Spacecraft Attitude Control in Hamiltonian Framework

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Abstract

The objective of this paper is to give a design scheme for attitude control algorithms of a generic spacecraft. Along with the system model formulated in the Hamilton's canonical form the algorithm uses information about a required potential energy and a dissipative term. The control action is the sum of the gradient of the potential energy and the dissipative force. It is shown that this control law makes the system uniformly asymptotically stable to the desired reference point. Three problems were addressed in the paper: spacecraft stabilization in the inertial frame, libration damping with the use of electromagnetic coils and a slew maneuver with an additional objective of avoiding undesirable regions e.g. causing blindness of optical sensors.

1 INTRODUCTION

The subject of control of mechanical systems here also spacecraft has always been in very focus of control engineering. The recent advances of computer technology, vastly increasing computational power, availability of symbolic software tool-boxes have initiated a tremendous research effort within nonlinear control methods. Probably the most influential has been the geometric control methods as presented in [1], [2], [3]; passivity based control in [4],[5], [6]; nonlinear H_∞ in [7], [8].

The paper comprises a further development of the work reported in [9] and [2], ch. 12, dealing with stabilization of hamiltonian systems. The great impact on this paper had the geometrical description of the physical mechanics in [10]. A further influence on this work had the articles [11] and [12] studying canonical transformation from the ordinary three-dimensional physical space of Euler angles to the four dimensional space of the unit quaternion. This approach is used in this paper to model the rotational motion of a rigid body in the Hamilton's canonical equations.

The idea of this paper is very intuitive and consist of the following steps. A dynamical system is modeled and its desired performance is specified using two hamiltonians: one of the original system and the second of the desired one. The desired system inherits the kinetic energy of the original one, but the potential energy has to comply with the requirements on the feedback system, e.g. the minimum of the potential energy shall be reached at the reference point. Additionally a dissipation term is incorporated which defines the time response of the closed loop system. A control action is designed such that the feedback system coincides with the desired one. Three problems are addressed here: spacecraft stabilization in the inertial frame, libration damping with the use of electromagnetic coils and a slew maneuver with an additional objective of avoiding some regions e.g. defined by certain bright objects causing blindness of optical sensors.

2 CONTROL OF HAMILTONIAN SYSTEMS

A problem of stabilization to a reference, a certain point in the phase plane, is the topic of this section. Two systems will be considered: one corresponding to the actual plant and a system which is counterpart to the control objectives. The latter will be called the system of objectives. The motion of the plant and the system of objectives are described using Hamilton's canonical form. The formulation of kinetic energy for both systems is the same, however the potential energy of the system of objectives is expressed such that the reference is stable. If in addition dissipation is added the reference becomes asymptotically stable. The control action is now chosen such that the flow of the plant coincides with the flow of the system of objectives.

2.1 System Canonical Form

It is assumed in this work that the system is conservative, hence the equations of motion can be formulated in Hamilton's canonical form. Having the Lagrange function L

$$L(\mathbf{q},\dot{\mathbf{q}}) = T(\mathbf{q},\dot{\mathbf{q}}) - U(\mathbf{q}), \tag{1}$$

the hamiltonian is given by

$$H(\mathbf{q},\mathbf{p}) = \langle \mathbf{p},\dot{\mathbf{q}}\rangle - L(\mathbf{q},\dot{\mathbf{q}}), \tag{2}$$

where $\langle \mathbf{p},\dot{\mathbf{q}}\rangle = \mathbf{p}^T\dot{\mathbf{q}}$ denotes a scalar product in the Euclidean space, the generalized momentum $\mathbf{p}$ is calculated using the equality $\mathbf{p} = \partial L/\partial \dot{\mathbf{q}}$. The control action is regarded as the generalized external force $\mathbf{M}_p$ acting upon the system. Following the lines of [13] pp. 316 the Hamilton's canonical equations are

$$\begin{aligned} \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{M}_p \\ \dot{\mathbf{q}} &= \frac{\partial H}{\partial \mathbf{p}}. \end{aligned} \tag{3}$$

2.2 System of Objectives

The equation of motions for the system of objectives will be expressed in the Hamilton's canonical form as well. It is assumed that the system of objectives inherits the kinetic energy T, whereas an extra energy contribution V is added to the original potential energy U

$$U_d(\mathbf{q}) = U(\mathbf{q}) + V(\mathbf{q}). \tag{4}$$

The function $V(\mathbf{q})$ is designed such that the reference $\mathbf{q}_0$ becomes stable, i.e. the minimum of the potential energy U_d is reached at $\mathbf{q}_0$.

The equations of motion for the system of objectives are formulated using the lagrangian L_o

$$L_o(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - U_d(\mathbf{q}), \tag{5}$$

The system of objectives is not asymptotically stable yet, but it is stable in Lyapunov sense, which can be shown applying a Lyapunov function

$$v(t) = H_o(t) - U_d(\mathbf{q}_0), \tag{6}$$

where

$$H_o(\mathbf{q}(t), \mathbf{p}(t)) = \langle \mathbf{p}, \dot{\mathbf{q}} \rangle - L_o(\mathbf{q}, \dot{\mathbf{q}}). \tag{7}$$

The function $v(t)$ is constant and positive definite around $\mathbf{q}_0$. In order to get asymptotic stability a dissipation term is added in the system of objectives. The work of the dissipation force has to be negative semidefinite. In general the work of the field $\mathbf{M}_d$ on the path l is defined as follows

$$W = \int_l \langle \mathbf{M}_d, \mathbf{dq} \rangle, \tag{8}$$

and the time derivative of the work is now

$$\dot{W} = \langle \mathbf{M}_d, \dot{\mathbf{q}} \rangle. \tag{9}$$

Proposition 1 *Consider a system given by the Hamilton's canonical equations*

$$\begin{aligned} \dot{\mathbf{p}} &= -\frac{\partial H_o}{\partial q} + \mathbf{M}_d \\ \dot{\mathbf{q}} &= \frac{\partial H_o}{\partial p}, \end{aligned} \tag{10}$$

where $H_o(\mathbf{q}, \mathbf{p}) = \langle \mathbf{p}, \dot{\mathbf{q}} \rangle - T(\mathbf{q}, \dot{\mathbf{q}}) + U_d(\mathbf{q})$ *is the hamiltonian,* T *is the kinetic energy of the plant,* U_d *is lpdf (locally positive definite function) around* $\mathbf{q}_0$ *given by Eq. (4), and the work* W *done by the vector field* $\mathbf{M}_d$ *is given by*

$$W = \int_l \langle \mathbf{M}_d, \mathbf{dq} \rangle \tag{11}$$

If the time derivative dW/dt *is lndf (locally negative definite function) then the system (10) is locally uniformly asymptotically stable. If* U_d *is radially unbounded pdf (positive definite function) and* W *is ndf (negative definite function) then the asymptotic stability is global.*

Proof of Proposition 1 *Consider a Lyapunov candidate function* $v(t) = H_o(t)$. *The time derivative of* H_o *is*

$$\dot{H}_o = \left(\frac{\partial H_o}{\partial \mathbf{p}}\right)^T \dot{\mathbf{p}} + \left(\frac{\partial H_o}{\partial \mathbf{q}}\right)^T \dot{\mathbf{q}}. \tag{12}$$

Using the Hamilton's canonical equations

$$\dot{H}_o = -\left(\frac{\partial H_o}{\partial \mathbf{p}}\right)^T \frac{\partial H_o}{\partial \mathbf{q}} + \left(\frac{\partial H_o}{\partial \mathbf{p}}\right)^T \mathbf{M}_p \tag{13}$$

$$+ \left(\frac{\partial H_o}{\partial \mathbf{q}}\right)^T \frac{\partial H_o}{\partial \mathbf{p}} = \dot{\mathbf{q}}^T \mathbf{M}_d = \dot{W} \tag{14}$$

which is lndf (ndf), thus the equilibrium $(\mathbf{q}, \mathbf{p}) = (\mathbf{q}_0, \mathbf{0})$ *is locally (globally) uniformly asymptotically stable.* □

2.3 Control Synthesis

The objective of the control design in the framework of the Hamilton's formalism is to generate $\mathbf{M}_p$ such that the equations of motion for the plant and the system of objectives are equivalent. In other words, the control action has to compensate for two terms: one originating from the function V in Eq. (4) and the second contributing from the work W in Eq. (11). This is summarized in the following theorem:

Theorem 1 *Consider a plant given in Hamilton's canonical form (3). The control action*

$$\mathbf{M}_p = -\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} + \mathbf{M}_d, \tag{15}$$

where V *is given by Eq. (4),* U_d *is lpdf around* $\mathbf{q}_0$ *and the time derivative of the work* $dW/dt = \langle \mathbf{M}_d, \dot{\mathbf{q}} \rangle$ *is lndf around* $\mathbf{q}_0$ *then the feedback system is locally uniformly asymptotically stable to the reference* $\mathbf{q}_0$. *If* U_d *is radially unbounded pdf and* $dW/dt = \langle \mathbf{M}_d, \dot{\mathbf{q}} \rangle$ *is ndf around* $\mathbf{q}_0$ *then the uniform asymptotic stability is global.*

Proof of Theorem 1 *If the control action (15) is substituted in Eq. (3) for* $\mathbf{M}_p$ *the equations of motion for the plant are identical to the system given by Eq. (10). The hypothesis of Proposition 1 are satisfied and hence the feedback system governed by Eq. (15) is uniformly asymptotically stable.* □

The control action in Eq. (15) consists of two terms. The first one determines sensitivity of the closed loop system towards disturbances, whereas the second decides the length of the settling time. For a conservative system the disturbance force has to perform a work $W = U(\mathbf{q}_1) - U(\mathbf{q}_0)$ to change the potential energy from the level $U(\mathbf{q}_0)$ to $U(\mathbf{q}_1)$. Thereby, the larger the gradient $\partial V(\mathbf{q})/\partial \mathbf{q}$ the larger work necessary to move the plant from the point $\mathbf{q}_0$ to $\mathbf{q}_1$. The dissipation $dW(t)/dt$ is related to the amount of energy dissipated by the controller in a certain fixed time T, thus it corresponds to the response time. This control structure can be compared with a standard PD controller used for linear systems.

3 MOTION CONTROL OF A RIGID BODY

A method for the control synthesis presented in the last section is readily applicable in the systems where the dynamics and kinematics are represented in E^{2n}, however e.g. motion involving the rotation is typically not expressed in the canonical form. The dynamics are given by the Euler equation in E^3, whereas the most natural description of the kinematics is given by the elements of $SO_3(\mathbb{R})$ or by the unit quaternion, an element of S^3.

3.1 Rigid Body Canonical Form

The subject of finding a transformation to Hamilton's canonical form is often addressed in the literature of modern celestial mechanics. [11] studied the canonical transformation $y = f(x)$ of the state space $\mathbf{y} \in \mathbb{R}^{2n}$ to $\mathbf{x} \in \mathbb{R}^{2m}$ with $m > n$. In the current paper only a special case $m = n + 1$ is investigated, since the results can be applied to the rotational motion of a rigid body in function of the unit quaternion $\mathbf{q} := \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T \in S^3$ and the conjugate momenta $\mathbf{p} := \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix}^T$. Interested reader is refered to [12] for more detailed study on this topic.

The kinetic energy of a rigid body rotation is a function of the instantaneous angular velocity $\boldsymbol{\omega}$

$$T = \frac{1}{2}\boldsymbol{\omega}^T \mathbf{J} \boldsymbol{\omega}, \tag{16}$$

where $\mathbf{J}$ is the inertia tensor. The angular velocity vector may be regarded as an element of the quaternion vector space $\boldsymbol{\Omega} := \begin{bmatrix} 0 & \boldsymbol{\omega}^T \end{bmatrix}^T \in E \times E^3$, and the Equation (16) takes the form

$$T = \frac{1}{2}\boldsymbol{\Omega}^T \mathbf{J}^* \boldsymbol{\Omega}, \tag{17}$$

where $\mathbf{J}^*$ is a block diagonal matrix

$$\mathbf{J}^* = \begin{bmatrix} J_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}. \tag{18}$$

The element J_0 takes in general an arbitrary nonsingular value. Using the standard quaternion parameterizations of kinematics

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{Q}(\mathbf{q})\boldsymbol{\Omega}, \text{ where } \mathbf{Q}(\mathbf{q}) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \tag{19}$$

the kinetic energy is

$$T = 2\mathbf{q}^T \mathbf{Q}(\dot{\mathbf{q}}) \mathbf{J}^* \mathbf{Q}^T(\dot{\mathbf{q}})\mathbf{q}. \tag{20}$$

The lagrangian for the rigid body motion is now $L(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q})$, where T is given by Eq. (20). Applying the hamiltonian $H(\mathbf{q}, \mathbf{p}) = \langle \mathbf{p}, \dot{\mathbf{q}} \rangle - L(\mathbf{q}, \dot{\mathbf{q}})$ and Eq. (3) the canonical equations are formulated

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{1}{4}\mathbf{Q}(\mathbf{q})\mathbf{J}^{*-1}\mathbf{Q}^T(\mathbf{q})\mathbf{p} \qquad (21) \\ \dot{\mathbf{p}} &= -\frac{1}{4}\mathbf{Q}(\mathbf{p})\mathbf{J}^{*-1}\mathbf{Q}^T(\mathbf{p})\mathbf{q} - \frac{\partial U(\mathbf{q})}{\partial \mathbf{q}} + \mathbf{M}_p. \end{aligned}$$

For the rotational motion the control action is a torque denoted here by $\mathbf{M}_c$. On a spacecraft the control torque is generated by a set of actuators such as gas jets, momentum/reaction wheels, electromagnetic coils. The work is invariant of a canonical transformation. To calculate $\mathbf{M}_p$ for a given torque $\mathbf{M}_c$ the time derivative of the work of $\mathbf{M}_p$ as in Eq. (9) and the work of the field $\mathbf{M}_c$ are used:

$$\dot{\mathbf{q}}^T(t)\mathbf{M}_p(t) = \dot{W}(t) = \boldsymbol{\omega}^T(t)\mathbf{M}_c(t). \tag{22}$$

Applying Eqs. (22) and (19) the time derivative of the work is

$$\dot{W}(t) = 2\dot{\mathbf{q}}^T(t)\mathbf{Q}\left(\mathbf{q}(t)\right)\mathbf{M}(t), \text{ where } \mathbf{M} = \begin{bmatrix} 0 & \mathbf{M}_c^T \end{bmatrix}^T \tag{23}$$

hence

$$\mathbf{M}_p(t) = 2\mathbf{Q}(\mathbf{q}(t))\mathbf{M}(t) \tag{24}$$

or equivalently

$$\mathbf{M}_p(t) = 2\mathbf{R}(\mathbf{M}_c(t))\mathbf{q}(t), \tag{25}$$

where

$$\mathbf{R}(\mathbf{M}_c) = \begin{bmatrix} 0 & -M_1 & -M_2 & -M_3 \\ M_1 & 0 & M_3 & -M_2 \\ M_2 & -M_3 & 0 & M_1 \\ M_3 & M_2 & -M_1 & 0 \end{bmatrix} \tag{26}$$

Comparing Eq. (24) and the kinematics (19) a very important observation is made that $\mathbf{M}_p$ lies on the tangent space $TS^3{}_q$. Thereby, M_p computed from Eq. (15) is not necessarily producible by a physical actuator. It will be illustrated in Section 4 that a damping term belonging to $TS^3{}_q$ is reasonably easy to design as a linear combination of the vector fields of $TS^3{}_q$. The task becomes more involved if we wish to satisfy $\partial V(\mathbf{q})/\partial \mathbf{q} \in TS^3{}_q$. It will be shown in the theorem below that to generate a stable control action it is enough to use the orthogonal projection P_T of $\partial V(\mathbf{q})/\partial \mathbf{q}$ on the tangent space, $TS^3{}_q$.

Theorem 2 *Consider a plant given in Hamilton's canonical form (3). The control action*

$$\mathbf{M}_p = -P_T \frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} + \mathbf{M}_d, \qquad (27)$$

where P_T is the orthogonal projection on the tangent space TM_q, V is given by Eq. (4), U_d is lpdf around $\mathbf{q}_0$ and the time derivative of the work $\dot{W} = \langle \mathbf{M}_d, \dot{\mathbf{q}} \rangle$ is lndf around $\mathbf{q}_0$. Then the feedback system is locally uniformly asymptotically stable to the reference $\mathbf{q}_0$. If U_d is radially unbounded pdf and $\dot{W} = \langle \mathbf{M}_d, \dot{\mathbf{q}} \rangle$ is ndf around $\mathbf{q}_0$ then the uniform asymptotic stability is global.

Proof of Theorem 2 *Take as the Lyapunov candidate function $v(t) = H(t) + V(t)$, where H is given by Eqs. (1) and (2). The time derivative of $v(t)$ is*

$$\begin{aligned} \dot{v} &= -\left(\frac{\partial H}{\partial \mathbf{p}}\right)^T \frac{\partial H}{\partial \mathbf{q}} + \left(\frac{\partial H}{\partial \mathbf{p}}\right)^T \mathbf{M}_p \left(\frac{\partial H}{\partial \mathbf{q}}\right)^T \frac{\partial H}{\partial \mathbf{p}} \\ &+ \left(P_T \frac{\partial H}{\partial \mathbf{q}}\right)^T \dot{\mathbf{q}} + \left((\mathrm{id} - \mathrm{P_T}) \frac{\partial \mathrm{H}}{\partial \mathbf{q}}\right)^T \dot{\mathbf{q}}, \end{aligned} \qquad (28)$$

but $\dot{\mathbf{q}} \in TM_q$ and P_T is the orthogonal projection on TM_q thus the last term in Eq (28) is zero and

$$\dot{v} = \dot{\mathbf{q}}^T \mathbf{M}_d = \dot{W}. \qquad (29)$$

which is lndf thus the equilibrium $\mathbf{q}_0, \mathbf{0}$ is uniformly asymptotically stable. □

4 Spacecraft Attitude Control

The theoretical findings developed in the preceding chapters will be implemented for the spacecraft attitude control. Three topics are addressed: spacecraft stabilization in the inertial frame, libration damping with the use of electromagnetic coils and a slew maneuver with an objective imposed of avoiding certain undesirable orientations.

4.1 Stabilization in Inertial Frame

A spacecraft motion in the inertial coordinate system was provided in Eq. (21) with the potential energy $U = 0$. The control objective is to correct the attitude to the reference $\mathbf{q}_{ref}$.

The proposed system of objectives is

$$H_d(\mathbf{q}, \mathbf{p}, \mathbf{M}_d) = H(\mathbf{q}, \mathbf{p}) + V(\mathbf{q}) + \langle \mathbf{M}_d, \mathbf{q} \rangle, \qquad (30)$$

where H corresponds to the hamiltonian of the rigid body motion with $U = 0$. The potential energy $V(\mathbf{q}) = 2k_1(1 - q_0)$, where k_1 is a positive constant and q_0 is the scalar part of the quaternion

$$\begin{bmatrix} q_0 & \tilde{\mathbf{q}}^T \end{bmatrix}^T := \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T = \mathbf{Q}^T(\mathbf{q}_{ref})\mathbf{q}. \qquad (31)$$

The dissipation force $\mathbf{M}_d = \mathbf{K}_2 \dot{\mathbf{q}}$ is chosen, where the matrix $\mathbf{K}_2$ is negative definite. Now, the work W dissipates the energy, since $\dot{W} = \langle \mathbf{K}_2 \dot{\mathbf{q}}, \dot{\mathbf{q}} \rangle$ is ndf. According to Theorem 2 the control (27) is uniformly asymptoticly stable. The closed form of the control law is derived using the following equality

$$\mathbf{M}_v \equiv -P_T \frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} = -\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} + (\mathrm{id} - \mathrm{P_T}) \frac{\partial \mathrm{V}(\mathbf{q})}{\partial \mathbf{q}}, \qquad (32)$$

thus

$$\begin{aligned} \mathbf{M}_v &= 2k_1 \left(\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T - q_0 \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T \right) \\ &= 2k_1 \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 & -q_0 q_1 & -q_0 q_2 & -q_0 q_3 \end{bmatrix}^T \end{aligned} \qquad (33)$$

but

$$\mathbf{M}_c(t) = 1/2 \mathbf{Q}^T(\mathbf{q}(t))(\mathbf{M}_v(t) + \mathbf{M}_d(t)) \qquad (34)$$

therefore

$$\mathbf{M}_c = -k_1 \tilde{\mathbf{q}} + \frac{1}{2} \mathbf{Q}^T(\mathbf{q}) \mathbf{K}_2 \dot{\mathbf{q}}. \qquad (35)$$

If the matrix $\mathbf{K}_2$ is substituted by a negative scalar $4k_2$ a more compact form of Eq. (35) can be calculated applying the relation in the kinematics (19)

$$\mathbf{M}_c = -k_1 \tilde{\mathbf{q}} + k_2 \boldsymbol{\omega}, \qquad (36)$$

where $\boldsymbol{\omega}$ is the angular velocity of the spacecraft.

Remark 1 *If the potential energy $V(\mathbf{q}) = 2k_1(1 - q_0)$ is replace by $V(\mathbf{q}) = 2k_1(1 + q_0)$ with the minimum at $q_0 = -1$ then the control law*

$$\mathbf{M}_c = k_1 \tilde{\mathbf{q}} + k_2 \boldsymbol{\omega} \qquad (37)$$

assures that the equilibrium $\mathbf{q} = -\mathbf{q}_{ref}$ is asymptotically stable. At this point it is important to notice that both $\mathbf{q}_{ref}$ and $-\mathbf{q}_{ref}$ define the same physical orientation.

4.2 Slew Maneuver with Region Avoidance

The control objective of the three-axis attitude control addressed in the preceding subsection is extended here to an additional design objective: During the slew maneuver a certain orientation is prohibited. This scenario is encountered when the attitude is acquired from the star camera; looking towards the sun causes blindness of the CCD chip.

A spacecraft motion is once more given by Eq. (21) with the potential energy $U = 0$. The orientation of the **o**bstacle in the **i**nertial frame is given by the quaternion ${}^{o}_{i}\mathbf{q}$, whereas the **r**eference in the **i**nertial frame is specified by ${}^{r}_{i}\mathbf{q}$. The **s**pacecraft's attitude in the **i**nertial coordinate system is provided by the star camera, $\mathbf{q} \equiv {}^{s}_{i}\mathbf{q}$.

The potential function in this case study is shaped such that its minimum is at the reference and the maximum at the obstacle attitude. The potential energy proposed is

$$V(t) = 2 - {}^{s}_{r}\mathbf{q}_0 - {}^{s}_{o}\mathbf{q}_0^2, \tag{38}$$

where ${}^{s}_{r}\mathbf{q}_0$ is the scalar part of the quaternion ${}^{s}_{r}\mathbf{q} = {}^{r}_{i}\mathbf{q}^*\mathbf{q}$ [1], and ${}^{s}_{r}\mathbf{q}_0$ is the scalar part of ${}^{s}_{o}\mathbf{q} = {}^{o}_{i}\mathbf{q}^*\mathbf{q}$, hence

$$\begin{aligned} {}^{s}_{r}\mathbf{q}_0 &= \mathbf{n}_r^T\mathbf{q} \\ {}^{s}_{o}\mathbf{q}_0 &= \mathbf{n}_o^T\mathbf{q}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} \mathbf{n}_r &= \begin{bmatrix} {}^{s}_{r}q_0 & {}^{s}_{r}q_1 & {}^{s}_{r}q_2 & {}^{s}_{r}q_3 \end{bmatrix}^T \\ \mathbf{n}_o &= \begin{bmatrix} {}^{s}_{o}q_0 & {}^{s}_{o}q_1 & {}^{s}_{o}q_2 & {}^{s}_{o}q_3 \end{bmatrix}^T. \end{aligned} \tag{40}$$

Applying the potential energy in Eq. (38) and the dissipation force $\mathbf{M}_d = \mathbf{K}\dot{\mathbf{q}}$ to the generic control law (15) gives

$$\begin{aligned} \mathbf{M}_p &= \mathbf{n}_r + 2\mathbf{n}_o^T\mathbf{q}\,\mathbf{n}_r - \mathbf{n}_r^T\mathbf{q}\,\mathbf{q} - 2\left(\mathbf{n}_o^T\mathbf{q}\right)^2\,\mathbf{q} \\ &+ 1/2\mathbf{K}\mathbf{Q}(\mathbf{q})\boldsymbol{\Omega}, \end{aligned} \tag{41}$$

and the control torque is $\mathbf{M}_c(t) = 1/2\mathbf{Q}^T(\mathbf{q}(t))\,\mathbf{M}_p(t)$.

4.3 Libration Damping

A very cost and energy effective control principle for a gravity gradient stabilized satellite is to use the electromagnetic coils for spacecraft actuation. The concept is that the interaction between the Earth's magnetic field and a magnetic field generated by the coil results in a mechanical torque. This is expressed by the formula

$$\mathbf{M}_c(t) = \mathbf{m}(t) \times \mathbf{B}(t), \tag{42}$$

i.e. the control torque $\mathbf{M}_c$ is the vector product of the magnetic moment $\mathbf{m}$ generated in the coils and the geomagnetic field vector $\mathbf{B}$. The motion of a spacecraft on a low Earth orbit can be very concisely described by the following hamiltonian, for more details see [14]

$$H = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{J}\boldsymbol{\omega} + \frac{3}{2}\omega_o^2\mathbf{k}^T\mathbf{J}\mathbf{k} - \frac{1}{2}\omega_o^2\mathbf{j}^T\mathbf{J}\mathbf{j}, \tag{43}$$

where $\boldsymbol{\omega}$ is the angular velocity of the spacecraft principal frame relative to the LVLH coordinate system [2], ω_o is the mean motion, $\mathbf{j}, \mathbf{k}$ are the unit vectors along the y and z axes of LVLH. It is assumed in Eq. (43) that the principal axes of the spacecraft are such that the maximum moment of inertia is about the y axis, and the minimum about the z axis.

A closer look at the potential energy, the last two terms in Eq. (43), reviles that the system has four stable equilibria

$$\{\boldsymbol{\omega}, \mathbf{j}, \mathbf{k}\} = \{\mathbf{0}, \pm\mathbf{1}_j, \pm\mathbf{1}_k\}, \tag{44}$$

where $\mathbf{1}_j = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $\mathbf{1}_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. In other words the equilibria are such that the spacecraft principal and LVLH y axes coincide or point in the opposite directions and the z axes of the spacecraft and LVLH frames coincide or are opposite. If one of those equilibria is the system's desired reference then it is sufficient to use a control providing pure damping.

In [15] the following control law was proposed

$$\mathbf{m}(t) = \mathbf{H}\boldsymbol{\omega} \times \mathbf{B}, \tag{45}$$

where $\mathbf{H}$ is a positive definite matrix. Magnetic torquing following Eq. (45) obviously introduces time dependency in the equations of satellite motion. This time variation is periodic by nature, which arises from two superimposed periodic fluctuations of the geomagnetic field. One is due to revolution of the satellite around the Earth and the second due to rotation of the Earth.

To show asymptotic stability of the suggested control law it is enough to calculate the time derivative of the work done by the field $\mathbf{M}_c(t) = \mathbf{m}(t) \times \mathbf{B}(t)$

$$\dot{W}(t) = \boldsymbol{\omega}^T(t)\mathbf{M}_c(t) = -\boldsymbol{\omega}^T\mathbf{S}^T(\mathbf{B}(t))\mathbf{S}(\mathbf{B}(t))\mathbf{H}\boldsymbol{\omega}, \tag{46}$$

where $\mathbf{S}(\mathbf{B})$ is a 3 by 3 skew symmetric matrix representing the vector product operator: $\mathbf{B}\times$. From Eq. (46) it is seen that $\dot{W}(t)$ is only negative semidefinite. However, two observations can be here employed. The first is that the Earth's magnetic field is periodic,

[1] $\mathbf{q}^*$ denotes $\mathbf{q}$ conjugated

[2] Local-Vertical-Local-Horizontal Coordinate System (LVLH) is a right orthogonal coordinate system with the origin at the spacecraft's center of mass. The z axis (local vertical) is parallel to the radius vector and points from the spacecraft center of mass to the center of the Earth. The positive y axis is pointed in the direction of the negative angular momentum vector. The x axis (local horizontal) completes the right orthogonal coordinate system.

and the second that the largest invariant set contained in the set $\{\boldsymbol{\omega} : \dot{W} = 0\}$ is $\boldsymbol{\omega} \equiv 0$. Thus applying Krasovskii-LaSalle theorem [5] pp.178 and Theorem 1 the system is proved to be asymptotically stable to one of the attractors $\{\boldsymbol{\omega}, \mathbf{j}, \mathbf{k}\} = \{\mathbf{0}, \pm\mathbf{1}_j, \pm\mathbf{1}_k\}$.

5 CONCLUSION

An elegant scheme for control design of mechanical systems was proposed in this work. The desired feedback dynamics was specified in a Hamilton's canonical form. The designer has to define a desired potential energy with minimum at the reference point and a dissipative term. The resultant controller is uniformly asymptoticly stable. The results were applied to the rotational motion of a rigid body in function of the unit quaternion and its conjugate momenta. Three problems were successfully tackled in the paper: spacecraft stabilization in the inertial frame, libration damping with the use of electromagnetic coils and a slew maneuver with an additional objective of avoiding undesirable regions.

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