Rafał Wiśniewski

Lecture Notes on Modeling of a Spacecraft

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Aalborg Universitet

Afdeling for Proceskontrol
Fr. Bajers Vej 7c
DK-9220 Aalborg

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1. Introduction to Lecture Notes

The objective of this lecture notes is to establish mathematical model of a spacecraft on a low earth orbit. The material presented in this notes is used in the course Modeling of Mechanical Systems at Aalborg University, Department of Control Engineering.

Motion of a spacecraft is described by a set of nonlinear ordinary differential equations. Its solution: angular momentum and the satellite orientation remain on some geometrical surface. This motion can be described with respect to certain coordinate systems, which are fixed or revoluting in the inertial frame. The spacecraft is considered as a rigid body which orientation is corrected to a reference frame by a feedback system. The definition of the coordinate systems used throughout this book are provided in Chapter 2.

The spacecraft motion can be divided into two parts: kinematics and dynamics. The kinematics characterizes relation between satellite's angular velocity and its orientation in space, the attitude. The minimum number of parameters locally identifying the attitude is three (Euler angles), however, four parameters are necessary for global attitude representation (a unit quaternion). Various attitude representations for spacecraft application are provided in Chapter 3.

The dynamics describes dependence between external torques and the spacecraft's angular velocity. The external torques are disturbances and a control torque. The control torque originates from the interaction between the Earth magnetic field and magnetic field generated in the magnetorquers. The main disturbances are the gravity gradient torque, aerodynamic drag and electromagnetic torque of the spacecraft's electronics. The dynamic equations of motion are provided in Chapter 5. The explicit formulas for kinetic and potential energy of a LEO satellite are given in Chapter 6.

2. Coordinate Systems

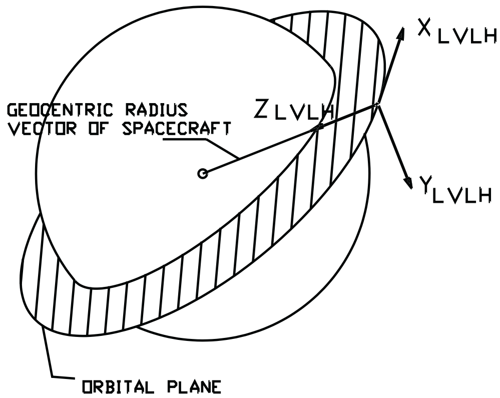

Fig. 2.1. Local-Vertical-local-Horizontal Coordinate System (LVLH)

The motion of a spacecraft is related to four coordinate systems: Principal Coordinate System (PCS), built on the spacecraft principal axes, a Spacecraft Body Coordinate System (SCB) corresponding to the satellite structure, a Local-Vertical-Local-Horizontal Coordinate System (LVLH) referring to the current position of the satellite in orbit, and an Earth Centered Inertial Coordinate System (SCI), an inertial frame with the origin in the Earth's center of mass. The formal definitions of these coordinate systems are

- **Principal Coordinate System (PCS)** is a right orthogonal coordinate system built on the Spacecraft's principal axes with the origin placed in the center of mass. The y axis is the axis of the maximum moment of inertia, the x axis is the intermediate, and the z axis is the minimum ($I_{\mathrm{y}} > I_{\mathrm{x}} > I_{\mathrm{z}}$). A vector $\boldsymbol{v}$ resolved in PCS is denoted by $\boldsymbol{v}_{\mathrm{p}}$.
- **Spacecraft Body Coordinate System (SCB)** is a right orthogonal coordinate system fixed in the spacecraft structure with the origin in the center of mass. It defines the orientation of attitude determination and control hardware (attitude sensors and actuators). A vector $\boldsymbol{v}$ resolved in PCS is denoted by $\boldsymbol{v}_{\mathrm{b}}$.

- **Local-Vertical-Local-Horizontal Coordinate System (LVLH)** is a right orthogonal coordinate system with the origin at the spacecraft's center of mass. The z axis (local vertical) is parallel to the radius vector and points from the spacecraft center of mass to the center of the Earth . The positive y axis is pointed in the direction of the negative angular momentum vector. The x axis (local horizontal) completes the right orthogonal coordinate system. The positive x axis lies in the orbital plane in the direction of the velocity vector (only identical to the velocity vector for perfectly circular orbits), Fig. 2.1. A vector $\boldsymbol{v}$ resolved in LVLH is denoted by $\boldsymbol{v}_{\mathrm{o}}$.
- **Earth Centered Inertial Coordinate System (ECI)** is the frame with the origin in the Earth's center. The z axis is parallel to the rotation axis of the Earth and points towards the North Pole. The x axis is parallel to the line connecting the center of the Earth with Vernal Equinox and points towards Vernal Equinox (Vernal Equinox is the point where the ecliptic crosses the Earth equator going from South to North on the first day of spring), Fig. 2.2. A vector $\boldsymbol{v}$ resolved in ECI is denoted by $\boldsymbol{v}_{\mathrm{i}}$.

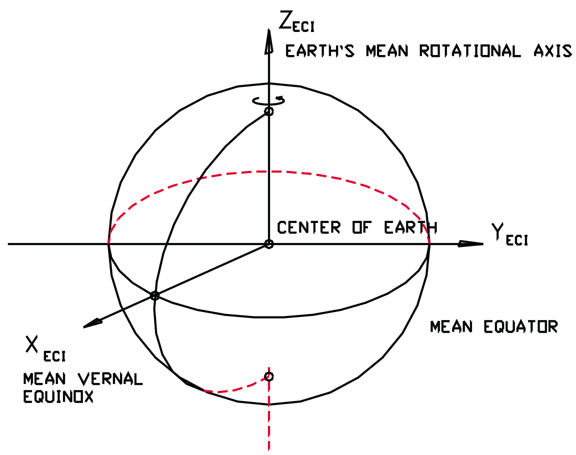

Fig. 2.2. Earth Centered Coordinate System (ECI)

3. Attitude Representations

The fundamental problem of the attitude representation is to specify an orientation of a coordinate system fixed in a spacecraft {s} with respect to a reference coordinates {r}. The orientation can be parameterized by several methods: a rotation matrix, a unit quaternion, Euler angles. The most natural attitude description is given by the rotation matrix, which is composed of a unit vectors of {s} projected on {r}. This representation uses nine components with 6 constrains, thus the attitude can be locally specified by three parameters. The Euler angles seems to be most physically appealing three parameter representation of attitude. The minimum number of parameters necessary to give a global representation is four. For this purpose a unit quaternion can be used, which consists of four components with an amplitude constraint.

3.1 Rotation Matrix

Consider a triad of unit vectors spanning the coordinate system {r}, and call them $\boldsymbol{i}_{\mathrm{r}}$, $\boldsymbol{j}_{\mathrm{r}}$, and $\boldsymbol{k}_{\mathrm{r}}$. The relations between these vector are

$$\boldsymbol{i}_{\mathrm{r}} \times \boldsymbol{j}_{\mathrm{r}} = \boldsymbol{k}_{\mathrm{r}}, \quad \boldsymbol{j}_{\mathrm{r}} \times \boldsymbol{k}_{\mathrm{r}} = \boldsymbol{i}_{\mathrm{r}} \text{ and } \boldsymbol{k}_{\mathrm{r}} \times \boldsymbol{i}_{\mathrm{r}} = \boldsymbol{j}_{\mathrm{r}}. \tag{3.1}$$

The basic problem is now to describe the orientation of this triad relative

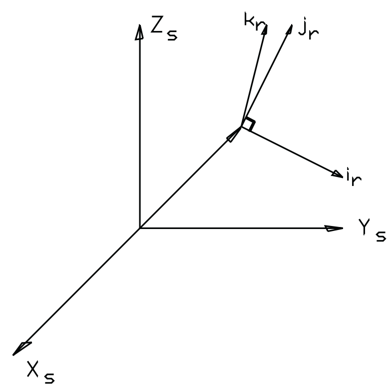

Fig. 3.1. The basic problem of attitude representation is to describe the orientation of the triad $\boldsymbol{i}_{\mathrm{r}}$, $\boldsymbol{j}_{\mathrm{r}}$, $\boldsymbol{k}_{\mathrm{r}}$ relative the coordinate system {s}.

the coordinate system {s}. This orientation is characterized completely by specifying the components of $\boldsymbol{i}_{\mathrm{r}}$, $\boldsymbol{j}_{\mathrm{r}}$, and $\boldsymbol{k}_{\mathrm{r}}$ along the x, y and z axes of {s}, see Fig. 3.1

$$(\boldsymbol{i}_{\mathrm{r}})_{\mathrm{s}} = \begin{bmatrix} i_{\mathrm{rx}} \\ i_{\mathrm{ry}} \\ i_{\mathrm{rz}} \end{bmatrix}, \quad (\boldsymbol{j}_{\mathrm{r}})_{\mathrm{s}} = \begin{bmatrix} j_{\mathrm{rx}} \\ j_{\mathrm{ry}} \\ j_{\mathrm{rz}} \end{bmatrix}, \quad (\boldsymbol{k}_{\mathrm{r}})_{\mathrm{s}} = \begin{bmatrix} k_{\mathrm{rx}} \\ k_{\mathrm{ry}} \\ k_{\mathrm{rz}} \end{bmatrix}. \tag{3.2}$$

Any vector $\boldsymbol{v}$ in {r} is a linear combination of the triad $\boldsymbol{i}_{\mathrm{r}}$, $\boldsymbol{j}_{\mathrm{r}}$, $\boldsymbol{k}_{\mathrm{r}}$

$$\boldsymbol{v}_{\mathrm{r}} = v_{\mathrm{x}}\ \boldsymbol{i}_{\mathrm{r}} + v_{\mathrm{y}}\ \boldsymbol{j}_{\mathrm{r}} + v_{\mathrm{z}}\ \boldsymbol{k}_{\mathrm{r}}. \tag{3.3}$$

Since the components of the triad along the axes of {s} are known the vector $\boldsymbol{v}$ in {s} is

$$\boldsymbol{v}_{\mathrm{s}} = v_{\mathrm{x}}\,(\boldsymbol{i}_{\mathrm{r}})_{\mathrm{s}} + v_{\mathrm{y}}\,(\boldsymbol{j}_{\mathrm{r}})_{\mathrm{s}} + v_{\mathrm{z}}\,(\boldsymbol{k}_{\mathrm{r}})_{\mathrm{s}} = \begin{bmatrix}(\boldsymbol{i}_{\mathrm{r}})_{\mathrm{s}} & (\boldsymbol{j}_{\mathrm{r}})_{\mathrm{s}} & (\boldsymbol{k}_{\mathrm{r}})_{\mathrm{s}}\end{bmatrix} \boldsymbol{v}_{\mathrm{r}}. \tag{3.4}$$

The 3 by 3 matrix $\mathsf{A} = \begin{bmatrix}(\boldsymbol{i}_{\mathrm{r}})_{\mathrm{s}} & (\boldsymbol{j}_{\mathrm{r}})_{\mathrm{s}} & (\boldsymbol{k}_{\mathrm{r}})_{\mathrm{s}}\end{bmatrix}$ is called the rotation matrix. It maps vectors from the coordinate system {r} to {s}. To suppress the notation the vectors $(\boldsymbol{i}_{\mathrm{r}})_{\mathrm{s}}$, $(\boldsymbol{j}_{\mathrm{r}})_{\mathrm{s}}$, $(\boldsymbol{k}_{\mathrm{r}})_{\mathrm{s}}$ will be denoted by $\boldsymbol{i}_{\mathrm{r}}$, $\boldsymbol{j}_{\mathrm{r}}$, $\boldsymbol{k}_{\mathrm{r}}$. Actually the projection on the {s} coordinates is the only interesting one, since $(\boldsymbol{i}_{\mathrm{r}})_{\mathrm{r}}$, $(\boldsymbol{j}_{\mathrm{r}})_{\mathrm{r}}$, $(\boldsymbol{k}_{\mathrm{r}})_{\mathrm{r}}$ is trivial.

There are six constraints imposed on the rotation matrix A. The first three comes from the fact the the length of vectors $\boldsymbol{i}_{\mathrm{r}}$, $\boldsymbol{j}_{\mathrm{r}}$, $\boldsymbol{k}_{\mathrm{r}}$ is one. The remaining three are due to mutual orthogonality of the triad, Eq. (3.1). These constraints in the language of the matrix product can be summarized in equation

$$\mathsf{A}\mathsf{A}^{\mathrm{T}} = \mathsf{A}^{\mathrm{T}}\mathsf{A} = \mathsf{E}. \tag{3.5}$$

It means that the inverse of a rotation matrix is given by its transposition.

3.2 Algebraic Properties of Rotation Matrix

A real square matrix A is called orthogonal if $\mathsf{A}^{\mathrm{T}} = \mathsf{A}^{-1}$, that the rotation matrix is a real orthogonal 3 by 3 matrix. It is well known from the linear algebra that if A is an n by n orthogonal matrix, then

1. for any column n-vector $\boldsymbol{v}$, $\parallel \mathsf{A}\boldsymbol{v} \parallel = \parallel \boldsymbol{v} \parallel$,
2. for any n-vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, $\langle \mathsf{A}\boldsymbol{v}, \mathsf{A}\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$, where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^{\mathrm{n}}$.

The first part of the statement follows from the second. The second can be shown using the definition of the inner product in $\mathbb{R}^n$

$$\langle \mathsf{A}\boldsymbol{v}, \mathsf{A}\boldsymbol{w}\rangle = (\mathsf{A}\boldsymbol{v})^{\mathrm{T}}\mathsf{A}\boldsymbol{w} = \boldsymbol{v}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}}\mathsf{A}\boldsymbol{w} = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = \langle \boldsymbol{v}, \boldsymbol{w}\rangle \tag{3.6}$$

Part 1. says that applying an orthogonal matrix to a vector does not change its length (it just rotates it about the origin). The part 2. says that the angles between vectors does not change when an orthogonal matrix is applied. Therefore, if A denotes a 3 by 3 orthogonal matrix (rotation matrix) and $\boldsymbol{v}, \boldsymbol{w}$ are a 3-vectors the following rule for calculation of vector products follows

$$(\mathsf{A}\boldsymbol{v}) \times (\mathsf{A}\boldsymbol{w}) = \mathsf{A}(\boldsymbol{v} \times \boldsymbol{w}) \tag{3.7}$$

In general every orthogonal matrix has determinant 1 or -1, since for an orthogonal matrix A

$$(\det(\mathsf{A})\det(\mathsf{A}))^2 = \det(\mathsf{A})\det(\mathsf{A}^T) = \det(\mathsf{A}\mathsf{A}^T) = \det(\mathsf{1}) = 1 \tag{3.8}$$

The rotation matrix A is an 3 by 3 orthogonal matrix with determinant equal to 1

$$\det \mathsf{A} = \langle \boldsymbol{i}_\mathrm{r}, (\boldsymbol{j}_\mathrm{r} \times \boldsymbol{k}_\mathrm{r}) \rangle = 1, \tag{3.9}$$

since the triad $\boldsymbol{i}_\mathrm{r}$, $\boldsymbol{j}_\mathrm{r}$, $\boldsymbol{k}_\mathrm{r}$ defines the right orthogonal coordinate system.

The final but very important remark is that the rotation matrices form a group. The binary operation is the same as the matrix multiplication, the identity is the identity matrix, and the inverse complies with the matrix inverse. The group of the rotation matrices, i.e. the orthogonal matrices with determinant 1 is called special orthogonal group $\mathrm{SO}_3(\mathbb{R})$. The number three stands for 3 by 3, and $\mathbb{R}$ for real matrices.

3.3 Attitude Parameterization by Quternion

Rotation of coordinate systems can be described by means of a unit quaternion. A salient feature of the unit quaternions is that they provide a convenient product rule for successive rotations and a simple form of kinematics. Before a definition of a unit quaternion and its physical interpretation will be given, notion of a quaternion will be introduced.

Consider a four dimensional Euclidean space, $\mathbb{E}^4$, with the usual definition of the scalar product of $\boldsymbol{v}$ and $\boldsymbol{w}$, $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^T \boldsymbol{w}$, and the norm $\| \boldsymbol{v} \| = \langle \boldsymbol{v}, \boldsymbol{v} \rangle^{\frac{1}{2}}$. Furthermore, let $\boldsymbol{e}$ denote a unit vector in $\mathbb{E}^4$, and let $\mathbb{E}$ be an orthogonal complement to the vector space spanned by $\boldsymbol{e}$. Now, any vector $\tilde{\boldsymbol{q}}$ in $\mathbb{E}^4$ can be uniquely expressed as

$$\tilde{\boldsymbol{q}} = q_0 \boldsymbol{e} + \boldsymbol{q}, \tag{3.10}$$

where $q_0 \in \mathbb{R}$, and $\boldsymbol{q} \in \mathbb{E}$.

$\mathbb{E}$ is three dimensional and every vector can be represented as a linear combination of the triad of mutually perpendicular unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. Therefore, an element of $\mathbb{E}^4$ is given by

$$\tilde{\mathbf{q}} = q_0 \boldsymbol{e} + q_1 \boldsymbol{i} + q_2 \boldsymbol{j} + q_3 \boldsymbol{k}. \tag{3.11}$$

Now a definition from (Jurdjevic 1997) of a quaternion can be adopted. Quaternions are elements of $\mathbb{E}^4$ endowed with the vector structure of $\mathbb{E}^4$ and the multiplication

$$\tilde{\boldsymbol{q}}\tilde{\boldsymbol{s}} = (q_0 s_0 - \langle \boldsymbol{q}, \boldsymbol{s} \rangle)\boldsymbol{e} + q_0 \boldsymbol{s} + s_0 \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{s} \tag{3.12}$$

for any $\tilde{\boldsymbol{q}} = q_0 \boldsymbol{e} + \boldsymbol{q}$ and $\tilde{\boldsymbol{s}} = s_0 \boldsymbol{e} + \boldsymbol{s}$

Notice that

$$\tilde{\boldsymbol{q}}\boldsymbol{e} = \boldsymbol{e}\tilde{\boldsymbol{q}}, \tag{3.13}$$

therefore $\boldsymbol{e}$ is the multiplicative identity in the space of quaternions. The conjugate of quaternion $\tilde{\boldsymbol{q}}^*$ is

$$\tilde{\boldsymbol{q}}^* = q_0\boldsymbol{e} - \boldsymbol{q} \tag{3.14}$$

The following properties of quaternions are very useful

$$(\tilde{\boldsymbol{q}} + \tilde{\boldsymbol{s}})^* = \tilde{\boldsymbol{q}}^* + \tilde{\boldsymbol{s}}^* \tag{3.15}$$

conjugate of the sum of two quaternions is the sum of their conjugates,

$$(\tilde{\boldsymbol{q}}\tilde{\boldsymbol{s}})^* = \tilde{\boldsymbol{s}}^*\tilde{\boldsymbol{q}}^* \tag{3.16}$$

conjugate of the product of two quaternions is the product of their conjugates,

$$\tilde{\boldsymbol{q}}\tilde{\boldsymbol{q}}^* = (q_0^2 + \parallel \boldsymbol{q} \parallel^2)\boldsymbol{e} \tag{3.17}$$

product of a quaternion and its conjugate is $\boldsymbol{e}$ multiplied by its length,

$$\parallel \tilde{\boldsymbol{q}}\tilde{\boldsymbol{s}} \parallel = \parallel \tilde{\boldsymbol{q}} \parallel \parallel \tilde{\boldsymbol{s}} \parallel \tag{3.18}$$

norm of the product of two quaternions is the product of their norms,

$$\tilde{\boldsymbol{q}}^{-1} = \frac{1}{\parallel \tilde{\boldsymbol{q}} \parallel^2}\tilde{\boldsymbol{q}}^* \tag{3.19}$$

The inverse of the quaternion is its conjugate divided by the square of its length.

Any complex number $x+iy$ can be represented as a quaternion $\tilde{\boldsymbol{q}} = x\boldsymbol{e}+y\boldsymbol{i}$, where x is the scalar part of $\tilde{\boldsymbol{q}}$, and $y\boldsymbol{i}$ is the vector part. Actually whole triad $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ can be identified with quaternions having zero scalar part. The mutual orthogonality and unit length of the triad imply the following quaternion product

$$\boldsymbol{i}\boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j}\boldsymbol{k} = \boldsymbol{i},\ \boldsymbol{k}\boldsymbol{i} = \boldsymbol{j} \tag{3.20}$$

and

$$\boldsymbol{i}^2 = -\boldsymbol{e},\ \boldsymbol{j}^2 = -\boldsymbol{e},\ \boldsymbol{k}^2 = -\boldsymbol{e}. \tag{3.21}$$

A salient feature of the quaternion space is that it provides common framework for representing the vectors of the three dimensional Euclidean space (for which the scalar part is zero) and the scalars (for which the vector part is zero).

The property in Eq. (3.19), says that the inverse of a quaternion is equal to its conjugate divided by the length. If all the quaternions are limited to those with unit length, the inverse is exactly equal to the conjugate. The unit quaternions (quaternions with unit length) can be interpreted geometrically as a sphere in $\mathbb{E}^4$

$$\mathbb{S}^3 = \{\tilde{\boldsymbol{q}} : \| \tilde{\boldsymbol{q}} \| = 1\}, \tag{3.22}$$

and algebraicly as a group with the binary operation of quaternion product and the identity element equal to $\boldsymbol{e}$.

In the last part of this subsection we will pay closer attention to a representation of a rotation by a unit quaternion. Consider a mapping $\mathsf{A} : \mathbb{S}^2 \to \mathrm{SO}_3(\mathbb{R})$

$$\mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{x}) = \tilde{\boldsymbol{q}}\boldsymbol{x}\tilde{\boldsymbol{q}}^*, \tag{3.23}$$

where $\boldsymbol{x}$ is quaternion representation of a vector in $\mathbb{R}^3$. To check that $\mathsf{A}(\tilde{\boldsymbol{q}})$ is rotation matrix, it is necessary to check that it preserves the lengths of transforming vectors

$$\| \mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{x}) \| = \| \tilde{\boldsymbol{q}}\boldsymbol{x}\tilde{\boldsymbol{q}}^* \| = \| \tilde{\boldsymbol{q}} \| \| \boldsymbol{x} \| \| \tilde{\boldsymbol{q}}^* \| = \| \boldsymbol{x} \| \tag{3.24}$$

and that the orientation is preserved

$$\begin{aligned}
\mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{x} \times \boldsymbol{y}) &= \mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{x}\boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{e}) = \tilde{\boldsymbol{q}}(\boldsymbol{x}\boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{e})\tilde{\boldsymbol{q}}^* \\
&= \mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{x})\mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{y}) + \langle \mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{x}), \mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{y}) \rangle \boldsymbol{e} \\
&= \mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{x}) \times \mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{y})
\end{aligned} \tag{3.25}$$

In the equation above $\boldsymbol{x}$ and $\boldsymbol{y}$ are both vectors in the three dimensional Euclidean space and quaternions with zero scalar parts.

The mapping of the identity quaternion $\boldsymbol{e}$ by $\mathsf{A}(\cdot)$ gives the identity matrix, but this is not the only quaternion which maps to identity matrix, since both $\mathsf{A}(\boldsymbol{e})(\boldsymbol{x}) = \boldsymbol{x}$ and $\mathsf{A}(-\boldsymbol{e})(\boldsymbol{x}) = \boldsymbol{x}$. Therefore, the mapping $\mathsf{A}(\tilde{\boldsymbol{q}})$ is a homomorphism, with the kernel consisting of $\{-\boldsymbol{e}, \boldsymbol{e}\}$. Now, $\mathbb{S}^3$ is a double cover of $\mathrm{SO}_3(\mathbb{R})$, two elements of $\mathbb{S}^3$, $\tilde{\boldsymbol{q}}$ and $-\tilde{\boldsymbol{q}}$ give the same element of $SO_3(\mathbb{R})$, $\mathsf{A}(\tilde{\boldsymbol{q}})$. $\mathsf{A}(\tilde{\boldsymbol{q}})$ is a real orthogonal matrix, therefore, a notation $\mathsf{A}(\tilde{\boldsymbol{q}})\boldsymbol{x}$ (the product of a matrix by a compatible vector), instead of $\mathsf{A}(\tilde{\boldsymbol{q}})(\boldsymbol{x})$, is adopted in the sequel as it is more natural.

3.4 Unit Quaternion as Rotation Representation

The construction of the unit quaternion arises from the Euler's theorem that the general displacement of a rigid body with one point fixed is a rotation about some axis. Furthermore, the real orthogonal matrix (rotation matrix) always has an eigenvalue +1. The eigenvector corresponding to this eigenvalue is the axis of rotation. Thus, the rotation of coordinate systems can be uniquely described by a unit vector, $\boldsymbol{\epsilon} = \begin{bmatrix}\epsilon_1 & \epsilon_2 & \epsilon_3\end{bmatrix}^T$, giving an axis of rotation, and an angle of rotation ϕ.

A unit quaternion, $\tilde{\boldsymbol{q}}$ can be interpreted as a combination of the components of the unit vector and the angle of rotation

$$\begin{aligned} q_0 &= \cos\frac{\phi}{2} \\ q_1 &= \epsilon_1 \sin\frac{\phi}{2} \\ q_2 &= \epsilon_2 \sin\frac{\phi}{2} \\ q_3 &= \epsilon_3 \sin\frac{\phi}{2}. \end{aligned} \tag{3.26}$$

As mentioned, the same attitude can be described by two different unit quaternions $\mathbf{q}$ and $-\mathbf{q}$, the first is given for the angle of rotation ϕ, and the latter for the angle $2\pi + \phi$.

The product of quaternions in Eq. (3.12) provides a simple methods for calculation of successive rotations. Consider three coordinate systems fixed in a spacecraft {s}, a reference {r} and inertial {i} coordinate systems. Let the quaternions specifying orientation of {s} in {r} and {r} in {i} be ${}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}$ and ${}^{\mathrm{r}}_{\mathrm{i}}\tilde{\boldsymbol{q}}$ respectively then the rotation of {s} in the coordinate system {i} is

$${}^{\mathrm{s}}_{\mathrm{i}}\tilde{\boldsymbol{q}} = {}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}\,{}^{\mathrm{r}}_{\mathrm{i}}\tilde{\boldsymbol{q}}. \tag{3.27}$$

It is sometimes more convenient to use the quaternion product in a matrix notation

$${}^{\mathrm{s}}_{\mathrm{i}}\tilde{\boldsymbol{q}} = \mathsf{R}({}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}})\,{}^{\mathrm{r}}_{\mathrm{i}}\tilde{\boldsymbol{q}}, \tag{3.28}$$

where

$$\mathsf{R}(\tilde{\boldsymbol{q}}) = \begin{bmatrix} -q_0 & -q_1 & -q_2 & q_3 \\ -q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \tag{3.29}$$

The following equalities involving mapping R are true

$$\mathsf{R}(\tilde{\boldsymbol{q}})\mathsf{R}^T(\tilde{\boldsymbol{q}}) = \mathsf{R}^T(\tilde{\boldsymbol{q}})\mathsf{R}(\tilde{\boldsymbol{q}}) = \tilde{\boldsymbol{q}}^T\tilde{\boldsymbol{q}}\mathsf{E}. \tag{3.30}$$

The relation between the unit quaternion and the rotation matrix was already discussed in Eq. (3.23). The construction of the rotation matrix from the coordinate system {r} to {s} was provided in Eq. (3.4)

$${}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A} = \begin{bmatrix} \boldsymbol{i}_{\mathrm{r}} & \boldsymbol{j}_{\mathrm{r}} & \boldsymbol{k}_{\mathrm{r}} \end{bmatrix}, \tag{3.31}$$

where $\boldsymbol{i}_{\mathrm{r}}$, $\boldsymbol{j}_{\mathrm{r}}$, $\boldsymbol{k}_{\mathrm{r}}$, see Fig. 3.1, are the unit vectors of the x, y, and z axes of {r} projected on the coordinates of {s}. Now, the triad of the unit vectors can be parameterized by the unit quaternion, ${}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}$, and represented in a straightforward form

$$
\begin{aligned}
\boldsymbol{i}_{\mathrm{r}} &= \left[\; q_0^2+q_1^2-q_2^2-q_3^2 \quad 2(q_1q_2-q_3q_0) \quad 2(q_1q_3+q_2q_0) \;\right]^{\mathrm{T}}, \\
\boldsymbol{j}_{\mathrm{r}} &= \left[\; 2(q_1q_2+q_3q_0) \quad q_0^2-q_1^2+q_2^2-q_3^2 \quad 2(q_2q_3-q_1q_0) \;\right]^{\mathrm{T}}, \qquad (3.32) \\
\boldsymbol{k}_{\mathrm{r}} &= \left[\; 2(q_1q_3-q_2q_0) \quad 2(q_2q_3+q_1q_0) \quad q_0^2-q_1^2-q_2^2+q_3^2 \;\right]^{\mathrm{T}}.
\end{aligned}
$$

In the last part of this subsection the formulas for successive rotations will be provided. Having matrix representation of the unit quaternion, successive rotations can be described by the following equation

$$
{}^{\mathrm{s}}_{\mathrm{i}}\mathsf{A} = \mathsf{A}({}^{\mathrm{s}}_{\mathrm{i}}\tilde{\boldsymbol{q}}) \;=\; \mathsf{A}({}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}})\mathsf{A}({}^{\mathrm{r}}_{\mathrm{i}}\tilde{\boldsymbol{q}}) \;=\; {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}\;{}^{\mathrm{r}}_{\mathrm{i}}\mathsf{A}. \qquad (3.33)
$$

Frequently, it is necessary to use the inverse transformation of a rotation, It is given by the conjugate of the unit quaternion ${}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}} \;=\; {}^{\mathrm{r}}_{\mathrm{s}}\tilde{\boldsymbol{q}}^*$, Eq. (3.19) and the transpose of a rotation matrix ${}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A} \;=\; {}^{\mathrm{r}}_{\mathrm{s}}\mathsf{A}^{\mathrm{T}}$, Eq (3.5).

3.5 Euler Angles

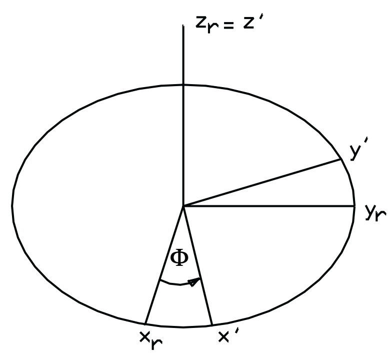

Fig. 3.2. The full description of an attitude can be given by three Euler angles. The sequence used is started by rotating the initial coordinate system {r} about the z_{r} axis by an angle ϕ.

One can carry out transformation from the coordinate system {r} to {s} by means of successive rotations performed in a specific sequence. The full description of this orientation can be given by three counterclockwise angles, known as Euler angles. The convention employed here conforms with (Wertz 1990). The sequence used is started by rotating the initial coordinate system, {r} about the z_{r} axis by an angle ϕ. The resultant coordinate system is spanned on the x', y', and z' axes, see Fig. 3.2. This coordinate system is denoted by {r$'$}. The axes z_{r} of {r} and z' of {r$'$} coincide. In the second step the coordinate system {r$'$} is rotated about the x' axis by an angle θ in Fig. 3.3, and the resultant coordinates are labeled x'', y'', z'' and the frame

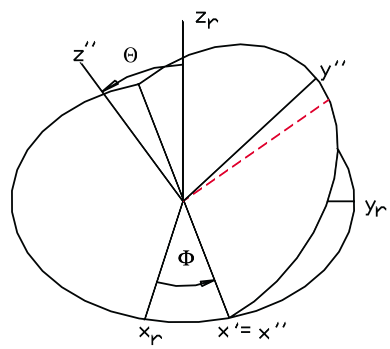

Fig. 3.3. In the second step the coordinate system $\{r'\}$ spanned on the x', y', z' axes is rotated about the x' axis by an angle θ.

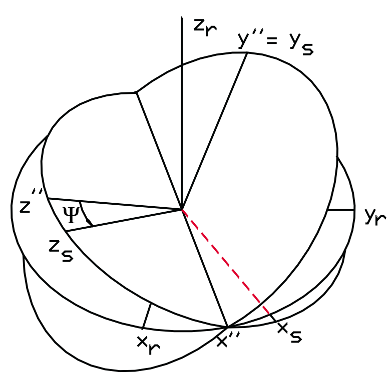

Fig. 3.4. In the third and last stage $\{r''\}$ spanned on the x'', y'', z'' axes is rotated about y'' axis by an angle ψ.

is denoted by $\{\mathrm{r}''\}$. Now, the x' axis of $\{\mathrm{r}'\}$ and x'' of $\{\mathrm{r}''\}$ coincide. Finally $\{\mathrm{r}''\}$ is rotated about y'' axis by an angle ψ, see Fig 3.2. The result is the desired coordinate system $\{\mathrm{s}\}$. The angles ϕ, θ, ψ are often referred to as pitch, roll, and yaw, respectively,

The matrix describing rotation from $\{\mathrm{r}\}$ to $\{\mathrm{s}\}$ is a product of successive rotations about z axis of $\{\mathrm{r}\}$, x' of $\{\mathrm{r}'\}$, and y" of $\{\mathrm{r}''\}$

$$ {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A} \; = \; {}^{\mathrm{s}}_{\mathrm{r}''}\mathsf{A}\, {}^{\mathrm{r}''}_{\mathrm{r}'}\mathsf{A}\, {}^{\mathrm{r}'}_{\mathrm{r}}\mathsf{A}. \tag{3.34} $$

The matrix ${}^{\mathrm{r}'}_{\mathrm{r}}\mathsf{A}$ represents rotation about z_{r} axis by ϕ and it is equal to

$$ {}^{\mathrm{r}'}_{\mathrm{r}}\mathsf{A} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.35} $$

Notice that the last row is $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, which means that the rotation does not change the third axis. The remaining two rotation matrices are

$$ {}^{\mathrm{r}''}_{\mathrm{r}'}\mathsf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \text{ and } {}^{\mathrm{s}}_{\mathrm{r}''}\mathsf{A} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}. \tag{3.36} $$

Finally the multiplication of the aforementioned matrices gives

$$ {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A} = \begin{bmatrix} \cos\psi\cos\phi - \sin\theta\sin\psi\sin\phi & \cos\psi\sin\phi + \sin\theta cin\psi\cos\phi & -\cos\theta sin\psi \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\psi\cos\phi + \sin\theta\cos\psi\sin\phi & \sin\psi sin\phi - \sin\theta\cos\psi\cos\phi & \cos\theta\cos\psi \end{bmatrix} \tag{3.37} $$

4. Kinematics of Rigid Body

Three types of the attitude parameterization were discussed in the last section: the rotation matrix, unit quaternion, three Euler angles. The orientation of a spacecraft changes as time progresses. This changes are dependent on the angular velocity of the vehicle. A mathematical description of the relation between the body's orientation in space and its angular velocity comprises kinematics. The focus is put on kinematics for rotation matrix and unit quaternion in this section. The import feature of these representations is that both of them give singularity-free kinematics. The solution trajectories live on differential manifolds: in the case of rotation matrix the trajectory is on $\mathrm{SO}_3(\mathbb{R})$, in the case of unit quaternion on $\mathbb{S}^3$. The kinematics parameterized by Euler angles can be viewed as a local projection on $\mathbb{R}^3$ of the vector fields describing the ordinary differential equations of quaternionic (or given by the rotation matrix) kinematics. In this book Euler angles (pitch, roll, and yaw) are only use for interpretation of control results, and another local projection $\Pi : \mathbb{S}^3 \to \mathbb{R}^3$ is used

$$\Pi(\tilde{\boldsymbol{q}}) = \boldsymbol{q} = q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}. \tag{4.1}$$

Description of kinematics parameterized by Euler parameters can be found for instance in (Wertz 1990).

4.1 Kinematics Parameterized by Rotation Matrix

Consider rotation of the coordinate system {s} in {r}. At time t {s} has an orientation described by the rotation matrix ${}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}(t)$ after infinitely short period of time Δt it has a new orientation given by ${}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}(t+\Delta t)$. The rotation from the coordinate system $\{\mathrm{s}\}_{\mathrm{t}}$ ({s} at time t) to $\{\mathrm{s}\}_{\mathrm{t}+\Delta\mathrm{t}}$ ({s} at time $t+\Delta t$) is given by

$${}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}(t+\Delta t) = \mathsf{A}\ {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}(t) \tag{4.2}$$

Following the argument in (Goldstein 1950), if rotation of a coordinate system is infinitely small the matrix of transformation is the sum of the identity matrix and some small perturbation

$$\mathsf{A} = \mathsf{E} + \epsilon. \tag{4.3}$$

Having the matrix A a vector $\boldsymbol{v}$ given at time t can be calculated after time Δt

$$\boldsymbol{v}(t+\Delta t) = \mathsf{A}\boldsymbol{v}(t), \tag{4.4}$$

and the change of the vector $\boldsymbol{v}$ is

$$\Delta\boldsymbol{v}(t) = \boldsymbol{v}(t+\Delta t) - \boldsymbol{v}(t) = \epsilon\boldsymbol{v}(t). \tag{4.5}$$

The same changes of $\boldsymbol{v}$ can be analyzed from a geometrical point of view, see Fig 4.1. A vector $\boldsymbol{v}(t)$ is rotated about a unit vector $\boldsymbol{n}$ by an angle $\Delta\phi$. The resultant vector is $\boldsymbol{v}(t+\Delta t)$. The vector $\boldsymbol{v}(t+\Delta t)$ is the sum of vectors $\overrightarrow{OA} + \overrightarrow{AD} + \overrightarrow{DC}$. The vector $\overrightarrow{OA}$ is calculated from the inner product of $\boldsymbol{n}$ and $\boldsymbol{v}(t)$, $\overrightarrow{OA} = \langle\boldsymbol{n}, \boldsymbol{v}(\mathrm{t})\rangle\,\boldsymbol{n}$. The vector $\overrightarrow{AD}$ is in the direction of $\overrightarrow{AB}$ and its magnitude is equal to $\|\ \overrightarrow{AC}\ \|\ \cos\Delta\phi$. The vector $\overrightarrow{DC}$ is in the direction of $\boldsymbol{v}(t)\times\boldsymbol{n}$ and its amplitude is $\|\ \overrightarrow{AC}\ \|\ \sin\Delta\phi$. Finally

$$\begin{aligned}\boldsymbol{v}(t+\Delta t) &= \langle\boldsymbol{n}, \boldsymbol{v}(t)\rangle\,\boldsymbol{n} + ((\boldsymbol{v}(t) - \langle\boldsymbol{n}, \boldsymbol{v}(t)\rangle)\cos\Delta\phi + (\boldsymbol{v}(t)\times\boldsymbol{n})\sin\Delta\phi \\ &= \boldsymbol{v}(t)\cos\Delta\phi + \langle\boldsymbol{n}, \boldsymbol{v}(t)\rangle\,\boldsymbol{n}\,(1-\cos\Delta) + (\boldsymbol{v}(t)\times\boldsymbol{n})\sin\Delta\phi\end{aligned} \tag{4.6}$$

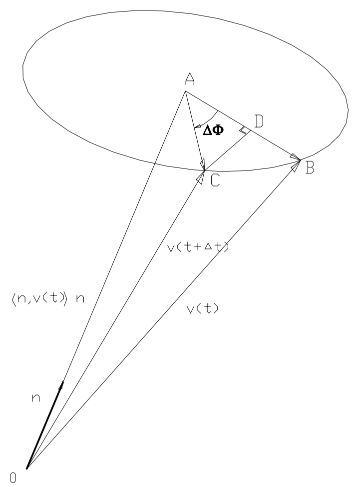

Fig. 4.1. A vector $\boldsymbol{v}(t)$ is rotated about a unit vector $\boldsymbol{n}$ by an angle $\Delta\phi$.

Eq. (4.6) for an infinitely small rotation ($\sin\Delta\phi \approx \Delta\phi,\ \cos\Delta\phi \approx 1$) takes the form

$$\Delta\boldsymbol{v}(t) = \boldsymbol{v}(t) \times \boldsymbol{n}\Delta\phi \tag{4.7}$$

or if the body's velocity is defined as $\boldsymbol{\Omega} = \boldsymbol{n}\Omega$ (the angular velocity has direction $\boldsymbol{n}$ and the magnitude Ω), where $\Delta\phi = \Omega\Delta t$, then

$$\Delta\boldsymbol{v}(t) = \boldsymbol{v}(t) \times \boldsymbol{\Omega}\Delta t, \tag{4.8}$$

Define an isomorphism $\mathsf{S} : \mathbb{R}^3 \to \mathrm{SP}_3(\mathbb{R})$, where $\mathrm{SP}_3(\mathbb{R})$ denotes is a symplectic group of antisymmetric real 3 by 3 matrices

$$\mathsf{S}(\boldsymbol{v}) = \begin{bmatrix} 0 & v_{\mathrm{z}} & -v_{\mathrm{y}} \\ -v_{\mathrm{z}} & 0 & v_{\mathrm{x}} \\ v_{\mathrm{y}} & -v_{\mathrm{x}} & 0 \end{bmatrix}. \tag{4.9}$$

Now, Eq. (4.8) can be represented in the matrix form

$$\Delta\boldsymbol{v}(t) = \mathsf{S}(\boldsymbol{\Omega})\boldsymbol{v}(t)\Delta t. \tag{4.10}$$

Properties of the matrix A will be investigated in the sequel. The inverse of A is

$$\mathsf{A}^{-1} = \mathsf{E} - \epsilon, \tag{4.11}$$

since $\mathsf{A}\mathsf{A}^{-1} = \mathsf{E}$ and

$$\mathsf{A}\mathsf{A}^{-1} = (\mathsf{E} + \epsilon)(\mathsf{E} - \epsilon) = \mathsf{E} + \epsilon - \epsilon = \mathsf{E} \tag{4.12}$$

In Eq. (4.12) we used the fact that ϵ^2 is zero matrix since ϵ is infinitely small and a linear approximation is considered. Now the orthogonality implies that the inverse coincides with the transposition thus

$$\mathsf{E} - \epsilon = \mathsf{A}^{-1} = \mathsf{A}^{\mathrm{T}} = \mathsf{E} + \epsilon^{\mathrm{T}} \Rightarrow \epsilon^{\mathrm{T}} = -\epsilon \tag{4.13}$$

In the most general form an antisymmetric matrix can be written as

$$\epsilon = \begin{bmatrix} 0 & \theta_{\mathrm{z}} & -\theta_{\mathrm{z}} \\ -\theta_{\mathrm{z}} & 0 & \theta_{\mathrm{z}} \\ \theta_{\mathrm{y}} & -\theta_{\mathrm{x}} & 0 \end{bmatrix}. \tag{4.14}$$

Comparing the equation above with Eq. (4.5) it is concluded that $\Delta \boldsymbol{v}(t) = \boldsymbol{v} \times \boldsymbol{\Theta}$, where $\boldsymbol{\Theta} = \begin{bmatrix} \theta_{\mathrm{x}} & \theta_{\mathrm{y}} & \theta_{\mathrm{z}} \end{bmatrix}^{\mathrm{T}}$, and additionally using Eqs. (4.8) and (4.10)

$$\boldsymbol{\Theta} = \boldsymbol{\Omega} \Delta t \text{ or } \epsilon = \mathsf{S}(\boldsymbol{\Omega}) \Delta t. \tag{4.15}$$

It has been shown that the rotation matrix changes as time progresses according to Eq. (4.2). A small change of the rotation matrix is modeled by a perturbation matrix ϵ. It has been demonstrated that ϵ is an antisymmetric matrix with components coinciding with components of $\boldsymbol{\Omega}\Delta t$. Using this information the equation of kinematics can be formulated. The kinematic equation is the derivative of ${}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}$ with respect to time

$$\frac{d{}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}}{dt} = \lim_{\Delta t \to 0} \frac{{}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}(t + \Delta t) - {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}(t)}{\Delta t}, \tag{4.16}$$

but from Eq. (4.2)

$$\frac{d{}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{\epsilon}}{\Delta t} {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A} = \lim_{\Delta t \to 0} \frac{\mathsf{S}(\boldsymbol{\Omega})\Delta t}{\Delta t} {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A} = \mathsf{S}(\boldsymbol{\Omega})\, {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}. \tag{4.17}$$

The vector $\boldsymbol{\Omega}$ lies along axis of the infinitesimal rotation occurring between t and $t + \delta t$. This direction is known as instantaneous axis of rotation. The vector $\boldsymbol{\Omega}$ will be denoted as $\boldsymbol{\Omega}_{\mathrm{sr}}$ in the sequel, since it specifies the angular velocity of the coordinate system $\{\mathrm{s}\}$ in $\{\mathrm{r}\}$. Remember that the angular velocity vector $\boldsymbol{\Omega}_{\mathrm{sr}}$ is resolved in $\{\mathrm{s}\}$ coordinates, thus more formal notation would be $(\boldsymbol{\Omega}_{\mathrm{sr}})_{\mathrm{s}}$, but it seems to be superfluous, since the angular velocity always will be resolve in $\{\mathrm{s}\}$ coordinates in this book.

The final remarks is about the rate of change of a vector. Consider a vector $\boldsymbol{v}_{\mathrm{s}}$ resolved in $\{\mathrm{s}\}$ coordinates

$$\boldsymbol{v}_{\mathrm{s}} = {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}\boldsymbol{v}_{\mathrm{r}} \tag{4.18}$$

Its rate of change can be calculated using the chain rule

$$\frac{d}{dt}\boldsymbol{v}_{\mathrm{s}} = \left(\frac{d}{dt}{}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}\right)\boldsymbol{v}_{\mathrm{r}} + {}^{\mathrm{s}}_{\mathrm{r}}\mathsf{A}\left(\frac{d}{dt}\boldsymbol{v}_{\mathrm{r}}\right) = \mathsf{S}(\boldsymbol{\Omega}_{\mathrm{sr}})\boldsymbol{v}_{\mathrm{s}} + \left(\frac{d}{dt}\boldsymbol{v}_{\mathrm{r}}\right)_{\mathrm{s}}. \tag{4.19}$$

4.2 Kinematics Parameterized by Unit Quaternion

Consider once again a rotation of the coordinate system $\{\mathrm{s}\}$ in $\{\mathrm{r}\}$. At time t the rotation is given by the unit quaternion ${}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}(t)$. After infinitesimal time interval Δt the rotation is changed and given by ${}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}(t+\Delta t)$. The rotation from coordinate system $\{\mathrm{s}\}_{\mathrm{t}}$ to $\{\mathrm{s}\}_{\mathrm{t}+\Delta \mathrm{t}}$ is described by

$$ {}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}(t) \;=\; \tilde{\boldsymbol{q}}\;{}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}(t+\Delta t). \tag{4.20} $$

The physical interpretation of the unit quaternion was introduced in Subsec. 3.4

$$ \tilde{\boldsymbol{q}} = \cos\frac{\Delta\phi}{2}\boldsymbol{e} + \sin\frac{\Delta\phi}{2}\boldsymbol{\epsilon}, \tag{4.21} $$

where a unit vector $\boldsymbol{\epsilon}$ defines the axis and $\Delta\phi$ is the angle of rotation. Remark that the scalars are quaternions with vector parts equal to zero, and the vectors have zero scalar parts. For infinitesimal rotation angle the quaternion is given by

$$ \tilde{\boldsymbol{q}} = \boldsymbol{e} + \frac{\Delta\phi}{2}\boldsymbol{\epsilon} = \boldsymbol{e} + \frac{\Omega\Delta t}{2}\boldsymbol{\epsilon}. \tag{4.22} $$

Denote the angular velocity in the direction $\boldsymbol{\epsilon}$ with magnitude Ω by $\boldsymbol{\Omega}_{\mathrm{sr}}$ then

$$ \tilde{\boldsymbol{q}} = \boldsymbol{e} + \frac{\boldsymbol{\Omega}_{\mathrm{sr}}\Delta t}{2}. \tag{4.23} $$

Now kinematic equation can be formulated

$$ \frac{d{}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}}{dt} = \lim_{\Delta t\to 0}\frac{{}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}(t+\Delta t) - {}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}(t)}{\Delta t}, \tag{4.24} $$

but from Eqs. (4.20) and (4.23)

$$ \frac{d{}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}}{dt} = \lim_{\Delta t\to 0}\frac{\boldsymbol{\Omega}_{\mathrm{sr}}\Delta t}{2\Delta t}{}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{\Omega}_{\mathrm{sr}}\;{}^{\mathrm{s}}_{\mathrm{r}}\tilde{\boldsymbol{q}}. \tag{4.25} $$

4.3 Kinematics of LEO Spacecraft

The general forms of kinematics for a rotating body were derived in the last two subsections. The kinematics for a LEO satellite is provided in this subsection. In fact, the only difference is that the coordinate system PCS takes place of $\{s\}$ and LVLH takes place of $\{r\}$, thus for instance the quaternionic kinematics is

$$ \frac{\mathrm{d}}{\mathrm{dt}}{}^{p}_{o}\tilde{\boldsymbol{q}} = \frac{1}{2}\mathsf{R}(\boldsymbol{\Omega}_{\mathrm{po}})^{p}_{o}\tilde{\boldsymbol{q}}, \tag{4.26} $$

where ${}^{p}_{o}\tilde{\boldsymbol{q}}$ is the unit quaternion describing rotation from LVLH to PCS, $\boldsymbol{\Omega}_{\mathrm{po}}$ is the angular velocity of PCS in LVLH, and $\mathsf{R}(\cdot)$ is given in Eq. (3.28). It is sometimes convenient to represent Eq. (4.26) by an equivalent formula expressed by separate integrations of the vector and the scalar part of the quaternion

$$\begin{aligned} \dot{q}_0 &= -\frac{1}{2}\boldsymbol{\Omega}_{\mathrm{po}}\boldsymbol{q}, \\ \dot{\boldsymbol{q}} &= \frac{1}{2}\boldsymbol{\Omega}_{\mathrm{po}}q_0 - \frac{1}{2}\boldsymbol{\Omega}_{\mathrm{po}} \times \mathbf{q}. \end{aligned} \tag{4.27}$$

The dynamics of a rigid body addressed in the next section corresponds the spacecraft rotation in an inertial coordinate system, therefore it is necessary to state en explicit formula for relation between $\boldsymbol{\Omega}_{\mathrm{po}}$ and $\boldsymbol{\Omega}_{\mathrm{pi}}$. The rotation of LVLH in ECI is about the axis normal to the orbital plane in the direction of $-\boldsymbol{j}_{\mathrm{o}}$. The rate of this rotation has the magnitude ω_{o} - here we made an approximation that the orbital plane is invariant, i.e. motion of line of nodes is disregarded

$$\boldsymbol{\Omega}_{\mathrm{po}} = \boldsymbol{\Omega}_{\mathrm{pi}} - \boldsymbol{\Omega}_{\mathrm{oi}} = \boldsymbol{\Omega}_{\mathrm{pi}} + \omega_{\mathrm{o}}(t)\boldsymbol{j}_{\mathrm{o}}. \tag{4.28}$$

The orbital rate, ω_o is constant for a circular orbit, but time varying for an elliptic one. Our interest is magnetic control of a spacecraft in a low Earth orbit with small eccentricity, therefore for control design the orbital rate will be considered as constant.

5. Dynamics of LEO Spacecraft

The rotational motion about the center of mass is described by the direct Newtonian approach. The solution is a set of three differential equations known as Euler's equation of motion. The dynamics relates torques acting on the spacecraft to the satellite's angular momentum. The Euler's equation in the inertial coordinates, ECI, is the well known formula

$$\left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{\mathrm{i}} = \boldsymbol{N}_{\mathrm{i}}, \tag{5.1}$$

where $\boldsymbol{N}$ is the sum of external torques acting on the body, $\boldsymbol{L}$ is the angular momentum, the subscript i is used because the derivative is with respect to ECI. The angular momentum is defined as

$$\boldsymbol{L} = \mathsf{I}\boldsymbol{\Omega}_{\mathrm{pi}}, \tag{5.2}$$

where I is the inertia tensor, its elements are the inertia coefficients. The inertia tensor has a remarkably convenient form in PCS. In PCS the coordinates are spanned on the spacecraft's principal axes, and the inertia tensor is a diagonal 3 by 3 matrix. The derivative in Eqs. (5.1) can be represented in body fixed coordinates. Eq. (4.19) can be used for this purpose

$$\frac{d}{dt}\boldsymbol{L}_{\mathrm{p}} = \mathsf{S}(\boldsymbol{\Omega}_{\mathrm{pi}})\boldsymbol{L}_{\mathrm{p}} + \left(\frac{d}{dt}\boldsymbol{L}_{\mathrm{i}}\right)_{\mathrm{p}} = \mathsf{S}(\boldsymbol{\Omega}_{\mathrm{pi}})\boldsymbol{L}_{\mathrm{p}} + \boldsymbol{N}_{\mathrm{p}} \tag{5.3}$$

Now, the dynamic equation of motion for a rigid spacecraft in LEO is

$$\mathsf{I}\dot{\boldsymbol{\Omega}}_{\mathrm{pi}}(t) = -\boldsymbol{\Omega}_{\mathrm{pi}}(t) \times \mathsf{I}\boldsymbol{\Omega}_{\mathrm{pi}}(t) + \boldsymbol{N}_{\mathrm{ctrl}}(t) + \boldsymbol{N}_{\mathrm{gg}}(t) + \boldsymbol{N}_{\mathrm{dis}}(t). \tag{5.4}$$

In the equation above the subscript p is dropped. $\boldsymbol{N}_{\mathrm{ctrl}}(t)$ is the control torque, $\boldsymbol{N}_{\mathrm{gg}}(t)$ is the gravity gradient torque and $\boldsymbol{N}_{\mathrm{dis}}(t)$ is the disturbance torques.

5.1 Magnetic torque

Magnetic control torque is generated by an interaction of the geomagnetic field with the magnetorquer current $i(t)$ which gives rise to a magnetic moment $m(t)$

$$m(t) = n_{coil}\ i_{coil}(t)\ A_{coil}. \tag{5.5}$$

The electromagnetic coils are mounted mutually perpendicular and their placement is defined in SCB, thus the vector representing entire magnetic moment producible by all three coils is given in SCB. The transformation from SCB to PCB (in which the Euler equation is given) is necessary, hence $\boldsymbol{m}_{\mathrm{p}} = {}^{\mathrm{p}}_{\mathrm{b}}\mathsf{A}\boldsymbol{m}_{\mathrm{b}}$. The control torque acting on the satellite is

$$\boldsymbol{N}_{\mathrm{ctrl}}(t) = \boldsymbol{m}(t) \times \boldsymbol{B}(t), \tag{5.6}$$

where $\boldsymbol{B}(t)$ is the magnetic flux vector of the Earth. The magnetic moment, $\boldsymbol{m}$, is considered as the control signal in the following. In Eq. (5.6) the subscript p is dropped.

5.2 Geomagnetic Field

The geomagnetic field is essentially that of a magnetic dipole. The south pole is in the northern hemisphere at about $79°N$ latitude and $290°E$ longitude. There are certain deviations from the dipole model called anomalies. The largest anomalies are encountered over Brazil and Siberia. The exact nature

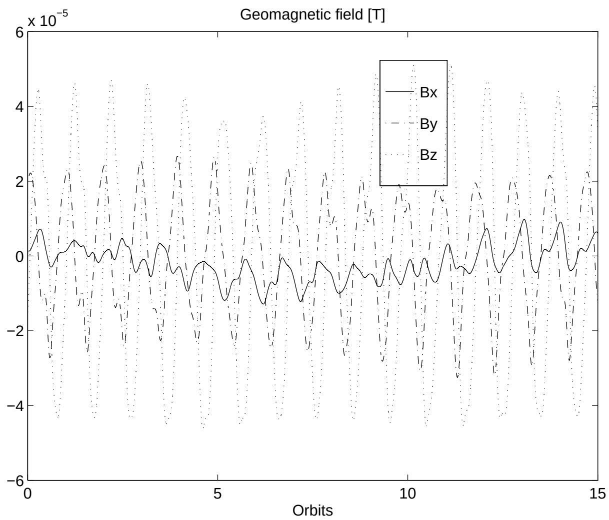

Fig. 5.1. The geomagnetic field vector in LVLH for a polar orbit propagated by the 10th order spherical harmonic model during a period of 24 h.

of the geomagnetic field generator is unknown, however the solution can be conveniently expressed in spherical harmonics (Langel 1987).

$$V(r, \theta, \phi) = a \sum_{n=1}^{k} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} \left(g_{\mathrm{n}}^{\mathrm{m}} \cos m\phi + h_{\mathrm{n}}^{\mathrm{m}} \sin m\phi\right) P_{\mathrm{n}}^{\mathrm{m}}(\theta), \qquad (5.7)$$

where a, r, θ, ϕ are the equatorial radius of the Earth, the geocentric distance, coelevation, east longitude from Greenwich, respectively. The functions $P_{\mathrm{n}}^{\mathrm{m}}(\theta)$ are the associated Legendre function, see (Langel 1987). The set of Gaussian coefficients $g_{\mathrm{n}}^{\mathrm{m}}$, $h_{\mathrm{n}}^{\mathrm{m}}$ is specified by the International Reference Field (IGRF)

Consider a near polar orbit. The geomagnetic field observed in this orbit, seen in LVLH, has large the x and z components, while the y component is comparatively small. The orbit position is well approximated as fixed in ECI, thus the rotation of the Earth is visible via fluctuations of the geomagnetic field vector's y component with frequency 1/24 1/hour. An example of geomagnetic field variation on orbit is given in Fig. 5.1. The geomagnetic field has been computed using 10th order spherical harmonic model ($k = 10$ in Eq. (5.7)).

5.3 Momentum/Reaction Wheels

A spacecraft equipped with momentum or reaction wheels is no longer a rigid body, since the wheels are rotating. This provides an extra angular momentum which contributes to the total momentum. The total angular momentum of the spacecraft and the weels is

$$\boldsymbol{L} = \mathsf{I}\Omega_{\mathrm{pi}} + \boldsymbol{h}_{\mathrm{w}}, \tag{5.8}$$

where $\boldsymbol{h}_{\mathrm{w}}$ is the angular momentum of the wheels. The following equation follows after substituting Eq. (5.8) into Eq. (5.9)

$$\mathsf{I}\dot{\boldsymbol{\Omega}}_{\mathrm{pi}}(t) = -\boldsymbol{\Omega}_{\mathrm{pi}}(t) \times (\mathsf{I}\boldsymbol{\Omega}_{\mathrm{pi}}(t) + \boldsymbol{h}_{\mathrm{w}}(t)) - \frac{d\boldsymbol{h}_{\mathrm{w}}(t)}{dt} + \boldsymbol{N}_{\mathrm{gg}}(t) + \boldsymbol{N}_{\mathrm{dis}}(t). \tag{5.9}$$

Now the factor $\frac{d\boldsymbol{h}_{\mathrm{w}}(t)}{dt}$ is treated as the control input.

5.4 Gas Jet

A gass jet generates a thrust force at the place it is mounted to the spacecraft structure. If the distance from the centre of mass to the jet is nonzero, a torque may originate. The torque produced by the gas jet is given by

$$\boldsymbol{N}_{\mathrm{jet}} = \boldsymbol{r} \times \boldsymbol{F}_{\mathrm{jet}}, \tag{5.10}$$

where $\boldsymbol{r}$ is the vector from the center of mass to the thruster, $\boldsymbol{F}_{\mathrm{jet}}$ is the force generated by the thruster.

The gass jets is characterized by the minimum opening and closing times, i.e. the delay times from commanded start/stop to the instant when the trust begins to buildup/decay. Additionally the thrust profile includes the rise/fall time, which is the time required to establish steady state propellant flow. More details can be found in (Wertz 1990).

5.5 Gravitational Torque

Gravitational torque is fundamental component for the model of a LEO spacecraft motion. The gravitational field in space is not uniform, therefore the gravitational forces acting on specific parts of the spacecraft construction are different. Integration of this effects over the whole body gives the gravitational torque. If one makes the assumptions that

- Only gravitational field of the Earth is considered.
- The Earth possesses a spherically symmetric mass distribution.
- The spacecraft is small compared to its distance from the center of the Earth.
- The spacecraft consists of a single body

then the model of the gravitational torque according to (Hughes 1986) is

$$\boldsymbol{N}_{\mathrm{gg}} = \frac{3\mu}{R_{\mathrm{cm}}^3}(\boldsymbol{R}_{\mathrm{cm}} \times \mathsf{I}^c \boldsymbol{R}_{\mathrm{cm}}), \tag{5.11}$$

where μ is the Earth gravitational constant, R_{cm} is the distance from the center of the Earth to the spacecraft's center of gravity (R_{cm} is a subject of variation, when an elliptic orbit is considered), $\boldsymbol{R}_{\mathrm{cm}}$ is the unit vector from the center of the Earth to the spacecraft's center of mass, nadir. Observe that nadir is equivalent to the unit vector on the z axis of LVLH, $\boldsymbol{k}_{\mathrm{o}}$, and the constant $\frac{\mu}{R_{cm}^3} = \omega_{\mathrm{o}}^2$, where ω_{o} is the orbital rate. Now, the gravitational torque is

$$\boldsymbol{N}_{\mathrm{gg}} = 3\omega_o^2(\boldsymbol{k}_{\mathrm{o}} \times \mathsf{I}\ \boldsymbol{k}_{\mathrm{o}}). \tag{5.12}$$

5.6 Disturbance Torques

There are three main sources of disturbance torques acting on LEO spacecraft: the radiation pressure, residual magnetic moment and aerodynamic drag.

5.6.1 Radiation Pressure

The pressure is due to solar radiation incident on a spacecraft surface. The radiation pressure depends on the the inverse of the square of the distance from the Sun. It is roughly independent on the altitude and its order of magnitude is 10^{-5} Nm. Solar radiation torque varies dependent on the geometry and optical properties of the spacecraft's surfaces, furthermore it is different for different spacecraft's orientations relative to the Sun. Explicit formulas for solar pressure torque are provided in (Cappellari 1976).

5.6.2 Residual Magnetic Moment

The disturbance magnetic torque is originated from the interaction between residual magnetic field of the spacecraft and the magnetic field of the Earth. There are three main sources of the magnetic torque: spacecraft magnetic moments, eddy currents and hysteresis. The spacecraft magnetic moments generate a torque according to Eq. (5.6), where $\boldsymbol{m}$ is now the sum of residual magnetic moments rather than the control signal. Spinning motion of the spacecraft induces the eddy currents, which are interacting with the magnetic field of the Earth giving a torque. In a permeable material rotating in the geomagnetic field energy is dissipated to heat due to frictional motion of the magnetic domain. The energy loss over one rotation gives rise to the disturbance torque due to hysteresis. The order of magnitude of the magnetic disturbance torque can reach 10^{-4} Nm. More details and references to other works on modeling of the magnetic disturbance torque can be found in (Wertz 1990).

5.6.3 Aerodynamic Torque

The aerodynamic drag is the main disturbance torque acting on LEO spacecraft. Its magnitude can be as large as 10^{-1} Nm for en orbits with 100 km altitude. The interaction of the upper atmosphere molecules with satellite's surface introduces an aerodynamic torque. Assuming that the energy of the molecules is totally absorbed on impact with the spacecraft, the force $\mathrm{d}\boldsymbol{f}_{\mathrm{aero}}$ on a surface element $\mathrm{d}A$ is described by

$$\mathrm{d}\boldsymbol{f}_{\mathrm{aero}} = -\frac{1}{2}\mathrm{C_D}\rho \mathrm{v}^2(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v}\mathrm{dA}, \tag{5.13}$$

where $\boldsymbol{n}$ is an outward normal to the surface, $\boldsymbol{v}$ is the unit vector in the direction of the translational velocity of the surface element relative to the incident stream of the molecules. The atmospheric density is denoted by ρ, and the drag coefficient by C_{D}.

The total aerodynamic torque is determined by integration over the total spacecraft surface. One can approximate the satellite structure by a collection of simple geometrical figures. The total aerodynamic torque is the sum of the torques acting on individual parts of the satellite

$$\boldsymbol{N}_{\mathrm{aero}} = \sum_{i=1}^{k} \boldsymbol{r}_{\mathrm{i}} \times \boldsymbol{F}_{\mathrm{i}}, \tag{5.14}$$

where $\boldsymbol{r}_{\mathrm{i}}$ is the vector from the spacecraft center of mass to the center of pressure of the ith element.

To simplify the expression in Eqs. (5.14) and (5.13) assume that the satellite is modeled as a number of plane surfaces. The aerodynamic torque then becomes

$$\boldsymbol{N}_{\mathrm{aero}} = \frac{1}{2}C_{\mathrm{D}}\rho v^2 \sum_{i=1}^{k} A_i(\boldsymbol{n}_i\cdot\boldsymbol{v})\boldsymbol{v}\times\boldsymbol{r}_i \tag{5.15}$$

where A_i is the surface areas. Eq. (5.15) can be furthermore decomposed into a sum of 3 surfaces: A_1 perpendicular to the x-axis of PCS, the cross section surface perpendicular to the y-axis of PCS, A_2, and the cross section surface perpendicular to the z-axis of PCS, A_3.

$$\begin{aligned}\boldsymbol{N}_{\mathrm{aero}} &= \frac{1}{2}C_{\mathrm{D}}\rho v^2 \left(A_1\left([1\ 0\ 0]^{\mathrm{T}}\cdot\boldsymbol{i}_{\mathrm{o}}\right)\boldsymbol{i}_{\mathrm{o}}\times\boldsymbol{r}_1\right.\\ &+ A_2\left([0\ 1\ 0]^{\mathrm{T}}\cdot\boldsymbol{i}_{\mathrm{o}}\right)\boldsymbol{i}_{\mathrm{o}}\times\boldsymbol{r}_2\\ &+ \left.A_3\left([0\ 0\ 1]^{\mathrm{T}}\cdot\boldsymbol{i}_{\mathrm{o}}\right)\boldsymbol{i}_{\mathrm{o}}\times\boldsymbol{r}_3\right).\end{aligned} \tag{5.16}$$

5.6.4 Atmospheric model

The most commonly known model for atmospheric density is Jacchia-Roberts atmospheric model, see (Cappellari 1976). The model includes semiannual, semidiurnal, terdiurnal, and diurnal variations. The atmospheric density is determined as a function of the satellite altitude and the exospheric temperature. The exospheric temperature is parameterized by the daily average 10.7-centimetre solar flux, $F_{10.7}$, as observed in the solar observatory at Ottawa, Canada, and a geomagnetic activity index: the geomagnetic planetary index, K_{p}.

6. Rotational Energy of Rigid Body

The model of the spacecraft motion given by the Euler equation is based on the momentum conservation theorem. For some control applications it is more convenient to use energy of rotating body, instead. The energy of a spacecraft is defined with respect to LVLH, which is the reference coordinate system for attitude control.

6.1 Kinetic Energy

The standard kinetic energy of the satellite is a quadratic form reflecting the spacecraft's velocity in an inertial coordinate system. In this study we focus only on the rotation of the spacecraft with respect to the reference coordinate system, LVLH.

The total angular velocity of the spacecraft (or rather the angular velocity of the coordinate system PCS) relative to ECI is a sum of the spacecraft's angular velocity with respect to LVLH and the angular velocity of the spacecraft's revolution about the Earth (the orbital rate). If the portion due to the orbital rate is disregarded then kinetic energy of the rotary motion is

$$E_{\mathrm{kin}}(t) = \frac{1}{2}\boldsymbol{\Omega}_{\mathrm{po}}^{\mathrm{T}}(t)\mathsf{I}\boldsymbol{\Omega}_{\mathrm{co}}(t). \tag{6.1}$$

6.2 Potential Energy

The potential energy due to the gravity gradient is minimum ($E_{gg} = 0$) when the z axis of PCS (the axis of minimum moment of inertia) is ideally aligned with the z axis of LVLH, since there is no gravity gradient acting on the spacecraft. Its maximum value is reached when the spacecraft attitude is such that the z axis of LVLH coincides with the y axis of PCS (the z axes of PCS and LVLH are orthogonal). The potential energy associated with the gravity gradient can be then considered as a measure of the inclination angle between z axes of PCS and LVLH. It is formulated as

$$E_{\mathrm{gg}}(t) = \frac{3}{2}\omega_{\mathrm{o}}^2 \left(\boldsymbol{k}_{\mathrm{o}}^{\mathrm{T}}(t) \mathsf{I} \boldsymbol{k}_{\mathrm{o}}(t) - I_{\mathrm{z}} \right), \tag{6.2}$$

where the vector $\boldsymbol{k}_{\mathrm{o}}$ is a unit vector along the z axis of LVLH. It is important to note that Eq. (6.2) is resolved in PCS, thus I is a constant diagonal matrix and $\boldsymbol{k}_{\mathrm{o}}(t)$ is time varying. The moment of inertia about the z axis, I_{z}, is subtracted from the left hand side of Eq. (6.2) to make the minimum energy equal to zero.

It is assumed in Eq. (6.2) that the orbit is near circular and thus the orbital rate, ω_o is well approximated by a constant. As an example take an orbit with eccentricity $e = 0.025$ (apogee 850 km, perigee 450 km), one can conclude that for this orbit ω_o is constant within 3 percent.

The potential energy has also a component originating from the revolution of the satellite about the Earth. Consider the summand $\boldsymbol{\Omega}_{\mathrm{pi}}(t) \times \mathsf{I}\boldsymbol{\Omega}_{\mathrm{pi}}(t)$ in the equation of dynamics (5.4). Using Eq. (4.28) this term can be rewritten as

$$\begin{aligned} \boldsymbol{\Omega}_{\mathrm{pi}}(t) \times \mathsf{I}\boldsymbol{\Omega}_{\mathrm{pi}}(t) &= \boldsymbol{\Omega}_{\mathrm{po}}(t) \times \mathsf{I}\boldsymbol{\Omega}_{\mathrm{po}}(t) - \omega_{\mathrm{o}}\boldsymbol{j}_{\mathrm{o}}(t) \times \mathsf{I}\boldsymbol{\Omega}_{\mathrm{po}}(t) \\ &- \omega_{\mathrm{o}}\boldsymbol{\Omega}_{\mathrm{po}}(t) \times \mathsf{I}\boldsymbol{j}_{\mathrm{o}}(t) + \omega_{\mathrm{o}}^2\boldsymbol{j}_{\mathrm{o}}(t) \times \mathsf{I}\boldsymbol{j}_{\mathrm{o}}(t), \end{aligned} \tag{6.3}$$

where $\boldsymbol{j}_{\mathrm{o}}$ is a unit vector on the y axis of LVLH, but since the whole equation is represented in the PCS axes $\boldsymbol{j}_{\mathrm{o}}(t)$ is time varying.

The summand $\omega_{\mathrm{o}}^2\mathbf{j}_{\mathrm{o}}(t) \times \mathsf{I}\boldsymbol{j}_{\mathrm{o}}(t)$ is not dependent on the satellite's angular velocity, and hence gives a contribution only to the potential energy. Therefore, the potential energy due to the revolution of the satellite about the Earth is

$$E_{\mathrm{gyro}}(t) = \frac{1}{2}\omega_{\mathrm{o}}^2 \left(I_{\mathrm{y}} - \boldsymbol{j}_{\mathrm{o}}^{\mathrm{T}} \mathsf{I} \boldsymbol{j}_{\mathrm{o}} \right). \tag{6.4}$$

The minimum of this energy ($E_{\mathrm{gyro}} = 0$) is obtained when the y axis of PCS is aligned with the y axis of LVLH, and maximum when the z axis of PCS coincides with the y axis of LVLH.

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