Advanced Derivative

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1 Bond and Forward contract

1.1 Interest Rates

Asm. 1.1.1. We asssume that there exists a default free money market account

- default-free
- liquid (borrowing ragte = lending rate)
- everyone can access equally

The interest rates associating with the default free m.m. (money market) are called risk-free rates.

1.1.1 Types of Accrual (利息)

Suppose one invests cash amount A at $t = 0$ for T years.
V : The amount of cash to be returned in T years time.

1. Annual compounding with interest rate R_1 per annum
   $V = A(1 + R_1)^T$
2. Semiannual compounding with interest rate R_2 per annum
   $V = A(1 + \frac{R_2}{2})^{2T}$
3. m-times compounding with interest rate R_m per annum
   $V = A(1 + \frac{R_m}{m})^{mT}$ (typiccaly we have $m = 1, 2, 4, 12, 52, 365$)
4. continuous compounding with interest rate r per annum
   $V = \lim_{m\to\infty} A(1 + \frac{r}{m})^{mT} = Ae^{rT}$

Relation among different compounding conventions:

No-Arbitrage $\rightarrow$ for any m,

$$e^{rT} = \left(1 + \frac{R_m}{m}\right)^{mT} \tag{1.1}$$

$$\Leftrightarrow \quad r = m \ln\left(1 + \frac{R_m}{m}\right), \quad R_m = m(e^{r/m} - 1). \tag{1.2}$$

at a given time, $R_1(T)$: T dependent.

1.1.2 Zero Rate and Bonds

Def. 1.1.1. (zero rate) The T-year zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for T-years without any intermidiate coupon payment.

Ex. 1.1.1. 5-year zero rate = 5 % per annum. (continuous compounding)
\$ 100 を deposit at $t = 0 \rightarrow$ (5 years later) $100e^{0.05\times5} \approx 128.40$

Present value (PV) $\quad P.V. = 128.40 \times e^{-0.05\times5} = 100$

Def. 1.1.2. (ZCB : zero coupon bond) T-year zero coupon bond is a bond which pays an unit amount of cash in T-year time without any coupon payment (assume risk-free, liquid).

If R is T-year zero rate, current price of T-year zero coupon Bond is given by :

$$P(0,T) = e^{-RT} \tag{1.3}$$

(0: current time, T: maturity), we call "discount factor".

Ex. 1.1.2. Suppose we have (at $t = 0$)

- $T = 0.5$, $R = 5.0\%$ (zero coupon rate)
- $T = 1.0$, $R = 5.8\%$
- $T = 1.5$, $R = 6.4\%$
- $T = 2.0$, $R = 6.8\%$

fixed coupon bond

- maturity: $T = 2$
- coupon payments: 6% per annum semiannually.
- principal: \$100

$$\begin{aligned}\text{Bond price} &= 3P(0,0.5) + 3P(0,1.0) + 3P(0,1.5) + 103P(0,2)\\ &= 3 \times e^{-5\%\times0.5} + 3 \times e^{-5.8\%\times1.0} + 3 \times e^{-6,4\%\times1.5} + 103 \times e^{-6.8\%\times2.0}\\ &\approx 98.39\end{aligned} \tag{1.4}$$

1.1.3 Yield (平均利率，平均利回り，平均 discount rate)

Def. 1.1.3. (Bond's Yield) A bond's yield is the single discount rate that when applied to eery cash flow, gives the market bond price.

Ex. 1.1.3. using Ex. 1.1.2, suppose market price = 98.39
Then the yield for the bond y is given by solving

$$3e^{-0.5y} + 3e^{-1.0y} + 3e^{-1.5y} + 103e^{-2.0y} = 98.39 \tag{1.5}$$
$$\Rightarrow y \simeq 6.76\% \quad \text{(continuous compounding)} \tag{1.6}$$

Def. 1.1.4. (Par Yield) The par yield for a certain maturity is the coupon rate that makes the bond price equal to its principal.

Ex. 1.1.4. using Ex. 1.1.2, par yield C for 2-year coupon bond is given by:

$$\frac{C}{2}e^{-0.050\cdot 0.5} + \frac{C}{2}e^{-0.058\cdot 1.0} + \frac{C}{2}e^{-0.064\cdot 1.5} + \left(100 + \frac{C}{2}\right)e^{-0.068\cdot 2.0} = 100 \tag{1.7}$$
$$\Rightarrow C \simeq 6.87\% \tag{1.8}$$

(par yield for $T = 2$, semiannual coupon payments)
In general,

- T-year bond
- m-time coupon payment per annum
- par yield C

$$\sum_{n=1}^{mT} \frac{C}{m} P\left(0, \frac{n}{m}\right) + 100P(0,T) = 100 \tag{1.9}$$
$$\Rightarrow C = \frac{100(1 - P(0,T))}{A}, \quad A = \sum_{n=1}^{mT} \frac{1}{m} P\left(0, \frac{n}{m}\right) \tag{1.10}$$

1.1.4 Duration

fixed coupond bond:

- (cash flow at T_i) $= C_i \quad (i = 1, ..., n)$
- C_i: coupon (+ principal at maturity)

Suppose its yield is given by y (continuous compounding).
Bond price:

$$B = \sum_{i=1}^{n} C_i e^{-yT_i} \tag{1.11}$$

The duration of the Bond:

$$D := -\frac{1}{B}\frac{d}{dy}B = -\left(\frac{dB/dy}{B}\right) = \frac{1}{B}\sum_{i=1}^{n} C_i T_i e^{-yT_i} \tag{1.12}$$

Ex. 1.1.5. zero coupon Bond

$$(C_i = 0)_{i=1,2,\ldots,n-1},\ C_n = 1, \quad B = e^{-yT_n} \tag{1.13}$$

$$\Rightarrow D = \frac{1}{B}C_n T_n e^{-yT_n} = T_n \tag{1.14}$$

(* Duration の長短により，金利に対する反応度の違いがわかる．)

Suppose the yield changes small amount Δy, ($y \to y + \Delta y, \quad B \to B + \Delta B$)

$$\frac{\Delta B}{B} = -D\Delta y + o(\Delta y) \tag{1.15}$$

(abbr.)

Suppose $D = 10$ (10 year), yield: $\Delta y = +0.1\%$ (10 basis points, b.p.):

$$\frac{\Delta B}{B} \approx -10 \times 0.1\% = -1\% = -0.01 \tag{1.16}$$

1.1.5 Modified Duration

yield (m-time compounding) $\hat{y}$

the same bond:

$$B = \sum_{i=1}^{n} C_i \left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \tag{1.17}$$

modified duration:

$$\begin{aligned} D^* &:= -\frac{1}{B}\frac{dB}{d\hat{y}} = \frac{1}{B}\sum_{i=1}^{n}\frac{C_i T_i}{1+\hat{y}/m}\left(1+\frac{\hat{y}}{m}\right)^{-mT_i} \\ &= \frac{1}{B(1+\hat{y}/m)}\sum_{i=1}^{n} C_i T_i \left(1+\frac{\hat{y}}{m}\right)^{-mT_i} \\ &= \frac{1}{B(1+\hat{y}/m)}\sum_{i=1}^{n} C_i T_i e^{-yT_i} = \frac{D}{1+\hat{y}/m} \quad \text{(no arbitrage)} \end{aligned} \tag{1.18}$$

* duration の議論は cash flow が一方向のときのみ使える．Insurance では通用しないので注意.

1.1.6 Convecity)

y : yiled (continuous compounding)

$$C := \frac{1}{B}\frac{d^2B}{dy^2} = \frac{1}{B}\sum_{i=1}^{n} C_i T_i^2 e^{-yT_i} \tag{1.19}$$

$y \to y + \Delta y, \quad B \to B + \Delta B$:

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2 B}{dy^2}(\Delta y)^2 + o(\Delta y^2) \tag{1.20}$$

$$\Rightarrow \frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2 + o(\Delta y^2) \tag{1.21}$$

(Duration matching : abbr.)

1.2 Forward Contract

1.2.1 Forward Price

Def. 1.2.1. (Forward Contract) A forward contract with maturity T is a bilateral binding promise(agreement) such that at time $t = T(> 0)$, the two parties exchange:

- the cash amount given by the time T realization of a certain index (such as a stock price) with the fixed amount of cash (cash delivery)
- the unit amount of asset (such as a share of an equity) with fixed amount of cash (physical delivery=現物)

Def. 1.2.2. (Forward Price) A forward price F at the current time ($t = 0$) (契約時) of the underlying index X is the amount of cash K that make the present value of the forward contract exchanging X_t and K at T zero. (Forward contract has $P.V. = 0$, with $K = F$.)
* F は契約時に支払う額ではないことに注意 (元手は不要)
* K such that P.V. of the fwd contract = 0

Ex. 1.2.1. Consider a forward contract on a non-dividend paying stock, with mat. T.
$X_T = S_T$ (stock price at T), exchange $F \leftrightarrow S_T$ (at T).

Asm. 1.2.1. Stock market is liquid, zero-coupon bond is liquid.

the forward price at $t = 0$ is given by:

$$F = \frac{S_0}{P(0,T)} = e^{rT}S_0 \tag{1.22}$$

- r : zero-rate for mat T, continuous, compounding
- $P(0,T)$: zero coupon bond price

Prf. 1.2.1. replication strategy

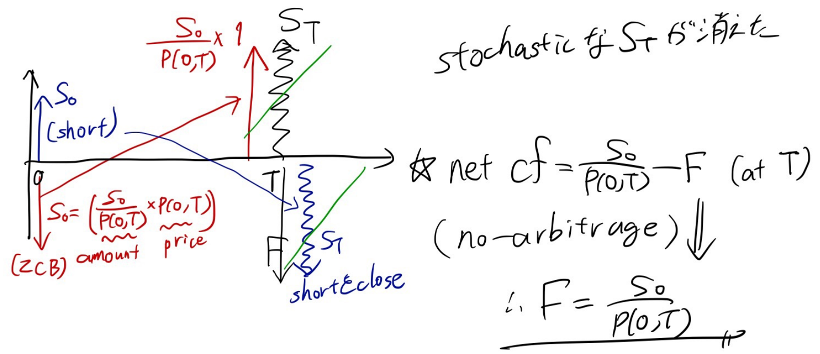
- Enter the fwd contract to get one share of stock (S_T) by paying F at T ($t = 0$ で enter する際に元手は不要)
- Sell one share of stock at $t = 0$ to get S_0 (short position)
- Use S_0 to buy ZCB (zero coupon bond) by the amount $S_0/P(0,T)$

- Pay F and receive S_T at T (return S_T to lender)
- Receive $S_0/P(0,T)$ from ZCB lender

(cash flow illustration below)

if $F \neq \dfrac{S_0}{P(0,T)} \Rightarrow$ arbitrage (no risk, arbitrary positive return)

No-arbitrage $\Rightarrow F = \dfrac{S_0}{P(0,T)} = e^{rT} S_0$ (F は stochastic ではない)



Ex. 1.2.2. Same as Ex.1.2.1 but now the stock pays <u>continuous dividend</u>, with dividend rate y ($y \in \mathbb{R}$, constant)

One share of the stock pays $S_t y dt$ for the interval $[t, t+dt]$ for any $t \geq 0$.

<u>forward price at $t = 0$</u>

$$F = \frac{S_0}{P(0,T)} e^{-yT} = S_0 e^{(r-y)T} \tag{1.23}$$

r : zero-rate for mat T at $t = 0$

Suppose we heve N_t shares at t, dividend paid in $[t, t+dt]$: $S_t N_t y dt$

$\Rightarrow$ reinvest $\Delta N_t = N_t y dt$

if one reinvests the whole dividend payment,

$$\frac{dN_t}{dt} = N_t y \Rightarrow N_t = N_0 e^{yt} \quad \text{for all} \quad t \geq 0 \tag{1.24}$$

Therefore, if one wants $N_T = 1$, N_0 is to be e^{-yT}.

Prf. 1.2.2. replication strategy

($t = 0$)

- Enter the fwd contract to receive F and deliver one share stock at T (Ex.1.2.1 と逆の party)

- Sell $\dfrac{S_0 e^{-yT}}{P(0,T)}$ amount of ZCB with maturity T
- Buy e^{-yT} shares of stock

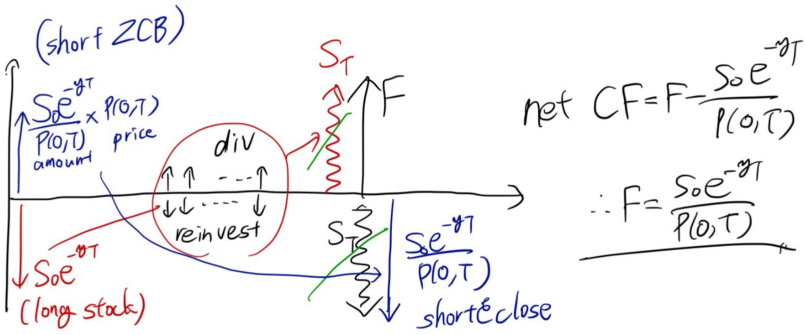
(always)

- Reinvest every div. payment to the stock

($t = T$)

- Receive F, deliver one share of stock
- Return $\dfrac{S_0 e^{-yT}}{P(0,T)}$ to the ZCB lender

(cash flow illustration below)



イメージ:

$$\text{P.V}(\text{receive}\, S_T \,\text{at}\, T) = F \times P(0,T) = \frac{S_0}{P(0,T)} \times P(0,T) = S_0 \tag{1.25}$$

random な cash flow の P.V. を計算するときは，Forward Price を求めて，$P(0,T)$ を乗じてやればよい (ただし市場が liquid,replicatable なときのみ)

⇒ 後に risk-nuetral の下で計算すればわざわざ replication を考える必要がなくなる

1.2.2 Mark to Market of a fwd contract

* P.V.(at $t = 0$){fwd contract} は 0 だが，時間が進むにつれて，P.V. は変化する．

Suppose we entered the fwd contract to receive X_T in exchange for a fixed amount of cash F_0 (F_0 : fwd price at $t = 0$). P.V.($t = 0$)= 0
at time $t \in (0,T)$, suppose fwd price is given by F_t. We want to know P.V.($t > 0$).
Ans.:

$$\text{P.V.}(t) = P(t,T)(F_t - F_0) \tag{1.26}$$

Prf. 1.2.3. enter the new fwd contract at $t = t$ (pays X_T and receives F_t at $t = T$)
* F_t の t は「時刻 t に contract に enter した」という意味.
(abbr.)

$$\begin{aligned} \text{P.V.}(\text{at}\, t)\{\text{new + original fwd contracts}\} &= P(t,T)(F_t - F_0) \\ &= 0 + \text{P.V.}(\text{at}\, t)\{\text{original}\} \end{aligned} \tag{1.27}$$

1.2.3 Put-Call Parity

Def. 1.2.3. (Call Option and Put Option) A call (respectively, put) option on a certain index X with expiry T and strike K is the binomial?? contract to pay the holder of the option the cash amount equal to $\max(X_T - K, 0)$, (resp, $\max(K - X_T, 0)$)

Let C (resp, P) be the call (resp, put) option orice (at $t = 0$). We have the following put-call parity:

$$C - P = P(0,T)(F_X - K) \tag{1.28}$$

- F_X : the fwd price of X with mat. T (時刻 $t = X$ ではなく，underlying asset X をもとにした fwd price at $t = 0$)

Prf. 1.2.4. cash flow at T:

$$\max(X_T, K, 0) - \max(K - X_T, 0) = X_T - K \tag{1.29}$$

$$\tag{1.30}$$

Present value of above is given by:

$$C - P = P(0,T)(F_X - K) \tag{1.31}$$

motivation:

- liquidity の問題
- call, put の一方が求まれば，もう一方をすぐに求められる
- PDE の計算は put の方が簡単 (because of boundary condition)

1.3 Forward Rate Agreement and Interest Rate Swap

*金利スワップは not tradable

1.3.1 Simple Rate and Day-Count Convention

Suppose T_i specifies the date $D_i = D(d_i, m_i, y_i)$

1. Actual/365

$$\delta(T_0, T_1) = \frac{D_1 - D_0}{365} \tag{1.32}$$

2. Actual/360

$$\delta(T_0, T_1) = \frac{D_1 - D_0}{360} \tag{1.33}$$

3. 30/360

$$\delta(T_0, T_1) = \frac{\max(30 - d_0, 0) + \min(d_1, 30) + 360(y_1 - y_0) + 30(m_1 - m_0 - 1)}{360} \tag{1.34}$$

4. actual/actual considering leap year? 365 or 366

Def. 1.3.1. (Simple Rate) A (risk-free) simple rate (not compound) $L(T_{i-1}, T_i)$ with day-count $\delta(T_{i-1}, T_i)$ is the interest rate with accrual convention defined in such a way that, when one invest N amount of cash at T_{i-1}, then he receives $N(1 + \delta_i L(T_{i-1}, T_i))$ at time T_i. $L(T_{i-1}, T_i)$ is the zero coupon rate at T_{i-1} for $[T_{i-1}, T_i]$ with corresponding day-count convention.

*accrual: ??

1.3.2 Forward Rate Agreement (FRA)

Def. 1.3.2. Forward Rate Agreement A FRA is a binding(義務の) contract with the two parties (lender and borrower) agreeing to let a certain fixed rate K act on a prefixed notional amount N, over a future period $[T_M, T_N]$.

*notional: 想定される?

Def. 1.3.3. Forward Rate A forward rate F for the period $[T_M, T_N]$ with day-count $\delta = \delta(T_M, T_N)$ is the fixed rate K with the some day-count convention that makes the present value of the FRA zero.

off course,

$$F = L(T_M, T_N) \quad (\text{at}\, T_M) \tag{1.35}$$

*L : simple rate

Lem. 1.3.1. Let $\delta = \delta(T_M, T_N)$. Then the forward rate F at $t = 0$ fot the period $[T_M, T_N]$ is given by

$$F = \frac{1}{\delta}\left(\frac{P(0, T_M)}{P(0, T_N)} - 1\right) \tag{1.36}$$

(*asm: liquid, no-arbitrage)

Prf. 1.3.1. replication strategy

- Enter the FRA of rate F to borrow unit amount of cash for $[T_M, T_N]$
- Sell one ZCB with mat. T_N (short)
- Buy ZCB with mat. T_N with principal amount $\dfrac{P(0,T_M)}{P(0,T_N)}$
- At T_M, borrow unit amount of cash through FRA and use it to return ZCB
- At T_N, receive the principal $\dfrac{P(0,T_M)}{P(0,T_N)}$, and pays $(1+\delta F)$

(abbr.)

Net cash flow at T_N: $\left(\dfrac{P(0,T_M)}{P(0,T_N)}\right) - (1+\delta F)$

If we require no-arbitrage,

$$\left(\frac{P(0,T_M)}{P(0,T_N)}\right) - (1+\delta F) = 0 \quad \Rightarrow \quad F = \frac{1}{\delta}\left(\frac{P(0,T_M)}{P(0,T_N)} - 1\right) \tag{1.37}$$

We write the above F as

$$F(0,T_M,T_N) = \frac{1}{\delta}\left(\frac{P(0,T_M)}{P(0,T_N)} - 1\right) \tag{1.38}$$

In general. at $t < T_M$,

$$F(t,T_M,T_N) = \frac{1}{\delta}\left(\frac{P(t,T_M)}{P(t,T_N)} - 1\right) \tag{1.39}$$

$t \uparrow T_M$:

$$F(T_M,T_M,T_N) = \frac{1}{\delta}\left(\frac{1}{P(T_M,T_N)} - 1\right) = L(T_M,T_N) \tag{1.40}$$

$(\because)$

$$1 = P(T_M,T_N)\{1+\delta L(T_M,T_N)\} \tag{1.41}$$

at T_M invest 1, at T_N return $1+\delta L(T_M,T_N)$
(otherwise there exist arbitrage opportunities.)
(abbr.)

Ex. 1.3.1. Q) P.V.(at 0) $\{\text{receive } 1+\delta L(T_M,T_N)\}$? $\to$ A) P.V.=0
$(\because)$ Suppose that we are at $t = T_M$, $L(T_M,T_N)$ ... known

$$\text{P.V.}(\text{at}\, T_M) = -1 + P(T_M,T_N)(1+\delta L(T_M,T_N)) = 0 \tag{1.42}$$

将来のある時点 $(t = T_M)$ で P.V.$= 0$ なら，さかのぼった $t = 0$ でも当然 P.V.$= 0$

$$P.V.(\text{at}\,0)\{\text{receive}\,1+\delta F(0,T_M,T_N)\,\text{at}\,T_N\} = P.V.(\text{at}\,0)\{\text{receive}\,1+\delta L(T_M,T_N)\} \tag{1.43}$$

$$(\therefore) \quad P(0,T_N)F(0,T_M,T_N) = P.V.(\text{at}\,0)\{\text{receive}\,L(T_M,T_N)\} \tag{1.44}$$

1.3.3 Fixed vs Floating Interest Swap

Fix a time partition $0 = T_0 < T_1 < ... < T_M$

Def. 1.3.4. (Spot-start Swap) A spot-start swap with maturity T_M and notional amount N is the contract in which one party (receiver) receives cash amount $NK\Delta_i$ (K fixed) and pays the stochastic amount $NL(T_{i-1}, T_i)\delta_i$ at every T_i, $i = \{1, 2, ..., M\}$. The other party (payer) has the opposit cash flow. Here,

$$\begin{cases} \Delta_i & = \Delta(T_{i-1}, T_i) \quad \text{(fixed)} \\ \delta_i & = \delta(T_{i-1}, T_i) \quad \text{(floating)} \end{cases} \tag{1.45}$$

are the day counts fot a fixed and floating payments, respectively.

(abbr.)

Def. 1.3.5. (Swap Rate) The (spot) swap rate for the maturity T_M is the fixed rate K that makes the present value of the swap zero.

P.V. of the swap rate:

$$PV_{fix} = NK \sum_{i=1}^{M} P(0, T_i)\Delta_i \tag{1.46}$$

$$\begin{aligned} PV_{float} &= N \sum_{i=1}^{M} P(0, T_i)F(0, T_{i-1}, T_i)\delta_i = N \sum_{i=1}^{M} P(0, T_i) \left(\frac{P(0, T_{i-1})}{P(0, T_i)} - 1 \right) \\ &= N \sum_{i=1}^{M} (P(0, T_i) - P(0, T_i)) = N(1 - P(0, T_M)) \end{aligned} \tag{1.47}$$

Swap rate K:

$$K = \frac{1 - P(0, T_M)}{\sum_{i=1}^{M} P(0, T_i)\Delta_i} := S(0; T_0, T_M) \tag{1.48}$$

* Economic meaning of swap rate:

$$S(0; T_0, T_M) = \frac{\sum_{i=1}^{M} P(0, T_i)F(0, T_{i-1}, T_i)\delta_i}{\sum_{i=1}^{M} \Delta_i P(0, T_i)} \tag{1.49}$$

Let us approximate as

$$P(0, T_i) \approx 1,\ \delta_i \approx \Delta_i \quad \text{for all}\, i \tag{1.50}$$

$$\Rightarrow S(0; T_0, T_M) \approx \frac{\sum_{i=1}^{M} F(0, T_{i-1}, T_i)}{M} \tag{1.51}$$

...average of fwd rates!

Def. 1.3.6. (Forward Swap) A forward swap is the swap which starts at some future time. Fixed rate (fixed at $t = 0$) which make the P.V. of the swap is called the forward swap rate.

Ex. 1.3.2. A forward swap for the period $[T_M, T_N]$
$\Rightarrow$ cash flow exchanges at T_i, $i = \{M+1, ..., N\}$
$NK\Delta_i \leftrightarrow NL(T_{i+1}, T_i)\delta_i$
Let fixed rate be K, notional $= 1$

$$PV_{fix} = NK \sum_{i=M+1}^{N} P(0, T_i)\Delta_i \tag{1.52}$$

$$PV_{float} = N \sum_{i=M+1}^{N} P(0, T_i)F(0, T_{i-1}, T_i)\delta_i = P(0, T_M) - P(0, T_N) \tag{1.53}$$

Fwd Swap Rate:

$$S(0; T_M, T_N) = \frac{P(0, T_M) - P(0, T_N)}{\sum_{i=M+1}^{N} P(0, T_i)\Delta_i} \tag{1.54}$$

1.3.4 Relation to the fixed coupon bond

Consider the spot-start swap for $[T_0 = 0, T_N]$ (notional$= 0$)

$$PV_{float} = \sum_{i=1}^{N} \delta_i P(0, T_i)F(0, T_{i-1}, T_i) = 1 - P(0, T_N) \tag{1.55}$$

(abbr.)
Bond-Swap, Fixed vs Floating swap, ???
defined $i(t)$: index, $i \in \{0, ..., N\}$ s.t. $t \in [T_i, T_{i+1})$ $\quad T_{i(t)} \le t < T_{i(t)+1}$
<u>current time t</u>

$$\begin{aligned}
&\text{P.V.(floating leg?+ final principal)}\\
=&P(t, T_{i(t)+1})\delta_{i(t)+1}L(T_{i(t)}, T_{i(t)+1}) + \sum_{j=i(t)+2}^{N} P(t, T_j)\delta_j F(t, T_{j-1}, T_j) + P(t, T_N)\\
=&P(t, T_{i(t)+1})\delta_{i(t)+1}L(T_{i(t)}, T_{i(t)+1}) + \sum_{j=i(t)+2}^{N} P(t, T_j)(1 + \delta_{i(t)+1}L(T_i, T_{i+1})) + P(t, T_N)\\
=&P(t, T_{i(t)+1})(1 + \delta_{i(t)+1}L(T_i, T_{i+1})) \approx 1
\end{aligned} \tag{1.56}$$

Thus, floating leg + final principal $\approx$ IR-RISK 0
IR-Swap Risk $\approx$ fixed leg + final principal payment
$\Leftrightarrow$ fixed coupon Bond

1.3.5 Yield Curve Construction (Simplified...)

Asm. 1.3.1. There are market quotes of spot-starting swaps with swap rate $\{S_n\}_{n=1}^N$ with corresponding maturities $\{T_n\}_{n=1}^N$

$$S_n : S(0; T_0, T_n), \quad T_0 = 0 \tag{1.57}$$

Ex. 1.3.3. 3 month. $0 = T_0 < T_1 < ... < T_N$

We want to get $\{P(0,T_n)\}_{n=1}^N$ which are consistent with the swap quotes.

1) Determin $P(0,T_1)$

$$S_1\Delta_1 P(0,T_1) = P(0,T_0) - P(0,T_1) \tag{1.58}$$

$$S_1 = \frac{1 - P(0,T_1)}{\Delta_1 P(0,T_1)} \tag{1.59}$$

$$P(0,T_1) = \frac{P(0,T_0)}{1 + \Delta_1 S_1} = \frac{1}{1 + \Delta_1 S_1} \tag{1.60}$$

2) Suppose we have obtained $\{P(0,T_n)\}_{n=1}^{m-1}$. Consider T_m-maurity swap:

$$S_m\Delta_m P(0,T_m) + S_m \sum_{n=1}^{m-1} \Delta_n P(0,T_n) = 1 - P(0,T_m) \tag{1.61}$$

$$\Rightarrow (*) \quad P(0,T_m) = \frac{1 - S_m\{\sum_{n=1}^{m-1} \Delta_n P(0,T_n)\}}{1 + \Delta_m S_m} \tag{1.62}$$

using

$$(*) \quad S_m = S(0; T_0, T_m) = \frac{\sum_{n=1}^{m} \Delta_n P(0,T_n)}{1 - P(0,T_m)} \tag{1.63}$$

yield curve ... to be interpolated

1.3.6 Market-to-Market fo a forward swap

Suppose has swap starting T_M with maturity T_N as a receiver (long bond party) with the fixed rate X, notional L. Suppose the current ($t = 0$) market quotes is given by $S(0, T_M, T_N)$.

$$\begin{aligned} PV(t=0) &= LX \sum_{n=M+1}^{N} \Delta_n P(0,T_n) - L \sum_{n=M+1}^{N} P(0,T_n)\delta_n P(0,T_n)F(0,T_{n-1},T_n) \\ &= L \sum_{n=M+1}^{N} \Delta_n P(0,T_n)(X - S(0,T_M,T_N)) \end{aligned} \tag{1.64}$$

bond の receiver は金利が下がったら嬉しい

1.3.7 Approximation of a fwd swap rate

$$
\begin{aligned}
S(0,T_M,T_N) = \frac{P(0,T_M)-P(0,T_N)}{\sum_{i=M+1}^{N}\Delta_i P(0,T_i)} &= \frac{\sum_{i=M+1}^{N}\delta_i F(0,T_{i-1},T_i)P(0,T_i)}{\sum_{i=M+1}^{N}\Delta_i P(0,T_i)} \\
&\approx \frac{1}{N-M}\sum_{i=M+1}^{N} F(0,T_{i-1},F_i) \quad (as\,\delta_i \approx \Delta_i,\; P(0,T_i)=1)
\end{aligned} \tag{1.65}
$$

$0 < T_M < T_N$:

$$
NS(0;T_0,T_N) \approx MS(0;T_0,T_M) + (N-M)S(0;T_M,T_N) \tag{1.66}
$$

$NS(0;T_0,T_N) \approx \{[T_0,T_N]$ の fwd rate の sum$\}$

$$
\begin{aligned}
(\therefore)S(0;T_M,T_N) &\approx \frac{NS(0;T_0,T_M)-MS(0;T_0,T_M)}{N-M} \\
&\approx \frac{T_N}{T_N-T_M}S(0;T_0,T_N) - \frac{T_M}{T_N-T_M}S(0;T_0,T_M)
\end{aligned} \tag{1.67}
$$

1.3.8 Deltas

* market quotes (input) spot-swap rates $(S_n)_{n=1}^{N} \Rightarrow P(0,T) \Rightarrow$ pricing...

Delas(PVO 1s)

P.V. of receive swap $[T_M,T_N]$. Notional: L, fixed rate: X

$$
P.V.(t=0) = L\sum_{i=M+1}^{N}\Delta_i P(0,T_i)(X - S(0;T_M,T_N)) \tag{1.68}
$$

Suppose the market change induces

$$
S(0;T_M,T_N) \;\to\; S(0;T_M,T_N) + \delta S \tag{1.69}
$$

then, change of the P.V. :

$$
\begin{aligned}
\delta P.V. =&L\sum_{i=M+1}^{N}\Delta_i(\delta P(0,T_i))(X - S(0;T_M,T_N)) \\
&+ L\sum_{i=M+1}^{N}\Delta_i P(0,T_i)(-\delta S_M,N) + \text{higher order}
\end{aligned} \tag{1.70}
$$

1st term order $\sim 1R^2$

2nd term order $\sim 1R^1$

$||$1st term$|| << ||$2nd term$||$

$$\delta P.V. \approx L \sum_{i=M+1}^{N} \Delta_i P(0,T_i)(-\delta S_M, N) \tag{1.71}$$

* $(-\delta S_M, N)$: fwd swap rate の変化

$$S(0;T_M,T_N) \approx \frac{T_N}{T_N - T_M} S(0;T_0,T_N) - \frac{T_M}{T_N - T_M} S(0;T_0,T_M) \tag{1.72}$$

$$\delta S_{M,N} \approx \frac{T_N}{T_N - T_M}\delta S_N - \frac{T_M}{T_N - T_M}\delta S_M \tag{1.73}$$

- δS_N : change of $S(0,T_0,T_N)$
- δS_M : change of $S(0,T_0,T_M)$

$$\begin{aligned}
&\delta P.V.(\text{fwd swap}\,(T_M,T_N)) \\
&\approx -L\sum_{i=M+1}^{N}\Delta_i P(0,T_i)\left\{\frac{T_N}{T_N-T_M}\delta S_N - \frac{T_M}{T_N-T_M}\delta S_M\right\} \\
&\approx -L(T_N-T_M)\times\frac{1}{T_N-T_M}(T_N\delta S_N - T_M\delta S_M) \quad (\text{approx. } P(0,T_i)\approx 1) \\
&= -L(T_N\delta S_N - T_M\delta S_M)
\end{aligned} \tag{1.74}$$

(* day-count convention のズレは無視)

- spot-start swap
- maturity T_N
- Notional L

receiver:

$$\delta P.V. = -L \times T_N \times \delta S_N \tag{1.75}$$