Attitude Control for Magnetic Actuated Satellite

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Absract

Magnetic actuation utilizes the mechanic torque that is the result of interaction of the current in a coil with an external magnetic field. A main obstacle is, however, that torques can only be produced perpendicular to the magnetic field. This actuation principle has been a topic of research since earliest satellites were launched. Magnetic control has been applied for nutation damping for gravity gradient stabilized satellites, and for velocity decrease for satellites without appendages.

This paper deals with three-axis stabilization of a gravity gradient low earth orbit satellite. The problem of controlling the spacecraft attitude using only magnetic torquing is realized in the form of the sliding mode control. Stability analysis reveals that this control law is stable for satellites, which moments of inertia are of the same order of magnitude. Furthermore a proportional-derivative feedback cross product with the local geomagnetic field vector is proposed. It is proven that this control law is globally asymptotically stable. Simulations have shown that the control law is a good candidate for cheap onboard attitude control

Keywords

Attitude control, magnetic torquing, sliding control, Lyapunov stability theory

1 Introduction

Magnetic torquing is attractive for small, comparatively cheap satellite missions. Magnetic control systems are lightweight, require low power and are inexpensive. There is broad literature covering the area of satellite magnetic control. The 3-axis stabilization using magnetic torquing only for a gravity gradient stabilized satellite is a subject of Musser and Ward (1989) and Wisniewski (1995*b*). Both approaches deal with linear analysis of the satellite motion and 3-axis control is only obtained if a significant gravity gradient exists. The available literature on nonlinear control for 3-axis stabilization of satellites includes Cavallo *et al.* (1993). In this paper two magnetic coils and a reaction wheel are used. A sliding mode attitude controller for a magnetic actuated satellite without appendages is addressed in Wisniewski (1994*b*) and Wisniewski (1994*a*). Purely magnetic control for a low earth orbit satellite with optional moments of inertia is addressed in Wisniewski and Blanke (1996).

The difficulty with the magnetic control is due to the fact that the mechanical control torque is always perpendicular to the geomagnetic field vector. The system is thus only controllable in two directions if geomagnetic field is fixed at any instant in time. The observation that the geomagnetic field is time dependent in an orbit fixed coordinate system is used in the paper.

This study gives an application of the sliding control to the magnetic actuation of a spacecraft. The emphasis is on the sliding condition development. The stability analysis shows that the sliding control is only stable for satellites, which the principal moments of inertia are of the same order of magnitude. An angular velocity continuous feedback controller covers the second part of this article. The satellite trajectory converges toward one of four locally asymptotically stable equilibria. A modification of this controller introduced in the last section makes three of those equilibria unstable, while the forth is now globally asymptotically stable. Simulations using a complete nonlinear model of the satellite kinematics and dynamics together with interacting environmental torques are used to illustrate the results.

This study was initiated as a part of the Danish Ørsted satellite project. The Ørsted satellite is a 60 kg auxiliary payload to be launched in March 1997 into 450 x 850 km orbit with 96 degrees inclination. The satellite is equipped with an 8 m long gravity gradient boom. The two primary science objectives of the mission are to measure the main geomagnetic field and study its interaction with the solar wind plasma. The 3-axis stabilization is required for the boom deployment preparation phase and the science observation mission phase. The attitude, such that the boom is upside-down in the science observation phase may likely occur due to the aerodynamic torque, therefore the means to achieve global stability of the attitude controller is desired.

2 Satellite Model Description

A rigid spacecraft on low earth polar orbit is considered in this paper. Three perpendicular magnetic coils are mounted onboard. The information of the satellite attitude and angular velocity is available through an onboard attitude determination system, see Bak *et al.* (1996).

2.1 Nomenclature

${}^c\boldsymbol{\Omega}_{cw}$	Angular velocity vector Control CS relative to World CS seen in Control CS.
${}^c\boldsymbol{\Omega}_{co}$	Angular velocity vector Control CS relative to Orbit CS seen in Control CS.
ω_o	Orbital rate.
$\mathbf{I}$	Inertia tensor, moments of inertia about principal axes, $\mathbf{I} = diag([I_x\ I_y\ I_z]^T)$, where $I_x \geq I_y \geq I_z$.
${}^c\mathbf{i}_o,\ {}^c\mathbf{j}_o\ {}^c\mathbf{k}_o$	Unit vector along x-, y-, z-axis of Orbit CS seen from Control CS, respectively.
${}^c\mathbf{N}_{ctrl}$	Control torque resolved in Control CS.
${}^c\mathbf{N}_{ggl}$	Gravity gradient torque resolved in Control CS.
${}^c\mathbf{N}_{dis}$	Disturbance torques resolved in Control CS.
${}^c\mathbf{m}$	Magnetic moment in Control CS.
${}^c\mathbf{B}$	Magnetic field vector in Control CS.
${}^c_o\mathbf{q} = [\mathbf{q}^T\ q_4]^T$	Quaternion representing Control CS relative to Orbit CS. It consists of a vector

	part, $\mathbf{q}$, and a scalar part, q_4.
$\mathbf{A}\left({}^c_o\mathbf{q}\right)$	Attitude cosine direction matrix based on ${}^c_o\mathbf{q}$.
n_{coil}	Number of coil windings.
A_{coil}	Coil area.
i_{coil}	Current in coil.
$\mathbf{s}$	Sliding variable.
$\mathbf{S}$	Sliding manifold.
$\mathbf{E}$	Identity matrix

2.2 Formulation of Equation of Motion

The mathematical model of a satellite is described by the dynamic equations and the kinematic equations of motion, see (Wertz, 1990). The dynamics relates torques acting on the satellite to the satellite's angular velocity in World CS, definitions of the coordinate systems used in this paper are incorporated in Appendix A . The kinematics provides integration of the angular velocity . In this paper the attitude is parameterized by four components of a quaternion describing rotation of the Control CS in the Orbit CS. The quaternion representation is described in Appendix B.

The dynamic equations of motion of a satellite considered as rigid body is

$$\mathbf{I}\dot{\mathbf{\Omega}}_{cw}(t) = -{}^c\mathbf{\Omega}_{cw}(t) \times \mathbf{I}{}^c\mathbf{\Omega}_{cw}(t) + {}^c\mathbf{N}_{ctrl}(t) + {}^c\mathbf{N}_{gg}(t) + {}^c\mathbf{N}_{dis}(t). \tag{2.1}$$

Control torque is generated by an interaction of the geomagnetic field with the magnetorquer current $i(t)$ which gives rise to a magnetic moment $m(t)$

$$m(t) = n_{coil}\; i_{coil}(t)\; A_{coil}. \tag{2.2}$$

The control torque acting on the satellite is then

$${}^c\mathbf{N}_{ctrl}(t) = {}^c\mathbf{m}(t) \times {}^c\mathbf{B}(t). \tag{2.3}$$

Gravity gradient torque is given by

$${}^c\mathbf{N}_{gg}(t) = 3\omega_o^2({}^c\mathbf{k}_o \times \mathbf{I}\; {}^c\mathbf{k}_o). \tag{2.4}$$

The disturbance torque is mainly due to the aerodynamic drag, see Wisniewski (1995*a*).

The kinematics describes the body's orientation in space and is obtained through integration of the angular velocity. The kinematic equations are expressed by separate integrations of the vector and the scalar part of the attitude quaternion

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{1}{2}{}^c\mathbf{\Omega}_{co} q_4 + \frac{1}{2}{}^c\mathbf{\Omega}_{co} \times \mathbf{q}, \\ \dot{q}_4 &= -\frac{1}{2}{}^c\mathbf{\Omega}_{co} \cdot \mathbf{q}. \end{aligned} \tag{2.5}$$

The relation between satellite angular velocity in World CS and angular velocity w.r.t. Orbit CS is obtained by

$${}^c\mathbf{\Omega}_{co} = {}^c\mathbf{\Omega}_{cw} - \omega_o{}^c\mathbf{i}_o. \tag{2.6}$$

The orbital rate, ω_o is constant for a circular orbit, but time varying for an elliptic orbit. The eccentricity of the Ørsted orbit is comparatively small and the variation of ω_o are rejectable.

Concluding, a satellite motion is characterized by the 7th order nonlinear differential equation. The quaternion representation of the attitude provides one redundant equation in kinematics. The advantage is the the description of the kinematics lacks from singularity, what is the case for Euler angle parameterization of the attitude.

3 Sliding Mode Controller Design

The satellite trajectory is expected to be in the vicinity of the reference for the most of the operational time, but there are certain transition or contingency phases, that the satellite motion can not be considered as rotation in the neighbourhood of a reference and the nonlinear terms in Eqs. 2.1 to 2.6 can become dominant. The problem is, thus, inherent nonlinear and nonlinear control methods are needed. An example of those transition phases is when the satellite is released from a launcher, and it can be expected to have a random tumbling motion in space with known bounds on the angular velocity. Moreover, after the boom deployment the resultant attitude may be boom upside-down.

A sliding mode controller is implemented for the attitude corrections using magnetic torquing. Full attitude information in the form of the attitude quaternion, ${}^{c}_{o}\mathbf{q}$, and the satellite angular velocity with respect to the Orbit CS, ${}^{c}\mathbf{\Omega}_{co}$, are used as feedback signals. The objective of the attitude control is to turn the satellite such that the Control CS coincides with the Orbit CS, i.e. $\mathbf{\Omega}_{co}$ converges to $\mathbf{0}$, the vector part of the attitude quaternion, $\mathbf{q}$, converges to $\mathbf{0}$, and the scalar part, q_4, approaches 1.

The design strategy of the sliding mode controller consists of two steps, see (Utkin, 1992), (Slotine and Li, 1991):

1. Sliding manifold design.

2. Sliding condition design.

Consider a manifold, a 3 dimensional hyperplane, in the state space of a 6th order system $[{}^{c}\mathbf{\Omega}_{co}\ \mathbf{q}]^T$. The sliding manifold is designed in such a way that the satellite trajectory, if on the hyperplane, converges to the reference. However, the satellite motion is not confined to the 3 dimentional hyperplane in general. Therefore a control law forcing the satellite motion toward the manifold is necessary for achieving stable satellite motion. The sliding condition keeps decreasing the distance from the state to the sliding manifold, such that every solution ${}^{c}\mathbf{\Omega}_{cw}$, $\mathbf{q}$ originating outside the sliding manifold tends to it. Now the manifold is an invariant set of the satellite motion and the trajectory of the system converges to the reference.

The result of the sliding condition design will be a desired control torque. When the desired control torque is implemented the satellite trajectory converges to the sliding manifold, but due to the satellite motion is stable on the sliding manifold, its trajectory converges to the reference. However the magnetic actuated satellite possesses one serious obstacle: the magnetic control torque is confined to lie perpendicular to the geomagnetic field vector and may not comply with the control torque which is desired to turn the satellite toward the sliding manifold. With the geomagnetic field varying along an orbit this implies, e.g., that over the earth's poles the rotation about the z-axis of the Orbit CS is totally uncontrollable.

3.1 Sliding Manifold Design

It will be shown the satellite motion on a certain 3 dimensional hyperplane in the 6-dimensional state space of the vector part of the attitude quaternion, $\mathbf{q}$, and the satellite angular velocity, ${}^{c}\mathbf{\Omega}_{co}$, is stable.

First, let a sliding variable, $\mathbf{s}$, be defined as in Eq. 3.7

$$\mathbf{s} = \mathbf{I}^{c}\mathbf{\Omega}_{co} + \mathbf{\Lambda}_{q}\mathbf{q}, \tag{3.7}$$

where $\mathbf{\Lambda}_{q}$ is a positive definite matrix.

The sliding manifold is the subspace of the state space, where the sliding variable equals $\mathbf{0}$

$$\mathbf{S} = \{\mathbf{q}, {}^{c}\mathbf{\Omega}_{co} : \mathbf{s} = \mathbf{0}\}. \tag{3.8}$$

The definition of the sliding variable, $\mathbf{s}$, in Eq. 3.7 guarantees convergence of $\mathbf{q}$ to zero and q_4 to 1 with an exponential rate. To prove this statement, consider a Lyapunov candidate function

$$v_q = \mathbf{q}^T\mathbf{q} + (1 - q_4)^2, \tag{3.9}$$

is equivalent to

$$v_q = 2(1 - q_4), \tag{3.10}$$

since $\mathbf{q}^T\mathbf{q} + q_4^2 = 1$, see Appendix B.

The time derivative of the Lyapunov candidate function is calculated applying the kinematics in Eq. 2.5

$$\dot{v}_q = \mathbf{q}^T\ {}^{c}\mathbf{\Omega}_{co}, \tag{3.11}$$

thus

$$\dot{v}_q = -\mathbf{q}^T\mathbf{I}^{-1}\mathbf{\Lambda}_{q}\mathbf{q}. \tag{3.12}$$

The time derivative of the Lyapunov function is negative definite, since $\mathbf{\Lambda}_{q}$ is the positive definite matrix. According to Lyapunov's direct method, the equilibrium ${}^{c}_{o}\mathbf{q} = [0\ 0\ 0\ 1]^T,\ {}^{c}\mathbf{\Omega}_{co} = \mathbf{0}$ is asymptotically stable if the satellite is on the sliding manifold, $\mathbf{s}$.

The 3 dimensional hyperplane, Eq. 3.8, in 6 dimensional space $[{}^{c}\mathbf{\Omega}_{co}^T\ \mathbf{q}^T]^T$ is sufficient to describe the motion of the satellite (7th order differential equation) in the sliding mode. Notice that the equilibrium ${}^{c}_{o}\mathbf{q} = [0\,0\,0\,-1]^T,\ {}^{c}\mathbf{\Omega}_{co} = \mathbf{0}$ is unstable even though ${}^{c}_{o}\mathbf{q} = [0\,0\,0\,-1]^T$ and ${}^{c}_{o}\mathbf{q} = [0\,0\,0\,1]^T$ represent the same attitude (Control CS coincides with Orbit CS). Furthermore if the sliding variable is defined as

$$\mathbf{s} = \mathbf{I}\mathbf{\Omega}_{co} - \mathbf{\Lambda}_{q}\mathbf{q}, \tag{3.13}$$

it is possible to show using the Lyapunov candidate function

$$v_q = \mathbf{q}^T\mathbf{q} + (1 + q_4)^2, \tag{3.14}$$

that the equilibrium ${}^{c}_{o}\mathbf{q} = [0\ 0\ 0\ \ -1]^T,\ {}^{c}\mathbf{\Omega}_{co} = \mathbf{0}$ is asymptotically stable and the equilibrium ${}^{c}_{o}\mathbf{q} = [0\ 0\ 0\ 1]^T,\ {}^{c}\mathbf{\Omega}_{co} = \mathbf{0}$ is unstable.

3.2 Sliding Condition Development

The objective of the analysis is to derive the desired control torque turning the satellite trajectory toward the sliding manifold. The satellite motion is described in the space of the sliding variable, $\mathbf{s}$. A salient feature of this approach is that the reduced 3rd order system is considered. The representation of the satellite motion in the space of the sliding variable is calculated by differentiation of the sliding variable $\mathbf{s}(t)$ w.r.t. time.

$$\dot{\mathbf{s}} = \mathbf{I}^c\dot{\mathbf{\Omega}}_{cw} - \omega_o{}^c\dot{\mathbf{i}}_o + \mathbf{\Lambda}_q\dot{\mathbf{q}} \tag{3.15}$$

The derivatives of the satellite angular velocity and the attitude quaternion are calculated according to the equations of kinematics and dynamics, Eqs. 2.5 and 2.1

$$\dot{\mathbf{s}} = -{}^c\mathbf{\Omega}_{cw}\times\mathbf{I}^c\mathbf{\Omega}_{cw}+3\omega_o^2{}^c\mathbf{k}_o\times\mathbf{I}^c\mathbf{k}_o-\omega_o\mathbf{I}({}^c\mathbf{i}_o\times{}^c\mathbf{\Omega}_{co})+\frac{1}{2}\mathbf{\Lambda}_q({}^c\mathbf{\Omega}_{co}q_4+{}^c\mathbf{\Omega}_{co}\times\mathbf{q})+{}^c\mathbf{N}_{ctrl}. \tag{3.16}$$

Assume that the satellite trajectory is on the sliding manifold. The equivalent control is the control necessary to keep the satellite on the sliding manifold. In other words if the control torque is equal to the equivalent control then the time derivative of the sliding variable equals zero. If the satellite is not on the sliding manifold, the desired control torque equals the sum of the equivalent control and a part making the sliding variable converge to $\mathbf{0}$:

$$\mathbf{N}_{des} = \mathbf{N}_{eq} - \mathbf{\Lambda}_s\mathbf{s}, \tag{3.17}$$

the equivalent control torque, $\mathbf{N}_{eq}$ is

$$\mathbf{N}_{eq} = {}^c\mathbf{\Omega}_{cw}\times\mathbf{I}^c\mathbf{\Omega}_{cw} - 3\omega_o^2({}^c\mathbf{k}_o\times\mathbf{I}^c\mathbf{k}_o)+\omega_o\mathbf{I}({}^c\mathbf{i}_o\times{}^c\mathbf{\Omega}_{co}) - \frac{1}{2}\mathbf{\Lambda}_q({}^c\mathbf{\Omega}_{co}q_4+{}^c\mathbf{\Omega}_{co}\times\mathbf{q}), \tag{3.18}$$

and $\mathbf{\Lambda}_s$ is a positive definite matrix.

If the control torque were producible in each direction the desired control, $\mathbf{N}_{des}$ could be substituted in Eq. 3.16 for the control torque, $\mathbf{N}_{ctrl}$. Thus the time derivative of the sliding variable, $\mathbf{s}$ would be:

$$\dot{\mathbf{s}} = -\mathbf{\Lambda}_s\mathbf{s}. \tag{3.19}$$

The system described by differential equation in Eq. 3.19 is stable, hence the sliding condition is fulfilled. Unfortunately, the magnetic generated torque is perpendicular to the local geomagnetic field vector and can only partly comply with Eq. 3.17.

3.3 Modified Sliding Condition

The satellite appears uncontrollable if fixed at any instant of time due to the magnetic torque vector is constrained to always lie perpendicular to the local geomagnetic field vector. A modified sliding condition for magnetic stabilized satellites is discussed in this subsection.

The desired control torque is projected on a vector defined by the components of a sliding variable, i.e. the desired control torque is resolved into two components: perpendicular and parallel to the sliding variable vector. Magnetic generated torque is due to compensate only the component parallel to the sliding variable vector.

Consider a problem of the orthogonal projection of the desired control torque, $\mathbf{N}_{des}(t)$ onto the instant sliding variable vector, $\mathbf{s}(t)$ (see Fig. 3.1). The desired control torque, $\mathbf{N}_{des}(t)$ has two

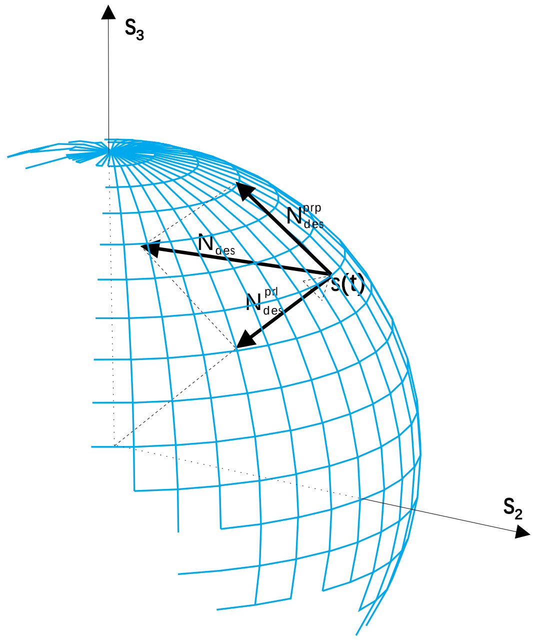

Fig. 3.1. Desired control torque resolved in s-space.

components: parallel, $\mathrm{N}_{des}^{prl}(t)$, and perpendicular, $\mathrm{N}_{des}^{prp}(t)$, to the vector $\mathbf{s}(t)$.

The control torque, N_{ctrl} needs only compensate N_{des}^{prl}, since N_{des}^{prp} does not decreases the distance from the satellite trajectory to the sliding manifold. This control principle has an intuitive interpretation. The component N_{des}^{prl} is responsible for diminishing of the sphere radius in Fig. 3.1, whereas N_{des}^{prp} is responsible for movement on the sphere surface (sphere radius remains unchanged).

The same results are acquired as a result of theoretical analysis and are formalized in Theorem 3.1.

Theorem 3.1 *The control torque that compensates N_{des}^{prl} makes the distance from the state $[{}^{c}\mathbf{\Omega}_{co}(t)\ \mathbf{q}(t)]^{T}$ to the sliding manifold in Eqs. 3.8 and 3.7 converge to zero, and the sliding condition is satisfied.*

Proof of Theorem 3.1 *Barbalat's Lemma, see (Slotine and Li, 1991) is used to prove that the manifold S is an invariant set in the s-space. Construct a Lyapunov function:*

$$v_s = \frac{1}{2}\mathbf{s}^T\mathbf{s}. \tag{3.20}$$

The motion in the s-space is described by the equation:

$$\dot{\mathbf{s}} = -\mathrm{N}_{eq} + \mathrm{N}_{ctrl}, \tag{3.21}$$

but the control torque compensates N_{des}^{prl}, thus

$$\dot{\mathbf{s}} = -\mathbf{\Lambda}_s\mathbf{s} + \mathrm{N}^{prp}, \tag{3.22}$$

where N^{prp} is a rest vector perpendicular to the vector $\mathbf{s}(t)$. Finally time derivative of the Lyapunov function is given by

$$\dot{v}_s = \mathbf{s}^T(-\mathbf{\Lambda}_s\mathbf{s} + \mathrm{N}^{prp}) = -\mathbf{s}\mathbf{\Lambda}_s\mathbf{s} \tag{3.23}$$

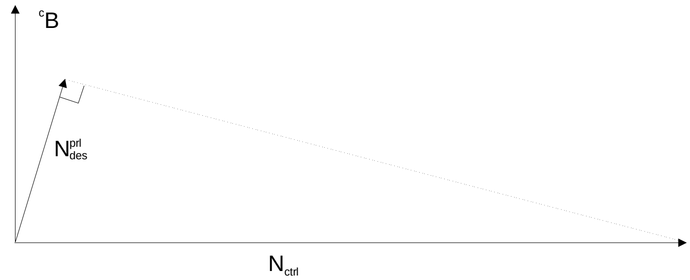

Fig. 3.2. Large $\mathbf{N}_{ctrl}$ is necessary to compensate small $\mathbf{N}_{des}^{prl}$.

The time derivative of Lyapunov function is uniformly continuous and negative semidefinite, hence the conditions of Barbalat's lemma are fulfilled. ■

The control law, which has the objective to compensate $\mathbf{N}_{des}^{prl}$ applying the magnetic actuation is only feasible when the geomagnetic field is not ideally parallel to the sliding variable, $\mathbf{s}$. Furthermore, if ${}^{c}\mathbf{B}$ and $\mathbf{s}$ are near to parallel the amplitude of the control signal may be very large, since the large control torque $\mathbf{N}_{ctrl}$ is desired to compensate even small $\mathbf{N}_{des}^{prl}$, see Fig. 3.2. In practice, the magnetic moment is confined, thus the ideal compensation of $\mathbf{N}_{des}^{prl}$ is not possible. An approximate compensation is introduced

$$
{}^{c}\mathbf{m} = \frac{{}^{c}\mathbf{B} \times \mathbf{N}_{des}^{prl}}{||{}^{c}\mathbf{B}||^{2}}, \tag{3.24}
$$

where

$$
\mathbf{N}_{des}^{prl} = \frac{\mathbf{N}_{des} \cdot \mathbf{s}}{||\mathbf{s}||^{2}}\mathbf{s}. \tag{3.25}
$$

Notice that the control law in Eq. 3.24 well compensates $\mathbf{N}_{des}^{prl}$, when ${}^{c}\mathbf{B}$ is perpendicular or nearly perpendicular to $\mathbf{s}$, and produces small control torque when ${}^{c}\mathbf{B}$ and $\mathbf{s}$ are close to parallel.

The control law based on the approximate compensation of the desired control torque in Eq. 3.24 is observed to be locally stable for small values of the gain $\mathbf{\Lambda}_{q}$. Additional global stability property is gained when the principal moments of inertia are of the same order of magnitude (the satellite is in the boom stowed configuration). In this case the feedback generated according to Eq. 3.24 consists of the cross product of the angular velocity ${}^{c}\mathbf{\Omega}_{co}$ with the local geomagnetic field vector, ${}^{c}\mathbf{B}$, plus small perturbation of the satellite attitude. This phenomenon inspired further work on proportional-derivative feedback cross product with the local geomagnetic field vector treated in the next section.

3.4 Influence of Modeling Errors

The influence of modeling errors are considered in this section. The sliding control introduced in the previous section provides the desired control torque that equals the sum of the equivalent control and the part making the sliding variable convergent to zero. The satellite motion model is provided with a certain parametric uncertainty, e.g. the moments of inertia are determined with finite accuracy. It follows that the equivalent control is not perfectly known, and an application of the desired control torque makes the sliding variable convergent to some constant vector $\mathbf{s}_{0}$, instead of convergent to zero. Hence the satellite attitude converges to the reference with a bias vector $\mathbf{q}_{0}$.

Let the sliding variable, $\mathbf{s}(t)$ converge to a vector $\mathbf{s}_{0}$, then the satellite trajectory is on a manifold given by

$$\mathbf{I}^c\boldsymbol{\Omega}_{co} + \boldsymbol{\Lambda}_q\mathbf{q} - \mathbf{s}_0 = \mathbf{0}. \tag{3.26}$$

The objectives of the following analysis is to provide approximate value of the bias vector $\mathbf{q}_0$. The vector $\mathbf{s}_0$ is considered to be small, therefore it is appropriate to analyse the satellite motion in the vicinity of the reference. The satellite angular velocity, ${}^c\boldsymbol{\Omega}_{co}$ and the first three components of the attitude quaternion, $\mathbf{q}$ are small, while the scalar component q_4 may be approximated by 1, see physical interpretation of a quaternion Wertz (1990). Now, applying the equations of kinematics in Eq. 2.5 the approximation of the time derivative of $\mathbf{q}$ is calculated

$$\dot{\mathbf{q}} = \frac{1}{2}{}^c\boldsymbol{\Omega}_{co}, \tag{3.27}$$

and the linear approximation of the sliding manifold is

$$2\mathbf{I}\dot{\mathbf{q}} + \boldsymbol{\Lambda}_q\mathbf{q} - \mathbf{s}_0 = \mathbf{0}. \tag{3.28}$$

Finally, the satellite motion on the sliding manifold is given by

$$\mathbf{q} = \boldsymbol{\Lambda}_q\mathbf{s}_0 + e^{-\frac{1}{2}\mathbf{I}^{-1}\boldsymbol{\Lambda}_q(t-t_0)}\mathbf{q}(t_0), \tag{3.29}$$

thus the bias $\mathbf{q}_0$ equals

$$\mathbf{q}_0 = \boldsymbol{\Lambda}_q\mathbf{s}_0. \tag{3.30}$$

Note that the larger the components of the matrix $\boldsymbol{\Lambda}_q$ are the larger the bias, $\mathbf{q}_0$ is.

3.5 Simulation Validation of Sliding Mode Attitude Control

The sliding mode attitude controller was evaluated by the Monte Carlo simulation for the Ørsted satellite with boom stowed configuration. A circular orbit with inclination of 96 degrees has been simulated. The geomagnetic field has been computed using 8th order spherical harmonic model. Inertia tensor of the Ørsted satellite was equal $I = diag([3.4278\ 2.9038\ 1.2750])$. Various initial values of the angular velocity and the attitude were tested. The controller was evaluated for the initial values of the attitude both in the neighbourhood of the reference and for z-principal axis pointing up-side down w.r.t. z-axis of the Orbit CS.

The control parameters were found empirically:

$$\boldsymbol{\Lambda}_s = diag([0.003\ 0.003\ 0.003]),\ \boldsymbol{\Lambda}_q = diag([0.002\ 0.002\ 0.002]).$$

Figure 3.3 shows simulation of the angular velocity and the attitude quaternion. The attitude is represented by pitch, roll, and yaw angles in Figure 3.4. The initial value of pitch is 60 deg, roll is 100 deg, and yaw is -100 deg. Initial angular velocity ${}^c\boldsymbol{\Omega}_{co}$ is $[-0.002\ 0.002\ 0.002]^T$ rad/sec. Already after 2 orbits pitch, roll and yaw are within 10 deg. The plot of the magnetic moment used for attitude correction depicted in Figure 3.4 shows that the sliding mode attitude controller is power efficient.

The satellite motion is influenced by the aerodynamic drag and eccentricity of the orbit.

Simulation of the Ørsted satellite motion with disturbance torques acting on the satellite structure is depicted in Figure 3.5. The sliding mode attitude controller keeps the steady state attitude error within ± 3 deg.

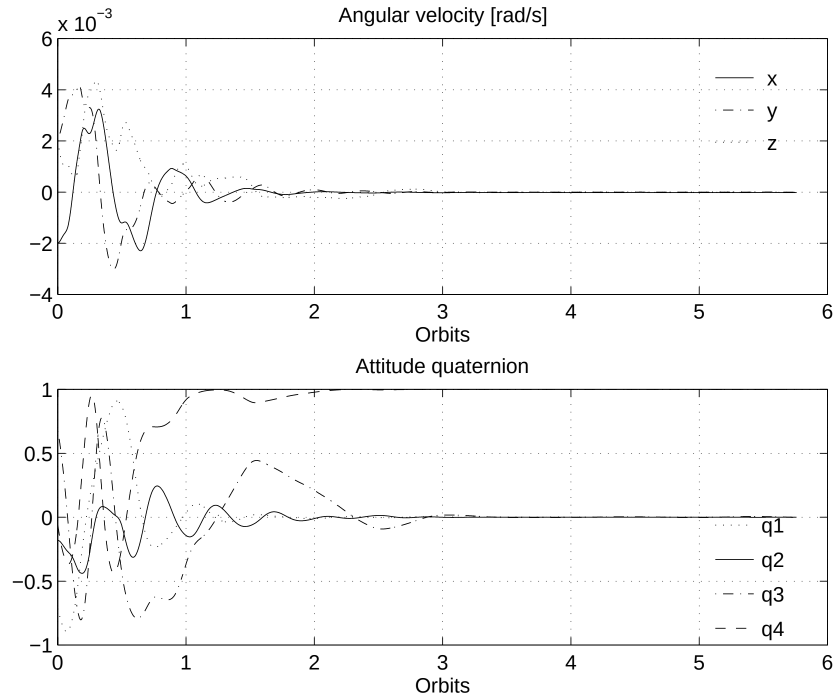

Fig. 3.3. The angular velocity, ${}^{c}\mathbf{\Omega}_{co}$ and the attitude quaternion,${}^{c}_{o}\mathbf{q}$. The attitude quaternion converges to the reference $[0\ 0\ 0\ 1]^T$.

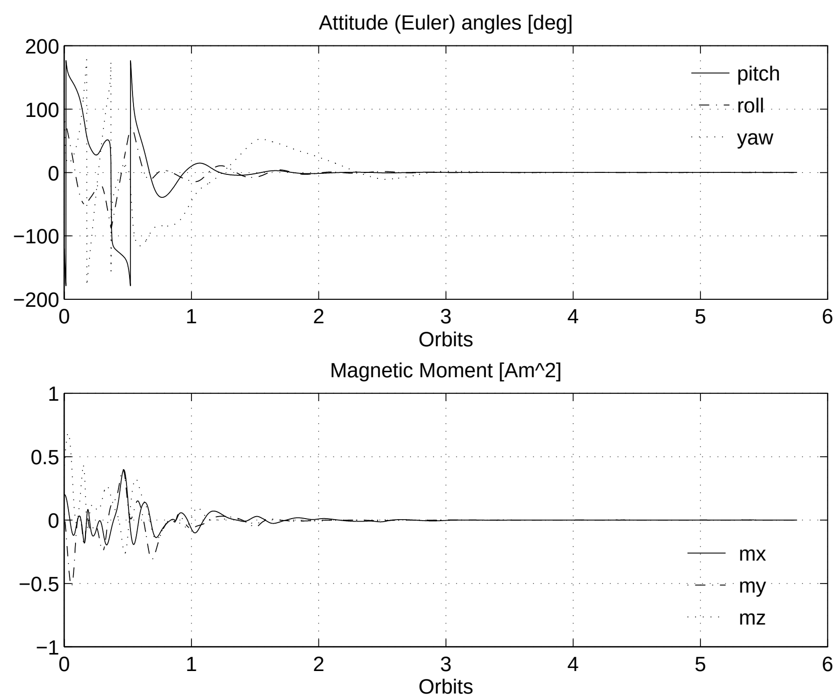

Fig. 3.4. The attitude is represented by the Euler angles. Plot of magnetic moment shows effect utilization of the sliding mode attitude controller.

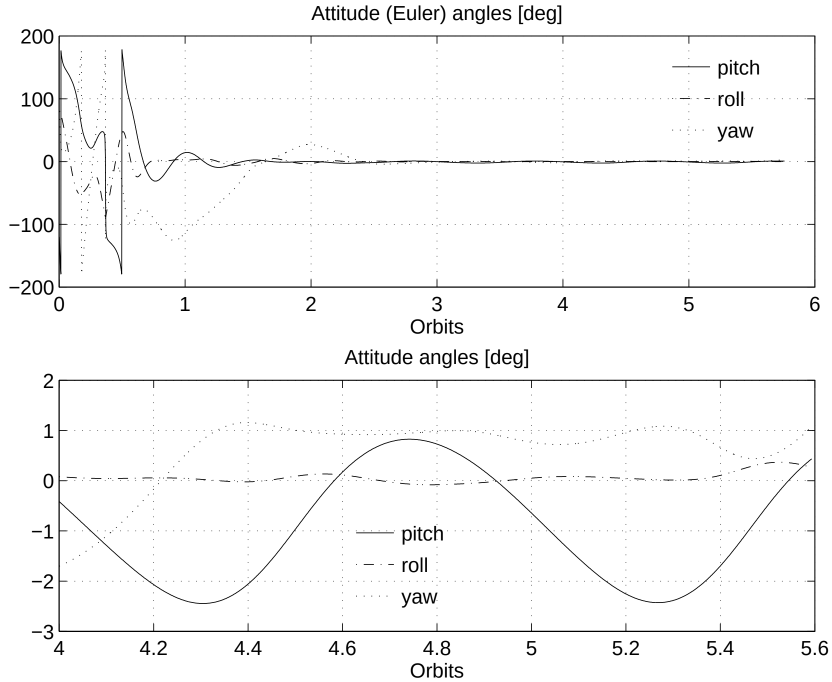

Fig. 3.5. Angular velocity and attitude quaternion of the Ørsted satellite on elliptical orbit. Motion is influenced by the aerodynamic drag.

4 Attitude Stability at Large

The control torque is always perpendicular to the geomagnetic field vector, therefore it is desirable that the magnetic moment is also perpendicular to geomagnetic field vector, since only this component has actual influence on the control torque. Consider the following control law

$$ {}^{c}\mathbf{m}(t) = {}^{c}\mathbf{B}(t) \times \ \mathbf{H}{}^{c}\mathbf{\Omega}_{co}(t), \tag{4.31} $$

where $\mathbf{H}$ is a 3×3 positive definite constant matrix. Global stability of control law in Eq. 4.31 is expressed in the following theorem.

Theorem 4.1 *Consider the control law in Eq. 4.31 then the satellite, Eq. 2.1 to 2.6 has 4 asymptotically stable local equilibria:*

$$ \{({}^{c}\mathbf{\Omega}_{co},\ {}^{c}\mathbf{k}_{o},\ {}^{c}\mathbf{i}_{o}) : (\mathbf{0},\ \pm^{o}\,\mathbf{k}_{o},\ \pm^{o}\,\mathbf{i}_{o})\}. $$

Proof of Theorem 4.1 *Consider a scalar function expressing total energy of the satellite. The total energy of the system is the sum of the kinetic energy of rotary motion, the potential energy generated by the gravity gradient and the energy originating from rotation of the satellite around the earth.*

$$ E_{tot} = E_{kin} + E_{gg} + E_{gyro}. $$

This leads to

$$ \begin{aligned} E_{tot} = {} & \tfrac{1}{2}{}^{c}\mathbf{\Omega}_{co}^{T}\mathbf{I}{}^{c}\mathbf{\Omega}_{co} + \tfrac{3}{2}\omega_o^2({}^{c}\mathbf{k}_o^T\mathbf{I}{}^{c}\mathbf{k}_o - I_z) \\ & + \tfrac{1}{2}\omega_o^2(I_x - {}^{c}\mathbf{i}_o^T\mathbf{I}{}^{c}\mathbf{i}_o), \end{aligned} \tag{4.32} $$

where I_x is the satellite maximum moment of inertia, and I_z is the minimum.

Now, the time derivative of E_{tot} is shown to be positive semidefinite.

$$\dot{E}_{tot} = {}^c\mathbf{\Omega}_{co}^T \mathbf{I}^c\dot{\mathbf{\Omega}}_{co} + 3\omega_o^2 {}^c\mathbf{k}_o^T \mathbf{I}^c\dot{\mathbf{k}}_o - \omega_o^2 {}^c\mathbf{i}_o^T \mathbf{I}^c\dot{\mathbf{i}}_o \tag{4.33}$$

Eqs. 2.1 to 2.6 are substituted into Eq. 4.33 yielding

$$\begin{aligned}\dot{E}_{tot} = {}^c\mathbf{\Omega}_{co}^T(-{}^c\mathbf{\Omega}_{cw} \times \mathbf{I}^c\mathbf{\Omega}_{cw} + 3\omega_o^2\ {}^c\mathbf{k}_o^T \times \\ \mathbf{I}^c\mathbf{k}_o + {}^c\mathbf{N}_{ctrl}) - \omega_o {}^c\mathbf{\Omega}_{co}^T \mathbf{I}({}^c\mathbf{i}_o \times {}^c\mathbf{\Omega}_{co}) \\ +3\omega_o^2 {}^c\mathbf{k}_o^T \mathbf{I}({}^c\mathbf{k}_o \times {}^c\mathbf{\Omega}_{co}) - \omega_o^2 {}^c\mathbf{i}_o^T \mathbf{I}({}^c\mathbf{i}_o \times {}^c\mathbf{\Omega}_{co}).\end{aligned} \tag{4.34}$$

Since

$$\begin{aligned}{}^c\mathbf{\Omega}_{co}^T({}^c\mathbf{\Omega}_{cw} \times \mathbf{I}^c\mathbf{\Omega}_{cw}) = \omega_o {}^c\mathbf{\Omega}_{co}^T({}^c\mathbf{i}_o \times \mathbf{I}^c\mathbf{\Omega}_{co}) \\ +\omega_o^2\ {}^c\mathbf{\Omega}_{co}^T({}^c\mathbf{i}_o \times \mathbf{I}^c\mathbf{i}_o),\end{aligned} \tag{4.35}$$

Eq. 4.35 is reduced to

$$\dot{E}_{tot} = {}^c\mathbf{\Omega}_{co}^T \mathbf{N}_{ctrl}. \tag{4.36}$$

If proposed control law 4.31 is applied then Eq. 4.36 becomes

$$\dot{E}_{tot} = -{}^c\mathbf{\Omega}_{co}^T \tilde{\mathbf{B}}\tilde{\mathbf{B}}\mathbf{H}^c\mathbf{\Omega}_{co}. \tag{4.37}$$

Here $\tilde{\mathbf{B}}$ is the skew symmetric matrix representing a cross product operator: ${}^c\mathbf{B}\times$. $\tilde{\mathbf{B}}\tilde{\mathbf{B}}$ is semipositive definite and $\mathbf{H}$ is positive definite. The derivative of the total energy is thus positive semidefinite. Moreover $E_{tot}(t)$ is limited by the initial value $E_{tot}(t_0)$. $\ddot{E}_{tot}(t)$ is a continuous function of ${}^c\mathbf{B}(t)$ and ${}^c\mathbf{\Omega}_{co}(t)$ which are bounded (${}^c\mathbf{\Omega}_{co}(t) < {}^c\mathbf{\Omega}_{co}(t_0)$). Thus $\dot{E}_{tot}(t)$ is uniformly continuous in time. According to the Barbalat's lemma, see (Slotine and Li, 1991), lemma 4.2, $\lim_{t\to\infty} \dot{E}_{tot}(t) = 0$.

It will be proven that $E_{tot}(t)$ converges to 0 by contradiction. Let $E_{tot}(t)$ not converge to 0. Then Eq. 4.37 implies that $\dot{E}_{tot}(t) = 0$ and $E_{tot}(t)$ is constant. Hence the vector ${}^c\mathbf{B}(t)$ is parallel to ${}^c\mathbf{\Omega}_{co}(t)$ or ${}^c\mathbf{B}(t)$ is parallel to $\mathbf{\Lambda}_q {}^c\mathbf{\Omega}_{co}(t)$ for each t. It can be expressed by

$$\forall_{t\geq 0}\ {}^c\mathbf{B}(t) = \alpha {}^c\mathbf{\Omega}_{co}(t)\ \vee\ {}^c\mathbf{B}(t) = \beta \mathbf{\Lambda}_\Omega {}^c\mathbf{\Omega}_{co}(t), \tag{4.38}$$

where α and β are some non zero constants. Differentiating ${}^c\mathbf{\Omega}_{co}(t)$ and ${}^c\mathbf{B}(t)$ in Eq. 4.38 with respect to time gives

$$\begin{aligned}\forall_{t\geq 0}\ {}^o\dot{\mathbf{B}}(t) = \alpha \mathbf{A}^T({}^c_o\mathbf{q})\ {}^c\dot{\mathbf{\Omega}}_{co}(t) \\ \vee\ {}^o\dot{\mathbf{B}}(t) = \beta \mathbf{A}^T({}^c_o\mathbf{q})(\mathbf{\Lambda}_\Omega {}^c\dot{\mathbf{\Omega}}_{co}(t) - \mathbf{\Lambda}_\Omega {}^c\mathbf{\Omega}_{co} \times {}^c\mathbf{\Omega}_{co}),\end{aligned} \tag{4.39}$$

since

$${}^c\dot{\mathbf{B}}(t) = {}^c\mathbf{B}(t) \times {}^c\mathbf{\Omega}_{co}(t) + \mathbf{A}({}^c_o\mathbf{q}) {}^o\dot{\mathbf{B}}(t). \tag{4.40}$$

The time propagation of the angular velocity, ${}^c\mathbf{\Omega}_{co}$, is given by Eqs. 2.1, 2.4, and 2.6 for ${}^c\mathbf{N}_{ctrl} = \mathbf{0}$. Notice that r.h.s of Eq. 4.38 is not dependent on ${}^o\mathbf{B}(t)$ nor ${}^o\dot{\mathbf{B}}(t)$. ${}^o\mathbf{B}$ and ${}^o\dot{\mathbf{B}}$ are given by the geomagnetic field model in Wertz (1990). The time propagation of the geomagnetic field in the Orbit CS is depicted in Fig. 4.6. The development of ${}^c\dot{\mathbf{\Omega}}_{co}(t)$ with zero control torque (because of Eq. 4.38) is given by Eqs. 2.1 and 2.4. It follows that Eq. 4.39 is not valid. This shows the contradiction. ■

Remark 4.1 *The control law in Eq. 4.31 can be used for three-axis magnetic stabilization of the satellite in a neighbourhood of one of 4 equilibria stated in theorem 4.1 if $I_x > I_y > I_z$.*

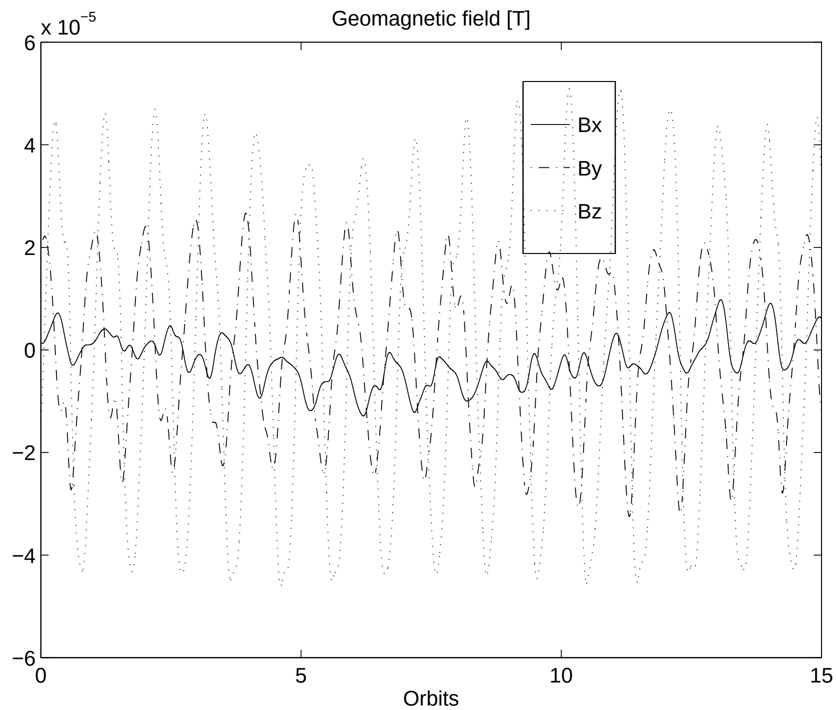

Fig. 4.6. Time propagation of geomagnetic field in Orbit CS.

5 Local Attitude Stability

Three-axis attitude control at equilibrium $(\mathbf{0},\ {}^o\mathbf{k}_o,\ {}^o\mathbf{i}_o)$ is addressed in this section. The control law in Eq. 4.31 can give any of four attitude equilibria in theorem 4.1. This is not operationally acceptable. Therefore a new control law is introduced

$$^c\mathbf{m}(t) = {}^c\mathbf{B}(t) \times \mathbf{H}{}^c\mathbf{\Omega}_{co}(t) - \epsilon{}^c\mathbf{B}(t) \times {}^c_o\mathbf{q}(t), \tag{5.41}$$

where $\mathbf{H}$ is a positive definite constant matrix and ϵ is a constant scalar. In Eq. 5.41 a small perturbation in attitude was added comparing with the control law, Eq. 4.31. For small ϵ the satellite is stable in the neighbourhood of reference $(\mathbf{0},\ {}^o\mathbf{k}_o,\ {}^o\mathbf{i}_o)$, since the differential equations describing motion of the satellite are well posed. The question is, how large ϵ shall be that the system is still locally stable.

The system is first linearized. The attitude quaternion is linearized in a special way as a multiplication of a small perturbation by the quaternion in reference $(\mathbf{0},\ {}^o\mathbf{k}_o,\ {}^o\mathbf{i}_o)$.

$${}^c_o\mathbf{q}_{ref} = \mathbf{q}_{ref} \cdot [\delta q^T\ 1]^T, \tag{5.42}$$

where $\mathbf{q}_{ref} = [0\ 0\ 0\ 1]^T$ and $\delta q = [\delta q_1\ \delta q_2\ \delta q_3]^T$.

The linearized equation of motion, i.e. Eqs. 2.1 to 2.6 is

$$\frac{d}{dt}\begin{bmatrix} \delta\Omega \\ \delta q \end{bmatrix} = \tilde{\mathbf{A}}\begin{bmatrix} \delta\Omega \\ \delta q \end{bmatrix} + \tilde{\mathbf{B}}(t)(\mathbf{H}\delta\Omega - \epsilon\delta q), \tag{5.43}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & -2k\sigma_x & 0 & 0 \\ 0 & 0 & \omega_o\sigma_y & 0 & 2k\sigma_y & 0 \\ 0 & \omega_o\sigma_z & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \omega_o \\ 0 & 0 & \frac{1}{2} & 0 & -\omega_o & 0 \end{bmatrix},$$

$$\sigma_x = \frac{I_y - I_z}{I_x},\ \sigma_y = \frac{I_z - I_x}{I_y},\ \sigma_z = \frac{I_x - I_y}{I_z},$$

$$\tilde{\mathbf{B}} = \mathbf{I}^{-1} \begin{bmatrix} {}^oB_y^2 + {}^oB_z^2 & {}^oB_x{}^oB_y & {}^oB_x{}^oB_z \\ {}^oB_x{}^oB_y & {}^oB_x^2 + {}^oB_z^2 & {}^oB_y{}^oB_z \\ {}^oB_x{}^oB_z & {}^oB_y{}^oB_z & {}^oB_x^2 + {}^oB_y^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Observing that the geomagnetic field is periodic with one orbit period, local stability of the satellite is analyzed with the use of Floquet theory Mohler (1991).

Consider the following form of equation (5.43)

$$\frac{d\mathbf{x}}{dt} = \hat{\mathbf{A}}(\epsilon, t)\mathbf{x}, \tag{5.44}$$

where $\hat{\mathbf{A}}(\epsilon, t) = \hat{\mathbf{A}}(\epsilon, t+T)$, for $T = \frac{2\pi}{\omega_o}$.

It follows from Floquet theory that the solution to the system in Eq. 5.44 consists of periodically modulated exponential matrix functions, and 5.44 is asymptotically stable if all eigenvalues of a certain matrix, $\mathbf{R}$ defined below, have negative real parts. The transition matrix $\mathbf{F}(t, \tau)$ for the system in Eq. 5.44 with fixed ϵ is

$$\mathbf{F}(t_0 + T, t_0, \epsilon) = e^{R(\epsilon)T}. \tag{5.45}$$

Now

$$det[\mathbf{I}\rho - \mathbf{R}] = 0 \tag{5.46}$$

implies that characteristic exponents ρ_i are

$$Re\{\rho_i\} < 0, \quad i = 1, ..., n, \tag{5.47}$$

or that

$$det[\mathbf{I}\lambda - \mathbf{e}^{RT}] = 0 \tag{5.48}$$

implies that characteristic multipliers λ_i are

$$|\lambda_i| < 1, \quad i = 1, ..., n. \tag{5.49}$$

It was proven in section 4 that the satellite motion is locally stable at the reference $(\mathbf{0},\ {}^o\mathbf{k}_o,\ {}^o\mathbf{i}_o)$ for $\epsilon = 0$. Now we find all ϵ for which the linearized satellite system in Eq. 5.43, with a certain fixed value of matrix $\mathbf{H}$, is stable by plotting a locus of characteristic multipliers. An example of characteristic multipliers locus is depicted in Fig. 5.7, where $\epsilon = 0.6\ 10^{-3}$ is the limit of stability.

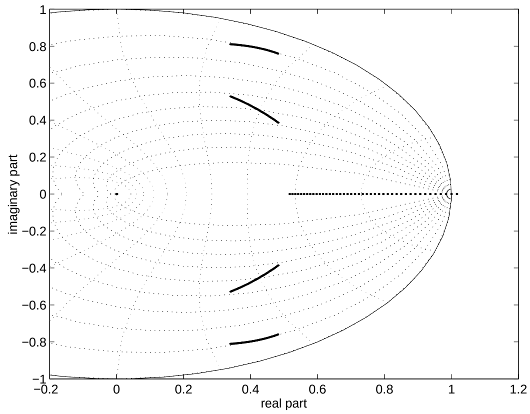

Fig. 5.7. Locus of characteristic multiplier $\lambda(\epsilon)$ for ϵ in $[0, 0.6\,10^{-3}]$, $\mathbf{H} = 0.3\mathbf{E}$.

6 Globally Stabilizing Controller

The control law in Eq. 5.41 was shown to be locally stable. A globally stable controller is the ultimate goal. The main obstacle is again the cross product with the geomagnetic field vector.

6.1 Idealized Quaternion Feedback

If it were possible to produce a control torque proportional to the quaternion error, a globally stable controller would result. This is shown in the following theorem.

Theorem 6.1 *The control law*

$$ {}^c\mathbf{m}(t) = {}^c\mathbf{B}(t) \times \mathbf{H}\,{}^c\mathbf{\Omega}_{co}(t) - \epsilon\,{}^c_o\mathbf{q}(t), \tag{6.50}$$

where $\mathbf{H}$ is a positive definite constant matrix, ϵ is a positive scalar, makes the system globally asymptotically stable at the reference $(\mathbf{0},\ {}^o\mathbf{k}_o,\ {}^o\mathbf{i}_o)$.

Proof of Theorem 6.1 *Total energy in the system is slightly modified from Eq. 4.32, proof of theorem 4.1 and equal*

$$\begin{aligned} E_{tot} = {} & \tfrac{1}{2}{}^c\mathbf{\Omega}_{co}^T \mathbf{I}\,{}^c\mathbf{\Omega}_{co} + \tfrac{3}{2}\omega_o^2({}^c\mathbf{k}_o^T \mathbf{I}\,{}^c\mathbf{k}_o - I_z) \\ & + \tfrac{1}{2}\omega_o^2(I_x - {}^c\mathbf{i}_o^T \mathbf{I}\,{}^c\mathbf{i}_o) + \epsilon(q_1^2 + q_2^2 \\ & + q_3^2 + (1 - q_4)^2). \end{aligned} \tag{6.51}$$

The attitude quaternion satisfies the constraint equation $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$, thus

$$\begin{aligned} E_{tot} = {} & \tfrac{1}{2}{}^c\mathbf{\Omega}_{co}^T \mathbf{I}\,{}^c\mathbf{\Omega}_{co} + \tfrac{3}{2}\omega_o^2({}^c\mathbf{k}_o^T \mathbf{I}\,{}^c\mathbf{k}_o - I_z) \\ & + \tfrac{1}{2}\omega_o^2(I_x - {}^c\mathbf{i}_o^T \mathbf{I}\,{}^c\mathbf{i}_o) + 2\epsilon(1 - q_4). \end{aligned} \tag{6.52}$$

The time derivative of Eq. 6.52 gives

$$\dot{E}_{tot} = {}^c\mathbf{\Omega}_{co}^T \mathbf{N}_{ctrl} + \epsilon\,{}^c\mathbf{\Omega}_{co}^T\,{}^c_o\mathbf{q}. \tag{6.53}$$

Applying the control law defined in Eq. 6.50, $\dot{E}_{tot}$ is

$$\dot{E}_{tot} = -{}^c\mathbf{\Omega}_{co}^T \tilde{\mathbf{B}}\tilde{\mathbf{B}}\mathbf{H}\,{}^c\mathbf{\Omega}_{co}. \tag{6.54}$$

This complies with Eq. 4.37. Thus the satellite with control law, Eq. 6.50 is globally asymptotically stable at the reference $(\mathbf{0},\ {}^o\mathbf{k}_o,\ {}^o\mathbf{i}_o)$. ■

6.2 Quaternion Feedback with Magnetic Torquing

Achievable control with magnetorquers involves the cross product with the geomagnetic field. The purpose of this subsection is to derive a global stabilizing controller under this limitation.

The proposed feedback with magnetic torquing is

$$ {}^{c}\mathbf{m}(t) = {}^{c}\mathbf{B}(t) \times \mathbf{H}{}^{c}\mathbf{\Omega}_{co}(t) + \epsilon(t){}^{c}\mathbf{B}(t) \times {}^{c}_{o}\mathbf{q}(t), \tag{6.55} $$

where $\mathbf{H}$ is a positive definite constant matrix and $\epsilon(t)$ is a piecewise continuous positive scalar function satisfying

$$ \begin{aligned} \epsilon(t) = const > 0,\ t \in (kT, (k+1)T),\ k = 1, 2, ... \\ \epsilon(kT) > \epsilon((k+1)T) > 0 \end{aligned} \tag{6.56} $$

is addressed in the sequel.

Theorem 6.2 *Consider the control law in Eq. 6.55 then the satellite, Eqs. 2.1 to 2.6 has 4 asymptotically stable local equilibria:*
$\{({}^{c}\mathbf{\Omega}_{co},\ {}^{c}\mathbf{k}_{o},\ {}^{c}\mathbf{i}_{o}) : (\mathbf{0},\ \pm^{o}\ \mathbf{k}_{o},\ \pm^{o}\ \mathbf{i}_{o})\}$.

Proof of Theorem 6.2 *For simplicity of notation the equations of satellite Eqs. 2.1 to 2.6 actuated according to Eq. 4.31 are represented by*

$$ \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \tag{6.57} $$

and the equations of satellite motion actuated according to Eq. 6.55, for constant $\epsilon(t) = \epsilon(kT)$*, are*

$$ \dot{\mathbf{x}}(t) = \mathbf{f}_k(\mathbf{x}(t), t). \tag{6.58} $$

Moreover, the differential equation 6.57 has solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ *for the initial condition* $\mathbf{x}(t_0) = \mathbf{x}_0$*, and Eq. 6.58 has solution* $\mathbf{x}_k(t, t_0, \mathbf{x}_0)$ *for the initial condition* $\mathbf{x}_k(t_0) = \mathbf{x}_0$.

Since the kinematic and dynamic differential equations are Lipschitz, the following is true

$$ if\ lim_{k\to\infty}\mathbf{f}_k(\mathbf{x}(t), t) = \mathbf{f}(\mathbf{x}(t), t)\ then $$

$$ lim_{k\to\infty}\mathbf{x}_k(t, t_0, \mathbf{x}_0) = \mathbf{x}(t, t_0, \mathbf{x}_0), $$

thus

$$ if\ lim_{t\to\infty}\mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{y}_f\ then $$

$$ lim_{t\to\infty}\mathbf{x}_k(t, t_0, \mathbf{x}_0) = \mathbf{y}_f. $$

This means, if $lim_{t\to\infty}\epsilon(t) = 0$*, each trajectory of the satellite actuated by the control law in Eq. 6.55 converges to one of the equilibria:*
$\{({}^{c}\mathbf{\Omega}_{co},\ {}^{c}\mathbf{k}_{o},\ {}^{c}\mathbf{i}_{o}) : (\mathbf{0},\ \pm^{o}\ \mathbf{k}_{o},\ \pm^{o}\ \mathbf{i}_{o})\}$. ■

It was stated in theorem 6.2 that the satellite with control law, Eq. 6.55 has 4 asymptotically stable equilibria. Local analysis demonstrated that the equilibrium $(\mathbf{0},\ {}^{o}\mathbf{k}_{o},\ {}^{o}\mathbf{i}_{o})$ is locally stable if $\epsilon(t) < \tilde{\epsilon}$, where $\tilde{\epsilon}$ is the maximum value of ϵ in Eq. 5.41 providing local stability. On the other hand if $\epsilon(t)$

is large enough in Euclidean norm sense so that the quaternion feedback is the most significant component on the r.h.s. of Eq. 2.1, then vectors $\mathbf{I}^c\mathbf{\Omega}_{co}(t)$ and $\mathbf{q}(t)$ become parallel

$$\mathbf{I}^c\dot{\mathbf{\Omega}}_{co}(t) \approx -\epsilon(t)^c\mathbf{B} \times (^c\mathbf{B} \times \mathbf{q}(t)). \tag{6.59}$$

Equation (6.59) implies that for large $\epsilon(t) > 0$

$$^c\mathbf{\Omega}_{co}^T(^c\mathbf{B} \times (^c\mathbf{B} \times \mathbf{q},(t))) \leq 0, \tag{6.60}$$

and

$$^c\mathbf{\Omega}_{co}^T\mathbf{q} \leq 0, \tag{6.61}$$

since I is positive definite, and $\tilde{\mathbf{B}}\tilde{\mathbf{B}}$ is positive semidefinite.

If Eq. 6.60 is always satisfied, then the satellite is asymptotically stable at the reference $(\mathbf{0},\ ^o\mathbf{k}_o,\ ^o\mathbf{i}_o)$. Proof of this statement is similar to the proof of asymptotic stability of the control law in Eq. 6.50. The total energy in the system is defined by Eq. 6.52, where $2\epsilon(1-q_4)$ is substituted for

$$2\epsilon(t_0)(1-q_4) \sup_{t\in[0,T)} |eig(\tilde{\mathbf{B}}^T\tilde{\mathbf{B}})|,$$

thus time derivative of E_{tot} is negative definite for each nonzero $^c\mathbf{\Omega}_{co}$ and $\mathbf{q}$.

From the analysis carried out so far it follows that for $\epsilon(t_0)$ large enough and $\Delta = |\epsilon((k+1)T) - \epsilon(kt)|$ small enough, the satellite trajectory diverges from equilibria:

$$(\mathbf{0},\ -^o\mathbf{k}_o,\ ^o\mathbf{i}_o),\ (\mathbf{0},\ -^o\mathbf{k}_o,\ -^o\mathbf{i}_o),\ (\mathbf{0},\ ^o\mathbf{k}_o,\ -^o\mathbf{i}_o)$$

toward equilibrium $(\mathbf{0},\ ^o\mathbf{k}_o,\ ^o\mathbf{i}_o)$, which is in turn locally stable.

Simulations results have confirmed our hypothesis that the satellite actuated according to Eq. 6.55 is globally asymptotically stable. This control algorithm is especially useful when a boom is in the upside-down position. In practical implementation for the Ørsted satellite $\epsilon(t)$ was chosen continuous and convergent to some constant $\epsilon(t),\ \hat{\epsilon} \leq \tilde{\epsilon}$.

The similarity between the locally and globally stable controllers is striking. The only difference is the gain factor $\epsilon(t)$. This makes implementation exceptionally simple: 1) if the boom is in upright position, apply control law in Eq. 6.55 with constant $\epsilon(t),\ \hat{\epsilon} \leq \tilde{\epsilon}$, 2) if the boom is detected to be upside-down, apply Eq. 6.55 with $\epsilon(t)$ gradually decreasing to $\hat{\epsilon}$.

6.3 Simulation Results

The global stabilizing controller developed in this article has been implemented for the Ørsted satellite in the operational phase. The satellite has the following moments of inertia: $I_{xx} = 178.3\,kgm^2,\ I_{yy} = 177.8\,kgm^2,\ I_{zz} = 1.3\,kgm^2$.

Simulation results are shown in Figs. 6.8 to 6.10. The initial conditions are such that the satellite is in upside-down position corresponding to the equilibrium $(\mathbf{0},\ -^o\mathbf{k}_o,\ ^o\mathbf{i}_o)$. The controller is quite convincing. It takes less than half an orbit to turn the satellite up, and it is stabilized to the operational region within 6 orbits. This is rather satisfactory considering that the available mechanical torque is less than $10^{-3}\ Nm$, which is only three times as much as the magnitude of the maximum gravity gradient torque for this satellite.

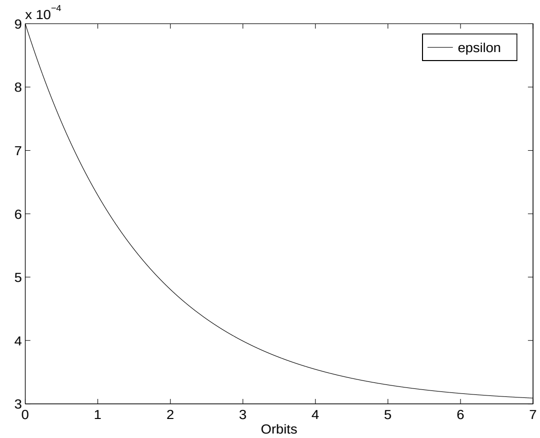

Fig. 6.8. ϵ converges to $\hat{\epsilon} = 0.0003,\ \mathbf{H} = 0.3\mathbf{E}$.

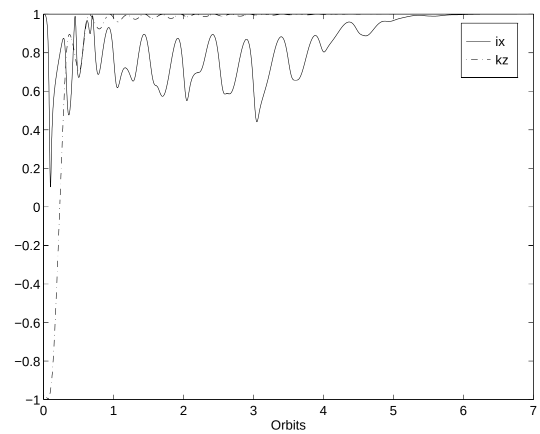

Fig. 6.9. ${}^{c}\mathbf{k}_{oz}$ characterizes convergence of ${}^{c}\mathbf{k}_{o}$ toward ${}^{o}\mathbf{k}_{o}$ (if ${}^{c}\mathbf{k}_{oz} < 0$ satellite is in upside-down position), while ${}^{c}\mathbf{i}_{ox}$ characterizes convergence of ${}^{c}\mathbf{i}_{o}$ toward ${}^{o}\mathbf{i}_{o}$

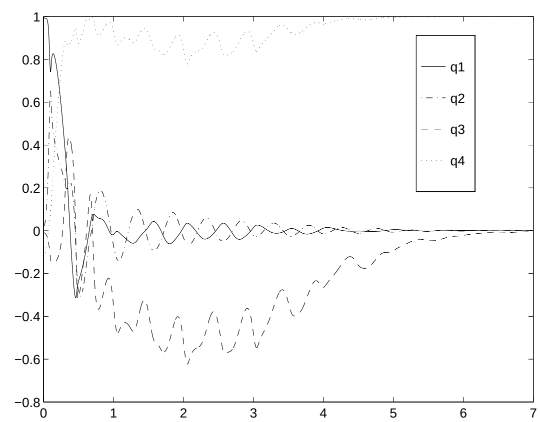

Fig. 6.10. The attitude quaternion, ${}^{c}_{o}\mathbf{q}$ converges to $[0\ 0\ 0\ 1]^{T}$ from an upside-down condition.

7 Conclusions

The sliding control law for three-axis stabilization of a tumbling satellite was described and analyzed in this paper. This analysis inspired further work on the proportional-derivative feedback cross product with the local geomagnetic field. Both locally and globally stabilizing controllers were proposed, and a rigorous stability analysis was carried out. This work is believed to contribute to the development of proportional-derivative feedback control based only on magnetic torquing for low earth orbit satellites. Simulation results showed the proficiency of the new controller in the upside-down configuration, a worst case situation for the satellite.

Acknowledgment

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A Coordinate Systems

- Control CS is a right orthogonal coordinate system built on the principal axes of the satellite with the origin placed in the centre of mass. The x-axis is the axis of the maximum moment of inertia, and the z-axis is the minimum.
- Orbit CS is a right orthogonal coordinate system fixed in the centre of mass of the satellite. The z-axis points at zenith (is aligned with the centre of the Earth and points away from the Earth), the x-axis points in the orbital plane normal direction and its sense coincides with the sense of the orbital angular velocity vector.
- World CS is an inertial right orthogonal coordinate system with origin in the centre of mass of the satellite. The z-axis is parallel to the rotation axis of the Earth and point towards the North Pole. The x-axis is parallel to the line connecting the centre of the Earth with Vernal Equinox and points toward Vernal Equinox. Vernal Equinox is the point where the ecliptic crosses the Earth equator going from South to North on the first day of spring.

B Quaternions

Information included in this appendix is based on Wertz (1990).

Rotation of coordinate systems can by described by means of a quaternion. A vital feature of quaternions is that they provide convenient product rule for successive rotations and simple form of kinematics lacking singularities.

The four parameters (q_1, q_2, q_3, q_4) form the components of the quaternion, $\tilde{\mathbf{q}}$, as follows

$$\tilde{\mathbf{q}} \equiv q_4 + i\, q_1 + i\, q_2 + i\, q_3, \tag{2.62}$$

where i, j, and k are the hyper imaginary numbers satisfying the condition

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ {}'ij = -ji &= k \\ jk = -kj &= i \\ ki = -ik &= j. \end{aligned} \tag{2.63}$$

The first three components, $\mathbf{q} = [q_1, q_2, q_3]^T$, form the vector part of the quaternion and the quantity, q_4, is the scalar part. The inverse of $\tilde{\mathbf{q}}$ is defined as

$$\tilde{\mathbf{q}}^* \equiv q_4 - i\, q_1 - i\, q_2 - i\, q_3. \tag{2.64}$$

The norm of quaternion $\tilde{\mathbf{q}} = [\mathbf{q}^T\ q_4]^T$ is defined as

$$|\tilde{\mathbf{q}}| \equiv \sqrt{\tilde{\mathbf{q}}^* \tilde{\mathbf{q}}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}. \tag{2.65}$$

The construction of the unit quaternion arises from the claim that the rotation of coordinate systems can be uniquely described by a unit vector, $\mathbf{e} = [e_1\ e_2\ e_3]^T$ giving an axis of rotation as well as its sense, and an angle of rotation ϕ. A quaternion, $\tilde{\mathbf{q}}$, incorporates four parameters:

$$\begin{aligned} q_1 &\equiv e_1 sin\tfrac{\phi}{2} \\ q_2 &\equiv e_2 sin\tfrac{\phi}{2} \\ q_3 &\equiv e_3 sin\tfrac{\phi}{2} \\ q_4 &\equiv cos\tfrac{\phi}{2}. \end{aligned} \tag{2.66}$$

Notice that norm of the quaternion defined according to Eq. 2.66 is 1. Furthermore the same attitude can given by two quaternions q and -q, the first is given when the angle of rotation is ϕ, and the latter $2\pi + \phi$.

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