Advanced Derivative

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July 24, 2018

1 Bond and Forward Contract

1.1 Interest Rates

Asm. 1.1.1. We asssume that there exists a default free money market account

- default-free
- liquid (borrowing ragte = lending rate)
- everyone can access equally

The interest rates associating with the default free m.m. (money market) are called risk-free rates.

1.1.1 Types of Accrual (利息)

Suppose one invests cash amount A at $t = 0$ for T years.
V : The amount of cash to be returned in T years time.

1. Annual compounding with interest rate R_1 per annum
   $V = A(1 + R_1)^T$
2. Semiannual compounding with interest rate R_2 per annum
   $V = A(1 + \frac{R_2}{2})^{2T}$
3. m-times compounding with interest rate R_m per annum
   $V = A(1 + \frac{R_m}{m})^{mT}$ (typiccaly we have $m = 1, 2, 4, 12, 52, 365$)
4. continuous compounding with interest rate r per annum
   $V = \lim_{m\to\infty} A(1 + \frac{r}{m})^{mT} = Ae^{rT}$

Relation among different compounding conventions:

No-Arbitrage $\rightarrow$ for any m,

$$e^{rT} = \left(1 + \frac{R_m}{m}\right)^{mT} \tag{1.1}$$

$$\Leftrightarrow \quad r = m \ln\left(1 + \frac{R_m}{m}\right), \quad R_m = m(e^{r/m} - 1). \tag{1.2}$$

at a given time, $R_1(T)$: T dependent.

1.1.2 Zero Rate and Bonds

Def. 1.1.1. (zero rate) The T-year zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for T-years without any intermidiate coupon payment.

Ex. 1.1.1. 5-year zero rate = 5 % per annum. (continuous compounding)
\$ 100 を deposit at $t = 0 \rightarrow$ (5 years later) $100e^{0.05\times 5} \approx 128.40$

Present value (PV) $P.V. = 128.40 \times e^{-0.05\times 5} = 100$

Def. 1.1.2. (ZCB : zero coupon bond) T-year zero coupon bond is a bond which pays an unit amount of cash in T-year time without any coupon payment (assume risk-free, liquid).

If R is T-year zero rate, current price of T-year zero coupon Bond is given by :

$$P(0, T) = e^{-RT} \tag{1.3}$$

(0: current time, T: maturity), we call "discount factor".

Ex. 1.1.2. Suppose we have (at $t = 0$)

- $T = 0.5$, $R = 5.0\%$ (zero coupon rate)
- $T = 1.0$, $R = 5.8\%$
- $T = 1.5$, $R = 6.4\%$
- $T = 2.0$, $R = 6.8\%$

fixed coupon bond

- maturity: $T = 2$
- coupon payments: 6% per annum semiannually.
- principal: \$100

$$\begin{aligned}\text{Bond price} &= 3P(0, 0.5) + 3P(0, 1.0) + 3P(0, 1.5) + 103P(0, 2) \\ &= 3 \times e^{-5\%\times 0.5} + 3 \times e^{-5.8\%\times 1.0} + 3 \times e^{-6,4\%\times 1.5} + 103 \times e^{-6.8\%\times 2.0} \\ &\approx 98.39\end{aligned} \tag{1.4}$$

1.1.3 Yield (平均利率，平均利回り，平均 discount rate)

Def. 1.1.3. (Bond's Yield) A bond's yield is the single discount rate that when applied to eery cash flow, gives the market bond price.

Ex. 1.1.3. using Ex. 1.1.2, suppose market price = 98.39
Then the yield for the bond y is given by solving

$$3e^{-0.5y} + 3e^{-1.0y} + 3e^{-1.5y} + 103e^{-2.0y} = 98.39 \tag{1.5}$$
$$\Rightarrow y \simeq 6.76\% \quad \text{(continuous compounding)} \tag{1.6}$$

Def. 1.1.4. (Par Yield) The par yield for a certain maturity is the coupon rate that makes the bond price equal to its principal.

Ex. 1.1.4. using Ex. 1.1.2, par yield C for 2-year coupon bond is given by:

$$\frac{C}{2}e^{-0.050\cdot 0.5} + \frac{C}{2}e^{-0.058\cdot 1.0} + \frac{C}{2}e^{-0.064\cdot 1.5} + \left(100 + \frac{C}{2}\right)e^{-0.068\cdot 2.0} = 100 \tag{1.7}$$
$$\Rightarrow C \simeq 6.87\% \tag{1.8}$$

(par yield for $T = 2$, semiannual coupon payments)

In general,

- T-year bond
- m-time coupon payment per annum
- par yield C

$$\sum_{n=1}^{mT} \frac{C}{m} P\left(0, \frac{n}{m}\right) + 100P(0,T) = 100 \tag{1.9}$$
$$\Rightarrow C = \frac{100(1 - P(0,T))}{A}, \quad A = \sum_{n=1}^{mT} \frac{1}{m} P\left(0, \frac{n}{m}\right) \tag{1.10}$$

1.1.4 Duration

fixed coupond bond:

- (cash flow at T_i) $= C_i \quad (i = 1, ..., n)$
- C_i: coupon (+ principal at maturity)

Suppose its yield is given by y (continuous compounding).
Bond price:

$$B = \sum_{i=1}^{n} C_i e^{-yT_i} \tag{1.11}$$

The duration of the Bond:

$$D := -\frac{1}{B}\frac{d}{dy}B = -\left(\frac{dB/dy}{B}\right) = \frac{1}{B}\sum_{i=1}^{n} C_i T_i e^{-yT_i} \tag{1.12}$$

Ex. 1.1.5. zero coupon Bond

$$(C_i = 0)_{i=1,2,\dots,n-1},\, C_n = 1, \quad B = e^{-yT_n} \tag{1.13}$$

$$\Rightarrow D = \frac{1}{B} C_n T_n e^{-yT_n} = T_n \tag{1.14}$$

(* Duration の長短により，金利に対する反応度の違いがわかる．)

Suppose the yield changes small amount Δy, ($y \to y + \Delta y, \quad B \to B + \Delta B$)

$$\frac{\Delta B}{B} = -D\Delta y + o(\Delta y) \tag{1.15}$$

(abbr.)

Suppose $D = 10$ (10 year), yield: $\Delta y = +0.1\%$ (10 basis points, b.p.):

$$\frac{\Delta B}{B} \approx -10 \times 0.1\% = -1\% = -0.01 \tag{1.16}$$

1.1.5 Modified Duration

yield (m-time compounding) $\hat{y}$

the same bond:

$$B = \sum_{i=1}^{n} C_i \left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \tag{1.17}$$

modified duration:

$$\begin{aligned} D^* : &= -\frac{1}{B}\frac{dB}{d\hat{y}} = \frac{1}{B}\sum_{i=1}^{n} \frac{C_i T_i}{1 + \hat{y}/m}\left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \\ &= \frac{1}{B(1 + \hat{y}/m)}\sum_{i=1}^{n} C_i T_i \left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \\ &= \frac{1}{B(1 + \hat{y}/m)}\sum_{i=1}^{n} C_i T_i e^{-yT_i} = \frac{D}{1 + \hat{y}/m} \quad \text{(no arbitrage)} \end{aligned} \tag{1.18}$$

* duration の議論は cash flow が一方向のときのみ使える．Insurance では通用しないので注意.

1.1.6 Convecity)

y : yiled (continuous compounding)

$$C := \frac{1}{B}\frac{d^2 B}{dy^2} = \frac{1}{B}\sum_{i=1}^{n} C_i T_i^2 e^{-yT_i} \tag{1.19}$$

$y \to y + \Delta y, \quad B \to B + \Delta B$:

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2 B}{dy^2}(\Delta y)^2 + o(\Delta y^2) \tag{1.20}$$

$$\Rightarrow \frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2 + o(\Delta y^2) \tag{1.21}$$

(Duration matching : abbr.)

1.2 Forward Contract

1.2.1 Forward Price

Def. 1.2.1. (Forward Contract) A forward contract with maturity T is a bilateral binding promise(agreement) such that at time $t = T (> 0)$, the two parties exchange:

- the cash amount given by the time T realization of a certain index (such as a stock price) with the fixed amount of cash (cash delivery)
- the unit amount of asset (such as a share of an equity) with fixed amount of cash (physical delivery=現物)

Def. 1.2.2. (Forward Price) A forward price F at the current time ($t = 0$) (契約時) of the underlying index X is the amount of cash K that make the present value of the forward contract exchanging X_t and K at T zero. (Forward contract has $P.V. = 0$, with $K = F$.)
* F は契約時に支払う額ではないことに注意 (元手は不要)
* K such that P.V. of the fwd contract $= 0$

Ex. 1.2.1. Consider a forward contract on a non-dividend paying stock, with mat. T.
$X_T = S_T$ (stock price at T), exchange $F \leftrightarrow S_T$ (at T).

Asm. 1.2.1. Stock market is liquid, zero-coupon bond is liquid.

the forward price at $t = 0$ is given by:

$$F = \frac{S_0}{P(0,T)} = e^{rT} S_0 \tag{1.22}$$

- r : zero-rate for mat T, continuous, compounding
- $P(0,T)$: zero coupon bond price

Prf. 1.2.1. replication strategy

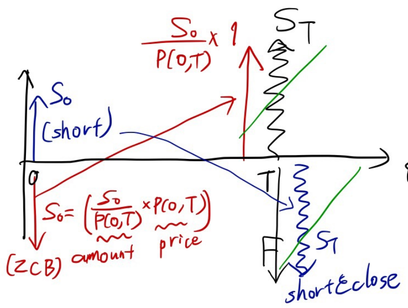
- Enter the fwd contract to get one share of stock (S_T) by paying F at T ($t = 0$ で enter する際に元手は不要)
- Sell one share of stock at $t = 0$ to get S_0 (short position)
- Use S_0 to buy ZCB (zero coupon bond) by the amount $S_0/P(0,T)$

- Pay F and receive S_T at T (return S_T to lender)
- Receive $S_0/P(0,T)$ from ZCB lender

(cash flow illustration below)

if $F \neq \dfrac{S_0}{P(0,T)} \Rightarrow$ arbitrage (no risk, arbitrary positive return)

No-arbitrage $\Rightarrow F = \dfrac{S_0}{P(0,T)} = e^{rT}S_0$ (F は stochastic ではない)



stochastic な S_T が消えた

☆ net cf $= \dfrac{S_0}{P(0,T)} - F$ (at T)

(no-arbitrage) $\Downarrow$

$\therefore$ $F = \dfrac{S_0}{P(0,T)}$ //

Ex. 1.2.2. Same as Ex.1.2.1 but now the stock pays continuous dividend, with dividend rate y ($y \in \mathbb{R}$, constant)

One share of the stock pays $S_t y dt$ for the interval $[t, t+dt]$ for any $t \geq 0$.

forward price at $t = 0$

$$F = \frac{S_0}{P(0,T)} e^{-yT} = S_0 e^{(r-y)T} \tag{1.23}$$

r : zero-rate for mat T at $t = 0$

Suppose we heve N_t shares at t, dividend paid in $[t, t+dt]$: $S_t N_t y dt$

$\Rightarrow$ reinvest $\Delta N_t = N_t y dt$

if one reinvests the whole dividend payment,

$$\frac{dN_t}{dt} = N_t y \Rightarrow N_t = N_0 e^{yt} \quad \text{for all} \quad t \geq 0 \tag{1.24}$$

Therefore, if one wants $N_T = 1$, N_0 is to be e^{-yT}.

Prf. 1.2.2. replication strategy

$(t = 0)$

- Enter the fwd contract to receive F and deliver one share stock at T (Ex.1.2.1 と逆の party)

- Sell $\dfrac{S_0 e^{-yT}}{P(0,T)}$ amount of ZCB with maturity T
- Buy e^{-yT} shares of stock

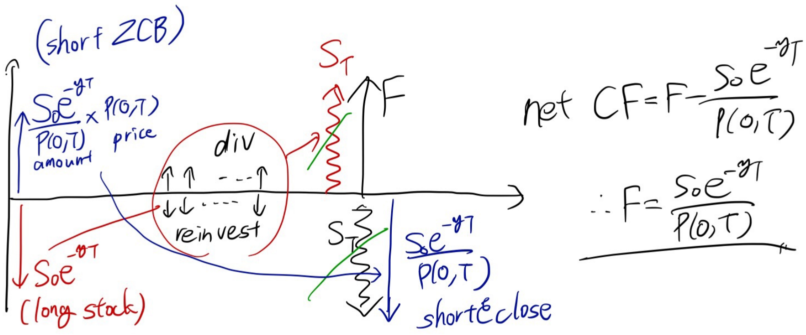
(always)

- Reinvest every div. payment to the stock

($t = T$)

- Receive F, deliver one share of stock
- Return $\dfrac{S_0 e^{-yT}}{P(0,T)}$ to the ZCB lender

(cash flow illustration below)



イメージ:

$$\text{P.V}(\text{receive}\, S_T\, \text{at}\, T) = F \times P(0,T) = \frac{S_0}{P(0,T)} \times P(0,T) = S_0 \tag{1.25}$$

random な cash flow の P.V. を計算するときは，Forward Price を求めて，$P(0,T)$ を乗じてやればよい (ただし市場が liquid,replicatable なときのみ)

⇒ 後に risk-nuetral の下で計算すればわざわざ replication を考える必要がなくなる

1.2.2 Mark to Market of a fwd contract

* P.V.(at $t = 0$){fwd contract} は 0 だが，時間が進むにつれて，P.V. は変化する．

Suppose we entered the fwd contract to receive X_T in exchange for a fixed amount of cash F_0 (F_0 : fwd price at $t = 0$). P.V.($t = 0$)$= 0$
at time $t \in (0,T)$, suppose fwd price is given by F_t. We want to know P.V.($t > 0$).
Ans.:

$$\text{P.V.}(t) = P(t,T)(F_t - F_0) \tag{1.26}$$

Prf. 1.2.3. enter the new fwd contract at $t = t$ (pays X_T and receives F_t at $t = T$)
* F_t の t は「時刻 t に contract に enter した」という意味.
(abbr.)

$$\begin{aligned} \text{P.V.}(\text{at}\, t)\{\text{new} + \text{original fwd contracts}\} &= P(t,T)(F_t - F_0) \\ &= 0 + \text{P.V.}(\text{at}\, t)\{\text{original}\} \end{aligned} \tag{1.27}$$

1.2.3 Put-Call Parity

Def. 1.2.3. (Call Option and Put Option) A call (respectively, put) option on a certain index X with expiry T and strike K is the binomial?? contract to pay the holder of the option the cash amount equal to $\max(X_T - K, 0)$, (resp, $\max(K - X_T, 0)$)

Let C (resp, P) be the call (resp, put) option orice (at $t = 0$). We have the following put-call parity:

$$C - P = P(0,T)(F_X - K) \tag{1.28}$$

- F_X : the fwd price of X with mat. T (時刻 $t = X$ ではなく，underlying asset X をもとにした fwd price at $t = 0$)

Prf. 1.2.4. cash flow at T:

$$\max(X_T, K, 0) - \max(K - X_T, 0) = X_T - K \tag{1.29}$$
$$\tag{1.30}$$

Present value of above is given by:

$$C - P = P(0,T)(F_X - K) \tag{1.31}$$

motivation:

- liquidity の問題
- call, put の一方が求まれば，もう一方をすぐに求められる
- PDE の計算は put の方が簡単 (because of boundary condition)

1.3 Forward Rate Agreement and Interest Rate Swap

*金利スワップは not tradable

1.3.1 Simple Rate and Day-Count Convention

Suppose T_i specifies the date $D_i = D(d_i, m_i, y_i)$

1. Actual/365

$$\delta(T_0, T_1) = \frac{D_1 - D_0}{365} \tag{1.32}$$

2. Actual/360

$$\delta(T_0, T_1) = \frac{D_1 - D_0}{360} \tag{1.33}$$

3. 30/360

$$\delta(T_0, T_1) = \frac{\max(30 - d_0, 0) + \min(d_1, 30) + 360(y_1 - y_0) + 30(m_1 - m_0 - 1)}{360} \tag{1.34}$$

4. actual/actual considering leap year? 365 or 366

Def. 1.3.1. (Simple Rate) A (risk-free) simple rate (not compound) $L(T_{i-1}, T_i)$ with day-count $\delta(T_{i-1}, T_i)$ is the interest rate with accrual convention defined in such a way that, when one invest N amount of cash at T_{i-1}, then he receives $N(1 + \delta_i L(T_{i-1}, T_i))$ at time T_i. $L(T_{i-1}, T_i)$ is the zero coupon rate at T_{i-1} for $[T_{i-1}, T_i]$ with corresponding day-count convention.

*accrual: ??

1.3.2 Forward Rate Agreement (FRA)

Def. 1.3.2. (Forward Rate Agreement) A FRA is a binding(義務の) contract with the two parties (lender and borrower) agreeing to let a certain fixed rate K act on a prefixed notional(想定元本) amount N, over a future period $[T_M, T_N]$.

*notional: 想定される?

Def. 1.3.3. (Forward Rate) A forward rate F for the period $[T_M, T_N]$ with day-count $\delta = \delta(T_M, T_N)$ is the fixed rate K with the some day-count convention that makes the present value of the FRA zero.

off course,

$$F = L(T_M, T_N) \quad (\text{at}\, T_M) \tag{1.35}$$

*L : simple rate

Lem. 1.3.1. Let $\delta = \delta(T_M, T_N)$. Then the forward rate F at $t = 0$ fot the period $[T_M, T_N]$ is given by

$$F = \frac{1}{\delta}\left(\frac{P(0, T_M)}{P(0, T_N)} - 1\right) \tag{1.36}$$

(*asm: liquid, no-arbitrage)

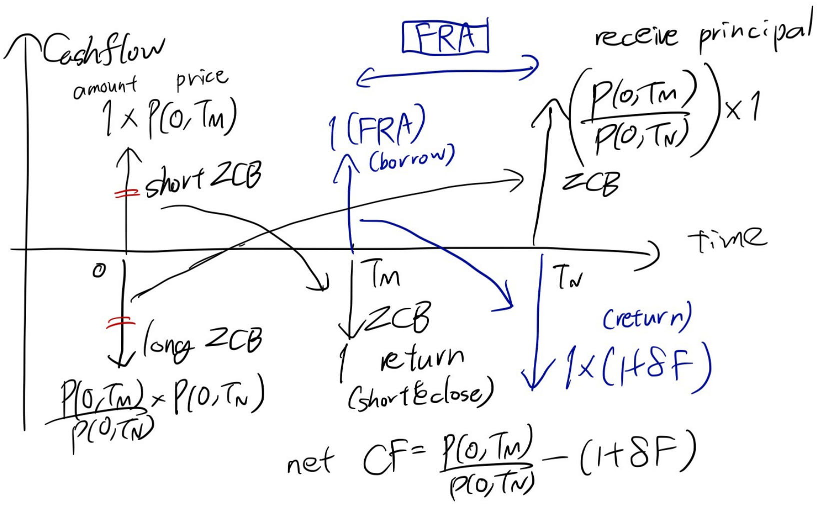
Prf. 1.3.1. replication strategy

- Enter the FRA of rate F to borrow unit amount of cash for $[T_M, T_N]$
- Sell one ZCB with mat. T_N (short)
- Buy ZCB with mat. T_N with principal amount $\dfrac{P(0,T_M)}{P(0,T_N)}$
- At T_M, borrow unit amount of cash through FRA and use it to return ZCB
- At T_N, receive the principal $\dfrac{P(0,T_M)}{P(0,T_N)}$, and pays $(1+\delta F)$

Net cash flow at T_N: $\left(\dfrac{P(0,T_M)}{P(0,T_N)}\right) - (1+\delta F)$

If we require no-arbitrage,

$$\left(\frac{P(0,T_M)}{P(0,T_N)}\right) - (1+\delta F) = 0 \quad \Rightarrow \quad F = \frac{1}{\delta}\left(\frac{P(0,T_M)}{P(0,T_N)} - 1\right) \tag{1.37}$$



We write the above F as

$$F(0,T_M,T_N) = \frac{1}{\delta}\left(\frac{P(0,T_M)}{P(0,T_N)} - 1\right) \tag{1.38}$$

In general. at $t < T_M$,

$$F(t,T_M,T_N) = \frac{1}{\delta}\left(\frac{P(t,T_M)}{P(t,T_N)} - 1\right) \tag{1.39}$$

$t \uparrow T_M$:

$$F(T_M,T_M,T_N) = \frac{1}{\delta}\left(\frac{1}{P(T_M,T_N)} - 1\right) = L(T_M,T_N) \tag{1.40}$$

$(\because)$

$$1 = P(T_M, T_N)\{1 + \delta L(T_M, T_N)\} \tag{1.41}$$

at T_M invest 1, at T_N return $1 + \delta L(T_M, T_N)$
(otherwise there exist arbitrage opportunities.)
(abbr.)

Ex. 1.3.1. Q) P.V.(at 0) {receive $1 + \delta L(T_M, T_N)$} ? $\rightarrow$ A) P.V.=0
$(\because)$ Suppose that we are at $t = T_M$, $L(T_M, T_N)$ ... known

$$\text{P.V.}(\text{at}\, T_M) = -1 + P(T_M, T_N)(1 + \delta L(T_M, T_N)) = 0 \tag{1.42}$$

将来のある時点 $(t = T_M)$ で P.V.$= 0$ なら，さかのぼった $t = 0$ でも当然 P.V.$= 0$

$$P.V.(\text{at}\,0)\{\text{receive}\,1 + \delta F(0, T_M, T_N)\,\text{at}\,T_N\} = P.V.(\text{at}\,0)\{\text{receive}\,1 + \delta L(T_M, T_N)\} \tag{1.43}$$

$$(\therefore) \quad P(0, T_N)F(0, T_M, T_N) = P.V.(\text{at}\,0)\{\text{receive}\,L(T_M, T_N)\} \tag{1.44}$$

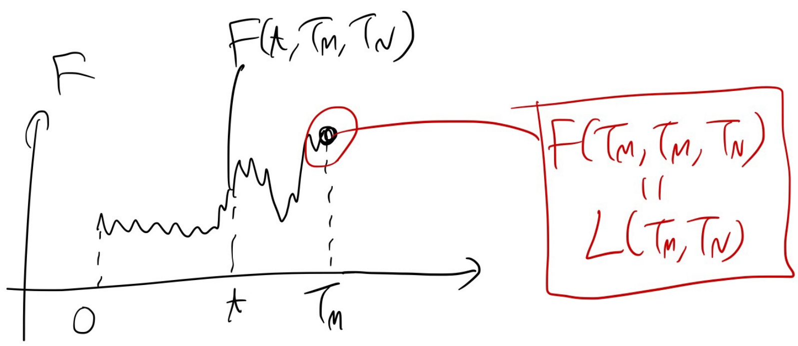
1.3.3 Fixed vs Floating Interest Swap

Fix a time partition $0 = T_0 < T_1 < ... < T_M$

Def. 1.3.4. (Spot-start Swap) A spot-start swap with maturity T_M and notional amount N is the contract in which one party (receiver) receives cash amount $NK\Delta_i$ (K fixed) and pays the stochastic amount $NL(T_{i-1}, T_i)\delta_i$ at every T_i, $i = \{1, 2, ..., M\}$. The other party (payer) has the opposit cash flow. Here,

$$\begin{cases} \Delta_i & = \Delta(T_{i-1}, T_i) \quad \text{(fixed)} \\ \delta_i & = \delta(T_{i-1}, T_i) \quad \text{(floating)} \end{cases} \tag{1.45}$$

are the day counts fot a fixed and floating payments, respectively.



Def. 1.3.5. (Swap Rate) The (spot) swap rate for the maturity T_M is the fixed rate K that makes the present value of the swap zero.

P.V. of the swap rate:

$$PV_{fix} = NK\sum_{i=1}^{M} P(0,T_i)\Delta_i \tag{1.46}$$

$$\begin{aligned} PV_{float} &= N\sum_{i=1}^{M} P(0,T_i)F(0,T_{i-1},T_i)\delta_i = N\sum_{i=1}^{M} P(0,T_i)\left(\frac{P(0,T_{i-1})}{P(0,T_i)} - 1\right) \\ &= N\sum_{i=1}^{M}(P(0,T_i) - P(0,T_i)) = N(1 - P(0,T_M)) \end{aligned} \tag{1.47}$$

Swap rate K:

$$K = \frac{1 - P(0,T_M)}{\sum_{i=1}^{M} P(0,T_i)\Delta_i} := S(0;T_0,T_M) \tag{1.48}$$

* Economic meaning of swap rate:

$$S(0;T_0,T_M) = \frac{\sum_{i=1}^{M} P(0,T_i)F(0,T_{i-1},T_i)\delta_i}{\sum_{i=1}^{M}\Delta_i P(0,T_i)} \tag{1.49}$$

Let us approximate as

$$P(0,T_i) \approx 1,\ \delta_i \approx \Delta_i \quad \text{for all}\, i \tag{1.50}$$

$$\Rightarrow S(0;T_0,T_M) \approx \frac{\sum_{i=1}^{M} F(0,T_{i-1},T_i)}{M} \tag{1.51}$$

...average of fwd rates!

Def. 1.3.6. (Forward Swap) A forward swap is the swap which starts at some future time. Fixed rate (fixed at $t = 0$) which make the P.V. of the swap is called the forward swap rate.

Ex. 1.3.2. A forward swap for the period $[T_M, T_N]$
$\Rightarrow$ cash flow exchanges at T_i, $i = \{M+1, ..., N\}$
$NK\Delta_i \leftrightarrow NL(T_{i-1},T_i)\delta_i$

Let fixed rate be K, notional $= 1$

$$PV_{fix} = NK\sum_{i=M+1}^{N} P(0,T_i)\Delta_i \tag{1.52}$$

$$PV_{float} = N\sum_{i=M+1}^{N} P(0,T_i)F(0,T_{i-1},T_i)\delta_i = P(0,T_M) - P(0,T_N) \tag{1.53}$$

Fwd Swap Rate:

$$S(0;T_M,T_N) = \frac{P(0,T_M) - P(0,T_N)}{\sum_{i=M+1}^{N} P(0,T_i)\Delta_i} \tag{1.54}$$

1.3.4 Relation to the fixed coupon bond

Consider the spot-start swap for $[T_0 = 0, T_N]$ (notional= 0)

$$PV_{float} = \sum_{i=1}^{N} \delta_i P(0,T_i) F(0, T_{i-1}, T_i) = 1 - P(0, T_N) \tag{1.55}$$

(abbr.)

Bond-Swap, Fixed vs Floating swap, ???

defined $i(t)$: index, $i \in \{0, ..., N\}$ s.t. $t \in [T_i, T_{i+1})$ $\quad T_{i(t)} \le t < T_{i(t)+1}$

current time t

$$\begin{aligned}
&\text{P.V.(floating leg?+ final principal)}\\
=&P(t, T_{i(t)+1})\delta_{i(t)+1} L(T_{i(t)}, T_{i(t)+1}) + \sum_{j=i(t)+2}^{N} P(t, T_j)\delta_j F(t, T_{j-1}, T_j) + P(t, T_N)\\
=&P(t, T_{i(t)+1})\delta_{i(t)+1} L(T_{i(t)}, T_{i(t)+1}) + \sum_{j=i(t)+2}^{N} P(t, T_j)(1 + \delta_{i(t)+1} L(T_i, T_{i+1})) + P(t, T_N)\\
=&P(t, T_{i(t)+1})(1 + \delta_{i(t)+1} L(T_i, T_{i+1})) \approx 1
\end{aligned} \tag{1.56}$$

Thus, floating leg + final principal $\approx$ IR-RISK 0

IR-Swap Risk $\approx$ fixed leg + final principal payment

$\Leftrightarrow$ fixed coupon Bond

1.3.5 Yield Curve Construction (Simplified...)

Asm. 1.3.1. There are market quotes of spot-starting swaps with swap rate $\{S_n\}_{n=1}^{N}$ with corresponding maturities $\{T_n\}_{n=1}^{N}$

$$S_n : S(0; T_0, T_n), \quad T_0 = 0 \tag{1.57}$$

Ex. 1.3.3. 3 month. $0 = T_0 < T_1 < ... < T_N$

We want to get $\{P(0, T_n)\}_{n=1}^{N}$ which are consistent with the swap quotes.

1) Determin $P(0, T_1)$

$$S_1 \Delta_1 P(0, T_1) = P(0, T_0) - P(0, T_1) \tag{1.58}$$

$$S_1 = \frac{1 - P(0, T_1)}{\Delta_1 P(0, T_1)} \tag{1.59}$$

$$P(0, T_1) = \frac{P(0, T_0)}{1 + \Delta_1 S_1} = \frac{1}{1 + \Delta_1 S_1} \tag{1.60}$$

2) Suppose we have obtained $\{P(0,T_n)\}_{n=1}^{m-1}$. Consider T_m-maurity swap:

$$S_m\Delta_m P(0,T_m) + S_m\sum_{n=1}^{m-1}\Delta_n P(0,T_n) = 1 - P(0,T_m) \tag{1.61}$$

$$\Rightarrow (*) \quad P(0,T_m) = \frac{1 - S_m\{\sum_{n=1}^{m-1}\Delta_n P(0,T_n)\}}{1+\Delta_m S_m} \tag{1.62}$$

using

$$(*) \quad S_m = S(0;T_0,T_m) = \frac{\sum_{n=1}^{m}\Delta_n P(0,T_n)}{1-P(0,T_m)} \tag{1.63}$$

yield curve ... to be interpolated

1.3.6 Market-to-Market fo a forward swap

Suppose has swap starting T_M with maturity T_N as a receiver (long bond party) with the fixed rate X, notional L. Suppose the current ($t=0$) market quotes is given by $S(0,T_M,T_N)$.

$$\begin{aligned} PV(t=0) &= LX\sum_{n=M+1}^{N}\Delta_n P(0,T_n) - L\sum_{n=M+1}^{N}P(0,T_n)\delta_n P(0,T_n)F(0,T_{n-1},T_n) \\ &= L\sum_{n=M+1}^{N}\Delta_n P(0,T_n)(X - S(0,T_M,T_N)) \end{aligned} \tag{1.64}$$

bond の receiver は金利が下がったら嬉しい

1.3.7 Approximation of a fwd swap rate

$$\begin{aligned} S(0,T_M,T_N) &= \frac{P(0,T_M)-P(0,T_N)}{\sum_{i=M+1}^{N}\Delta_i P(0,T_i)} = \frac{\sum_{i=M+1}^{N}\delta_i F(0,T_{i-1},T_i)P(0,T_i)}{\sum_{i=M+1}^{N}\Delta_i P(0,T_i)} \\ &\approx \frac{1}{N-M}\sum_{i=M+1}^{N}F(0,T_{i-1},F_i) \quad (as\,\delta_i\approx\Delta_i,\ P(0,T_i)=1) \end{aligned} \tag{1.65}$$

$0<T_M<T_N$:

$$NS(0;T_0,T_N) \approx MS(0;T_0,T_M) + (N-M)S(0;T_M,T_N) \tag{1.66}$$

$NS(0;T_0,T_N) \approx \{[T_0,T_N]$ の fwd rate の sum$\}$

$$\begin{aligned} (\because) S(0;T_M,T_N) &\approx \frac{NS(0;T_0,T_M) - MS(0;T_0,T_M)}{N-M} \\ &\approx \frac{T_N}{T_N-T_M}S(0;T_0,T_N) - \frac{T_M}{T_N-T_M}S(0;T_0,T_M) \end{aligned} \tag{1.67}$$

1.3.8 Deltas

* market quotes (input) spot-swap rates $(S_n)_{n=1}^N \Rightarrow P(0,T) \Rightarrow$ pricing...

Delas(PVO 1s)

P.V. of receive swap $[T_M, T_N]$. Notional: L, fixed rate: X

$$P.V.(t=0) = L \sum_{i=M+1}^{N} \Delta_i P(0,T_i)(X - S(0;T_M,T_N)) \tag{1.68}$$

Suppose the market change induces

$$S(0;T_M,T_N) \ \rightarrow \ S(0;T_M,T_N) + \delta S \tag{1.69}$$

then, change of the P.V. :

$$\begin{aligned} \delta P.V. =& L \sum_{i=M+1}^{N} \Delta_i(\delta P(0,T_i))(X - S(0;T_M,T_N)) \\ &+ L \sum_{i=M+1}^{N} \Delta_i P(0,T_i)(-\delta S_M, N) + \text{higher order} \end{aligned} \tag{1.70}$$

1st term order $\sim 1R^2$

2nd term order $\sim 1R^1$

$||$1st term$|| << ||$2nd term$||$

$$\delta P.V. \approx L \sum_{i=M+1}^{N} \Delta_i P(0,T_i)(-\delta S_M, N) \tag{1.71}$$

* $(-\delta S_M, N)$: fwd swap rate の変化

$$S(0;T_M,T_N) \approx \frac{T_N}{T_N - T_M} S(0;T_0,T_N) - \frac{T_M}{T_N - T_M} S(0;T_0,T_M) \tag{1.72}$$

$$\delta S_{M,N} \approx \frac{T_N}{T_N - T_M} \delta S_N - \frac{T_M}{T_N - T_M} \delta S_M \tag{1.73}$$

- δS_N : change of $S(0,T_0,T_N)$
- δS_M : change of $S(0,T_0,T_M)$

$$
\begin{aligned}
&\delta P.V.(\text{fwd swap}\,(T_M, T_N)) \\
\approx& -L \sum_{i=M+1}^{N} \Delta_i P(0,T_i) \left\{ \frac{T_N}{T_N - T_M}\delta S_N - \frac{T_M}{T_N - T_M}\delta S_M \right\} \\
\approx& -L(T_N - T_M) \times \frac{1}{T_N - T_M}(T_N \delta S_N - T_M \delta S_M) \quad (\text{approx.}\, P(0,T_i) \approx 1) \\
=& -L(T_N \delta S_N - T_M \delta S_M)
\end{aligned}
\tag{1.74}
$$

(* day-count convention のズレは無視)

- spot-start swap
- maturity T_N
- Notional L

receiver:

$$
\delta P.V. = -L \times T_N \times \delta S_N \tag{1.75}
$$

2 Binomial Model

多期間モデルは連続モデルへの直観を与える．PDE の数値計算との関連もあり．

2.1 One-Period Binomial Model

2.1.1 Model Description

There are two points in time $t = 0, T$. Two tradable assets:

- Bond (risk-free asset)

$$B_0 = 1 \tag{2.1}$$
$$B_T = e^{rT} \tag{2.2}$$

  (deterministic)

  r: zero rate for $[0, T]$ at $T = 0$

- Stock (risky asset)

$$S_0 = s\,(> 0) \tag{2.3}$$
$$S_T = \begin{cases} su & \text{with prob.}\ P_u \\ sd & \text{with prob.}\ P_d \end{cases} \tag{2.4}$$
$$P_u > 0,\ P_d > 0,\ P_u + P_d = 1,\ 0 < d < u \tag{2.5}$$

  We write for simplicity:

$$S_T = sZ \tag{2.6}$$
$$Z = \begin{cases} u & \text{with prob.}\ P_u \\ d & \text{with prob.}\ P_d \end{cases} \tag{2.7}$$

  under emprical measure $\mathbb{P}$, $\mathbb{P}(\{up\}) = P_u$, $\mathbb{P}(\{down\}) = P_d$

Portfolio : $h(x, y) \quad x, y \in \mathbb{R}$

- x: number of position for the bond
- y: number of position for the stock

the value process of the portfolio h:

$$V_t^h = xB_t + yS_t \quad \text{in general} \tag{2.8}$$
$$V_0^h = x + ys \tag{2.9}$$
$$V_T^h = xe^{rT} + ysZ \tag{2.10}$$

Def. 2.1.1. An arbitrage portfolio us a portfolio with the properties:

$$V_0^h = 0,\ P(V_T^h \geq 0) = 1,\ P(V_T^h > 0) > 0 \tag{2.11}$$

* no-arbitrage $\Leftrightarrow$ existence of risk neutral measure (we'll see later)

Prop. 2.1.1. The one-period binomial model is arbitrage free iff (=if and only if) the following condition holds:

$$0 < d < e^{rT} < u \tag{2.12}$$

Prf. 2.1.1. (proof of above prop.)

- (necessity = only if): Suppose 2.12 does not hold.
  1. $e^{rT} \leq d < u$
     (a) Sell the bond s units
     (b) Buy one unit of the stock
     $h(x, y) = (-s, 1)$, net cash flow $= 0$ at $t = 0$. Then,
     $$V_T^h = -se^{rT} + sZ = s(-e^{rT} + Z) \tag{2.13}$$
     It's clear that $V_T^h \geq 0$, $P(V_T^h \geq 0) = 1$.
     $$P(V_T^h > 0) = P(Z = u) = P_u > 0 \tag{2.14}$$
     $\rightarrow$ arbitrage.
  2. $d < u \leq e^{rT}$

     (a) Sell one unit of the s stock
     (b) Buy the bond s units
     $h(x, y) = (s, -1)$, net cash flow $= 0$ at $t = 0$. Then,
     $$V_T^h = se^{rT} - sZ = s(e^{rT} - Z) \tag{2.15}$$
     It's clear that $V_T^h \geq 0$, $P(V_T \geq 0) = 1$.
     $$P(V_T^h > 0) = P(Z = d) = P_d > 0 \tag{2.16}$$
     $\rightarrow$ arbitrage.
- (sufficiency = if): Suppose 2.12 holds.
  Assume $V_0^h = 0$ then $x + ys = 0 \Leftrightarrow ys = -x$.
  $$V_T^h = xe^{rT} + ysZ = x(e^{rT} - Z) \tag{2.17}$$
  $$P(V_T^h \geq 0) < 1 \tag{2.18}$$

  $\rightarrow$ no-arbitrage.

2.1.2 Risk-neutral Probability Measure

Suppose $d < e^{rT} < u$ holds,
$\Rightarrow$ one can find q_u, q_d > 0 s.t.

$$\begin{cases} q_u + q_d = 1 \\ q_d u + q_d d = e^{rT} \end{cases} \tag{2.19}$$

$$\Rightarrow \begin{cases} q_u = \dfrac{e^{rT} - d}{u - d} \quad (> 0) \\ q_d = \dfrac{u - e^{rT}}{u - d} \quad (> 0) \end{cases} \tag{2.20}$$

We define a new (and not emprical) probability measure $\mathbb{Q}$ such that

$$\mathbb{Q}(Z = u) = q_u \tag{2.21}$$

$$\mathbb{Q}(Z = d) = q_d \tag{2.22}$$

It's interesting to observe that

$$\begin{aligned} e^{-rT}\mathrm{E}^{\mathbb{Q}}[S_T] &= e^{-rT}(q_u \times su + q_d \times sd) \\ &= se^{-rT}(q_u u + q_d d) = s\,(= S_0) \end{aligned} \tag{2.23}$$

In general,

$$\{\ e^{-rT} S_t\}_{t=0,T} \quad \ldots\, \mathbb{Q}\text{-martingale} \tag{2.24}$$

Def. 2.1.2. (Risk-neutral Measure) The probability measure $\mathbb{Q}$ with associated probability (q_u, q_d) satisfying the condition eq(2.12) is called the risk-neutral measure.

Prop. 2.1.2. The one-period binomial model just explained is arbitrage free iff there exists a risk-neutral measure $\mathbb{Q}$ ($q_u > 0$, $q_d > 0$)

Prf. 2.1.2. arbitrage free $\Leftrightarrow d < e^{rT} < u$ ($\because$ Prop. 2.1.1)

$$\mathbb{Q} = (q_u, q_d) \tag{2.25}$$

$$q_u = \frac{e^{rT} - d}{u - d}, \quad q_d = \frac{u - e^{rT}}{u - d} \tag{2.26}$$

$$q_u + q_d = 1 \tag{2.27}$$

If such $\mathbb{Q}$ exists, then

$$\exists q_u, q_d\ > 0 \quad \text{s.t.} \begin{cases} q_u + q_d = 1 \\ q_u u + q_d d = e^{rT} \end{cases} \tag{2.28}$$

$$\Rightarrow d < e^{rT} < u \tag{2.29}$$

2.1.3 Risk-neutral Pricing

Def. 2.1.3. (Contingent Claim) A contingent claim is any random cash flow X_T at T of the form $X_T = \Phi(S_T)$ with some function Φ.

Ex. 2.1.1. (call option with strike K)

$$\Phi(S_T) = \max(S_T - K, 0) = (S_T - K)^+ \tag{2.30}$$

Def. 2.1.4. (Replicable / Complete) A given contingent claim X is said to be replicatable (or perfectly hedgeable) if there exists a portfolio h such that $V_T^h = X_T$ with probability 1 (under $\mathbb{P}$). In this case, we call h a replicating portfolio of X. If all contingent claim are replicable, we say the market is complete. Otherwise the market is incomplete.

Prop. 2.1.3. Suppose that a contingent claim X is replicable by portfolio h. Then, any price at $t = 0$ of the claim X other than V_0^h will lead to an arbitrage opportunity.

Prf. 2.1.3. Suppose h is given by $h = (x, y)$. Suppose $V_0^h \neq X_0$.

1. $V_0^h < X_0$
    - Short the contingent claim X (one gets X_0)
    - Construct a portfolio $h(x, y) : V_0^h = x + sy$
    - Buy $(X_0 - V_0^h)$ units of bond

    portfolio $h' = (x + X_0 - V_0^h, y)$ + short position of X
    net cash flow at T:

    $$V_T^{h'} - X_T = (X_0 - V_0^h)e^{rT} + V_T^h - X_T = (X_0 - V_0^h)e^{rT} > 0 \tag{2.31}$$

    $\rightarrow$ arbitrage.
2. $V_0^h > X_0$
    - Buy the contingent claim X
    - Short the replicating portfolio $h(x, y) : V_0^h = x + sy$
    - Buy $(V_0^h - X_0)$ units of bond

    portfolio $h'' = (V_0^h - X_0 - x, -y)$ + long position of X
    net cash flow at T:

    $$V_T^{h''} + X_T = (V_0^h - X_0)e^{rT} - V_T^h + X_T = (V_0^h - X_0)e^{rT} > 0 \tag{2.32}$$

    $\rightarrow$ arbitrage.

Prop. 2.1.4. The binomial model is complete.

Prf. 2.1.4. Consider a general contingent claim X whose payoff at T is $\Phi(S_T)$. $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ function.

Ex. 2.1.2. Call option $\Phi(S_T) = \max(S_T - K, 0), \quad K \in \mathbb{R}$
It suffices to construct a strategy $h = (x, y)$ s.t.

$$V_T^h = \begin{cases} \Phi(su) & \text{if } Z = u \\ \Phi(sd) & \text{if } Z = d \end{cases} \tag{2.33}$$

* terminal value が複製元デリバティブと同じになるような portfolio h を探す.

$$\begin{cases} e^{rT}x + suy = \Phi(su) \\ e^{rT}x + sdy = \Phi(sd) \end{cases} \tag{2.34}$$

$$\Leftrightarrow \begin{cases} x = e^{-rT}\dfrac{u\Phi(sd) - d\Phi(su)}{u-d} \\ y = \dfrac{1}{s}\dfrac{\Phi(su) - \Phi(sd)}{u-d} \end{cases} \tag{2.35}$$

Option Pricing (Assume there is no-arbitrage opportunity)
Consider the same contingent claim X with payoff $\Phi(S_T)$
The price X_0 at $t = 0$ of the claim is given by $X_0 = V_0^h$ ($\because$ Prop.2.1.3).

$$\begin{aligned} X_0 = x + sy &= e^{-rT}\left\{\frac{u\Phi(sd) - d\Phi(su)}{u-d} + e^{rT}\frac{\Phi(su) - \Phi(sd)}{u-d}\right\} \\ &= e^{-rT}\left\{\frac{e^{rT} - d}{u-d}\Phi(su) + \frac{u - e^{rT}}{u-d}\Phi(sd)\right\} \end{aligned} \tag{2.36}$$

$$X_0 = e^{-rT}\{q_u\Phi(su) + q_d\Phi(sd)\} = e^{-rT}\mathrm{E}^{\mathbb{Q}}[\Phi(S_T)] \tag{2.37}$$

$(P_u, P_d) \leftrightarrow (q_u, q_d)$

2.2 The Multi-Period Binomial Model

2.2.1 Model Description

$0 = t_0 < t_1 < ... < t_N = T$
time partition:

$$t_i - t_{i-1} = \Delta t = \frac{T}{N} \quad \text{for } \forall i \tag{2.38}$$

each t_i $i = \{0, ..., N\}$, one can trade securities.
Securities:

- Bond (risk-free, non-random)
  Bond price process:

$$B_0 = 1,\ B_n = e^{r\Delta t}B_{n-1} = e^{rn\Delta t} \tag{2.39}$$

  Risk-free rate $r\ (\geq 0)$: constant

- Stock

  Stock price process:

$$S_0 = s,\ S_n = S_{n-1}Z_n \tag{2.40}$$

$$\begin{cases} P(Z_n = u) = P_u \\ P(Z_n = d) = P_d \end{cases} \quad \text{for every } n \tag{2.41}$$

  Assumption:

$$0 < d < u,\ P_u > 0,\ P_d > 0,\ P_u + P_d = 1 \tag{2.42}$$

  2^N : number of all securities up to t_N

Def. 2.2.1. A portfolio strategy is defined as a process

$$h_{t_i} = \begin{pmatrix} x_{t_i} \\ y_{t_i} \end{pmatrix}, \quad i = \{0, ..., N\} \tag{2.43}$$

h_{t_i} is the position fot (Bond, Stock) for the period $[t_i, t_{i+1})$, newly taken at t_i and kept unchanged until t_{i+1}. h_{t_i} can be dependent only on $(S_0, ..., S_{t_{i-1}}$.

Portfolio value at t_i:

$$V_{t_i}^h = h_{t_i}^T \begin{pmatrix} B_{t_i} \\ S_{t_i} \end{pmatrix} = x_{t_i}B_{t_i} + y_{t_i}S_{t_i} \tag{2.44}$$

(* T : transposition)

For simplicity, we sometimes write:

$$V_i^h = x_iB_i + y_iS_i \tag{2.45}$$

Def. 2.2.2. (Self-Financing) A portfolio strategy h is said to be self-financing if the following condition holds for every time step i:

$$h_i^T \begin{pmatrix} B_i \\ S_i \end{pmatrix} = h_{i-1}^T \begin{pmatrix} B_i \\ S_i \end{pmatrix} \tag{2.46}$$

* $t = t_i$ での portfolio 組み換え

If the strategy h is self-financing,

$$\begin{aligned} \Delta V_i^h := V_i^h - V_{i-1}^h &= (x_{i-1}B_i + y_{i-1}S_i) - (x_{i-1}B_{i-1} + y_{i-1}S_{i-1}) \\ &= x_{i-1}(B_i - B_{i-1}) + y_{i-1}(S_i - S_{i-1}) \\ &= x_{i-1}\Delta B_i + y_{i-1}\Delta S_i \end{aligned} \tag{2.47}$$

2.2.2 Risk-neutral Pricing

Asm. 2.2.1. We assume $d < e^{r\Delta t} < u$

Def. 2.2.3. The risk-neutral probability measure $\mathbb{Q}$ (associated with (q_u, q_d)) is defined as a probability measure satisfying

$$S_i = e^{-r\Delta t}\mathrm{E}^{\mathbb{Q}}[S_{i+1}|S_i] \quad \text{for every } i = \{0, ..., N-1\} \tag{2.48}$$

The existence and uniqueness can be shown as follows:

$$\begin{cases} q_u + q_d = 1 \\ uq_u + dq_d = e^{r\Delta t} \end{cases} \tag{2.49}$$

$$\Leftrightarrow \begin{cases} q_u = \dfrac{e^{r\Delta t} - d}{u - d} \quad (> 0) \\ q_d = \dfrac{u - e^{r\Delta t}}{u - d} \quad (> 0) \end{cases} \qquad \text{by Asm.2.2.1} \tag{2.50}$$

Prop. 2.2.1. Suppose that a contingent claim X is replicable by the self-financing portfolio h. Then the price of the claim X at any time t_i $0 \le i \le N$ must be equal to $V^h_{t_i}$, otherwise there exists an arbitrage opportunity.

Goal:

Thm. 2.2.1. The multi-period binomial model satisfying assumption 2.2.1 is complete, i.e. cash-flow of any contingent claim can be perfectly replicated.
(by self-financinf portfolio strategy $(h_{t_i})_{i\ge 0}$)

*************以降 未完*************

Preparetion:

- European Option
- expiry: $t_N = T$
- option holder receives (or pay if negative) $\Phi(S_N)\,(= \Phi(S_T))$
- $\Phi : \mathbb{R} \to \mathbb{R}$

state at t_n ... $(n+1)$ 個
number of up movement : $(0, ..., n)$
We label it by (n, k), $k \in \{0, ..., n\}$, $S_n(k) = su^k d^{n-k}$
Construction of self-financing strategy at t_N. Replicating portfolio must satisfy:

$$V_N(k) = \Phi(S_N(k)) = \Phi(su^k d^{n-k}) \quad \text{for every } k = 0, ..., N \tag{2.51}$$

at $t_N - 1$

3 Itô Formula and Option Valuation

3.1 Probability Space

Def. 3.1.1. Let Ω be a non-empty set, and let $\mathcal{F}$ be a collection of subsets of Ω. We say thatt $\mathcal{F}$ is a σ-algebra (or σ-field) if it satisfies:

1. empty set $\phi \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c = (\Omega \backslash A) \in \mathcal{F}$
3. Suppose there is a sequence of sets $\{A_i\}_{i=1,\dots}$ and $A_i \in \mathcal{F}$ for all i, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Remarks:

- (1)(2)$\Rightarrow$ $\Omega \in \mathcal{F}$
- (3)(2)$\Rightarrow$ $\cap_{i=1}^{\infty} A_i = (\cup_{i=1}^{\infty} A_i^c)^c \in \mathcal{F}$

* σ(sigma) means 'countable infinity' (可算無限)

Ex. 3.1.1.
- $\Omega = \{\omega_1, \omega_2\}$
- $\mathcal{F} = \{\phi, \{\omega_1, \omega_2\}\}$
- $\mathcal{F} = \{\phi, \{\omega_1, \omega_2\}, \{\omega_1\}, \{\omega_2\}\}$

Def. 3.1.2. Let Ω be a non-empty set, and let $\mathcal{F}$ be a σ-algebra on Ω. A probability measure $\mathbb{P}$ is a function that assigns a number in $[0,1]$ to every set $A \in \mathcal{F}$, called probability of A $\mathbb{P}(A)$ and satisfies:

1. $\mathbb{P}(\Omega) = 1$
2. $(A_i)_{i \geq 1}$, $A_i \in \mathcal{F}$, $A_i \cap A_j = \phi \quad if\,(i \neq j)$

then $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

topology (collection of all open sets)
(略)

3.2 Property of Stochastic Integral

Gaussian Distribution

one-dimensional Gaussian distribution (mean 0, variance 1) $N(0,1)$...standard normal prob density: $\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$

for $a > 0$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \tag{3.1}$$

This implies

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = (-1)^n \int_{-\infty}^{\infty} \left(\frac{\partial^n}{\partial a^n}\right) e^{-ax^2} dx = (-1)^n \left(\frac{\partial^n}{\partial a^n} \sqrt{\frac{\pi}{a}}\right) \tag{3.2}$$

$(X_t)_{0 \le t \le T}$: $\mathbb{F}$-adopted,
$\{X_t(\omega) \in B\} \in \mathcal{F}_t \quad \forall t$,
$B \in \mathcal{B}(\mathbb{R}^n)$
$\Rightarrow$ If we have $\mathcal{F}_t$ information, we know the ? X_t of

Def. 3.2.1. Let $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t \ge 0})$ be a filtered probability space. W is a continuous $\mathbb{F}$-adopted process that satisfy

- $W(0) = 0$
- $W(u) - W(t)$ is independent from $\mathcal{F}_t$ (and in particular $W(s) - W(r)$) whenever $r < s \le t < u$
- $W(t) - W(s) \sim N(0, t-s)$ in distribution
  if $d > 1$ then, for each i, $W^i(t) - W^i(s) \sim N(0, t-s)$, W^i, W^j are independent.

Here $N(0, v)$ is the gaussian distribution of mean 0, variance v. Then we call W a $(\mathbb{P}; \mathbb{F})$-Brownian motion.

Some propositions of Brownian motion (one-dimensional case)

Set $s < t$, $\Delta t = t - s$, $\Delta W := W(t) - W(s)$

$\Rightarrow$

$$\mathrm{E}[\Delta W] = 0 \tag{3.3}$$
$$\mathrm{E}[(\Delta W)^2] = \mathrm{V}[\Delta W] = t - s = \Delta t \tag{3.4}$$

$$\begin{aligned} \mathrm{V}[(\Delta W)^2] = \mathrm{E}[(\Delta W)^4] - \mathrm{E}[(\Delta W)^w]^2 &= \int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}} dx - (\Delta t)^2 \\ &= (\Delta t)^2 \int_{-\infty}^{\infty} \frac{z^4}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - (\Delta t)^2 = 3(\Delta t)^2 - (\Delta t)^2 = 2(\Delta t)^2 \end{aligned} \tag{3.5}$$

Consider the limit $\Delta \to dt$, then $(\Delta W)^2 \to (dW)^2 = dt$ (variance$\to 0$ as dt$\to 0$)

Def. 3.2.2. Denote $\mathcal{P}$ the vector space of maps $H : \Omega \times [0, T] \to R$ of the form

$$H_t(\omega) = \sum_{i=1}^{n} h_{i-1}(\omega) \mathbb{1}_{(t_{i-1}, t_i]}(t) \tag{3.6}$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < ... < t = T$,

3.3 Partial Differential Equation for Option Pricing

Work in the physical(emprical) measure

n-risky asset price process $S_t = (S_t^1, ..., S_t^n)$

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dW_t \tag{3.7}$$

$$b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \tag{3.8}$$

$$\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d} \tag{3.9}$$

$$\tag{3.10}$$

W: d-dimensional B.M. under the physical measure

money market

$$\beta(t) = \exp\left(\int_0^t r(s)ds\right) \quad (s \geq 0) \tag{3.11}$$

$r(s)$: non-random

Let us consider a European Option which pays $\Psi(S_T)$ at T $(\Psi : \mathbb{R}^n \to \mathbb{R})$

(ex. $\Psi(S_t) = \max(S_t^1 - K, 0)$ call on the 1st asset)

Suppose the option price at $t \in [0, T]$ is given by some function:

$c : [0, T] \times \mathbb{R}^n \to \mathbb{R}$,

$c \in \mathbb{C}^{1,2}$ (or smooth)

such that the price at t is $c(t, S_t)$

Itô-formula implies

$$dC(t, S_t) = \frac{\partial}{\partial t}c(t, S_t)dt + \sum_{i=1}^{n} c(t, s_t)dS_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 c(t, S_t)}{\partial x_i \partial x_j}(\sigma\sigma^T)_{ij}(t, S_t)dt \tag{3.12}$$

Treat tye contingent claim as another treadable risky security. (オプションを取引可能資産に加える)

- η_t, $0 \leq t \leq$: number of position for the option
- π_t^i, $0 \leq t \leq$, $i = 1, ..., n$: number of position for the ith stock
- ζ_t, $0 \leq t \leq$: number of position for β

Portfolio strategy:

$$h_t = (\eta_t, \pi_t, \zeta_t) = (\eta_t, \pi_t^1, ..., \pi_t^n, \zeta_t) \tag{3.13}$$

$$V_t^h = \eta_t c(t, S_t) + \pi_t \cdot S_t + \zeta_t\beta(t) = \eta_t c(t, S_t) + \sum_{i=1}^{n} \pi_t^i S_t^i + \zeta_t\beta(t) \tag{3.14}$$

Self-financing condition

$$
\begin{aligned}
dV_t^h =& \eta_t dc(t,S_t) + \pi_t \cdot dS_t + \zeta_t d\beta(t) \\
=& \eta_t dc(t,S_t) + \pi_t \cdot dS_t + r(t)(V_t^h - \eta_t c(t,S_t) - \pi_t \cdot S_t)dt \qquad (3.15) \\
=& r(t)(V_t^h - \pi_t \cdot S_t)dt + \sum_{i=1}^{n}(\eta_t \frac{\partial}{\partial x_i} c(t,S_t) + \pi_t^i)dS_t^i \\
&+ \eta_t \left(\frac{\partial}{\partial t} c(t,S_t) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(t,S_t)\frac{\partial^2 c(t,S_t)}{\partial x_i \partial x_j} - r(t)c(t,S_t) \right) dt \qquad (3.16)
\end{aligned}
$$

Choose the following strategy h such that

$$
\pi_t^i = \eta_t \frac{\partial}{\partial x_i} c(t,S_t) \quad i = 1,...,n \, (0 \le t \le T) \tag{3.17}
$$

(delta neutral position)

Then, for such a h,

$$
\begin{aligned}
dV_t^j = \eta_t \{ & \underline{\frac{\partial}{\partial t} c(t,S_t) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(t,S_t)\frac{\partial^2 c(t,S_t)}{\partial x_i \partial x_j}} \\
& \underline{+ r(t)\sum_{i=1}^{n}(S_t^i \frac{\partial}{\partial x_i} c(t,S_t) - r(t)c(t,S_t)} \, \}dt + r(t)V_t^h dt \qquad (3.18)
\end{aligned}
$$

There is no term of stochastic integral dW. Specify the trading strategy such that

$$
\eta_t = \begin{cases} (\eta_t > 0) & \text{when}\{...\} \ge 0 \\ (\eta_t < 0) & \text{when}\{...\} < 0 \end{cases} \tag{3.19}
$$

example:

$$
\begin{cases} 1 & \text{when}\{...\} \ge 0 \\ -1 & \text{when}\{...\} < 0 \end{cases} \tag{3.20}
$$

$r(t)$ より高いリターンを無リスクで得られる!?

V_t^h :

- higher return rate $> r(t)$
- risk-free (no dW-term)

$\Rightarrow$ arbitrage

For the absense of arbitrage, we mus have $\{...\} = 0$

As a result, we have got the next pricing PDE:

$$
\frac{\partial}{\partial t} c(t,x) + r(t)\sum_{i=1}^{n} x^i \frac{\partial}{\partial x_i} c(t,x) + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(t,x)\frac{\partial^2 c(t,x)}{\partial x_i \partial x_j} - r(t)c(t,x) = 0 \tag{3.21}
$$

$$
\text{B.C.} \quad c(T,x) = \Psi(s) \quad \text{for } x \in \mathbb{R}^n \tag{3.22}
$$

for $(t,x) \in [0,T] \times \mathbb{R}^n$

dW-term を消すような strategy をつくる

$\Rightarrow r(t)V_t^h dt$ 以外の項がゼロになる必要がある (no-arbitrage)

$\Rightarrow$ We got PDE.

Black-Scholes: special case $n = d = 1$

$$\sigma(t,x) = \sigma x, \quad r(t) \equiv r \geq 0 \tag{3.23}$$

3 次元以上だと計算が大変 (だった)

Suppose $c(t,x) \in \mathbb{C}^{1,2}([0,T] \times \mathbb{R}^n)$, solves the last PDE. Consider the self-financing portfolio given by $(\eta, \pi, \zeta) = (-1, \pi, \zeta)$.

(short-position of the option, stock, money market)

$$\begin{aligned} dV_t =& r(t)(V_t - \pi_t - \zeta_t)dt + \sum_{i=1}^n \left(-\frac{\partial}{\partial x_i} c(t, S_t) + \pi_t^i \right) dS_t^i \\ & - \left(\frac{\partial}{\partial t} c(t,S_t) + \frac{1}{2} \sum_{i,j=1}^n (\sigma\sigma^T)_{ij}(t,S_t) \frac{\partial^2 c(t,S_t)}{\partial x_i \partial x_j} - r(t)c(t,S_t) \right) dt \end{aligned} \tag{3.24}$$

$$dV_t = r(t)(V_t - \pi_t - \zeta_t)dt + \sum_{i=1}^n \left(-\frac{\partial}{\partial x_i} c(t, S_t) + \pi_t^i \right) dS_t^i + r(t) \sum_{i=1}^n S_t^i \frac{\partial c(t,S_t)}{\partial x_i} dt \tag{3.25}$$

Since $c(t,x)$ satisfies the PDE, choose π^* given by

$$\pi^{*i}(t) = \frac{\partial}{\partial x_i} c(t, S_t) \quad i = 1, ..., n \tag{3.26}$$

as π_t.

$\Rightarrow$ for such π^*

$$dV_t = r(t)V_t dt \tag{3.27}$$

$\therefore$ If $V_0 = 0$, then $V_t = 0$ for all $t \in [0,T]$

This means the portfolio starting from a cash value $c(0, S_0)$ with dynamic position:

$$\left(\pi^{*i}(t), \frac{c(t,S_t) - \pi_t^* \cdot S_t}{\beta(t)} \right) \tag{3.28}$$

for stock and money market.

(perfectly replicates not only the option payoff $c(T, S_T) = \Psi(S_T)$ but also the intermediate option value $c(t, S_t)$.

3.4 Relation to the Risk-Neutral Pricing

Suppose that there is a probability measure Q equivalent to P such that

$$dS(t) = r(t)S(t)dt + \sigma(t, S_t)dW_t^Q \tag{3.29}$$

W^Q : Brownian motion under the measure Q

3.5 BS Option Formula

$r, \sigma > 0$: const.
one-dimensional S, W^Q

Suppose the stock price follows a geometric Brownian motion (log-normal process):

$$dS_t = S_t(rd_t + \sigma dW_t^Q) \tag{3.30}$$

or equivalently

$$S_t = S_0 + \int_0^t S_u(rdu + \sigma dW_u^Q) \tag{3.31}$$

Call Option with strike K (> 0)

$$\Psi(S_T) = \max(S_T - K, 0) \tag{3.32}$$

option price at $t = 0$:

$$C = e^{-rT}\mathrm{E}^Q[\max(S_T - K, 0)] \tag{3.33}$$

$S_T > 0$ for all $t \geq 0$ if $S_0 > 0$

Apply Itt$\hat{\text{o}}$ formula to $\ln(S_t)$

$$d\ln(S_t) = \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t^Q \tag{3.34}$$

Integrates the both hands for $[0, T]$

$$\ln S_t - \ln S_0 = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^Q \tag{3.35}$$

$$S_T = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^Q\right) \tag{3.36}$$

Note that $W_T^Q \approx (dist)\sqrt{T}z$

z $N(0, 1)$: standard normal (mean:0, variance:1)

z has the density $\phi(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$C = e^{-rT} \int_{-\infty}^{\infty} \max\left(S_0 \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) T + \sigma\sqrt{T} z \right\} - K, 0 \right) \phi(z) dz \tag{3.37}$$

$$C = e^{-rT} \int_{-\infty}^{\infty} \max\left(F e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}z} - K, 0 \right) \phi(z) dz, \quad (F = e^{rT} S_0 = \frac{S_0}{P(0,T)}) \tag{3.38}$$

Let us define

$$d2 := \frac{1}{\sigma\sqrt{T}} \left(\left(\frac{F}{K} \right) - \frac{\sigma^2}{2} T \right) \tag{3.39}$$

Then,

$$F e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}z} \geq K \quad \text{when } z \geq -d_2 \tag{3.40}$$

Therefore,

$$C = e^{-rT} \int_{-d_2}^{\infty} \left(F e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}z} - K \right) \phi(z) dz \tag{3.41}$$

- 1st term:

$$\begin{aligned} &F e^{-rT} e^{-\frac{\sigma^2}{2}T} \int_{-d_2}^{\infty} e^{\sigma\sqrt{T}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = F e^{-rT} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - \sigma\sqrt{T})^2} dz \\ =&F e^{-rT} \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z'^2} dz' = F e^{-rT} N(d_1) \end{aligned} \tag{3.42}$$

$$d_1 := d_2 + \sigma\sqrt{T} = \frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{F}{K} \right) + \frac{\sigma^2}{2} T \right) \tag{3.43}$$

- 2nd term:

$$-K e^{-rT} \int_{-d_2}^{\infty} \phi(z) dz = -K e^{-rT} N(d_2) \tag{3.44}$$

Put option?

$$P = e^{-rT} \mathrm{E}^Q [\max(K - S_T, 0)] \tag{3.45}$$

$$\Psi(S_T) = \max(K - S_T, 0) \tag{3.46}$$

Call と同様の計算は面倒...

Put-Call parity

$$\max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K \tag{3.47}$$

$$\begin{aligned} \therefore\ C - P &= e^{-rT} \mathrm{E}^Q [S_T - K] = e^{-rT} \mathrm{E}^Q [S_T] - K e^{-rT} \\ &= \mathrm{E}^Q [\beta(T)^{-1} S_T] - K e^{-rT} \end{aligned} \tag{3.48}$$

$(\beta(t)^{-1}S_t)_{0\le t\le T}$ ... Q-martingale

$$\because\ d(\beta(t)^{-1}S_t) = \beta(t)^{-1}\{r(t)S_t dt + dSt\} = \beta(t)^{-1}\sigma S_t dW_t^Q \tag{3.49}$$

Therefore,

$$C - P = \beta(0)^{-1}S_0 - Ke^{-rT} = S_0 - Ke^{-rT} \tag{3.50}$$

$$\begin{aligned} P &= C - S_0 + Ke^{-rT} = e^{-rT}(FN(d_1) - KN(d_2) - F + K) \\ &= -e^{-rT}F(1 - N(d_1)) + e^{-rT}K(1 - N(d_2)) \\ &= e^{-rT}\{KN(-d_2) - FN(-d_1)\} \end{aligned} \tag{3.51}$$

3.6 Opriont Formula for a Stock with Dividend Yield

dividend payment $[t, t+dt]$ $q_(t)S_t dt$ q:dividend yield

If we reinvest all the dividends to the same stock, one share at $t = 0$

$\Rightarrow$ $\exp\left(\int_0^t q(s)ds\right)$ share at t

$\because$ Number of share $n(t)$ follows $\dfrac{d}{dt}n(t) = q(t)n(t)$

For no-arbitrage, $\left(S(t)e^{\int_0^t q(s)ds}\right)_{0\le t\le T}$ must follow the dynamics corresponding to the stock price without dividend payment.

$\Rightarrow$ $\left(S(t)e^{\int_0^t q(s)ds}\right)_{0\le t\le T}$ must have the drift rate $r(t)$.

under Q, dS must follow:

$$dS_t = (r(t) - q(t))S_t dt + \sigma(t, S_t)dW_t^Q \tag{3.52}$$

$$\begin{aligned} d(S_t e^{\int_0^t q(s)ds}) &= e^{\int_0^t q(s)ds}(q(t)dt + dS_t) \\ &= e^{\int_0^t q(s)ds}(r(t)S_t dt + \sigma(t, S_t)dW_t^Q) \end{aligned} \tag{3.53}$$

dividend yield

$$dS_t = S_t(r_t - q_t)dt + S_t\sigma_t dW_t^Q \tag{3.54}$$

For simplicity, $r, q, \sigma > 0$: constants

$$d\ln S_t = \left(r - q - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t^Q \tag{3.55}$$

$$S_t = \tag{3.56}$$

lllll

Ex. 3.6.1. (FX)
$(X_t)_{0\leq t\leq T}$: the price of unit amount of a foreign currency in terms of the domestic currency
X_T : asset price, dividend yield $r^f(t)$
(foreign interest rate)

$$dX_t = (r^d(t) - r^f(t))X_t + \sigma(t, X_t)dW_t^Q \tag{3.57}$$

($X > 0$ を保証するもの)

Def. 3.6.1. (Implied volatility) The implied volatility of an option contract is the volatility constant σ which gives the market price of the option when σ is used in B.S. model.

(illustration)

3.6.1 Market Option Quotes and the Implied Distribution

For simplicity, assume $r > 0$ is constant, Q exists.
Then, a call option price is given by

$$c(K) = e^{-rT}\mathrm{E}^Q[(S_T - K)^+] \tag{3.58}$$

expiry T : fixed, strike K : floating
Suppose S_T has a smooth probability density function $g(x) : \mathbb{R} \to \mathbb{R}$ under Q. Then,

$$c(K) = e^{-rT}\int_K^\infty (x - K)g(x)dx \tag{3.59}$$

Then,

$$\begin{aligned}\frac{\partial c}{\partial K} &= -e^{-rT}(x-K)g(x)dx|_{x=K} = -e^{-rT}\int_K^\infty g(x)dx \\ &= -e^{-rT}\int_K^\infty g(x)dx \qquad (3.60)\\ \frac{\partial^2 c}{\partial K^2} &= e^{-rT}g(K) \qquad (3.61)\end{aligned}$$

Implied volatility の価格変化から S_T の density を得る：

$$g(K) = e^{rT}\frac{\partial^2 c}{\partial K^2} \tag{3.62}$$

3.6.2 Greeks

Sensitivity of option (or any derivative contract) price with respect to various parameters (S, t, σ, etc.)
Option price $\Pi(t, S_t)$ at $(t.S_t)$

- Delta:

$$\frac{\partial}{\partial x}\Pi(t,x)|_{x=S_t} \tag{3.63}$$

- Gamma:

$$\frac{\partial^2}{\partial x^2}\Pi(t,x)|_{x=S_t} \tag{3.64}$$

- Vega(Kappa):

$$\frac{\partial}{\partial \sigma}\Pi \tag{3.65}$$

- Cross Gamma:

$$\frac{\partial^2 \Pi}{\partial x_i \partial x_j} \tag{3.66}$$

Cross Gamma は P/L-attirbution を調べるのに使える．

complete market には十分な流動性が必要．unheagable な product に対する線形な価格評価 (risk-neutral pricing) をするのは恐ろしい．仮定を置かないと正当化されない．本当は，各事業者がリスク効用関数 (risk-averseness) を設定し，それに応じて最適化問題として解くべきである（が，大変なのでやらない).