Stabilization of Rotational Motion with Application to Spacecraft Attitude Control

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Abstract

The objective of this paper is to develop a control scheme for stabilization of a hamiltonian system. The method generalizes the results available in the literature on motion control in the Euclidean space to an arbitrary differential manifold equipped with a metric. This modification is essencial for global stabilization of a rotary motion.

Along with a model of the system formulated in the Hamilton's canonical form the algorithm uses information about a required potential energy and a dissipation term. The control action is the sum of the gradient of the potential energy and the dissipation force. It is shown that this control law makes the system uniformly asymptotically stable to the desired reference point. The concept is very straightforward in the Euclidean space, however a global rotation control can not be tackled. An additional modification is made to address a system which flow lies on a Riemannian manifold. The Lyapunov stability theory is adapted and reformulated to fit to the new framework of Riemannian manifolds. To illustrate the results a spacecraft attitude control problem is considered. Firstly, a global canonical representation for the spacecraft motion is found, then three spacecraft control problems are addressed: stabilization in the inertial frame, magnetic libration damping for the gravity gradient stabilization and a slew maneuver with obstacle avoidance.

Keywords

Hamiltonian methods, nonlinear control, stability theory, attitude control

I. INTRODUCTION

The work reported in this article develops a control algorithm for a hamiltonian system. The resultant design scheme provided offers a good framework for solving motion control problems of mechanical systems performing both translatory and/or rotary types of motion. The special focus is on the illustration of the theoretical findings in the examples of the rotational motion control of a rigid body: a spacecraft.

The subject of control of mechanical systems has always been in very focus of control engineering. The recent advances of computer technology, vastly increasing computational power, availability of symbolic software tool-boxes have initiated a tremendous research effort within nonlinear control methods. Probably the most influential has been the geometric control methods as presented in [1], [2], [3]; passivity based control in [4],[5], [6]; nonlinear H_∞ in [7], [8].

The paper comprises a further development of the work reported in [9] and [2], ch. 12, dealing with stabilization of hamiltonian systems. This paper generalizes the previous results in the Euclidean space to Riemannian manifolds. This modification is crucial for global stabilization of a rotary motion. The great impact on this paper had the geometrical description of the physical mechanics in [10]. A further influence on this work had the articles [11] and [12] studying canonical transformation from the ordinary three-dimensional physical space of Euler angles to the four dimensional space of the unit quaternion. This approach is used in this paper to model the rotational motion of a rigid body in the Hamilton's canonical equations.

The idea of this paper is very intuitive and consist of the following steps. A dynamical system is modeled and its desired performance is specified using two hamiltonians: one of the original system and the second of the desired one. The desired system inherits the kinetic energy of the original one, but the potential energy has to comply with the requirements on the feedback system, e.g. the minimum of the potential energy shall be reached at the reference point. Additionally a dissipation term is incorporated which defines the time response of the closed loop system. A control action is designed such that the feedback system coincides with the desired one. The stability analysis used in this paper are based the Lyapunov second method, however it is necessary to reformulate the standard notions defined on the Euclidean space to a metric space, here a Riemannian manifold. The theoretical findings are applied to the attitude control of a spacecraft, a particularly interesting case study, as the dynamics is described in E^3, whereas the kinematics expresses the time propagation of the attitude matrix, an element of the special orthogonal group $SO_3(\mathbb{R})$. In the article the unit quaternion parameterization of the attitude is used as the minimal representation providing singularity free kinematics. Thorough the whole paper a number of examples are given to illustrate the concepts used. A remarkable contribution of this article is the validation of the theoretical method for the spacecraft three-axis attitude control. Three problems are addressed here: spacecraft stabilization in the inertial frame, libration damping with the use of electromagnetic coils and a slew maneuver with an additional objective of avoiding some regions e.g. defined by certain bright objects causing blindness of optical sensors.

The paper is organized as follows. Section II starts with preliminaries of the Riemannian manifolds, then the standard notions of stability and their proofs are formulated using the concept of Riemannian metric. The main result is Theorem 4 in Section III, which gives a control algorithm for a hamiltonian system. The control scheme is implemented for the attitude control of a spacecraft in Section IV.

II. STABILITY ON RIEMANNIAN MANIFOLDS

The motion of a system on a Riemannian manifold is considered in this paper. Before going into details of the controller synthesis some basic properties of Riemannian manifolds will be shortly reviewed, [13]. In the second part of this section standard theorems of stability will be reformulated in the framework of the Riemannian manifold.

The central attribute of the Riemannian manifold is that it is a metric space, with the metric $d(\boldsymbol{q}_1, \boldsymbol{q}_2)$ defined as the infimum of the length of the curves joining the points $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. In this contstruction a notion of a field of bilinear forms will be introduced.

Definition 1: A field Φ on a manifold M consits of a function assigning to each point $\boldsymbol{q}$ of M a smooth bilinear form Φ_q on TM_q, $\Phi_q : TM_q \times TM_q \to \mathbb{R}$

Now the Riemanian manifold is defined in the following way.

Definition 2: A manifold M is called a Riemannian manifold if there is defined a field of symmetric, positive definite, bilinear forms Φ; Φ is called the Riemannian metric.

The vital feature of the Riemannian manifold is that the tangent space TM_q with the scalar product defined by the Riemannian metric $\Phi_q(\boldsymbol{X}_q, \boldsymbol{Y}_q)$ is made into a Euclidean space. The length L of a curve $t \mapsto \boldsymbol{q}(t), a \leq t \leq b$ such that $\boldsymbol{q}_1 = \boldsymbol{q}(a)$ and $\boldsymbol{q}_2 = \boldsymbol{q}(b)$ is defined as the value of the integral

$$L = \int_a^b \left(\Phi \left(\frac{d\boldsymbol{q}}{dt}, \frac{d\boldsymbol{q}}{dt} \right) \right)^{\frac{1}{2}} dt. \tag{1}$$

If all curves from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ are considered, it is possible to choose one whose length reaches the infimum denoted by $d(\boldsymbol{q}_1, \boldsymbol{q}_2)$. The function $d : M \times M \to \mathbb{R}_+$ is a metric. It is symmetric and nonnegative by construction. To check the triangle inequality it is enough to observe that a curve from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ and a curve form $\boldsymbol{q}_2$ to $\boldsymbol{q}_3$ can be joined to give a curve from $\boldsymbol{q}_1$ to $\boldsymbol{q}_3$, whose length is the sum of the lengths of the latter two.

Theorem 1: A connected Riemannian manifold M is a metric space with the metric $d : M \times M \to \mathbb{R}_+$ defined by $d(\boldsymbol{q}_1, \boldsymbol{q}_2)$.

Example 1: In Section IV two Riemannian manifolds will be of particular interest for spacecraft control: the groups of the unit quaternions, S^3, and the special orthogonal group $SO_3(\mathbb{R})$. On the sphere S^3, the Riemannian metric is the scalar product, $\Phi(\dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = \langle \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}} \rangle$, and the metric $d(\boldsymbol{q}_1, \boldsymbol{q}_2)$ is the length of the arc of the great circle connecting $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. The Riemannian metric on $SO(3)$ is $\Phi(\dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = \mathrm{tr}(\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}^{\mathrm{T}})$, and the metric is given by the Rodrigues formula

$$d(\boldsymbol{q}_1, \boldsymbol{q}_2) = \left| \arccos \left(\frac{1}{2} \mathrm{tr}(\boldsymbol{q}_2 \boldsymbol{q}_1^{\mathrm{T}}) - 1) \right) \right|. \tag{2}$$

This is the angle of the rotation about the eigen axis of the matrix $\boldsymbol{q}_2 \boldsymbol{q}_1^T$ corresponding to the eigenvalue 1.

The customary concept of stability in the literature on nonlinear control e.g. [14], [5] is formulated using features of the metric in the Euclidean space. The generalization of these results to the systems defined on a Riemannian manifolds involves only replacing the Euclidean metric with an abstract metric. The the standard definitions of positive definiteness sounds:

Definition 3: A function $V : \mathbb{R}_+ \times M \to \mathbb{R}$ is said to be a locally positive definite function (lpdf) around an equilibrium $\boldsymbol{q}_0$ if it is continuous, $V(t, \boldsymbol{q}_0) = 0 \; \forall t \in \mathbb{R}_+$, and there exist a constant $r > 0$ and a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ of class K [1] such that

$$\alpha(d(\boldsymbol{q}, \boldsymbol{q}_0)) \leq V(t, \boldsymbol{q}), \; \forall t \geq 0, \; d(\boldsymbol{q}, \boldsymbol{q}_0) < r. \tag{3}$$

V is a locally negative definite function (lndf) around $\boldsymbol{q}_0$ if $-V$ is an lpdf around $\boldsymbol{q}_0$. V is locally descrescent around $\boldsymbol{q}_0$ if there exists a constant $r > 0$ and a function β of class K such that

$$V(t, \boldsymbol{q}) \leq \beta(d(\boldsymbol{q}, \boldsymbol{q}_0)), \; \forall t \geq 0, \; d(\boldsymbol{q}, \boldsymbol{q}_0) < r. \tag{4}$$

All the properties stated in the definition are global if they are valid for every $\boldsymbol{q}$ on the manifold M.

Remark 1: Notice that if the manifold M is compact then there exist a constant r such that $d(\boldsymbol{q}, \boldsymbol{q}_0) < r \; \forall \boldsymbol{q} \in M$ and if the conditions stated in Definition 3 are valid for this r the local properties become global.

Example 2: Consider a function $V(t) = 1 - q_0(t)$, where $q_0(t)$ is the scalar part of the unit quaternion. In the framework of Euclidean space the function $V(t)$ is not definite, but on S^3 it is positive definite around the group identity element $\boldsymbol{e} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$. According to physical interpretation in [15] $q_0 = \cos \frac{1}{2}\phi$, where $\phi \in \langle 0, \pi \rangle$ is the metric on

[1]continuous and strictly increasing; details in [5] pp. 144

S^3, i.e. the angle of rotation about the instantaneous axis of rotation. It is seen that V is of class K and $V(\boldsymbol{e}) = 0$ for each r, and hence globally positive definite function; shortly positive definite function (pdf).

We conclude this section with the formulation of the Lyapunov direct method on Riemannian manifolds.

Theorem 2: The equilibrium $\boldsymbol{q}_0$ is locally uniformly asymptotically stable if there exists a C^1 descrescent lpdf V such that $-\dot{V}$ is lpdf.

Proof of Theorem 2: Analyzing the standard proof for uniform asymptotic stability in $\mathbb{R}^n$ with metric $\| \boldsymbol{x} - \boldsymbol{x}_0 \|$, e.g. as in [5] Theorem 25, it is concluded the proof is also true for Theorem 2, i.e. for the Riemannian manifold M with metric $d(\boldsymbol{x}, \boldsymbol{x}_0)$. □

In a similar manner a global version of Theorem 2 can be formulated by replacing lpdf with pdf. In fact most of the "Lyapunov like" theorems including those for non-autonomous systems may be formulated on the Riemannian manifolds, e.g. Krasovskii-LaSalle theorem for periodic systems [5] pp.178-179 used in the example of the magnetic attitude control in Subsection V-C sounds

Theorem 3: Suppose the system is periodic [2] and there exists a C^1 function $V : \mathbb{R}_+ \times M \to \mathbb{R}$ having the same period as the system such that $V(t)$ is lpdf, $\dot{V}(t) \leq 0$. Define $R = \{\boldsymbol{q} \in M : \exists t \geq 0 \text{ such that } \dot{V}(t) = 0\}$ and suppose that the largest invariant set in R is the trivial trajectory $q = q_0$. Then the equilibrium q_0 is uniformly asymptotically stable.

Proof of Theorem 3: The proof is a direct consequence of Lemma 71 in [5] by noticing that the proof of this lemma can be analogously carried out for an arbitrary Riemannian manifold. □

III. CONTROL OF HAMILTONIAN SYSTEMS

A problem of stabilization to a reference, a certain point in the phase plane, is the topic of this section. Two systems will be considered: one corresponding to the actual plant and a system which is counterpart to the control objectives. The latter will be called the system of objectives. The motion of the plant and the system of objectives are described using Hamilton's canonical form. The formulation of kinetic energy for both systems is the same, however the potential energy of the system of objectives is expressed such that the reference is stable. If in addition dissipation is added the reference becomes asymptotically stable. The control action is now chosen such that the flow of the plant coincides with the flow of the system of objectives.

A. System Canonical Form

The system with no control action is assumed to be conservative. Furthermore the holonomic constrains are imposed on the system. The flow of such a system evolves on a certain Riemannian manifold M. The potential energy of the system is $U(\boldsymbol{q})$, where $\boldsymbol{q} \in M$. The kinetic energy may be defined using Riemannian metric, i.e. a positive definite quadratic form Φ on tangent space TM_q as in [10],

$$T = \frac{1}{2}\Phi(\boldsymbol{v}, \boldsymbol{v}), \text{ where } \boldsymbol{v} \in TM_q. \tag{5}$$

Remark 2: The tangent space TM_q is final dimensional, its dimension is the same as the dimension of the manifold M, hence the bilinear form $\Phi(\boldsymbol{v}, \boldsymbol{v})$ can be represented as $\boldsymbol{v}^T \boldsymbol{\Psi} \boldsymbol{v}$, where the matrix $\boldsymbol{\Psi}$ is positive definite, and $\Phi(\boldsymbol{v}, \boldsymbol{v})$ can be written as standard Euclidean scalar product

$$\Phi(\boldsymbol{v}, \boldsymbol{v}) = \langle \boldsymbol{\Psi}^T \boldsymbol{v}, \boldsymbol{v} \rangle. \tag{6}$$

The system with no control action is considered first. Having the Lagrange function L

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q}), \tag{7}$$

the hamiltonian is given by

$$H(\boldsymbol{q}, \boldsymbol{p}) = \langle \boldsymbol{p}, \dot{\boldsymbol{q}} \rangle - L(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{8}$$

where $\langle \boldsymbol{p}, \dot{\boldsymbol{q}} \rangle = \boldsymbol{p}^T \dot{\boldsymbol{q}}$ denotes a scalar product in the Euclidean space, the generalized momentum $\boldsymbol{p}$ is calculated using the equality $\boldsymbol{p} = \partial L / \partial \dot{\boldsymbol{q}}$. Now the equations of motion are represented in Hamilton's canonical equations

$$\begin{aligned} \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{q}} \\ \dot{\boldsymbol{q}} &= \frac{\partial H}{\partial \boldsymbol{p}}. \end{aligned} \tag{9}$$

[2]the vector fields describing motion of the system are periodic $\boldsymbol{f}_i(\boldsymbol{q}, t) = \boldsymbol{f}_i(\boldsymbol{q}, t + T)$

Remark 3: The Hamilton's function formulated in the paper does not depend explicitly on time, hence $H(\boldsymbol{q}, \boldsymbol{p}) = const$

Remark 4: The term

$$\langle \boldsymbol{p}, \dot{\boldsymbol{q}} \rangle - T(\boldsymbol{q}, \dot{\boldsymbol{q}}), \text{ where } \boldsymbol{p} = \frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} \tag{10}$$

in the Hanilton's function is lpdf around $\dot{\boldsymbol{q}} = \boldsymbol{0}$ since locally can be represented as

$$\langle \boldsymbol{p}, \dot{\boldsymbol{q}} \rangle - T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \left(\frac{\partial v(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} \right)^T \boldsymbol{\Psi} \frac{\partial v(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} \dot{\boldsymbol{q}}. \tag{11}$$

For majority of the mechanical systems the kinetic energy can be expressed in the form, [10]

$$T = \dot{\boldsymbol{q}}^T \boldsymbol{\Omega}^T(\boldsymbol{q}) \boldsymbol{\Omega}(\boldsymbol{q}) \dot{\boldsymbol{q}}, \tag{12}$$

hence the term (10) and Hamilton's function are pdf.

The control action is regarded as the generalized external force $\boldsymbol{M}_p$ acting upon the system. Following the lines of [16] pp. 316 the Hamilton's canonical equations are

$$\begin{aligned} \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{q}} + \boldsymbol{M}_p \\ \dot{\boldsymbol{q}} &= \frac{\partial H}{\partial \boldsymbol{p}}. \end{aligned} \tag{13}$$

B. System of Objectives

The equation of motions for the system of objectives will be formulated in the Hamilton's canonical form. It is assumed that the system of objectives inherits the kinetic energy T, whereas an extra contribution a function $V : M \to \mathbb{R}$ is added to the original potential energy U

$$U_d(\boldsymbol{q}) = U(\boldsymbol{q}) + V(\boldsymbol{q}). \tag{14}$$

The function $V(\boldsymbol{q})$ is designed such that the reference $\boldsymbol{q}_0$ becomes stable, i.e. the minimum of the potential energy U_d is reached at $\boldsymbol{q}_0$.

The equations of motion for the system of objectives are given by Eqs. (8) and (9), where the lagrangian is $L = L_o$

$$L_o(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U_d(\boldsymbol{q}), \tag{15}$$

The system of objectives is not asymptotically stable yet, but it is stable in Lyapunov sense, which can be shown applying a Lyapunov function

$$v(t) = H_o(t) - U_d(\boldsymbol{q}_0), \tag{16}$$

where

$$H_o(\boldsymbol{q}(t), \boldsymbol{p}(t)) = \langle \boldsymbol{p}, \dot{\boldsymbol{q}} \rangle - L_o(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{17}$$

The function $v(t)$ is constant and positive definite around $\boldsymbol{q}_0$. In order to get asymptotic stability a dissipation term is added in the system of objectives. The work of the dissipation force has to be negative semidefinite. In general the work of the field $\boldsymbol{M}_d$ on the path l is defined as follows

$$W = \int_l \langle \boldsymbol{M}_d, \boldsymbol{dq} \rangle, \tag{18}$$

and the time derivative of the work is now

$$\dot{W} = \langle \boldsymbol{M}_d, \dot{\boldsymbol{q}} \rangle. \tag{19}$$

Proposition 1: Consider a system given by the Hamilton's canonical equations

$$\begin{aligned} \dot{\boldsymbol{p}} &= -\frac{\partial H_o}{\partial q} + \boldsymbol{M}_d \\ \dot{\boldsymbol{q}} &= \frac{\partial H_o}{\partial p}, \end{aligned} \tag{20}$$

where $H_o(\boldsymbol{q},\boldsymbol{p}) = \langle \boldsymbol{p}, \dot{\boldsymbol{q}} \rangle - T(\boldsymbol{q}, \dot{\boldsymbol{q}}) + U_d(\boldsymbol{q})$ is the hamiltonian, T is the kinetic energy of the plant, U_d is lpdf around $\boldsymbol{q}_0$ given by Eq. (14), and the work W done by the vector field $\boldsymbol{M}_d$ is given by

$$W = \int_l \langle \boldsymbol{M}_d, \boldsymbol{dq} \rangle \tag{21}$$

If the time derivative dW/dt is lndf then the system (20) is locally uniformly asymptotically stable

Remark 5: If U_d is pdf and dW/dt is ndf then asymptotic stability in Proposition 1 is global.

Example 3: As an example of a dissipation term take $\boldsymbol{M}_d = \boldsymbol{K}\dot{\boldsymbol{q}}$, where the matrix $\boldsymbol{K}$ is negative definite. According to Eq. (18). The work done by $\boldsymbol{M}_d$ is equivalent to

$$W = \int_{t_0}^{t} \langle \boldsymbol{K}\dot{\boldsymbol{q}}, \dot{\boldsymbol{q}} \rangle dt, \tag{22}$$

hence the time derivative is $\dot{W} = \dot{\boldsymbol{q}}^T \boldsymbol{K} \dot{\boldsymbol{q}}$, which is ndf.

Proof of Proposition 1: Consider a Lyapunov candidate function $v(t) = H_o(t)$, which is lpdf according to Remark 4. The time derivative of H_o is

$$\dot{H}_o = \left(\frac{\partial H_o}{\partial \boldsymbol{p}}\right)^T \dot{\boldsymbol{p}} + \left(\frac{\partial H_o}{\partial \boldsymbol{q}}\right)^T \dot{\boldsymbol{q}}. \tag{23}$$

Using the Hamilton's canonical equations

$$\dot{H}_o = -\left(\frac{\partial H_o}{\partial \boldsymbol{p}}\right)^T \frac{\partial H_o}{\partial \boldsymbol{q}} + \left(\frac{\partial H_o}{\partial \boldsymbol{p}}\right)^T \boldsymbol{M}_p + \left(\frac{\partial H_o}{\partial \boldsymbol{q}}\right)^T \frac{\partial H_o}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}}^T \boldsymbol{M}_d = \dot{W} \tag{24}$$

which is lpdf, thus according to Theorem 2 for $(\boldsymbol{q},\boldsymbol{p}) \in M$ the equilibrium $(\boldsymbol{q}_0, \boldsymbol{0})$ is uniformly asymptotically stable. □
Proposition 1 can be reformulated for periodic systems if the time derivative of the work done by $\boldsymbol{M}_d$ is negative semidefinite. The proof is very similar, the only difference is that Theorem 3 is used instead of Theorem 2.

Corollary 1: Consider the system (20) as in Proposition 1. Let the field $\boldsymbol{M}_d(t)$ is periodic with the period T ($M_d(t) = M_d(t+T)$) and the time derivative of work W done by the vector field $\boldsymbol{M}_d$ in Eq. (21) is locally negative semidefinite. Then the system is locally uniformly stable.

C. Control Synthesis

The objective of the control design in the framework of the Hamilton's formalism is to generate $\boldsymbol{M}_p$ such that the equations of motion for the plant and the system of objectives are equivalent. In other words, the control action has to compensate for two terms: one originating from the function V in Eq. (14) and the second contributing from the work W in Eq. (21). The main results of this paper are summarized in the following theorem:

Theorem 4: Consider a plant given in Hamilton's canonical form (13). The control action

$$\boldsymbol{M}_p = -\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}} + \boldsymbol{M}_d, \tag{25}$$

where V is given by Eq. (14), U_d is lpdf around $\boldsymbol{q}_0$ and the time derivative of the work $dW/dt = \langle \boldsymbol{M}_d, \dot{\boldsymbol{q}} \rangle$ is lndf around $\boldsymbol{q}_0$ then the feedback system is locally uniformly asymptotically stable to the reference $\boldsymbol{q}_0$. If U_d is pdf and $dW/dt = \langle \boldsymbol{M}_d, \dot{\boldsymbol{q}} \rangle$ is ndf around $\boldsymbol{q}_0$ then the uniform asymptotic stability is global.

Proof of Theorem 4: If the control action (25) is substituted in Eq. (13) for $\boldsymbol{M}_p$ the equations of motion for the plant are identical to the system given by Eq. (20). The hypothesis of Proposition 1 are satisfied and hence the feedback system governed by Eq. (25) is locally uniformly asymptotically stable. Using Remark 5 instead of Proposition 1 global stability is proved. □

The control action in Eq. (25) consists of two terms. The first one determines sensitivity of the closed loop system towards disturbances, whereas the second decides the length of the settling time. For a conservative system the disturbance force has to perform a work $W = U(\boldsymbol{q}_1) - U(\boldsymbol{q}_0)$ to change the potential energy from the level $U(\boldsymbol{q}_0)$ to $U(\boldsymbol{q}_1)$. Thereby, the larger the gradient $\partial V(\boldsymbol{q})/\partial \boldsymbol{q}$ the larger work necessary to move the plant from the point $\boldsymbol{q}_0$ to $\boldsymbol{q}_1$. The dissipation $dW(t)/dt$ is related to the amount of energy dissipated by the controller in a certain fixed time T, thus it corresponds to the response time. This control structure can be compared with a standard PD controller used for linear systems.

IV. MOTION CONTROL OF A RIGID BODY

A method for the control synthesis presented in the last section is readily applicable in the systems where the dynamics and kinematics are represented in E^{2n}, however e.g. motion involving the rotation is typically not expressed in the canonical form. The dynamics are given by the Euler equation in E^3, whereas the most natural description of the kinematics is given by the elements of the group (also the Riemannian manifold) $SO_3(\mathbb{R})$ or by the unit quaternion, an elemnt of S^3.

A. Rigid Body Canonical Form

The subject of finding a transformation to Hamilton's canonical form is often addressed in the literuture of modern celestial mechanics. [11] studied the canonical transformation $y = f(x)$ of the state space $\boldsymbol{y} \in \mathbb{R}^{2n}$ to $\boldsymbol{x} \in \mathbb{R}^{2m}$ with $m > n$. In this paper only a special case $m = n + 1$ is investigated, since the results can be applied to the rotational motion of a rigid body in function of the unit quaternion $\boldsymbol{q} := \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T \in S^3$ and the conjugate momenta $\boldsymbol{p} := \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix}^T$. Interested reader is refered to [12] for more detailed study on this topic.

The kinetic energy of a rigid body rotation is a function of the instantaneous angular velocity $\boldsymbol{\omega}$

$$T = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{J} \boldsymbol{\omega}, \tag{26}$$

where $\boldsymbol{J}$ is the inertia tensor. The angular velocity vector may be regarded as an element of the quaternion vector space $\boldsymbol{\Omega} := \begin{bmatrix} 0 & \boldsymbol{\omega}^T \end{bmatrix}^T \in E \times E^3$, and the Equation (26) takes the form

$$T = \frac{1}{2}\boldsymbol{\Omega}^T \boldsymbol{J}^* \boldsymbol{\Omega}, \tag{27}$$

where $\boldsymbol{J}^*$ is a block diagonal matrix

$$\boldsymbol{J}^* = \begin{bmatrix} J_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J} \end{bmatrix}. \tag{28}$$

The element J_0 takes in general an arbitrary nonsingular value. Using the standard quaternion parameterizations of kinematics

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{Q}(\boldsymbol{q})\boldsymbol{\Omega}, \text{ where } \boldsymbol{Q}(\boldsymbol{q}) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \tag{29}$$

the kinetic energy is

$$T = 2\boldsymbol{q}^T \boldsymbol{Q}(\dot{\boldsymbol{q}}) \boldsymbol{J}^* \boldsymbol{Q}^T(\dot{\boldsymbol{q}})\boldsymbol{q}. \tag{30}$$

The lagrangian for the rigid body motion is now $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q})$, where T is given by Eq. (30). Applying the hamiltonian $H(\boldsymbol{q}, \boldsymbol{p}) = \langle \boldsymbol{p}, \dot{\boldsymbol{q}} \rangle - L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and Eq. (13) the canonical equations are formulated

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{1}{4}\boldsymbol{Q}(\boldsymbol{q})\boldsymbol{J}^{*-1}\boldsymbol{Q}^T(\boldsymbol{q})\boldsymbol{p} \\ \dot{\boldsymbol{p}} &= -\frac{1}{4}\boldsymbol{Q}(\boldsymbol{p})\boldsymbol{J}^{*-1}\boldsymbol{Q}^T(\boldsymbol{p})\boldsymbol{q} - \frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} + \boldsymbol{M}_p. \end{aligned} \tag{31}$$

For the rotational motion the control action is a torque denoted here by $\boldsymbol{M}_c$. On a spacecraft the control torque is generated by a set of actuators such as gas jets, momentum/reaction wheels, electromagnetic coils. The work is invariant of a canonical transformation. To calculate $\boldsymbol{M}_p$ for a given torque $\boldsymbol{M}_c$ the time derivative of the work of $\boldsymbol{M}_p$ defined in Eq. (19) and $\boldsymbol{M}_c$ are used:

$$\dot{\boldsymbol{q}}^T(t)\boldsymbol{M}_p(t) = \dot{W}(t) = \boldsymbol{\omega}^T(t)\boldsymbol{M}_c(t). \tag{32}$$

Applying Eqs. (32) and (29) the time derivative of the work is

$$\dot{W}(t) = 2\dot{\boldsymbol{q}}^T(t)\boldsymbol{Q}\left(\boldsymbol{q}(t)\right)\boldsymbol{M}(t), \text{ where } \boldsymbol{M} = \begin{bmatrix} 0 & \boldsymbol{M}_c^T \end{bmatrix}^T \tag{33}$$

hence

$$\boldsymbol{M}_p(t) = 2\boldsymbol{Q}(\boldsymbol{q}(t))\boldsymbol{M}(t) \tag{34}$$

or equivalently

$$
\boldsymbol{M}_p(t) = 2\boldsymbol{R}(\boldsymbol{M}_c(t))\boldsymbol{q}(t), \text{ where } \boldsymbol{R}(\boldsymbol{M}_c) = \begin{bmatrix} 0 & -M_1 & -M_2 & -M_3 \\ M_1 & 0 & M_3 & -M_2 \\ M_2 & -M_3 & 0 & M_1 \\ M_3 & M_2 & -M_1 & 0 \end{bmatrix} \tag{35}
$$

Comparing Eq. (34) and the kinematics (29) a very important observation is made that $\boldsymbol{M}_p$ lies on the tangent space ${TS^3}_q$. Thereby, M_p computed from Eq. (25) is not necessarily producible by a physical actuator. As seen in Example 3 a damping term belonging to ${TS^3}_q$ is reasonably easy to design as a linear combination of the vector fields. The task becomes more involved if we wish to satisfy $\partial V(\boldsymbol{q})/\partial\boldsymbol{q} \in {TS^3}_q$. It will be shown in the theorem below that to generate a stable control action it is enough to use the orthogonal projection P_T of $\partial V(\boldsymbol{q})/\partial\boldsymbol{q}$ on the tangent space, ${TS^3}_q$.

Theorem 5: Consider a plant given in Hamilton's canonical form (13). The control action

$$
\boldsymbol{M}_p = -P_T \frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}} + \boldsymbol{M}_d, \tag{36}
$$

where P_T is the orthogonal projection on the tangent space TM_q, V is given by Eq. (14), U_d is lpdf around $\boldsymbol{q}_0$ and the time derivative of the work $\dot{W} = \langle \boldsymbol{M}_d, \dot{\boldsymbol{q}} \rangle$ is lndf around $\boldsymbol{q}_0$. Then the feedback system is locally uniformly asymptotically stable to the reference $\boldsymbol{q}_0$. If U_d is pdf and $\dot{W} = \langle \boldsymbol{M}_d, \dot{\boldsymbol{q}} \rangle$ is ndf around $\boldsymbol{q}_0$ then the uniform asymptotic stability is global.

Proof of Theorem 5: Take as the Lyapunov candidate function $v(t) = H(t)+V(t)$, where H is given by Eqs. (7) and (8). The time derivative of $v(t)$ is

$$
\dot{v} = -\left(\frac{\partial H}{\partial \boldsymbol{p}}\right)^T \frac{\partial H}{\partial \boldsymbol{q}} + \left(\frac{\partial H}{\partial \boldsymbol{p}}\right)^T \boldsymbol{M}_p + \left(\frac{\partial H}{\partial \boldsymbol{q}}\right)^T \frac{\partial H}{\partial \boldsymbol{p}} + \left(P_T \frac{\partial H}{\partial \boldsymbol{q}}\right)^T \dot{\boldsymbol{q}} + \left((\mathrm{id} - \mathrm{P_T})\frac{\partial \mathrm{H}}{\partial \boldsymbol{q}}\right)^T \dot{\boldsymbol{q}}, \tag{37}
$$

but $\dot{\boldsymbol{q}} \in TM_q$ and P_T is the orthogonal projection on TM_q thus the last term in Eq (37) is zero and

$$
\dot{v} = \dot{\boldsymbol{q}}^T \boldsymbol{M}_d = \dot{W}. \tag{38}
$$

which is lndf thus according to Theorem 2 the equilibrium $\boldsymbol{q}_0$ is uniformly asymptotically stable. □

Example 4: Consider a potential function in Example (2): $V(\boldsymbol{q}) = 1 - q_4$, and the zero dissipation term $\boldsymbol{M}_d = 0$. Then

$$
\boldsymbol{M}_p \equiv -P_T \frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}} = -\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}} + (\mathrm{id} - \mathrm{P_T}) \frac{\partial \mathrm{V}(\boldsymbol{q})}{\partial \boldsymbol{q}}, \tag{39}
$$

thus

$$
\begin{aligned}
\boldsymbol{M}_p &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T - q_0 \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T \\
&= \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 & -q_0 q_1 & -q_0 q_2 & -q_0 q_3 \end{bmatrix}^T
\end{aligned} \tag{40}
$$

but $\boldsymbol{M}_c(t) = 1/2\boldsymbol{Q}^T(\boldsymbol{q}(t))\,\boldsymbol{M}_p(t)$ therefore $\boldsymbol{M}_c(t) = -1/2\begin{bmatrix} q_1(t) & q_2(t) & q_3(t) \end{bmatrix}^T$.

V. Spacecraft Attitude Control

The theoretical findings developed in the preceding chapters will be implemented to the spacecraft attitude control issue. Three topics are addressed: spacecraft stabilization in the inertial frame, libration damping with the use of electromagnetic coils and a slew maneuver with an objective imposed of avoiding certain undesirable orientations.

A. Stabilization in Inertial Frame

A spacecraft motion in the inertial coordinate system was provided in Eq. (31) with the potential energy $U = 0$. The control objective is to correct the attitude to the reference $\boldsymbol{q}_{ref}$.

The proposed system of objectives is

$$
H_d(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{M}_d) = H(\boldsymbol{q}, \boldsymbol{p}) + V(\boldsymbol{q}) + \langle \boldsymbol{M}_d, \boldsymbol{q} \rangle, \tag{41}
$$

where H corresponds to the hamiltonian of the rigid body motion with $U = 0$. The potential energy $V(\boldsymbol{q}) = 2k_1(1-q_0)$, where k_1 is a positive constant and q_0 is the scalar part of the quaternion

$$
\begin{bmatrix} q_0 & \tilde{\boldsymbol{q}}^T \end{bmatrix}^T := \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T = \boldsymbol{Q}^T(\boldsymbol{q}_{ref})\boldsymbol{q}. \tag{42}
$$

The function V is positive definite around $\boldsymbol{q}_{ref}$, see Example 2. The dissipation force $\boldsymbol{M}_d = \boldsymbol{K}_2\dot{\boldsymbol{q}}$ is chosen, where the matrix $\boldsymbol{K}_2$ is negative definite. Now, the work W dissipates the energy, since $\dot{W} = \langle \boldsymbol{K}_2\dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}\rangle$ is ndf. According to Theorem 5 the control (36) is uniformly asymptoticaly stable. The closed form of the control law is derived using Example 3 and 4

$$\boldsymbol{M}_c = -k_1\tilde{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{Q}^T(\boldsymbol{q})\boldsymbol{K}_2\dot{\boldsymbol{q}}. \tag{43}$$

If the matrix $\boldsymbol{K}_2$ is substituted by a negative scalar $4k_2$ a more compact form of Eq. (43) can be calculated applying the relation in the kinematics (29)

$$\boldsymbol{M}_c = -k_1\tilde{\boldsymbol{q}} + k_2\boldsymbol{\omega}, \tag{44}$$

where $\boldsymbol{\omega}$ is the angular velocity of the spacecraft.

Remark 6: If the potential energy $V(\boldsymbol{q}) = 2k_1(1 - q_0)$ is replace by $V(\boldsymbol{q}) = 2k_1(1 + q_0)$ with the minimum at $q_0 = -1$ then the control law

$$\boldsymbol{M}_c = k_1\tilde{\boldsymbol{q}} + k_2\boldsymbol{\omega} \tag{45}$$

assures that the equilibrium $\boldsymbol{q} = -\boldsymbol{q}_{ref}$ is asymptotically stable. At this point it is important to notice that both $\boldsymbol{q}_{ref}$ and $-\boldsymbol{q}_{ref}$ define the same physical orientation.

B. Slew Maneuver with Region Avoidance

The control objective of the three-axis attitude control addressed in the preceding subsection is extended here to an additional design objective: During the slew maneuver a certain orientation is prohibited. This scenario is encountered when the attitude is acquired from the star camera; looking towards the sun causes blindness of the CCD chip.

A spacecraft motion is once more given by Eq. (31) with the potential energy $U = 0$. The orientation of the obstacle in the inertial frame is given by the quaternion ${}^{o}_{i}\boldsymbol{q}$, whereas the reference is specified by ${}^{r}_{i}\boldsymbol{q}$. The spacecraft's attitude in the inertial coordinate system is provided by the star camera, $\boldsymbol{q} \equiv {}^{s}_{i}\boldsymbol{q}$.

The potential function in this case study is shaped such that its minimum is at the reference and the maximum at the obstacle attitude. The potential energy proposed is

$$V(t) = 2 - {}^{s}_{r}\boldsymbol{q}_0 - {}^{s}_{o}\boldsymbol{q}_0^2, \tag{46}$$

where ${}^{s}_{r}\boldsymbol{q}_0$ is the scalar part of the quaternion ${}^{s}_{r}\boldsymbol{q} = {}^{r}_{i}\boldsymbol{q}^*\boldsymbol{q}$ [3], and ${}^{s}_{r}\boldsymbol{q}_0$ is the scalar part of ${}^{s}_{o}\boldsymbol{q} = {}^{o}_{i}\boldsymbol{q}^*\boldsymbol{q}$, hence

$$\begin{aligned} {}^{s}_{r}\boldsymbol{q}_0 &= \boldsymbol{n}_r^T\boldsymbol{q} \\ {}^{s}_{o}\boldsymbol{q}_0 &= \boldsymbol{n}_o^T\boldsymbol{q}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} \boldsymbol{n}_r &= \begin{bmatrix} {}^{s}_{r}q_0 & {}^{s}_{r}q_1 & {}^{s}_{r}q_2 & {}^{s}_{r}q_3 \end{bmatrix}^T \\ \boldsymbol{n}_o &= \begin{bmatrix} {}^{s}_{o}q_0 & {}^{s}_{o}q_1 & {}^{s}_{o}q_2 & {}^{s}_{o}q_3 \end{bmatrix}^T. \end{aligned} \tag{48}$$

Applying the potential energy in Eq. (46) and the dissipation force $\boldsymbol{M}_d = \boldsymbol{K}\dot{\boldsymbol{q}}$ to the generic control law (25) gives

$$\boldsymbol{M}_p = \boldsymbol{n}_r + 2\boldsymbol{n}_o^T\boldsymbol{q}\;\boldsymbol{n}_r - \boldsymbol{n}_r^T\boldsymbol{q}\;\boldsymbol{q} - 2\left(\boldsymbol{n}_o^T\boldsymbol{q}\right)^2\;\boldsymbol{q} + 1/2\boldsymbol{K}\boldsymbol{Q}(\boldsymbol{q})\boldsymbol{\Omega}, \tag{49}$$

and the control torque is $\boldsymbol{M}_c(t) = 1/2\boldsymbol{Q}^T(\boldsymbol{q}(t))\,\boldsymbol{M}_p(t)$.

C. Libration Damping

A very cost and energy effective control principle for a gravity gradient stabilized satellite is to use the electromagnetic coils for spacecraft actuation. The concept is that the interaction between the Earth's magnetic field and a magnetic field generated by the coil results in a mechanical torque. This is expressed by the formula

$$\boldsymbol{M}_c(t) = \boldsymbol{m}(t) \times \boldsymbol{B}(t), \tag{50}$$

i.e. the control torque $\boldsymbol{M}_c$ is the vector product of the magnetic moment $\boldsymbol{m}$ generated in the coils and the geomagnetic field vector $\boldsymbol{B}$. The motion of a spacecraft on a low Earth orbit can be very concisely described by the following hamiltonian, for more details see [17]

$$H = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} + \frac{3}{2}\omega_o^2\boldsymbol{k}^T\boldsymbol{J}\boldsymbol{k} - \frac{1}{2}\omega_o^2\boldsymbol{j}^T\boldsymbol{J}\boldsymbol{j}, \tag{51}$$

[3] $\boldsymbol{q}^*$ denotes $\boldsymbol{q}$ conjugated

where $\boldsymbol{\omega}$ is the angular velocity of the spacecraft principal frame relative to the LVLH coordinate system [4], ω_o is the mean motion, $\boldsymbol{j}, \boldsymbol{k}$ are the unit vectors along the y and z axes of LVLH. It is assumed in Eq. (51) that the principal axes of the spacecraft are such that the maximum moment of inertia is about the y axis, and the minimum about the z axis.

A closer look at the potential energy, the last two terms in Eq. (51), reviles that the system has four stable equilibria

$$\{\boldsymbol{\omega}, \boldsymbol{j}, \boldsymbol{k}\} = \{\mathbf{0}, \pm\mathbf{1}_j, \pm\mathbf{1}_k\}, \tag{52}$$

where $\mathbf{1}_j = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $\mathbf{1}_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. In other words the equilibria are such that the spacecraft principal and LVLH y axes coincide or point in the opposite directions and the z axes of the spacecraft and LVLH frames coincide or are opposite. If one of those equalibria is the system's desired reference then it is sufficient to use a control providing pure damping.

In [18] the following control law was proposed

$$\boldsymbol{m}(t) = \boldsymbol{H}\boldsymbol{\omega} \times \boldsymbol{B}, \tag{53}$$

where $\boldsymbol{H}$ is a positive definite matrix. Magnetic torquing following Eq. (53) obviously introduces time dependency in the equations of satellite motion. This time variation is periodic by nature, which arises from two superimposed periodic fluctuations of the geomagnetic field. One is due to revolution of the satellite around the Earth and the second due to rotation of the Earth.

To show asymptotic stability of the suggested control law it is enough to calculate the time derivative of the work done by the field $\boldsymbol{M}_c(t) = \boldsymbol{m}(t) \times \boldsymbol{B}(t)$

$$\dot{W}(t) = \boldsymbol{\omega}^T(t)\boldsymbol{M}_c(t) = -\boldsymbol{\omega}^T \boldsymbol{S}^T(\boldsymbol{B}(t))\boldsymbol{S}(\boldsymbol{B}(t))\boldsymbol{H}\boldsymbol{\omega}, \tag{54}$$

where $\boldsymbol{S}(\boldsymbol{B})$ is a 3 by 3 skew symmetric matrix representing the vector product operator: $\boldsymbol{B}\times$. From Eq. (54) it is seen that $\dot{W}(t)$ is only negative semidefinite. However, two observations can be here employed. The first is that the Earth's magnetic field is periodic, and the second that the largest invariant set contained in the set $\{\boldsymbol{\omega} : \dot{W} = 0\}$ is $\boldsymbol{\omega} \equiv 0$. Thus applying Corollary 1 the system is proved to be asymptotically stable to one of the attractors $\{\boldsymbol{\omega}, \boldsymbol{j}, \boldsymbol{k}\} = \{\mathbf{0}, \pm\mathbf{1}_j, \pm\mathbf{1}_k\}$.

VI. CONCLUSION

An ellegant scheme for control design of mechanical systems was proposed in this work. The desired feedback dynamics was specified in a Hamilton's canonical form. The designer has to define a desired potential energy with minimum at the reference point and a dissipative term. The resultant controller is uniformly asymptoticaly stable. The algorithm can be implemented for systems which trajectories lies on an arbitrary Riemannian manifold. Special care was taken to redefine the standard notions of stability to fit to the geometric framework used in this paper. The results were applied to the rotational motion of a rigid body in function of the unit quaternion and the its conjugate momenta. Three problems were addressed in the paper: spacecraft stabilization in the inertial frame, libration damping with the use of electromagnetic coils and a slew maneuver with an additional objective of avoiding undesirable regions e.g. causing blindness of optical sensors.

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