AE 423 Space Technology I

Chapter 2 Satellite Dynamics

2.1 Introduction

In this chapter we review some dynamics relevant to satellite dynamics and we establish some of the basic properties of satellite dynamics.

2.2 Dynamics of a Rigid Body

Consider the rigid body B with mass center B*. Reference frame B is fixed in the body. Given the element of mass dm whose location relative to B* is defined by

$$\vec{r} = r_1\vec{b}_1 + r_2\vec{b}_2 + r_3\vec{b}_3 \tag{2.1}$$

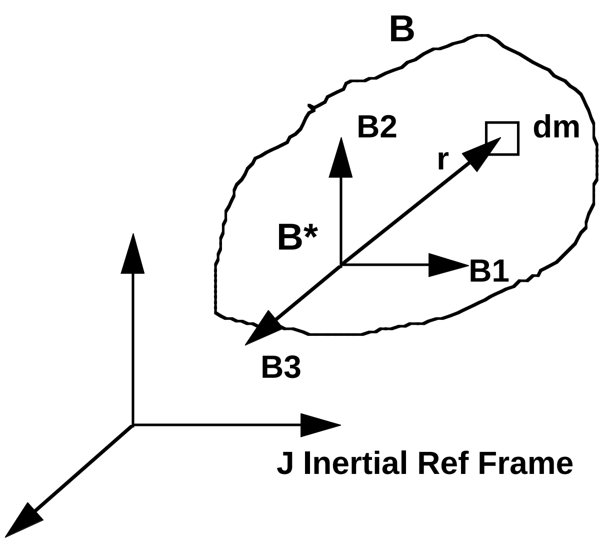


Figure 2.1 Reference Frame and Mass Element

The moment of inertia matrix is defined by

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \tag{2.2}$$

where

$$
\begin{aligned}
&I_{11} = \int_B \left(r_2^2 + r_3^2\right) dm, I_{22} = \int_B \left(r_1^2 + r_3^2\right) dm, I_{33} = \int_B \left(r_1^2 + r_2^2\right) dm \\
&I_{ij} = -\int_B r_i r_j dm, i \neq j
\end{aligned} \tag{2.3}
$$

Note that *I* is symmetric.

We will always use the center of mass as the reference point.

The angular momentum is

$$
\begin{aligned}
\vec{H}^{B/J} &= I\vec{\omega}^{B/J} \\
H_1^{B/J} &= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\
H_2^{B/J} &= I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\
H_1^{B/J} &= I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3
\end{aligned} \tag{2.4}
$$

The rotational kinetic energy is

$$
E_k = \frac{1}{2}\vec{\omega}^{B/J} \bullet \vec{H}^{B/J} \tag{2.5}
$$

The equations of motion are:

$$
\frac{{}^J d\vec{H}^{B/J}}{dt} = \vec{T} \tag{2.6}
$$

where $\vec{T}$ is the sum of the torques on the body. $\vec{H}^{B/J}$ will always be the inertial angular momentum so we will drop the J.

$$\frac{{}^{J}d\vec{H}^{B}}{dt} = \frac{{}^{B}d\vec{H}^{B}}{dt} + \vec{\omega}^{B/J} \times \vec{H}^{B} = \vec{T} \tag{2.7}$$

In scalar form these are

$$\begin{aligned} \dot{H}_1 + \omega_2\omega_3\left(H_3 - H_2\right) &= T_1 \\ \dot{H}_2 + \omega_1\omega_3\left(H_1 - H_3\right) &= T_2 \\ \dot{H}_3 + \omega_1\omega_2\left(H_2 - H_1\right) &= T_3 \end{aligned} \tag{2.8}$$

We derived these equations for a rigid body but they apply to any system of particles and/or rigid bodies as long as the torques are computed about the reference point. **In this course our reference point for the angular momentum will always be the center of mass of the system.**

If the axes are principal axes then

$$H_1^B = I_{11}\omega_1,\; H_2^B = I_{22}\omega_2,\; H_3^B = I_{33}\omega_3 \tag{2.9}$$

and the equations of motion become

$$\begin{aligned} I_{11}\dot{\omega}_1 + \omega_2\omega_3\left(I_{33} - I_{22}\right) &= T_1 \\ I_{22}\dot{\omega}_2 + \omega_3\omega_1\left(I_{11} - I_{33}\right) &= T_2 \\ I_{33}\dot{\omega}_3 + \omega_1\omega_2\left(I_{22} - I_{11}\right) &= T_3 \end{aligned} \tag{2.10}$$

These are called Euler's equations of motion for a rigid body.

<u>Torque Free Motion of a Body of Revolution</u>

We want to consider spin about the axis of symmetry of an axisymmetric body when there are no torques. Let B_3 be the axis of symmetry, then

$$\begin{aligned} &T_1 = T_2 = T_3 = 0 \\ &I_{11} = I_{22} = I_T, I_{33} = I_s \end{aligned} \tag{2.11}$$

Define

$$\sigma = I_s \,/\, I_T \tag{2.12}$$

Substituting the above in Eqs. (2.10) gives

$$\begin{aligned} &\dot{\omega}_1 + \omega_2\omega_3(\sigma - 1) = 0 \\ &\dot{\omega}_2 - \omega_1\omega_3(\sigma - 1) = 0 \\ &\dot{\omega}_3 = 0, \omega_3 = \omega_s \end{aligned} \tag{2.13}$$

Thus, the spin rate is a constant. Let

$$\lambda = (\sigma - 1)\omega_s \tag{2.14}$$

λ is called the nutation frequency.

$$\begin{aligned} &\dot{\omega}_1 = -\lambda\omega_2 \\ &\dot{\omega}_2 = \lambda\omega_1 \\ &\ddot{\omega}_1 + \lambda^2\omega_1 = 0 \end{aligned} \tag{2.15}$$

The solution is

$$\begin{aligned} &\omega_1 = \omega_{10}\cos\lambda t - \omega_{20}\sin\lambda t \\ &\omega_2 = \omega_{10}\sin\lambda t + \omega_{20}\cos\lambda t \\ &\omega_1 = \omega_T\cos(\lambda t + \alpha) \\ &\omega_2 = \omega_t\sin(\lambda t + \alpha) \\ &\omega_T = \left(\omega_{10}^2 + \omega_{20}^2\right)^{1/2}, \tan\alpha = \frac{\omega_{20}}{\omega_{10}} \end{aligned} \tag{2.16}$$

α is a phase angle and ω_T is called the transverse angular velocity.

Now determine the angular velocity as a function of the Euler angles defined in Fig. 2.2.

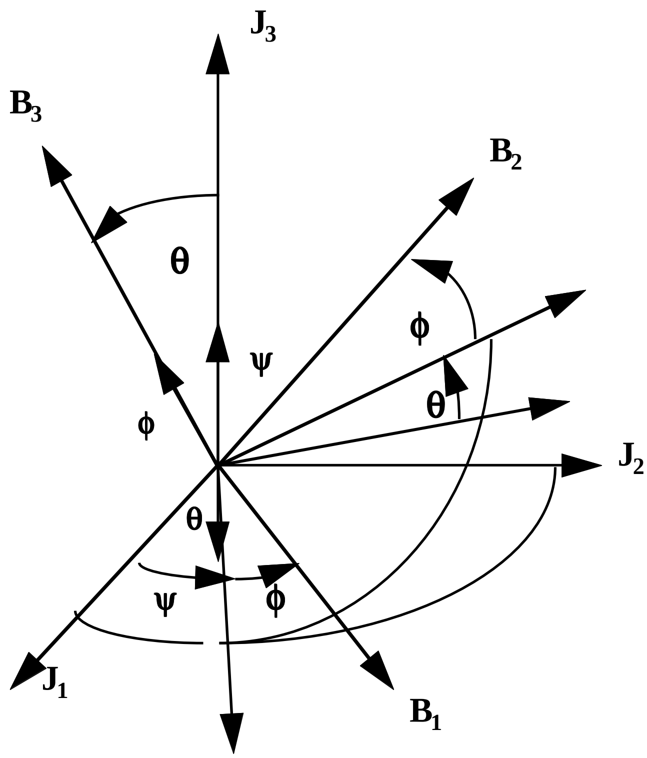


Figure 2.2 Euler Angles

Since there are no external torques the total angular momentum H is constant. We will let the direction of the J_3 axis be along the angular momentum vector. Then

$$\begin{aligned}\vec{\omega}^B &= \left(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi\right)\vec{b}_1 \\ &\quad + \left(-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi\right)\vec{b}_2 + \left(\dot{\phi} + \dot{\psi}\cos\theta\right)\vec{b}_3\end{aligned} \tag{2.17}$$

$$\begin{aligned}\dot{\psi} &= \left(\omega_1\sin\phi + \omega_2\cos\phi\right)/\sin\theta \\ \dot{\theta} &= \left(\omega_1\cos\phi - \omega_2\sin\phi\right) \\ \dot{\phi} &= \omega_3 - \dot{\psi}\cos\theta\end{aligned} \tag{2.18}$$

Also, since the spin rate, ω_S , is a constant, the component of H along the B_3 axis, H_3, is constant. With J_3 as the total angular momentum

$$\cos\theta = H_3 / H = I_s\omega_s / H \tag{2.19}$$

Thus, θ is constant and is called the nutation angle.

$$\begin{aligned} \omega_1^2 + \omega_2^2 = \omega_T^2 = \dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta \\ \dot{\phi} + \dot{\psi}\cos\theta = \omega_s \end{aligned} \tag{2.20}$$

Since θ is constant, $\dot{\psi}$ *and* $\dot{\phi}$ must be constant.

$$\begin{aligned} \dot{\psi}\sin\theta = \omega_1 \sin\phi + \omega_2\cos\phi \\ \dot{\psi}\sin\theta = \omega_T \sin(\lambda t + \alpha + \phi) \end{aligned} \tag{2.21}$$

Therefore,

$$\begin{aligned} \lambda t + \alpha + \phi = \text{constant} \\ \dot{\phi} = -\lambda = -(\sigma - 1)\omega_s \\ \dot{\psi} = \sigma\omega_s / \cos\theta \end{aligned} \tag{2.22}$$

Consider the plane formed by the J_3 (angular momentum vector) and the B_3 (spin) axis. This plane is called the nutation plane. The nutation plane rotates at the rate $\dot{\psi} = \sigma\omega_s / \cos\theta$. From the following figure

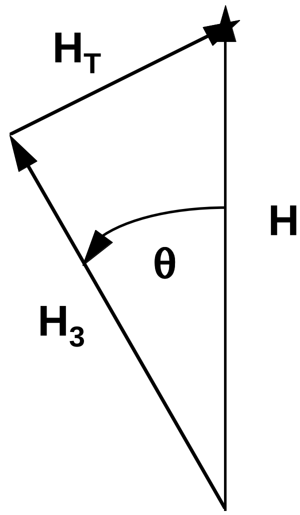


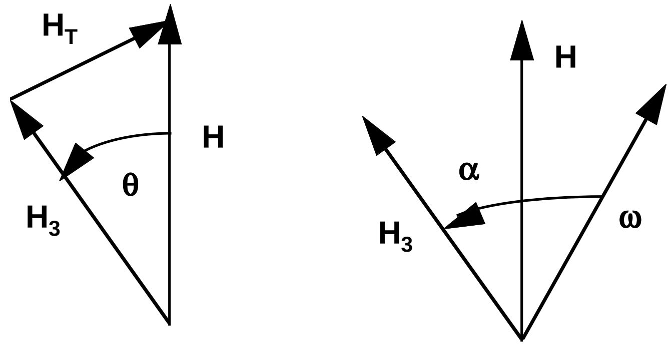
Figure 2.3 Nutation Plane

$$\vec{H} = H\vec{j}_3 = H_3\vec{b}_3 + \vec{H}_T$$
$$\vec{H}_T = I_T\left(\omega_1\vec{b}_1 + \omega_2\vec{b}_2\right) = I_T\vec{\omega}_T$$

Thus, $\vec{\omega}_T$ must be in the plane formed by the angular momentum vector and the spin axis. This is why it is called the transverse angular velocity.

Our control system will have to perform two functions. The first will be to damp out the nutation, that is, reduce the angle θ to zero. Then, if the angular momentum vector is not in the desired direction, e.g., along the orbit normal, it will have to move the angular momentum vector to its desired direction. We will address these issues in Chapter 4.

Now consider the following figure



$$\tan\theta = \frac{H_T}{H_s} = \frac{I_T\omega_T}{I_s\omega_s} = \frac{\omega_T}{\sigma\omega_s}$$
$$\tan\alpha = \frac{\omega_T}{\omega_s} = \sigma\tan\theta \qquad (2.23)$$

The angular velocity vector is in the nutation plane, the plane formed by the spin axis and the angular momentum vector. This plane rotates at the rate $\dot{\psi} = \sigma\omega_s / \cos\theta$. Since $\dot{\phi} = -\lambda = -(\sigma - 1)\omega_s$ an observer fixed to the body would see himself rotating in the negative direction relative to the nutation plane when $\sigma > 1$, and in the positive direction when $\sigma < 1$.

2.3 Stability of Spin

We want to investigate the stability of the spin of a rigid body about a principal axis. The body is no longer axisymmetric as in the previous section. Let the rigid body be rotating about the B_3 axis and let the three axes be principal axes. We want to determine the stability of this spin. In state space format

$$\dot{\vec{\omega}} = \vec{f}(\vec{\omega})$$
$$\dot{f}_1 = \omega_2\omega_3\left(I_{22} - I_{33}\right) / I_{11}$$
$$\dot{f}_2 = \omega_1\omega_3\left(I_{33} - I_{11}\right) / I_{22} \qquad (2.24)$$
$$\dot{f}_3 = \omega_1\omega_2\left(I_{11} - I_{22}\right) / I_{33}$$

An equilibrium point of this system is any one of the ω_i unequal to zero, and the other two ω_i equal to zero, i.e., spin about any of the axes. We are considering spin about the B_3 axis. Expand Eq. (3.23) in a Taylor series about the equilibrium point and retain only the linear terms in the ω_i.

$$\begin{aligned}
\omega_1 &= 0 + \delta\omega_1 \\
\omega_2 &= 0 + \delta\omega_2 \\
\omega_3 &= \omega_s + \delta\omega_3
\end{aligned}$$

The equations of motion become

$$\begin{aligned}
\delta\dot{\omega}_1 &= \left[\omega_s\left(I_{22} - I_{33}\right) / I_{11}\right]\delta\omega_2 \\
\delta\dot{\omega}_2 &= \left[\omega_s\left(I_{33} - I_{11}\right) / I_{22}\right]\delta\omega_1 \qquad (2.25) \\
\delta\dot{\omega}_3 &= 0
\end{aligned}$$

Differentiating the first equation and substituting for $\delta\dot{\omega}_2$ gives

$$\delta\ddot{\omega}_1 + \left[\omega_s^2\left(I_{33} - I_{11}\right)\left(I_{33} - I_{22}\right) / \left(I_{11}I_{22}\right)\right]\delta\omega_1 = 0 \qquad (2.26)$$

If the coefficient multiplying $\delta\omega_1$ is positive the motion is described by a harmonic oscillator, thus it is stable if $I_{33} > I_{11}$ *and* $I_{33} > I_{22}$ *or* $I_{33} < I_{11}$ *and* $I_{33} < I_{22}$. Otherwise it is unstable. Therefore, we get the result that spin about the axis of minimum or maximum moment of inertia is stable, and spin about the axis of middle moment of inertia is unstable. We will see in Chapter 4 that if there is any energy dissipation in the body, which always is the case, then spin about the axis of minimum moment of inertia is also unstable. A heuristic argument showing this to be the case is now presented. For spin about a principal axis the kinetic energy is

$$E_k = \frac{1}{2} I\omega^2 = \frac{1}{2}\frac{H^2}{I}$$

Since the energy dissipation is internal (no external forces or torques) the angular momentum is constant. Spin about a principal axis is an equilibrium configuration and energy cannot dissipate. However, let there be a small perturbation from this configuration. Then energy will dissipate and the system will seek out the equilibrium configuration with the minimum kinetic energy. This is spin about the axis of maximum moment of inertia. This fact, although well known to astronomers, was not known by some of the early spacecraft designers and became known by a satellite becoming unstable. The time constant for the unstable motion may be minutes or hours depending upon the source of dissipation.

2.4 Dynamics of a Satellite with Reaction or Momentum Wheels

Consider a satellite with a reaction or momentum wheel with its spin axis defined in the B-frame by the unit vector $\vec{n}$. The angular momentum of the system is

$$\vec{H} = \vec{H}_B + \vec{h} \tag{2.27}$$

where $\vec{H}_B$ is the angular momentum of the rigid body and $\vec{h}$ is the angular momentum of the wheel relative to the body and is given by

$$\vec{h} = h\vec{n} \tag{2.28}$$

Equations (2.8) are applicable since they are for a system, not just a rigid body. However, since there are four unknowns, the wheel speed relative to the body and the three components of the angular velocity or angular momentum, we still need another equation. If the wheel is axisymmetric then we can write

$$\dot{h} = T_M \tag{2.29}$$

where T_M is the wheel motor torque applied to the wheel. An equal and opposite torque is applied to the satellite. Thus, the equations of motion of a system with N wheels are

$$\begin{aligned} &\dot{H}_1 + \omega_2 H_3 - \omega_3 H_2 = T_1 \\ &\dot{H}_2 + \omega_3 H_1 - \omega_1 H_3 = T_2 \\ &\dot{H}_3 + \omega_1 H_2 - \omega_2 H_1 = T_3 \\ &\dot{h}_i = T_{M\,i}, = 1, 2, \ldots, N \end{aligned} \tag{2.30}$$

The satellite or body angular momentum is

$$\begin{aligned} &\dot{H}_1 + \omega_2 \omega_3 \left(H_3 - H_2\right) = T_1 \\ &H_1 = H_{B1} + \sum_{i=1}^{N} h_i \vec{n}_i \cdot \vec{e}_1 \\ &H_2 = H_{B2} + \sum_{i=1}^{N} h_i \vec{n}_i \cdot \vec{e}_2 \\ &H_3 = H_{B3} + \sum_{i=1}^{N} h_i \vec{n}_i \cdot \vec{e}_3 \end{aligned} \tag{2.31}$$

From the satellite angular momentum we obtain the angular velocity.

$$\vec{\omega} = I^{-1} \vec{H}_B$$

In this course we will use a (1,2,3)=(yaw, roll, pitch) sequence of rotations. The angular velocity is

$$\begin{aligned} &\vec{\omega} = R_3(\theta) R_2(\phi) R_1(\psi) \begin{pmatrix} \dot{\psi} \\ 0 \\ n \end{pmatrix} + R_3(\theta) R_2(\phi) \begin{pmatrix} 0 \\ \dot{\phi} \\ 0 \end{pmatrix} + R_3(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} \\ &\vec{\omega} = \begin{pmatrix} \dot{\psi}\cos\theta\cos\phi - \dot{\phi}\sin\theta \\ \dot{\psi}\sin\theta\cos\phi + \dot{\phi}\cos\theta \\ \dot{\theta} + \dot{\psi}\sin\phi \end{pmatrix} + n \begin{pmatrix} -\sin\theta\sin\psi - \cos\theta\sin\phi\cos\psi \\ \cos\theta\sin\psi - \sin\theta\sin\phi\cos\psi \\ \cos\phi\cos\psi \end{pmatrix} \end{aligned} \tag{2.32}$$

where n is the orbital angular velocity or mean motion. This term is needed since we are referencing the attitude with respect to a rotating (Earth pointing) reference frame. The angular rates are

$$\begin{aligned}
\dot{\phi} &= \left(-\omega_1 \sin\theta + \omega_2 \cos\theta - n\sin\psi\right) \\
\dot{\psi} &= \left(\omega_1 \cos\theta + \omega_2 \sin\theta + n\sin\phi\cos\psi\right) / \cos\phi \\
\dot{\theta} &= \omega_3 - n\cos\phi\cos\psi - \dot{\psi}\sin\phi \\
\dot{\theta} &= \omega_3 - \left(\omega_1 \cos\theta + \omega_2 \sin\theta\right)\tan\phi - n\cos\psi / \cos\phi
\end{aligned} \tag{2.33}$$

Note the singularity when $\psi = \pm 90\,\deg$. There is always a singularity with any set of Euler angles.

The process is given the external torques and the motor torques:

1. Integrate Eqs. (2.30) to obtain the system angular momentum and reaction wheel angular momenta.
2. Solve for the body or satellite angular momentum with Eq. (2.31).
3. Solve for the angular velocity using Eq. (2.32).
4. Solve for the angular rates using Eq. (2.33) and integrate to obtain the attitude.

In the class project we will neglect the mean motion term. This is justified because the term is so small for geosynchronous satellites.

2.4 Disturbance Torques

2.4.1 Solar Pressure Torques (Agrawal, pg 133)

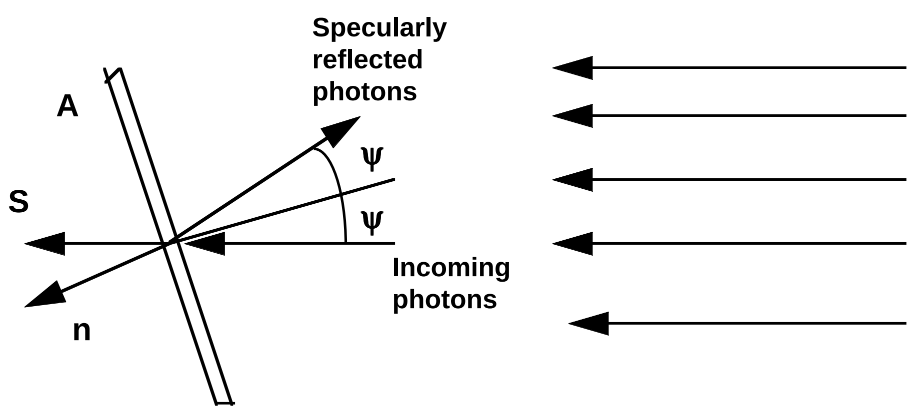
The torque resulting from solar pressure is the major long-term disturbance torque for geosynchronous spacecraft. The dominant torque on LEO spacecraft is the gravity gradient or aerodynamic. The solar radiation force results from the impingement of photons on the spacecraft. A fraction, ρ_S, are specularly reflected, a fraction, ρ_d, are diffusely reflected and a fraction, ρ_a, are absorbed by the surface.

In the figure below there is a surface of area A on which the photons are impinging. The normal to the surface is $\vec{n}$ and the unit vector from the sun to the spacecraft is $\vec{S}$. The angle between the unit vectors $\vec{n}$ and $\vec{S}$ is ψ. The solar flux is

$$\vec{P} = P\vec{S}$$

The force created by the absorbed photons results from the transfer of momentum of the photons to the spacecraft and is given by

$$\vec{F}_a = \rho_a PA(\vec{n} \cdot \vec{S})\vec{S} \tag{2.34}$$



where P is the solar radiation pressure. Note that the force is in the direction along the sun line. When the photons are specularly reflected they transfer twice the momentum and the direction is normal to the surface. There is no momentum transfer tangent to the surface.

$$\vec{F}_s = 2\rho_s PA(\vec{n} \cdot \vec{S})^2 \vec{n} \tag{2.35}$$

For that portion that is diffusely reflected the photon's momentum may be considered stopped at the surface and reradiated uniformly into the hemisphere. Thus the force has a component due to the transfer of momentum and a component due to the reradiation. Since it is reradiated uniformly the reradiation component will be normal to the surface. The force is

$$\vec{F}_d = \rho_d PA\left(\vec{n} \cdot \vec{S}\right)\left(\vec{S} + \frac{2}{3}\vec{n}\right) \tag{2.36}$$

The total solar radiation force is

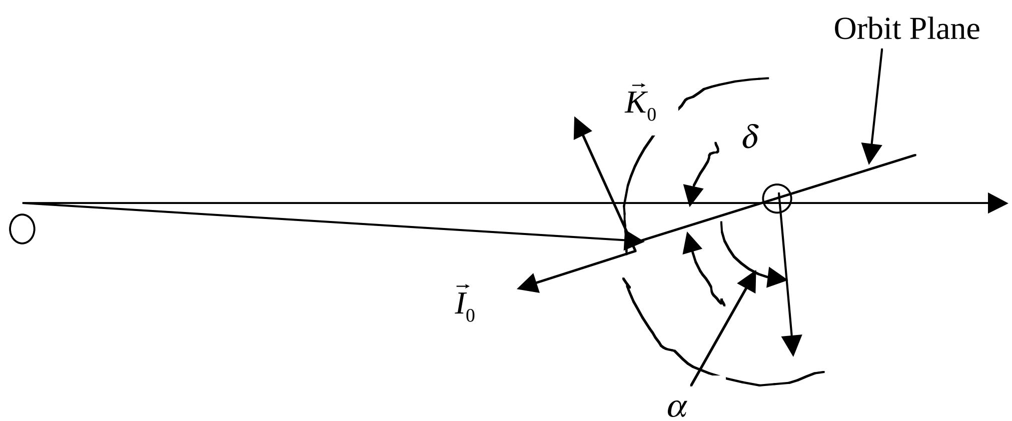
$$\vec{F}_S = PA\left(\vec{n} \cdot \vec{S}\right)\left[(1-\rho_s)\vec{S} + 2\left(\rho_s\left(\vec{n} \cdot \vec{S}\right) + \frac{1}{3}\rho_d\right)\vec{n}\right] \tag{2.37}$$

where $\rho_a + \rho_s + \rho_d = 1$ has been used. The solar pressure is usually assumed to be constant and to have the value P=4.644*10^{-6} N/m^2. Let

$$\begin{aligned} &\vec{F} = PA\left(\vec{n} \cdot \vec{S}\right)\left(F_{S1}\vec{S} + F_{S2}\vec{n}\right) \\ &F_{S1} = (1-\rho_s), F_{S2} = 2\left(\rho_s\left(\vec{n} \cdot \vec{S}\right) + \frac{1}{3}\rho_d\right) \end{aligned} \tag{2.38}$$

Referring to the figure below and with δ as the declination of the sun, positive from the vernal equinox to the autumnal equinox, and α as the orbit angle measured from local noon, the unit vector to the sun is

$$\vec{S} = -\cos\delta\vec{I}_0 - \sin\delta\vec{K}_0 = -\cos\delta\cos\alpha\vec{o}_1 + \cos\delta\sin\alpha\vec{o}_2 - \sin\delta\vec{o}_3 \tag{2.39}$$



The (O_1, O_2, O_3) are the axes of the orbital frame and coincide with the (I_0, J_0, K_0) axes when $\alpha = 0$. For momentum bias and three axis stabilized spacecraft the major portion of the solar pressure torque is usually due to the solar panels that track the sun. Usually the solar panels only have a single gimbal, which is about the 3-axis or pitch axis. Thus, they are always within 23 degrees of the Sun and the loss of power from not pointing directly at the sun is offset by not having the added complexity and weight of a double gimbal system to directly track the Sun. Thus,

$$
\begin{aligned}
&\vec{I}_0 = \cos\alpha\vec{o}_1 - \sin\alpha\vec{o}_2 \\
&\vec{J}_0 = \sin\alpha\vec{o}_1 + \cos\alpha\vec{o}_2 \\
&\vec{K}_0 = \vec{o}_3 \\
&\vec{n} = -\vec{I}_0 = -\cos\alpha\vec{o}_1 + \sin\alpha\vec{o}_2
\end{aligned} \tag{2.40}
$$

Then

$$
\begin{aligned}
&\vec{F} = PA\left[-K_2\vec{K}_0 - K_1\vec{I}_0\right] \\
&K_1 = \left(F_{S1}\cos\delta + F_{S2}\right)\cos\delta, K_2 = F_{S1}\cos\delta\sin\delta
\end{aligned} \tag{2.41}
$$

In the orbital frame the solar pressure force is

$$
\vec{F} = -PA\left(K_1\cos\alpha\vec{o}_1 - K_1\sin\alpha\vec{o}_2 + K_2\vec{o}_3\right) \tag{2.42}
$$

Assuming no attitude errors the vector from the center of mass to the center of pressure of A is $\vec{r}_{cp} = x_{cp}\vec{o}_1 + y_{cp}\vec{o}_2 + z_{cp}\vec{o}_3$. Then the solar pressure torque is

$$\vec{T}_S = \vec{r}_{cp} \times \vec{F}_S$$

$$\vec{T}_S = PA\begin{pmatrix} -y_{cp}K_2 - z_{cp}K_1 \sin\alpha \\ -z_{cp}K_1 \cos\alpha + x_{cp}K_2 \\ x_{cp}K_1 \sin\alpha + y_{cp}K_1 \cos\alpha \end{pmatrix} \tag{2.43}$$

In the $\left(\vec{I}_0, \vec{J}_0, \vec{K}_0\right)$ system which rotates at approximately 1 deg/day the solar radiation torque is

$$\vec{T}_S = PA\begin{pmatrix} -y_{cp}K_2 \cos\alpha - x_{cp}K_2 \sin\alpha \\ K_2\left(x_{cp} \cos\alpha - y_{cp} \sin\alpha\right) - z_{cp}K_1 \\ x_{cp}K_1 \sin\alpha + y_{cp}K_1 \cos\alpha \end{pmatrix} \tag{2.44}$$

Note that the pitch torque is periodic, hence its effect over one orbit is zero. However, the roll and yaw torques have secular components which mean that over one orbit there will be a net change in the spacecraft angular momentum. To determine the critical or design conditions one must evaluate the torques at equinox and solstice to determine the maximum conditions. This will be addressed in Chapter 3.

2.4.2 Gravity Gradient Torque

The gravity gradient torque on a rigid body in a circular orbit of radius R_0 and orbital angular velocity ω_0 can be expressed as

$$\vec{T}_G = 3\omega_0^2 \vec{o}_1 \times \bar{I} \bullet \vec{o}_1 \tag{2.45}$$

where $\bar{I}$ is the inertia dyadic

$$\bar{I} = \begin{pmatrix} I_{11}\vec{b}_1\vec{b}_1 & I_{12}\vec{b}_1\vec{b}_2 & I_{11}\vec{b}_1\vec{b}_3 \\ I_{21}\vec{b}_2\vec{b}_1 & I_{22}\vec{b}_2\vec{b}_2 & I_{23}\vec{b}_2\vec{b}_3 \\ I_{31}\vec{b}_3\vec{b}_1 & I_{32}\vec{b}_3\vec{b}_2 & I_{33}\vec{b}_3\vec{b}_3 \end{pmatrix} \tag{2.46}$$

To perform the dot and cross products $\vec{o}_1$ must be transformed into the body frame. With

$$R^{BO} = (R_{ij})$$
$$\vec{o}_1 = R_{11}\vec{b}_1 + R_{21}\vec{b}_2 + R_{31}\vec{b}_3 \tag{2.47}$$

Two cases of interest are the disturbance torques due to products of inertia and those resulting from an attitude error when the axes are principal axes.

<u>Products of Inertia</u>

With $R_{21} = R_{31} = 0, R_{11} = 1$ and $\psi = \phi = \theta = 0$ the gravity gradient torque is

$$\vec{T}_G = 3\omega_0^2\left(I_{12}\vec{b}_3 - I_{13}\vec{b}_2\right) \tag{2.48}$$

Note that there is no yaw torque. Also note that this is a constant torque which can only be countered in the long term with external torques, e.g. thrusters.

<u>Principal axes</u>

The gravity gradient torque in this case becomes

$$\vec{T}_G = 3\omega_0^2\begin{bmatrix} R_{21}R_{31}\left(I_{33} - I_{22}\right) \\ R_{31}R_{11}\left(I_{11} - I_{33}\right) \\ R_{21}R_{11}\left(I_{22} - I_{11}\right) \end{bmatrix} \tag{2.49}$$

For the 1-2-3 rotation

$$\begin{aligned} R_{11} &= \cos\theta\cos\phi \\ R_{21} &= \sin\theta\cos\phi \\ R_{31} &= \sin\phi \end{aligned} \tag{2.50}$$

$$\vec{T}_G = 3\omega_0^2 \begin{bmatrix} (I_{33} - I_{22})\sin\theta\sin\phi\cos\phi \\ (I_{11} - I_{33})\cos\theta\sin\phi\cos\phi \\ (I_{22} - I_{11})\sin\theta\cos\theta\cos^2\phi \end{bmatrix} \tag{2.51}$$

For small angles

$$\vec{T}_G = 3\omega_0^2 \begin{bmatrix} 0 \\ (I_{11} - I_{33})\phi \\ (I_{11} - I_{22})\theta \end{bmatrix} \tag{2.52}$$

Note that the yaw torque is zero. This is why there is very little control of yaw when using gravity gradient. The only control torque is from the nonlinear coupling of roll and pitch.