# FLEXIBLE SPACECRAFT REORIENTATIONS USING GIMBALED MOMENTUM WHEELS *

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## Abstract

We study the reorientations of flexible spacecraft using momentum exchange devices. A new concise form of the equations of motion for a spacecraft with gimbaled momentum wheels and flexible appendages is presented. The derivation results in a set of vector nonlinear first order differential equations with gimbal torques and spin axis torques as the control inputs. Feedback control laws which result in smooth reorientations are sought with the goal of minimizing structural excitations. We pay special attention to a class of maneuvers wherein the magnitude of the momentum in the wheel cluster is held constant, resulting in a so-called "stationary platform maneuver." The advantage of this maneuver is that the platform angular velocity remains small throughout, thereby reducing the excitation of the appendages.

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# INTRODUCTION

Momentum exchange devices are typically categorized as either momentum wheels (fixed spin axis) or control moment gyros (fixed spin speed). The gimbaled momentum wheel (GMW) is simply a combination of the two, i.e., a control moment gyro (CMG) with a controllable speed wheel. We begin the paper by presenting a new set of equations of motion for a rigid spacecraft with $N$ gimbaled momentum wheels. A distinct advantage of this new set of equations is the explicit dependence upon the spin axis and gimbal axis input torques, which are arguably the control variables for the system. Certain restrictions on the GMW equations permit simplification to the momentum wheel or CMG case. New areas of interest in astronautics, such as simultaneous attitude control and energy storage,[1] might benefit from this new form of the general equations.

We then turn to the use of momentum storage devices to reorient a spacecraft from one rest condition to another (i.e. zero angular velocity at start and end of the maneuver). The rest condition (a stationary platform) is only possible when all of the angular momentum is contained in the cluster of GMWs. In the case of the gyrostat, Hall has shown that variation of the wheel speeds along a specific manifold in the space of rotor momenta results in a spacecraft reorientation which keeps body angular velocity small. This so-called stationary platform condition on the rotor momenta holds the magnitude of the angular momentum in the cluster constant. This *stationary platform surface* is a hyper-ellipsoid in the N-dimensional space of rotor momenta. Choosing control torques which maintain the stationary platform surface is relatively simple, and leads to maneuvers that maintain a small platform angular velocity.

The concept applies to the CMG case as well. The manifold here, however, lies in the space of gimbal angles, which is $2\pi$ periodic in all directions. Only certain (modulo $2\pi$) combinations of gimbal angles meet the requirement for constant cluster momentum magnitude. Unfortunately, the stationary platform surface for the CMG cluster is rather complicated.

We discuss the challenges of maintaining the stationary platform surface (for both cases) while satisfying the kinematical requirements of a reorientation maneuver. A Lyapunov control law is modified to allow reorientations which simultaneously stay close to the stationary platform condition. The benefits of these reorientation maneuvers for flexible spacecraft are demonstrated by two examples.

This problem involves several dynamical models that have received extensive attention in the literature. Hughes[2] provided an excellent development of the equations of motion for rigid satellites with momentum wheels and simple flexible components. Hall[3] extended the development to a system of multiple wheels and derived a torque control law which led to a so-called stationary platform maneuver. Further investigations of the stationary platform maneuvers are described in Schultz[4] and Hall.[5,6] Margulies and Aubrun[7] provided the basic geometric theory essential to understand the capabilities and limitations of single gimbal CMG clusters. A useful development of the full equations of motion for spacecraft with CMGs was provided by Oh and Vadali.[8] They developed effective feedback control laws and showed that with gyro inertias included, the control laws must provide gimbal acceleration commands (instead of gimbal rate commands). They also developed a method for avoiding singularities in the case of a redundant set of CMGs. Junkins and Kim[9] provided the tools for our development of the full system of flexible satellite equations, while Meirovitch and Stemple[10] provided the foundation for application of Lagrange's Equations in quasicoordinate form.

## EQUATIONS OF MOTION

Consider a rigid body in which $N$ gimbaled momentum wheels are imbedded (see Figure 1). Each GMW is composed of a flywheel mounted in a gimbal frame and incorporates the variable speed of a momentum wheel and the gimbal arrangement of a typical single gimbal CMG.

The GMWs are designated $W_1, W_2, \ldots, W_N$ and the rigid platform is identified as $B$. The platform is in general not symmetric. A reference frame, $\mathcal{F}_{\mathbf{b}}$, is established in the body which

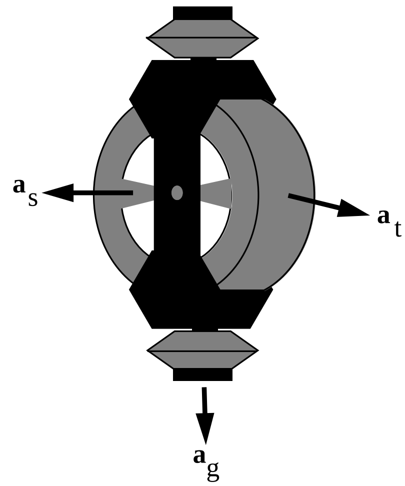


Figure 1: A Gimbaled Momentum Wheel

has basis $(\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3)$. The body is free to translate and rotate with respect to an inertially fixed reference frame, $\mathcal{F}_{\mathbf{i}}$, with basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3)$. The wheels spin about their individual axes of symmetry which are expressed as the unit vectors $\vec{\mathbf{a}}_{s1}, \vec{\mathbf{a}}_{s2}, \ldots, \vec{\mathbf{a}}_{sN}$. The directions of the spin axis unit vectors vary with the gimbal angles. The gimbal axes are always orthogonal to the spin axes and are denoted by the unit vectors $\vec{\mathbf{a}}_{g1}, \vec{\mathbf{a}}_{g2}, \ldots, \vec{\mathbf{a}}_{gN}$. A third set of unit vectors given by $\vec{\mathbf{a}}_{t1}, \vec{\mathbf{a}}_{t2}, \ldots, \vec{\mathbf{a}}_{tN}$ (subscript representing *transverse*) where $\vec{\mathbf{a}}_{tj} = \vec{\mathbf{a}}_{sj} \times \vec{\mathbf{a}}_{gj}$ will prove useful in the derivation.

We define a matrix $\mathbf{A}_s$ such that the columns of $\mathbf{A}_s$ are the column matrices $\mathbf{a}_{sj}$ $(j = 1 \ldots N)$ which specify the orientation of the spin axes of the wheels, $W_j$ $(j = 1 \ldots N)$, in the vehicle body frame $\mathcal{F}_{\mathbf{b}}$. That is

$$\mathbf{A}_s = \begin{bmatrix} \mathbf{a}_{s1} & \mathbf{a}_{s2} & \cdots & \mathbf{a}_{sN} \end{bmatrix} \tag{1}$$

The matrices $\mathbf{A}_g$ and $\mathbf{A}_t$ are defined similarly. Whereas $\mathbf{A}_g$ is a constant matrix, the matrices $\mathbf{A}_s$ and $\mathbf{A}_t$ depend on the gimbal angles.

The moment of inertia for the spacecraft is assumed constant except for the change caused by variation in the gimbal angles. It is also assumed that the center of mass of the spacecraft is fixed in the body and does not vary with gimbal angles. The inertia dyadic $\vec{\mathbf{I}}$ is formed from the body inertia dyadic plus the parallel axis contributions of the wheels. It is given by

$$\vec{\mathbf{I}} = \vec{\mathbf{I}}_{\mathrm{B}} + \sum_{j=1}^{N} m_j(\vec{\mathbf{r}}_j \cdot \vec{\mathbf{r}}_j \vec{\mathbf{1}} - \vec{\mathbf{r}}_j \vec{\mathbf{r}}_j) \tag{2}$$

where $m_j$ is the mass and $\vec{\mathbf{r}}_j$ the fixed location of the center of mass of the $j$-th GMW.

Note that we have adopted the convenient and descriptive notation wherein a physical vector, such as $\vec{\mathbf{r}}_j$, is identified by the arrow over the character, while the representation of the vector's components in a particular reference frame are absent the arrow, for example $\mathbf{r}_j$.

We define the terms $I_{sj}$, $I_{gj}$, and $I_{tj}$ to be the total spin axis inertia, the total gimbal axis inertia, and the total transverse axis inertia of the $j$-th GMW (including the gimbal frame). The total spin axis inertia of the GMW is the sum of the gimbal frame inertia and wheel inertia. We split it into the terms $I_{swj}$ and $I_{sgj}$ so that

$$I_{sj} = I_{swj} + I_{sgj}$$

We form $\mathbf{I}_{sw}$ as a diagonal matrix composed of the spin axis moments of inertia of the GMW wheels:

$$\mathbf{I}_{sw} = \mathrm{diag}(I_{sw1}, I_{sw2}, \ldots, I_{swN}) \tag{3}$$

Four other $N \times N$ inertia matrices, $\mathbf{I}_s$, $\mathbf{I}_g$, and $\mathbf{I}_t$, and $\mathbf{I}_{sg}$, are defined in a similar manner.

The linear momentum of the system expressed in body frame components is given by

$$\mathbf{p} = m\mathbf{v} + \boldsymbol{\omega}^{\times}\mathbf{c} \tag{4}$$

where $\mathbf{v}$ is the velocity of the origin of $\mathcal{F}_{\mathbf{b}}$ and $\mathbf{c}$ is the first mass moment of the body/GMW system about the origin of $\mathcal{F}_{\mathbf{b}}$. The angular velocity of the body reference frame with respect to inertial space is $\boldsymbol{\omega}$. The cross notation represents the skew symmetric form

$$\boldsymbol{\omega}^{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{5}$$

The system angular momentum can be expressed as

$$\vec{\mathbf{h}} = \vec{\mathbf{I}} \cdot \vec{\boldsymbol{\omega}} + \vec{\mathbf{c}} \times \vec{\mathbf{v}} + \sum_{j=1}^{N} \vec{\mathbf{h}}_{aj} \tag{6}$$

where $\vec{\mathbf{h}}_{aj}$ is the *absolute* angular momentum of the $j$-th GMW about its own center of mass. Instead of grouping the GMW contributions to the angular momentum by GMW (as did Oh and Vadali[8]), we decompose the GMW contributions to angular momentum into components in the spin, gimbal, and transverse directions. This is expressed as (using body frame components)

$$\mathbf{h} = \mathbf{I}\boldsymbol{\omega} + \mathbf{c}^{\times}\mathbf{v} + \mathbf{A}_s\mathbf{h}_{sa} + \mathbf{A}_g\mathbf{h}_{ga} + \mathbf{A}_t\mathbf{h}_{ta} \tag{7}$$

The new terms $\mathbf{h}_{sa}$, $\mathbf{h}_{ga}$, and $\mathbf{h}_{ta}$ are $N \times 1$ column matrices which represent the components of absolute angular momentum of the GMWs about their spin axes, gimbal axes, and spin axes respectively.

One term in Equation (7) deserves special attention. The angular momentum of a GMW about its spin axis is a combination of angular momentum due to the flywheel itself, plus a contribution due to the gimbal frame. We simply split $\mathbf{h}_{sa}$ into two terms as

$$\mathbf{h}_{sa} = \mathbf{h}_{swa} + \mathbf{h}_{sga} \tag{8}$$

where $\mathbf{h}_{swa}$ is the $N \times 1$ column matrix of absolute angular momenta of the wheels about their spin axes, and $\mathbf{h}_{sga}$ is the $N \times 1$ column matrix of absolute angular momenta of the gimbal frames about the GMW spin axes.

The absolute angular momentum components may be expressed in terms of the platform angular velocity and *relative* angular momenta of the GMWs. The relationships are

$$\begin{aligned}
\mathbf{h}_{swa} &= \mathbf{I}_{sw}\mathbf{A}_s^{\mathrm{T}}\boldsymbol{\omega} + \mathbf{h}_{swr} && (9)\\
\mathbf{h}_{sga} &= \mathbf{I}_{sg}\mathbf{A}_s^{\mathrm{T}}\boldsymbol{\omega} + \mathbf{h}_{sgr} && (10)\\
\mathbf{h}_{ga} &= \mathbf{I}_{g}\mathbf{A}_g^{\mathrm{T}}\boldsymbol{\omega} + \mathbf{h}_{gr} && (11)\\
\mathbf{h}_{ta} &= \mathbf{I}_{t}\mathbf{A}_t^{\mathrm{T}}\boldsymbol{\omega} + \mathbf{h}_{tr} && (12)
\end{aligned}$$

The motion of the GMW gimbal is constrained to rotation about the gimbal axis, so there can be no motion of the GMW relative to the platform in the transverse direction, nor can the gimbal rotate relative to the platform about the spin axis. Therefore $\mathbf{h}_{tr} = \mathbf{h}_{sgr} = \mathbf{0}$

which implies that we can rewrite Equation (7) as

$$\mathbf{h} = (\mathbf{I} + \mathbf{A}_t\mathbf{I}_t\mathbf{A}_t^{\mathrm{T}} + \mathbf{A}_s\mathbf{I}_{sg}\mathbf{A}_s^{\mathrm{T}})\boldsymbol{\omega} + \mathbf{c}^{\times}\mathbf{v} + \mathbf{A}_s\mathbf{h}_{swa} + \mathbf{A}_g\mathbf{h}_{ga} \tag{13}$$

Note that the inertia-like matrix multiplying $\boldsymbol{\omega}$ in Equation (13) is not necessarily constant.

We now present the equations of motion for a rigid body with GMWs in terms of the system linear and angular momenta, as well as the angular momentum components of each GMW about the spin and gimbal axes. Because the orientation of the GMW is important (as it gives the direction of the spin momentum vector), an equation of motion is also required for the gimbal angle.

The dynamical equations are a $3N + 6$ set given by

$$\begin{aligned}
\dot{\mathbf{h}} &= \mathbf{h}^{\times}\boldsymbol{\omega} + \mathbf{p}^{\times}\mathbf{v} + \mathbf{g}_e \\
\dot{\mathbf{p}} &= \mathbf{p}^{\times}\boldsymbol{\omega} + \mathbf{f}_e \\
\dot{\mathbf{h}}_{swa} &= \mathbf{g}_w \\
\dot{\mathbf{h}}_{ga} &= ((\mathbf{I}_t - \mathbf{I}_{sg})\mathbf{A}_s{}^{\mathrm{T}}\boldsymbol{\omega} - \mathbf{h}_{swa}) \star (\mathbf{A}_t{}^{\mathrm{T}}\boldsymbol{\omega}) + \mathbf{g}_g \\
\dot{\boldsymbol{\delta}} &= \mathbf{I}_g^{-1}\mathbf{h}_{ga} - \mathbf{A}_g^{\mathrm{T}}\boldsymbol{\omega}
\end{aligned} \tag{14}$$

where $\mathbf{g}_e$, $\mathbf{f}_e$, $\mathbf{g}_w$, and $\mathbf{g}_g$ represent the external torque, external force, the spin axis torques, and the gimbal torques respectively. The $N \times 1$ column matrix $\boldsymbol{\delta}$ is formed from the GMW gimbal angles. The operator $\star$ represents term by term multiplication of the two adjacent $N \times 1$ column matrices. The $\star$ operation could be carried out alternatively as

$$\mathbf{u} \star \mathbf{v} = \mathrm{diag}(\mathbf{u})\mathbf{v} = \mathrm{diag}(\mathbf{v})\mathbf{u} \tag{15}$$

The system velocities and momenta are related in matrix form as

$$\begin{bmatrix} \mathbf{h} \\ \mathbf{p} \\ \mathbf{h}_{ga} \\ \mathbf{h}_{swa} \end{bmatrix} = \begin{bmatrix} \mathbf{J} & \mathbf{c}^{\times} & \mathbf{A}_g\mathbf{I}_g & \mathbf{A}_s\mathbf{I}_{sw} \\ -\mathbf{c}^{\times} & m\mathbf{1} & \mathbf{0}^{(3\times N)} & \mathbf{0}^{(3\times N)} \\ \mathbf{I}_g\mathbf{A}_g^{\mathrm{T}} & \mathbf{0}^{(N\times 3)} & \mathbf{I}_g & \mathbf{0}^{(N\times N)} \\ \mathbf{I}_{sw}\mathbf{A}_s^{\mathrm{T}} & \mathbf{0}^{(N\times 3)} & \mathbf{0}^{(N\times N)} & \mathbf{I}_{sw} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v} \\ \dot{\boldsymbol{\delta}} \\ \boldsymbol{\omega}_{swr} \end{bmatrix} = \boldsymbol{\Phi}_{gmw} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v} \\ \dot{\boldsymbol{\delta}} \\ \boldsymbol{\omega}_{swr} \end{bmatrix} \tag{16}$$

where

$$\mathbf{J} = \mathbf{I} + \mathbf{A}_t\mathbf{I}_t\mathbf{A}_t^{\mathrm{T}} + \mathbf{A}_s\mathbf{I}_s\mathbf{A}_s^{\mathrm{T}} + \mathbf{A}_g\mathbf{I}_g\mathbf{A}_g^{\mathrm{T}} \tag{17}$$

For the rigid spacecraft with GMWs, $\mathbf{J}$ represents the inertia of the entire spacecraft.

We define some new terms to aid in the numerical computation of the $3 \times N$ matrices $\mathbf{A}_s$ and $\mathbf{A}_t$. From the $N \times 1$ vector, $\boldsymbol{\delta}$, of gimbal angles, we compute the $N \times N$ matrices $\boldsymbol{\Delta}^{\mathrm{c}}$ and $\boldsymbol{\Delta}^{\mathrm{s}}$ where

$$\boldsymbol{\Delta}^{\mathrm{c}} = \mathrm{diag}(\cos \boldsymbol{\delta}) \tag{18}$$

$$\boldsymbol{\Delta}^{\mathrm{s}} = \mathrm{diag}(\sin \boldsymbol{\delta}) \tag{19}$$

and $\cos \boldsymbol{\delta}$ and $\sin \boldsymbol{\delta}$ are column matrices of the cosines and sines taken term by term of the column matrix $\boldsymbol{\delta}$. By defining the matrices $\mathbf{A}_{s0}$ and $\mathbf{A}_{t0}$ as the values of $\mathbf{A}_s$ and $\mathbf{A}_t$ when the gimbal angles are all zero, $\mathbf{A}_s$ and $\mathbf{A}_t$ may be written as functions of gimbal angles using the expressions

$$\mathbf{A}_s = \mathbf{A}_{s0}\boldsymbol{\Delta}^{\mathrm{c}} - \mathbf{A}_{t0}\boldsymbol{\Delta}^{\mathrm{s}} \tag{20}$$

$$\mathbf{A}_t = \mathbf{A}_{t0}\boldsymbol{\Delta}^{\mathrm{c}} + \mathbf{A}_{s0}\boldsymbol{\Delta}^{\mathrm{s}} \tag{21}$$

For single-gimbal GMWs, $\mathbf{A}_g = \mathbf{A}_{g0}$ is fixed, so $\dot{\mathbf{A}}_g = \mathbf{0}$. The rates of change of $\mathbf{A}_s$ and $\mathbf{A}_t$, however, can be shown to be

$$\dot{\mathbf{A}}_s = -\mathbf{A}_t \mathrm{diag}(\dot{\boldsymbol{\delta}}) \tag{22}$$

$$\dot{\mathbf{A}}_t = \mathbf{A}_s \mathrm{diag}(\dot{\boldsymbol{\delta}}) \tag{23}$$

The previous set of equations must be appended with a set of equations to describe the kinematics. We use quaternions, so that the four equations

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{G}(\mathbf{q})\boldsymbol{\omega} \tag{24}$$

where

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \tag{25}$$

are added to describe completely the dynamics and kinematics of a maneuver.

The addition of an Euler-Bernoulli appendage to the rigid spacecraft results in one new equation of motion. The new equation is a hybrid differential equation, and the continuous deflection of the appendage can be discretized by the assumed modes method. This results in the addition of a new set of differential equations, one for each appendage of the form

$$\dot{\mathbf{h}}_\nu = a_1\mathbf{m}_1 + a_2\mathbf{m}_2 + a_3\mathbf{M}_3\boldsymbol{\nu} - \mathbf{K}\boldsymbol{\nu} \tag{26}$$

$$\dot{\boldsymbol{\nu}} = \boldsymbol{\mu} \tag{27}$$

The generalized appendage momentum $\mathbf{h}_\nu$, and the modal displacement $\boldsymbol{\nu}$ are of dimension $m \times 1$ where $m$ is the number of assumed mode shapes. The variable $\boldsymbol{\mu}$ represents the modal velocity. The coefficients $a_1$, $a_2$, and $a_3$ above are functions of the appendage geometry and system velocities. The matrices $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{M}_3$, and $\mathbf{K}$ are constant matrices determined by the beam properties and the assumed mode shapes. The details of this development may be found in Ford.[11]

If we attach $p$ appendages to the flexible spacecraft model, then we have a system of $10 + 3N + 2mp$ total first order nonlinear differential equations describing the dynamics and kinematics of the system. Note that a spacecraft with 3 GMWs and 3 flexible appendages will yield a set of 25 differential equations if we keep only the first mode of oscillation. As with the rigid body case, it is possible to express the relationship between momenta and velocities as

$$\begin{bmatrix} \mathbf{h} \\ \mathbf{p} \\ \mathbf{h}_{ga} \\ \mathbf{h}_{swa} \\ \mathbf{h}_\nu \end{bmatrix} = \boldsymbol{\Phi}_{sys} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v} \\ \dot{\boldsymbol{\delta}} \\ \boldsymbol{\omega}_{swr} \\ \mu \end{bmatrix} \tag{28}$$

The coefficient matrix $\boldsymbol{\Phi}_{sys}$ is a function of $\boldsymbol{\delta}$ and $\boldsymbol{\nu}$. Equation (28) must be solved at each time step, since the velocities are needed in Equations (14), (24), (26), and (27).

The spacecraft with GMWs is a generalization of the spacecraft with momentum wheels, in which case the gimbal angles are always zero. Setting $\dot{\boldsymbol{\delta}} = \mathbf{0}$ leads to a set of $10 + N + 2mp$ equations. Similarly, for the CMG case, $\mathbf{h}_{swr}$ is constant, and the equations of motion reduce to a set of order $10 + 2N + 2mp$.

## SPACECRAFT REORIENTATIONS

We now turn to the task of controlling the reorientation of a spacecraft using momentum exchange devices. In general, the spacecraft may already possess significant angular momentum stored in the cluster of momentum storage devices. We assume that the spacecraft initially has no angular velocity with respect to inertial space, and our goal is to take up a new orientation with the spacecraft again at rest. The problem is therefore a "rest-to-rest" maneuver, with the initial and final orientations specified. With no external torques, the total angular momentum of the spacecraft/cluster system remains fixed in inertial space. At the final orientation, therefore, the cluster momentum has a unique orientation in the body coordinate frame, a property not shared by *zero-momentum* spacecraft.

We begin this section with a description of the requirements of a suitable control law and discuss the problems associated with reorienting a spacecraft containing momentum devices. We then describe a reorientation profile termed the *stationary platform maneuver*[6] which has the advantage of generally keeping angular velocities low during the reorientation. A Lyapunov control law developed by Oh and Vadali[8] is presented, with a momentum exchange cluster as the control device and where the control input is the rate of change of cluster momentum. This law is modified and used to reorient the spacecraft while remaining close to the stationary platform condition.

### Reorientations Using a Momentum Exchange Cluster

We begin by writing the equations of motion for a spacecraft containing a momentum exchange cluster in a general form. The total angular momentum of a spacecraft about its mass center is

$$\mathbf{h} = \mathbf{J}\boldsymbol{\omega} + \mathbf{h}_c \tag{29}$$

where $\mathbf{J}$ is an inertia-like matrix which will be defined in a suitable manner relevant to the momentum exchange device under consideration. We leave open the possibility of $\mathbf{J}$ being variable. The cluster momentum $\mathbf{h}_c$ includes the spin momentum of the GMWs relative to inertial space and the angular momentum associated with the GMW gimbal rate. For a

GMW cluster, for example, we add a subscript to $\mathbf{J}$ and $\mathbf{J}_{gmw}$ takes the form

$$\mathbf{J}_{gmw} = \mathbf{I} + \mathbf{A}_t \mathbf{I}_t \mathbf{A}_t^{\mathrm{T}} + \mathbf{A}_s \mathbf{I}_{sg} \mathbf{A}_s^{\mathrm{T}} \tag{30}$$

and the cluster momentum is

$$\mathbf{h}_c = \mathbf{A}_s \mathbf{h}_{swa} + \mathbf{A}_g \mathbf{h}_{ga} \tag{31}$$

Assuming no external torques, the equation of motion for the spacecraft represented by Equation (29) is

$$\dot{\boldsymbol{\omega}} = \mathbf{J}^{-1} \left[ -\boldsymbol{\omega}^{\times} (\mathbf{J}\boldsymbol{\omega} + \mathbf{h}_c) - \dot{\mathbf{J}}\boldsymbol{\omega} - \dot{\mathbf{h}}_c \right] \tag{32}$$

where $\dot{\mathbf{J}}$ is determined by the gimbal rates. Note that $\dot{\mathbf{J}}$ is zero for the momentum wheel case. The reorientation profile can be controlled as desired through the input term $\mathbf{J}^{-1}\dot{\mathbf{h}}_c$.

In the general case of the GMW, we have

$$\begin{aligned} \dot{\mathbf{h}}_c &= \dot{\mathbf{A}}_s \mathbf{h}_{swa} + \mathbf{A}_s \dot{\mathbf{h}}_{swa} + \mathbf{A}_g \dot{\mathbf{h}}_{ga} \\ &= -\mathbf{A}_t \mathrm{diag}(\dot{\boldsymbol{\delta}}) \mathbf{h}_{swa} + \mathbf{A}_s \dot{\mathbf{h}}_{swa} + \mathbf{A}_g \dot{\mathbf{h}}_{ga} \end{aligned} \tag{33}$$

Keep in mind that we have control over the last two terms of Equation (33) using the torque inputs $\mathbf{g}_w$ and $\mathbf{g}_g$ (see Equations (14)). Note, however, that the first term is dependent on the gimbal rates, which are *states* of the system. This term can dominate for the case of large $\mathbf{h}_{swa}$, and is the term which provides the so-called torque amplification property of the single gimbal CMG.

For the momentum wheel case, the gimbals are fixed ($\dot{\boldsymbol{\delta}} = \mathbf{0}$) and

$$\dot{\mathbf{h}}_{ga} = \dot{\mathbf{h}}_{gr} - \mathbf{I}_g \mathbf{A}_g^{\mathrm{T}} \dot{\boldsymbol{\omega}} = -\mathbf{I}_g \mathbf{A}_g^{\mathrm{T}} \dot{\boldsymbol{\omega}} \tag{34}$$

so that it is convenient to lump the gimbal axis inertia into $\mathbf{J}$ and define

$$\mathbf{J}_{mw} = \mathbf{I} + \mathbf{A}_t \mathbf{I}_t \mathbf{A}_t^{\mathrm{T}} + \mathbf{A}_s \mathbf{I}_{sg} \mathbf{A}_s^{\mathrm{T}} + \mathbf{A}_g \mathbf{I}_g \mathbf{A}_g^{\mathrm{T}} \tag{35}$$

so that

$$\dot{\mathbf{h}}_c = \mathbf{A}_s \dot{\mathbf{h}}_{swa} = \mathbf{A}_{s0} \mathbf{g}_w \tag{36}$$

Similarly, for the CMG, recall that

$$\mathbf{h}_{swa} = \mathbf{h}_{swr} + \mathbf{I}_{sw}\mathbf{A}_s^{\mathrm{T}}\boldsymbol{\omega} \tag{37}$$

so defining

$$\mathbf{J}_{cmg} = \mathbf{I} + \mathbf{A}_t\mathbf{I}_t\mathbf{A}_t^{\mathrm{T}} + \mathbf{A}_s\mathbf{I}_s\mathbf{A}_s^{\mathrm{T}} + \mathbf{A}_g\mathbf{I}_g\mathbf{A}_g^{\mathrm{T}} \tag{38}$$

then

$$\begin{aligned} \dot{\mathbf{h}}_c &= -\mathbf{A}_t \mathrm{diag}(\dot{\boldsymbol{\delta}})\mathbf{h}_{swr} + \mathbf{A}_g\dot{\mathbf{h}}_{gr} \\ &= -\mathbf{A}_t \mathrm{diag}(\mathbf{h}_{swr})\dot{\boldsymbol{\delta}} + \mathbf{A}_g\mathbf{I}_g\ddot{\boldsymbol{\delta}} \end{aligned} \tag{39}$$

This demonstrates that the torque generated by a cluster of CMGs has a component about the gimbal axes (due to gimbal acceleration) and a transverse torque output due to gimbal rates. The torque input $\mathbf{g}_g$ can be computed given the desired $\ddot{\boldsymbol{\delta}}$.

## A Lyapunov Feedback Control Law

In this section, we develop the stabilizing Lyapunov control law due to Oh and Vadali,[8] but using the variables of the current development. We define a positive definite Lyapunov function

$$V = k_0(\mathbf{q} - \mathbf{q}_f)^{\mathrm{T}}(\mathbf{q} - \mathbf{q}_f) + \frac{1}{2}(\boldsymbol{\omega} - \boldsymbol{\omega}_f)^{\mathrm{T}}\mathbf{J}(\boldsymbol{\omega} - \boldsymbol{\omega}_f) \tag{40}$$

where $k_0$ is a positive scalar constant, $\mathbf{q}$ is the $4 \times 1$ quaternion vector and $\boldsymbol{\omega}$ is the $3 \times 1$ angular velocity vector. The constant terms $\mathbf{q}_f$ and $\boldsymbol{\omega}_f$ are the desired final values for the reorientation maneuver. Though we intend to focus on rest-to-rest to maneuvers, we carry $\boldsymbol{\omega}_f$ through the development for completeness.

The derivative of the Lyapunov function is

$$\dot{V} = -(\boldsymbol{\omega} - \boldsymbol{\omega}_f)^{\mathrm{T}}\left(-k_0\mathbf{G}^{\mathrm{T}}(\mathbf{q}_f)\mathbf{q} + \mathbf{J}\dot{\boldsymbol{\omega}}_f - \mathbf{J}\dot{\boldsymbol{\omega}} + \frac{1}{2}\dot{\mathbf{J}}\boldsymbol{\omega}_f - \frac{1}{2}\dot{\mathbf{J}}\boldsymbol{\omega}\right) \tag{41}$$

We see that $\dot{V}$ can be guaranteed negative semidefinite when we ensure that

$$-k_0\mathbf{G}^{\mathrm{T}}(\mathbf{q}_f)\mathbf{q} + \mathbf{J}\dot{\boldsymbol{\omega}}_f - \mathbf{J}\dot{\boldsymbol{\omega}} - \frac{1}{2}\dot{\mathbf{J}}\boldsymbol{\omega} + \frac{1}{2}\dot{\mathbf{J}}\boldsymbol{\omega}_f = \mathbf{K}_1(\boldsymbol{\omega} - \boldsymbol{\omega}_f) \tag{42}$$

where $\mathbf{K}_1$ is a positive definite gain matrix. Substitution of Equation (32) leads to the relationship

$$\dot{\mathbf{h}}_c = \mathbf{K}_1(\boldsymbol{\omega} - \boldsymbol{\omega}_f) + k_0\mathbf{G}^{\mathrm{T}}(\mathbf{q}_f)\mathbf{q} - \mathbf{J}\dot{\boldsymbol{\omega}}_f - \boldsymbol{\omega}^{\times}(\mathbf{J}\boldsymbol{\omega} + \mathbf{h}_c) - \frac{1}{2}\dot{\mathbf{J}}(\boldsymbol{\omega} + \boldsymbol{\omega}_f) \tag{43}$$

In general, the magnitude of $k_0$ determines the speed of the reorientation, since it directly multiplies the quaternion error in deciding the magnitude of the required torque. It should be apparent from Equation (43) that choosing $k_0 = 0$ and $\mathbf{K}_1 \neq \mathbf{0}$ will simply drive $\boldsymbol{\omega} \rightarrow \mathbf{0}$ without any consideration for the quaternion error. Conversely, choosing $\mathbf{K}_1 = \mathbf{0}$ and $k_0 \neq 0$ will drive the system to the final quaternions without consideration of the desired target angular velocity, $\boldsymbol{\omega}_f$, with a high probability of an overshoot. The constant $\mathbf{K}_1$ should therefore be selected in consideration of the magnitude of $k_0$. It can be shown that the system is critically damped in the linear range (near $\mathbf{q} = \mathbf{q}_f$) when the diagonals of $\mathbf{K}_1$ are chosen as[8]

$$K_{1_{ii}} = \sqrt{2I_{ii}k_0} \tag{44}$$

It is possible to show this system is globally asymptotically stable when $\dot{\mathbf{h}}_c$ is chosen as in Equation (43).

## Control Using Cluster Devices

For a rigid spacecraft with momentum wheels, then $\dot{\mathbf{J}}_{mw} = \mathbf{0}$, and assuming we wish to accomplish a reorientation for which $\boldsymbol{\omega}_f = \mathbf{0}$ and $\dot{\boldsymbol{\omega}}_f = \mathbf{0}$, then $\dot{V} \leq 0$ when

$$\dot{\mathbf{h}}_c = \mathbf{K}_1\boldsymbol{\omega} + k_0\mathbf{G}^{\mathrm{T}}(\mathbf{q}_f)\mathbf{q} - \boldsymbol{\omega}^{\times}(\mathbf{J}\boldsymbol{\omega} + \mathbf{h}_c) \tag{45}$$

Equation (36) provides the relation for choosing the momentum wheel spin axis torques, $\mathbf{g}_w$. For the case of three non-coplanar momentum wheels ($\mathbf{A}_{s0}$ non-singular), a unique solution for $\mathbf{g}_w$ exists. For a redundant set of momentum wheels, an infinite number of solutions exists. It is common to choose the minimum norm solution for the torques given by

$$\mathbf{g}_w = \mathbf{A}_s^{\mathrm{T}}(\mathbf{A}_s\mathbf{A}_s^{\mathrm{T}})^{-1}\dot{\mathbf{h}}_c \tag{46}$$

Unfortunately, the case of the CMG cluster is not quite so simple. With changing gimbal angles, $\mathbf{J}_{cmg}$ is not constant, and even after computing an $\dot{\mathbf{h}}_c$ to stabilize the system, Equation (39) causes complications in choosing the control input. We again assume that $\boldsymbol{\omega}_f = \mathbf{0}$ and $\dot{\boldsymbol{\omega}}_f = \mathbf{0}$. We rearrange Equation (43) slightly to the form

$$\dot{\mathbf{h}}_c + \frac{1}{2}\dot{\mathbf{J}}_{cmg}\boldsymbol{\omega} = \mathbf{K}_1\boldsymbol{\omega} + k_0\mathbf{G}^{\mathrm{T}}(\mathbf{q}_f)\mathbf{q} - \boldsymbol{\omega}^{\times}(\mathbf{J}_{cmg}\boldsymbol{\omega} + \mathbf{h}_c) \tag{47}$$

so that all terms which are a function of gimbal rates and accelerations are on the left hand side. Then Equation (47) may be written as

$$\mathbf{B}\ddot{\boldsymbol{\delta}} + \mathbf{D}\dot{\boldsymbol{\delta}} = \mathbf{K}_1\boldsymbol{\omega} + k_0\mathbf{G}^{\mathrm{T}}(\mathbf{q}_f)\mathbf{q} - \boldsymbol{\omega}^{\times}(\mathbf{J}_{cmg}\boldsymbol{\omega} + \mathbf{h}_c) \tag{48}$$

where the matrix coefficient $\mathbf{B}$ is simply (from Equation (39))

$$\mathbf{B} = \mathbf{A}_g\mathbf{I}_g \tag{49}$$

After some considerable algebraic manipulations, we find

$$\begin{aligned} \mathbf{D} \;=\; & -\mathbf{A}_t\mathrm{diag}(\mathbf{h}_{swr}) \\ & +\frac{1}{2}\Big[\; (\mathbf{a}_{s1}\mathbf{a}_{t1}^{\mathrm{T}} + \mathbf{a}_{t1}\mathbf{a}_{s1}^{\mathrm{T}})\boldsymbol{\omega} \quad (\mathbf{a}_{s2}\mathbf{a}_{t2}^{\mathrm{T}} + \mathbf{a}_{t2}\mathbf{a}_{s2}^{\mathrm{T}})\boldsymbol{\omega} \quad \cdots \\ & \quad \cdots \quad (\mathbf{a}_{sN}\mathbf{a}_{tN}^{\mathrm{T}} + \mathbf{a}_{tN}\mathbf{a}_{sN}^{\mathrm{T}})\boldsymbol{\omega} \;\Big](\mathbf{I}_t - \mathbf{I}_s) \end{aligned} \tag{50}$$

In general, because $\mathbf{h}_{swr}$ is large, the contributions to $\mathbf{D}$ from the first term of Equation (50) are of much greater magnitude than those of the second term (especially for small $\boldsymbol{\omega}$ or when $\mathbf{I}_t \approx \mathbf{I}_s$).

The system will be stable when Equation (48) is satisfied. Of course, $\dot{\boldsymbol{\delta}}$ is actually determined by the integration of $\ddot{\boldsymbol{\delta}}$ and is therefore a state of the system. The advantage of single gimbal control moment gyros, however, lies in the torque amplification resulting from the gimbal rates. Normally, we seek to drive the gimbals so that

$$\mathbf{D}\dot{\boldsymbol{\delta}} = \mathbf{K}_1\boldsymbol{\omega} + k_0\mathbf{G}^{\mathrm{T}}(\mathbf{q}_f)\mathbf{q} - \boldsymbol{\omega}^{\times}(\mathbf{J}_{cmg}\boldsymbol{\omega} + \mathbf{h}_c) \tag{51}$$

as closely as possible, while simply tolerating the resulting torque generated by the gimbal accelerations. A solution to Equation (51) gives the desired gimbal rates, which we denote as $\dot{\boldsymbol{\delta}}_{des}$. Oh and Vadali showed that by choosing

$$\ddot{\boldsymbol{\delta}} = \mathbf{K}_\delta(\dot{\boldsymbol{\delta}}_{des} - \dot{\boldsymbol{\delta}}) \tag{52}$$

where $\mathbf{K}_\delta$ is positive definite, we can keep $\dot{\boldsymbol{\delta}}$ close to that required to satisfy Equation (51), ensuring that we are taking advantage of the torques generated from the gimbal rates.

As in the case of momentum wheels, a redundant set of CMGs provides us with an underdetermined problem. We then seek a solution to the problem

$$\dot{\boldsymbol{\delta}}_{des} = \mathbf{D}^\dagger \dot{\mathbf{h}}_c \tag{53}$$

The superscript $\dagger$ is used in Equation (52) to indicate that when $\mathbf{D}$ is not invertible then a suitable pseudoinverse should be used. This occurs when $\mathbf{D}$ is not square, but may also occur when the gimbals are in a singular configuration. The singularity problem is not addressed in this paper, but gets considerable attention in Oh and Vadali[8] as well as in Ford and Hall.[12]

## THE STATIONARY PLATFORM CONDITION

*Stationary platform maneuvers* (SPMs) are a class of gyrostat maneuvers investigated by Hall.[6] A spacecraft in which all of the angular momentum is contained in the momentum exchange devices will have zero angular velocity. A necessary (but not sufficient) condition for this state is that the momentum cluster have an angular momentum magnitude equal to the total system angular momentum magnitude. By controlling the rotors of a gyrostat in such a way that the maneuver remains near a branch of equilibrium motions for which the platform angular velocity is zero, then the angular velocity of the platform will remain small throughout the maneuver provided the rotor torques are small.

The low angular velocity resulting from execution of the SPM condition for a spacecraft with momentum wheels naturally leads to the question of its utility in the reorientation of flexible spacecraft. High angular velocities during a rest-to-rest maneuver imply high angular accelerations, which naturally tend to excite oscillations of any flexible appendages.

Existence of the SPM for a spacecraft with momentum wheels also raises the question of its existence for a spacecraft with control moment gyros. While following a stationary platform path with momentum wheels translates into following a hyper-ellipsoid in the space of wheel speeds, translation of the stationary platform condition into gimbal angle space for the CMG is considerably more complicated, as illustrated below.

## Stationary Condition for the GMW

We start by investigating the case where all of the angular momentum is contained in the momentum storage devices. That is, when $\boldsymbol{\omega} = \mathbf{0}$ in Equation (13). We also assume for this development that the linear momentum, $\mathbf{p}$, is constant, and without loss of generality equal to zero. This yields the stationary platform condition given by

$$\mathbf{h} = \mathbf{A}_s \mathbf{h}_{swa} + \mathbf{A}_g \mathbf{h}_{ga}$$

All combinations of $\mathbf{h}_{swa}$ and $\mathbf{h}_{ga}$ which maintain a constant $\mathbf{h}$ will maintain a stationary platform. Note, however, that if $\mathbf{h}_{ga} \neq \mathbf{0}$, the gimbal rates are nonzero which implies $\mathbf{A}_s$ is changing. This, in turn, requires varying spin axis torques to maintain a fixed $\mathbf{h}$. Explicitly, we must have

$$\dot{\mathbf{h}} = \mathbf{A}_s \dot{\mathbf{h}}_{swa} + \dot{\mathbf{A}}_s \mathbf{h}_{swa} + \mathbf{A}_g \dot{\mathbf{h}}_{ga} = \mathbf{0} \tag{54}$$

Since $\boldsymbol{\omega} = \mathbf{0}$, then Equation (54) becomes

$$\mathbf{A}_s \mathbf{g}_w - \mathbf{A}_t \mathrm{diag}\dot{\boldsymbol{\delta}} \mathbf{h}_{swa} + \mathbf{A}_g \mathbf{g}_g = \mathbf{0} \tag{55}$$

which might be useful in the case where we desire to reorient the gimbals and wish to supply spin axis torques to maintain zero angular velocity.

We are most interested in the cases where $\mathbf{h}_{ga}$ is zero (the gimbal rates are zero). In this case, we need only satisfy the condition

$$\mathbf{h} = \mathbf{A}_s \mathbf{h}_{swa} \tag{56}$$

Using the expression for $\mathbf{A}_s$ in terms of the gimbal angles, then for a stationary platform with zero gimbal rates the total angular momentum must be

$$\mathbf{h} = [\mathbf{A}_{s0}\boldsymbol{\Delta}^c - \mathbf{A}_{t0}\boldsymbol{\Delta}^s]\mathbf{h}_{swa} \tag{57}$$

Equation (57) can be put in an alternate form by defining a new $N \times N$ matrix to be

$$\mathbf{H}_{swa} = \mathrm{diag}(\mathbf{h}_{swa}) \tag{58}$$

so that we also have

$$\mathbf{h} = \mathbf{A}_{s0}\mathbf{H}_{swa}\cos\boldsymbol{\delta} - \mathbf{A}_{t0}\mathbf{H}_{swa}\sin\boldsymbol{\delta} \tag{59}$$

With multiple GMWs in the spacecraft, it might prove beneficial to exchange the momentum contributed by the individual GMWs with each other while maintaining the angular momentum of the system. We return to Equation (57) which gives $\mathbf{h}$ as a function of the vector $\mathbf{h}_{swa}$ and the gimbal angles $\boldsymbol{\delta}$. Taking the partial derivative of $\mathbf{h}$ with respect to the vector of spin axis momenta, we have

$$\frac{\partial \mathbf{h}}{\partial \mathbf{h}_{swa}} = \mathbf{A}_{s0}\boldsymbol{\Delta}^c - \mathbf{A}_{t0}\boldsymbol{\Delta}^s \tag{60}$$

An expression for $\mathbf{h}$ in alternate form is Equation (59). The partial derivative of $\mathbf{h}$ with respect to the gimbal angles is therefore given by

$$\frac{\partial \mathbf{h}}{\partial \boldsymbol{\delta}} = -\mathbf{A}_{s0}\mathbf{H}_{swa}\boldsymbol{\Delta}^s - \mathbf{A}_{t0}\mathbf{H}_{swa}\boldsymbol{\Delta}^c$$

which can also be written as

$$\frac{\partial \mathbf{h}}{\partial \boldsymbol{\delta}} = -(\mathbf{A}_{s0}\boldsymbol{\Delta}^s + \mathbf{A}_{t0}\boldsymbol{\Delta}^c)\mathbf{H}_{swa} \tag{61}$$

A differential change in the body angular momentum vector then, can be expressed as

$$d\mathbf{h} = \frac{\partial \mathbf{h}}{\partial \mathbf{h}_{swa}} d\mathbf{h}_{swa} + \frac{\partial \mathbf{h}}{\partial \boldsymbol{\delta}} d\boldsymbol{\delta} \tag{62}$$

so that to maintain a constant angular momentum vector in body coordinates, any incremental change in $\boldsymbol{\delta}$ should be accompanied by an appropriate change in $\mathbf{h}_{swa}$ (or vice versa) to ensure $d\mathbf{h} = \mathbf{0}$. That is, we constrain the variation through the equation

$$(\mathbf{A}_{s0}\mathbf{\Delta}^{c} - \mathbf{A}_{t0}\mathbf{\Delta}^{s})d\mathbf{h}_{swa} = (\mathbf{A}_{s0}\mathbf{\Delta}^{s} + \mathbf{A}_{t0}\mathbf{\Delta}^{c})\mathbf{H}_{swa}d\boldsymbol{\delta} \tag{63}$$

If the coefficient matrices are square (and nonsingular), then Equation (63) has a unique solution. If not, then either a least squares or minimum norm solution could be used. Note also that the coefficient matrices $(\partial\mathbf{h}/\partial\mathbf{h}_{swa})$ and $(\partial\mathbf{h}/\partial\boldsymbol{\delta})$ are $3 \times N$, which means they have a nullspace of dimension at least $N-3$. Variation of either $\mathbf{h}_{swa}$ or $\boldsymbol{\delta}$ in the direction of the nullspace of the corresponding coefficient matrix will also assure no variation in $\mathbf{h}$.

When Equation (57) is satisfied, the platform will be stationary. We are interested, however, in reorientation of the body. Note that the value of $\mathbf{h}$ is free to vary as the body rotates, since $\mathbf{h}$ is the angular momentum expressed in body coordinates. The magnitude of $\mathbf{h}$, however, is fixed if we assume no external torques are acting on the system. That is,

$$\mathbf{h}^{\mathrm{T}}\mathbf{h} = h^2 = \text{constant} \tag{64}$$

when the external torque is zero. We define the *stationary platform condition* as the condition which exists when the magnitude of the cluster momentum equals the magnitude of the total momentum. That is

$$\mathbf{h}_c^{\mathrm{T}}\mathbf{h}_c = h^2 = \text{constant} \tag{65}$$

Hall showed that, in the case of a gyrostat, maintaining the stationary platform condition during a slow maneuver results in a reorientation which keeps the angular velocity low.

## Stationary Condition for the Momentum Wheel

In the case of a cluster of momentum wheels, then we simply consider the restriction of GMWs to fixed gimbal axes. The condition of Equation (65) implies that

$$h^2 = \text{constant} = \mathbf{h}_{swa}^{\mathrm{T}}\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{s0}\mathbf{h}_{swa} \tag{66}$$

For a spacecraft containing $N$ momentum wheels, Equation (66) describes a hyper-ellipsoid of dimension $N$ in the space spanned by $(h_{swa1}, 0, \ldots, 0)$,$(0, h_{swa2}, \ldots, 0)$,. . .,$(0, \ldots, 0, h_{swaN})$, which we refer to as $\mathbf{h}_{swa}$-space. A requirement for a stationary platform is that the wheel momenta lie on the hyper-ellipsoid in $\mathbf{h}_{swa}$-space. We accomplish this by noting that the differentiation of Equation (66) results in the dynamical constraint

$$\mathbf{h}_{swa}^{\mathrm{T}}\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{s0}\dot{\mathbf{h}}_{swa} = \mathbf{h}_{swa}^{\mathrm{T}}\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{s0}\mathbf{g}_{w} = 0 \tag{67}$$

Of course, Equation (67) is satisfied when the wheel torques lie in the null space of $\mathbf{A}_{s0}$, but such "null motion" torques cause no motion. Thus, the applied torques must satisfy

$$(\mathbf{A}_{s0}\mathbf{g}_{w})^{\mathrm{T}}(\mathbf{A}_{s0}\mathbf{h}_{swa}) = 0 \tag{68}$$

## Stationary Condition for the Control Moment Gyro

If we assume all of the GMWs have the same spin momentum, then

$$\mathbf{H}_{swa} = h_{swa}\mathbf{1} \tag{69}$$

where $\mathbf{1}$ is the $N \times N$ identity matrix. Also, with $\boldsymbol{\omega} = \mathbf{0}$, then the rotor relative angular momenta are identical to the absolute momenta. This means the stationary platform angular momentum vector must be

$$\mathbf{h} = h_{swr}(\mathbf{A}_{s0}\cos\boldsymbol{\delta} - \mathbf{A}_{t0}\sin\boldsymbol{\delta}) \tag{70}$$

To satisfy the stationary platform condition, we need

$$\cos\boldsymbol{\delta}^{\mathrm{T}}\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{s0}\cos\boldsymbol{\delta} + \sin\boldsymbol{\delta}^{\mathrm{T}}\mathbf{A}_{t0}^{\mathrm{T}}\mathbf{A}_{t0}\sin\boldsymbol{\delta} - 2\cos\boldsymbol{\delta}^{\mathrm{T}}\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{t0}\sin\boldsymbol{\delta} = (h/h_{swr})^2 = \text{constant} \tag{71}$$

where we have used the fact that the transpose of a scalar is itself. We will refer to the satisfaction of Equation (71) as the *CMG stationary platform condition* and the set of all $\boldsymbol{\delta}$ for which it is satisfied as the *stationary platform surface*, a surface which resides in $\boldsymbol{\delta}$-space.

We define the function $F$ to be

$$F \equiv \frac{1}{2}\cos\boldsymbol{\delta}^{\mathrm{T}}\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{s0}\cos\boldsymbol{\delta} + \frac{1}{2}\sin\boldsymbol{\delta}^{\mathrm{T}}\mathbf{A}_{t0}^{\mathrm{T}}\mathbf{A}_{t0}\sin\boldsymbol{\delta} - \cos\boldsymbol{\delta}^{\mathrm{T}}\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{t0}\sin\boldsymbol{\delta} \tag{72}$$

A depiction of the three-dimensional surface described by Equation (72) when $\mathbf{A}_{s0}$ is the $3 \times 3$ identity matrix ($F = 1/2$) is in Figure 2.

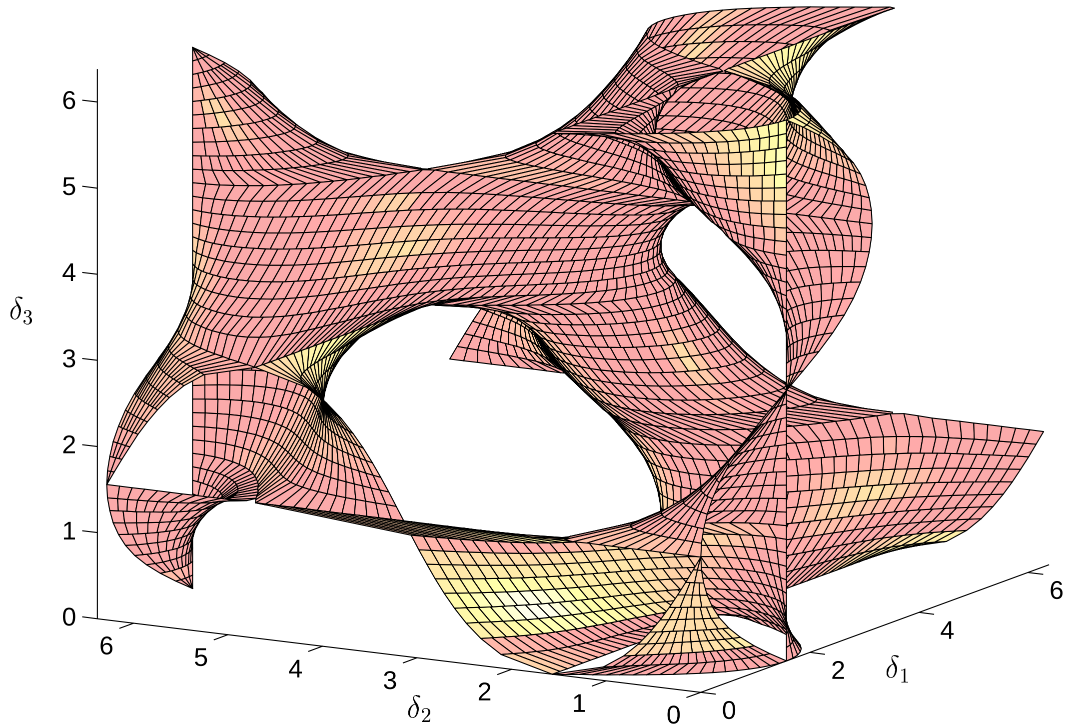


Figure 2: A CMG Stationary Platform Surface ($F = 1/2$)

Note that $F$ is a function of only the gimbal angles. A differential change in $F$ is strictly a result of a differential change in the gimbal angles. Thus

$$dF = \left(\frac{dF}{d\boldsymbol{\delta}}\right) d\boldsymbol{\delta} \tag{73}$$

Differential movement of the gimbals in the null space of $(dF/d\boldsymbol{\delta})$ will therefore maintain the stationary platform condition. Maintaining the gimbal velocity $\dot{\boldsymbol{\delta}}$ in the null space of $(dF/d\boldsymbol{\delta})$ will accomplish the same. We also note that

$$\dot{F} = \left(\frac{dF}{d\boldsymbol{\delta}}\right) \dot{\boldsymbol{\delta}} \tag{74}$$

If we assume it is possible to control the gimbal angle rates directly, then the stationary platform condition is maintained by ensuring $\dot{F} = 0$.

Differentiation of the function $F$ yields the matrix expression

$$\left(\frac{dF}{d\boldsymbol{\delta}}\right) = -\cos(\boldsymbol{\delta}^{\mathrm{T}})\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{s0}\boldsymbol{\Delta}^{s} + \sin(\boldsymbol{\delta}^{\mathrm{T}})\mathbf{A}_{t0}^{\mathrm{T}}\mathbf{A}_{t0}\boldsymbol{\Delta}^{c} - \cos(\boldsymbol{\delta}^{\mathrm{T}})\mathbf{A}_{s0}^{\mathrm{T}}\mathbf{A}_{t0}\boldsymbol{\Delta}^{c} + \sin(\boldsymbol{\delta}^{\mathrm{T}})\mathbf{A}_{t0}^{\mathrm{T}}\mathbf{A}_{s0}\boldsymbol{\Delta}^{s} \tag{75}$$

which is a $1 \times N$ matrix with a null space of dimension $N-1$. For a cluster of 3 control moment gyros, for example, the stationary platform can be maintained by remaining on the two-dimensional stationary platform surface.

In reality, gimbal rates are integrals of gimbal accelerations and cannot be controlled directly. Assuming we have the capability to provide a desired torque to the gimbal axis, however, gimbal accelerations, $\ddot{\boldsymbol{\delta}}$, can be controlled precisely. We ask then what the gimbal accelerations must be to maintain the SPC.

Differentiation of Equation (74) produces

$$\ddot{F} = \left(\frac{d\dot{F}}{d\boldsymbol{\delta}}\right)\dot{\boldsymbol{\delta}} + \left(\frac{dF}{d\boldsymbol{\delta}}\right)\ddot{\boldsymbol{\delta}} \tag{76}$$

so that if we desire to hold $\ddot{F} = 0$, the gimbal accelerations may be computed by choosing gimbal accelerations such that

$$\left(\frac{dF}{d\boldsymbol{\delta}}\right)\ddot{\boldsymbol{\delta}} = -\left(\frac{d\dot{F}}{d\boldsymbol{\delta}}\right)\dot{\boldsymbol{\delta}} \tag{77}$$

is satisfied. The coefficient of $\dot{\boldsymbol{\delta}}$ in Equation (77) can be computed as

$$\begin{aligned}\left(\frac{d\dot{F}}{d\boldsymbol{\delta}}\right) &= \dot{\boldsymbol{\delta}}^{\mathrm{T}}(\boldsymbol{\Delta}^c\mathbf{A}_{t0} - \boldsymbol{\Delta}^s\mathbf{A}_{s0})(\mathbf{A}_{s0}\boldsymbol{\Delta}^s + \mathbf{A}_{t0}\boldsymbol{\Delta}^c) \\ &\quad +(\cos\boldsymbol{\delta}^{\mathrm{T}}\mathbf{A}_{s0} + \sin\boldsymbol{\delta}^{\mathrm{T}}\mathbf{A}_{t0})(\mathbf{A}_{s0}\boldsymbol{\Delta}^c - \mathbf{A}_{t0}\boldsymbol{\Delta}^s)\mathrm{diag}\dot{\boldsymbol{\delta}}\end{aligned} \tag{78}$$

A minimum norm solution or other technique could be used to solve Equation (77), yielding gimbal accelerations which would maintain the stationary platform condition.

It is conceivable that in some instances we may be faced with a system state in which the stationary platform condition has been violated. That is

$$F \neq \frac{1}{2}\left(\frac{h}{h_{swr}}\right)^2$$

or else the rate of change of $F$ as expressed by Equation (74) is not zero, implying that the trajectory departs the stationary platform surface. To return to the stationary surface, we observe that by choosing $\ddot{\boldsymbol{\delta}}$ so that the second order equation

$$\ddot{F} + 2\zeta\omega_n\dot{F} + \omega_n^2\Delta F = 0 \tag{79}$$

where

$$\Delta F = F - \frac{1}{2}\left(\frac{h}{h_{swr}}\right)^2 \tag{80}$$

has damped roots, we drive the stationary platform error exponentially back to zero. The damping and natural frequency may be selected as desired. Specifically, choosing a $\ddot{\boldsymbol{\delta}}$ which satisfies

$$\left(\frac{\partial F}{\partial \boldsymbol{\delta}}\right)\ddot{\boldsymbol{\delta}} = -\left(\left(\dot{\overline{\frac{\partial F}{\partial \boldsymbol{\delta}}}}\right) + 2\zeta\boldsymbol{\omega}_n\left(\frac{\partial F}{\partial \boldsymbol{\delta}}\right)\right)\dot{\boldsymbol{\delta}} - \omega_n^2\left(F - \frac{1}{2}\left(\frac{h}{h_{swr}}\right)^2\right) \tag{81}$$

will drive $\Delta F \to 0$, and return the cluster to the stationary platform condition should a deviation exist.

## Lyapunov Control with Stationary Platform Weighting

We now attempt to enforce the stationary platform condition while applying a Lyapunov control law. Since $\dot{\mathbf{h}}_c$ may contain components in the direction of $\mathbf{h}_c$, we seek to drive $\mathbf{h}_c$ back to its original magnitude whenever a deviation is present. A Lyapunov function based on the error in the magnitude of $\mathbf{h}_c$ (deviation from the magnitude of $\mathbf{h}$) is

$$V_h = \frac{1}{4}(\mathbf{h}_c^{\mathrm{T}}\mathbf{h}_c - \mathbf{h}^{\mathrm{T}}\mathbf{h})^2 \tag{82}$$

Note $V_h = 0$ only when the magnitude of $\mathbf{h}_c$ is equal to the magnitude of $\mathbf{h}$. The derivative of $V_h$ is

$$\dot{V}_h = (\mathbf{h}_c^{\mathrm{T}}\mathbf{h}_c - \mathbf{h}^{\mathrm{T}}\mathbf{h})\mathbf{h}_c^{\mathrm{T}}\dot{\mathbf{h}}_c \tag{83}$$

so that we can ensure $\dot{V}_h \leq 0$ by choosing

$$\mathbf{h}_c^{\mathrm{T}}\dot{\mathbf{h}}_c = k_2(\mathbf{h}^{\mathrm{T}}\mathbf{h} - \mathbf{h}_c^{\mathrm{T}}\mathbf{h}_c), \quad k_2 \geq 0 \tag{84}$$

The minimum norm solution of Equation (84) is

$$\dot{\mathbf{h}}_c = k_2(\mathbf{h}^{\mathrm{T}}\mathbf{h} - \mathbf{h}_c^{\mathrm{T}}\mathbf{h}_c)\frac{\mathbf{h}_c}{\mathbf{h}_c^{\mathrm{T}}\mathbf{h}_c} \tag{85}$$

Now consider the ramifications of adding (85) to the control law of Equation (43) so that

$$\begin{aligned} \dot{\mathbf{h}}_c \quad &= \quad \mathbf{K}_1(\boldsymbol{\omega} - \boldsymbol{\omega}_f) + k_0\mathbf{G}^{\mathrm{T}}(\mathbf{q}_f)\mathbf{q} - \mathbf{J}\dot{\boldsymbol{\omega}}_f - \boldsymbol{\omega}^{\times}(\mathbf{J}\boldsymbol{\omega} + \mathbf{h}_c) \\ &\quad -\frac{1}{2}\dot{\mathbf{J}}(\boldsymbol{\omega} + \boldsymbol{\omega}_f) + k_2(\mathbf{h}^{\mathrm{T}}\mathbf{h} - \mathbf{h}_c^{\mathrm{T}}\mathbf{h}_c)\frac{\mathbf{h}_c}{\mathbf{h}_c^{\mathrm{T}}\mathbf{h}_c} \end{aligned} \tag{86}$$

A sufficient condition to ensure that Equation (86) still guarantees stability of the closed loop system is that[11]

$$\|\mathbf{J}\boldsymbol{\omega}\| \leq 2\|\mathbf{h}_c\| \tag{87}$$

For low angular velocities or large cluster momentum, Equation (87) will be satisfied.

The control law of Equation (86) does allow for deviation from the SPC to satisfy the kinematics, but also attempts to return the vector $\mathbf{h}_c$ to the stationary platform value when a deviation does occur. The aggressiveness of the system in attempting to maintain the SPC is determined by $k_2$. When $k_2 = 0$, the controller will make no attempt to maintain the SPC.

We make one additional observation. Since it is only required that $k_2 \geq 0$, it is permissible to allow it to vary. For example, using a variable $k_2$ such that

$$k_2 = \tilde{k}_2(1 - q_0) \tag{88}$$

ensures that $k_2 \geq 0$ and relaxes the stationary platform constraint just as the body arrives at the destination quaternion (assuming $\mathbf{q}_f = [\ 1 \quad 0 \quad 0 \quad 0\ ]^{\mathrm{T}}$).

We now demonstrate these control law modifications with two numerical examples.

## MANEUVER EXAMPLES

Reorientation maneuvers could be qualitatively compared by examining the histories of the quaternions and angular velocities. For a flexible satellite, however, two variables of primary interest are the time of reorientation and appendage deflections during the maneuver. A scalar function which provides a measure of total appendage deflection is the system potential energy.

We emphasize that the Lyapunov control law presented here is strictly valid only for the rigid spacecraft. Stability analysis of the system with flexible appendages is not investigated here. We choose appendages which are small compared to the mass of the rigid body to which they are attached so that they do not significantly affect the dynamics of the rigid body reorientation.

The satellite model data are given in Table 1, and represent a relatively small spacecraft. In the momentum wheel case, the relative spin momenta represent the starting values, whereas for the CMG case, the spin momenta are fixed. This model is taken directly from Oh and Vadali,[8] except that three one meter long appendages modeled as Euler-Bernoulli beams are cantilevered from the spacecraft. Appendage 1 is attached at the coordinates $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and extends in the $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ direction. The appendage is allowed flexure only in the plane normal to $\hat{\mathbf{b}}_3$. Appendages 2 and 3 are attached along the $\hat{\mathbf{b}}_2$ and $\hat{\mathbf{b}}_3$ axes in an identical manner, and are allowed flexure orthogonal to the $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ directions respectively. Each appendage has a mass of 0.1 $kg$ and a stiffness selected to provide a first mode natural frequency of approximately 0.1 Hz. The complete set of equations of motion for a spacecraft containing gimbaled momentum wheels and flexible Euler-Bernoulli appendages are presented in Ford.[11]

Table 1: SATELLITE HUB PHYSICAL DATA

| Item | Value | Units |
|---|---|---|
| $m$ | 100 | $kg$ |
| $\mathbf{I}$ | $\text{diag}\{86.215, 85.070, 113.565\}$ | $kg$-$m^2$ |
| $\mathbf{I}_{sw}$ | $\text{diag}\{0.05, 0.05, 0.05\}$ | $kg$-$m^2$ |
| $\mathbf{I}_{tw}$ | $\text{diag}\{0.03, 0.03, 0.03\}$ | $kg$-$m^2$ |
| $\mathbf{I}_{tg}$ | $\text{diag}\{0.01, 0.01, 0.01\}$ | $kg$-$m^2$ |
| $\mathbf{I}_{sg}$,$\mathbf{I}_{gg}$ | $\text{diag}\{0, 0, 0\}$ | $kg$-$m^2$ |
| $\mathbf{a}_{s10}$ | $[1, 0, 0]^{\mathrm{T}}$ | |
| $\mathbf{a}_{s20}$ | $[0, 1, 0]^{\mathrm{T}}$ | |
| $\mathbf{a}_{s30}$ | $[0, 0, 1]^{\mathrm{T}}$ | |
| $\mathbf{a}_{g1}$ | $[0, 1, 0]^{\mathrm{T}}$ | |
| $\mathbf{a}_{g2}$ | $[0, 0, 1]^{\mathrm{T}}$ | |
| $\mathbf{a}_{g3}$ | $[1, 0, 0]^{\mathrm{T}}$ | |
| $\mathbf{h}_{swr}$ | $[7.2, 7.2, 7.2]^{\mathrm{T}}$ | $kg$-$m^2/sec$ |

First we demonstrate the effects of the stationary platform maneuver using a cluster of momentum wheels by comparing it to a direct maneuver. We examine the effects of changing the spin momenta in a linear fashion between the initial values and the unique final values. Since the angular momentum in the inertial coordinate frame is constant, the final value of angular momentum in the body frame is

$$\mathbf{h}_{final} = \mathbf{C}_{final}\mathbf{C}_0^{\mathrm{T}}\mathbf{h}_0 \tag{89}$$

so that we choose

$$\dot{\mathbf{h}}_c = \frac{\mathbf{h}_{final} - \mathbf{h}_0}{t_{man}} \tag{90}$$

where $t_{man}$ is the desired maneuver time.

In the case of the stationary platform maneuver, we desire to maintain the magnitude of $\mathbf{h}_c$ constant while changing its direction steadily to the final value $\mathbf{h}_{final}$. The angular change of the cluster angular momentum in the body frame is given by

$$\theta_{spm} = \arccos\left(\frac{\mathbf{h}_0^{\mathrm{T}}\mathbf{h}_{final}}{\mathbf{h}_0^{\mathrm{T}}\mathbf{h}_0}\right) \tag{91}$$

We define a plane in the body coordinate frame which contains $\mathbf{h}_c$ and $\mathbf{h}_{final}$ by a normal vector $\hat{\mathbf{e}}_{spm}$ where

$$\hat{\mathbf{e}}_{spm} = \frac{\mathbf{h}_c^{\times}\mathbf{h}_{final}}{||\mathbf{h}_c^{\times}\mathbf{h}_{final}||} \tag{92}$$

The cluster momentum $\mathbf{h}_c$ will vary along a constant arc from $\mathbf{h}_0$ to $\mathbf{h}_{final}$ if we maintain

$$\dot{\mathbf{h}}_c = \hat{\mathbf{e}}_{spm}^{\times}\mathbf{h}_c\left(\frac{\theta_{spm}}{t_{man}}\right) \tag{93}$$

Figure 3 show the results of the direct and stationary platform maneuver for the small satellite when the cluster momentum is reoriented about the $(1, -2, 3)$ body axis through an angle of 2.856 radians using a cluster of 3 momentum wheels.

We set $t_{man} = 500sec$ and show the dynamics for an extra 100 seconds after completion of the maneuver. Note the drastic difference between the angular velocities (and consequently, the quaternion histories). Whereas the rotor momenta follow only a slightly different profile, the resultant reorientation is much better behaved for the stationary platform maneuver. It is evident from the potential energy history that the appendage deflections during the direct maneuver are larger.

Unfortunately, neither the open loop stationary platform maneuver nor the direct maneuver solve the kinematics problem. While the cluster momentum components in the body

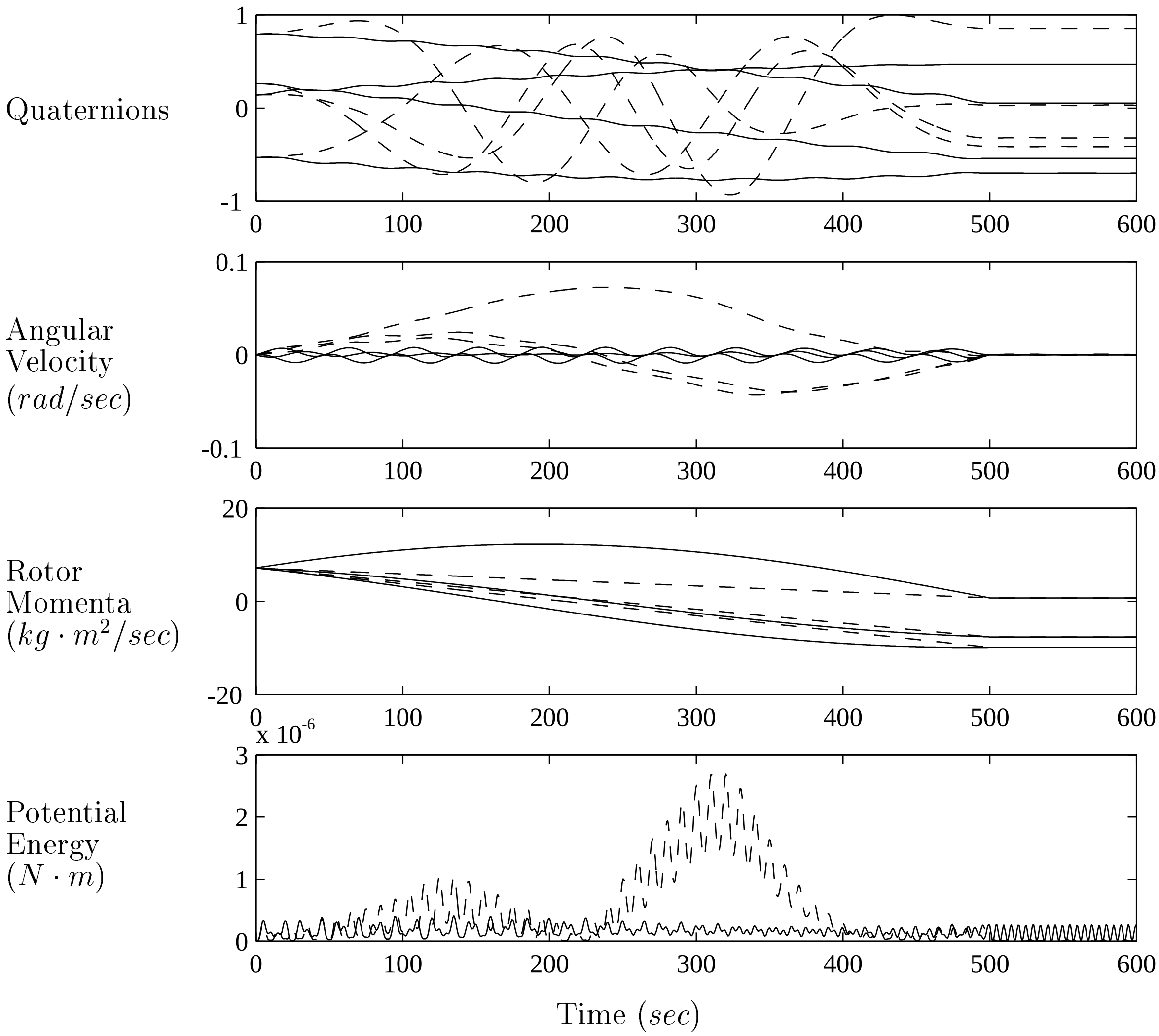


Figure 3: Reorientation Parameters for the Direct (- - -) and Stationary Platform Maneuver (—) Using Momentum Wheels

frame are unique for a specified final orientation, achieving the required final cluster momentum only satisfies the kinematics to within a rotation about the angular momentum vector.

We present in Figures 4 and 5 the results of a Lyapunov law reorientation using a cluster of 3 control moment gyros. We also present the results of a Lyapunov law reorientation which includes the stationary platform weighting. Equations (81) and (85) were both used to remain close to the stationary platform surface. In both maneuvers the quaternion gain was selected as $k_0 = 0.35$ and $\mathbf{K}_1$ was computed using Equation (44). For the weighted maneuver, the weighting gain was $k_2 = 0.1$. Because the Lyapunov law commands maximum torque at

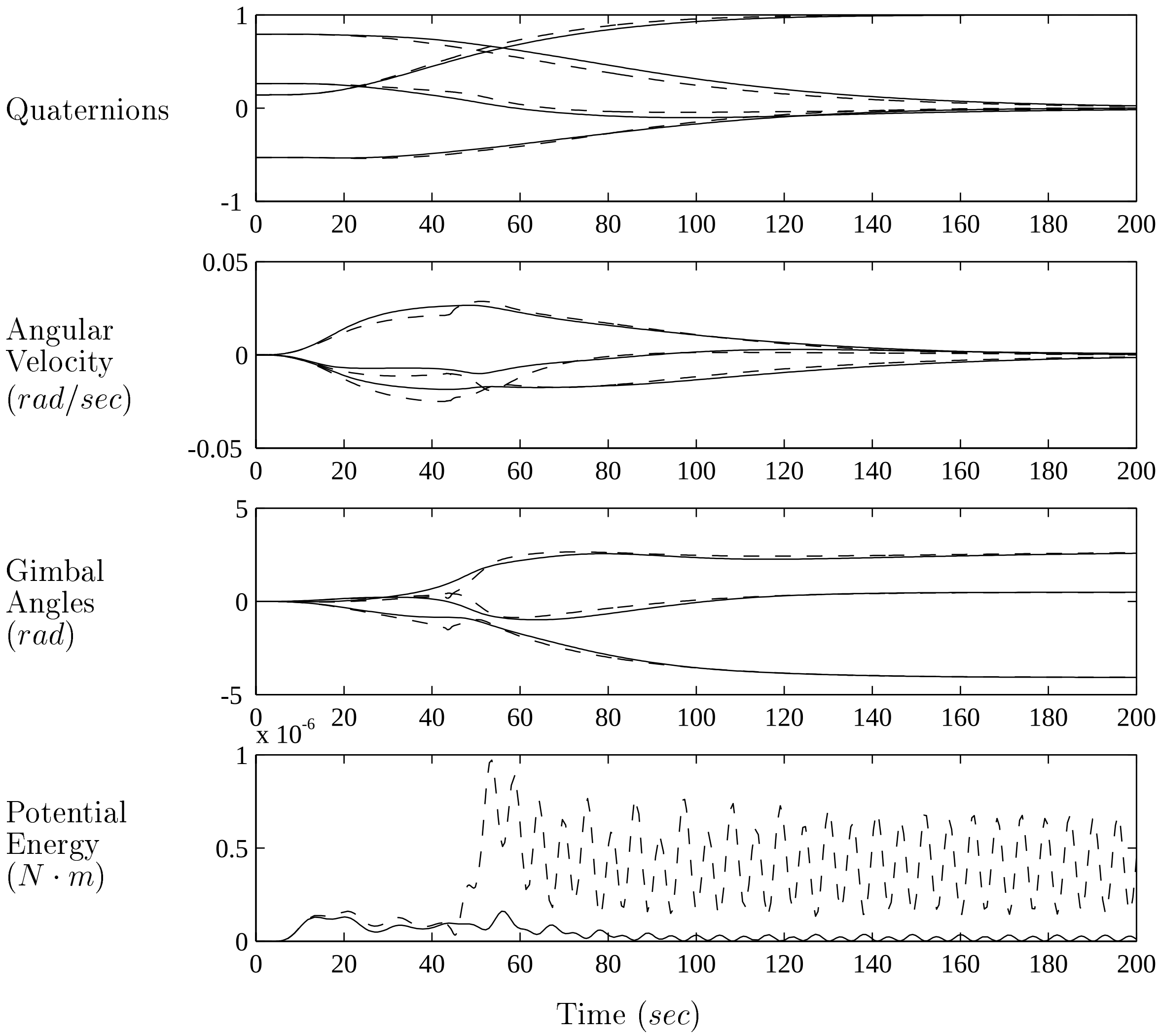


Figure 4: Reorientation Parameters for the Lyapunov Maneuver (- - -) and Lyapunov Maneuver with Stationary Platform Weighting (—) Using CMGs

$t = 0$, a polynomial smoothing term was used during the first 20 seconds of the reorientation to avoid exciting the appendages with a step input.

Though the reorientations are accomplished in roughly the same amount of time, the paths taken by the gimbal angles are different enough to create a notably different effect on the flexible appendages. A significant contribution to the appendage excitations in this case is the brush with an external singularity that occurs at approximately $t = 50$ seconds, resulting in high gimbal rates.

## CONCLUSIONS

This paper presents a new and concise form of the equations of motion for a spacecraft

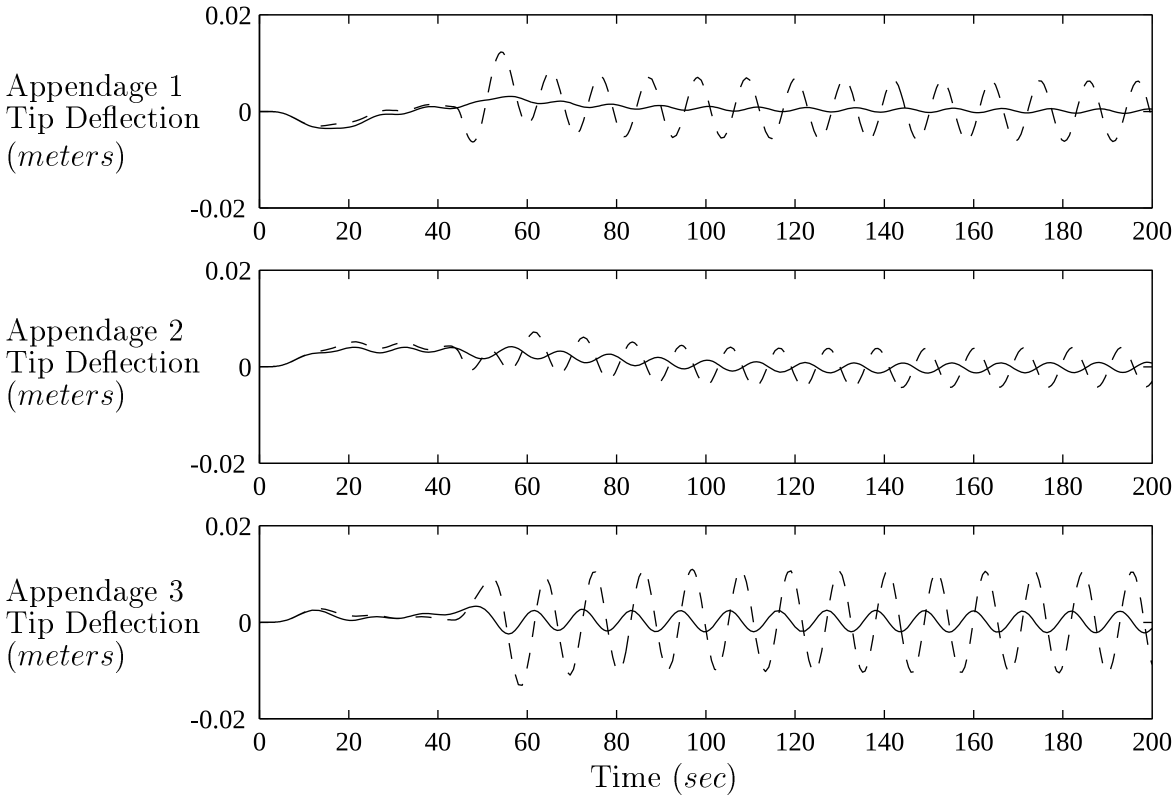


Figure 5: Appendage Deflections for the Lyapunov Maneuver (- - -) and Lyapunov Maneuver with Stationary Platform Weighting (—) Using CMGs

with a cluster of gimbaled momentum wheels and a set of flexible Euler-Bernoulli appendages. These equations may be specialized to the equations for the momentum wheel or control moment gyro cluster. The Lyapunov control law proposed by Oh and Vadali is expressed in terms of the control inputs and variables of this development. The stationary platform condition for the gimbaled momentum wheel is developed, and specialized to the control moment gyro cluster. Examples of stationary platform maneuvers using a momentum wheel cluster and a control moment gyro cluster are presented. In these cases, the maneuvers which remain closer to the stationary platform condition cause smaller deformations of the flexible appendages.

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