Kepler elements $\longrightarrow$ Cartesian position and velocity

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Problem: Given 6 Kepler elements $(a, e, I, \omega, \Omega, M)$, find the corresponding inertial position $\boldsymbol{r}$ and velocity $\dot{\boldsymbol{r}}$.

Solution (Kaula,1966): First get the eccentric anomaly E from the mean anomaly M by iteratively solving Kepler's equation:

$$E - e\sin E = M \;\Rightarrow\; E_{i+1} = e\sin E_i + M \text{ , with starting value } E_0 = M \tag{1}$$

Next, get the position and the velocity in the q-frame, which has its z-axis perpendicular to the orbital plane and its x-axis pointing to the perigee:

$$\boldsymbol{q} = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{pmatrix} \;,\; \dot{\boldsymbol{q}} = \frac{na}{1 - e\cos E}\begin{pmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{pmatrix} \tag{2}$$

In case the true anomaly ν is given in the original problem instead of the mean anomaly M, the vectors $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are obtained by:

$$\boldsymbol{q} = \begin{pmatrix} r\cos\nu \\ r\sin\nu \\ 0 \end{pmatrix} \;,\; \dot{\boldsymbol{q}} = \frac{na}{\sqrt{1-e^2}}\begin{pmatrix} -\sin\nu \\ e + \cos\nu \\ 0 \end{pmatrix} \tag{3}$$

with

$$r = \frac{a(1-e^2)}{1 + e\cos\nu} \tag{4}$$

The transformation from inertial frame to the q-frame is performed by the rotation sequence $R_3(\omega)R_1(I)R_3(\Omega)$. So, vice versa, the inertial position and velocity are obtained by the reverse transformations:

$$\boldsymbol{r} = R_3(-\Omega)R_1(-I)R_3(-\omega)\boldsymbol{q} \tag{5}$$

$$\dot{\boldsymbol{r}} = R_3(-\Omega)R_1(-I)R_3(-\omega)\dot{\boldsymbol{q}} \tag{6}$$

Cartesian position and velocity $\longrightarrow$ Kepler elements

Problem: Given a satellite's inertial position $\boldsymbol{r}$ and velocity $\dot{\boldsymbol{r}}$, find the corresponding Kepler elements $(a, e, I, \omega, \Omega, M)$.

Solution (Kaula,1966): The angular momentum vector per unit mass is normal to the orbital plane. It defines the inclination I and right ascension of the ascending node Ω:

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \tag{7}$$

$$\tan \Omega = \frac{h_1}{-h_2} \tag{8}$$

$$\tan I = \frac{\sqrt{h_1^2 + h_2^2}}{h_3} \tag{9}$$

Rotate $\boldsymbol{r}$ into the p-frame in the orbital plane now and derive the argument of latitude u:

$$\boldsymbol{p} = R_1(I) R_3(\Omega) \boldsymbol{r} \tag{10}$$

$$\tan u = \tan(\omega + \nu) = \frac{p_2}{p_1} \tag{11}$$

The semi-major axis a comes from the total energy and requires the scalar velocity $v = |\dot{\boldsymbol{r}}|$. The eccentricity e needs the scalar angular momentum $h = |\boldsymbol{h}|$:

$$T - V = \frac{v^2}{2} - \frac{GM}{r} = -\frac{GM}{2a} \tag{12}$$

$$a = \frac{GM\, r}{2GM - r v^2} \tag{13}$$

$$e = \sqrt{1 - \frac{h^2}{GM\, a}} \tag{14}$$

In order to extract the eccentric anomaly E, we need to know the radial velocity first:

$$\dot{r} = \frac{\boldsymbol{r} \cdot \dot{\boldsymbol{r}}}{r} \tag{15}$$

$$\cos E = \frac{a - r}{ae} \tag{16}$$

$$\sin E = \frac{r \dot{r}}{e \sqrt{GM\, a}} \tag{17}$$

The true anomaly is obtained from the eccentric one:

$$\tan \nu = \frac{\sqrt{1 - e^2} \sin E}{\cos E - e} \tag{18}$$

Subtracting ν from the argument of latitude u yields the argument of perigee ω. Finally, Kepler's equation provides the mean anomaly:

$$E - e \sin E = M \tag{19}$$