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THE FINITE DIFFERENCE METHOD AT ARBITRARY IRREGULAR GRIDS AND ITS APPLICATION IN APPLIED MECHANICS

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Abstract—Presented modification of the FDM enables local condensation of the mesh and easy discretisation of the boundary conditions in the case of an arbitrary shape of the domain. As a result an essential reduction of the required number of nodal points may usually be achieved. Thus the FDM can be competitive in some fields to the finite element method.

Several troubles arising from the use of an irregulr mesh have been solved, the most important being elimination of singular or ill-conditioned stars, and a successful way of automatic discretization of boundary conditions was proposed. As a result of this complete automatization of the FDM has been reached. Many problems connected with the theory of the method proposed and its computer implementation have been discussed.

FIDAM code—a system of computer programs designed for the solution of two-dimensional, linear and nonlinear, elliptic problems and three-dimensional parabolic problems is presented.

Problems to be solved should be layed down in a local formulation as a set of partial derivative equations of the second order with boundary conditions not exceeding the same order.

Various particular problems of applied mechanics and physics such as: torsion of bars, plane elasticity problems, deflections of plates and membranes, fluid flow and temperature distribution have been solved using the FIDAM procedures. For the solution of nonlinear problems various iterative methods have been adopted—with special attention payed to the self-correcting method where the Newton–Raphson and incremental procedures are particular cases.

The present version of the method allows fully automatic calculations to be carried out as in advanced programs of the finite element method and may be preferred in non-linear, optimization and time-dependent problems.

INTRODUCTION

The rapid development of computer technology observed since the early sixties, has had a significant influence on the revaluation of existing numerical methods and in turn has caused the development of new ones, the first of which is the Finite Element Method (FEM). It has now become the most popular method due to its numerous advantages such as universality, versality and facility in computer implementation [21, 31].

The fascination for FEM, however, caused by its enormous seccesses or simply by fashion, has resulted in a relative stagnation in some other methods especially the Finite Difference Method (FDM).

When compared to the classical version of the FDM at regular meshes [5, 7, 26], the FEM was proved to be much more successful in treatment of boundary conditions, especially at irregular domains and in local condensations of nodes which improve accuracy in zones of rapidly growing gradients of the solution. Using an arbitrary irregular mesh of nodal points, however, one can overcome these difficulties simultaneously preserving the basic advantages of the FDM. On the other hand some new problems arise, mainly associated with the automatic generation of well-conditioned FD formulas.

Though the idea of irregular grids is not new [5, 7, 20] a possibility of practical calculation was dependent on computer technique development. Evolution of irregular grids starts from the grid being partially regular in subdomains, (Fig. 1a, MacNeal [20]), then irregular, but with restricted topology, (Fig. 1b, Frey [8]) to arbitrary irregular mesh. The basis of the method was published in the early seventies by Jensen [10]. He considered a six-point scheme (star). Using Taylor's series expansions he obtained FD formulas approximating derivatives up to the second order. The main disadvantage of his approach was frequent singularity or ill-conditioning of a control scheme. Several authors tried to develop an automatic procedure which avoids incorrect stars and thus improving the accuracy of the FD formulas. Perrone and Kao [23] suggested, that additional nodes in a star should be considered and an averaging process for the generation of FD coefficients applied. Acceptable stars were obtained due to an application of a geometrical criterion for the selection of nodes. In Kurowski's and Szmelter's papers [12, 29] general triangulation of the region is first made. Stars consist then of nodes situated in the vertices of all triangles having a common central point.

A similar idea was used in papers by Tribillo, Cendrowicz and Kaczkowski [3, 11, 30] which deals with some practical applications of the method and computer implementation.

A different approach was presented by Frey [8]. Using the concept of isoparametric elements [31] he introduced a flexible stencil of arbitrary shape mapped into a regular rectangular mesh.

A curvilinear coordinate system was introduced to transform the whole region into a rectangle. All FD

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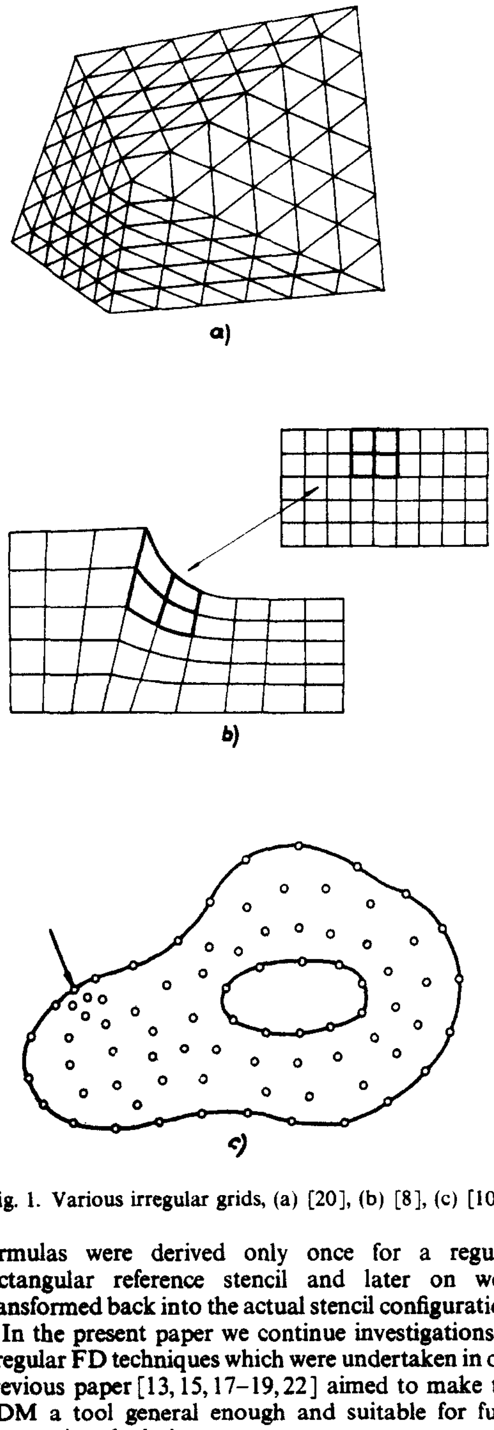


Fig. 1. Various irregular grids, (a) [20], (b) [8], (c) [10].

formulas were derived only once for a regular rectangular reference stencil and later on were transformed back into the actual stencil configuration.

In the present paper we continue investigations of irregular FD techniques which were undertaken in our previous paper [13, 15, 17–19, 22] aimed to make the FDM a tool general enough and suitable for fully automatic calculations.

Several problems caused by an irregular mesh are discussed, such as

- mesh generation;
- automatic search for nodes in the neighbourhood of the considered nodal point;
- an optimal choice of a star and the avoidance of singular as well as ill-conditioned schemas;
- classification of stars;
- automatic generation of FD formulas;
- a new "optimal" way of generating the FD formulas has been introduced;
- for two-dimensional, second order problems the nine-point control scheme has been used, but the procedure remains applicable for other higher order problems;
- reasonable "assignment of an area" to individual nodal points;
- discretization of boundary conditions including derivative terms;
- application to non-elliptic equations;
- solution of linear and nonlinear algebraic equations.

The set of computer programs, called FIDAM (Finite Difference–Arbitrary Mesh) in ALGOL 60 was prepared and tested by calculating both linear and nonlinear problems in applied mechanics. The programs are fully automatic such as those based on the FEM and are especially designed for the solution of large systems where a small computer is used.

2. MESH GENERATION

The generation of an irregular mesh for the FDM may be done in the same way as in the FEM so various automatic mesh generators [2, 12, 14, 27, 32] can be adopted here. On the other hand it is recommended for optimization of the procedure to take into account some special features of the FD technique

- stars of nodes are usually "larger" than elements (i.e. contain more nodes) so the mesh should be as regular as possible—to minimize the number of different FD formulas;
- the ranges of stars are coextensive thus the boundary between two regions is not a single line of nodes; if the entire domain is divided into several identical regions then additional fictitious nodes should be introduced into the mesh [13] to assure proper continuity of the solution; auxiliary nodes outside the boundary are also usually introduced for boundary conditions with derivative terms;
- in regular subdomains a coincidence of a mesh type, selection of stars and FD formulas generation should be secured, e.g. for assumed nine point stars a rectangular mesh should be generated and a regular triangular—for seven point stars.

The automatic generation of a mesh for the FEM is often done in two stages [12, 27]; the generation of nodes and then the generation of elements. In the FDM the second stage, usually more troublesome, may be omitted, as it was done in the presented FIDAM code. Partitioning of the domain into elements is useful mainly for the assignment of an area to each node (see Section 9).

3. AUTOMATIC SELECTION OF STARS

All points in the control scheme are called "a star" of nodes. The number and the position of nodes in each star are the decisive factors affecting FD formula approximation.

Jensen [10] selected the nodes for the star according to their distance from the "central" node. The criterion is simple but it fails very often due to the irregular density of nodes (Fig. 2). The "eight segments" criterion (Fig. 3) suggested by Perrone and Kao [23] results in well selected stars, but it appears to be too rigorous, overly complicated and consumes too much computer time.

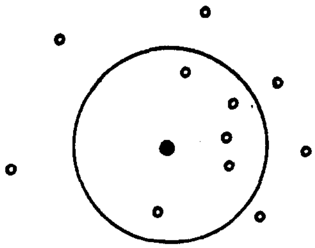


Fig. 2. Selection of stars via distance criterion.

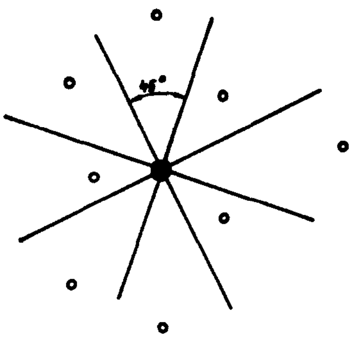


Fig. 3. Selection of stars used in [23].

In the present paper an analogous, but much simpler and quicker procedure has been successfully applied. The domain around the central point is divided into four quadrants, where the two nearest points in each quadrant are included into the star (Fig. 4). The control for a node is performed simply by computing its distance from the central node and comparing the signs of the local coordinates of the node.

To minimize computing time selection of the star is executed in several steps—in each step less nodes are examined but with a more precise criterion. For a very large number of nodes in the mesh a three-step algorithm is recommended

selecting several groups of nodes in the vicinity of the central node;
selecting 20–30 nodes via the distance criterion;
applying one of the various possible geometrical criteria which selects the required number of nodes for the star.

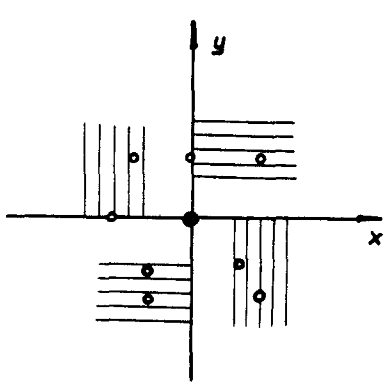


Fig. 4. Four quadrants criterion.

4. CLASSIFICATION OF STARS

For each star of nodes the FD formulas should be generated. It is obvious, however, that some stars with identical FD formulas may occur. To avoid duplication of formulas and for minimization of storage area, the classes of stars are defined by a set of local nodal coordinates.

Thus FD formulas are generated only once for all identical stars. Local and global coordinates of nodes are stored as integer variables giving a considerable saving of storage and computer time with a small loss of accuracy in locating nodes.

5. DIFFERENCE COEFFICIENTS FOR IRREGULAR MESHES

FD formulas for each star should be generated automatically. There exist some computer programs for these purposes but only for regular (rectangular) grids [6]. For irregular grids a Taylor series expansion or polynomial (Lagrange) interpolation may be used.

For any sufficiently differentiable function $f(x, y)$ in a given domain the Taylor series expansion around a point (x_0, y_0) can be used

$$f = f_0 + h\frac{\partial f_0}{\partial x} + k\frac{\partial f_0}{\partial y} + \frac{h^2}{2}\frac{\partial^2 f_0}{\partial x^2} + \frac{k^2}{2}\frac{\partial^2 f_0}{\partial y^2} + kh\frac{\partial^2 f_0}{\partial x\,\partial y} + O(\Delta^3), \qquad (1)$$

where

$$f = f(x, y), \quad f_0 = f(x_0, y_0),$$
$$h = x - x_0, \quad k = y - y_0, \quad \Delta = \sqrt{h^2 + k^2}.$$

Writing eqn (1) for each of the nodes in the star, we derive the set of linear equations ($m \geqslant 5$)

$$[A]\{Df\} - \{f\} = \{0\}, \qquad (2)$$

with

$$[A] = \begin{bmatrix} h_1 & k_1 & h_1^2/2 & k_1^2/2 & h_1 \cdot k_2 \\ h_2 & \cdots & \cdots & \cdots & \cdots \\ \vdots & & & & \\ \vdots & & & & \\ h_m & & & & \end{bmatrix}, \qquad (3)$$

$$\{f\}^T = \{f_1 - f_0, f_2 - f_0, \ldots, f_m - f_0\},$$

where the five unknown derivatives at the point (x_0, y_0) are

$$\{Df\}^T = \left\{\frac{\partial f_0}{\partial y}, \frac{\partial f_0}{\partial y}, \frac{\partial^2 f_0}{\partial x^2}, \frac{\partial^2 f_0}{\partial y^2}, \frac{\partial^2 f_0}{\partial x\,\partial y}\right\}.$$

The main difficulty in a successful application of such an approach is to avoid singular or an ill-conditioned matrix $[A]$ so as to obtain acceptable derivatives $\{Df\}$.

The minimal number of nodes required for determining the $\{Df\}$ vector may be increased ($m > 5$)

in order to improve the accuracy in approximating the derivatives. Equation (2) then forms an overdetermined set of linear equations. The solution may be obtained by minimization of the norm

$$\mathbb{B} = \sum_{i=1}^{m} \left[\left(f_0 - f_i + \frac{\partial f_0}{\partial x} h_i + \cdots \right) \frac{1}{\Delta_i^3} \right]^2 = \text{min.} \tag{4}$$

Writing

$$\frac{\partial \mathbb{B}}{\partial \{Df\}} = 0 \tag{4a}$$

we come to a set of five equations with five unknowns.

The weighting coefficients $(1/\Delta_i^3)$ are inversely related to the distance from the corresponding node to the central node of the star and also to the assumed accuracy of approximation. For example, a regular square mesh and a 9 point star this method produces derivative values more accurately than the corresponding classical FD formulas or those generated by the averaging process [23] (Fig. 5). Also in the case of an irregular mesh a better solution is obtained. For details and a general approach to higher order problems see [19]. In the averaging process four unsymmetrical matrices are inversed. For the suggested method only one inversion of a symmetrical matrix is necessary, so it therefore performs the generation of FD formulas about 5 times faster.

The value of m depends on the mesh, especially in the typical (regular) region. As far as possible, in order to save computer time, we assume a square or rectangular mesh, the 9 point stars are suggested so as to obtain regular symmetrical FD formulas.

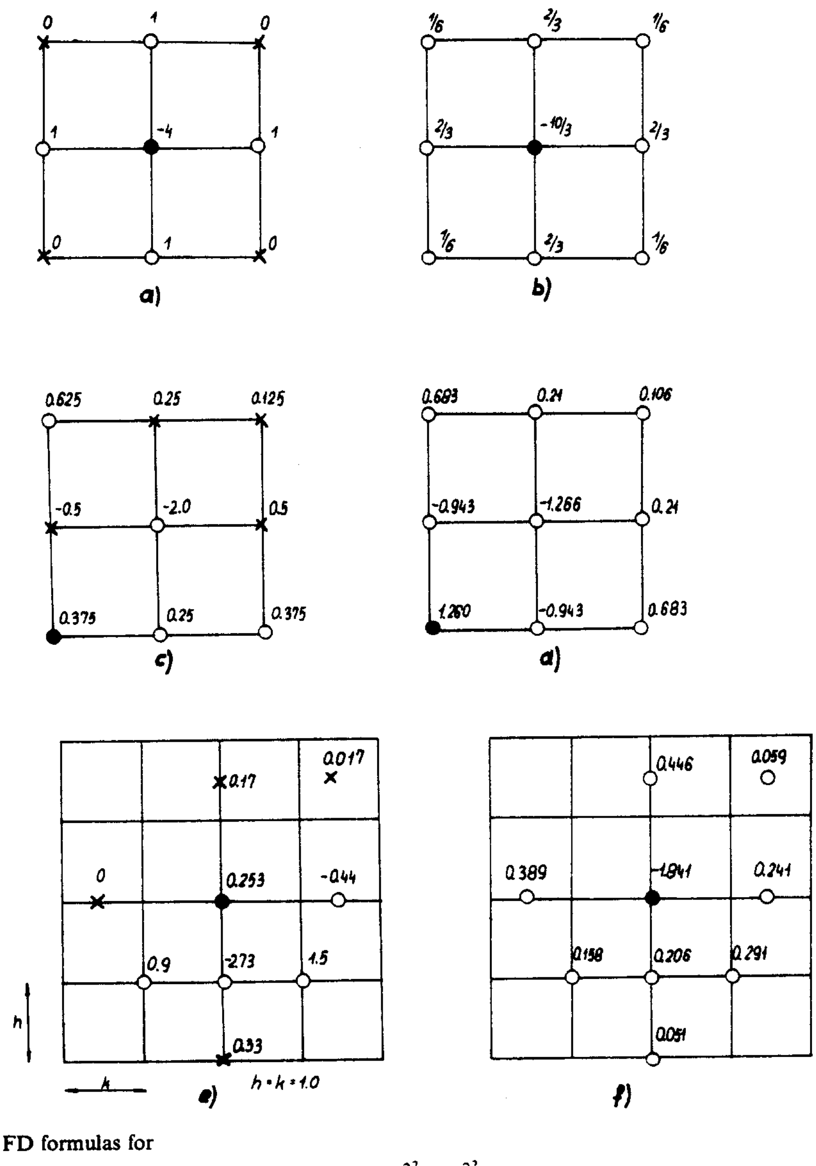


Fig. 5. FD formulas for

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(a), (c), (e) averaging process; (b), (d), (f) Eqn (4); ● central node; ○ main nodes in averaging process.

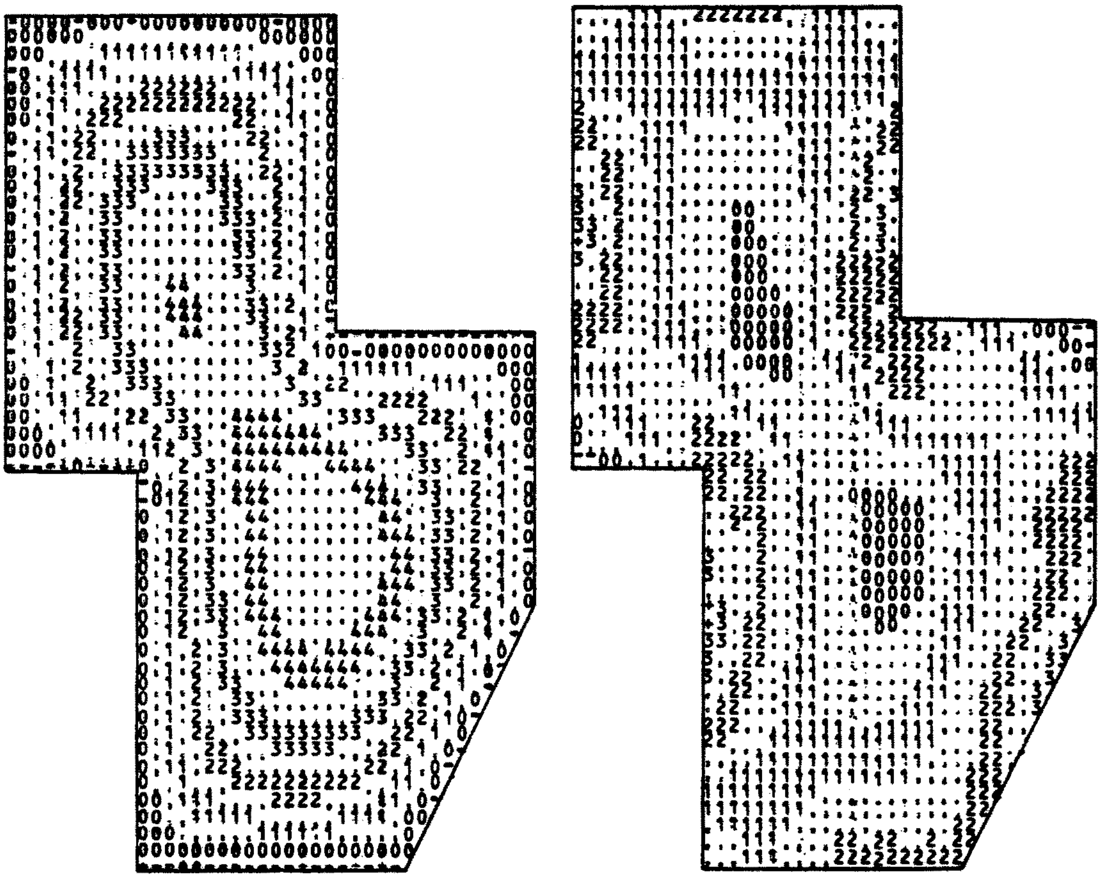


Fig. 6. The solution of Poisson's problem $\nabla^2\phi = -1$;
(a) contours of ϕ function, (b) contours of grad ϕ.

6. CONVERGENCE TESTS

The convergence of the presented method has been shown experimentally in some numerical tests. They have been carried out with the use of several similar programs differing in selection of nodes and in the algorithm used to generate FD formulas.

As an example, Poisson's equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -1 \tag{5}$$

for the domain shown in Fig. 6 has been solved. The solution (Fig. 6a) may be a stress function for the torsion of a prismatic bar. The corresponding stress distribution is mapped in Fig. 6b.

Figure 7 illustrates the convergence of the solution against the number of nodes for 3 analogous programs

(a) selection of the star based on distance criterion only and generating FD formulas by the averaging process;
(b) selection based on distance criterion and minimization procedure used for FD formulas;
(c) selection of stars and generation of FD formulas suggested in this paper.

For program (a) and 20 inner nodes the solution was unstable. An "exact" solution was obtained by the classical FDM with about 2500 unknowns.

Other tests [13, 19] also showed program (c) to be the best.

7. APPROXIMATION OF BOUNDARY CONDITIONS

An interesting idea for the discretization of boundary conditions in regular mesh for irregular domain was published by Hunt [9]. By using irregular grids it may be solved much easier. An arbitrary grid allows us to situate the nodes on boundary lines, so the boundary condition in the form

$$u(x, y) = f(x, y),$$

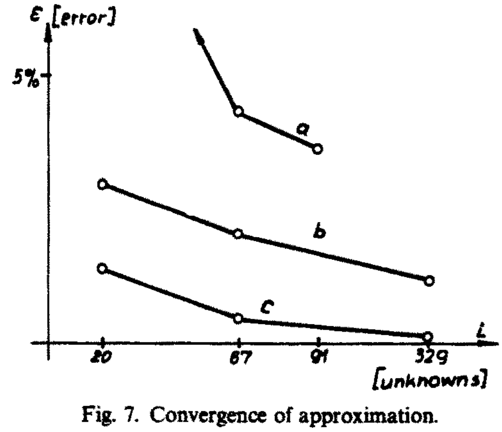


Fig. 7. Convergence of approximation.

where $u(x, y)$ is any given function, may be introduced into the solution obviously. Some troubles arise when the boundary condition contains differential terms.

To avoid "non-balanced" stars, selected usually for boundary nodes, additional nodes outside the domain may be introduced into the mesh. The number of these auxiliary nodes should be equal to the number of boundary nodes. For each boundary node then two sets of FD equations have to be generated: the first one corresponding to boundary conditions and the second one as for inner nodes.

As an example a temperature distribution for the square cross-section prismatic bar with a uniformly heated boundary along $y = \pm 1$ line and a constant zero temperature along $x = \pm 1$ line (Fig. 8) has been analyzed.

A better solution (Fig. 9) was observed when additional nodes were introduced, though both results were acceptable with the exception of a small region near the vertices $x = \pm y = \pm 1$, where the boundary conditions were contradictory. Although, some reports [1] have mentioned that auxiliary nodes cause essential modification of results, e.g. introduce an additional form of vibration.

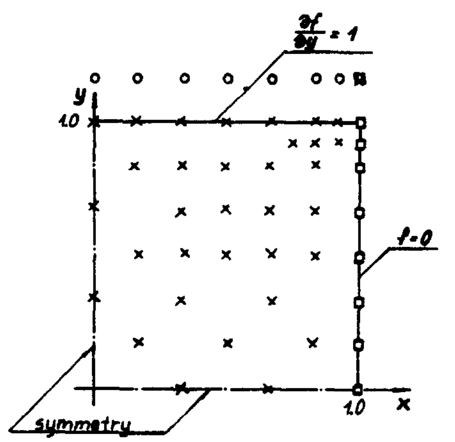


Fig. 8. A mesh for temperature distribution problem.

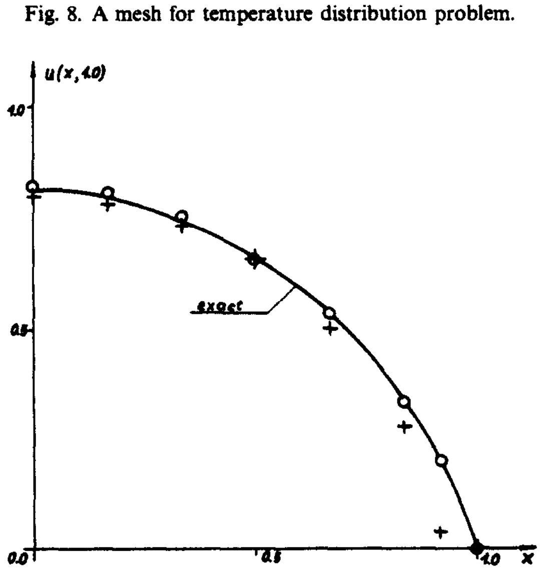


Fig. 9. Temperature distribution along the $y = 1$ line; + 33 inner and boundary nodes, ○ 7 auxiliary nodes.

It is generally observed that the accuracy of the solution is much better if a higher order approximation of derivatives for boundary conditions is assumed. In FIDAM programs second order FD formulas (4) are used also for boundary conditions with first order derivatives only.

For boundary conditions containing second order derivatives (e.g. for a set of differential equations such as for a plate $\nabla^2 W = M$, $\nabla^2 M = -\rho/D$ with a free-supported edge) only balanced stars are recommended.

For unbalanced stars the best estimation of the second order derivative is located not on the boundary but in the domain, between the nodes (Fig. 10). This is caused by the "constant curvature" of the star where the second order Taylor's expansion is used. The result of this unbalancing is a systematic error of the solution.

8. AN AID FOR INTERPRETATION OF RESULTS

In many problems the function used in differential equations is not the one of main interest. For example, the solution of a plane elasticity problem when the Airy function $\phi(x, y)$ may be applied, which in fact has no practical meaning, the stress distribution can be calculated from the following differential formulas:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{+\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = \frac{-\partial^2 \phi}{\partial x\, \partial y}.$$

So the calculation of derivatives for the solution is a very important step in order to interpret the results and it is advisable to perform it automatically through the computer program. In the FIDAM code the FD coefficients, obtained by formula (4), are stored in a drum memory, thus they can be used for evaluating derivatives. Therefore, any required function u in the form

$$u = L_u f, \tag{6}$$

f—solution of the entire differential equation,
L_u—differential operator of a second order,

may be calculated, and a print-out for each nodal point is obtained.

In most practical cases the number of nodes and nodal values in FD solutions are so numerous that any manual calculation would be practically impossible. The most appropriate form of results is a graph. An image of contour lines is recommended, because it can be produced by any computer, even one not equipped with a plotter. The "map" of a function may be printed on a line printer in a fast and easy way.

For this purpose the considered function should be defined in a continuous form. In the present paper a fast method of approximation for an irregular net of nodal points has been suggested. It uses the same minimization procedure as in the generation of FD formulas. The norm $\mathbb{B}$ (eqn 4) is determined for any point in a domain, thus f_0 is not a nodal value and it should be evaluated together with all derivatives $\{Df\}$. Due to weighting factors $1/\Delta_i^3$ the presented method gives us not only approximation in the sense of minimum deviation of given values but the interpolation of a function, i.e. if a point tends to a node then the limit of f_0 is exactly the nodal value.

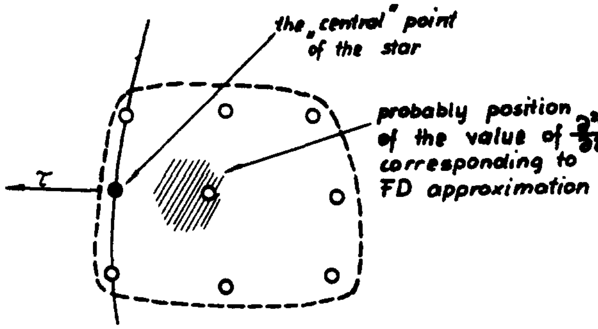


Fig. 10. Position of the value of $\partial^2 f/\partial t^2$; ●—assumed in FD approximation, hatching—real.

Of course the number of considered nodes m have to be much smaller than the total number of nodes, thus for two close points A and B the corresponding sets of nodal points may vary (Fig. 11). Discontinuities caused by this fact may be limited to a required level by selecting more nodes for the stars, due to weighting factors $1/\Delta_i^3$ which minimize the influence of distant nodes. In FIDAM, m was set to 12 and discontinuities were neglectably small. An illustration of this technique is shown in Fig. 12.

9. ASSIGNMENT OF AN AREA

In local nonhomogeneous problems as well as in the variational approach it is necessary to define a domain assigned to each node.

The simplest proposal is to assign to each node an area where the size of this area depends on the distance between the nodes and/or the area of a polygon which circumscribes the star. Then a reasonable coefficient is applied to ensure a proper summation value of the areas F_i. The value of this coefficient should be determined so that the sum of the areas equals the area of the whole domain. For example, the equation

$$F_i = \alpha \left(\sum_{j=1}^{m} \Delta_{ij}^{\nu} \right)^{2/\nu} \tag{7}$$

with $\nu = -2$ was found to give acceptable results in some tests [13]. For application to shell analysis a modified FD method [5] is developed which considers separate nodes and separate integrating areas for

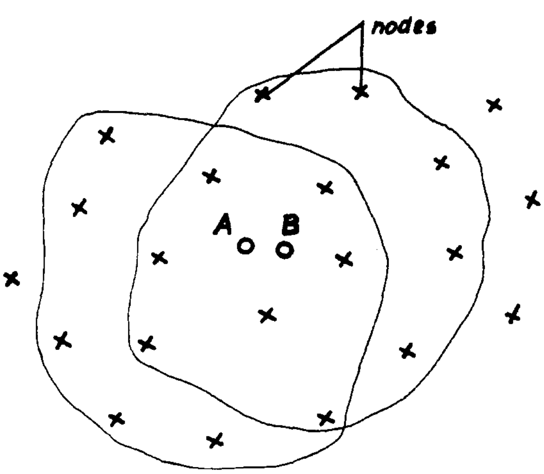


Fig. 11. Various stars for two close points A, B.

various derivative terms and functions used. In the technique discussed here all FD formulas are symmetrical (except some near the boundary) thus such distinction does not have to be used.

Another, more precise proposal is based on dividing the entire domain into triangles (Fig. 13). The triangles are divided into three parts, each one assigned to an apex. Partition of the sides of the triangle by symmetrical lines is suitable for acute-angled triangles but for obtuse angles further analysis is necessary (Fig. 14).

The main imperfection of these two approaches is the independence of the area assigned and the shape of the star. In fact, the area should be assigned to the star and not to the nodes.

Applying selection criteria which is related to the area assignment for a star seems to give a proper solution.

10. APPLICATION TO NON-ELLIPTIC EQUATIONS

Time-dependent problems are described by differential equations of a parabolic or hyperbolic type. It then becomes necessary to ensure the stability of the generated FD formulas. So far there have been no theoretical investigations carried out in this field in the case of an irregular mesh. Moreover, the Perrone and Kao procedure for a square mesh yields unstable results for the parabolic problem

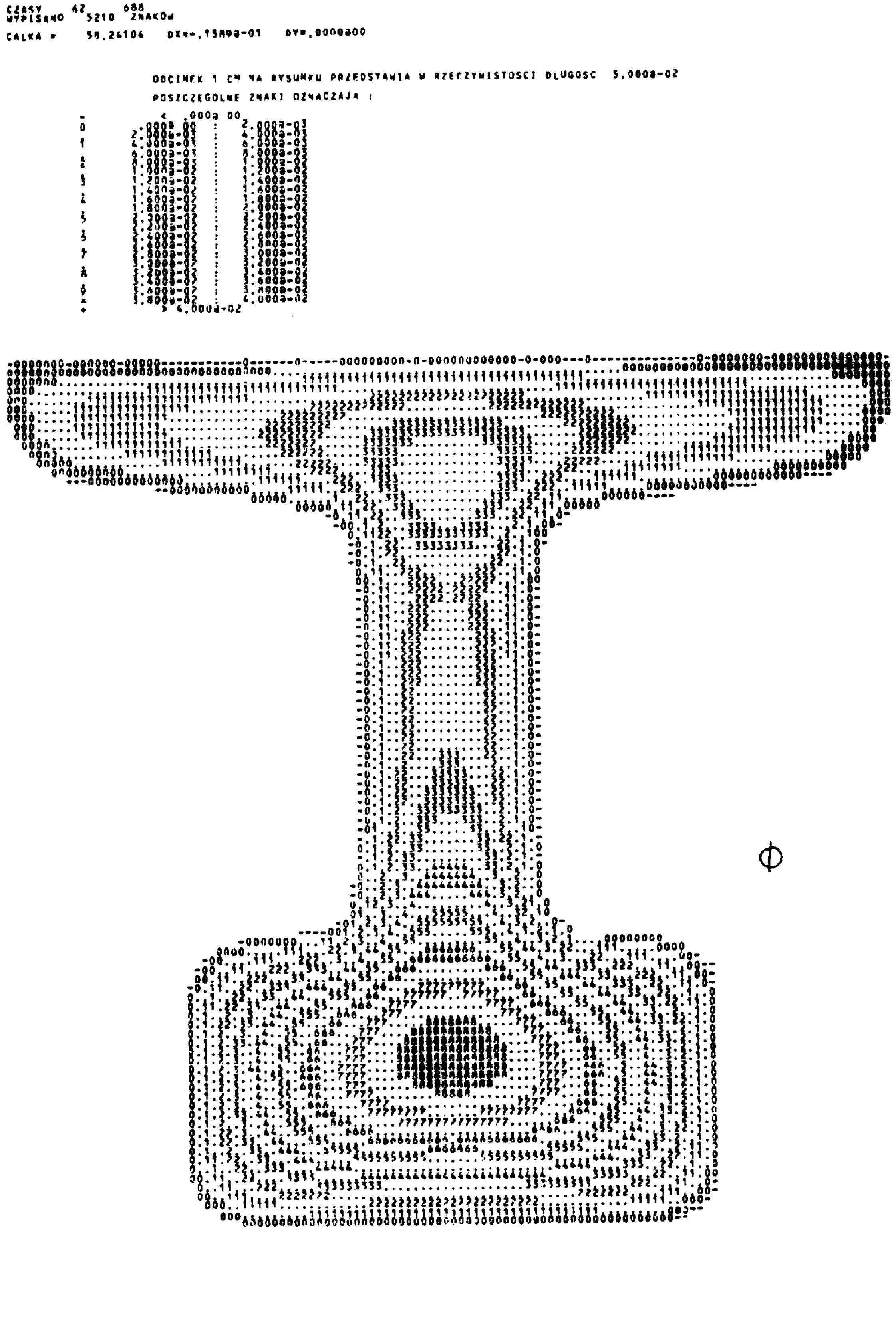
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}.$$

Applying a mesh which is not time dependent, it is possible to avoid such difficulties. In this case we can introduce into calculations stable formulas for time-dependent derivatives. Thus the expression for the space derivatives (in x, y coordinates) may be generated automatically. In space coordinates the differential equation usually remains elliptical, therefore the investigation of stability may be omitted.

As an illustration of the method presented, two possible approximations for the time-dependent problem of heat transfer

$$\nabla^2 f = \frac{\partial f}{\partial t} \tag{8}$$

have been tested (Fig. 15). They are a three-dimensional extension of FD formulas reported in [24]. The first "open" formula becomes stable only for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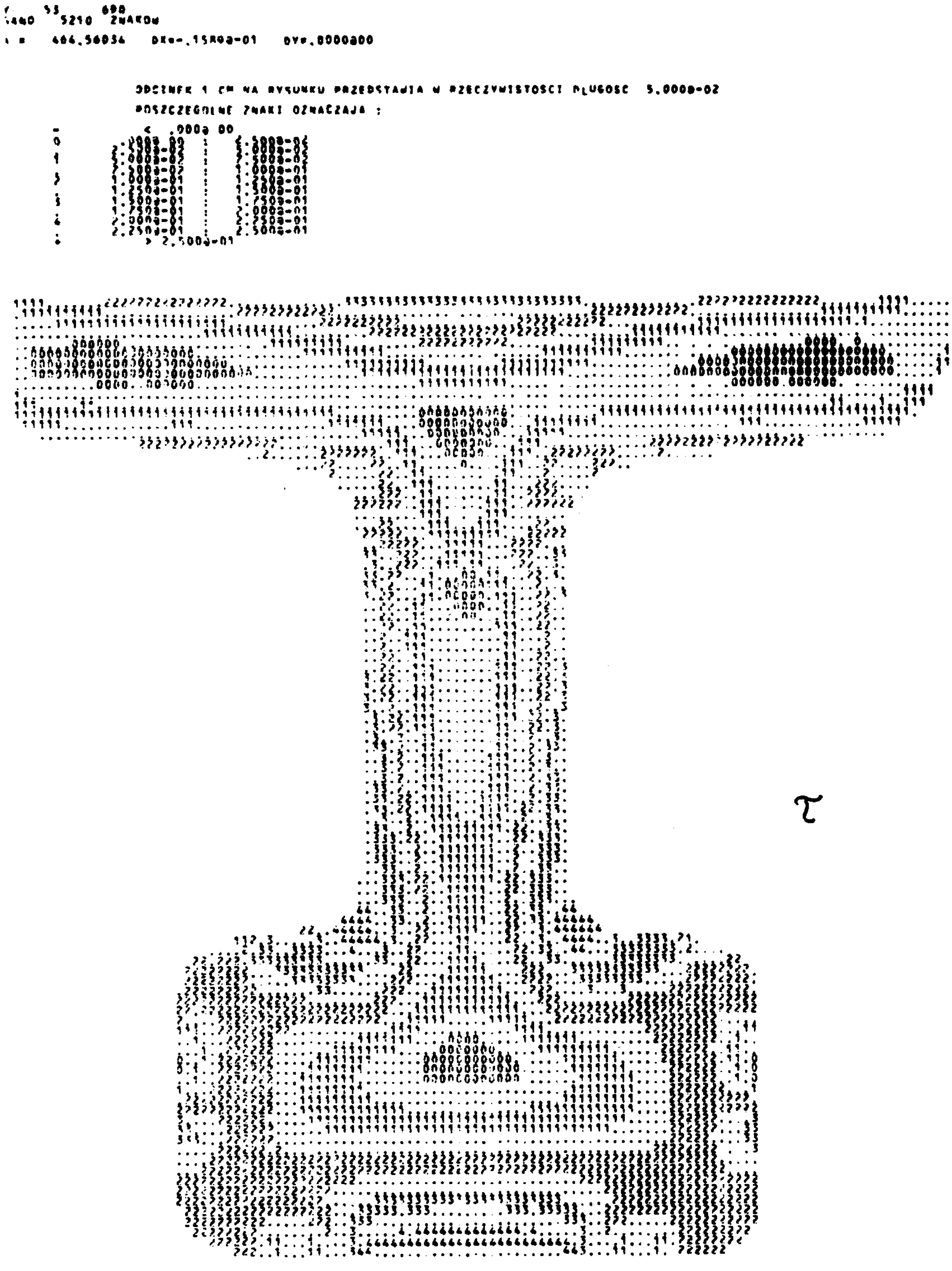


Fig. 12. Contours of (a) Prandtl function, (b) its gradient in twisted bar of rail—shaped cross-section.

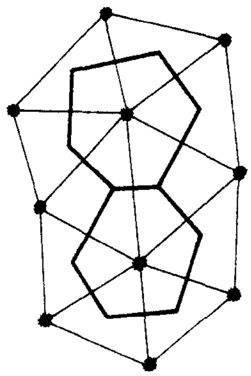


Fig. 13. Assignment of an area (symmetrical lines in triangles).

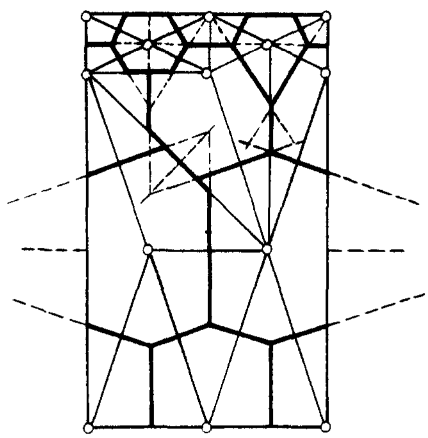


Fig. 14. Partition of obtuse-angled triangles.

a reasonably small value Δt (mesh size in time). An acceptable Δt should be evaluated according to the minimum distance between two nodes of an analyzed mesh.

A larger Δt produces unstable results around the region of the finest grid (Fig. 16). It was proved [13] that for square grids the minimization procedure (Fig. 5b) enlarges the stable time step by 50%.

The second scheme (Fig. 15b) is an extension of the Crank–Nicholson method. Its application requires more calculations in order to solve the set of linear equations for each time step. It allows us, however, to assume larger time mesh spaces. Runchall [25] published other applicable formulas for regular meshes, which can be adopted here, as well as some comparative studies.

11. NONLINEAR PROBLEMS

The FDM is universally applicable to both linear and nonlinear problems. For nonlinear problems various iterative procedures can be applied. The Newton–Raphson and selfcorrecting procedures converge more quickly than other approaches [28]. They have been adopted here for the FDM at arbitrary irregular grids.

The "slope matrix" was found by means of previously calculated coefficients of FD formulas. For each step of iteration an auxiliary linear differential equation was introduced which can be derived from the nonlinear differential equation considered [15]. The algorithm consists only of procedures which are used in linear problems, so the nonlinear FIDAM code despite it being quite simple works very effectively.

For the solution of linear FD equations at each step of nonlinear iteration, the iterative method seems to be very profitable. It was observed, unfortunately, that overrelaxation or Gauss–Seidel methods very often diverge. Because of this, a "projection" method which is always convergent has been introduced. Some improvements of this method have been tested [16]. The method may be also useful for linear elliptical problems if boundary conditions and/or additional terms corresponding to the first derivatives positive definite character of FD linear equations.

Using FD formulation for some nonlinear problems which have a non-continuous solution (such as elastic–plastic torsion of a bar) leads to increased errors in the vicinity of the discontinuity lines (e.g. line 3 in Fig. 19). In a problem of such type they are caused by the discontinuous slope of the proper solution at this boundary (in Section 5 it was shown that the solution was assumed to be smooth enough up to the second derivatives).

The arbitrary mesh enabled us to improve the results by inserting additional nodes along the elastic–plastic boundary and introducing appropriate boundary conditions. If the assumed elastic boundary appears not to be correct, then the mesh should be redefined during computation.

12. THE FIDAM PROGRAMS

The first version of the FIDAM code was designed for a small computer ("Odra 1204") with relatively fast random access to drum storage. The use of Algol 1204, which strongly limits the structure of binary subprograms, was the reason that FIDAM consists of a set of separate programs.

Some of the programs do not require any input data using mass storage for intermediate results.

The storage size allows us to analyse problems with 1000 nodes and 2000 degrees of freedom. If the

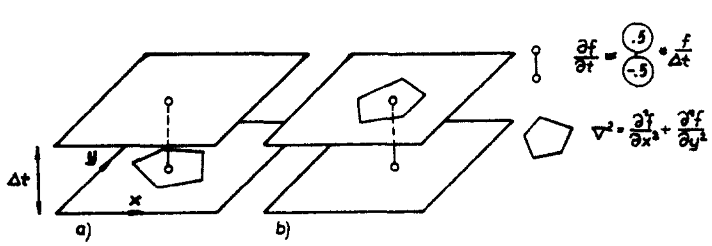


Fig. 15. Time integration schemes, (a) "open", (b) Crank–Nicholson type.

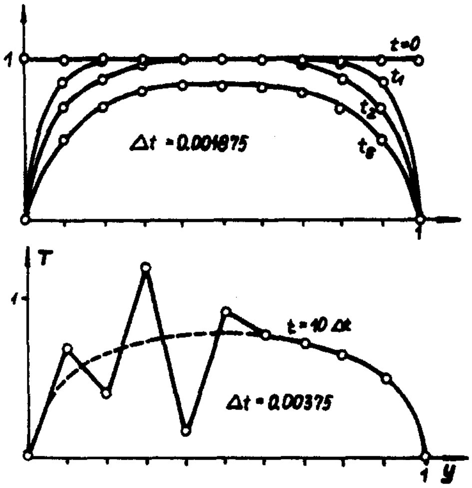


Fig. 16. Stable and partially unstable results of heat flow problem.

Gauss–Seidel procedure converges then, taking these limitations into account, a solution to the problem may be obtained within a reasonable time limit.

On the other hand, if there is a slow or non convergence of the Gauss–Seidel procedure then the projection method or frontal analyses can be applied where a new limiting factor for the number of nodes used is the bandwidth.

The following programs are used in the FIDAM set:

dan—for semiautomatic generation of a mesh;
ww—for the generation of FD formulas;
itg, ipd, paf—for the solution of linear equations by various methods (overrelaxation, projection or the frontal method for nonsymmetric equations);
wyd—calculates additional functions (eqn 6);
map—prints "maps" of results;
przyg, rozw, nmp—for the solution of nonlinear problems;
par—for parabolic equations;
pns—calculates "the starting solution" for the iterative method which is the solution derived using a coarse mesh.

In the mass storage four sets of data are located:

the node coordinates;
FD formula coefficients;
a set of FD equations;
the solution and additional functions.

For medium size problems which may be solved by FIDAM code, only coefficients of FD formulas should be located on random access device, remaining data may reside in sequential files. For large problems and in the case of slow mass storages the use of a file with coefficients of FD formulas and classification of stars (Section 4) may be inefficient. Then FD formulas can be generated for each requirement separately (e.g. for the calculation of derivatives). The structure of the FIDAM code may be easily adopted using modern minicomputers, which are comparable to the "Odra 1204".

13. NUMERICAL RESULTS

Table 1 describes various problems of applied mechanics, which were solved with the use of FIDAM code.

The plain elasticity problem of the tension of a notched plate was solved where equivalent stresses according to the Huber–Mises–Hencky criterion was mapped (Fig. 17). A time-dependent temperature distribution for a rail-shaped bar is illustrated in Fig. 18. The physical nonlinear problem of the torsion of an elastic–plastic bar has been solved. Figure 19 illustrates both the elastic–plastic boundaries for several steps of the torsion process and a shear stress distribution. Geometrical nonlinearity was analysed in the case of a perfect membrane (Fig. 20).

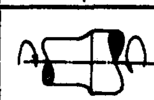
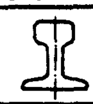
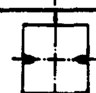
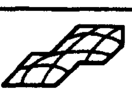
These and other solution (see Table 1) gives us an overall view of the possible applications of the FIDAM, not only for those mentioned but also for a wide class of problems in the field of applied mechanics.

14. CONCLUSIONS

Future development of the presented set of programs would enable a considerable increase in the method's application.

The automatic generation of an irregular mesh, selection of stars and assignment of an area should be investigated. Interrelationship between these three steps would allow us to improve on accuracy and save

Table 1. Applications FIDAM to some problems in mechanics

problem	region	type of boundary value problem
torsion of a prismatic bar		elliptic equation
torsion of axially-symmetric bar		elliptic equation with variable coefficients
plate deflections		biharmonic equation
stationary heat flow	0 0 0 0 0 0 0 0 0 0 0	elliptic equation
non-stationary heat flow		parabolic equation
plane stress analysis		the set of elliptic equations
elasto-plastic torsion of a bar		physical nonlinearity
large deflection of membrane		geometrical nonlinearity

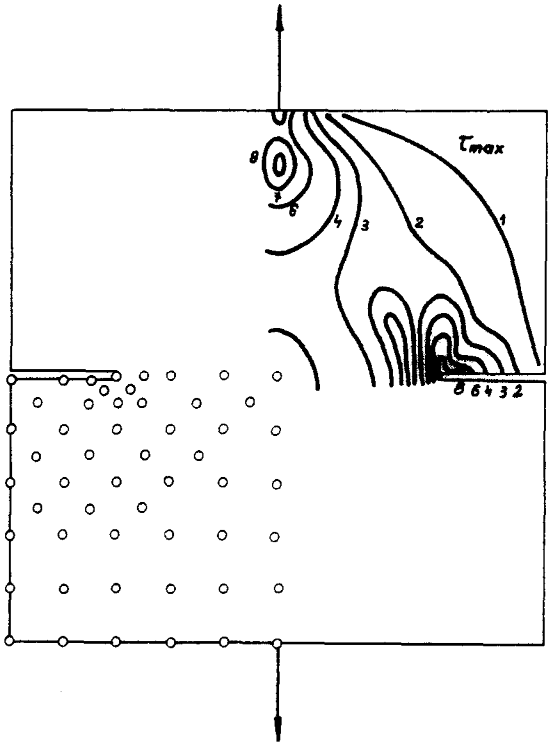


Fig. 17. Tension of a notched plate.

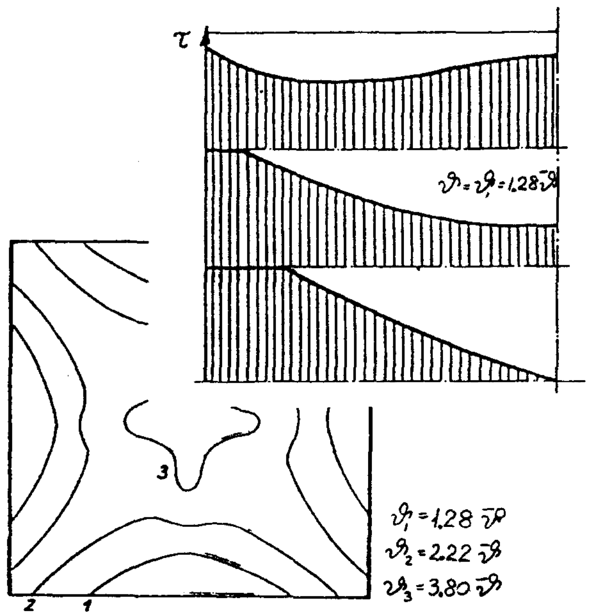


Fig. 19. Torsion of prismatic bar: elasto-plastic boundaries and shearing stresses.

a considerable amount of computer time. Further research is also required for analysis based on the variational approach. The main advantage of this formulation is based on the fact that in the energy expression, the highest order of the derivatives is only one-half of that obtained in the local approach.

Application of a multilocal version of FDM [5] to an irregular mesh may be a very promising modification to the presented method. Further tests and investigations are also necessary for the application of the method to non-elliptic problems. Because of the possibility in modifying a mesh during the course of calculations an interesting use of an irregular mesh arises. Such modification allows for an increase in accuracy (e.g. automatic optimization of the number of nodes used). A set of computer programs which design the mesh and improve the solution in the course of several iterative steps could be the final achievement.

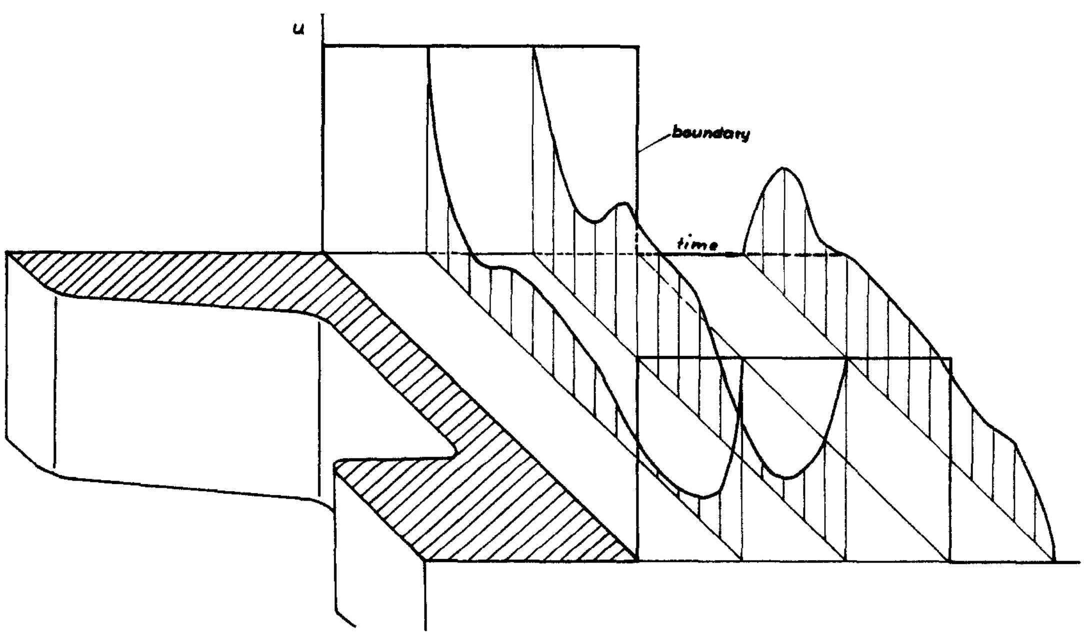


Fig. 18. Time dependent heat flow.

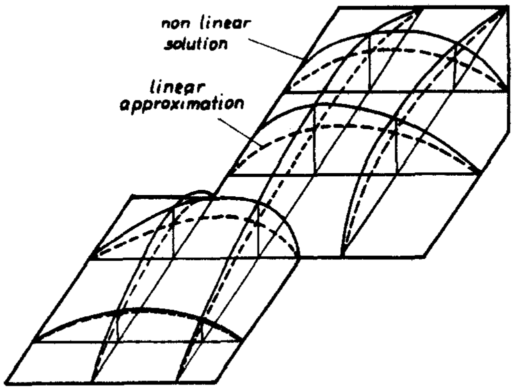


Fig. 20. Large deflections of perfect membrane.

The method permits us to solve various problems of applied mechanics. Due to successful solutions of the difficulties discussed above, the FDM at irregular meshes is universal enough to be competitive with the FEM, especially for problems with physical or geometrical nonlinearities, optimization and time and/or temperature dependence.

REFERENCES

1. D. Bushnell, B. O. Almroth and F. Brogan, Finite difference energy method for nonlinear shell analysis. *Comp. Structures* 3(1), 361–388 (1971).
2. A. Bykat, Automatic generation of triangular grid: I—subdivision of a general polygon into convex subregions. II—triangulation of convex polygons. *IJ. NME* 6(10), 1329–1342 (1976).
3. J. Cendrowicz and R. Tribiłło, Variational approach to the static analysis of plates of an arbitrary shape, (in Polish). *Archiwum Inżynierii Lądowej* 3(24) (1978).
4. K. P. Chuang and A. S. Veletsos, A study of two approximate methods of analyzing cylindrical shell roofs. *Struct. Res.* Series No. 258. University of Illinois, Urbana, Ill (Oct. 1962).
5. L. Collatz, *Numerische Behandlung von Differentialgleichungen.* Springer-Verlag, Berlin–Göttingen–Heidelberg (1955).
6. Z. J. Csendes, A Fortran program to generate finite difference formulas. *IJ.NME* 3(9), 581–599 (1975).
7. G. E. Forsythe and W. R. Wasow, *Finite—Difference Methods for Partial Differential Equations.* Wiley, New York (1960).
8. W. H. Frey, Flexible finite—difference stencils from isoparametric finite elements. *IJ.NME* 11, 1653–1665 (1977).
9. B. Hunt, Finite difference approximation of boundary conditions, along irregular boundaries. *IJ. NME* 12, 229–235 (1978).
10. P. S. Jensen, Finite difference techniques for variable grids. *Comp. Structures* 2, 17–29 (1972).
11. Z. Kaczkowski and R. Tribiłło, A generalization of the finite difference method (in Polish). *Archiwum Inżynierii Lądowej* 2(21), 287–293 (1975).
12. Z. Kurowski, The solution of set of linear partial differential equations by FDM (in Polish). Ph.D. thesis, Wojskowa Akademia Techniczna, Warsaw (1977).
13. T. Liszka, The finite difference method at arbitrary irregular meshes and advantages of its applications in mechanics (in Polish). Ph.D. thesis, Politechnika Krakowska, Cracow (1978).
14. T. Liszka, An automatic generation of arbitrary meshes in two-dimensional regions (in Polish). Proceedings of the 4th Conf. on Comp. Meth in Struct. Mech., Koszalin, Poland (April 1979).
15. T. Liszka and J. Orkisz, A finite difference method of solution of nonlinear problems in mechanics. Proceedings of the 3rd Conf. on Comp. Meth. in Struct. Mech., Opole, Poland, 47–56 (May 1977).
16. T. Liszka and J. Orkisz, An always convergent iterative method of solution of simultaneous linear algebraic equations (in Polish). Proceedings of the 3rd Conf. on Comp. Meth. in Struct. Mech., Opole, Poland, 37–45 (May 1977).
17. T. Liszka and J. Orkisz, Finite difference method of arbitrary irregular meshes in nonlinear problems of applied mechanics. IV Structural Mechanics in Reactor Technology, San Francisco (Aug. 1977).
18. T. Liszka and J. Orkisz, System FIDAM—the finite difference method at arbitrary meshes for nonlinear analysis. 2nd intern. Conf. on Comp. Meth. in Nonlinear Mechanics, 23–26 March, Austin, U.S.A. (1979).
19. T. Liszka, J. Orkisz and M. Zyczkowski, A method of generation of the optimal difference formulas for arbitrary irregular grids. To be sent to Int. J. Num. Meth. in Engng. (1979).
20. R. H. MacNeal, An asymmetrical finite difference network. *Q. Appl. Math.* 11, 295–310 (1953).
21. J. T. Oden, Finite element of nonlinear continua. McGraw-Hill, New York (1972).
22. J. Orkisz, Computer approach to the finite difference method (in Polish). Proceedings of the 4th Conf. on Comp. Meth. in Struct. Mech., Koszalin (April 1979).
23. N. Perrone and R. Kao, A general finite difference method for arbitrary meshes. *Comp. Structures* 5, 45–58 (1975).
24. R. D. Richtmyer and K. W. Morton, Difference Method for Initial Value Problems. Wiley and Sons, New York (1967).
25. A. K. Runchal, Comparative criteria for finite-difference formulations for problems of fluid flow, *IJ. NME* 11, 1667–1679 (1977).
26. A. A. Samarski, Theory of Difference Shemas (in Russian). Nauka, Printing House, Moscow (1977).
27. R. D. Shaw and R. G. Pitchen, Modifications to the Suhara–Fukuda method of network generation. *IJ. NME* 12, 93–99 (1978).
28. J. A. Stricklin and W. E. Haisler, Formulation and solution procedures for nonlinear structural analysis, *Comp. Structures* 7, 125–136 (1977).
29. J. Szmelter and Z. Kurowski, A complete program for solving systems of linear partial differential equations in plain domains (in Polish). Proceedings of the 3rd Conf. on Comp. Meth. in Struct. Mech., Opole, Poland, 237–247 (1977).
30. R. Tribiłło, Application of algebraic structures to a generalized finite difference method (in Polish). Politechnika Białostocka, Zeszyt Naukowy nr 9, Białystok (1976).
31. O. C. Zienkiewicz, The Finite Element Method in Engineering Science. McGraw-Hill, London (1971).
32. O. C. Zienkiewicz and D. V. Phillips, An automatic mesh generation scheme for plane and curved surfaces by isoparametric coordinates. *IJ. NME* 3, 519–528 (1971).