Parametric & Nonparametric Statistics Project

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1 Introduction

2 Preliminaries

In the project below, we will use the following parameters:

- $\mathcal{N} = 9$ (first name: 'Aleksandr', 9 letters)
- $\mathcal{S} = 9$ (last name: 'Smoliakov', 9 letters)
- $\mathcal{I}_1 = 5$ (last digit of study book number)
- $\mathcal{I}_2 = 8$ (second last digit of study book number)

Let $G_1, \ldots, G_m$ be given distribution functions and $p_1, \ldots, p_m$ be probabilities that sum to 1. The distribution function G defined by

$$G(u) := p_1 G_1(u) + \cdots + p_m G_m(u) = \sum_{k=1}^{m} p_k G_k(u), \quad u \in \mathbb{R}$$

is called a mixture of distribution functions $G_1, \ldots, G_m$ with probabilities (or weights) $p_1, \ldots, p_m$.

G is the distribution function of the random variable Z generated in the following way:

1. Choose $k \in \{1, \ldots, m\}$ at random with probabilities (or weights) $p_1, \ldots, p_m$. The chosen number is denoted by k^*.
2. Generate a random variable $Z^*_{k^*}$ according to the distribution function G_{k^*} and assign $Z \leftarrow Z^*_{k^*}$.

In this task, we will have $m = 2$, so the algorithm for generating Z is as follows:

$$Z \leftarrow Z^*_{1+k^*} \quad k^* \sim \text{Binomial}(1, p_2), \quad Z^*_k \sim G_k \ (k = 1, 2).$$

Let

$$\mathcal{G}(\Theta) = \{G(\cdot|\boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta\}$$

be a given parametric family of absolutely continuous parametric functions $G(\cdot|\boldsymbol{\theta})$ with the respective distribution densities $g(\cdot|\boldsymbol{\theta})$ dependent on the unknown parameter $\boldsymbol{\theta} \in \Theta$. It is assumed that $\boldsymbol{\theta}$ is two-dimensional, i.e., $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$.

2.1 Parametric Family Selection

Using the assigned formula $\ell := \left\lfloor \frac{\mathcal{I}_2 + 2.5}{2} \right\rfloor$, we find $\ell = 5$. Thus, we will use the parametric family $\mathcal{G}_5(\Theta)$ in this task.

$\mathcal{G}_5(\Theta)$ contains distribution functions of random variables uniformly distributed on $[\theta_1, \theta_2]$, where $\theta_1 < \theta_2$.

It can be expressed as:

$$G(u|\boldsymbol{\theta}) = \begin{cases} 0 & u < \theta_1 \\ \frac{u - \theta_1}{\theta_2 - \theta_1} & \theta_1 \le u \le \theta_2 \\ 1 & u > \theta_2 \end{cases}$$

with $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ and $\theta_1 < \theta_2$.

3 Task 1: Testing Goodness-of-Fit

3.1 Basic Distribution Function

The problem gives a specific basic parameter:

$$\boldsymbol{\theta}_0 = (-\mathcal{N}, \mathcal{S} + 4) = (-9, 13).$$

Thus, the basic distribution function is:

$$G_0(u) = \mathcal{G}_5(u|\boldsymbol{\theta}_0) = U(-9, 13).$$

For a uniform distribution $U(a, b)$:

- Mean: $\mu = \frac{a+b}{2}$
- Variance: $v^2 = \frac{(b-a)^2}{12}$

For $G_0 = U(-9, 13)$:

$$\mu_0 = \frac{-9 + 13}{2} = \frac{4}{2} = 2$$

$$v_0^2 = \frac{22^2}{12} = \frac{484}{12} = \frac{121}{3} \approx 40.3333$$

3.2 Finding Mixture Distributions

We are given the following equations for the mixture distributions G_1 and G_2:

$$\mu_0 = \mu(\boldsymbol{\theta}_1), \quad \mathcal{N}v_0^2 = v^2(\boldsymbol{\theta}_1).$$

$$\mu_0 + 2v_0 = \mu(\boldsymbol{\theta}_2), \quad v_0^2 = \mathcal{S}v^2(\boldsymbol{\theta}_2).$$

3.2.1 Determining G_1

First we determine G_1. We have:

$$\mu_0 = \mu(\theta_1), \quad \mathcal{N}v_0^2 = v^2(\theta_1).$$

It is given that $\mu(\theta_1) = \mu_0 = 2$.
Plugging in $\mathcal{N} = 9$ and $v_0^2 = \frac{121}{3}$, we get:

$$v^2(\boldsymbol{\theta}_1) = \mathcal{N}v_0^2 = 9 \times \frac{121}{3} = 363.$$

Let $G_1(u) = U(a_1, b_1)$. For a uniform distribution:

$$\mu(\boldsymbol{\theta}_1) = \frac{a_1 + b_1}{2}, \quad v^2(\boldsymbol{\theta}_1) = \frac{(b_1 - a_1)^2}{12}.$$

Since $\mu(\boldsymbol{\theta}_1) = 2$:

$$\frac{a_1 + b_1}{2} = 2 \implies a_1 + b_1 = 4.$$

Since $v^2(\boldsymbol{\theta}_1) = 363$:

$$\frac{(b_1 - a_1)^2}{12} = 363 \implies b_1 - a_1 = \sqrt{(b_1 - a_1)^2} = \sqrt{4356} = 66.$$

Solving the system:

$$a_1 + b_1 = 4, \quad b_1 - a_1 = 66.$$

Adding the two equations:

$$2b_1 = 70 \implies b_1 = 35.$$

$$a_1 = 4 - 35 = -31.$$

Thus:

$$\boldsymbol{\theta}_1 = (-31, 35) \implies G_1(u) = U(-31, 35).$$

3.2.2 Determining G_2

Repeating the process for G_2. We have:

$$\mu_0 + 2v_0 = \mu(\boldsymbol{\theta}_2), \quad v_0^2 = \mathcal{S}v^2(\boldsymbol{\theta}_2).$$

Given $\mu_0 = 2$ and $v_0^2 = 40.3333$, we have:

$$\mu(\boldsymbol{\theta}_2) = \mu_0 + 2v_0 = 2 + 2 \times \sqrt{40.3333} = 2 + 2 \times 6.3509 = 2 + 12.7018 = 14.7018.$$

Also:

$$v_0^2 = \mathcal{S}v^2(\boldsymbol{\theta}_2) \implies 40.3333 = 9v^2(\boldsymbol{\theta}_2) \implies v^2(\boldsymbol{\theta}_2) = \frac{40.3333}{9} \approx 4.4815.$$

For $G_2(u) = U(a_2, b_2)$:

$$\frac{a_2 + b_2}{2} = 14.7018 \implies a_2 + b_2 = 29.4036.$$

$$\frac{(b_2 - a_2)^2}{12} = 4.4815 \implies (b_2 - a_2)^2 = 4.4815 \times 12 = 53.7777.$$

$$b_2 - a_2 = \sqrt{53.7777} \approx 7.3333.$$

Solving the system:

$$a_2 + b_2 = 29.4036, \quad b_2 - a_2 = 7.3333.$$

Adding the two equations:

$$2b_2 = 36.7369 \implies b_2 = 18.3685.$$

$$a_2 = 29.4036 - 18.3685 = 11.0351.$$

Thus:

$$\boldsymbol{\theta}_2 = (-11.0351, 18.3685) \implies G_2(u) = U(11.0351, 18.3685).$$

3.3 Computing p_1 and p_2

Given:

$$\tau = \frac{1}{1 + I_1}, \quad I_1 = 5 \implies \tau = \frac{1}{6}.$$

$$\alpha_1 = 0.1, \quad \alpha_2 = 0.01.$$

$$p_1 = (\alpha_1)^{1-\tau}(\alpha_2)^{\tau} = (0.1)^{5/6}(0.01)^{1/6} \approx 0.06813.$$

Then:

$$p_2 = \frac{5p_1}{\sqrt{S}} = \frac{5 \times 0.06813}{\sqrt{9}} = \frac{0.3406}{3} \approx 0.1135.$$

3.4 Determining Mixture Distributions

We consider testing:

$$H_0 : F_Y = G_0 \textbf{ versus } H' : F_Y \neq G_0$$

We will compare the empirical distribution of samples generated from:

1. $F_Y = (1 - p_1)G_0 + p_1 G_1$, i.e. a mixture of G_0 and G_1.
2. $F_Y = (1 - p_2)G_0 + p_2 G_2$, i.e. a mixture of G_0 and G_2.

The tests are conducted for sample sizes:

$$N_1 = 10 \times (2 + \mathcal{N}) = 10 \times (2 + 9) = 110,$$

$$N_2 = 100 \times (2 + \mathcal{N}) = 100 \times (2 + 9) = 1100.$$

3.5 Goodness-of-Fit Tests

We will use the Kolmogorov-Smirnov test for the given samples $(Y_t)_{t=1}^n$:

The test statistic is:

$$D_n = \sup_u |F_n(u) - F(u)|,$$

where F_n is the empirical distribution function (EDF) based on the sample and F is the theoretical distribution function. In this case, $F = G_0$.

Since:

$$F_Y(u) = (1 - p_k)G_0(u) + p_k G_k(u),$$

we have:

$$F_Y(u) - G_0(u) = p_k[G_k(u) - G_0(u)],$$

for $k = 1$ or $k = 2$.

Thus, the maximum difference between F_Y and G_0 is:

$$\sup_u |F_Y(u) - G_0(u)| = p_k \sup_u |G_k(u) - G_0(u)|.$$

We need $\sup_u |G_1(u) - G_0(u)|$ and $\sup_u |G_2(u) - G_0(u)|$.

3.5.1 G_1 vs. G_0

$G_0 = U(-9, 13)$, so:

$$G_0(u) = \begin{cases} 0 & u < -9 \\ \frac{u+9}{22} & -9 \leq u \leq 13 \\ 1 & u > 13 \end{cases}$$

$G_1 = U(-31, 35)$, so:

$$G_1(u) = \begin{cases} 0 & u < -31 \\ \frac{u+31}{66} & -31 \leq u \leq 35 \\ 1 & u > 35 \end{cases}$$

To find $\sup |G_1(u) - G_0(u)|$, we investigate ranges of u piecewise between the breakpoints of the two functions.

1. For $u < -31$: $G_0(u) = G_1(u) = 0$, so the difference is 0.
2. For $-31 \leq u < -9$: $G_0(u) = 0$, $G_1(u) = \frac{u+31}{66}$. The difference is $\frac{u+31}{66}$, which is increasing as u approaches -9, where it is $\frac{-9+22}{66} = \frac{1}{3}$.

3. For $-9 \le u < 13$: $G_0(u) = \frac{u+9}{22}$, $G_1(u) = \frac{u+31}{66}$. The difference is $\frac{u+31}{66} - \frac{u+9}{22} = \frac{2u-4}{66}$, which is increasing from $-\frac{1}{3}$ at -9 to $\frac{1}{3}$ at 13.

4. For $13 \le u < 35$: $G_0(u) = 1$, $G_1(u) = \frac{u+31}{66}$. The difference is $\frac{u+31}{66} - 1 = \frac{u-35}{66}$, which is increasing from $-\frac{1}{3}$ at 13 to 0 at 35.

5. For $u \ge 35$: $G_0(u) = G_1(u) = 1$, so the difference is 0.

The maximum absolute difference is $\frac{1}{3}$ at the endpoints of the range $[-9, 13]$.
Hence:

$$\sup_u |G_1(u) - G_0(u)| = 1/3 \approx 0.3333.$$

For the mixture, taking $p_1 \approx 0.06813$:

$$\sup_u |F_Y(u) - G_0(u)| = p_1 \times 0.3333 = 0.06813 \times 0.3333 \approx 0.02271.$$

3.5.2 G_2 vs. G_0

Repeating the process for G_2:
$G_0 = U(-9, 13)$, so:

$$G_0(u) = \begin{cases} 0 & u < -9 \\ \frac{u+9}{22} & -9 \le u \le 13 \\ 1 & u > 13 \end{cases}$$

$G_2 = U(11.0351, 18.3685)$, so:

$$G_2(u) = \begin{cases} 0 & u < 11.0351 \\ \frac{u-11.0351}{7.3333} & 11.0351 \le u \le 18.3685 \\ 1 & u > 18.3685 \end{cases}$$

To find $\sup |G_2(u) - G_0(u)|$, we investigate ranges of u piecewise between the breakpoints of the two functions.

1. For $u < -9$: $G_0(u) = G_2(u) = 0$, so the difference is 0.

2. For $-9 \le u < 11.0351$: $G_0(u) = \frac{u+9}{22}$, $G_2(u) = 0$. The difference is $\frac{u+9}{22}$, which is increasing as u approaches 11.0351, where it is $\frac{11.0351+9}{22} \approx 0.9107$.

3. For $11.0351 \le u < 13$: $G_0(u) = \frac{u+9}{22}$, $G_2(u) = \frac{u-11.0351}{7.3333}$. The difference is $\frac{u-11.0351}{7.3333} - \frac{u+9}{22} = \frac{3u-33.1053}{22} - \frac{u+9}{22} = \frac{2u-42.1053}{22}$, which is increasing from -0.9107 at 11.0351 to $\frac{13-42.1053}{22} \approx -0.7321$ at 13.

4. For $13 \le u < 18.3685$: $G_0(u) = 1$, $G_2(u) = \frac{u-11.0351}{7.3333}$. The difference is $\frac{u-11.0351}{7.3333} - 1 = \frac{u-18.3685}{7.3333}$, which is increasing from $-\frac{18.3685-13}{7.3333} = -\frac{5.3685}{7.3333} \approx -0.7321$ at 13 to 0 at 18.3685.

5. For $u \ge 18.3685$: $G_0(u) = G_2(u) = 1$, so the difference is 0.

The maximum absolute difference is ≈ 0.9107 at 11.0351.
Hence:

$$\sup_u |G_2(u) - G_0(u)| \approx 0.9107.$$

For the mixture, taking $p_2 \approx 0.1135$:

$$\sup_u |F_Y(u) - G_0(u)| = p_2 \times 0.9107 = 0.1135 \times 0.9107 \approx 0.1034.$$

3.6 Critical Values and Detection Probability

Under H_0, the Kolmogorov-Smirnov test critical values at significance $\alpha_1 = 0.1$ and $\alpha_2 = 0.01$ are approximately:

$$D_{N,\alpha_1} \approx \frac{1.22}{\sqrt{N}} \quad \text{for} \quad \alpha_1 = 0.1,$$

$$D_{N,\alpha_2} \approx \frac{1.63}{\sqrt{N}} \quad \text{for} \quad \alpha_2 = 0.01.$$

For $N = 110$:

$$D_{110,0.1} \approx \frac{1.22}{\sqrt{110}} \approx 0.1163 \quad \text{and} \quad D_{110,0.01} \approx \frac{1.63}{\sqrt{110}} \approx 0.1554.$$

- For G_1: $\sup |F_Y - G_0| \approx 0.02271 < 0.1163 < 0.1554$. Thus, at $N = 110$, it's unlikely we reject H_0. $p > 0.1$
- For G_2: $\sup |F_Y - G_0| \approx 0.1034 < 0.1163 < 0.1554$. There is some chance to reject at a higher α level. $p > 0.1$ (but it's closer to the borderline)

For $N = 1100$:

$$D_{1100,0.1} \approx \frac{1.22}{\sqrt{1100}} \approx 0.03678 \quad \text{and} \quad D_{1100,0.01} \approx \frac{1.63}{\sqrt{1100}} \approx 0.04915.$$

- For G_1: $\sup |F_Y - G_0| \approx 0.02271 < 0.03678 < 0.04915$. Even with 1100 samples, we will likely not reject H_0 at $\alpha = 0.1$. $p > 0.1$
- For G_2: $\sup |F_Y - G_0| \approx 0.03678 < 0.04915 < 0.1034$. We will almost certainly reject H_0 at $\alpha = 0.01$. $p < 0.01$

Thus, the results of the Kolmogorov-Smirnov test are as follows:

Mixture	**Sample Size**	**p-value**	**Result**
G_1	110	> 0.1	No rejection
G_1	1100	> 0.1	No rejection
G_2	110	> 0.1	No rejection
G_2	1100	< 0.01	Rejection at $\alpha = 0.01$

3.7 Conclusions

By analytically comparing the theoretical distributions, we have:

- **Mixture with G_1:**
  $\sup |F_Y - G_0| \approx 0.02271$.
  Even at $N = 1100$, we will likely not reject H_0 at $\alpha = 0.1$. The p-value is higher than 0.1.
- **Mixture with G_2:**
  $\sup |F_Y - G_0| \approx 0.1034$.
  We will almost certainly reject H_0 at $\alpha = 0.01$ even with 1100 samples. The p-value is < 0.01. We will likely not reject with 110 samples at $\alpha = 0.1$, but the p-value may be close.

It is evident that for Kolmogorov-Smirnov tests, the magnitude of the deviation from G_0 and the sample size play the decisive role.

As $N \to \infty$, if $F_Y \neq G_0$, the empirical distribution F_N converges to F_Y, and thus D_N converges to $\sup_u |F_Y(u) - G_0(u)|$.

4 Task 2: Applications of Bootstrap Technique

In this section we will * test Complex Goodness of Fit Hypothesis, * check bootstrap consistency, * compare bootstrap confidence interval construction methods.

We will use the same parametric family $\mathcal{G}_5(\Theta)$ and the same distributions G_0, G_1, G_2 as in Task 1.

$$G_0(u) = U(-9, 13), \quad G_1(u) = U(-31, 35), \quad G_2(u) = U(11.0351, 18.3685).$$

In this section we will use the sample sizes from Task 1, $N_1 = 110$ and $N_2 = 1100$.

The bootstrap sample size will be $B = 100 \times N$, where N is the original sample size.

4.1 Testing Goodness-of-Fit by Bootstrap

First we will test the Complex Goodness of Fit Hypothesis which asserts that the unknown distribution function F_Y belongs to the parametric family $\mathcal{G}(\Theta)$:

$$H_0 : F_Y \in \mathcal{G}(\Theta) \textbf{ versus } H' : F_Y \notin \mathcal{G}(\Theta).$$

We will make use of the parametric bootstrap technique to test this hypothesis. The test statistic is the Kolmogorov-Smirnov test statistic and the significance levels are $\alpha = 0.1$.

4.2 Checking Bootstrap Consistency

4.3 Bootstrap Confidence Intervals