Advanced Derivative

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1 Bond and Forward Contract

1.1 Interest Rates

Asm. 1.1.1. We asssume that there exists a default free money market account

- default-free
- liquid (borrowing ragte = lending rate)
- everyone can access equally

The interest rates associating with the default free m.m. (money market) are called risk-free rates.

1.1.1 Types of Accrual (利息)

Suppose one invests cash amount A at $t = 0$ for T years.
V : The amount of cash to be returned in T years time.

1. Annual compounding with interest rate R_1 per annum
   $V = A(1 + R_1)^T$
2. Semiannual compounding with interest rate R_2 per annum
   $V = A(1 + \frac{R_2}{2})^{2T}$
3. m-times compounding with interest rate R_m per annum
   $V = A(1 + \frac{R_m}{m})^{mT}$ (typiccaly we have $m = 1, 2, 4, 12, 52, 365$)
4. continuous compounding with interest rate r per annum
   $V = \lim_{m\to\infty} A(1 + \frac{r}{m})^{mT} = Ae^{rT}$

Relation among different compounding conventions:

No-Arbitrage $\rightarrow$ for any m,

$$e^{rT} = \left(1 + \frac{R_m}{m}\right)^{mT} \tag{1.1}$$

$$\Leftrightarrow \quad r = m \ln\left(1 + \frac{R_m}{m}\right), \quad R_m = m(e^{r/m} - 1). \tag{1.2}$$

at a given time, $R_1(T)$: T dependent.

1.1.2 Zero Rate and Bonds

Def. 1.1.1. (zero rate) The T-year zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for T-years without any intermidiate coupon payment.

Ex. 1.1.1. 5-year zero rate = 5 % per annum. (continuous compounding)
\$ 100 を deposit at $t = 0 \rightarrow$ (5 years later) $100e^{0.05\times 5} \approx 128.40$

Present value (PV) $\quad P.V. = 128.40 \times e^{-0.05\times 5} = 100$

Def. 1.1.2. (ZCB : zero coupon bond) T-year zero coupon bond is a bond which pays an unit amount of cash in T-year time without any coupon payment (assume risk-free, liquid).

If R is T-year zero rate, current price of T-year zero coupon Bond is given by :

$$P(0, T) = e^{-RT} \tag{1.3}$$

(0: current time, T: maturity), we call "discount factor".

Ex. 1.1.2. Suppose we have (at $t = 0$)

- $T = 0.5$, $R = 5.0\%$ (zero coupon rate)
- $T = 1.0$, $R = 5.8\%$
- $T = 1.5$, $R = 6.4\%$
- $T = 2.0$, $R = 6.8\%$

fixed coupon bond

- maturity: $T = 2$
- coupon payments: 6% per annum semiannually.
- principal: \$100

$$\begin{aligned}
\text{Bond price} &= 3P(0, 0.5) + 3P(0, 1.0) + 3P(0, 1.5) + 103P(0, 2) \\
&= 3 \times e^{-5\%\times 0.5} + 3 \times e^{-5.8\%\times 1.0} + 3 \times e^{-6{,}4\%\times 1.5} + 103 \times e^{-6.8\%\times 2.0} \\
&\approx 98.39
\end{aligned} \tag{1.4}$$

1.1.3 Yield (平均利率，平均利回り，平均 discount rate)

Def. 1.1.3. (Bond's Yield) A bond's yield is the single discount rate that when applied to eery cash flow, gives the market bond price.

Ex. 1.1.3. using Ex. 1.1.2, suppose market price = 98.39

Then the yield for the bond y is given by solving

$$3e^{-0.5y} + 3e^{-1.0y} + 3e^{-1.5y} + 103e^{-2.0y} = 98.39 \tag{1.5}$$

$$\Rightarrow y \simeq 6.76\% \quad \text{(continuous compounding)} \tag{1.6}$$

Def. 1.1.4. (Par Yield) The par yield for a certain maturity is the coupon rate that makes the bond price equal to its principal.

Ex. 1.1.4. using Ex. 1.1.2, par yield C for 2-year coupon bond is given by:

$$\frac{C}{2}e^{-0.050\cdot 0.5} + \frac{C}{2}e^{-0.058\cdot 1.0} + \frac{C}{2}e^{-0.064\cdot 1.5} + \left(100 + \frac{C}{2}\right)e^{-0.068\cdot 2.0} = 100 \tag{1.7}$$

$$\Rightarrow C \simeq 6.87\% \tag{1.8}$$

(par yield for $T = 2$, semiannual coupon payments)

In general,

- T-year bond
- m-time coupon payment per annum
- par yield C

$$\sum_{n=1}^{mT} \frac{C}{m} P\left(0, \frac{n}{m}\right) + 100P(0,T) = 100 \tag{1.9}$$

$$\Rightarrow C = \frac{100(1 - P(0,T))}{A}, \quad A = \sum_{n=1}^{mT} \frac{1}{m} P\left(0, \frac{n}{m}\right) \tag{1.10}$$

1.1.4 Duration

fixed coupond bond:

- (cash flow at T_i) $= C_i \quad (i = 1, ..., n)$
- C_i: coupon (+ principal at maturity)

Suppose its yield is given by y (continuous compounding).

Bond price:

$$B = \sum_{i=1}^{n} C_i e^{-yT_i} \tag{1.11}$$

The duration of the Bond:

$$D := -\frac{1}{B}\frac{d}{dy}B = -\left(\frac{dB/dy}{B}\right) = \frac{1}{B}\sum_{i=1}^{n} C_i T_i e^{-yT_i} \tag{1.12}$$

Ex. 1.1.5. zero coupon Bond

$$(C_i = 0)_{i=1,2,\dots,n-1},\ C_n = 1, \quad B = e^{-yT_n} \tag{1.13}$$

$$\Rightarrow D = \frac{1}{B} C_n T_n e^{-yT_n} = T_n \tag{1.14}$$

(* Duration の長短により，金利に対する反応度の違いがわかる．)

Suppose the yield changes small amount Δy, $(y \to y + \Delta y, \quad B \to B + \Delta B)$

$$\frac{\Delta B}{B} = -D\Delta y + o(\Delta y) \tag{1.15}$$

(abbr.)

Suppose $D = 10$ (10 year), yield: $\Delta y = +0.1\%$ (10 basis points, b.p.):

$$\frac{\Delta B}{B} \approx -10 \times 0.1\% = -1\% = -0.01 \tag{1.16}$$

1.1.5 Modified Duration

yield (m-time compounding) $\hat{y}$

the same bond:

$$B = \sum_{i=1}^{n} C_i \left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \tag{1.17}$$

modified duration:

$$\begin{aligned} D^* : &= -\frac{1}{B}\frac{dB}{d\hat{y}} = \frac{1}{B}\sum_{i=1}^{n} \frac{C_i T_i}{1 + \hat{y}/m}\left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \\ &= \frac{1}{B(1+\hat{y}/m)} \sum_{i=1}^{n} C_i T_i \left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \\ &= \frac{1}{B(1+\hat{y}/m)} \sum_{i=1}^{n} C_i T_i e^{-yT_i} = \frac{D}{1+\hat{y}/m} \quad \text{(no arbitrage)} \end{aligned} \tag{1.18}$$

* duration の議論は cash flow が一方向のときのみ使える．Insurance では通用しないので注意.

1.1.6 Convecity)

y : yiled (continuous compounding)

$$C := \frac{1}{B}\frac{d^2B}{dy^2} = \frac{1}{B}\sum_{i=1}^{n} C_i T_i^2 e^{-yT_i} \tag{1.19}$$

$y \to y + \Delta y, \quad B \to B + \Delta B$:

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}(\Delta y)^2 + o(\Delta y^2) \tag{1.20}$$

$$\Rightarrow \frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2 + o(\Delta y^2) \tag{1.21}$$

(Duration matching : abbr.)

1.2 Forward Contract

1.2.1 Forward Price

Def. 1.2.1. (Forward Contract) A forward contract with maturity T is a bilateral binding promise(agreement) such that at time $t = T(> 0)$, the two parties exchange:

- the cash amount given by the time T realization of a certain index (such as a stock price) with the fixed amount of cash (cash delivery)
- the unit amount of asset (such as a share of an equity) with fixed amount of cash (physical delivery=現物)

Def. 1.2.2. (Forward Price) A forward price F at the current time ($t = 0$) (契約時) of the underlying index X is the amount of cash K that make the present value of the forward contract exchanging X_t and K at T zero. (Forward contract has $P.V. = 0$, with $K = F$.)
* F は契約時に支払う額ではないことに注意 (元手は不要)
* K such that P.V. of the fwd contract $= 0$

Ex. 1.2.1. Consider a forward contract on a non-dividend paying stock, with mat. T.
$X_T = S_T$ (stock price at T), exchange $F \leftrightarrow S_T$ (at T).

Asm. 1.2.1. Stock market is liquid, zero-coupon bond is liquid.

the forward price at $t = 0$ is given by:

$$F = \frac{S_0}{P(0,T)} = e^{rT}S_0 \tag{1.22}$$

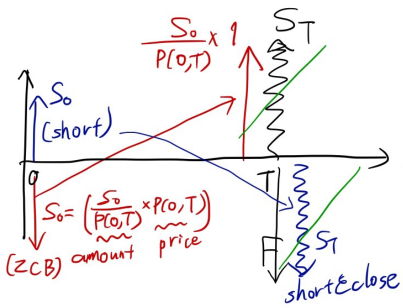
- r : zero-rate for mat T, continuous, compounding
- $P(0,T)$: zero coupon bond price

Prf. 1.2.1. replication strategy

- Enter the fwd contract to get one share of stock (S_T) by paying F at T ($t = 0$ で enter する際に元手は不要)
- Sell one share of stock at $t = 0$ to get S_0 (short position)
- Use S_0 to buy ZCB (zero coupon bond) by the amount $S_0/P(0,T)$

- Pay F and receive S_T at T (return S_T to lender)
- Receive $S_0/P(0,T)$ from ZCB lender

(cash flow illustration below)
if $F \neq \dfrac{S_0}{P(0,T)} \Rightarrow$ arbitrage (no risk, arbitrary positive return)
No-arbitrage $\Rightarrow F = \dfrac{S_0}{P(0,T)} = e^{rT}S_0$ (F は stochastic ではない)



stochastic な S_T が消えた

☆ net cf $= \dfrac{S_0}{P(0,T)} - F$ (at T)

(no-arbitrage) $\Downarrow$

$\therefore F = \dfrac{S_0}{P(0,T)}$ //

Ex. 1.2.2. Same as Ex.1.2.1 but now the stock pays continuous dividend, with dividend rate y ($y \in \mathbb{R}$, constant)

One share of the stock pays $S_t y dt$ for the interval $[t, t+dt]$ for any $t \geq 0$.
forward price at $t = 0$

$$F = \frac{S_0}{P(0,T)}e^{-yT} = S_0 e^{(r-y)T} \tag{1.23}$$

r : zero-rate for mat T at $t = 0$

Suppose we heve N_t shares at t, dividend paid in $[t, t+dt]$: $S_t N_t y dt$
$\Rightarrow$ reinvest $\Delta N_t = N_t y dt$

if one reinvests the whole dividend payment,

$$\frac{dN_t}{dt} = N_t y \Rightarrow N_t = N_0 e^{yt} \quad \text{for all} \quad t \geq 0 \tag{1.24}$$

Therefore, if one wants $N_T = 1$, N_0 is to be e^{-yT}.

Prf. 1.2.2. replication strategy
($t = 0$)

- Enter the fwd contract to receive F and deliver one share stock at T (Ex.1.2.1 と逆の party)

- Sell $\dfrac{S_0 e^{-yT}}{P(0,T)}$ amount of ZCB with maturity T
- Buy e^{-yT} shares of stock

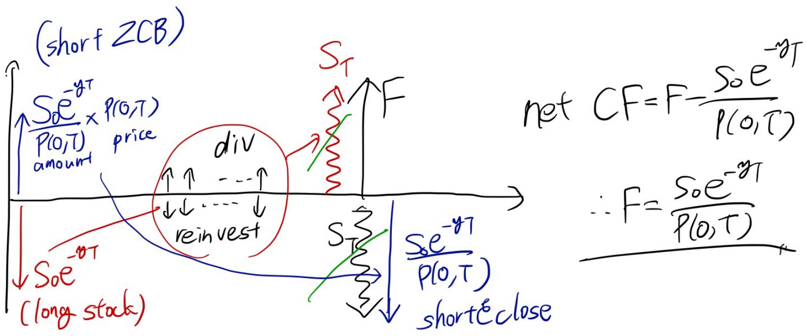
(always)

- Reinvest every div. payment to the stock

($t = T$)

- Receive F, deliver one share of stock
- Return $\dfrac{S_0 e^{-yT}}{P(0,T)}$ to the ZCB lender

(cash flow illustration below)



イメージ:

$$\text{P.V(receive } S_T \text{ at } T) = F \times P(0,T) = \frac{S_0}{P(0,T)} \times P(0,T) = S_0 \tag{1.25}$$

random な cash flow の P.V. を計算するときは，Forward Price を求めて，$P(0,T)$ を乗じてやればよい (ただし市場が liquid,replicatable なときのみ)

⇒ 後に risk-nuetral の下で計算すればわざわざ replication を考える必要がなくなる

1.2.2 Mark to Market of a fwd contract

* P.V.(at $t = 0$){fwd contract} は 0 だが，時間が進むにつれて，P.V. は変化する．

Suppose we entered the fwd contract to receive X_T in exchange for a fixed amount of cash F_0 (F_0 : fwd price at $t = 0$). P.V.($t = 0$)= 0
at time $t \in (0, T)$, suppose fwd price is given by F_t. We want to know P.V.($t > 0$).
Ans.:

$$\text{P.V.}(t) = P(t,T)(F_t - F_0) \tag{1.26}$$

Prf. 1.2.3. enter the new fwd contract at $t = t$ (pays X_T and receives F_t at $t = T$)
* F_t の t は「時刻 t に contract に enter した」という意味.
(abbr.)

$$\begin{aligned}\text{P.V.}(\text{at}\,t)\{\text{new + original fwd contracts}\} &= P(t,T)(F_t - F_0)\\ &= 0 + \text{P.V.}(\text{at}\,t)\{\text{original}\}\end{aligned} \tag{1.27}$$

1.2.3 Put-Call Parity

Def. 1.2.3. (Call Option and Put Option) A call (respectively, put) option on a certain index X with expiry T and strike K is the binomial?? contract to pay the holder of the option the cash amount equal to $\max(X_T - K, 0)$, (resp, $\max(K - X_T, 0)$)

Let C (resp, P) be the call (resp, put) option orice (at $t = 0$). We have the following put-call parity:

$$C - P = P(0,T)(F_X - K) \tag{1.28}$$

- F_X : the fwd price of X with mat. T (時刻 $t = X$ ではなく，underlying asset X をもとにした fwd price at $t = 0$)

Prf. 1.2.4. cash flow at T:

$$\max(X_T, K, 0) - \max(K - X_T, 0) = X_T - K \tag{1.29}$$
$$\tag{1.30}$$

Present value of above is given by:

$$C - P = P(0,T)(F_X - K) \tag{1.31}$$

motivation:

- liquidity の問題
- call, put の一方が求まれば，もう一方をすぐに求められる
- PDE の計算は put の方が簡単 (because of boundary condition)

1.3 Forward Rate Agreement and Interest Rate Swap

*金利スワップは not tradable

1.3.1 Simple Rate and Day-Count Convention

Suppose T_i specifies the date $D_i = D(d_i, m_i, y_i)$

1. Actual/365

$$\delta(T_0, T_1) = \frac{D_1 - D_0}{365} \tag{1.32}$$

2. Actual/360

$$\delta(T_0, T_1) = \frac{D_1 - D_0}{360} \tag{1.33}$$

3. 30/360

$$\delta(T_0, T_1) = \frac{\max(30 - d_0, 0) + \min(d_1, 30) + 360(y_1 - y_0) + 30(m_1 - m_0 - 1)}{360} \tag{1.34}$$

4. actual/actual considering leap year? 365 or 366

Def. 1.3.1. (Simple Rate) A (risk-free) simple rate (not compound) $L(T_{i-1}, T_i)$ with day-count $\delta(T_{i-1}, T_i)$ is the interest rate with accrual convention defined in such a way that, when one invest N amount of cash at T_{i-1}, then he receives $N(1 + \delta_i L(T_{i-1}, T_i))$ at time T_i. $L(T_{i-1}, T_i)$ is the zero coupon rate at T_{i-1} for $[T_{i-1}, T_i]$ with corresponding day-count convention.

*accrual: ??

1.3.2 Forward Rate Agreement (FRA)

Def. 1.3.2. Forward Rate Agreement A FRA is a binding(義務の) contract with the two parties (lender and borrower) agreeing to let a certain fixed rate K act on a prefixed notional amount N, over a future period $[T_M, T_N]$.

*notional: 想定される?

Def. 1.3.3. Forward Rate A forward rate F for the period $[T_M, T_N]$ with day-count $\delta = \delta(T_M, T_N)$ is the fixed rate K with the some day-count convention that makes the present value of the FRA zero.

off course,

$$F = L(T_M, T_N) \quad (\text{at}\, T_M) \tag{1.35}$$

*L : simple rate

Lem. 1.3.1. Let $\delta = \delta(T_M, T_N)$. Then the forward rate F at $t = 0$ fot the period $[T_M, T_N]$ is given by

$$F = \frac{1}{\delta}\left(\frac{P(0, T_M)}{P(0, T_N)} - 1\right) \tag{1.36}$$

(*asm: liquid, no-arbitrage)

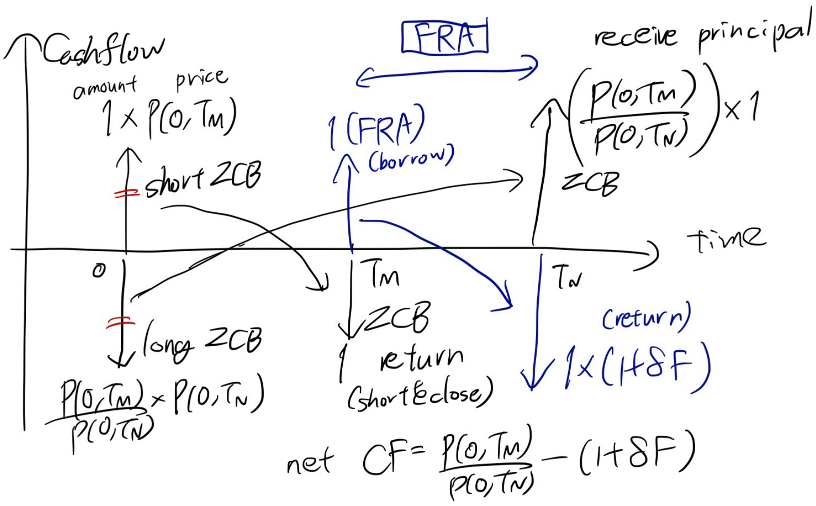
Prf. 1.3.1. replication strategy

- Enter the FRA of rate F to borrow unit amount of cash for $[T_M, T_N]$
- Sell one ZCB with mat. T_N (short)
- Buy ZCB with mat. T_N with principal amount $\dfrac{P(0,T_M)}{P(0,T_N)}$
- At T_M, borrow unit amount of cash through FRA and use it to return ZCB
- At T_N, receive the principal $\dfrac{P(0,T_M)}{P(0,T_N)}$, and pays $(1+\delta F)$

Net cash flow at T_N: $\left(\dfrac{P(0,T_M)}{P(0,T_N)}\right) - (1+\delta F)$

If we require no-arbitrage,

$$\left(\frac{P(0,T_M)}{P(0,T_N)}\right) - (1+\delta F) = 0 \quad \Rightarrow \quad F = \frac{1}{\delta}\left(\frac{P(0,T_M)}{P(0,T_N)} - 1\right) \tag{1.37}$$



We write the above F as

$$F(0,T_M,T_N) = \frac{1}{\delta}\left(\frac{P(0,T_M)}{P(0,T_N)} - 1\right) \tag{1.38}$$

In general. at $t < T_M$,

$$F(t,T_M,T_N) = \frac{1}{\delta}\left(\frac{P(t,T_M)}{P(t,T_N)} - 1\right) \tag{1.39}$$

$t \uparrow T_M$:

$$F(T_M,T_M,T_N) = \frac{1}{\delta}\left(\frac{1}{P(T_M,T_N)} - 1\right) = L(T_M,T_N) \tag{1.40}$$

$(\because)$

$$1 = P(T_M, T_N)\{1 + \delta L(T_M, T_N)\} \tag{1.41}$$

at T_M invest 1, at T_N return $1 + \delta L(T_M, T_N)$
(otherwise there exist arbitrage opportunities.)
(abbr.)

Ex. 1.3.1. Q) P.V.(at 0) {receive $1 + \delta L(T_M, T_N)$} ? $\to$ A) P.V.=0
$(\because)$ Suppose that we are at $t = T_M$, $L(T_M, T_N)$ ... known

$$\text{P.V.}(\text{at}\, T_M) = -1 + P(T_M, T_N)(1 + \delta L(T_M, T_N)) = 0 \tag{1.42}$$

将来のある時点 $(t = T_M)$ で P.V.$= 0$ なら，さかのぼった $t = 0$ でも当然 P.V.$= 0$

$$P.V.(\text{at}\, 0)\{\text{receive}\, 1 + \delta F(0, T_M, T_N)\, \text{at}\, T_N\} = P.V.(\text{at}\, 0)\{\text{receive}\, 1 + \delta L(T_M, T_N)\} \tag{1.43}$$

$$(\therefore) \quad P(0, T_N)F(0, T_M, T_N) = P.V.(\text{at}\, 0)\{\text{receive}\, L(T_M, T_N)\} \tag{1.44}$$

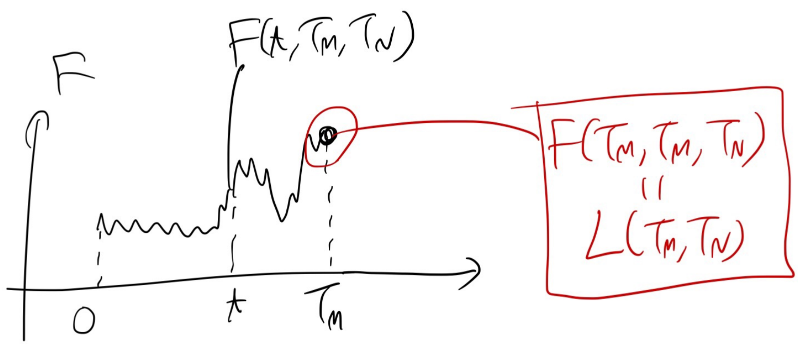
1.3.3 Fixed vs Floating Interest Swap

Fix a time partition $0 = T_0 < T_1 < ... < T_M$

Def. 1.3.4. (Spot-start Swap) A spot-start swap with maturity T_M and notional amount N is the contract in which one party (receiver) receives cash amount $NK\Delta_i$ (K fixed) and pays the stochastic amount $NL(T_{i-1}, T_i)\delta_i$ at every T_i, $i = \{1, 2, ..., M\}$. The other party (payer) has the opposit cash flow. Here,

$$\begin{cases} \Delta_i & = \Delta(T_{i-1}, T_i) \quad \text{(fixed)} \\ \delta_i & = \delta(T_{i-1}, T_i) \quad \text{(floating)} \end{cases} \tag{1.45}$$

are the day counts fot a fixed and floating payments, respectively.



Def. 1.3.5. (Swap Rate) The (spot) swap rate for the maturity T_M is the fixed rate K that makes the present value of the swap zero.

P.V. of the swap rate:

$$PV_{fix} = NK \sum_{i=1}^{M} P(0,T_i)\Delta_i \tag{1.46}$$

$$\begin{aligned} PV_{float} &= N \sum_{i=1}^{M} P(0,T_i)F(0,T_{i-1},T_i)\delta_i = N \sum_{i=1}^{M} P(0,T_i)\left(\frac{P(0,T_{i-1})}{P(0,T_i)} - 1\right) \\ &= N \sum_{i=1}^{M} (P(0,T_i) - P(0,T_i)) = N(1 - P(0,T_M)) \end{aligned} \tag{1.47}$$

Swap rate K:

$$K = \frac{1 - P(0,T_M)}{\sum_{i=1}^{M} P(0,T_i)\Delta_i} := S(0;T_0,T_M) \tag{1.48}$$

* Economic meaning of swap rate:

$$S(0;T_0,T_M) = \frac{\sum_{i=1}^{M} P(0,T_i)F(0,T_{i-1},T_i)\delta_i}{\sum_{i=1}^{M} \Delta_i P(0,T_i)} \tag{1.49}$$

Let us approximate as

$$P(0,T_i) \approx 1,\ \delta_i \approx \Delta_i \quad \text{for all}\, i \tag{1.50}$$

$$\Rightarrow S(0;T_0,T_M) \approx \frac{\sum_{i=1}^{M} F(0,T_{i-1},T_i)}{M} \tag{1.51}$$

...average of fwd rates!

Def. 1.3.6. (Forward Swap) A forward swap is the swap which starts at some future time. Fixed rate (fixed at $t = 0$) which make the P.V. of the swap is called the forward swap rate.

Ex. 1.3.2. A forward swap for the period $[T_M, T_N]$
$\Rightarrow$ cash flow exchanges at T_i, $i = \{M+1, ..., N\}$
$NK\Delta_i \leftrightarrow NL(T_{i-1},T_i)\delta_i$

Let fixed rate be K, notional $= 1$

$$PV_{fix} = NK \sum_{i=M+1}^{N} P(0,T_i)\Delta_i \tag{1.52}$$

$$PV_{float} = N \sum_{i=M+1}^{N} P(0,T_i)F(0,T_{i-1},T_i)\delta_i = P(0,T_M) - P(0,T_N) \tag{1.53}$$

Fwd Swap Rate:

$$S(0;T_M,T_N) = \frac{P(0,T_M) - P(0,T_N)}{\sum_{i=M+1}^{N} P(0,T_i)\Delta_i} \tag{1.54}$$

1.3.4 Relation to the fixed coupon bond

Consider the spot-start swap for $[T_0 = 0, T_N]$ (notional= 0)

$$PV_{float} = \sum_{i=1}^{N} \delta_i P(0,T_i) F(0,T_{i-1},T_i) = 1 - P(0,T_N) \tag{1.55}$$

(abbr.)

Bond-Swap, Fixed vs Floating swap, ???

defined $i(t)$: index, $i \in \{0,...,N\}$ s.t. $t \in [T_i, T_{i+1})$ $\quad T_{i(t)} \le t < T_{i(t)+1}$

current time t

$$\begin{aligned}
&\text{P.V.(floating leg?+ final principal)}\\
=&P(t,T_{i(t)+1})\delta_{i(t)+1}L(T_{i(t)},T_{i(t)+1}) + \sum_{j=i(t)+2}^{N} P(t,T_j)\delta_j F(t,T_{j-1},T_j) + P(t,T_N)\\
=&P(t,T_{i(t)+1})\delta_{i(t)+1}L(T_{i(t)},T_{i(t)+1}) + \sum_{j=i(t)+2}^{N} P(t,T_j)(1+\delta_{i(t)+1}L(T_i,T_{i+1})) + P(t,T_N)\\
=&P(t,T_{i(t)+1})(1+\delta_{i(t)+1}L(T_i,T_{i+1})) \approx 1
\end{aligned} \tag{1.56}$$

Thus, floating leg + final principal $\approx$ IR-RISK 0

IR-Swap Risk $\approx$ fixed leg + final principal payment

$\Leftrightarrow$ fixed coupon Bond

1.3.5 Yield Curve Construction (Simplified...)

Asm. 1.3.1. There are market quotes of spot-starting swaps with swap rate $\{S_n\}_{n=1}^N$ with corresponding maturities $\{T_n\}_{n=1}^N$

$$S_n : S(0;T_0,T_n), \quad T_0 = 0 \tag{1.57}$$

Ex. 1.3.3. 3 month. $0 = T_0 < T_1 < ... < T_N$

We want to get $\{P(0,T_n)\}_{n=1}^N$ which are consistent with the swap quotes.

1) Determin $P(0,T_1)$

$$S_1 \Delta_1 P(0,T_1) = P(0,T_0) - P(0,T_1) \tag{1.58}$$

$$S_1 = \frac{1 - P(0,T_1)}{\Delta_1 P(0,T_1)} \tag{1.59}$$

$$P(0,T_1) = \frac{P(0,T_0)}{1+\Delta_1 S_1} = \frac{1}{1+\Delta_1 S_1} \tag{1.60}$$

2) Suppose we have obtained $\{P(0,T_n)\}_{n=1}^{m-1}$. Consider T_m-maurity swap:

$$S_m \Delta_m P(0,T_m) + S_m \sum_{n=1}^{m-1} \Delta_n P(0,T_n) = 1 - P(0,T_m) \tag{1.61}$$

$$\Rightarrow (*) \quad P(0,T_m) = \frac{1 - S_m\{\sum_{n=1}^{m-1} \Delta_n P(0,T_n)\}}{1 + \Delta_m S_m} \tag{1.62}$$

using

$$(*) \quad S_m = S(0;T_0,T_m) = \frac{\sum_{n=1}^{m} \Delta_n P(0,T_n)}{1 - P(0,T_m)} \tag{1.63}$$

yield curve ... to be interpolated

1.3.6 Market-to-Market fo a forward swap

Suppose has swap starting T_M with maturity T_N as a receiver (long bond party) with the fixed rate X, notional L. Suppose the current ($t = 0$) market quotes is given by $S(0, T_M, T_N)$.

$$\begin{aligned} PV(t=0) &= LX \sum_{n=M+1}^{N} \Delta_n P(0,T_n) - L \sum_{n=M+1}^{N} P(0,T_n)\delta_n P(0,T_n) F(0,T_{n-1},T_n) \\ &= L \sum_{n=M+1}^{N} \Delta_n P(0,T_n)(X - S(0,T_M,T_N)) \end{aligned} \tag{1.64}$$

bond の receiver は金利が下がったら嬉しい

1.3.7 Approximation of a fwd swap rate

$$\begin{aligned} S(0,T_M,T_N) &= \frac{P(0,T_M) - P(0,T_N)}{\sum_{i=M+1}^{N} \Delta_i P(0,T_i)} = \frac{\sum_{i=M+1}^{N} \delta_i F(0,T_{i-1},T_i) P(0,T_i)}{\sum_{i=M+1}^{N} \Delta_i P(0,T_i)} \\ &\approx \frac{1}{N-M} \sum_{i=M+1}^{N} F(0,T_{i-1},F_i) \quad (as\, \delta_i \approx \Delta_i,\ P(0,T_i) = 1) \end{aligned} \tag{1.65}$$

$0 < T_M < T_N$:

$$NS(0;T_0,T_N) \approx MS(0;T_0,T_M) + (N-M)S(0;T_M,T_N) \tag{1.66}$$

$NS(0;T_0,T_N) \approx \{[T_0,T_N]$ の fwd rate の sum$\}$

$$\begin{aligned} (\therefore) S(0;T_M,T_N) &\approx \frac{NS(0;T_0,T_M) - MS(0;T_0,T_M)}{N-M} \\ &\approx \frac{T_N}{T_N - T_M} S(0;T_0,T_N) - \frac{T_M}{T_N - T_M} S(0;T_0,T_M) \end{aligned} \tag{1.67}$$

1.3.8 Deltas

* market quotes (input) spot-swap rates $(S_n)_{n=1}^{N} \Rightarrow P(0,T) \Rightarrow$ pricing...

Delas(PVO 1s)

P.V. of receive swap $[T_M, T_N]$. Notional: L, fixed rate: X

$$P.V.(t=0) = L \sum_{i=M+1}^{N} \Delta_i P(0,T_i)(X - S(0;T_M,T_N)) \tag{1.68}$$

Suppose the market change induces

$$S(0;T_M,T_N) \;\rightarrow\; S(0;T_M,T_N) + \delta S \tag{1.69}$$

then, change of the P.V. :

$$\begin{aligned}\delta P.V. =& L \sum_{i=M+1}^{N} \Delta_i(\delta P(0,T_i))(X - S(0;T_M,T_N)) \\ &+ L \sum_{i=M+1}^{N} \Delta_i P(0,T_i)(-\delta S_M, N) + \text{higher order}\end{aligned} \tag{1.70}$$

1st term order $\sim 1R^2$

2nd term order $\sim 1R^1$

$||$1st term$||$ << $||$2nd term$||$

$$\delta P.V. \approx L \sum_{i=M+1}^{N} \Delta_i P(0,T_i)(-\delta S_M, N) \tag{1.71}$$

* $(-\delta S_M, N)$: fwd swap rate の変化

$$S(0;T_M,T_N) \approx \frac{T_N}{T_N - T_M} S(0;T_0,T_N) - \frac{T_M}{T_N - T_M} S(0;T_0,T_M) \tag{1.72}$$

$$\delta S_{M,N} \approx \frac{T_N}{T_N - T_M} \delta S_N - \frac{T_M}{T_N - T_M} \delta S_M \tag{1.73}$$

- δS_N : change of $S(0,T_0,T_N)$
- δS_M : change of $S(0,T_0,T_M)$

$$
\begin{aligned}
&\delta P.V.(\text{fwd swap}\,(T_M, T_N)) \\
&\approx -\,L \sum_{i=M+1}^{N} \Delta_i P(0,T_i) \left\{ \frac{T_N}{T_N - T_M}\delta S_N - \frac{T_M}{T_N - T_M}\delta S_M \right\} \\
&\approx -\,L(T_N - T_M) \times \frac{1}{T_N - T_M}(T_N \delta S_N - T_M \delta S_M) \quad (\text{approx.}\, P(0,T_i) \approx 1) \\
&= -\,L(T_N \delta S_N - T_M \delta S_M)
\end{aligned}
\tag{1.74}
$$

(* day-count convention のズレは無視)

- spot-start swap
- maturity T_N
- Notional L

receiver:

$$
\delta P.V. = -L \times T_N \times \delta S_N \tag{1.75}
$$

2 Binomial Model

多期間モデルは連続モデルへの直観を与える．PDE の数値計算との関連もあり．

2.1 One-Period Binomial Model

2.1.1 Model Description

There are two points in time $t = 0, T$. Two tradable assets:

- Bond (risk-free asset)

$$B_0 = 1 \tag{2.1}$$
$$B_T = e^{rT} \tag{2.2}$$

  (deterministic)

  r: zero rate for $[0, T]$ at $T = 0$

- Stock (risky asset)

$$S_0 = s\,(> 0) \tag{2.3}$$
$$S_T = \begin{cases} su & \text{with prob. } P_u \\ sd & \text{with prob. } P_d \end{cases} \tag{2.4}$$
$$P_u > 0,\ P_d > 0,\ P_u + P_d = 1,\ 0 < d < u \tag{2.5}$$

  We write for simplicity:

$$S_T = sZ \tag{2.6}$$
$$Z = \begin{cases} u & \text{with prob. } P_u \\ d & \text{with prob. } P_d \end{cases} \tag{2.7}$$

  under emprical measure $\mathbb{P}$, $\mathbb{P}(\{up\}) = P_u$, $\mathbb{P}(\{down\}) = P_d$

Portfolio : $h(x, y) \quad x, y \in \mathbb{R}$

- x: number of position for the bond
- y: number of position for the stock

the value process of the portfolio h:

$$V_t^h = xB_t + yS_t \quad \text{in general} \tag{2.8}$$
$$V_0^h = x + ys \tag{2.9}$$
$$V_T^h = xe^{rT} + ysZ \tag{2.10}$$

Def. 2.1.1. An arbitrage portfolio us a portfolio with the properties:

$$V_0^h = 0,\ P(V_T^h \geq 0) = 1,\ P(V_T^h > 0) > 0 \tag{2.11}$$

* no-arbitrage $\Leftrightarrow$ existence of risk neutral measure (we'll see later)

Prop. 2.1.1. The one-period binomial model is arbitrage free iff (=if and only if) the following condition holds:

$$0 < d < e^{rT} < u \tag{2.12}$$

Prf. 2.1.1. (proof of above prop.)

- (necessarity = only if): Suppose 2.12 does not hold.
  1. $e^{rT} \leq d < u$
     (a) Sell the bond s units
     (b) Buy one unit of the stock
     $h(x, y) = (-s, 1)$, net cash flow $= 0$ at $t = 0$. Then,
     $$V_T^h = -se^{rT} + sZ = s(-e^{rT} + Z) \tag{2.13}$$
     It's clear that $V_T^h \geq 0$, $P(V_T^h \geq 0) = 1$.
     $$P(V_T^h > 0) = P(Z = u) = P_u > 0 \tag{2.14}$$
     $\rightarrow$ arbitrage.
  2. $d < u \leq e^{rT}$

     (a) Sell one unit of the s stock
     (b) Buy the bond s units
     $h(x, y) = (s, -1)$, net cash flow $= 0$ at $t = 0$. Then,
     $$V_T^h = se^{rT} - sZ = s(e^{rT} - Z) \tag{2.15}$$
     It's clear that $V_T^h \geq 0$, $P(V_T \geq 0) = 1$.
     $$P(V_T^h > 0) = P(Z = d) = P_d > 0 \tag{2.16}$$
     $\rightarrow$ arbitrage.
- (sufficiency = if): Suppose 2.12 holds.
  Assume $V_0^h = 0$ then $x + ys = 0 \Leftrightarrow ys = -x$.
  $$V_T^h = xe^{rT} + ysZ = x(e^{rT} - Z) \tag{2.17}$$
  $$P(V_T^h \geq 0) < 1 \tag{2.18}$$

  $\rightarrow$ no-arbitrage.

2.1.2 Risk-neutral Probability Measure

Suppose $d < e^{rT} < u$ holds,
$\Rightarrow$ one can find q_u, q_d > 0 s.t.

$$\begin{cases} q_u + q_d = 1 \\ q_d u + q_d d = e^{rT} \end{cases} \tag{2.19}$$

$$\Rightarrow \begin{cases} q_u = \dfrac{e^{rT} - d}{u - d} \quad (> 0) \\ q_d = \dfrac{u - e^{rT}}{u - d} \quad (> 0) \end{cases} \tag{2.20}$$

We define a new (and not emprical) probability measure $\mathbb{Q}$ such that

$$\mathbb{Q}(Z = u) = q_u \tag{2.21}$$

$$\mathbb{Q}(Z = d) = q_d \tag{2.22}$$

It's interesting to observe that

$$\begin{aligned} e^{-rT}\mathrm{E}^{\mathbb{Q}}[S_T] &= e^{-rT}(q_u \times su + q_d \times sd) \\ &= se^{-rT}(q_u u + q_d d) = s\,(= S_0) \end{aligned} \tag{2.23}$$

In general,

$$\{\ e^{-rT} S_t\}_{t=0,T} \quad \ldots\, \mathbb{Q}\text{-martingale} \tag{2.24}$$

Def. 2.1.2. (Risk-neutral Measure) The probability measure $\mathbb{Q}$ with associated probability (q_u, q_d) satisfying the condition eq(2.12) is called the risk-neutral measure.

Prop. 2.1.2. The one-period binomial model just explained is arbitrage free iff there exists a risk-neutral measure $\mathbb{Q}$ ($q_u > 0$, $q_d > 0$)

Prf. 2.1.2. arbitrage free $\Leftrightarrow d < e^{rT} < u$ ($\because$ Prop. 2.1.1)

$$\mathbb{Q} = (q_u, q_d) \tag{2.25}$$

$$q_u = \frac{e^{rT} - d}{u - d}, \quad q_d = \frac{u - e^{rT}}{u - d} \tag{2.26}$$

$$q_u + q_d = 1 \tag{2.27}$$

If such $\mathbb{Q}$ exists, then

$$\exists q_u, q_d\ > 0 \quad \text{s.t.} \begin{cases} q_u + q_d = 1 \\ q_u u + q_d d = e^{rT} \end{cases} \tag{2.28}$$

$$\Rightarrow d < e^{rT} < u \tag{2.29}$$

2.1.3 Risk-neutral Pricing

Def. 2.1.3. (Contingent Claim) A contingent claim is any random cash flow X_T at T of the form $X_T = \Phi(S_T)$ with some function Φ.

Ex. 2.1.1. (call option with strike K)

$$\Phi(S_T) = \max(S_T - K, 0) = (S_T - K)^+ \tag{2.30}$$

Def. 2.1.4. (Replicable / Complete) A given contingent claim X is said to be replicatable (or perfectly hedgeable) if there exists a portfolio h such that $V_T^h = X_T$ with probability 1 (under $\mathbb{P}$). In this case, we call h a replicating portfolio of X. If all contingent claim are replicable, we say the market is complete. Otherwise the market is incomplete.

Prop. 2.1.3. Suppose that a contingent claim X is replicable by portfolio h. Then, any price at $t = 0$ of the claim X other than V_0^h will lead to an arbitrage opportunity.

Prf. 2.1.3. Suppose h is given by $h = (x, y)$. Suppose $V_0^h \neq X_0$.

1. $V_0^h < X_0$
    - Short the contingent claim X (one gets X_0)
    - Construct a portfolio $h(x, y) : V_0^h = x + sy$
    - Buy $(X_0 - V_0^h)$ units of bond

    portfolio $h' = (x + X_0 - V_0^h, y)$ + short position of X
    net cash flow at T:

    $$V_T^{h'} - X_T = (X_0 - V_0^h)e^{rT} + V_T^h - X_T = (X_0 - V_0^h)e^{rT} > 0 \tag{2.31}$$

    $\to$ arbitrage.
2. $V_0^h > X_0$
    - Buy the contingent claim X
    - Short the replicating portfolio $h(x, y) : V_0^h = x + sy$
    - Buy $(V_0^h - X_0)$ units of bond

    portfolio $h'' = (V_0^h - X_0 - x, -y)$ + long position of X
    net cash flow at T:

    $$V_T^{h''} + X_T = (V_0^h - X_0)e^{rT} - V_T^h + X_T = (V_0^h - X_0)e^{rT} > 0 \tag{2.32}$$

    $\to$ arbitrage.

Prop. 2.1.4. The binomial model is complete.

Prf. 2.1.4. Consider a general contingent claim X whose payoff at T is $\Phi(S_T)$. $\Phi : \mathbb{R} \to \mathbb{R}$ function.

Ex. 2.1.2. Call option $\Phi(S_T) = \max(S_T - K, 0), \quad K \in \mathbb{R}$
It suffices to construct a strategy $h = (x, y)$ s.t.

$$V_T^h = \begin{cases} \Phi(su) & \text{if } Z = u \\ \Phi(sd) & \text{if } Z = d \end{cases} \tag{2.33}$$

* terminal value が複製元デリバティブと同じになるような portfolio h を探す.

$$\begin{cases} e^{rT}x + suy = \Phi(su) \\ e^{rT}x + sdy = \Phi(sd) \end{cases} \tag{2.34}$$

$$\Leftrightarrow \begin{cases} x = e^{-rT}\dfrac{u\Phi(sd) - d\Phi(su)}{u-d} \\ y = \dfrac{1}{s}\dfrac{\Phi(su) - \Phi(sd)}{u-d} \end{cases} \tag{2.35}$$

Option Pricing (Assume there is no-arbitrage opportunity)
Consider the same contingent claim X with payoff $\Phi(S_T)$
The price X_0 at $t = 0$ of the claim is given by $X_0 = V_0^h$ ($\because$ Prop.2.1.3).

$$\begin{aligned} X_0 = x + sy &= e^{-rT}\left\{\frac{u\Phi(sd) - d\Phi(su)}{u-d} + e^{rT}\frac{\Phi(su) - \Phi(sd)}{u-d}\right\} \\ &= e^{-rT}\left\{\frac{e^{rT} - d}{u-d}\Phi(su) + \frac{u - e^{rT}}{u-d}\Phi(sd)\right\} \end{aligned} \tag{2.36}$$

$$X_0 = e^{-rT}\{q_u\Phi(su) + q_d\Phi(sd)\} = e^{-rT}\mathrm{E}^{\mathbb{Q}}[\Phi(S_T)] \tag{2.37}$$

$(P_u, P_d) \leftrightarrow (q_u, q_d)$

2.2 The Multi-Period Binomial Model

2.2.1 Model Description

$0 = t_0 < t_1 < ... < t_N = T$
time partition:

$$t_i - t_{i-1} = \Delta t = \frac{T}{N} \quad \text{for } \forall i \tag{2.38}$$

each t_i $i = \{0, ..., N\}$, one can trade securities.
Securities:

- Bond (risk-free, non-random)
  Bond price process:

$$B_0 = 1,\ B_n = e^{r\Delta t}B_{n-1} = e^{rn\Delta t} \tag{2.39}$$

  Risk-free rate $r\ (\geq 0)$: constant

- Stock

  Stock price process:

$$S_0 = s,\ S_n = S_{n-1} Z_n \tag{2.40}$$

$$\begin{cases} P(Z_n = u) = P_u \\ P(Z_n = d) = P_d \end{cases} \quad \text{for every } n \tag{2.41}$$

  Assumption:

$$0 < d < u,\ P_u > 0,\ P_d > 0,\ P_u + P_d = 1 \tag{2.42}$$

  2^N : number of all securities up to t_N

Def. 2.2.1. A portfolio strategy is defined as a process

$$h_{t_i} = \begin{pmatrix} x_{t_i} \\ y_{t_i} \end{pmatrix}, \quad i = \{0, ..., N\} \tag{2.43}$$

h_{t_i} is the position fot (Bond, Stock) for the period $[t_i, t_{i+1})$, newly taken at t_i and kept unchanged until t_{i+1}. h_{t_i} can be dependent only on $(S_0, ..., S_{t_{i-1}}$.

Portfolio value at t_i:

$$V^h_{t_i} = h^T_{t_i} \begin{pmatrix} B_{t_i} \\ S_{t_i} \end{pmatrix} = x_{t_i} B_{t_i} + y_{t_i} S_{t_i} \tag{2.44}$$

(* T : transposition)

For simplicity, we sometimes write:

$$V^h_i = x_i B_i + y_i S_i \tag{2.45}$$

Def. 2.2.2. (Self-Financing) A portfolio strategy h is said to be self-financing if the following condition holds for every time step i:

$$h^T_i \begin{pmatrix} B_i \\ S_i \end{pmatrix} = h^T_{i-1} \begin{pmatrix} B_i \\ S_i \end{pmatrix} \tag{2.46}$$

* $t = t_i$ での portfolio 組み換え

If the strategy h is self-financing,

$$\begin{aligned} \Delta V^h_i := V^h_i - V^h_{i-1} &= (x_{i-1} B_i + y_{i-1} S_i) - (x_{i-1} B_{i-1} + y_{i-1} S_{i-1}) \\ &= x_{i-1}(B_i - B_{i-1}) + y_{i-1}(S_i - S_{i-1}) \\ &= x_{i-1} \Delta B_i + y_{i-1} \Delta S_i \end{aligned} \tag{2.47}$$

2.2.2 Risk-neutral Pricing

Asm. 2.2.1. We assume $d < e^{r\Delta t} < u$

Def. 2.2.3. The risk-neutral probability measure $\mathbb{Q}$ (associated with (q_u, q_d)) is defined as a probability measure satisfying

$$S_i = e^{-r\Delta t}\mathrm{E}^{\mathbb{Q}}[S_{i+1}|S_i] \quad \text{for every } i = \{0, ..., N-1\} \tag{2.48}$$

The existence and uniqueness can be shown as follows:

$$\begin{cases} q_u + q_d = 1 \\ uq_u + dq_d = e^{r\Delta t} \end{cases} \tag{2.49}$$

$$\Leftrightarrow \begin{cases} q_u = \dfrac{e^{r\Delta t} - d}{u - d} \quad (> 0) \\ q_d = \dfrac{u - e^{r\Delta t}}{u - d} \quad (> 0) \end{cases} \qquad \text{by Asm.2.2.1} \tag{2.50}$$

Prop. 2.2.1. Suppose that a contingent claim X is replicable by the self-financing portfolio h. Then the price of the claim X at any time t_i $0 \leq i \leq N$ must be equal to $V^h_{t_i}$, otherwise there exists an arbitrage opportunity.

Goal:

Thm. 2.2.1. The multi-period binomial model satisfying assumption 2.2.1 is complete, i.e. cash-flow of any contingent claim can be perfectly replicated.
(by self-financinf portfolio strategy $(h_{t_i})_{i\geq 0}$)

**************以降 未完**************

Preparetion:

- European Option
- expiry: $t_N = T$
- option holder receives (or pay if negative) $\Phi(S_N)\,(= \Phi(S_T))$
- $\Phi : \mathbb{R} \to \mathbb{R}$

state at t_n ... $(n+1)$ 個
number of up movement : $(0, ..., n)$
We label it by (n, k), $k \in \{0, ..., n\}$, $S_n(k) = su^k d^{n-k}$
Construction of self-financing strategy at t_N. Replicating portfolio must satisfy:

$$V_N(k) = \Phi(S_N(k)) = \Phi(su^k d^{n-k}) \quad \text{for every } k = 0, ..., N \tag{2.51}$$

at $t_N - 1$

3 Itô Formula and Option Valuation

3.1 Probability Space