

Advanced Derivative

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1 Bond and Forward Contract

1.1 Interest Rates

Asm. 1.1.1. We assume that there exists a default free money market account

- default-free
- liquid (borrowing rate = lending rate)
- everyone can access equally

The interest rates associating with the default free m.m. (money market) are called risk-free rates.

1.1.1 Types of Accrual (利息)

Suppose one invests cash amount A at $t = 0$ for T years.

V : The amount of cash to be returned in T years time.

1. Annual compounding with interest rate R_1 per annum

$$V = A(1 + R_1)^T$$

2. Semiannual compounding with interest rate R_2 per annum

$$V = A(1 + \frac{R_2}{2})^{2T}$$

3. m-times compounding with interest rate R_m per annum

$$V = A(1 + \frac{R_m}{m})^{mT} \quad (\text{typically we have } m = 1, 2, 4, 12, 52, 365)$$

4. continuous compounding with interest rate r per annum

$$V = \lim_{m \rightarrow \infty} A(1 + \frac{r}{m})^{mT} = Ae^{rT}$$

Relation among different compounding conventions:

No-Arbitrage \rightarrow for any m ,

$$e^{rT} = \left(1 + \frac{R_m}{m}\right)^{mT} \quad (1.1)$$

$$\Leftrightarrow r = m \ln \left(1 + \frac{R_m}{m}\right), \quad R_m = m(e^{r/m} - 1). \quad (1.2)$$

at a given time, $R_1(T) : T$ dependent.

1.1.2 Zero Rate and Bonds

Def. 1.1.1. (zero rate) The T-year zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for T-years without any intermediate coupon payment.

Ex. 1.1.1. 5-year zero rate = 5 % per annum. (continuous compounding)

\$ 100 ￼ deposit at $t = 0 \rightarrow$ (5 years later) $100e^{0.05 \times 5} \approx 128.40$

Present value (PV) $P.V. = 128.40 \times e^{-0.05 \times 5} = 100$

Def. 1.1.2. (ZCB : zero coupon bond) T-year zero coupon bond is a bond which pays an unit amount of cash in T-year time without any coupon payment (assume risk-free, liquid).

If R is T-year zero rate, current price of T-year zero coupon Bond is given by :

$$P(0, T) = e^{-RT} \quad (1.3)$$

(0: current time, T : maturity), we call "discount factor".

Ex. 1.1.2. Suppose we have (at $t = 0$)

- $T = 0.5$, $R = 5.0\%$ (zero coupon rate)
- $T = 1.0$, $R = 5.8\%$
- $T = 1.5$, $R = 6.4\%$
- $T = 2.0$, $R = 6.8\%$

fixed coupon bond

- maturity: $T = 2$
- coupon payments: 6% per annum semiannually.
- principal: \$100

$$\begin{aligned} \text{Bond price} &= 3P(0, 0.5) + 3P(0, 1.0) + 3P(0, 1.5) + 103P(0, 2) \\ &= 3 \times e^{-5\% \times 0.5} + 3 \times e^{-5.8\% \times 1.0} + 3 \times e^{-6.4\% \times 1.5} + 103 \times e^{-6.8\% \times 2.0} \\ &\approx 98.39 \end{aligned} \quad (1.4)$$

1.1.3 Yield (平均利率, 平均利回り, 平均 discount rate)

Def. 1.1.3. (Bond's Yield) A bond's yield is the single discount rate that when applied to every cash flow, gives the market bond price.

Ex. 1.1.3. using Ex. 1.1.2, suppose market price = 98.39

Then the yield for the bond y is given by solving

$$3e^{-0.5y} + 3e^{-1.0y} + 3e^{-1.5y} + 103e^{-2.0y} = 98.39 \quad (1.5)$$

$$\Rightarrow y \simeq 6.76\% \quad (\text{continuous compounding}) \quad (1.6)$$

Def. 1.1.4. (Par Yield) The par yield for a certain maturity is the coupon rate that makes the bond price equal to its principal.

Ex. 1.1.4. using Ex. 1.1.2, par yield C for 2-year coupon bond is given by:

$$\frac{C}{2}e^{-0.050 \cdot 0.5} + \frac{C}{2}e^{-0.058 \cdot 1.0} + \frac{C}{2}e^{-0.064 \cdot 1.5} + \left(100 + \frac{C}{2}\right)e^{-0.068 \cdot 2.0} = 100 \quad (1.7)$$

$$\Rightarrow C \simeq 6.87\% \quad (1.8)$$

(par yield for $T = 2$, semiannual coupon payments)

In general,

- T-year bond
- m-time coupon payment per annum
- par yield C

$$\sum_{n=1}^{mT} \frac{C}{m} P\left(0, \frac{n}{m}\right) + 100P(0, T) = 100 \quad (1.9)$$

$$\Rightarrow C = \frac{100(1 - P(0, T))}{A}, \quad A = \sum_{n=1}^{mT} \frac{1}{m} P\left(0, \frac{n}{m}\right) \quad (1.10)$$

1.1.4 Duration

fixed coupon bond:

- (cash flow at T_i) = C_i ($i = 1, \dots, n$)
- C_i : coupon (+ principal at maturity)

Suppose its yield is given by y (continuous compounding).

Bond price:

$$B = \sum_{i=1}^n C_i e^{-yT_i} \quad (1.11)$$

The duration of the Bond:

$$D := -\frac{1}{B} \frac{dB}{dy} = -\left(\frac{dB/dy}{B}\right) = \frac{1}{B} \sum_{i=1}^n C_i T_i e^{-yT_i} \quad (1.12)$$

Ex. 1.1.5. zero coupon Bond

$$(C_i = 0)_{i=1,2,\dots,n-1}, C_n = 1, \quad B = e^{-yT_n} \quad (1.13)$$

$$\Rightarrow D = \frac{1}{B} C_n T_n e^{-yT_n} = T_n \quad (1.14)$$

(* Duration の長短により, 金利に対する反応度の違いがわかる.)

Suppose the yield changes small amount Δy , ($y \rightarrow y + \Delta y$, $B \rightarrow B + \Delta B$)

$$\frac{\Delta B}{B} = -D\Delta y + o(\Delta y) \quad (1.15)$$

(abbr.)

Suppose $D = 10$ (10 year), yield: $\Delta y = +0.1\%$ (10 basis points, b.p.):

$$\frac{\Delta B}{B} \approx -10 \times 0.1\% = -1\% = -0.01 \quad (1.16)$$

1.1.5 Modified Duration

yield (m-time compounding) \hat{y}

the same bond:

$$B = \sum_{i=1}^n C_i \left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \quad (1.17)$$

modified duration:

$$\begin{aligned} D^* &:= -\frac{1}{B} \frac{dB}{d\hat{y}} = \frac{1}{B} \sum_{i=1}^n \frac{C_i T_i}{1 + \hat{y}/m} \left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \\ &= \frac{1}{B(1 + \hat{y}/m)} \sum_{i=1}^n C_i T_i \left(1 + \frac{\hat{y}}{m}\right)^{-mT_i} \\ &= \frac{1}{B(1 + \hat{y}/m)} \sum_{i=1}^n C_i T_i e^{-yT_i} = \frac{D}{1 + \hat{y}/m} \quad (\text{no arbitrage}) \end{aligned} \quad (1.18)$$

* duration の議論は cash flow が一方向のときのみ使える. Insurance では通用しないので注意.

1.1.6 Convexity

y : yiled (continuous compounding)

$$C := \frac{1}{B} \frac{d^2 B}{dy^2} = \frac{1}{B} \sum_{i=1}^n C_i T_i^2 e^{-yT_i} \quad (1.19)$$

$y \rightarrow y + \Delta y, \quad B \rightarrow B + \Delta B:$

$$\Delta B = \frac{dB}{dy} \Delta y + \frac{1}{2} \frac{d^2 B}{dy^2} (\Delta y)^2 + o(\Delta y^2) \quad (1.20)$$

$$\Rightarrow \frac{\Delta B}{B} = -D \Delta y + \frac{1}{2} C (\Delta y)^2 + o(\Delta y^2) \quad (1.21)$$

(Duration matching : abbr.)

1.2 Forward Contract

1.2.1 Forward Price

Def. 1.2.1. (Forward Contract) A forward contract with maturity T is a bilateral binding promise (agreement) such that at time $t = T (> 0)$, the two parties exchange:

- the cash amount given by the time T realization of a certain index (such as a stock price) with the fixed amount of cash (cash delivery)
- the unit amount of asset (such as a share of an equity) with fixed amount of cash (physical delivery=現物)

Def. 1.2.2. (Forward Price) A forward price F at the current time ($t = 0$) (契約時) of the underlying index X is the amount of cash K that make the present value of the forward contract exchanging X_t and K at T zero. (Forward contract has $P.V. = 0$, with $K = F$.)

* F は契約時に支払う額ではないことに注意 (元手は不要)

* K such that P.V. of the fwd contract = 0

Ex. 1.2.1. Consider a forward contract on a non-dividend paying stock, with mat. T .

$X_T = S_T$ (stock price at T), exchange $F \leftrightarrow S_T$ (at T).

Asm. 1.2.1. Stock market is liquid, zero-coupon bond is liquid.

the forward price at $t = 0$ is given by:

$$F = \frac{S_0}{P(0, T)} = e^{rT} S_0 \quad (1.22)$$

- r : zero-rate for mat T , continuous, compounding
- $P(0, T)$: zero coupon bond price

Prf. 1.2.1. replication strategy

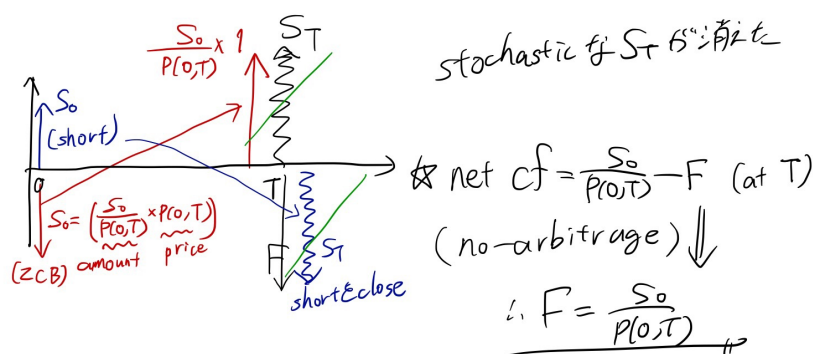
- Enter the fwd contract to get one share of stock (S_T) by paying F at T ($t = 0$ で enter する際に元手は不要)
- Sell one share of stock at $t = 0$ to get S_0 (short position)
- Use S_0 to buy ZCB (zero coupon bond) by the amount $S_0/P(0, T)$

- Pay F and receive S_T at T (return S_T to lender)
- Receive $S_0/P(0, T)$ from ZCB lender

(cash flow illustration below)

if $F \neq \frac{S_0}{P(0, T)} \Rightarrow$ arbitrage (no risk, arbitrary positive return)

No-arbitrage $\Rightarrow F = \frac{S_0}{P(0, T)} = e^{rT} S_0$ (F は stochastic ではない)



Ex. 1.2.2. Same as Ex.1.2.1 but now the stock pays continuous dividend, with dividend rate y ($y \in \mathbb{R}$, constant)

One share of the stock pays $S_t y dt$ for the interval $[t, t + dt]$ for any $t \geq 0$.

forward price at $t = 0$

$$F = \frac{S_0}{P(0, T)} e^{-yT} = S_0 e^{(r-y)T} \quad (1.23)$$

r : zero-rate for mat T at $t = 0$

Suppose we have N_t shares at t , dividend paid in $[t, t + dt]$: $S_t N_t y dt$

\Rightarrow reinvest $\Delta N_t = N_t y dt$

if one reinvests the whole dividend payment,

$$\frac{dN_t}{dt} = N_t y \Rightarrow N_t = N_0 e^{yt} \quad \text{for all } t \geq 0 \quad (1.24)$$

Therefore, if one wants $N_T = 1$, N_0 is to be e^{-yT} .

Prf. 1.2.2. replication strategy

($t = 0$)

- Enter the fwd contract to receive F and deliver one share stock at T (Ex.1.2.1 と逆の party)

- Sell $\frac{S_0 e^{-yT}}{P(0,T)}$ amount of ZCB with maturity T
- Buy e^{-yT} shares of stock

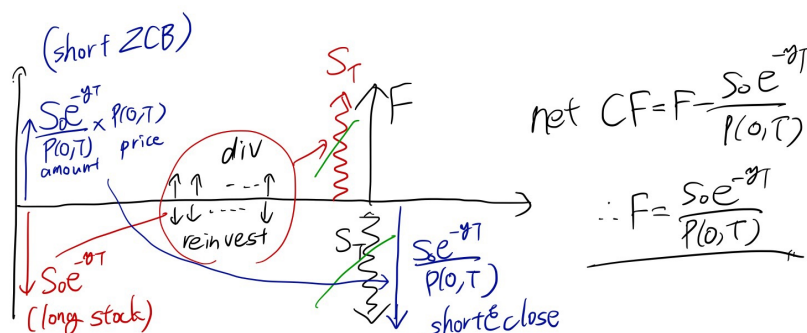
(always)

- Reinvest every div. payment to the stock

($t = T$)

- Receive F , deliver one share of stock
- Return $\frac{S_0 e^{-yT}}{P(0,T)}$ to the ZCB lender

(cash flow illustration below)



イメージ:

$$\text{P.V.}(\text{receive } S_T \text{ at } T) = F \times P(0, T) = \frac{S_0}{P(0, T)} \times P(0, T) = S_0 \quad (1.25)$$

random な cash flow の P.V. を計算するときは, Forward Price を求めて, $P(0, T)$ を乗じてやればよい (ただし市場が liquid, replicatable なときのみ)

⇒ 後に risk-neutral の下で計算すればわざわざ replication を考える必要がなくなる

1.2.2 Mark to Market of a Fwd Contract

* P.V.(at $t = 0$) {fwd contract} は 0 だが, 時間が進むにつれて, P.V. は変化する.

Suppose we entered the fwd contract to receive X_T in exchange for a fixed amount of cash F_0 (F_0 : fwd price at $t = 0$). P.V.($t = 0$) = 0

at time $t \in (0, T)$, suppose fwd price is given by F_t . We want to know P.V.($t > 0$).

Ans.:

$$\text{P.V.}(t) = P(t, T)(F_t - F_0) \quad (1.26)$$

Prf. 1.2.3. enter the new fwd contract at $t = t$ (pays X_T and receives F_t at $t = T$)

* F_t の t は「時刻 t に contract に enter した」という意味.

(abbr.)

$$\begin{aligned} \text{P.V.}(\text{at } t)\{\text{new} + \text{original fwd contracts}\} &= P(t, T)(F_t - F_0) \\ &= 0 + \text{P.V.}(\text{at } t)\{\text{original}\} \end{aligned} \quad (1.27)$$

1.2.3 Put-Call Parity

Def. 1.2.3. (Call Option and Put Option) A call (respectively, put) option on a certain index X with expiry T and strike K is the binomial?? contract to pay the holder of the option the cash amount equal to $\max(X_T - K, 0)$, (resp, $\max(K - X_T, 0)$)

Let C (resp, P) be the call (resp, put) option price (at $t = 0$). We have the following put-call parity:

$$C - P = P(0, T)(F_X - K) \quad (1.28)$$

- F_X : the fwd price of X with mat. T (時刻 $t = X$ ではなく, underlying asset X をもとにした fwd price at $t = 0$)

Prf. 1.2.4. cash flow at T :

$$\max(X_T, K, 0) - \max(K - X_T, 0) = X_T - K \quad (1.29)$$

$$(1.30)$$

Present value of above is given by:

$$C - P = P(0, T)(F_X - K) \quad (1.31)$$

motivation:

- liquidity の問題
- call, put の一方が求まれば, もう一方をすぐに求められる
- PDE の計算は put の方が簡単 (because of boundary condition)

1.3 Forward Rate Agreement and Interest Rate Swap

*金利スワップは not tradable

1.3.1 Simple Rate and Day-Count Convention

Suppose T_i specifies the date $D_i = D(d_i, m_i, y_i)$

1. Actual/365

$$\delta(T_0, T_1) = \frac{D_1 - D_0}{365} \quad (1.32)$$

2. Actual/360

$$\delta(T_0, T_1) = \frac{D_1 - D_0}{360} \quad (1.33)$$

3. 30/360

$$\delta(T_0, T_1) = \frac{\max(30 - d_0, 0) + \min(d_1, 30) + 360(y_1 - y_0) + 30(m_1 - m_0 - 1)}{360} \quad (1.34)$$

4. actual/actual considering leap year? 365 or 366

Def. 1.3.1. (Simple Rate) A (risk-free) simple rate (not compound) $L(T_{i-1}, T_i)$ with day-count $\delta(T_{i-1}, T_i)$ is the interest rate with accrual convention defined in such a way that, when one invest N amount of cash at T_{i-1} , then he receives $N(1 + \delta_i L(T_{i-1}, T_i))$ at time T_i . $L(T_{i-1}, T_i)$ is the zero coupon rate at T_{i-1} for $[T_{i-1}, T_i]$ with corresponding day-count convention.

*accrual: ??

1.3.2 Forward Rate Agreement (FRA)

Def. 1.3.2. (Forward Rate Agreement) A FRA is a binding(義務の) contract with the two parties (lender and borrower) agreeing to let a certain fixed rate K act on a prefixed notional(想定元本) amount N , over a future period $[T_M, T_N]$.

*notional: 想定される?

Def. 1.3.3. (Forward Rate) A forward rate F for the period $[T_M, T_N]$ with day-count $\delta = \delta(T_M, T_N)$ is the fixed rate K with the some day-count convention that makes the present value of the FRA zero.

off course,

$$F = L(T_M, T_N) \quad (\text{at } T_M) \quad (1.35)$$

* L : simple rate

Lem. 1.3.1. Let $\delta = \delta(T_M, T_N)$. Then the forward rate F at $t = 0$ for the period $[T_M, T_N]$ is given by

$$F = \frac{1}{\delta} \left(\frac{P(0, T_M)}{P(0, T_N)} - 1 \right) \quad (1.36)$$

(*asm: liquid, no-arbitrage)

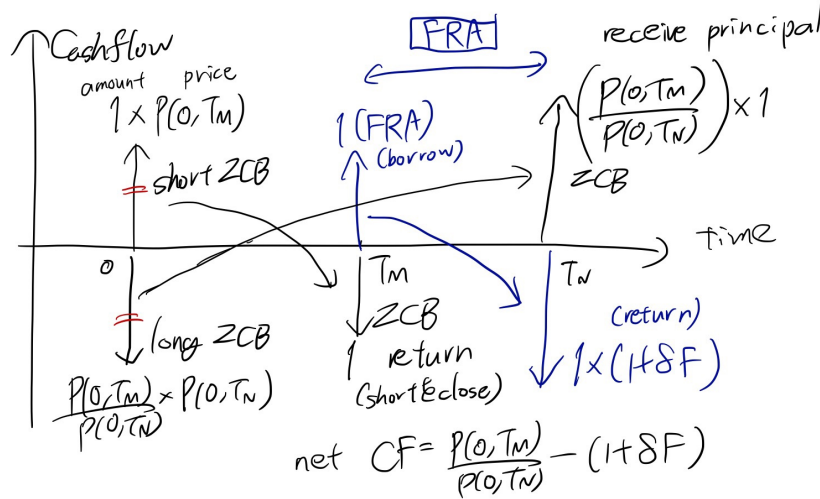
Prf. 1.3.1. replication strategy

- Enter the FRA of rate F to borrow unit amount of cash for $[T_M, T_N]$
- Sell one ZCB with mat. T_N (short)
- Buy ZCB with mat. T_N with principal amount $\frac{P(0, T_M)}{P(0, T_N)}$
- At T_M , borrow unit amount of cash through FRA and use it to return ZCB
- At T_N , receive the principal $\frac{P(0, T_M)}{P(0, T_N)}$, and pays $(1 + \delta F)$

Net cash flow at T_N : $\left(\frac{P(0, T_M)}{P(0, T_N)} \right) - (1 + \delta F)$

If we require no-arbitrage,

$$\left(\frac{P(0, T_M)}{P(0, T_N)} \right) - (1 + \delta F) = 0 \quad \Rightarrow \quad F = \frac{1}{\delta} \left(\frac{P(0, T_M)}{P(0, T_N)} - 1 \right) \quad (1.37)$$



We write the above F as

$$F(0, T_M, T_N) = \frac{1}{\delta} \left(\frac{P(0, T_M)}{P(0, T_N)} - 1 \right) \quad (1.38)$$

In general, at $t < T_M$,

$$F(t, T_M, T_N) = \frac{1}{\delta} \left(\frac{P(t, T_M)}{P(t, T_N)} - 1 \right) \quad (1.39)$$

$t \uparrow T_M$:

$$F(T_M, T_M, T_N) = \frac{1}{\delta} \left(\frac{1}{P(T_M, T_N)} - 1 \right) = L(T_M, T_N) \quad (1.40)$$

(\because)

$$1 = P(T_M, T_N)\{1 + \delta L(T_M, T_N)\} \quad (1.41)$$

at T_M invest 1, at T_N return $1 + \delta L(T_M, T_N)$
 (otherwise there exist arbitrage opportunities.)
 (abbr.)

Ex. 1.3.1. Q) P.V.(at 0) $\{\text{receive } 1 + \delta L(T_M, T_N)\}$? \rightarrow A) P.V.=0

(\because) Suppose that we are at $t = T_M$, $L(T_M, T_N)$... known

$$\text{P.V.}(\text{at } T_M) = -1 + P(T_M, T_N)(1 + \delta L(T_M, T_N)) = 0 \quad (1.42)$$

将来のある時点 ($t = T_M$) で P.V.= 0 なら, さかのぼった $t = 0$ でも当然 P.V.= 0

$$P.V.(\text{at } 0)\{\text{receive } 1 + \delta F(0, T_M, T_N) \text{ at } T_N\} = P.V.(\text{at } 0)\{\text{receive } 1 + \delta L(T_M, T_N)\} \quad (1.43)$$

$$(\because) P(0, T_N)F(0, T_M, T_N) = P.V.(\text{at } 0)\{\text{receive } L(T_M, T_N)\} \quad (1.44)$$

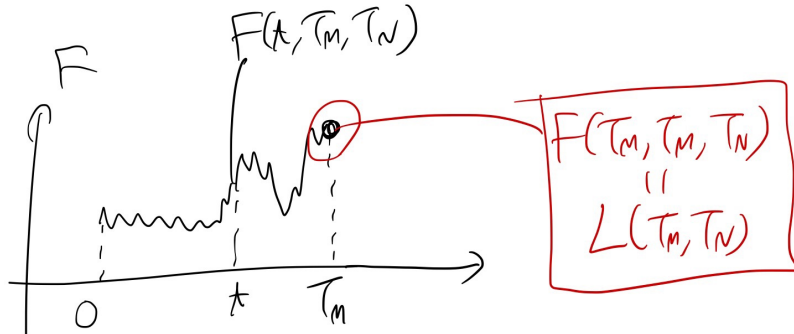
1.3.3 Fixed vs Floating Interest Swap

Fix a time partition $0 = T_0 < T_1 < \dots < T_M$

Def. 1.3.4. (Spot-start Swap) A spot-start swap with maturity T_M and notional amount N is the contract in which one party (receiver) receives cash amount $NK\Delta_i$ (K fixed) and pays the stochastic amount $NL(T_{i-1}, T_i)\delta_i$ at every T_i , $i = \{1, 2, \dots, M\}$. The other party (payer) has the opposite cash flow. Here,

$$\begin{cases} \Delta_i &= \Delta(T_{i-1}, T_i) \quad (\text{fixed}) \\ \delta_i &= \delta(T_{i-1}, T_i) \quad (\text{floating}) \end{cases} \quad (1.45)$$

are the day counts for a fixed and floating payments, respectively.



Def. 1.3.5. (Swap Rate) The (spot) swap rate for the maturity T_M is the fixed rate K that makes the present value of the swap zero.

P.V. of the swap rate:

$$PV_{fix} = NK \sum_{i=1}^M P(0, T_i) \Delta_i \quad (1.46)$$

$$\begin{aligned} PV_{float} &= N \sum_{i=1}^M P(0, T_i) F(0, T_{i-1}, T_i) \delta_i = N \sum_{i=1}^M P(0, T_i) \left(\frac{P(0, T_{i-1})}{P(0, T_i)} - 1 \right) \\ &= N \sum_{i=1}^M (P(0, T_{i-1}) - P(0, T_i)) = N(1 - P(0, T_M)) \end{aligned} \quad (1.47)$$

Swap rate K :

$$K = \frac{1 - P(0, T_M)}{\sum_{i=1}^M P(0, T_i) \Delta_i} := S(0; T_0, T_M) \quad (1.48)$$

* Economic meaning of swap rate:

$$S(0; T_0, T_M) = \frac{\sum_{i=1}^M P(0, T_i) F(0, T_{i-1}, T_i) \delta_i}{\sum_{i=1}^M \Delta_i P(0, T_i)} \quad (1.49)$$

Let us approximate as

$$P(0, T_i) \approx 1, \delta_i \approx \Delta_i \quad \text{for all } i \quad (1.50)$$

$$\Rightarrow S(0; T_0, T_M) \approx \frac{\sum_{i=1}^M F(0, T_{i-1}, T_i)}{M} \quad (1.51)$$

...average of fwd rates!

Def. 1.3.6. (Forward Swap) A forward swap is the swap which starts at some future time. Fixed rate (fixed at $t = 0$) which make the P.V. of the swap is called the forward swap rate.

Ex. 1.3.2. A forward swap for the period $[T_M, T_N]$

\Rightarrow cash flow exchanges at T_i , $i = \{M + 1, \dots, N\}$

$NK\Delta_i \leftrightarrow NL(T_{i-1}, T_i)\delta_i$

Let fixed rate be K , notional = 1

$$PV_{fix} = NK \sum_{i=M+1}^N P(0, T_i) \Delta_i \quad (1.52)$$

$$PV_{float} = N \sum_{i=M+1}^N P(0, T_i) F(0, T_{i-1}, T_i) \delta_i = P(0, T_M) - P(0, T_N) \quad (1.53)$$

Fwd Swap Rate:

$$S(0; T_M, T_N) = \frac{P(0, T_M) - P(0, T_N)}{\sum_{i=M+1}^N P(0, T_i) \Delta_i} \quad (1.54)$$

1.3.4 Relation to the Fixed Coupon Bond

Consider the spot-start swap for $[T_0 = 0, T_N]$ (notional= 0)

$$PV_{float} = \sum_{i=1}^N \delta_i P(0, T_i) F(0, T_{i-1}, T_i) = 1 - P(0, T_N) \quad (1.55)$$

(abbr.)

Bond-Swap, Fixed vs Floating swap, ???

defined $i(t) : \text{index, } i \in \{0, \dots, N\} \text{ s.t. } t \in [T_i, T_{i+1}) \quad T_{i(t)} \leq t < T_{i(t)+1}$

current time t

P.V.(floating leg?+ final principal)

$$\begin{aligned} &= P(t, T_{i(t)+1}) \delta_{i(t)+1} L(T_{i(t)}, T_{i(t)+1}) + \sum_{j=i(t)+2}^N P(t, T_j) \delta_j F(t, T_{j-1}, T_j) + P(t, T_N) \\ &= P(t, T_{i(t)+1}) \delta_{i(t)+1} L(T_{i(t)}, T_{i(t)+1}) + \sum_{j=i(t)+2}^N P(t, T_j) (1 + \delta_{i(t)+1} L(T_i, T_{i+1})) + P(t, T_N) \\ &= P(t, T_{i(t)+1}) (1 + \delta_{i(t)+1} L(T_i, T_{i+1})) \approx 1 \end{aligned} \quad (1.56)$$

Thus, floating leg + final principal \approx IR-RISK 0

IR-Swap Risk \approx fixed leg + final principal payment

\Leftrightarrow fixed coupon Bond

1.3.5 Yield Curve Construction (Simplified...)

Asm. 1.3.1. There are market quotes of spot-starting swaps with swap rate $\{S_n\}_{n=1}^N$ with corresponding maturities $\{T_n\}_{n=1}^N$

$$S_n : S(0; T_0, T_n), \quad T_0 = 0 \quad (1.57)$$

Ex. 1.3.3. 3 month. $0 = T_0 < T_1 < \dots < T_N$

We want to get $\{P(0, T_n)\}_{n=1}^N$ which are consistent with the swap quotes.

1) Determin $P(0, T_1)$

$$S_1 \Delta_1 P(0, T_1) = P(0, T_0) - P(0, T_1) \quad (1.58)$$

$$S_1 = \frac{1 - P(0, T_1)}{\Delta_1 P(0, T_1)} \quad (1.59)$$

$$P(0, T_1) = \frac{P(0, T_0)}{1 + \Delta_1 S_1} = \frac{1}{1 + \Delta_1 S_1} \quad (1.60)$$

2) Suppose we have obtained $\{P(0, T_n)\}_{n=1}^{m-1}$. Consider T_m -maturity swap:

$$S_m \Delta_m P(0, T_m) + S_m \sum_{n=1}^{m-1} \Delta_n P(0, T_n) = 1 - P(0, T_m) \quad (1.61)$$

$$\Rightarrow (*) \quad P(0, T_m) = \frac{1 - S_m \{\sum_{n=1}^{m-1} \Delta_n P(0, T_n)\}}{1 + \Delta_m S_m} \quad (1.62)$$

using

$$(*) \quad S_m = S(0; T_0, T_m) = \frac{\sum_{n=1}^m \Delta_n P(0, T_n)}{1 - P(0, T_m)} \quad (1.63)$$

yield curve ... to be interpolated

1.3.6 Mark-to-Market of a Forward Swap

Suppose has swap starting T_M with maturity T_N as a receiver (long bond party) with the fixed rate X , notional L . Suppose the current ($t = 0$) market quotes is given by $S(0, T_M, T_N)$.

*notional L は simple rate の $L(T_{i-1}, T_i)$ とは別物.

$$\begin{aligned} PV(t=0) &= LX \sum_{n=M+1}^N \Delta_n P(0, T_n) - L \sum_{n=M+1}^N P(0, T_n) \delta_n F(0, T_{n-1}, T_n) \\ &= L \left(\sum_{n=M+1}^N \Delta_n \right) P(0, T_N) (X - S(0, T_M, T_N)) \end{aligned} \quad (1.64)$$

bond の receiver は金利が下がったら嬉しい

1.3.7 Approximation of a Fwd Swap Rate

$$\begin{aligned} S(0, T_M, T_N) &= \frac{P(0, T_M) - P(0, T_N)}{\sum_{i=M+1}^N \Delta_i P(0, T_i)} = \frac{\sum_{i=M+1}^N \delta_i F(0, T_{i-1}, T_i) P(0, T_i)}{\sum_{i=M+1}^N \Delta_i P(0, T_i)} \\ &\approx \frac{1}{N - M} \sum_{i=M+1}^N F(0, T_{i-1}, T_i) \quad (\text{as } \delta_i \approx \Delta_i, P(0, T_i) = 1) \end{aligned} \quad (1.65)$$

... Swap rate is the average of fwd rates.

$$\begin{aligned} \sum_{i=M+1}^N F(0, T_{i-1}, T_i) &= \sum_{i=1}^N F(0, T_{i-1}, T_i) - \sum_{i=1}^M F(0, T_{i-1}, T_i) \\ &\approx NS(0; T_0, T_N) - MS(0; T_0, T_M) \quad (\because 1.50) \end{aligned} \quad (1.66)$$

$$0 < T_M < T_N:$$

$$NS(0; T_0, T_N) \approx MS(0; T_0, T_M) + (N - M)S(0; T_M, T_N) \quad (1.67)$$

$$NS(0; T_0, T_N) \approx \{[T_0, T_N] \text{ の fwd rate の sum}\}$$

$$\begin{aligned} \therefore S(0; T_M, T_N) &\approx \frac{NS(0; T_0, T_N) - MS(0; T_0, T_M)}{N - M} \\ &\approx \frac{T_N}{T_N - T_M} S(0; T_0, T_N) - \frac{T_M}{T_N - T_M} S(0; T_0, T_M) \end{aligned} \quad (1.68)$$

1.3.8 Delas (PV01s)

* market quotes (input) spot-swap rates $(S_n)_{n=1}^N \Rightarrow P(0, T) \Rightarrow \text{pricing...}$

Def. 1.3.7. (Deltas (PV01s)) Deltas are the price sensitivities of a trade (or portfolio) against the change of each swap rate S_n $n = 1, \dots, N$

P.V. of portfolio $\{S_n + 1b.p., (S_m)_{m \neq n}^N : \text{unchanged}\}$ – P.V. of portfolio $\{S_n, (S_n)_{n=1}^N\}$

ex. Deltas for a swap

Consider a fwd swap for $[T_M, T_N]$. Notional: L , fixed rate: X

P.V. of receiver swap $[T_M, T_N]$:

$$P.V.(t=0) = L \sum_{i=M+1}^N \Delta_i P(0, T_i) (X - S(0; T_M, T_N)) \quad (1.69)$$

Suppose the market change induces

$$S(0; T_M, T_N) \rightarrow S(0; T_M, T_N) + \delta S \quad (1.70)$$

then, change of the P.V. :

$$\begin{aligned} \delta P.V. &= L \sum_{i=M+1}^N \Delta_i (\delta P(0, T_i)) (X - S(0; T_M, T_N)) \\ &\quad + L \sum_{i=M+1}^N \Delta_i P(0, T_i) (-\delta S_{M,N}) + \text{higher order} \end{aligned} \quad (1.71)$$

1st term order $\sim IR^2$ (*IR:interest rate)

2nd term order $\sim IR^1$

$||\text{1st term}|| \ll ||\text{2nd term}||$

$$\delta P.V. \approx L \sum_{i=M+1}^N \Delta_i P(0, T_i) (-\delta S_{M,N}) \quad (1.72)$$

* $(-\delta S_{M,N})$: fwd swap rate の変化

$$S(0; T_M, T_N) \approx \frac{T_N}{T_N - T_M} S(0; T_0, T_N) - \frac{T_M}{T_N - T_M} S(0; T_0, T_M) \quad (1.73)$$

$$\delta S_{M,N} \approx \frac{T_N}{T_N - T_M} \delta S_N - \frac{T_M}{T_N - T_M} \delta S_M \quad (1.74)$$

- δS_N : change of $S(0, T_0, T_N)$
- δS_M : change of $S(0, T_0, T_M)$

$$\begin{aligned} & \delta P.V.(\text{fwd swap}(T_M, T_N)) \\ & \approx -L \sum_{i=M+1}^N \Delta_i P(0, T_i) \left\{ \frac{T_N}{T_N - T_M} \delta S_N - \frac{T_M}{T_N - T_M} \delta S_M \right\} \\ & \approx -L(T_N - T_M) \times \frac{1}{T_N - T_M} (T_N \delta S_N - T_M \delta S_M) \quad (\text{approx. } P(0, T_i) \approx 1) \\ & \approx -L(T_N \delta S_N - T_M \delta S_M) \end{aligned} \quad (1.75)$$

* spot-start swap with mat. T_N, T_M のみで表せる.

* day-count convention のズレは無視.

if $T_M = T_0$, replace $\delta S_M = 0$

spot-start swap, maturity T_N , Notional L , receiver:

$$\begin{aligned} \delta P.V. &= -L \times T_N \times \delta S_N \\ &\quad - (\text{元本}) \times (\text{duration}) \times (\text{yield の変化}) \end{aligned} \quad (1.76)$$

spot-start なら, 自分が enter した年限の swap rate の変化しか関係ない. 10 年 swap に入る限り, 5 年 swap や 20 年 swap の変化は関係ない.

→ ヘッジ戦略に使える.

2 Binomial Model

多期間モデルは連続モデルへの直観を与える．PDE の数値計算との関連もあり．

2.1 One-Period Binomial Model

2.1.1 Model Description

There are two points in time $t = 0, T$. Two tradable assets:

- Bond (risk-free asset)

$$B_0 = 1 \quad (2.1)$$

$$B_T = e^{rT} \quad (2.2)$$

(deterministic)

r : zero rate for $[0, T]$ at $T = 0$

- Stock (risky asset)

$$S_0 = s (> 0) \quad (2.3)$$

$$S_T = \begin{cases} su & \text{with prob. } P_u \\ sd & \text{with prob. } P_d \end{cases} \quad (2.4)$$

$$P_u > 0, P_d > 0, P_u + P_d = 1, 0 < d < u \quad (2.5)$$

We write for simplicity:

$$S_T = sZ \quad (2.6)$$

$$Z = \begin{cases} u & \text{with prob. } P_u \\ d & \text{with prob. } P_d \end{cases} \quad (2.7)$$

under empirical measure \mathbb{P} , $\mathbb{P}(\{up\}) = P_u$, $\mathbb{P}(\{down\}) = P_d$

Portfolio : $h(x, y) \quad x, y \in \mathbb{R}$

- x : number of position for the bond
- y : number of position for the stock

the value process of the portfolio h :

$$V_t^h = xB_t + yS_t \quad \text{in general} \quad (2.8)$$

$$V_0^h = x + ys \quad (2.9)$$

$$V_T^h = xe^{rT} + ysZ \quad (2.10)$$

Def. 2.1.1. An arbitrage portfolio is a portfolio with the properties:

$$V_0^h = 0, P(V_T^h \geq 0) = 1, P(V_T^h > 0) > 0 \quad (2.11)$$

* no-arbitrage \Leftrightarrow existence of risk neutral measure (we'll see later)

Prop. 2.1.1. The one-period binomial model is arbitrage free iff (=if and only if) the following condition holds:

$$0 < d < e^{rT} < u \quad (2.12)$$

Prf. 2.1.1. (proof of above prop.)

- (necessity = only if): Suppose 2.12 does not hold.

1. $e^{rT} \leq d < u$

(a) Sell the bond s units

(b) Buy one unit of the stock

$h(x, y) = (-s, 1)$, net cash flow = 0 at $t = 0$. Then,

$$V_T^h = -se^{rT} + sZ = s(-e^{rT} + Z) \quad (2.13)$$

It's clear that $V_T^h \geq 0, P(V_T^h \geq 0) = 1$.

$$P(V_T^h > 0) = P(Z = u) = P_u > 0 \quad (2.14)$$

\rightarrow arbitrage.

2. $d < u \leq e^{rT}$

(a) Sell one unit of the s stock

(b) Buy the bond s units

$h(x, y) = (s, -1)$, net cash flow = 0 at $t = 0$. Then,

$$V_T^h = se^{rT} - sZ = s(e^{rT} - Z) \quad (2.15)$$

It's clear that $V_T^h \geq 0, P(V_T^h \geq 0) = 1$.

$$P(V_T^h > 0) = P(Z = d) = P_d > 0 \quad (2.16)$$

\rightarrow arbitrage.

- (sufficiency = if): Suppose 2.12 holds.

Assume $V_0^h = 0$ then $x + ys = 0 \Leftrightarrow ys = -x$.

$$V_T^h = xe^{rT} + ysZ = x(e^{rT} - Z) \quad (2.17)$$

$$P(V_T^h \geq 0) < 1 \quad (2.18)$$

\rightarrow no-arbitrage.

2.1.2 Risk-neutral Probability Measure

Suppose $d < e^{rT} < u$ holds,

\Rightarrow one can find $q_u, q_d > 0$ s.t.

$$\begin{cases} q_u + q_d = 1 \\ q_d u + q_u d = e^{rT} \end{cases} \quad (2.19)$$

$$\Rightarrow \begin{cases} q_u = \frac{e^{rT} - d}{u - d} & (> 0) \\ q_d = \frac{u - e^{rT}}{u - d} & (> 0) \end{cases} \quad (2.20)$$

We define a new (and not empirical) probability measure \mathbb{Q} such that

$$\mathbb{Q}(Z = u) = q_u \quad (2.21)$$

$$\mathbb{Q}(Z = d) = q_d \quad (2.22)$$

It's interesting to observe that

$$\begin{aligned} e^{-rT} \mathbb{E}^{\mathbb{Q}}[S_T] &= e^{-rT} (q_u \times su + q_d \times sd) \\ &= s e^{-rT} (q_u u + q_d d) = s (= S_0) \end{aligned} \quad (2.23)$$

In general,

$$\{ e^{-rt} S_t \}_{t=0,T} \dots \mathbb{Q}\text{-martingale} \quad (2.24)$$

Def. 2.1.2. (Risk-neutral Measure) The probability measure \mathbb{Q} with associated probability (q_u, q_d) satisfying the condition eq(2.12) is called the risk-neutral measure.

Prop. 2.1.2. The one-period binomial model just explained is arbitrage free iff there exists a risk-neutral measure \mathbb{Q} ($q_u > 0, q_d > 0$)

Prf. 2.1.2. arbitrage free $\Leftrightarrow d < e^{rT} < u$ (\because Prop. 2.1.1)

$$\mathbb{Q} = (q_u, q_d) \quad (2.25)$$

$$q_u = \frac{e^{rT} - d}{u - d}, \quad q_d = \frac{u - e^{rT}}{u - d} \quad (2.26)$$

$$q_u + q_d = 1 \quad (2.27)$$

If such \mathbb{Q} exists, then

$$\exists q_u, q_d > 0 \quad \text{s.t.} \begin{cases} q_u + q_d = 1 \\ q_u u + q_d d = e^{rT} \end{cases} \quad (2.28)$$

$$\Rightarrow d < e^{rT} < u \quad (2.29)$$

2.1.3 Risk-neutral Pricing

Def. 2.1.3. (Contingent Claim) A contingent claim is any random cash flow X_T at T of the form $X_T = \Phi(S_T)$ with some function Φ .

Ex. 2.1.1. (call option with strike K)

$$\Phi(S_T) = \max(S_T - K, 0) = (S_T - K)^+ \quad (2.30)$$

Def. 2.1.4. (Replicable / Complete) A given contingent claim X is said to be replicable (or perfectly hedgeable) if there exists a portfolio h such that $V_T^h = X_T$ with probability 1 (under \mathbb{P}). In this case, we call h a replicating portfolio of X . If all contingent claim are replicable, we say the market is complete. Otherwise the market is incomplete.

Prop. 2.1.3. Suppose that a contingent claim X is replicable by portfolio h . Then, any price at $t = 0$ of the claim X other than V_0^h will lead to an arbitrage opportunity.

Prf. 2.1.3. Suppose h is given by $h = (x, y)$. Suppose $V_0^h \neq X_0$.

1. $V_0^h < X_0$

- Short the contingent claim X (one gets X_0)
- Construct a portfolio $h(x, y) : V_0^h = x + sy$
- Buy $(X_0 - V_0^h)$ units of bond

portfolio $h' = (x + X_0 - V_0^h, y)$ + short position of X

net cash flow at T :

$$V_T^{h'} - X_T = (X_0 - V_0^h)e^{rT} + V_T^h - X_T = (X_0 - V_0^h)e^{rT} > 0 \quad (2.31)$$

\rightarrow arbitrage.

2. $V_0^h > X_0$

- Buy the contingent claim X
- Short the replicating portfolio $h(x, y) : V_0^h = x + sy$
- Buy $(V_0^h - X_0)$ units of bond

portfolio $h'' = (V_0^h - X_0 - x, -y)$ + long position of X

net cash flow at T :

$$V_T^{h''} + X_T = (V_0^h - X_0)e^{rT} - V_T^h + X_T = (V_0^h - X_0)e^{rT} > 0 \quad (2.32)$$

\rightarrow arbitrage.

Prop. 2.1.4. The binomial model is complete.

Prf. 2.1.4. Consider a general contingent claim X whose payoff at T is $\Phi(S_T)$. $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ function.

Ex. 2.1.2. Call option $\Phi(S_T) = \max(S_T - K, 0)$, $K \in \mathbb{R}$

It suffices to construct a strategy $h = (x, y)$ s.t.

$$V_T^h = \begin{cases} \Phi(su) & \text{if } Z = u \\ \Phi(sd) & \text{if } Z = d \end{cases} \quad (2.33)$$

* terminal value が複製元デリバティブと同じになるような portfolio h を探す.

$$\begin{cases} e^{rT}x + suy = \Phi(su) \\ e^{rT}x + sdy = \Phi(sd) \end{cases} \quad (2.34)$$

$$\Leftrightarrow \begin{cases} x = e^{-rT} \frac{u\Phi(sd) - d\Phi(su)}{u - d} \\ y = \frac{1}{s} \frac{\Phi(su) - \Phi(sd)}{u - d} \end{cases} \quad (2.35)$$

Option Pricing (Assume there is no-arbitrage opportunity)

Consider the same contingent claim X with payoff $\Phi(S_T)$

The price X_0 at $t = 0$ of the claim is given by $X_0 = V_0^h$ (\because Prop.2.1.3).

$$\begin{aligned} X_0 = x + sy &= e^{-rT} \left\{ \frac{u\Phi(sd) - d\Phi(su)}{u - d} + e^{rT} \frac{\Phi(su) - \Phi(sd)}{u - d} \right\} \\ &= e^{-rT} \left\{ \frac{e^{rT} - d}{u - d} \Phi(su) + \frac{u - e^{rT}}{u - d} \Phi(sd) \right\} \end{aligned} \quad (2.36)$$

$$X_0 = e^{-rT} \{q_u \Phi(su) + q_d \Phi(sd)\} = e^{-rT} \mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] \quad (2.37)$$

$$(P_u, P_d) \leftrightarrow (q_u, q_d)$$

2.2 The Multi-Period Binomial Model

2.2.1 Model Description

$$0 = t_0 < t_1 < \dots < t_N = T$$

time partition:

$$t_i - t_{i-1} = \Delta t = \frac{T}{N} \quad \text{for } \forall i \quad (2.38)$$

each t_i $i = \{0, \dots, N\}$, one can trade securities.

Securities:

- Bond (risk-free, non-random)

Bond price process:

$$B_0 = 1, B_n = e^{r\Delta t} B_{n-1} = e^{rn\Delta t} \quad (2.39)$$

Risk-free rate $r (\geq 0)$: constant

- Stock

Stock price process:

$$S_0 = s, S_n = S_{n-1}Z_n \quad (2.40)$$

$$\begin{cases} P(Z_n = u) = P_u \\ P(Z_n = d) = P_d \end{cases} \quad \text{for every } n \quad (2.41)$$

Assumption:

$$0 < d < u, P_u > 0, P_d > 0, P_u + P_d = 1 \quad (2.42)$$

2^N : number of all securities up to t_N

Def. 2.2.1. A portfolio strategy is defined as a process

$$h_{t_i} = \begin{pmatrix} x_{t_i} \\ y_{t_i} \end{pmatrix}, \quad i = \{0, \dots, N\} \quad (2.43)$$

h_{t_i} is the position for (Bond, Stock) for the period $[t_i, t_{i+1})$, newly taken at t_i and kept unchanged until t_{i+1} . h_{t_i} can be dependent only on $(S_0, \dots, S_{t_{i-1}})$.

Portfolio value at t_i :

$$V_{t_i}^h = h_{t_i}^T \begin{pmatrix} B_{t_i} \\ S_{t_i} \end{pmatrix} = x_{t_i} B_{t_i} + y_{t_i} S_{t_i} \quad (2.44)$$

(* T : transposition)

For simplicity, we sometimes write:

$$V_i^h = x_i B_i + y_i S_i \quad (2.45)$$

Def. 2.2.2. (Self-Financing) A portfolio strategy h is said to be self-financing if the following condition holds for every time step i :

$$h_i^T \begin{pmatrix} B_i \\ S_i \end{pmatrix} = h_{i-1}^T \begin{pmatrix} B_i \\ S_i \end{pmatrix} \quad (2.46)$$

* $t = t_i$ での portfolio 組み換え

If the strategy h is self-financing,

$$\begin{aligned} \Delta V_i^h &:= V_i^h - V_{i-1}^h = (x_{i-1} B_i + y_{i-1} S_i) - (x_{i-1} B_{i-1} + y_{i-1} S_{i-1}) \\ &= x_{i-1} (B_i - B_{i-1}) + y_{i-1} (S_i - S_{i-1}) \\ &= x_{i-1} \Delta B_i + y_{i-1} \Delta S_i \end{aligned} \quad (2.47)$$

2.2.2 Risk-neutral Pricing

Asm. 2.2.1. We assume $d < e^{r\Delta t} < u$

Def. 2.2.3. The risk-neutral probability measure \mathbb{Q} (associated with (q_u, q_d)) is defined as a probability measure satisfying

$$S_i = e^{-r\Delta t} \mathbb{E}^{\mathbb{Q}}[S_{i+1}|S_i] \quad \text{for every } i = \{0, \dots, N-1\} \quad (2.48)$$

The existence and uniqueness can be shown as follows:

$$\begin{cases} q_u + q_d = 1 \\ uq_u + dq_d = e^{r\Delta t} \end{cases} \quad (2.49)$$

$$\Leftrightarrow \begin{cases} q_u = \frac{e^{r\Delta t} - d}{u - d} > 0 \\ q_d = \frac{u - e^{r\Delta t}}{u - d} > 0 \end{cases} \quad \text{by Asm.2.2.1} \quad (2.50)$$

Prop. 2.2.1. Suppose that a contingent claim X is replicable by the self-financing portfolio h . Then the price of the claim X at any time t_i $0 \leq i \leq N$ must be equal to $V_{t_i}^h$, otherwise there exists an arbitrage opportunity.

Goal:

Thm. 2.2.1. The multi-period binomial model satisfying assumption 2.2.1 is complete, i.e. cash-flow of any contingent claim can be perfectly replicated.

(by self-financing portfolio strategy $(h_{t_i})_{i \geq 0}$)

*****以降 未完*****

Preparation:

- European Option
- expiry: $t_N = T$
- option holder receives (or pay if negative) $\Phi(S_N)$ ($= \Phi(S_T)$)
- $\Phi : \mathbb{R} \rightarrow \mathbb{R}$

state at $t_n \dots (n+1)$ 個

number of up movement : $(0, \dots, n)$

We label it by (n, k) , $k \in \{0, \dots, n\}$, $S_n(k) = su^k d^{n-k}$

Construction of self-financing strategy at t_N . Replicating portfolio must satisfy:

$$V_N(k) = \Phi(S_N(k)) = \Phi(su^k d^{n-k}) \quad \text{for every } k = 0, \dots, N \quad (2.51)$$

at $t_N - 1$

3 Ito Formula and Option Valuation

3.1 Probability Space

Def. 3.1.1. Let Ω be a non-empty set, and let \mathcal{F} be a collection of subsets of Ω . We say that \mathcal{F} is a σ -algebra (or σ -field) if it satisfies:

1. empty set $\phi \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c = (\Omega \setminus A) \in \mathcal{F}$
3. Suppose there is a sequence of sets $\{A_i\}_{i=1, \dots}$ and $A_i \in \mathcal{F}$ for all i , then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Remarks:

- (1)(2) $\Rightarrow \Omega \in \mathcal{F}$
- (3)(2) $\Rightarrow \cap_{i=1}^{\infty} A_i = (\cup_{i=1}^{\infty} A_i^c)^c \in \mathcal{F}$

* σ (sigma) means 'countable infinity' (可算無限)

Ex. 3.1.1. • $\Omega = \{\omega_1, \omega_2\}$

- $\mathcal{F} = \{\phi, \{\omega_1, \omega_2\}\}$
- $\mathcal{F} = \{\phi, \{\omega_1, \omega_2\}, \{\omega_1\}, \{\omega_2\}\}$

Def. 3.1.2. Let Ω be a non-empty set, and let \mathcal{F} be a σ -algebra on Ω . A probability measure \mathbb{P} is a function that assigns a number in $[0, 1]$ to every set $A \in \mathcal{F}$, called probability of A $\mathbb{P}(A)$ and satisfies:

1. $\mathbb{P}(\Omega) = 1$
2. $(A_i)_{i \geq 1}, A_i \in \mathcal{F}, A_i \cap A_j = \phi \text{ if } (i \neq j)$

then $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

topology (collection of all open sets)

(略)

3.2 Property of Stochastic Integral

Gaussian Distribution

one-dimensional Gaussian distribution (mean 0, variance 1) $N(0, 1)$...standard normal prob density: $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

for $a > 0$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \quad (3.1)$$

This implies

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = (-1)^n \int_{-\infty}^{\infty} \left(\frac{\partial^n}{\partial a^n} \right) e^{-ax^2} dx = (-1)^n \left(\frac{\partial^n}{\partial a^n} \sqrt{\frac{\pi}{a}} \right) \quad (3.2)$$

$(X_t)_{0 \leq t \leq T} : \mathbb{F}$ -adopted,

$\{X_t(\omega) \in B\} \in \mathcal{F}_t \quad \forall t,$

$B \in \mathcal{B}(\mathbb{R}^n)$

\Rightarrow If we have \mathcal{F}_t information, we know the ? X_t of

Def. 3.2.1. Let $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ be a filtered probability space. W is a continuous \mathbb{F} -adopted process that satisfy

- $W(0) = 0$
- $W(u) - W(t)$ is independent from \mathcal{F}_t (and in particular $W(s) - W(r)$) whenever $r < s \leq t < u$
- $W(t) - W(s) \sim N(0, t - s)$ in distribution
if $d > 1$ then, for each i , $W^i(t) - W^i(s) \sim N(0, t - s)$, W^i, W^j are independent.

Here $N(0, v)$ is the gaussian distribution of mean 0, variance v . Then we call W a $(\mathbb{P}; \mathbb{F})$ -Brownian motion.

Some propositions of Brownian motion (one-dimensional case)

Set $s < t$, $\Delta t = t - s$, $\Delta W := W(t) - W(s)$

\Rightarrow

$$\mathbb{E}[\Delta W] = 0 \quad (3.3)$$

$$\mathbb{E}[(\Delta W)^2] = \mathbb{V}[\Delta W] = t - s = \Delta t \quad (3.4)$$

$$\begin{aligned} \mathbb{V}[(\Delta W)^2] &= \mathbb{E}[(\Delta W)^4] - \mathbb{E}[(\Delta W)^2]^2 = \int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}} dx - (\Delta t)^2 \\ &= (\Delta t)^2 \int_{-\infty}^{\infty} \frac{z^4}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - (\Delta t)^2 = 3(\Delta t)^2 - (\Delta t)^2 = 2(\Delta t)^2 \end{aligned} \quad (3.5)$$

Consider the limit $\Delta \rightarrow dt$, then $(\Delta W)^2 \rightarrow (dW)^2 = dt$ (variance $\rightarrow 0$ as $dt \rightarrow 0$)

Def. 3.2.2. Denote \mathcal{P} the vector space of maps $H : \Omega \times [0, T] \rightarrow R$ of the form

$$H_t(\omega) = \sum_{i=1}^n h_{i-1}(\omega) 1_{(t_{i-1}, t_i]}(t) \quad (3.6)$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_n = T$,

3.3 Partial Differential Equation for Option Pricing

Work in the physical(empirical) measure

n -risky asset price process $S_t = (S_t^1, \dots, S_t^n)$

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dW_t \quad (3.7)$$

$$b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (3.8)$$

$$\sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d} \quad (3.9)$$

$$(3.10)$$

W : d -dimensional B.M. under the physical measure

money market

$$\beta(t) = \exp \left(\int_0^t r(s)ds \right) \quad (s \geq 0) \quad (3.11)$$

$r(s)$: non-random

Let us consider a European Option which pays $\Psi(S_T)$ at T ($\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$)

(ex. $\Psi(S_t) = \max(S_t^1 - K, 0)$ call on the 1st asset)

Suppose the option price at $t \in [0, T]$ is given by some function:

$c : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$,

$c \in \mathbb{C}^{1,2}$ (or smooth)

such that the price at t is $c(t, S_t)$

Itô-formula implies

$$dC(t, S_t) = \frac{\partial}{\partial t} c(t, S_t)dt + \sum_{i=1}^n c(t, s_t) dS_t^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 c(t, S_t)}{\partial x_i \partial x_j} (\sigma \sigma^T)_{ij}(t, S_t)dt \quad (3.12)$$

Treat tye contingent claim as another treadable risky security. (オプションを取引可能資産に加える)

- $\eta_t, 0 \leq t \leq T$: number of position for the option
- $\pi_t^i, 0 \leq t \leq T, i = 1, \dots, n$: number of position for the i th stock
- $\zeta_t, 0 \leq t \leq T$: number of position for β

Portfolio strategy:

$$h_t = (\eta_t, \pi_t, \zeta_t) = (\eta_t, \pi_t^1, \dots, \pi_t^n, \zeta_t) \quad (3.13)$$

$$V_t^h = \eta_t c(t, S_t) + \pi_t \cdot S_t + \zeta_t \beta(t) = \eta_t c(t, S_t) + \sum_{i=1}^n \pi_t^i S_t^i + \zeta_t \beta(t) \quad (3.14)$$

Self-financing condition

$$\begin{aligned} dV_t^h &= \eta_t dc(t, S_t) + \pi_t \cdot dS_t + \zeta_t d\beta(t) \\ &= \eta_t dc(t, S_t) + \pi_t \cdot dS_t + r(t)(V_t^h - \eta_t c(t, S_t) - \pi_t \cdot S_t)dt \end{aligned} \quad (3.15)$$

$$\begin{aligned} &= r(t)(V_t^h - \pi_t \cdot S_t)dt + \sum_{i=1}^n (\eta_t \frac{\partial}{\partial x_i} c(t, S_t) + \pi_t^i) dS_t^i \\ &\quad + \eta_t \left(\frac{\partial}{\partial t} c(t, S_t) + \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^T)_{ij}(t, S_t) \frac{\partial^2 c(t, S_t)}{\partial x_i \partial x_j} - r(t)c(t, S_t) \right) dt \end{aligned} \quad (3.16)$$

Choose the following strategy h such that

$$\pi_t^i = \eta_t \frac{\partial}{\partial x_i} c(t, S_t) \quad i = 1, \dots, n \quad (0 \leq t \leq T) \quad (3.17)$$

(delta neutral position)

Then, for such a h ,

$$\begin{aligned} dV_t^j &= \eta_t \left\{ \frac{\partial}{\partial t} c(t, S_t) + \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^T)_{ij}(t, S_t) \frac{\partial^2 c(t, S_t)}{\partial x_i \partial x_j} \right. \\ &\quad \left. + r(t) \sum_{i=1}^n (S_t^i \frac{\partial}{\partial x_i} c(t, S_t) - r(t)c(t, S_t)) \right\} dt + r(t)V_t^h dt \end{aligned} \quad (3.18)$$

There is no term of stochastic integral dW . Specify the trading strategy such that

$$\eta_t = \begin{cases} (\eta_t > 0) & \text{when } \{\dots\} \geq 0 \\ (\eta_t < 0) & \text{when } \{\dots\} < 0 \end{cases} \quad (3.19)$$

example:

$$\begin{cases} 1 & \text{when } \{\dots\} \geq 0 \\ -1 & \text{when } \{\dots\} < 0 \end{cases} \quad (3.20)$$

$r(t)$ より高いリターンを無リスクで得られる!?

V_t^h :

- higher return rate $> r(t)$
- risk-free (no dW -term)

\Rightarrow arbitrage

For the absense of arbitrage, we must have $\{\dots\} = 0$

As a result, we have got the next pricing PDE:

$$\frac{\partial}{\partial t} c(t, x) + r(t) \sum_{i=1}^n x^i \frac{\partial}{\partial x_i} c(t, x) + \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^T)_{ij}(t, x) \frac{\partial^2 c(t, x)}{\partial x_i \partial x_j} - r(t)c(t, x) = 0 \quad (3.21)$$

$$\text{B.C. } c(T, x) = \Psi(s) \quad \text{for } x \in \mathbb{R}^n \quad (3.22)$$

for $(t, x) \in [0, T] \times \mathbb{R}^n$

dW -term を消すような strategy をつくる

$\Rightarrow r(t)V_t^h dt$ 以外の項がゼロになる必要がある (no-arbitrage)

\Rightarrow We got PDE.

Black-Scholes: special case $n = d = 1$

$$\sigma(t, x) = \sigma x, \quad r(t) \equiv r \geq 0 \quad (3.23)$$

3 次元以上だと計算が大変 (だった)

Suppose $c(t, x) \in \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^n)$, solves the last PDE. Consider the self-financing portfolio given by $(\eta, \pi, \zeta) = (-1, \pi, \zeta)$.

(short-position of the option, stock, money market)

$$\begin{aligned} dV_t = & r(t)(V_t - \pi_t - \zeta_t)dt + \sum_{i=1}^n \left(-\frac{\partial}{\partial x_i} c(t, S_t) + \pi_t^i \right) dS_t^i \\ & - \left(\frac{\partial}{\partial t} c(t, S_t) + \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^T)_{ij}(t, S_t) \frac{\partial^2 c(t, S_t)}{\partial x_i \partial x_j} - r(t)c(t, S_t) \right) dt \end{aligned} \quad (3.24)$$

$$dV_t = r(t)(V_t - \pi_t - \zeta_t)dt + \sum_{i=1}^n \left(-\frac{\partial}{\partial x_i} c(t, S_t) + \pi_t^i \right) dS_t^i + r(t) \sum_{i=1}^n S_t^i \frac{\partial c(t, S_t)}{\partial x_i} dt \quad (3.25)$$

Since $c(t, x)$ satisfies the PDE, choose π^* given by

$$\pi^{*i}(t) = \frac{\partial}{\partial x_i} c(t, S_t) \quad i = 1, \dots, n \quad (3.26)$$

as π_t .

\Rightarrow for such π^*

$$dV_t = r(t)V_t dt \quad (3.27)$$

\therefore If $V_0 = 0$, then $V_t = 0$ for all $t \in [0, T]$

This means the portfolio starting from a cash value $c(0, S_0)$ with dynamic position:

$$\left(\pi^{*i}(t), \frac{c(t, S_t) - \pi_t^* \cdot S_t}{\beta(t)} \right) \quad (3.28)$$

for stock and money market.

(perfectly replicates not only the option payoff $c(T, S_T) = \Psi(S_T)$ but also the intermediate option value $c(t, S_t)$).

3.4 Relation to the Risk-Neutral Pricing

Suppose that there is a probability measure Q equivalent to P such that

$$dS(t) = r(t)S(t)dt + \sigma(t, S_t)dW_t^Q \quad (3.29)$$

W^Q : Brownian motion under the measure Q

3.5 BS Option Formula

$r, \sigma > 0$: const.

one-dimensional S, W^Q

Suppose the stock price follows a geometric Brownian motion (log-normal process):

$$dS_t = S_t(rd_t + \sigma dW_t^Q) \quad (3.30)$$

or equivalently

$$S_t = S_0 + \int_0^t S_u(rdu + \sigma dW_u^Q) \quad (3.31)$$

Call Option with strike $K (> 0)$

$$\Psi(S_T) = \max(S_T - K, 0) \quad (3.32)$$

option price at $t = 0$:

$$C = e^{-rT} E^Q[\max(S_T - K, 0)] \quad (3.33)$$

$S_T > 0$ for all $t \geq 0$ if $S_0 > 0$

Apply Itô formula to $\ln(S_t)$

$$d \ln(S_t) = \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} (dS_t)^2 = \left(r - \frac{\sigma^2}{2} \right) dt + \sigma dW_t^Q \quad (3.34)$$

Integrates the both hands for $[0, T]$

$$\ln S_t - \ln S_0 = \left(r - \frac{\sigma^2}{2} \right) T + \sigma W_T^Q \quad (3.35)$$

$$S_T = S_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma W_T^Q \right) \quad (3.36)$$

Note that $W_T^Q \approx (dist)\sqrt{T}z$

$z \sim N(0, 1)$: standard normal (mean:0, variance:1)

z has the density $\phi(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}$

$$C = e^{-rT} \int_{-\infty}^{\infty} \max \left(S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} z \right\} - K, 0 \right) \phi(z) dz \quad (3.37)$$

$$C = e^{-rT} \int_{-\infty}^{\infty} \max \left(F e^{-\frac{\sigma^2}{2} T + \sigma \sqrt{T} z} - K, 0 \right) \phi(z) dz, \quad (F = e^{rT} S_0 = \frac{S_0}{P(0, T)}) \quad (3.38)$$

Let us define

$$d_2 := \frac{1}{\sigma \sqrt{T}} \left(\left(\frac{F}{K} \right) - \frac{\sigma^2}{2} T \right) \quad (3.39)$$

Then,

$$F e^{-\frac{\sigma^2}{2} T + \sigma \sqrt{T} z} \geq K \quad \text{when } z \geq -d_2 \quad (3.40)$$

Therefore,

$$C = e^{-rT} \int_{-d_2}^{\infty} \left(F e^{-\frac{\sigma^2}{2} T + \sigma \sqrt{T} z} - K \right) \phi(z) dz \quad (3.41)$$

• 1st term:

$$\begin{aligned} & F e^{-rT} e^{-\frac{\sigma^2}{2} T} \int_{-d_2}^{\infty} e^{\sigma \sqrt{T} z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz = F e^{-rT} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (z - \sigma \sqrt{T})^2} dz \\ & = F e^{-rT} \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z'^2} dz' = F e^{-rT} N(d_1) \end{aligned} \quad (3.42)$$

$$d_1 := d_2 + \sigma \sqrt{T} = \frac{1}{\sigma \sqrt{T}} \left(\ln \left(\frac{F}{K} \right) + \frac{\sigma^2}{2} T \right) \quad (3.43)$$

• 2nd term:

$$-K e^{-rT} \int_{-d_2}^{\infty} \phi(z) dz = -K e^{-rT} N(d_2) \quad (3.44)$$

Put option?

$$P = e^{-rT} \mathbb{E}^Q[\max(K - S_T, 0)] \quad (3.45)$$

$$\Psi(S_T) = \max(K - S_T, 0) \quad (3.46)$$

Call と同様の計算は面倒...

Put-Call parity

$$\max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K \quad (3.47)$$

$$\begin{aligned} \therefore C - P &= e^{-rT} \mathbb{E}^Q[S_T - K] = e^{-rT} \mathbb{E}^Q[S_T] - K e^{-rT} \\ &= \mathbb{E}^Q[\beta(T)^{-1} S_T] - K e^{-rT} \end{aligned} \quad (3.48)$$

$(\beta(t)^{-1}S_t)_{0 \leq t \leq T}$... Q -martingale

$$\because d(\beta(t)^{-1}S_t) = \beta(t)^{-1}\{r(t)S_t dt + dS_t\} = \beta(t)^{-1}\sigma S_t dW_t^Q \quad (3.49)$$

Therefore,

$$C - P = \beta(0)^{-1}S_0 - Ke^{-rT} = S_0 - Ke^{-rT} \quad (3.50)$$

$$\begin{aligned} P &= C - S_0 + Ke^{-rT} = e^{-rT}(FN(d_1) - KN(d_2) - F + K) \\ &= -e^{-rT}F(1 - N(d_1)) + e^{-rT}K(1 - N(d_2)) \\ &= e^{-rT}\{KN(-d_2) - FN(-d_1)\} \end{aligned} \quad (3.51)$$

3.6 Oport Formula for a Stock with Dividend Yield

dividend payment $[t, t + dt]$ $q(t)S_t dt$ q : dividend yield

If we reinvest all the dividends to the same stock, one share at $t = 0$

$\Rightarrow \exp\left(\int_0^t q(s)ds\right)$ share at t

\because Number of share $n(t)$ follows $\frac{d}{dt}n(t) = q(t)n(t)$

For no-arbitrage, $\left(S(t)e^{\int_0^t q(s)ds}\right)_{0 \leq t \leq T}$ must follow the dynamics corresponding to the stock price without dividend payment.

$\Rightarrow \left(S(t)e^{\int_0^t q(s)ds}\right)_{0 \leq t \leq T}$ must have the drift rate $r(t)$.

under Q , dS must follow:

$$dS_t = (r(t) - q(t))S_t dt + \sigma(t, S_t)dW_t^Q \quad (3.52)$$

$$\begin{aligned} d(S_t e^{\int_0^t q(s)ds}) &= e^{\int_0^t q(s)ds}(q(t)dt + dS_t) \\ &= e^{\int_0^t q(s)ds}(r(t)S_t dt + \sigma(t, S_t)dW_t^Q) \end{aligned} \quad (3.53)$$

dividend yield

$$dS_t = S_t(r_t - q_t)dt + S_t\sigma_t dW_t^Q \quad (3.54)$$

For simplicity, $r, q, \sigma > 0$: constants

$$d \ln S_t = \left(r - q - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t^Q \quad (3.55)$$

$$S_t = \quad (3.56)$$

lllll

Ex. 3.6.1. (FX)

$(X_t)_{0 \leq t \leq T}$: the price of unit amount of a foreign currency in terms of the domestic currency

X_T : asset price, dividend yield $r^f(t)$

(foreign interest rate)

$$dX_t = (r^d(t) - r^f(t))X_t + \sigma(t, X_t)dW_t^Q \quad (3.57)$$

($X > 0$ を保証するもの)

Def. 3.6.1. (Implied volatility) The implied volatility of an option contract is the volatility constant σ which gives the market price of the option when σ is used in B.S. model.

(illustration)

3.6.1 Market Option Quotes and the Implied Distribution

For simplicity, assume $r > 0$ is constant, Q exists.

Then, a call option price is given by

$$c(K) = e^{-rT} \mathbb{E}^Q[(S_T - K)^+] \quad (3.58)$$

expiry T : fixed, strike K : floating

Suppose S_T has a smooth probability density function $g(x) : \mathbb{R} \rightarrow \mathbb{R}$ under Q . Then,

$$c(K) = e^{-rT} \int_K^\infty (x - K)g(x)dx \quad (3.59)$$

Then,

$$\begin{aligned} \frac{\partial c}{\partial K} &= -e^{-rT} (x - K)g(x)dx|_{x=K} = -e^{-rT} \int_K^\infty g(x)dx \\ &= -e^{-rT} \int_K^\infty g(x)dx \end{aligned} \quad (3.60)$$

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K) \quad (3.61)$$

Implied volatility の価格変化から S_T の density を得る :

$$g(K) = e^{rT} \frac{\partial^2 c}{\partial K^2} \quad (3.62)$$

3.6.2 Greeks

Sensitivity of option (or any derivative contract) price with respect to various parameters (S, t, σ , etc.)

Option price $\Pi(t, S_t)$ at (t, S_t)

- Delta:

$$\frac{\partial}{\partial x} \Pi(t, x)|_{x=S_t} \quad (3.63)$$

- Gamma:

$$\frac{\partial^2}{\partial x^2} \Pi(t, x)|_{x=S_t} \quad (3.64)$$

- Vega(Kappa):

$$\frac{\partial}{\partial \sigma} \Pi \quad (3.65)$$

- Cross Gamma:

$$\frac{\partial^2 \Pi}{\partial x_i \partial x_j} \quad (3.66)$$

Cross Gamma は P/L-attribution を調べるのに使える。

complete market には十分な流動性が必要。unheagable な product に対する線形な価格評価 (risk-neutral pricing) をするのは恐ろしい。仮定を置かないと正当化されない。本当は、各事業者がリスク効用関数 (risk-averseness) を設定し、それに応じて最適化問題として解くべきである (が、大変なのでやらない)。