Model Independent Eigenaxis Maneuvers using Quaternion Feedback

Jonathan Lawton Randal W. Beard
Dept. Electrical and Computer Eng.
Brigham Young University
Provo, UT, 84602
{lawtonj, beard}@ee.byu.edu

Abstract

The main contribution of this paper is the derivation of a model independent feedback control law that causes a rigid body to execute a near eigenaxis maneuver. The degree of approximation is a function of the control gains and the initial attitude error. Using passivity techniques, the dependence of the control strategy on velocity information is eliminated. The technique used to derive the control strategy is a novel orthogonal decomposition of the unit quaternion into a component along the direction of the eigenaxis and a component that is orthogonal to that direction.

1 Introduction

A rigid body in any initial orientation can be rotated to a final orientation through a single rotation about a fixed axis. Such a rotation, called an eigenaxis rotation, constitutes the "shortest" rotation between two orientations. Although it has been shown that time optimal maneuvers may not be eigenaxis rotations [1], eigenaxis rotations can be used to generate near time optimal maneuvers [2]. Current strategies for eigenaxis rotations use a combination of feedback control and feedforward control terms. Wie, Weiss and Arapostathis [3] used feedback on the quaternion error combined with feedforward of the gyroscopic coupling to implement quaternion regulation via eigenaxis rotations. These regulation results were extended to the tracking problem by Weiss [4]. The tracking results are also model dependent and require cancelation of the gyroscopic coupling. Some effort has been made to derive a model independent approach to quaternion regulation via eigenaxis rotations. Cristi, Burl and Russo [5] developed adaptive control strategies to approximate the control found in [3]. Their approach still required approximate gyroscopic decoupling, i.e., feedforward control. Furthermore, the degree to which the rigid body motion approximates an eigenaxis rotation was not analyzed. No previous work has derived control laws for implementing eigenaxis rotations that are model independent and in feedback form. The contribution of this paper is to derive such a control strategy.

We factor an arbitrary rotation into a rotation about a fixed axis $\mathbf{u}$, where $\mathbf{u}$ is an arbitrary unit vector, followed by a rotation about an axis perpendicular to $\mathbf{u}$. By additionally including the orthogonal rotation term in the control law, we approximate an eigenaxis rotation.

In Section 2 we establish our notation and show that the Euclidean norm can be used to approximate the geodesic metric [6] which we use in our Lyapunov analysis. In Section 3 we present our quaternion factorization results. In Section 4 we use our quaternion factorization to derive the main result of this paper, which is the approximate eigenaxis rotation via feedback control. In Section 5 we derive a similar control law, using passivity-based control, that does not require velocity measurements. Section 6 illustrates the derived control strategies via simulation. Finally in Section 7 we summarize our results.

2 Approximate Geodesic Metric

It is useful to separate the quaternion into a vector component and a scalar component, i.e.,

$$\mathbf{q} = \begin{bmatrix} \mathcal{V}(\mathbf{q}) \\ \mathcal{S}(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \\ q_4 \end{bmatrix},$$

where $\mathcal{V}(\mathbf{q})$ is the vector component of $\mathbf{q}$ and $\mathcal{S}(\mathbf{q})$ is the scalar component of $\mathbf{q}$. The product or composition of two quaternions, $\mathbf{q}$ and $\mathbf{p}$, is given by the following relation

$$\mathbf{qp} = \begin{bmatrix} \mathcal{S}(\mathbf{q})\mathcal{V}(\mathbf{p}) + \mathcal{S}(\mathbf{p})\mathcal{V}(\mathbf{q}) + \mathcal{V}(\mathbf{q}) \times \mathcal{V}(\mathbf{p}) \\ \mathcal{S}(\mathbf{q})\mathcal{S}(\mathbf{p}) - \mathcal{V}(\mathbf{q})^T\mathcal{V}(\mathbf{p}), \end{bmatrix}, \quad (1)$$

where the operator $\times$ is the vector cross product operator. Due to the cross product term, composition is not commutative, i.e., $\mathbf{qp} \neq \mathbf{pq}$. The quaternion conjugate is defined by:

$$\mathbf{q}^* = \begin{bmatrix} -\mathcal{V}(\mathbf{q}) \\ \mathcal{S}(\mathbf{q}) \end{bmatrix} \qquad \text{(see [7])},$$

where $\mathbf{q}\mathbf{q}^* = \mathbf{q}^*\mathbf{q} = \mathbf{1} = (0,0,0,1)^T$.

Some additional properties of unit quaternions include:

$$\mathbf{q}^T\mathbf{p} = \mathcal{S}(\mathbf{q}^*\mathbf{p}) = \mathcal{S}(\mathbf{q}\mathbf{p}^*), \tag{2}$$
$$\mathcal{S}(\mathbf{q}\mathbf{p}) = \mathcal{S}(\mathbf{p}\mathbf{q}), \tag{3}$$
$$(\mathbf{q}\mathbf{p})^* = \mathbf{p}^*\mathbf{q}^*, \tag{4}$$
$$\|\mathbf{q}\mathbf{p}\| = \|\mathbf{q}\|\|\mathbf{p}\|. \tag{5}$$

A unit quaternion can be used to represent the rotation of a rigid body. The set of unit quaternions is given by $\mathcal{Q} = \{\mathbf{q} \in I\!R^4 | \mathbf{q}^T\mathbf{q} = 1\}$. It is shown in [7] that the set $\mathcal{Q}$ is closed under quaternion multiplication and quaternion conjugation, i.e., if $\mathbf{q}, \mathbf{p} \in \mathcal{Q}$ then $\mathbf{q}\mathbf{p}, \mathbf{q}^* \in \mathcal{Q}$.

The kinematic evolution for a rigid spacecraft is given by

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q}(t)\boldsymbol{\omega},$$

where $\boldsymbol{\omega} = (\omega^T, 0)^T$. The dynamics of the rotating rigid body are given by

$$J\dot{\omega} = -\omega \times J\omega + \tau,$$

where J is the moment of inertia of the rigid body, τ is the applied torque, and $\omega \in I\!R^3$ is the angular velocity represented as a vector.

The geodesic metric is given by

$$\rho(\mathbf{q}, \mathbf{p}) = 2|\arccos(\mathcal{S}(\mathbf{p}^*\mathbf{q}))|.$$

This metric is the minimum angle of rotation (i.e. the Euler angle) between a rigid body with attitude $\mathbf{q}$ and a rigid body with attitude $\mathbf{p}$ [6]. Unfortunately, the geodesic metric is seldom used directly because the arccos function is not compatible with Lyapunov theory due to its complicated derivative.

Fortunately, the Euclidean distance between two unit quaternions gives a good approximation to the geodesic metric. Lemma 2.1 relates the Euclidean distance to the geodesic metric.

Lemma 2.1 *If $\mathbf{q}$ and $\mathbf{p}$ are unit quaternions, then*

$$\frac{2}{\pi}\rho(\mathbf{q},\mathbf{p}) \le 2\|\mathbf{q}-\mathbf{p}\| \le \rho(\mathbf{q},\mathbf{p}). \tag{6}$$

Additionally, for small $\rho(\mathbf{q},\mathbf{p})$, $\rho(\mathbf{q},\mathbf{p}) \approx 2\|\mathbf{q}-\mathbf{p}\|$ in the sense that $\rho(\mathbf{q},\mathbf{p})$ is the first term of the Taylor Series approximation of $2\|\mathbf{q}-\mathbf{p}\|$.

Proof: See [8].

The next lemma will be used in Section 4 in calculating the time derivative of our Lyapunov functions.

Lemma 2.2 *Given*

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q}\boldsymbol{\omega}_q$$
$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{p}\boldsymbol{\omega}_p$$

then

$$\frac{d}{dt}\|\mathbf{q}-\mathbf{p}\|^2 = \mathcal{V}(\mathbf{p}^*\mathbf{q})^T(\omega_q - \omega_p).$$

Proof: See [8].

3 Orthogonal Axes Decomposition

In this section we show that any rotation can be decomposed into a rotation about a fixed axis $\mathbf{u}$ followed by a rotation about another axis perpendicular to $\mathbf{u}$. Theorem 3.1 shows that any unit quaternion can be factored into two unit quaternions representing this series of rotations.

Theorem 3.1 *Let the spacecraft attitude be given by $\mathbf{q}$. Given an arbitrary unit vector $\mathbf{u}$, if $\mathbf{q}_u$ is defined as*

$$\mathbf{q}_u = \begin{pmatrix} \mathcal{V}(\mathbf{q}_u) \\ \mathcal{S}(\mathbf{q}_u) \end{pmatrix} = \begin{pmatrix} \sin\left(\frac{\theta}{2}\right)\mathbf{u} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{7}$$

and $\mathbf{q}_R$ is defined as

$$\mathbf{q}_R = \mathbf{q}_u^*\mathbf{q}, \tag{8}$$

where θ is given by

$$\theta = 2\mathrm{atan2}(\mathbf{u}^T\mathcal{V}(\mathbf{q}), \mathcal{S}(\mathbf{q})), \tag{9}$$

then

1. $\mathbf{q} = \mathbf{q}_u\mathbf{q}_R$,
2. $\mathbf{u}^T\mathcal{V}(\mathbf{q}_R) = 0$,
3. $\mathcal{V}(\mathbf{q}_u)^T\mathcal{V}(\mathbf{q}_R) = 0$,
4. $\omega_u = \dot{\theta}\mathbf{u}$, *where ω_u is the angular velocity corresponding to $\mathbf{q}_u$.*

Proof: 1. The decomposition, $\mathbf{q} = \mathbf{q}_u\mathbf{q}_R$, follows if we multiply both sides of Equation (8) by $\mathbf{q}_u$ and use the fact that $\mathbf{q}_u\mathbf{q}_u^* = \mathbf{1}$.
2. By direct calculation of $\mathbf{q}_R = \mathbf{q}_u^*\mathbf{q}$, we get that

$$\mathcal{V}(\mathbf{q}_R) = \cos\left(\frac{\theta}{2}\right)\mathcal{V}(\mathbf{q}) - \mathcal{S}(\mathbf{q})\sin\left(\frac{\theta}{2}\right)\mathbf{u} - \sin\left(\frac{\theta}{2}\right)\mathbf{u}\times\mathcal{V}(\mathbf{q}).$$

Therefore we can calculate the quantity $\mathbf{u}^T\mathcal{V}(\mathbf{q}_R)$ as

$$\begin{aligned}\mathbf{u}^T\mathcal{V}(\mathbf{q}_R) =& \mathbf{u}^T\left(\cos\left(\frac{\theta}{2}\right)\mathcal{V}(\mathbf{q}) - \mathcal{S}(\mathbf{q})\sin\left(\frac{\theta}{2}\right)\mathbf{u}\right.\\ &\left. - \sin\left(\frac{\theta}{2}\right)\mathbf{u}\times\mathcal{V}(\mathbf{q})\right)\\ =& \left(\cos\left(\frac{\theta}{2}\right)\mathbf{u}^T\mathcal{V}(\mathbf{q}) - \mathcal{S}(\mathbf{q})\sin\left(\frac{\theta}{2}\right)\right)\\ =& \cos\left(\frac{\theta}{2}\right)\mathcal{S}(\mathbf{q})\left(\frac{\mathbf{u}^T\mathcal{V}(\mathbf{q})}{\mathcal{S}(\mathbf{q})} - \tan\left(\frac{\theta}{2}\right)\right)\\ =& 0.\end{aligned}$$

3. $\mathcal{V}(\mathbf{q}_u)^T\mathcal{V}(\mathbf{q}_R) = \sin(\theta/2)\mathbf{u}^T\mathcal{V}(\mathbf{q}_R) = 0.$
4. The unit quaternion $\mathbf{q}_u$ evolves according to

$$\dot{\mathbf{q}}_u = \frac{1}{2}\mathbf{q}_u\boldsymbol{\omega}_u.$$

Solving for $\boldsymbol{\omega}_u$ we find that

$$\begin{aligned}\boldsymbol{\omega}_u &= 2\mathbf{q}_u^*\dot{\mathbf{q}}_u\\ &= \dot{\theta}\begin{bmatrix}-\sin(\theta/2)\mathbf{u}\\ \cos(\theta/2)\end{bmatrix}\begin{bmatrix}\cos(\theta/2)\mathbf{u}\\ -\sin(\theta/2)\end{bmatrix}\\ &= \begin{bmatrix}\dot{\theta}\mathbf{u}\\ 0\end{bmatrix}.\end{aligned}$$

This can be written in vector notation as

$$\omega_u = \dot{\theta}\mathbf{u}.$$

■

In Lemma 3.1 the angular velocities corresponding to $\mathbf{q}_u$ and $\mathbf{q}_R$ are shown to be related to $\boldsymbol{\omega}$, the total angular velocity of the rigid body.

Lemma 3.1 *If $\boldsymbol{\omega}_u$ and $\boldsymbol{\omega}_R$ are the angular velocities associated with $\mathbf{q}_u$ and $\mathbf{q}_R$, i.e.,*

$$\begin{aligned}\dot{\mathbf{q}}_u &= \frac{1}{2}\mathbf{q}_u\boldsymbol{\omega}_u,\\ \dot{\mathbf{q}}_R &= \frac{1}{2}\mathbf{q}_R\boldsymbol{\omega}_R,\end{aligned}$$

then the total angular velocity $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \mathbf{q}_R^*\boldsymbol{\omega}_u\mathbf{q}_R + \boldsymbol{\omega}_R.$$

Proof: Since $\dot{\mathbf{q}} = (1/2)\mathbf{q}\boldsymbol{\omega}$, we can write

$$\begin{aligned}\boldsymbol{\omega} &= 2\mathbf{q}^*\dot{\mathbf{q}}\\ &= 2\mathbf{q}^*\frac{d}{dt}(\mathbf{q}_u\mathbf{q}_R)\\ &= 2\mathbf{q}^*[\dot{\mathbf{q}}_u\mathbf{q}_R + \mathbf{q}_u\dot{\mathbf{q}}_R]\\ &= \mathbf{q}_R^*\mathbf{q}_u^*[\mathbf{q}_u\boldsymbol{\omega}_u\mathbf{q}_R + \mathbf{q}_u\mathbf{q}_R\boldsymbol{\omega}_R]\\ &= \mathbf{q}_R^*\boldsymbol{\omega}_u\mathbf{q}_R + \boldsymbol{\omega}_R.\end{aligned}$$

■

4 Approximate Eigenaxis Maneuvers

The results of Theorem 3.1 can be used to derive a measure of how well a given rotation approximates an eigenaxis rotation. For an exact eigenaxis rotation $\mathbf{q}_R = \mathbf{1}$. Therefore to approximate an eigenaxis rotation we would like to maintain $\|\mathbf{q}_R - \mathbf{1}\|$ small during the entire maneuver.

Theorem 4.1 derives a control law that maintains $\|\mathbf{q}_R(t) - \mathbf{1}\|$ arbitrarily small provided that $\|\mathbf{q}_R(0) - \mathbf{1}\|$ is small enough and that the control gains are defined appropriately.

Theorem 4.1 *For all $\epsilon > 0$, there exists $\delta > 0$, and gains $k_1, k_2, d_1, d_2 > 0$ where $k_1 \neq k_2$ such that if*

1.

$$\begin{aligned}\tau = -k_1\mathcal{V}(\mathbf{q}) - k_2\mathcal{V}(\mathbf{q}_R) - d_1\omega&\\ - d_2(I - \mathbf{u}\mathbf{u}^T)\omega,&\end{aligned}\tag{10}$$

2. $\omega(0) = 0$,

3. $0 \leq \|\mathbf{q}_R(0) - \mathbf{1}\| \leq \delta$,

then

1. $\mathbf{q} \to 0$ *asymptotically,*

2. $\|\mathbf{q}_R(t) - \mathbf{1}\| \leq \epsilon$, $\forall t > 0$.

Proof: 1. Consider the Lyapunov function candidate

$$V = k_1\|\mathbf{q} - \mathbf{1}\|^2 + k_2\|\mathbf{q}_R - \mathbf{1}\|^2 + \frac{1}{2}\omega^T J\omega.$$

Differentiating V we get, from Lemma 2.2,

$$\dot{V} = k_1\omega^T\mathcal{V}(\mathbf{q}) + k_2\omega_R^T\mathcal{V}(\mathbf{q}_R) + \omega^T\tau.\tag{11}$$

From Lemma 3.1 we can solve for ω_R:

$$\omega_R = \omega - \mathbf{q}_R^*\omega_u\mathbf{q}_R,$$

which when dotted into $\mathcal{V}(\mathbf{q}_R)$ becomes

$$\begin{aligned}\mathcal{V}(\mathbf{q}_R)^T\omega_R &= \mathcal{V}(\mathbf{q}_R)^T(\omega - \mathbf{q}_R^*\omega_u\mathbf{q}_R)\\ &= \mathcal{V}(\mathbf{q}_R)^T\omega - \mathcal{S}(\mathbf{q}_R^*\boldsymbol{\omega}_u\mathbf{q}_R\mathbf{q}_R^*)\\ &= \mathcal{V}(\mathbf{q}_R)^T\omega - \mathcal{S}(\mathbf{q}_R^*\boldsymbol{\omega}_u)\\ &= \mathcal{V}(\mathbf{q}_R)^T\omega - \mathcal{V}(\mathbf{q}_R)^T\boldsymbol{\omega}_u\\ &= \mathcal{V}(\mathbf{q}_R)^T\omega\end{aligned}\tag{12}$$

since, from Theorem 3.1, ω_u is parallel to $\mathbf{u}$ and $\mathcal{V}(\mathbf{q}_R)$ is perpendicular to $\mathbf{u}$.

Substituting Equation (12) into Equation (11) gives

$$\begin{aligned}\dot{V} &= \omega^T(\tau + k_1\mathcal{V}(\mathbf{q}) + k_2\mathcal{V}(\mathbf{q}_R))\\ &= \omega^T(-d_1\omega - d_2(I_3 - \mathbf{u}\mathbf{u}^T)\omega)\\ &= -\omega^T(d_1 I_3 + d_2(I_3 - \mathbf{u}\mathbf{u}^T))\omega.\end{aligned}\tag{13}$$

The matrix $D = (d_1 I_3 + d_2(I_3 - \mathbf{u}\mathbf{u}^T))$ is symmetric. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be any two linearly independent vectors that are each perpendicular to $\mathbf{u}$. The vectors $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of D with eigenvalues $d_1, d_1 + d_2, d_1 + d_2$. Since D is symmetric with positive eigenvalues it is positive definite. Using this fact in conjunction with Equation (13) gives

$$\begin{aligned}\dot{V} &= -\omega^T D \omega \\ &\leq 0.\end{aligned}$$

Let $\Omega = \{(\mathbf{q}, \omega) | \dot{V} = 0\}$ and let $\bar{\Omega}$ be the largest invariant subset of Ω. When $\dot{V} = 0$ it follows that $\omega = 0$. Therefore on the set $\bar{\Omega}$, $\omega(t) \equiv 0$ and $\tau(t) \equiv 0$. Plugging this into the control law we get that on $\bar{\Omega}$,

$$k_1 \mathcal{V}(\mathbf{q}) + k_2 \mathcal{V}(\mathbf{q}_R) = 0. \tag{14}$$

Using Theorem 3.1 to write $\mathbf{q} = \mathbf{q}_u \mathbf{q}_R$, we obtain

$$\begin{aligned}\mathcal{V}(\mathbf{q}) = \mathcal{S}(\mathbf{q}_R)\mathcal{V}(\mathbf{q}_u) + \mathcal{S}(\mathbf{q}_u)\mathcal{V}(\mathbf{q}_R) \\ + \mathcal{V}(\mathbf{q}_u) \times \mathcal{V}(\mathbf{q}_R),\end{aligned}$$

which when substituted into Equation (14) gives

$$\begin{aligned}k_1 \mathcal{S}(\mathbf{q}_R)\mathcal{V}(\mathbf{q}_u) + (k_2 + k_1 \mathcal{S}(\mathbf{q}_u))\mathcal{V}(\mathbf{q}_R) \\ + k_1 \mathcal{V}(\mathbf{q}_u) \times \mathcal{V}(\mathbf{q}_R) = 0.\end{aligned} \tag{15}$$

If we multiply Equation (15) by each of $\mathcal{V}(\mathbf{q}_u)^T$, $\mathcal{V}(\mathbf{q}_R)^T$ and $(\mathcal{V}(\mathbf{q}_u) \times \mathcal{V}(\mathbf{q}_R))^T$, we get the following relations:

$$\mathcal{S}(\mathbf{q}_R)\|\mathcal{V}(\mathbf{q}_u)\|^2 = 0, \tag{16}$$
$$(k_2 + k_1 \mathcal{S}(\mathbf{q}_u))\|\mathcal{V}(\mathbf{q}_R)\|^2 = 0, \tag{17}$$
$$\|\mathcal{V}(\mathbf{q}_u) \times \mathcal{V}(\mathbf{q}_R)\|^2 = 0, \tag{18}$$

respectively. From Equation (18) noting that $\mathcal{V}(\mathbf{q}_R)^T \mathcal{V}(\mathbf{q}_u) = 0$, we get that $\mathcal{V}(\mathbf{q}_R)$ and $\mathcal{V}(\mathbf{q}_u)$ are simultaneously parallel and perpendicular. Either $\mathcal{V}(\mathbf{q}_R)$ or $\mathcal{V}(\mathbf{q}_u)$ must be identically equal to zero. If $\mathcal{V}(\mathbf{q}_R) = 0$ then $\mathcal{S}(\mathbf{q}_R) = \pm 1$ and from Equation (16) $\mathcal{V}(\mathbf{q}_u) = 0$. If on the other hand $\mathcal{V}(\mathbf{q}_u) = 0$ then $\mathcal{S}(\mathbf{q}_u) = \pm 1$ and provided that $k_1 \neq k_2$ Equation (17) gives us that $\mathcal{V}(\mathbf{q}_R) = 0$. Therefore we have that $\mathcal{V}(\mathbf{q}_u) = \mathcal{V}(\mathbf{q}_R) = 0$, $\mathcal{S}(\mathbf{q}_u) = \pm 1$ and $\mathcal{S}(\mathbf{q}_R) = \pm 1$. The product $\mathbf{q} = \mathbf{q}_u \mathbf{q}_R = \pm \mathbf{1}$. So $\bar{\Omega} = \{(\pm \mathbf{1}, 0)\}$. Both elements of $\bar{\Omega}$ correspond to the exact same orientation since

$$\mathbf{1}^* x \mathbf{1} = (-\mathbf{1})^* x (-\mathbf{1}).$$

Therefore by LaSalle's Invariance principle the attitude of the rigid body converges asymptotically to identity.

2. Since V is a Lyapunov function $\forall t > 0$, we get:

$$\begin{aligned}&k_2 \|\mathbf{q}_R(t) - \mathbf{1}\|^2 \\ &\leq k_1 \|\mathbf{q}(t) - \mathbf{1}\|^2 + k_2 \|\mathbf{q}_R(t) - \mathbf{1}\|^2 + \frac{1}{2}\omega(t)^T J \omega(t) \\ &= V(t) \\ &\leq V(0) \\ &= k_1 \|\mathbf{q}(0) - \mathbf{1}\|^2 + k_2 \|\mathbf{q}_R(0) - \mathbf{1}\|^2 + \frac{1}{2}\omega(0)^T J \omega(0) \\ &= k_1 \|\mathbf{q}(0) - \mathbf{1}\|^2 + k_2 \|\mathbf{q}_R(0) - \mathbf{1}\|^2 \\ &\leq k_1 \|\mathbf{q}(0) - \mathbf{1}\|^2 + k_2 \delta^2.\end{aligned} \tag{19}$$

We can solve Equation (19) to get

$$\|\mathbf{q}_R(t) - \mathbf{1}\| \leq \sqrt{\delta^2 + (k_1/k_2)\|\mathbf{q}(0) - \mathbf{1}\|^2}, \tag{20}$$

which is less than ϵ for suitable choices of δ and k_1/k_2. By making δ and k_1/k_2 small enough ϵ can be made arbitrarily small. ■

5 Passivity-Based Eigenaxis Rotations

In the case when the angular velocity is not directly available for feedback, we can use a passivity-based term similar to those found in [9, 10] to damp out the motion. Theorem 5.1 presents this control strategy.

Theorem 5.1 *For all $\epsilon > 0$ there exists $\delta > 0$, and gains $k_1, k_2 > 0$ where $k_1 \neq k_2$ such that if*

1. $$\begin{aligned}\dot{\xi} &= A\xi + B\mathbf{q}, \\ \mathbf{y} &= B^T P A \xi + B^T P B \mathbf{q}, \\ \tau &= -k_1 \mathcal{V}(\mathbf{q}) - k_2 \mathcal{V}(\mathbf{q}_R) - \mathcal{V}(\mathbf{q}^* y),\end{aligned}$$
2. $\|\mathbf{q}_R(0) - \mathbf{1}\| \leq \delta$,
3. $\omega(0) = 0$,
4. $A \in \mathbb{R}^{4\times 4}$ *Hurwitz,*
5. $B \in \mathbb{R}^{4\times 4}$ *full rank,*
6. $P \in \mathbb{R}^{4\times 4} > 0$ *solution to* $A^T P + PA = -Q \in \mathbb{R}^{4\times 4} < 0$,
7. $\xi(0) = -A^{-1} B \mathbf{q}(0)$,

then

1. $\mathbf{q} \to \mathbf{1}$ *asymptotically,*
2. $\|\mathbf{q}_R(t) - \mathbf{1}\| \leq \epsilon \ \forall t > 0$.

Proof: Note that the filter equation

$$\begin{aligned}\dot{\xi} &= A\xi + B\mathbf{q}\\ \mathbf{y} &= B^T PA\xi + B^T PB\mathbf{q}.\end{aligned}$$

can be expressed as

$$\begin{aligned}\dot{\bar{\xi}} &= A\bar{\xi} + B\dot{\mathbf{q}},\\ \mathbf{y} &= B^T P\bar{\xi},\end{aligned} \tag{21}$$

where $\bar{\xi} = \dot{\xi}$.

1. Consider the Lyapunov function candidate

$$V = k_1\|\mathbf{q} - \mathbf{1}\|^2 + k_2\|\mathbf{q}_R - \mathbf{1}\|^2 + \frac{1}{2}\omega^T J\omega + \bar{\xi}^T P\bar{\xi}.$$

Differentiating V and substituting in Equation (21) and using the same steps as in Section 4 we get:

$$\begin{aligned}\dot{V} &= \omega^T(\tau + k_1\mathcal{V}(\mathbf{q}) + k_2\mathcal{V}(\mathbf{q}_R)) - \bar{\xi}^T Q\bar{\xi} + 2\bar{\xi}PB\dot{\mathbf{q}}\\ &= \omega^T(\tau + k_1\mathcal{V}(\mathbf{q}) + k_2\mathcal{V}(\mathbf{q}_R)) - \bar{\xi}^T Q\bar{\xi} + \mathbf{y}^T(\mathbf{q}\boldsymbol{\omega})\\ &= \omega^T(\tau + k_1\mathcal{V}(\mathbf{q}) + k_2\mathcal{V}(\mathbf{q}_R)) - \bar{\xi}^T Q\bar{\xi} + \mathcal{S}(\mathbf{y}^*\mathbf{q}\boldsymbol{\omega})\\ &= \omega^T(\tau + k_1\mathcal{V}(\mathbf{q}) + k_2\mathcal{V}(\mathbf{q}_R)) - \bar{\xi}^T Q\bar{\xi} + \omega^T\mathcal{V}(\mathbf{q}^*\mathbf{y})\\ &= \omega^T(\tau + k_1\mathcal{V}(\mathbf{q}) + k_2\mathcal{V}(\mathbf{q}_R) + \mathcal{V}(\mathbf{q}^*\mathbf{y})) - \bar{\xi}^T Q\bar{\xi}\\ &= -\bar{\xi}^T Q\bar{\xi}\\ &\leq 0.\end{aligned}$$

Therefore V is a Lyapunov function. By LaSalle's invariance principle $\mathbf{q}, \omega$ will converge to the largest invariant subset of $\Omega = \{(\mathbf{q}, \omega)|\dot{V} = 0\}$. Let $\bar{\Omega}$ be the largest invariant subset of Ω. On the set $\bar{\Omega}$, $\bar{\xi}(t) \equiv 0$. This also implies that $\dot{\bar{\xi}}(t) \equiv 0$. From Equation (21), on this set, $\dot{\mathbf{q}}(t) \equiv 0$. Since $\boldsymbol{\omega} = 2\mathbf{q}^*\dot{\mathbf{q}}$ and $\tau = J\dot{\omega} + \omega \times J\omega$, it also follows that for all t, $\omega(t) \equiv \tau(t) \equiv 0$. Furthermore, since $\mathbf{y} = B^T P\bar{\xi}$, then on the set $\bar{\Omega}$, $\mathbf{y}(t) \equiv 0$. Therefore on the set $\bar{\Omega}$, $k_1\mathcal{V}(\mathbf{q}) + k_2\mathcal{V}(\mathbf{q}_R) = 0$. In the proof of Theorem 4.1 we showed that provided that $k_1 \neq k_2$, $\mathcal{V}(\mathbf{q}(t)) = 0$. By LaSalle's invariance principle, $\mathbf{q} \to \pm\mathbf{1}$.

2. The proof of the second statement is similar to that found in Theorem 4.1. ■

6 Simulations

We will compare the model dependent eigenaxis rotation of [3], the model independent eigenaxis rotation and the model independent passivity-based eigenaxis rotation. Suppose that the moment of inertia is given by

$$J = \begin{bmatrix} 1200 & 100 & -200\\ 100 & 2200 & 300\\ -200 & 300 & 3100 \end{bmatrix} kg\ m^2.$$

However, only an approximate $\tilde{J}$ is known where $J - \tilde{J}$ is zero mean with a variance of 100. Note that 100 is about 5% of the average value of J_{ii}.

Designing for a settling time of $50s$ and $\omega_n = 0.158$ rad/s, the model dependent control law will be

$$\tau = \omega \times \tilde{J} \times \omega - k\tilde{J}\mathcal{V}(\mathbf{q}) - d\tilde{J}\omega,$$

where $k = 0.05$ and $d = 0.316$.

The model independent control law is given by

$$\tau = -k_1\mathcal{V}(\mathbf{q}) - k_2\mathcal{V}(\mathbf{q}_R) - d_1\omega - d_2(I_3 - \mathbf{u}^T\mathbf{u})\omega,$$

where $k_1 = 115$, $d_1 = 736$, $k_2 = 100k_1$ and $d_2 = 100d_1$.

Finally we also implement the model independent passivity-based rotation by choosing k_1 and k_2 as before, and selecting $A = -10I_4$, $B = 54.26I_4$, and $Q = 100I_4$. The trajectories of $\mathbf{q}_i, \tau_i$ and ω_i are very similar to each other for the three simulations. This is not surprising since we have tuned the control laws to have similar responses. These plots are given in Figure 1, Figure 2, and Figure 3. The solid line (-) represents the model dependent control law. The dashed line (- -) is the model independent control law. The dashed dotted line (-.) is the model independent passivity-based control law.

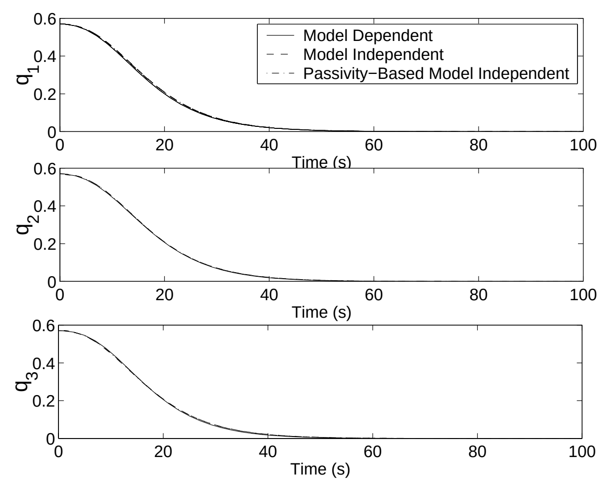

Figure 1: This figure compares the trajectory of $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ for each approximate eigenaxis rotation.

However there is a significant difference in the ability of each control law to execute an eigenaxis rotation. This is clearly illustrated in Figure 4. The model independent control law gives the best results. The model independent passivity-based control law also performs well, but does not do as good a job damping the non-eigenaxis motion. Finally the model dependent control law has the poorest results when the moment of inertia J is not precisely known.

7 Conclusions

The key result of this paper is the introduction of model independent eigenaxis rotations. This is a new

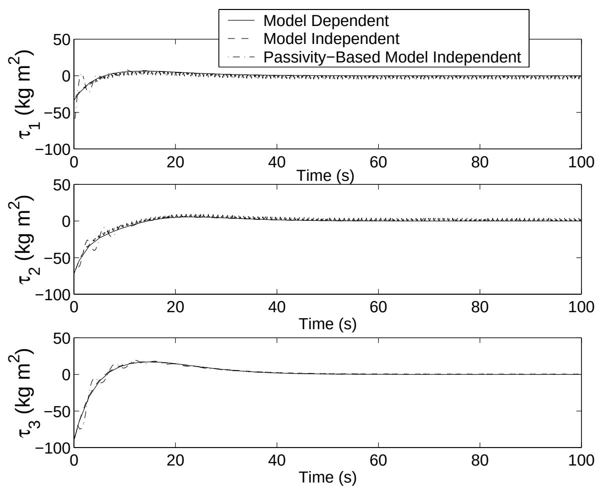

Figure 2: This figure compares the trajectory of τ_1, τ_2, τ_3 for each approximate eigenaxis rotation.

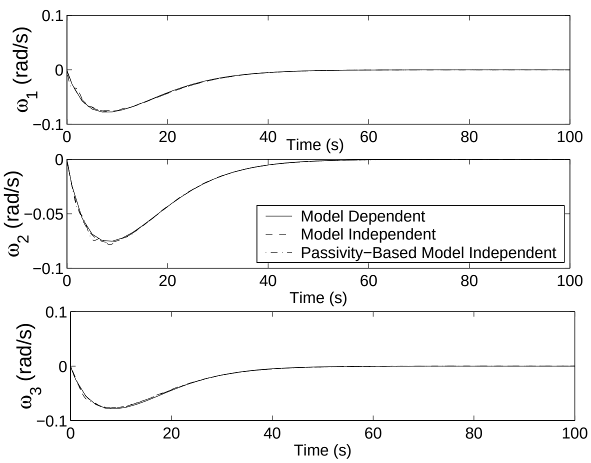

Figure 3: This figure compares the trajectory of $\omega_1, \omega_2, \omega_3$ for each approximate eigenaxis rotation.

result that uses feedback of the spacecraft attitude and feedback of the rotation necessary to return the rigid body to the eigenaxis. These results prove important in spacecraft formation flying because maintaining spacecraft alignment during a coordinated rotation requires eigenaxis rotations [8].

References

[1] K. D. Bilimoria and B. Wie, "Time-optimal three-axis reorientation of a rigid spacecraft," *Journal of Guidance, Control, and Dynamics*, vol. 16, pp. 446–452, May-June 1993.

[2] W. H. Steyn, "Near-minimum-time eigenaxis rotation maneuvers using reaction wheels," *Journal of Guidance, Control and Dynamics*, vol. 18, pp. 1184–1189, September-October 1995.

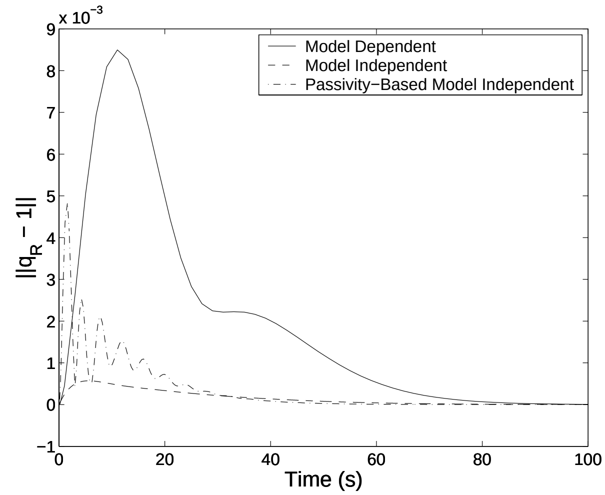

Figure 4: This figure compares the deviation from the eigenaxis, $\|\mathbf{q}_R - \mathbf{1}\|$, for each approximate eigenaxis rotation.

[3] B. Wie, H. Weiss, and A. Apapostathis, "Quaternion feedback regulator for spacecraft eigenaxis rotations," *AIAA Journal of Guidance and Control*, vol. 12, pp. 375–380, May 1989.

[4] H. Weiss, "Quaternion-based rate/attitude tracking system with application to gimbal attitude control," *Journal of Guidance, Control, and Dynamics*, vol. 16, pp. 609–616, July-August 1993.

[5] R. Cristi, J. Burl, and N. Russo, "Adaptive quaternion feedback regulation for eigenaxis rotations," *Journal of Guidance, Control, and Dynamics*, vol. 17, pp. 1287–1291, November-December 1994.

[6] C. Samson, M. Le Borgne, and B. Espiau, *Robot Control: A Task Function Approach.* Oxford Science Publications, 1991.

[7] L. Brand, *Vector and Tensor Analysis.* New York: John Willey & Sons, 1948.

[8] J. Lawton, *Multiple Spacecraft Formation Flying.* PhD thesis, Brigham Young University, December 2000.

[9] F. Lizarralde and J. Wen, "Attitude control without angular velocity measurement: A passivity approach," *IEEE Transaction On Automatic Control*, vol. 41, pp. 468–472, March 1996.

[10] P. Tsiotras, "Further passivity results for the attitude control problem," *IEEE Transaction On Automatic Control*, vol. 43, no. 11, pp. 1597–1600, 1998.