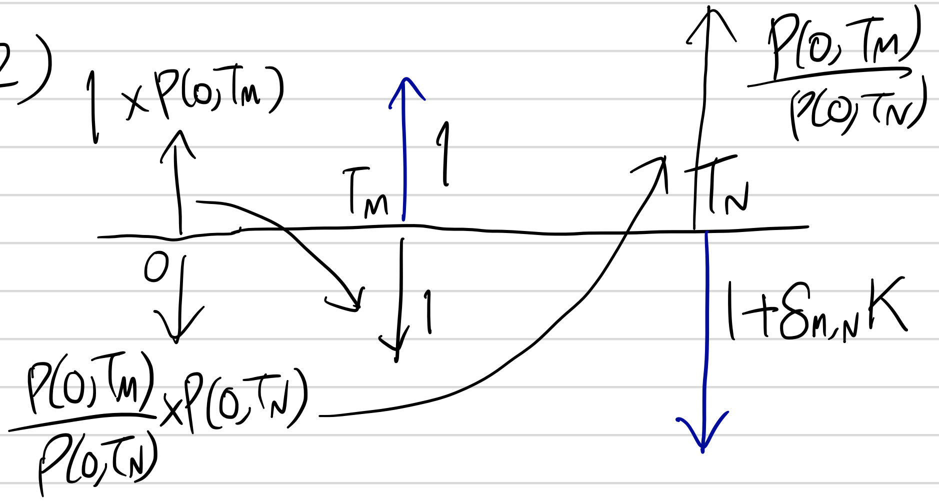
A forward rate agreement (FRA) is a binding contract between a lender and a borrower agreeing to let a certain fixed interest rate K act on a prefixed cash amount N over a certain future time interval $[T_M, T_N]$. The day-count fraction for the period is given by $\delta(T_M, T_N)$. The current time is $t = 0$ and assume $0 < T_M < T_N$. K is defined as a simple rate and there is no compounding in the time interval $[T_M, T_N]$.

(1.1): Explain the definition of the *forward rate* F for the above contract.

(1.1) F is the fixed rate K that makes P.V. of FRA zero. //

(1.2): Derive the forward rate F at $t = 0$ for the time interval $[T_M, T_N]$ based on a *replication strategy*, and express it by the risk-free zero-coupon bond prices and the day-count fraction. One must answer not only the result of F but also the details of the replication strategy.

(1.2)



replication strategy :)

at $t=0$ • Enter the FRA to borrow unite amount of cash for $[T_M, T_N]$

• Short one ZCB with mat. T_M

• Long $\frac{P(0,T_M)}{P(0,T_N)}$ amount of ZCB with mat. T_N

at $t=T_M$ • Borrow unit amount of cash and use it to close the short position

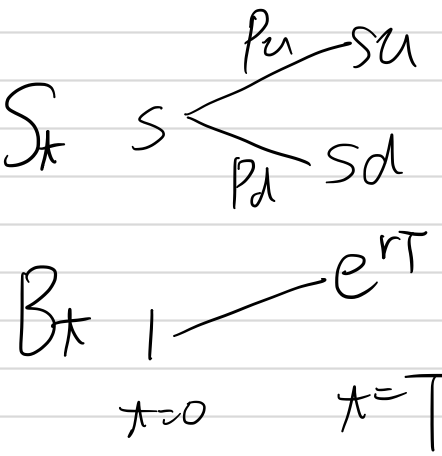
at $t=T_N$ • Receive $\frac{P(0,T_M)}{P(0,T_N)}$ and pay $1+\delta_{M,N}\boxed{F}$

$$\frac{P(0,T_M)}{P(0,T_N)} = 1+\delta_{M,N}F \iff F = \frac{1}{\delta_{M,N}}\left(\frac{P(0,T_M)}{P(0,T_N)} - 1\right)$$ //

Problem 2

Consider a one-period binomial stock model. The current time is 0 and the terminal time is T (> 0). There is a stock whose price at $t = 0$ is s. The stock price $S(T)$ at time T will be either $s \times u$ with probability p_u or $s \times d$ with probability p_d without any dividend payments. Here, p_u, p_d are the probabilities under the empirical measure. There also exists a risk-free bond whose price at $t = 0$ is 1 and will be e^{rT} at time T for sure. Here, all the parameters are constant satisfying $r, s, p_u, p_d > 0$, $0 < d < u$, and $p_u + p_d = 1$.

(2.1) Give q_u (the probability of the upward movement) and q_d (the probability of the downward movement) under the risk-neutral measure $\mathbb{Q}$ such that $\mathbb{E}^{\mathbb{Q}}[e^{-rT} S_T] = s$ is satisfied. Furthermore, give the conditions that makes $\mathbb{Q}$ well-defined, i.e., $q_u > 0, q_d > 0$ and $q_u + q_d = 1$.



$$\mathbb{E}^{\mathbb{Q}}[e^{-rT} S_T] = s$$

$$e^{-rT}(q_u \cdot su + q_d \cdot sd) = s$$

$$q_u u + (1 - q_u) d = e^{rT}$$

$$\therefore \begin{cases} q_u = \dfrac{e^{rT} - d}{u - d} \\ q_d = \dfrac{u - e^{rT}}{u - d} \end{cases}$$

//

conditions: $q_u > 0, q_d > 0$

$$\Longrightarrow d < e^{rT} < u$$

//

(2.2) Explain why the existence of the probability measure $\mathbb{Q}$ is necessary as well as sufficient for the absence of arbitrage in this one-period binomial stock model.

existence of $\mathbb{Q}$ (EMM) $\Leftrightarrow$ no-arbitrage を示す.

$\Leftrightarrow d < e^{rT} < u$

① ($\Rightarrow$ sufficiency)

$d < e^{rT} < u$

Assume a portfoilo h s.t. $V_0^h = 0$

then $x + yS = 0 \Leftrightarrow yS = -x$

Then, $V_T^h = xe^{rT} + ySZ = x(e^{rT} - Z)$

$$Z = \begin{cases} u & (P_u > 0) \\ d & (P_d > 0) \end{cases}$$

$\therefore P(V_T^h \geq 0) < 1 \Rightarrow$ no-arbitrage ⊡

② ($\Leftarrow$ necessarity)

no-arbitrage $\longrightarrow d < e^{rT} < u$ の対偶を示す

②-1. $e^{rT} \leq d < u$ と仮定

Portfolio $h = (-S, 1)$ $V_0^h = -S + 1 \times S = 0$

$V_T^h = -Se^{rT} + SZ = S(-e^{rT} + Z)$

$P(V_T^h \geq 0) = 1$, $P(V_T^h > 0) = P(Z = u) = P_u > 0$.

$\Rightarrow$ arbitrage

②-2. $d < u \leq e^{rT}$ と仮定

Portfolio $h = (S, -1)$ $V_0^h = S + (-1) \times S = 0$

$V_T^h = Se^{rT} - SZ = S(e^{rT} - Z)$

$P(V_T^h \geq 0) = 1$, $P(V_T^h > 0) = P(Z = d) = P_d > 0$

$\Rightarrow$ arbitrage

⊡

Problem 3

Let us consider the price dynamics of a non-dividend paying stock satisfying

$$dS(t) = rS(t)dt + \sigma S(t)dW_t^{\mathbb{Q}}, \quad S(0) = s$$

(3.1) Derive the stochastic differential equation for $\ln(S(t))$, i.e., calculate $d\ln(S(t))$ by Itô-formula.

(3.1) $f(x) = \ln x \qquad \frac{\partial f}{\partial t} = 0, \ \frac{\partial f}{\partial x} = \frac{1}{x}, \ \frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$

$$d(\ln S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\frac{dS_t^2}{S_t^2}$$

$$= r\,dt + \sigma dW_t^{\mathbb{Q}} - \frac{1}{2}\frac{1}{\cancel{S_t^2}}\left(\sigma^2 \cancel{S_t^2}\,dt\right) = \underline{\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t^{\mathbb{Q}}}$$

(3.2) Give a European call option price at time zero with the expiry T (> 0) and the strike price K (> 0). You do not have to give the derivation. 一応derivationをつけた.

fwd price

(3.2) $S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T^{\mathbb{Q}}} = {}^{☆}F e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Z} \qquad ☆F = \frac{S_0}{P(0,T)} = S_0 e^{rT}$

$Z \sim N(0,1)$

$$C = \mathbb{E}^{\mathbb{Q}}[e^{-rT}C_T] = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)_+]$$

$$= e^{-rT}\left(\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T \geq K\}}] - K\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{S_T \geq K\}}]\right)$$

$$= e^{-rT}\Big(F \underbrace{\mathbb{E}^{\mathbb{Q}}[e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Z}\mathbb{1}_{\{S_T \geq K\}}]}_{①} - K\underbrace{\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{S_T \geq K\}}]}_{②}\Big)$$

$$S_T \geq K \iff -\frac{\sigma^2}{2}T + \sigma\sqrt{T}Z \geq K/F$$

$$\iff Z \geq -\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{F}{K}\right) - \frac{\sigma^2}{2}T\right) = -d_-$$

$$① = \int_{-d_-}^{\infty} e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Z}\frac{1}{\sqrt{2\pi}}e^{-\frac{Z^2}{2}}dZ = \int_{-d_-}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(Z - \sigma\sqrt{T})^2}dZ$$

$$= \int_{-(d_- + \sigma\sqrt{T})}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Z'^2}dZ' = \int_{-\infty}^{d_- + \sigma\sqrt{T}}\phi(Z')dZ' = N(d_+)$$

$d_+ = d_- + \sigma\sqrt{T}$

$$② = \int_{-d_-}^{\infty}\phi(Z)dZ = \int_{-\infty}^{d_-}\phi(Z)dZ = N(d_-)$$

$$\underline{C = e^{-rT}(FN(d_+) - KN(d_-))}$$

解答の2行でok

(3.3) Derive a European put option price at time zero with the expiry T (> 0) and the strike price K (> 0) by using the put-call parity and the result of (3.2).

put-call parity:

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

$$\hookrightarrow c - p = e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T - K] = S_0 - Ke^{-rT}$$

$$p = c - S_0 + Ke^{-rT} = e^{-rT}(FN(d_+) - KN(d_-)) - Fe^{-rT} + Ke^{-rT}$$

$$= e^{-rT}\{-F(1 - N(d_+)) + K(1 - N(d_-))\}$$

$$= e^{-rT}\{KN(-d_-) - F(-d_+)\}$$

※ $d_\pm = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{F}{K}\right) \pm \frac{\sigma^2}{2}T\right) = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K} + \left(r \pm \frac{\sigma^2}{2}\right)T\right)$

$$\frac{rT}{\sigma\sqrt{T}} = \frac{r\sqrt{T}}{\sigma}$$

Problem 4 DC $r_d = 0$

Let the domestic currency be JPY (Japanese Yen) and assume its risk-free interest rate is zero. We denote NZD (New Zealand dollar) risk-free interest rate by r_f (> 0) and consider NZD as a foreign currency. We assume both interest rates are constant. The dynamics of NZD/JPY exchange rate $X(t)$ (i.e., the price of one NZD in terms of JPY at time t) is assumed to follow a geometric Brownian motion:

X

FC

1ドル X 円

$$dX(t) = (0 - r_f)X(t)dt + \sigma X(t)dW_t^{\mathbb{Q}}$$
$$X(0) = x$$

(in D.C.)

where σ is a positive constant and $\mathbb{Q}$ is the risk-neutral measure for JPY. Since the JPY interest rate is so low, a bank is offering a "seemingly attractive" structured deposit linked to NZD exchange rate for Japanese customers. It is designed as follows;

(i) At time 0, the customer deposit ¥ x (i.e., x units of JPY) to the bank. x 円

(ii) At the maturity T, the bank pays either (a) or (b) based on the realization of $X(T)$;

(a) $\left(xe^{RT}\right)$ in JPY (i.e. offering a domestic deposit with rate R) if $X(T) \geq x$ 円安

(b) $\left(e^{r_f T}\right)$ in NZD (i.e. offering a foreign deposit with rate r_f) if $X(T) < x$. 円高

Here, R is a constant and announced to the depositor (just) before the contract is made.

(4.1) Calculate the probability $\mathbb{Q}\left(X(T) \geq x\right)$, i.e. the probability of the events $\{X(T) \geq x\}$ under the risk-neutral measure. (You know the distribution of $X(T)$ under $\mathbb{Q}$!)

(4.1) $$X_T = x\exp\left\{\left(-r_f - \frac{\sigma^2}{2}\right)T + \sigma W_T^{\mathbb{Q}}\right\} = xe^{\left(-r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z}$$

$$X_T \geq x \Leftrightarrow \left(-r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z \geq \ln 1 = 0$$

$$Z \geq \frac{1}{\sigma\sqrt{T}}\left(r_f + \frac{\sigma^2}{2}\right)T = \frac{\sqrt{T}}{\sigma}\left(r_f + \frac{\sigma^2}{2}\right) = d_1$$

$$\mathbb{Q}(\{X_T \geq x\}) = \int_{d_1}^{\infty} \phi(z)dz = \int_{-\infty}^{-d_1} \phi(z)dz = \underline{N(-d_1)} \quad //$$

$$\left(\text{※} \; d_1 = \frac{r_f\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right)$$

?(4.2) Prove the following R^* makes the above deposit "fair" i.e., the present value of the bank payment at time T is equal to ¥ x;

$$R^* = \frac{1}{T}\ln\left(N\left(\frac{\sigma\sqrt{T}}{2} - \frac{r_f}{\sigma\sqrt{T}}\right)\Big/N\left(-\frac{\sigma\sqrt{T}}{2} - \frac{r_f}{\sigma\sqrt{T}}\right)\right)$$

with $N(\cdot)$ defined in Eq.(0.1).

dの次元と合わない

$\frac{r_f\sqrt{T}}{\sigma}$

$$\left(d_1 = \frac{r_f\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right)$$

(4.2) at $t=0$ deposit $-x$ //

$$\text{at } t=T \begin{cases} +x e^{RT} & X_T \geq x \\ +X_T e^{r_f T} & X_T < x \end{cases}$$

$X_T \geq x \iff z \geq d_1$

$X_T < x \iff z < d_1$

$$-x + \cancel{e^{-r_f T}}\,\mathbb{E}^{\mathbb{Q}}\left[x e^{RT}\mathbb{1}_{\{X_T \geq x\}} + X_T e^{r_f T}\mathbb{1}_{\{X_T < x\}}\right] = 0$$

$$\cancel{x} = \cancel{x} e^{RT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{X_T \geq x\}}\right] + \cancel{e^{r_f T}}\,\cancel{x}\,\cancel{e^{-r_f T}}\,\mathbb{E}^{\mathbb{Q}}\left[e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}z}\mathbb{1}_{\{X_T < x\}}\right]$$

$$1 = e^{RT}\int_{d_1}^{\infty}\phi(z)\,dz + \int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}dz$$

$$1 = e^{RT}\int_{-\infty}^{-d_1}\phi(z)\,dz + \int_{-\infty}^{d_1-\sigma\sqrt{T}}\phi(z')\,dz'$$

$$\left(d_2 = d_1 - \sigma\sqrt{T}\right)$$

$$1 = e^{RT}N(-d_1) + N(d_2)$$

$$e^{RT} = \frac{1-N(d_2)}{N(-d_1)} = \frac{N(-d_2)}{N(-d_1)}$$

$$\therefore R^* = \frac{1}{T}\ln\left(\frac{N(-d_2)}{N(-d_1)}\right) = \frac{1}{T}\ln\left(\frac{N\left(\frac{\sigma\sqrt{T}}{2} - \frac{r_f\sqrt{T}}{\sigma}\right)}{N\left(-\frac{\sigma\sqrt{T}}{2} - \frac{r_f\sqrt{T}}{\sigma}\right)}\right) \quad //$$

$$\left(d_1 = \frac{r_f\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2},\quad d_2 = \frac{r_f\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right)$$

☆3.3と比較, $\frac{r}{\sigma\sqrt{T}}$ はおかしい…

(4.3) One can see that R^* is positive since $N(\cdot)$ is a positive and increasing function. In fact, R^* can be very large for relatively short deposit period. What is an important risk that the depositors need to accept in return for a seemingly high return?

(4.3)

$$R^* = \frac{1}{T}\ln\left(\frac{N(-d_2)}{N(-d_1)}\right) \qquad d_{1,2} = \left(\frac{r_f}{\sigma} \pm \frac{\sigma}{2}\right)\sqrt{T}$$

R^* is (always) positive $\iff N(-d_2) > N(-d_1)$

$\iff -d_2 > -d_1$

$\because$ $N(\cdot)$ is increasing function

$\iff d_1 > d_2 \leftarrow$ obvious //

for small $|x|$

$$\ln(N(x)) \approx -\ln 2 + \sqrt{\frac{2}{\pi}}\,x \leftarrow \text{cf. H29 (1.8)}$$

$$R^* = \frac{1}{T}\left(\ln(N(-d_2)) - \ln(N(-d_1))\right)$$

$$\approx \frac{1}{T}\left(\sqrt{\frac{2}{\pi}}(-d_2) - \sqrt{\frac{2}{\pi}}(-d_1)\right)$$

$$= \frac{1}{T}\sqrt{\frac{2}{\pi}}(d_1 - d_2) = \frac{1}{T}\sqrt{\frac{2}{\pi}} \times \sigma\sqrt{T} = \sqrt{\frac{2}{\pi T}}\,\sigma + o(\sigma)$$ //

$T \searrow \implies R^* \nearrow$ //

important risk

high returnを享受する代わりに……??