Classical field theory via Cohesive homotopy types

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Abstract

We demonstrate how a refined formulation of classical mechanics and classical field theory that takes global effects properly into account (e.g. classical anomalies and the global descent to reduced phase spaces) is naturally given in "higher differential geometry". This is the context where smooth manifolds are refined to *smooth homotopy types*. We introduce and explain this higher differential geometry as we go along. At the same time we explain how the classical concepts of classical mechanics and classical field theory follow naturally from the abstract homotopy theory of *correspondences in higher slice toposes*.

The first part of the text is meant to serve the triple purpose of being an exposition of classical mechanics for homotopy type theorists, being an exposition of geometric homotopy theory for physicists, and finally to serve as the canonical example for the formulation of a local pre-quantum field theory which supports a localized quantization process to local quantum field theory in the sense of the cobordism hypothesis. On the way we also clarify some aspects of multisymplectic field theory by observing that the Hamilton-de Donder-Weyl field equation characterizes Maurer-Cartan elements in the L_∞-algebra of local observables.

The second part of the text discusses the relevant statements of differential cohomology in cohesive higher toposes in more detail.

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Contents

1 Introduction **3**

2 Hamilton-Jacobi-Lagrange mechanics via prequantized Lagrangian correspondences **5**
- 2.1 Phase spaces and symplectic manifolds . . . 5
- 2.2 Coordinate systems and the topos of smooth spaces . . . 7
- 2.3 Canonical transformations and Symplectomorphisms . . . 9
- 2.4 Trajectories and Lagrangian correspondences . . . 10
- 2.5 Observables, symmetries and the Poisson bracket Lie algebra . . . 14
- 2.6 Hamiltonian (time evolution) trajectories and Hamiltonian correspondences . . . 16
- 2.7 Noether symmetries and equivariant structure . . . 18
- 2.8 Gauge theory, smooth groupoids and higher toposes . . . 20
- 2.9 The kinetic action, pre-quantization and differential cohomology . . . 22
- 2.10 The classical action, the Legendre transform and Hamiltonian flows . . . 25
- 2.11 The classical action functional pre-quantizes Lagrangian correspondences . . . 27
- 2.12 Quantization, the Heisenberg group, and slice automorphism groups . . . 28
- 2.13 Integrable systems, moment maps and homomorphism into the Poisson bracket Lie algebra . 31
- 2.14 Classical anomalies and projective symplectic reduction . . . 32

3 De Donder-Weyl field theory via higher correspondences **33**
- 3.1 Local field theory Lagrangians and n-plectic smooth spaces . . . 33
- 3.2 Local observables, conserved currents and higher Poisson bracket homotopy Lie algebras . . . 37
- 3.3 Field equations of motion, higher Maurer-Cartan elements, and higher Lie integration . . . 40
- 3.4 Higher gauge theory, smooth ∞-groupoids and homotopy toposes . . . 43
- 3.5 Higher Chern-Simons-type boundary field theory and higher Chern-Weil theory . . . 43
- 3.6 Source terms, the off-shell Poisson bracket and holography with symplectic groupoids . . . 44

A Homotopy toposes and Cohesive homotopy types **48**

B Differential geometry via Smooth 0-types **52**
- B.1 Coordinate systems . . . 53
- B.2 Smooth 0-types . . . 56
- B.3 Differential forms . . . 62

C Higher geometry via Geometric homotopy types **67**
- C.1 Homotopy toposes of geometric ∞-groupoids . . . 68
- C.2 Higher geometric fiber bundles . . . 75
- C.3 Bisections of higher groupoids . . . 79
- C.4 Higher Atiyah groupoids . . . 79

D Differential cohomology via Cohesive homotopy types **84**
- D.1 Differential coefficients . . . 86
- D.2 Differential moduli . . . 91
- D.3 Higher quantomorphism- and Heisenberg-groupoids . . . 94
- D.4 Higher Courant groupoids . . . 97

1 Introduction

The theory of *classical mechanics* (e.g. [1]) is the mathematical theory of how macroscopic physical objects of the observable world (for instance dust, rocks and planets) move through space and time while interacting via forces between them. Similarly the theory of *classical field theory* (e.g. [20]) is the theory of how macroscopic field excitations (for instance of the electric field, the magnetic field, and the field of gravity) relate to space and time while interacting with each other and with physical objects.

As the name clearly suggests, classical mechanics and classical field theory are classical topics in mathematical physics, about whose foundations, it would seem, there is hardly much more to be said. However, the observable universe is fundamentally not described by classical field theory (when including microscopic objects and microscopic field excitations) but by quantum field theory (QFT), for which classical (or maybe better: pre-quantum) field theory is but the generating input datum. But QFT has lately seen considerable refinements in the formalization of its foundations (see [48] for a recent survey), notably in the mathematical formalization of the physical principle of (*causal*) *locality*. Namely the *cobordism hypthesis* [32] (see [3] for a review) asserts that the locality property – of at least those quantum field theories which are either topological or else expressible as ("holographic") boundary theories of topological theories – is encoded in a universal construction in "directed homotopy theory", called (∞, n)-category theory (see for instance [33, 3]). But this insight does reflect back on the formulation of the foundations of classical/pre-quantum field theory, because, as traditionally formulated, this does not support a quantization that would yield a genuinely local QFT in the sense of the cobordism hypothesis. Therefore some refined formulation of, or at least a new perspective on, classical field theory still seems to be needed, after all. This is what we are concerned with here.

In the physics literature roughly this issue is well-known as the problem of "non-covariance of canonical quantization". There are various proposals for how to refine the traditonal formulation of classical field theory such as to solve this problem, of which maybe the best developed is called "multiymplectic covariant field theory" (see e.g. [47]). This approach starts out with a natural generalization of the basic notion of *symplectic 2-forms* to higher degree forms, but it used to be unclear about on how the further ingredients of classical mechanics are generalized, such as notably what is to refine the *Poisson bracket* Lie algebra of observables as one passes to *local observables*.

In [15, 16, 43] it was shown that these problems find a solution – and that multisymplectic classical field theory finds a formulation that makes it compatible with the formulation of local quantum field theory via the cobordism hypothesis – if one understands geometric pre-quantization not in the traditional context of differential geometry (see B below for the basics of that), but in the context of *higher differential geometry* [50], the combination of differential geometry with *homotopy theory* (see [44] for the classical axiomatization of homotopy theory by model categories, [31] for the formulation by homotopy toposes and [59] for the modern perspective on homotopy theory as homotopy *type theory*). This is the context where smooth manifolds are allowed to be generalized first to smooth orbifolds (see for instance [39]) and then further to Lie groupoids (see for instance [36]), then further to smooth groupoids and to smooth moduli stacks and finally to smooth higher stacks (see [19]), hence to *smooth homotopy types* and more generally to *cohesive homotopy types* [50]. However, in these articles the relation to traditional descriptions of classical field theory was not discussed much. The present article is meant to expose the formulation of classical field theory via cohesive homotopy types more in detail.

We introduce and explain the relevant higher differential geometry as we go along. At the same time we explain how the classical concepts of classical mechanics and classical field theory follow naturally from just the abstract theory of "correspondences in higher slice toposes". Therefore the intended readership is twofold: on the one hand theoretical and mathematical physicists, and on the other hand homotopy-type heorists and homotopy type-theorists and topos theorists. Each group might take the following as an introduction to the topic from the other group's perspective. One purpose of this article is to explain how classical mechanics and classical field theory is naturally a subject in *both* specialities and to provide a dictionary to translate between the two points of view.

To put this statement in perspective, notice that *locally* and in fact *infinitesimally*, classical mechanics is to a large extent the theory of variational calculus. This *also* has a natural and useful formulation in homotopy theory and higher topos theory, namely the formulation that physicists know as *BV-formalism* (see e.g.[24]) and that mathematicians call *derived intersection theory* (e.g. [34]). This "derived" aspect of classical mechanics has found considerable attention in the literature in the last years (see for instance [**?**]). The higher global refinement that we discuss here however is complementary to this, and has not found due attention yet. Both aspects are complementary and naturally combine in "higher derived geometry", but to keep focus we do not particularly dwell much on the infinitesimal aspects here.

Notice that by far not all aspects of classical mechanics are local or even infinitesimal in nature. In particular the passage to genuine quantization – as opposed to just *formal* deformation quantization – crucially requires global structure to be specified. In the context of *geometric quantization* this extra global structure is traditionally called a *pre-quantization*, in order to reflect that it is more structure than typically considered in classical mechanics, but still just a pre-requisite to the actual quantization step. Following this, we will at times speak of our considerations here as being about pre-quantization. However, the distinction between "classical" and "pre-quantum" is not always clear cut. As we will demonstrate, pre-quantum mechanics is really just "classical mechanics globally done right". But the term "prequantum" may serve to emphasize the distinction to traditional but unduly local considerations.

The third purpose of the article, finally, is to provide the central motivating example for a research program that explores genuinely new territory, namely the *local* formulation of higher dimensional prequantum field theory, with *local* ("extended", "multi-tiered") understood and formalized as in the cobordism-hypothesis classification of local topological field theories [32].

Namely the cobordism theorem shows that local topological quantum field theories, including their boundary theories and generally their defect theories, are encoded by "de-transgressing" the spaces of quantum states assigned in codimension 1 to higher categorical analogs of spaces of states that are assigned in any codimension down to full codimension ("down to the point"). This formulation of local topological field theory is in effect a solution to the problem that physicists know as the "non-covariance of canonical quantization", namely the dependence of spaces of states in codimension one on choices of spatial (Cauchy-)hypersurfaces. In a "covariant" formulation of quantum field theory no such choices are involved, or else *all* such choices are considered coherently, equipped with a system of equivalences that relates them. This is what the cobordisms theorem effectively achieves for quantum field theories. For classical field theories we discuss this below in 3.

Notice then that most quantum field theories that we care about both in nature but also theoretically are not random examples of the axioms of quantum field theory (any of the sets of axioms) but are special in that they arise by a process of *quantization* from differential geometric data, namely the "classical" or "prequantum" data provided by spaces of fields equipped with action functionals. What has been missing is a "covariant" or "local" or "extended" or "multi-tiered" refinement of classical/prequantum field theory itself and a refinement of the quantization process that sends covariant/local/extended/multi-tiered prequantum field theories to covariant/local/extended/multi-tiered quantum field theories.

The description of classical mechanics that we provide here is meant to seamlessley provide such a refinement, namely "local prequantum field theory". This we discuss in a companion article [54], based on the discussion of higher prequantum geometry in [15]. Aspects of the general scheme of the resulting "local quantization" process have been discussed recently in [43]. A survey of the whole program is in [53].

For an introduction of the basic differential geometry needed here a nice textbook is [22]. For an introduction to all of the higher differential geometric concepts that we use here see [52, 50, 43]. A brief collection of pointers of the basic concepts of homotopy toposes and homotopy types is in A below. A discussion of the basic concepts of differential geoemtry in terms of the topos of smooth spaces is in B.

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2 Hamilton-Jacobi-Lagrange mechanics via prequantized Lagrangian correspondences

We show here how classical mechanics – Hamiltonian mechanics, Lagranian mechanics, Hamilton-Jacobi theory, see e.g. [1] – naturally arises from and is accurately captured by "pre-quantized Lagrangian correspondences". Since field theory is a refinement of classical mechanics, this serves also as a blueprint for the discussion of De Donder-Weyl-style classical field theory by higher correspondences below in 3, and more generally for the discussion of local prequantum field theory in [15, 43, 54].

The reader unfamiliar with classical mechanics may take the following to be a brief introduction to and indeed a systematic derivation of the central concepts of classical mechanics from the notion of correspondences in slice toposes. Conversely, the reader familiar with classical mechanics may take the translation of classical mechanics into correspondences in slice toposes as the motivating example for the formalization of prequantum field theory in [54]. The translation is summarized as a diagramatic dictionary below in 2.11.

The following sections all follow, in their titles, the pattern

Physical concept and *mathmatical formalization*

and each first recalls a naive physical concept, then motivates its mathematical formalization, then discusses this formalization and how it reflects back on the understanding of the physics.

Historical comment. Much of the discussion here is induced by just the notion of *pre-quantized Lagrangian correspondences*. The notion of plain Lagrangian correspondences (not pre-quantized) has been observed already in the early 1970s to usefully capture central aspects of Fourier transformation theory [25] and of classical mechanics [61], notably to unify the notion of Lagrangian subspaces of phase spaces with that of "canonical transformations", hence symplectomorphisms, betwee them. This observation has since been particularly advertized by Weinstein (e.g [62]), who proposed that some kind of *symplectic category* of symplectic manifolds with Lagrangian correspondences between them should be a good domain for a formalization of *quantization* along the lines of geometric quantization. Several authors have since discussed aspects of this idea. A recent review in the context of field theory is in [7].

But geometric quantization proper proceeds not from plain symplectic manifolds but from a lift of their symplectic form to a cocycle in differential cohomology, called a *pre-quantization* of the symplectic manifold. Therefore it is to be expected that some notion of pre-quantized Lagrangian correspondences, which put into correspondence these prequantum bundles and not just their underlying symplectic manifolds, is a more natural domain for geometric quantization, hence a more accurate formalization of pre-quantum geometry.

There is an evident such notion of prequantization of Lagrangian correspondences, and this is what we introduce and discuss in the following. While evident, it seems that it has previously found little attention in the literature, certainly not attention comparable to the fame enjoyed by Lagrangian correspondences. But it should. As we show now, classical mechanics globally done right is effectively identified with the study of prequantized Lagrangian correspondences.

2.1 Phase spaces and symplectic manifolds

Given a physical system, one says that its *phase space* is the space of its possible ("classical") histories or trajectories. Newton's second law of mechnanics says that trajectories of physical systems are (typically) determined by differential equations of *second* order, and therefore these spaces of trajectories are (typically) equivalent to initial value data of 0th and of 1st derivatives. In physics this data (or rather its linear dual) is referred to as the *canonical coordinates* and the *canonical momenta*, respectively, traditionally denoted by the symbols "q" and "p". Being coordinates, these are actually far from being canonical in the mathematical sense; all that has invariant meaning is, locally, the surface element $\mathbf{d}p \wedge \mathbf{d}q$ spanned by a change of coordinates and momenta.

Made precise, this says that a physical phase space is a sufficiently smooth manifold X which is equipped with a closed and non-degenerate differential 2-form $\omega \in \Omega^2_{\mathrm{cl}}(X)$, hence that phase spaces are *symplectic manifolds* (X, ω).

Example 2.1. The simplest nontrivial example is the phase space $\mathbb{R}^2 \simeq T^*\mathbb{R}$ of a single particle propagating on the real line. The standard coordinates on the plane are traditionally written $q, p \,:\, \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the symplectic form is the canonical volume form $\mathbf{d}q \wedge \mathbf{d}p$.

This is a special case of the following general and fundamental definition of *covariant phase spaces* (whose history is long and convoluted, two references being [64, 12]).

Example 2.2 (covariant phase space)**.** Let F be a smooth manifold – to be called the *field fiber* – and write $[\Sigma_1, F]$ for the manifold of smooth maps from the closed interval $\Sigma_1 := [0,1] \hookrightarrow \mathbb{R}$ into F (an infinite-dimensional Fréchet manifold). We think of F as a space of *spatial field configurations* and of $[\Sigma_1, F]$ as the space of *trajectories* or *histories* of spatial field configurations. Specifically, we may think of $[\Sigma_1, F]$ as the space of trajectories of a particle propagating in a space(-time) F.

A smooth function

$$L \,:\, [\Sigma_1, F] \longrightarrow \Omega^1(\Sigma_1)$$

to the space of differential 1-forms on Σ_1 is called a *local Lagrangian* of fields in F if for all $t \in \Sigma_1$ the assignment $\gamma \mapsto L_\gamma(t)$ is a smooth function of $\gamma(t), \dot\gamma(t), \ddot\gamma(t), \cdots$ (hence of the value of a curve $\gamma : \Sigma_1 \to F$ at t and of the values of all its derivatives at t). One traditionally writes

$$L \,:\, \gamma \mapsto L(\gamma, \dot\gamma, \ddot\gamma, \cdots) \wedge \mathbf{d}t$$

to indicate this. In cases of interest typically only first derivatives appear

$$L \,:\, \gamma \mapsto L(\gamma, \dot\gamma) \wedge \mathbf{d}t$$

and we concentrate on this case now for notational simplicity. Given such a local Lagrangian, the induced *local action functional* $S : [\Sigma_1, F] \to \mathbb{R}$ is the smooth function on trajectory space which is given by integrating the local Lagrangian over the interval:

$$S = \int_{\Sigma_1} L \,:\, [\Sigma_1, F] \xrightarrow{L} \Omega^1(\Sigma_1) \xrightarrow{\int_I} \mathbb{R}\,.$$

The *variational derivative* of the local Lagrangian is the smooth differential 2-form

$$\delta L \in \Omega^{1,1}([\Sigma_1, F] \times \Sigma_1)$$

on the product of trajectory space and parameter space, which is given by the expression

$$\begin{aligned} \delta L_\gamma &= \frac{\partial L}{\partial \gamma} \wedge \mathbf{d}t \wedge \delta\gamma + \frac{\partial L}{\partial \dot\gamma} \wedge \mathbf{d}t \wedge \frac{d}{dt}\delta\gamma \\ &= \underbrace{\left(\frac{\partial L}{\partial \gamma} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot\gamma}\right)}_{=:\mathrm{EL}_\gamma} \mathbf{d}t \wedge \delta\gamma + \frac{d}{dt} \underbrace{\left(\frac{\partial L}{\partial \dot\gamma} \wedge \delta\gamma\right)}_{=:\theta_\gamma} \mathbf{d}t \,. \end{aligned}$$

One says that $\mathrm{EL}_\gamma = 0$ (for all $t \in I$) is the *Euler-Lagrange equation of motion* induced by the local Lagrangian L, and that the 0-locus

$$X := \{\gamma \in [\Sigma_1, F] \mid \mathrm{EL}_\gamma = 0\} \hookrightarrow [\Sigma_1, F]$$

(also called the "shell") equipped with the 2-form

$$\omega := \delta\theta$$

is the *unreduced covariant phase space* (X, ω) induced by L.

Example 2.3. Consider the case that $F = \mathbb{R}$ and that the Lagrangian is of the form

$$\begin{aligned} L &:= L_{\mathrm{kin}} - L_{\mathrm{pot}} \\ &:= \left(\tfrac{1}{2}\dot{\gamma}^2 - V(\gamma)\right) \wedge \mathbf{d}t \end{aligned},$$

hence is a quadratic form on the first derivatives of the trajectory – called the *kinetic energy density* – plus any smooth function V of the trajectory position itself – called (minus) the *potential energy* density. Then the corresponding phase space is equivalent to $\mathbb{R}^2 \simeq T^*\mathbb{R}$ with the canonical coordinates identified with the initial value data

$$q := \gamma(0) \ , \ \ p = \dot{\gamma}$$

and with

$$\theta = p \wedge \mathbf{d}q$$

and hence

$$\omega = \mathbf{d}q \wedge \mathbf{d}p\,.$$

This is the phase space of example 2.1. Notice that the symplectic form here is a reflection entirely only of the kinetic action, independent of the potential action. This we come back to below in 2.9.

Remark 2.4. The differential 2-form ω on an unreduced covariant phase space in example 2.2 is closed, even exact, but in general far from non-degenerate, hence far from being symplectic. We may say that (X, ω) is a *pre-symplectic manifold*. This is because this differential form measures the reaction of the Lagrangian/action functional to variations of the fields, but the action functional may be *invariant* under some variation of the fields; one says that it has (*gauge-*)*symmetries*. To obtain a genuine symplectic form one needs to quotient out the flow of these symmetries from unreduced covariant phase space to obtain the *reduced* covariant phase space. This we turn to below in 2.7.

Remark 2.5. In the description of the mechanics of just particles, the Lagrangian L above has no further more fundamental description, it is just what it is. But in applications to n-dimensional *field* theory the differential 1-forms L and θ in example 2.2 arise themselves from integration of differential n-forms over space (Cauchy surfaces), hence from *transgression* of higher-degree data in higher codimension. This we describe in example 3.7 below. Since transgression in general loses some information, one should really work locally instead of integrating over Cauchy surfaces, hence work with the de-transgressed data and develop classical field theory for that. This we turn to below in 3 for classical field theory and then more generally for local prequantum field theory in [54].

2.2 Coordinate systems and the topos of smooth spaces

When dealing with spaces X that are equipped with extra structure, such as a closed differential 2-form $\omega \in \Omega^2_{\mathrm{cl}}(X)$, then it is useful to have a *universal moduli space* for these structures, and this will be central for our developments here. So we need a "smooth space" $\mathbf{\Omega}^2_{\mathrm{cl}}$ of sorts, characterized by the property that there is a natural bijection between smooth closed differential 2-forms $\omega \in \Omega^2_{\mathrm{cl}}(X)$ and smooth maps $X \longrightarrow \mathbf{\Omega}^2_{\mathrm{cl}}$. Of course such a universal moduli spaces of closed 2-forms does not exist in the category of smooth manifolds. But it does exist canonically if we slightly generalize the notion of "smooth space" suitably (the following is discussed in more detail below in B.2).

Definition 2.6. A *smooth space* or *smooth 0-type* X is

1. an assignment to each $n \in \mathbb{N}$ of a set, to be written $X(\mathbb{R}^n)$ and to be called the *set of smooth maps from* $\mathbb{R}^n$ *into* X,

2. an assignment to each ordinary smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ between Cartesian spaces of a function of sets $X(f) : X(\mathbb{R}^{n_2}) \to X(\mathbb{R}^{n_1})$, to be called the *pullback of smooth functions into* X *along* f;

such that

1. this assignment respects composition of smooth functions;

2. this assignment respect the covering of Cartesian spaces by open disks: for every good open cover $\{\mathbb{R}^n \simeq U_i \hookrightarrow \mathbb{R}^n\}_i$, the set $X(\mathbb{R}^n)$ of smooth functions out of $\mathbb{R}^n$ into X is in natural bijection with the set $\left\{(\phi_i)_i \in \prod_i X(U_i) \mid \forall_{i,j}\ \phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}\right\}$ of tuples of smooth functions out of the patches of the cover which agree on all intersections of two patches.

Remark 2.7. One may think of def. 2.6 as a formalization of the common idea in physics that we understand spaces by charting them with coordinate systems. A Cartesian space $\mathbb{R}^n$ is nothing but the standard n-dimensional coordinate system and one may think of the set $X(\mathbb{R}^n)$ above as the set of all possible ways (including all degenerate ways) of laying out this coordinate system in the would-be space X. Moreover, a function $f : \mathbb{R}^{n_1} \longrightarrow \mathbb{R}^{n_2}$ is nothing but a *coordinate transformation* (possibly degenerate), and hence the corresponding functions $X(f) : X(\mathbb{R}^{n_2}) \longrightarrow X(\mathbb{R}^{n_1})$ describe how the probes of X by coordinate systems change under coordinate transformations. Definition 2.6 takes the idea that any space in physics should be probe-able by coordinate systems in this way to the extreme, in that it *defines* a smooth spaces as a collection of probes by coordinate systems equipped with information about all possibe coordinate transformations.

The notion of smooth spaces is maybe more familiar with one little axiom added:

Definition 2.8. A smooth space X is called *concrete* if there exists a set $X_{\mathrm{disc}} \in \mathrm{Set}$ such that for each $n \in \mathbb{N}$ the set $X(\mathbb{R}^n)$ of smooth functions from $\mathbb{R}^n$ to X is a subset of the set of *all* functions from the underlying set of $\mathbb{R}^n$ to the set $X_{\mathrm{disc}} \in \mathrm{Set}$.

This definition of concrete smooth spaces (expressed slightly differently but equivalently) goes back to [8]. A comprehensive textbook account of differential geometry formulated with this definition of smooth spaces (called "diffeological spaces" there) is in [26].

While the formulation of def. 2.6 is designed to make transparent its geometric meaning, of course equivalently but more abstractly this says the following:

Definition 2.9. Write CartSp for the category of Cartesian spaces with smooth functions between them, and consider it equipped with the coverage (Grothendieck pre-topology), def. A.7, of good open covers. A *smooth space* or *smooth 0-type* is a sheaf on this site, def. A.10. The *topos of smooth 0-types* is the sheaf category, def. A.9,

$$\mathrm{Smooth0Type} := \mathrm{PSh}(\mathrm{CartSp})[\{\text{covering maps}\}^{-1}]\,.$$

In the following we will abbreviate the notation to

$$\mathbf{H} := \mathrm{Smooth0Type}\,.$$

For the discussion of presymplectic manifolds, we need the following two examples.

Example 2.10. Every smooth manifold $X \in \mathrm{SmoothManifold}$ becomes a smooth 0-type by the assignment

$$X : n \mapsto C^\infty(\mathbb{R}^n, X)\,.$$

(This defines in fact a concrete smooth space, def. 2.8, the underlying X_{disc} set being just the underlying set of points of the given manifold.) This construction extends to a full and faithful embedding of smooth manifolds into smooth 0-types

$$\mathrm{SmoothManifold} \hookrightarrow \mathbf{H}\;.$$

The other main example is in a sense at an opposite extreme in the space of all examples. It is given by smooth moduli space of *differential forms*. More details on this are below in B.3.

Example 2.11. For $p \in \mathbb{N}$, write $\mathbf{\Omega}^p_{\mathrm{cl}}$ for the smooth space given by the assignment

$$\mathbf{\Omega}^p_{\mathrm{cl}} : n \mapsto \Omega^p_{\mathrm{cl}}(\mathbb{R}^n)$$

and by the evident pullback maps of differential forms. Thes smooth spaces $\mathbf{\Omega}^n_{\mathrm{cl}}$ are *not* concrete, def. 2.8. In fact they are maximally non-concrete in that there is only a single smooth map $* \to \mathbf{\Omega}^n_{\mathrm{cl}}$ from the point into them. Hence the underlying point set of the smooth space $\mathbf{\Omega}^n_{\mathrm{cl}}$ looks like a singleton, and yet these smooth spaces are far from being the trivial smooth space: they admit many smooth maps $X \longrightarrow \mathbf{\Omega}^n_{\mathrm{cl}}$ from smooth manifolds of dimension at least n, as the following prop. 2.12 shows.

This solves the moduli problem for closed smooth differential forms:

Proposition 2.12. *For* $p \in \mathbb{N}$ *and* $X \in \mathrm{SmoothManifold} \hookrightarrow \mathrm{Smooth0Type}$*, there is a natural bijection*

$$\mathbf{H}(X, \mathbf{\Omega}^p_{\mathrm{cl}}) \simeq \Omega^p_{\mathrm{cl}}(X)\,.$$

So a presymplectic manifold (X, ω) is equivalently a map of smooth spaces of the form

$$\omega : \; X \longrightarrow \mathbf{\Omega}^2_{\mathrm{cl}} \;\; .$$

2.3 Canonical transformations and Symplectomorphisms

An equivalence between two phase spaces, hence a re-expression of the "canonical" coordinates and momenta, is called a *canonical transformation* in physics. Mathematically this is a *symplectomorphism*:

Definition 2.13. Given two (pre-)symplectic manifolds $(X_1, \omega - 1)$ and (X_2, ω) a *symplectomorphism*

$$f : (X_1, \omega_1) \longrightarrow (X_2, \omega_2)$$

is a diffeomorphism $f : X_1 \longrightarrow X_2$ of the underlying smooth spaces, which respects the differential forms in that

$$f^* \omega_2 = \omega_1\,.$$

The formulation above in 2.2 of pre-symplectic manifolds as maps into a moduli space of closed 2-forms yields the following equivalent re-formulation of symplectomorphisms, which is very simple in itself, but contains in it the seed of an important phenomenon:

Proposition 2.14. *Given two symplectic manifolds* (X_1, ω_1) *and* (X_2, ω_2)*, a symplectomorphism* $\phi : (X_1, \omega_1) \to (X_2, \omega_2)$ *is equivalently a commuting diagram of smooth spaces of the following form:*

$$\begin{tikzcd} X_1 \arrow[rr, "\phi"] \arrow[dr, "\omega_1"'] & & X_2 \arrow[dl, "\omega_2"] \\ & \mathbf{\Omega}^2_{\mathrm{cl}} & \end{tikzcd} \quad .$$

Situations like this are naturally interpreted in the *slice topos*:

Definition 2.15. For $A \in \mathbf{H}$ any smooth space, the *slice topos* $\mathbf{H}_{/A}$ is the category whose objects are objects $X \in \mathbf{H}$ equipped with maps $X \to A$, and whose morphisms are commuting diagrams in $\mathbf{H}$ of the form

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & A & \end{tikzcd} \quad .$$

Hence if we write SymplManifold for the category of smooth pre-symplectic manifolds and symplectomorphisms betwen them, then we have the following.

Proposition 2.16. *The construction of prop. 2.12 constitutes a full embedding*

$$\mathrm{SymplManifold} \hookrightarrow \mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}$$

of pre-symplectic manifolds with symplectomorphisms between them into the slice topos of smooth spaces over the smooth moduli space of closed differential 2-forms.

2.4 Trajectories and Lagrangian correspondences

A symplectomorphism clearly puts two symplectic manifolds "in relation" to each other. It turns out to be useful to say this formally. Recall:

Definition 2.17. For $X, Y \in \mathrm{Set}$ two sets, a relation R between elements of X and elements of Y is a subset of the Cartesian product set

$$R \hookrightarrow X \times Y\,.$$

More generally, for $X, Y \in \mathbf{H}$ two objects of a topos (such as the topos of smooth spaces), then a relation R between them is a subobject of their Cartesian product

$$R \hookrightarrow X \times Y\ .$$

In particular any function induces the relation "y is the image of x":

Example 2.18. For $f : X \longrightarrow Y$ a function, its *induced relation* is the relation which is exhibited by *graph* of f

$$\mathrm{graph}(f) := \{(x,y) \in X \times Y \mid f(x) = y\}$$

canonically regarded as a subobject

$$\mathrm{graph}(f) \hookrightarrow X \times Y\,.$$

Hence in the context of classical mechanics, in particular any symplectomorphism $f : (X_1, \omega_1) \longrightarrow (X_2, \omega_2)$ induces the relation

$$\mathrm{graph}(f) \hookrightarrow X_1 \times X_2\,.$$

Since we are going to think of f as a kind of"physical process", it is useful to think of the smooth space *graph*(f) here as the *space of trajectories* of that process. To make this clearer, notice that we may equivalently rewrite every relation $R \hookrightarrow X \times Y$ as a diagram of the following form:

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \quad = \quad \begin{tikzcd} & R \arrow[d] & \\ & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

reflecting the fact that every element $(x \sim y) \in R$ defines an element $x = i_X(x \sim y) \in X$ and an element $y = i_Y(x \sim y) \in Y$.

Then if we think of the space $R = \mathrm{graph}(f)$ of example 2.18 as being a space of trajectories starting in X_1 and ending in X_2, then we may read the relation as "there is a trajectory from an incoming configuration x_1 to an outgoing configuration x_2":

$$\begin{tikzcd} & \mathrm{graph}(f) \arrow[dl, "\mathrm{incoming}"'] \arrow[dr, "\mathrm{outgoing}"] & \\ X_1 & & X_2 \end{tikzcd} \quad .$$

Notice here that the defining property of a relation as a subset/subobject translates into the property of classical physics that there is *at most one trajectory* from some incoming configuration x_1 to some outgoing trajectory x_2 (for a fixed and small enough parameter time interval at least, we will formulate this precisely in the next section when we genuinely consider Hamiltonian correspondences).

In a more general context one could consider there to be several such trajectories, and even a whole smooth space of such trajectories between given incoming and outgoing configurations. Each such trajectory would "relate" x_1 to x_2, but each in a possible different way. We can also say that each trajectory makes x_1 *correspond* to x_2 in a different way, and that is the mathematical term usually used:

Definition 2.19. For $X, Y \in \mathbf{H}$ two spaces, a correspondence between them is a diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

with no further restrictions. Here Z is also called the *correspondence space*.

Observe that the graph of a function $f\colon X \to Y$ is, while defined differently, in fact equivalent to just the space X, the equivalence being induced by the map $x \mapsto (x, f(x))$

$$X \xrightarrow{\simeq} \mathrm{graph}(f)\,.$$

In fact the relation/correspondence which expresses "y is the image of f under x" may just as well be exhibited by the diagram

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}\,.$$

It is clear that this correspondence with correspondence space X should be regarded as being equivalent to the one with correspondence space $\mathrm{graph}(f)$. We may formalize this as follows

Definition 2.20. Given two correspondences $X \longleftarrow Z_1 \longrightarrow Y$ and $X \longleftarrow Z_2 \longrightarrow Y$ between the same objects in $\mathbf{H}$, then an equivalence between them is an equivalence $Z_1 \xrightarrow{\simeq} Z_2$ in $\mathbf{H}$ which fits into a commuting diagram of the form

$$\begin{tikzcd} & Z_1 \arrow[dl] \arrow[dr] \arrow[dd, "\simeq"] & \\ X & & Y \\ & Z_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Example 2.21. Given an function $f\colon X \longrightarrow Y$ we have the commuting diagram

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] \arrow[dd, "\simeq"] & \\ X & & Y \\ & \mathrm{graph}(f) \arrow[ul, "i_X"] \arrow[ur, "i_Y"'] & \end{tikzcd}$$

exhibiting an equivalence of the correspondence at the top with that at the bottom.

Correspondences between X any Y with such equivalences between them form a *groupoid*. Hence we write

$$\mathrm{Corr}\,(\mathbf{H})\,(X,Y) \in \mathrm{Grpd}\,.$$

Moreover, if we think of correspondences as modelling spaces of trajectories, then it is clear that their should be a notion of composition:

$$\left(\begin{tikzcd} & Y_1 \arrow[dl] \arrow[dr] & & Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 & & X_3 \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} & Y_1 \circ_{X_2} Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_3 \end{tikzcd}\right).$$

Heuristically, the composite space of trajectories $Y_1 \circ_{X_2} Y_2$ should consist precisely of those pairs of trajectories $(f,g) \in Y_1 \times Y_2$ such that the endpoint of f is the starting point of g. The space with this property is precisely the *fiber product* of Y_1 with Y_2 over X_2, denoted $Y_1 \underset{X_2}{\times} Y_2$ (also called the *pullback* of $Y_2 \longrightarrow X_2$ along $Y_1 \longrightarrow X_2$:

$$\left(\begin{tikzcd} & Y_1 \circ_{X_2} Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_3 \end{tikzcd}\right) = \left(\begin{tikzcd} & & Z_1 \underset{Y}{\times} Z_2 \arrow[dl] \arrow[dr] & & \\ & Z_1 \arrow[dl] \arrow[dr] & & Z_2 \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}\right).$$

Hence given a topos $\mathbf{H}$, correspondences between its objects form a category which composition the fiber product operation, where however the collection of morphisms between any two objects is not just a set, but is a groupoid (the groupoid of correspondences between two given objects and equivalences between them).

One says that correspondences form a $(2,1)$-*category*

$$\mathrm{Corr}(\mathbf{H}) \in (2,1)\mathrm{Cat}\,.$$

One reason for formalizing this notion of correspondences so much in the present context that it is useful now to apply it not just to the ambient topos $\mathbf{H}$ of smooth spaces, but also to its slice topos $\mathbf{H}_{/\mathbf{\Omega}^2_{cl}}$ over the universal moduli space of closed differential 2-forms.

To see how this is useful in the present context, notice the following

Proposition 2.22. *Let $\phi : (X_1, \omega_1) \to (X_2, \omega_2)$ be a symplectomorphism. Write*

$$(i_1, i_2) : \mathrm{graph}(\phi) \hookrightarrow X_1 \times X_2$$

for the graph of the underlying diffeomorphsm. This fits into a commuting diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} & \mathrm{graph}(\phi) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X_1 \arrow[dr, "\omega_1"'] \arrow[rr, phantom, "="] & & X_2 \arrow[dl, "\omega_2"] \\ & \Omega^2_{\mathrm{cl}} & \end{tikzcd}.$$

Conversely, a smooth function $\phi : X_1 \to X_2$ is a symplectomorphism precisely if its graph makes the above diagram commute.

Traditionally this is formalized as follows.

Definition 2.23. Given a symplectic manifold (X, ω), a submanifold $L \hookrightarrow X$ is called a *Lagrangian submanifold* if $\omega|_L = 0$ and if L has dimension $dim(L) = dim(X)/2$.

Definition 2.24. For (X_1, ω_1) and (X_2, ω_2) two symplectic manifolds, a correspondence $X_1 \xleftarrow{p_1} Y \xrightarrow{p_2} X_2$ of the underlying manifolds is a *Lagrangian correspondence* if the map $Y \to X_1 \times X_2$ exhibits a Lagrangian submanifold of the symplectic manifold given by $(X_1 \times X_2, p_2^*\omega_2 - p_1^*\omega_1)$.

Given two Lagrangian correspondence which intersect transversally over one adjacent leg, then their *composition* is the correspondence given by the intersection.

But comparison with def. 2.15 shows that Lagrangian correspondences are in fact plain correspondences, just not in smooth spaces, but in the slice $\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}$ of all smooth spaces over the universal smooth moduli space of closed differential 2-forms:

Proposition 2.25. *Under the identification of prop. 2.16 the construction of the diagrams in prop. 2.22 constitutes an injection of Lagrangian correspondence between (X_1, ω_1) and (X_2, ω_2) into the Hom-space* $\mathrm{Corr}\left(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}\right)((X_1, \omega_1), (X_2, \omega_2))$*. Moreover, composition of Lagrangian correspondence, when defined, coincides under this identification with the composition of the respective correspondences.*

Remark 2.26. The composition of correspondences in the slice topos is always defined. It may just happen the composite is given by a correspondence space which is a smooth space but not a smooth manifold. Or better, one may replace in the entire discussion the topos of smooth spaces with a topos of "derived" smooth spaces, modeled not on Cartesian spaces but on Cartesian dg-manifolds. This will then automatically make composition of Lagrangian correspondences take care of "transversal perturbations". Here we will not further dwell on this possibility. In fact, the formulation of Lagrangian correspondences and later of prequantum field theory by correspondences in toposes implies a great freedom in the choice of type of geometry in which set up everything. (The bare minimum condition on the topos $\mathbf{H}$ which we need to require is that it be *differentially cohesive* [50]).

It is also useful to make the following phenomenon explicit, which is the first incarnation of a recurring theme in the following discussions.

Proposition 2.27. *The category* $\mathrm{Corr}(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}})$ *is naturally a symmetric monoidal category, where the tensor product is given by*

$$(X_1, \omega_1) \otimes (X_2, \omega_2) = (X_1 \times X_2, \omega_1 + \omega_2) .$$

The tensor unit is $(*, 0)$. With respect to this tensor product, every object is dualizable, with dual object given by

$$(X, \omega)^v = (X, -\omega) .$$

Remark 2.28. Duality induces natural equivalences of the form

$$\mathrm{Corr}\left(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}\right)((X_1, \omega_1), (X_2, \omega_2),) \xrightarrow{\simeq} \mathrm{Corr}\left(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}\right)((*, 0), (X_1 \times X_2, \omega_2 - \omega_1),) .$$

Under this equivalence an isotropic (Lagrangian) correspondences which in $\mathbf{H}$ is given by a diagram as in prop. 2.22 maps to the diagram of the form

$$\begin{tikzcd}
 & \mathrm{graph}(\phi) \arrow[dl] \arrow[dr, "{(i_1, i_2)}"] & \\
* \arrow[dr, "0"'] \arrow[rr, phantom, "="] & & X_1 \times X_2 \arrow[dl, "\omega_2 - \omega_1"] \\
 & \mathbf{\Omega}^2_{\mathrm{cl}} &
\end{tikzcd} .$$

This makes the condition that the pullback of the difference $\omega_2 - \omega_1$ vanishes on the correspondence space more manifest. It is also the blueprint of a phenomenon that is important in the generalization to field theory in the sections to follow, where trajectories map to boundary conditions, and vice versa.

2.5 Observables, symmetries and the Poisson bracket Lie algebra

Given a phase space (X, ω) of some physical system, then a function $O : X \longrightarrow \mathbb{R}$ is an assignment of a value to every possible state (phase of motion) of that system. For instance it might assign to every phase of motion its position (measured in some units with respect to some reference frame), or its momentum, or its energy. The premise of classical physics is that all of these quantitites may in principle be observed in experiment, and therefore functions on phase space are traditionally called *classical observables*. Often this is abbreviated to just *observables* if the context is understood (the notion of observable in quantum mechanics and quantum field theory is more subtle, for a formalization of quantum observables in terms of correspondences in cohesive homotopy types see [43]).

While this is the immediate physics heuristics about what functions on phase space are are, it turns out that a central characteristic of mechanics and of field theory is an intimiate relation between the observables of a mechanical system and its *infinitesimal symmetry transformations*: an infinitesimal symmetry transformation of a phase space characterizes that observable of the system which is invariant under the symmetry transformation. Mathematically this relation is captured by a the structure of a Lie algebra on the vector space of all observables after relating them them to their *Hamiltonian vector fields*.

Definition 2.29. Given a symplectic manifold (X, ω) and a function $H : X \to \mathbb{R}$, its *Hamiltonian vector field* is the unique $v \in \Gamma(TX)$ which satisfies *Hamilton's equation of motion*

$$\mathbf{d}H = \iota_v \omega \,.$$

Example 2.30. For $(X, \omega) = (\mathbb{R}^2, \mathbf{d}q \wedge \mathbf{d}p)$ the 2-dimensional phase space form example 2.1, and for $t \mapsto (q(t), p(t)) \in X$ a curve, it is a Hamiltonian flow line if its tangent vectors $(\dot{q}(t), \dot{p}(t)) \in T_{(q(t),p(t))}\mathbb{R}^2 \simeq \mathbb{R}^2$ satisfy Hamilton's equations in the classical form:

$$\dot{q} = \frac{\partial H}{\partial p} \, ; \quad \dot{p} = -\frac{\partial H}{\partial q} \,.$$

Proposition 2.31. *Given a symplectic manifold (X, ω), every Hamiltonian vector field v is an* infinitesimal symmetry *of (X, ω) – an* infinitesimal symplectomorphism *– in that the Lie derivative of the symplectic form along v vanishes*

$$\mathcal{L}_v \omega = 0 \,.$$

Proof. Using Cartan's formula for the Lie derivative

$$\mathcal{L}_v = \mathbf{d} \circ \iota_v + \iota_v \circ \mathbf{d}$$

and the defining condition that the symplectic form is closed and that there is a function H with $\mathbf{d}H = \iota_v \omega$, one finds that the Lie derivative of ω along v is given by

$$\mathcal{L}_v \omega = \mathbf{d}\iota_v \omega + \iota_v \mathbf{d}\omega = \mathbf{d}^2 H = 0 \,.$$

□

Since infinitesimal symmetries should form a Lie algebra, this motivates the following definition.

Definition 2.32 (Poisson bracket for symplectic manifolds)**.** Let (X, ω) be a symplectic manifold. Given two functions $f, g \in C^\infty(X)$ with Hamiltonian vector fields v and w, def. 2.29, respectively, their *Poisson bracket* is the function obtained by evaluating the symplectic form on these two vector fields

$$\{f, g\} := \iota_w \iota_v \omega \,.$$

This operation

$$\{-,-\} : C^\infty(X) \otimes C^\infty(X) \longrightarrow C^\infty(X)$$

is skew symmetric and satisfies the Jacobi identity. Therefore

$$\mathfrak{pois}(X, \omega) := (C^\infty(X), \{-,-\})$$

is a Lie algebra (infinite dimensional in general), called the *Poisson bracket Lie algebra of classical observables* of the symplectic manifold X.

Remark 2.33. Below in 2.12 we indicate a general abstract characerization of the Poisson bracket Lie algebra (which is discussed in moreo detail below in D.3): it is the Lie algebra of "the automorphism group of any prequantization of (X, ω) in the higher slice topos over the moduli stack of circle-principal connections" [15]. To state this we first need the notion of *pre-quantization* which we come to below in 2.9. In the notation introduced there we will discuss in 2.12 that the Poisson bracket is given as

$$\mathfrak{pois}(X,\omega) = \mathrm{Lie}\left(\mathbf{Aut}_{/\mathbf{B}U(1)_{\mathrm{conn}}}(\nabla)\right) = \left\{ \begin{tikzcd} X \arrow[rr, "\simeq"] \arrow[dr, "\nabla"'] & \arrow[d, phantom, "\swArrow"] & X \arrow[dl, "\nabla"] \\ & \mathbf{B}U(1)_{\mathrm{conn}} & \end{tikzcd} \right\},$$

where ∇ denotes a pre-quantization of (X, ω).

This general abstract construction makes sense also for pre-symplectic manifolds and shows that the following slight generalization of the above traditonal definition is good and useful.

Definition 2.34 (Poisson bracket for pre-symplectic manifolds)**.** For (X, ω) a pre-symplectic manifold, denote by $\mathfrak{pois}(X, \omega)$ the Lie algebra whose underlying vector space is the space of pairs of Hamiltonians H with a *choice* of Hamiltonian vector field v

$$\{(v, H) \in \Gamma(TX) \otimes C^\infty(X) \mid \iota_v \omega = \mathbf{d}H\} \,,$$

and whose Lie bracket is given by

$$[(v_1, H_1), (v_2, H_2)] = ([v_1, v_2], \, \iota_{v_1 \wedge v_2} \omega) \,.$$

Remark 2.35. On a smooth manifold X there is a bijection between smooth vector fields and derivations of the algebra $C^\infty(X)$ of smooth functions, given by identifying a vector field v with the operation $v(-)$ of differentiating functions along v. Under this identification the Hamiltonian vector field v corresponding to a Hamiltonian H is identified with the derivation given by forming the Poisson bracket with H:

$$v(-) = \{H, -\} \;:\; C^\infty(X) \longrightarrow C^\infty(X) \,.$$

In applications in physics, given a phase space (X, ω) typically one smooth function $H : X \longrightarrow \mathbb{R}$, interpreted as the energy observable, is singled out and called *the* Hamiltonian. Its corresponding Hamiltonian vector field is then interpreted as giving the infinitesimal time evolution of the system, and this is where Hamilton's equations in def. 2.29 originate.

Definition 2.36. Given a phase space with Hamiltonian $((X, \omega), H)$, then any other classical $O \in C^\infty(X)$, it is called an *infinitesimal symmetry* of $((X, \omega), H)$ if the Hamiltonian vector field v_O of O preserves not just the symplectic form (as it automatically does by prop. 2.31) but also the given Hamiltonian, in that $\iota_{v_O} \mathbf{d} H = 0$.

Proposition 2.37 (symplectic Noether theorem)**.** *If a Hamiltonian vector field v_O is an infinitesimal symmetry of a phase space (X, ω) with time evolution H according to def. 2.36, then the corresponding Hamiltonian function $O \in C^\infty(X)$ is a conserved quantity along the time evolution, in that*

$$\iota_{v_H} \mathbf{d} O = 0 \,.$$

Conversely, if a function $O \in C^\infty(X)$ is preserved by the time evolution of a Hamiltonian H in this way, then its Hamiltonian vector field v_O is an infinitesimal symmetry of $((X, \omega), H)$.

Proof. This is immediate from the definition 2.29:

$$\begin{aligned}
\iota_{v_H} \mathbf{d} O &= \iota_{v_H} \iota_{v_O} \omega \\
&= -\iota_{v_O} \iota_{v_H} \omega \,. \\
&= \iota_{v_O} \mathbf{d} H
\end{aligned}$$

□

Remark 2.38. The utter simplicity of the proof of prop. 2.37 is to be taken as a sign of the power of the symplectic formalism in the formalization of physics, not as a sign that the statement itself is shallow. On the contrary, under a Legendre transform and passage from "Hamiltonian mechanics" to "Lagrangian mechanics" that we come to below in 2.11, the identification of symmetries with preserved observables in prop. 2.11 becomes the seminal *first Noether theorem*. See for instance [4] for a review of the Lagrangian Noether theorem and its symplectic version in the context of classical mechanics. Below in 3.2 we observe that the same holds true also in the full context of classical field theory, if only one refines Hamiltonian mechanics to its localization by Hamilton-de Donder-Weyl field theory. The full *n-plectic Noether theorem* (for all field theory dimensions n) is prop. 3.24 below.

In the next section we pass from infinitesimal Hamiltonian flows to their finite version, the Hamiltonian symplectomorphism.

2.6 Hamiltonian (time evolution) trajectories and Hamiltonian correspondences

We have seen so far transformations of phase space given by "canonical transformations", hence symplectomorphisms. Of central importance in physics are of course those transformations that are part of a smooth evolution group, notably for time evolution. These are the "canonical transformations" coming from a generating function, hence the symplectomorphisms which come from a Hamiltonian function (the energy function, for time evolution), the *Hamiltonian symplectomorphisms*. Below in 2.10 we see that this notion is implied by prequantizing Lagrangian correspondences, but here it is good to recall the traditional definition.

Definition 2.39. The flow of a Hamiltonian vector field is called the corresponding *Hamiltonian flow*.

Notice that by prop. 2.31 we have

Proposition 2.40. *Every Hamiltonian flow is a symplectomorphism.*

Those symplectomorphisms arising this way are called the *Hamiltonian symplectomorphisms*. Notice that the Hamiltonian symplectomorphism depends on the Hamiltonian only up to addition of a locally constant function.

Using the Poisson bracket $\{-,-\}$ induced by the symplectic form ω, identifying the derivation $\{H,-\}$: $C^\infty(X) \longrightarrow C^\infty(X)$ with the corresponding Hamiltonian vector field v by remark 2.35 and the exponent notation $\exp(t\{H,-\})$ with the Hamiltonian flow for parameter "time" $t \in \mathbb{R}$, we may write these Hamiltonian symplectomorphisms as

$$\exp(t\{H,-\}) \,:\, (X,\omega) \longrightarrow (X,\omega)\,.$$

It then makes sense to say that

Definition 2.41. A Lagrangian correspondence, def. 2.24, which is induced from a Hamiltonian symplectomorphism is a *Hamiltonian correspondences*

$$\left(\begin{tikzcd} & \mathrm{graph}\left(\exp\left(t\left\{H,-\right\}\right)\right) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X & & X \end{tikzcd} \right) \simeq \left(\begin{tikzcd} & X \arrow[dl, "="'] \arrow[dr, "\exp(t\{H,-\})"] & \\ X & & X \end{tikzcd} \right).$$

Remark 2.42. The smooth correspondence space of a Hamiltonian correspondence is naturally identified with the space of *classical trajectories*

$$\mathrm{Fields}^{\mathrm{class}}_{\mathrm{traj}}(t) := \mathrm{graph}\left(\exp(t)\{H,-\}\right)$$

in that

1. every point in the space corresponds uniquely to a trajectory of parameter time length t characterized as satisfying the equations of motion as given by Hamilton's equations for H;

2. the two projection maps to X send a trajectory to its initial and to its final configuration, respectively.

group structure is

Remark 2.43. By construction, Hamiltonian flows form a 1-parameter Lie group. By prop. 2.25 this group structure is preserved by the composition of the induced Hamiltonian correspondences.

It is useful to highlight this formally as follows.

Definition 2.44. Write $\mathrm{Bord}_1^{\mathrm{Riem}}$ for the category of 1-dimensional cobordisms equipped with Riemannian structure (hence with a real, non-negative length which is additive under composition), regarded as a symmetric monoidal category under disjoint union of cobordisms.

Then:

Proposition 2.45. *The Hamiltonian correspondences induced by a Hamiltonian function $H \,:\, X \longrightarrow \mathbb{R}$ are equivalently encoded in a smooth monoidal functor of the form*

$$\exp((-)\{H,-\}) \,:\, \mathrm{Bord}_1^{\mathrm{Riem}} \longrightarrow \mathrm{Corr}_1(\mathbf{H}_{/\Omega^2})\,,$$

where on the right we use the monoidal structure on correspondence of prop. 2.27.

Below the general discussion of prequantum field theory, such monoidal functors from cobordisms to correspondences of spaces of field configurations serve as the fundamental means of axiomatization. Whenever one is faced with such a functor, it is of particular interest to consider its value on *closed* cobordisms. Here in the 1-dimensional case this is the circle, and the value of such a functor on the circle would be called its (pre-quantum) *partition function.*

Proposition 2.46. *Given a phase space symplectic manifold (X, ω) and a Hamiltonian $H : X \longrightarrow \mathbb{R}$, them the prequantum evolution functor of prop. 2.45 sends the circle of circumference t, regarded as a cobordism from the empty 0-manifold to itself*

$$\begin{tikzcd} & S^1 & \\ \emptyset \arrow[ur, hook] & & \emptyset \arrow[ul, hook'] \end{tikzcd}$$

and equipped with the constant Riemannian metric of 1-volume t, to the correspondence

$$\begin{tikzcd} & \{x \in X \,|\, \exp(t\{H, -\})(x) = x\} \arrow[dl] \arrow[dr] & \\ * & & * \end{tikzcd}$$

which is the smooth space of H-Hamiltonian trajectories of (time) length t that are closed, hence that come back to their initial value, regarded canonically as a correspondence form the point to itself.

Proof. We can decompose the circle of length t as the compositon of

1. The coevaluation map on the point, regarded as a dualizable object $\mathrm{Bord}_1^{\mathrm{Riem}}$;
2. the interval of length t;
3. the evaluation map on the point.

The monoidal functor accordingly takes this to the composition of correspondences of

1. the coevaluation map on X, regarded as a dualizable object in $\mathrm{Corr}(\mathbf{H})$;
2. the Hamiltonian correspondence induced by $\exp(t\{H, -\})$;
3. the evaluation map on X.

As a diagram in $\mathbf{H}$, this is the following:

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta"] & & \mathrm{graph}(\exp(t\{H,-\})) \times X \arrow[dl] \arrow[dr] & & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ * & & X \times X & & X \times X & & * \end{tikzcd} .$$

By the definition of composition in $\mathrm{Corr}(\mathbf{H})$, the resulting composite correspondence space is the joint fiber product in $\mathbf{H}$ over these maps. This is essentially verbatim the diagrammatic definition of the space of closed trajectories of parameter length t. □

2.7 Noether symmetries and equivariant structure

So far we have considered smooth spaces equipped with differential forms, and correspondences between these. To find genuine classical mechanics and in particular find the notion of prequantization, we need to bring the notion of *gauge symmetry* into the picture. We introduce here symmetries in classical field theory following Noether's seminal analysis and then point out the crucial notion of *equivariance* of symplectic potentials necessary to give this global meaning. Below in 2.8 we see how building the *reduced phase space* by taking the symmetries into account makes the first little bit of "higher differential geometry" appear in classical field theory.

Definition 2.47. Given a local Lagrangian as in example 2.2 A *symmetry* of L is a vector field $v \in \Gamma(TPX)$ such that $\iota_v \delta L = 0$. It is called a *Hamiltonian symmetry* if restricted to phase space v is a Hamiltonian vector field, in that the contraction $\iota_v \omega$ is exact.

By definition of θ and EL in example 2.2, it follows that for v a symmetry, the 0-form

$$J_v := \iota_v \theta$$

is closed with respect to the time differential

$$\mathbf{d}_t J_v = 0\,.$$

Definition 2.48. The function J_v induced by a symmetry v is called the *conserved Noether charge* of v.

Example 2.49. For $Y = \mathbb{R}$ and $L = \frac{1}{2}\dot{\gamma}^2 \mathbf{d}t$ the vector field v tangent to the flow $\gamma \mapsto \gamma((-) + a)$ is a symmetry. This is such that $\iota_v \delta\gamma = \dot{\gamma}$. Hence the conserved quantity is $E := J_v = \dot{\gamma}^2$, the energy of the system. It is also a Hamiltonian symmetry.

Let then G be the group of Hamiltonian symmetries acting on $(\{\mathrm{EL} = 0\}, \omega = \delta\theta)$. Write $\mathfrak{g} = \mathrm{Lie}(G)$ for the Lie algebra of the Lie group. Given $v \in \mathfrak{g} = \mathrm{Lie}(G)$ identify it with the corresponding Hamiltonian vector field. Then it follows that the Lie derivative of θ is exact, hence that for every v one can find an h such that

$$\mathcal{L}_v \theta = \mathbf{d}h\,.$$

The choice of h here is a choice of identification that relates the phase space potential θ to itself under a different but equivalent perspective of what the phase space points are. Such choices of "gauge equivalences" are necessary in order to give the (pre-)symplectic form on the unreduced phase space an physical meaning in view of the symmetries of the system. Moreover, what is really necessary for this is a coherent choice of such gauge equivalences also for the "global" or "large" gauge transformations that may not be reached by exponentiating Lie algebra elements of the symmetry group G. Such a coherent choice of gauge equivalences on θ reflecting the symmetry of the physical system is mathematically called a *G-equivariant structure*.

Definition 2.50. Given a smooth space X equipped with the action $\rho : X \times G \longrightarrow X$ of a smooth group, and given a differential 1-form $\theta \in \Omega^1(X)$, and finally given a discrete subgroup $\Gamma \hookrightarrow \mathbb{R}$, then a *$G$-equivariant structure* on θ regarded as a $(\mathbb{R}/\Gamma)$-principal connection is

- for each $g \in G$ an equivalence

$$\eta_g :\ \theta \xrightarrow{\simeq} \rho(g)^*\theta$$

between θ and the pullback of θ along the action of g, hence a smooth function $\eta_g \in C^\infty(X, \mathbb{R}/\Gamma)$ with

$$\rho(g)^*\theta - \theta = \mathbf{d}\eta_g$$

such that

1. the assignment $g \mapsto \eta_g$ is smooth;
2. for all pairs $(g_1, g_2) \in G \times G$ there is an equality

$$\eta_{g_2}\eta_{g_1} = \eta_{g_2 g_1}\,.$$

Remark 2.51. Notice that the condition $\rho(g)^*\theta - \theta = \mathbf{d}\eta_g$ depends on η_g only modulo elements in the discrete group $\Gamma \hookrightarrow \mathbb{R}$, while the second condition $\eta_{g_2}\eta_{g_1} = \eta_{g_2 g_1}$ crucially depends on the actual representatives in $C^\infty(X, \mathbb{R}/\Gamma)$. For Γ the trivial group there is no difference, but in general it is unlikely that in this case the second condition may be satisfied. The second condition can in general only be satisfied modulo some subgroup of $\mathbb{R}$. Essentially the only such which yields a regular quotient is $\mathbb{Z} \hookrightarrow \mathbb{R}$ (or any non-zero rescaling of this), in which case

$$\mathbb{R}/\mathbb{Z} \simeq U(1)$$

is the circle group. This is the origin of the central role of *circle principal bundles* in field theory ("prequantum bundles"), to which we come below in 2.9.

The point of G-equivariant structure is that it makes the (pre-)symplectic potential θ "descend" to the quotient of X by G (the "correct quotient", in fact), which is the *reduced phase space*. To say precisely what this means, we now introduce the concept of smooth groupoids in 2.8.

Remark 2.52. This equivariance on local Lagrangian is one of the motivations for refining the discussion here to *local prequantum field theory* in [54]: By remark 2.5 for a genuine n-dimensional field theory, the Lagrangian 1-form L above is the transgression of an n-form Lagrangian on a moduli space of fields. In local prequantum field theory we impose an equivariant structure already on this de-transgressed n-form Lagrangian such that under transgression it then induces equivariant structures in codimension 1, and hence consistent phase spaces, in fact consistent prequantized phase spaces.

2.8 Gauge theory, smooth groupoids and higher toposes

The *gauge principle* is a deep principle of modern physics, which says that in general two configurations of a physical system may be nominally different and still be identified by a *gauge equivalence* between them. In homotopy type theory precisely this principle is what is captured by *intensional identity types* (see remark A.5). One class of example of such gauge equivalences in physics are the Noether symmetries induced by local Lagrangians which we considered above in 2.7. Gauge equivalences can be composed (and associatively so) and can be inverted. All physical statements respect this gauge equivalence, but it is wrong to identify gauge equivalent field configurations and pass to their sets of equivalence classes, as some properties depend on non-trivial auto-gauge transformations.

In mathematical terms what this says is precisely that field configurations and gauge transformations between them form what is called a *groupoid* or *homotopy 1-type*.

Definition 2.53. A *groupoid* $\mathcal{G}_\bullet$ is a set $\mathcal{G}_0$ – to be called its set of of *objects* or *configurations* – and a set $\mathcal{G}_1 = \left\{ \left(x_1 \xrightarrow{f} x_2 \right) | x_1, x_2 \in \mathcal{G}_0 \right\}$ – to be called the set of *morphisms* or *gauge transformations* – between these objects, together with a partial composition operation of morphisms over common objects

$$f_2 \circ f_1 : \; x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3$$

which is associative, and for which every object has a unit (the identity morphism $\mathrm{id}_x : x \to x$) and such that every morphism has an inverse.

The two extreme examples are:

Example 2.54. For X any set, it becomes a groupoid by considering for each object an identity morphism and no other morphisms.

Example 2.55. For G a group, there is a groupoid which we denote $\mathbf{B}G$ defined to have a single object $*$, one morphism from that object to itself for each element of the group

$$(\mathbf{B}G)_1 = \left\{ * \xrightarrow{g} * \mid g \in G \right\}$$

and where composition is given by the product operation in G.

The combination of these two examples which is of central interest here is the following.

Example 2.56. For X a set and G a group with an action $\rho : X \times G \longrightarrow X$ on X, the corresponding *action groupoid* or *homotopy quotient*, denoted $X//G$, is the groupoid whose objects are the elements of X, and whose morphisms are of the form

$$x_1 \xrightarrow{g} (x_2 = \rho(g)(x_1))$$

with composition given by the composition in G.

Remark 2.57. The homotopy quotient is a refinement of the actual quotient X/G in which those elements of X which are related by the G-action are actually *identified*. In contrast to that, the homotopy quotient makes element which are related by the action of the "gauge" group G be *equivalent without being equal*. Moreover it remember *how* two elements are equivalent, hence which "gauge transformation" relates them. This is most striking in example 2.55, which is in fact the special case of the homotopy quotient construction for the case that G acts on a single element:

$$\mathbf{B}G \simeq *//G .$$

Therefore given an unreduced phase space X as in 2.1 and equipped with an action of a gauge symmetry group as in 2.7, then the corresponding *reduced phase space* should be the homotopy quotient $X//G$, hence the space of fields with gauge equivalences between them. But crucially for physics, this is not just a discrete set of points with a discrete set of morphisms between them, as in the above definition, but in addition to the information about field configurations and gauge equivalences between them carries a *smooth structure*.

We therefore need a definition of *smooth groupoids*, hence of homotopy types which carry *differential geometric* structure. Luckily, the definition in 2.2 of smooth spaces immediately generalizes to an analogous definition of smooth groupoids.

First we need the following obvious notion.

Definition 2.58. Given two groupoids $\mathcal{G}_\bullet$ and $\mathcal{K}_\bullet$, a homomorphism $F_\bullet : \mathcal{G}_\bullet \longrightarrow \mathcal{K}_\bullet$ between them (called a *functor*) is a function $F_1 : \mathcal{G}_1 \longrightarrow \mathcal{K}_1$ between the sets of morphisms such that identity-morphisms are sent to identity morphisms and such that composition is respected.

Groupoids themselves are subject to a notion of gauge equivalence:

Definition 2.59. A functor $F_\bullet$ is called an *equivalence of groupoids* if its image hits every equivalence class of objects in $\mathcal{K}_\bullet$ and if for all $x_1, x_2 \in \mathcal{G}_0$ the map F_1 restricts to a bijection between the morphisms from x_1 to x_2 in $\mathcal{G}_\bullet$ and the morphisms between $F_0(x_1)$ and $F_0(x_2)$ in $\mathcal{K}_\bullet$.

With that notion we can express coordinate transformations between smooth groupoids and arrive at the following generalization of def. 2.6.

Definition 2.60. A *smooth groupoid* or *smooth homotopy 1-type* $X_\bullet$ is

1. an assignment to each $n \in \mathbb{N}$ of a groupoid, to be written $X_\bullet(\mathbb{R}^n)$ and to be called the *groupoid of smooth maps from $\mathbb{R}^n$ into X and gauge transformations between these*,
2. an assignment to each ordinary smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ between Cartesian spaces of a functor of groupoids $X(f) : X_\bullet(\mathbb{R}^{n_2}) \to X_\bullet(\mathbb{R}^{n_1})$, to be called the *pullback of smooth functions into X along f*;

such that both the components X_0 and X_1 form a smooth space according to def 2.6.

With this definition in hand we can now form the reduced phase space in a way that reflects both its smooth structure as well as its gauge-theoretic structure:

Example 2.61. Given a smooth space X and a smooth group G with a smooth action $\rho : X \times G \longrightarrow X$, then the *smooth homotopy quotient* of this action is the smooth groupoid, def. 2.60. which on each coordinate chart is the homotopy quotient, def. 2.56, of the coordinates of G acting on the coordinates of X, hence the assignment

$$X//G : \mathbb{R}^n \mapsto (X(\mathbb{R}^n)) // (G(\mathbb{R}^n)) .$$

Remark 2.62. In most of the physics literature only the infinitesimal approximation to the smooth homotopy quotient $X//G$ is considered, that however is famous: it is the *BRST complex* of gauge theory [24]. More in detail, to any Lie group G is associated a Lie algebra $\mathfrak{g}$, which is its "infinitesiamal approximation" in that it consists of the first order neightbourhood of the neutral element in G, equipped with the first linearized group structure, incarnated as the Lie bracket. In direct analogy to this, a smooth grouppoid such as $X//G$ has an infinitesimal approximation given by a *Lie algebroid*, a vector bundle on X whose fibers form the first order neighbourhood of the smooth space of morphisms at the identity morphisms. Moreover, Lie algebroids can equivalently be encoded dually by the algebras of functions on these first order neighbourhoods. These are differential graded-commutative algebras and the dgc-algebra associated this way to the smooth groupoid $X//G$ is what in the physics literature is known as the BRST complex.

To correctly capture the interplay between the differential geometric structure and the homotopy theoretic structure in this definition we have to in addition declare the following

Definition 2.63. A homomorphism $f_\bullet : X_\bullet \longrightarrow Y_\bullet$ of smooth groupoids is called a *local equivalence* if it is a *stalkwise* equivalence of groupoids, hence if for each Cartesian space $\mathbb{R}^n$ and for each point $x \in \mathbb{R}^n$, there is an open neighbourhood $\mathbb{R}^n \simeq U_x \hookrightarrow \mathbb{R}^n$ such that $F_\bullet$ restricted to this open neighbourhood is an equivalence of groupoids according to def. 2.59.

Definition 2.64. The $(2,1)$-*topos of smooth groupoids* is the homotopy theory obtained from the category $\mathrm{Sh}(\mathrm{CartSp}, \mathrm{Grpd})$ of smooth groupoids by universally turning the local equivalences into actual equivalences, def. A.13.

This refines the construciton of the topos of smooth spaces form before, and hence we find it convenient to use the same symbol for it:

$$\mathbf{H} := \mathrm{Sh}(\mathrm{CartSp}, \mathrm{Grpd})[\{\text{local equivalences}\}^{-1}] \,.$$

2.9 The kinetic action, pre-quantization and differential cohomology

The refinement of gauge transformations of differential 1-forms to coherent $U(1)$-valued functions which we have seen in the construction of the reduced phase space above in 2.7 also appears in physics from another angle, which is not explicitly gauge theoretic, but related to the global definition of the exponentiated action functional.

Given a pre-symplectic form $\omega \in \Omega^2_{\mathrm{cl}}(X)$, by the Poincaré lemma there is a good cover $\{U_i \hookrightarrow X\}_i$ and smooth 1-forms $\theta_i \in \Omega^1(U_i)$ such that $\mathbf{d}\theta_i = \omega_{|U_i}$. Physically such a 1-form is (up to a factor of 2) a choice of *kinetic energy density* called a *kinetic Lagrangian* L_{kin}:

$$\theta_i = 2L_{\mathrm{kin},i} \,.$$

Example 2.65. Consider the phase space $(\mathbb{R}^2,\ \omega = \mathbf{d}q \wedge \mathbf{d}p)$ of example 2.1. Since $\mathbb{R}^2$ is a contractible topological space we consider the trivial covering ($\mathbb{R}^2$ covering itself) since this is already a good covering in this case. Then all the $\{g_{ij}\}$ are trivial and the data of a prequantization consists simply of a choice of 1-form $\theta \in \Omega^1(\mathbb{R}^2)$ such that

$$\mathbf{d}\theta = \mathbf{d}q \wedge \mathbf{d}p \,.$$

A standard such choice is

$$\theta = -p \wedge \mathbf{d}q \,.$$

Then given a trajectory $\gamma\colon [0,1] \longrightarrow X$ which satisfies Hamilton's equation for a standard kinetic energy term, then $(p\mathbf{d}q)(\dot\gamma)$ is this kinetic energy of the particle which traces out this trajectory.

Given a path $\gamma : [0,1] \to X$ in phase space, its *kinetic action* S_{kin} is supposed to be the integral of $\mathcal{L}_{\mathrm{kin}}$ along this trajectory. In order to make sense of this in generality with the above locally defined kinetic Lagrangians $\{\theta_i\}_i$, there are to be transition functions $g_{ij} \in C^\infty(U_i \cap U_j, \mathbb{R})$ such that

$$\theta_j|_{U_j} - \theta_i|_{U_i} = \mathbf{d}g_{ij} \,.$$

If on triple intersections these functions satisfy

$$g_{ij} + g_{jk} = g_{ik} \quad \text{on } U_i \cap U_j \cap U_K$$

then there is a well defined action functional

$$S_{\text{kin}}(\gamma) \in \mathbb{R}$$

obtained by dividing γ into small pieces that each map to a single patch U_i, integrating θ_i along this piece, and adding the contribution of g_{ij} at the point where one switches from using θ_i to using θ_j.

However, requiring this condition on triple overlaps as an equation between $\mathbb{R}$-valued functions makes the local patch structure trivial: if this holds then one can find a single $\theta \in \Omega^1(X)$ and functions $h_i \in C^\infty(U_i, \mathbb{R})$ such that superficially pleasant effect that the action is $\theta_i = \theta|_{U_i} + \mathbf{d}h_i$. This has the simply the integral against this globally defined 1-form, $S_{\text{kin}} = \int_{[0,1]} \gamma^* L_{\text{kin}}$, but it also means that the pre-symplectic form ω is exact, which is not the case in many important examples.

On the other hand, what really matters in physics is not the action functional $S_{\text{kin}} \in \mathbb{R}$ itself, but the *exponentiated* action

$$\exp\left(\tfrac{i}{\hbar} S\right) \in \mathbb{R}/(2\pi\hbar)\mathbb{Z}\,.$$

For this to be well defined, one only needs that the equation $g_{ij} + g_{jk} = g_{ik}$ holds modulo addtion of an integral multiple of $h = 2\pi\hbar$, which is *Planck's constant*. If this is the case, then one says that the data $(\{\theta_i\}, \{g_{ij}\})$ defines equivalently

- a $U(1)$-principal connection;
- a degree-2 cocycle in ordinary differential cohomology

on X, with *curvature* the given symplectic 2-form ω.

Such data is called a *pre-quantization* of the symplectic manifold (X, ω). Since it is the exponentiated action functional $\exp(\frac{i}{\hbar} S)$ that enters the quantization of the given mechanical system (for instance as the integrand of a path integral), the prequantization of a symplectic manifold is indeed precisely the data necessary before quantization.

Therefore, in the spirit of the above discussion of pre-symplectic structures, we would like to refine the smooth moduli space of closed differential 2-forms to a moduli space of prequantized differential 2-forms.

Again this does naturally exist if only we allow for a good notion of "space". An additional phenomenon to be taken care of now is that while pre-symplectic forms are either equal or not, their pre-quantizations can be different and yet be *equivalent*:

because there is still a remaining freedom to change this data without changing the exponentiated action along a *closed* path: we say that a choice of functions $h_i \in C^\infty(U_i, \mathbb{R}/(2\pi\hbar)\mathbb{Z})$ defines an equivalence between $(\{\theta_i\}, \{g_{ij}\})$ and $(\{\tilde{\theta}_i\}, \{\tilde{g}_{ij}\})$ if $\tilde{\theta}_i - \theta_i = \mathbf{d}h_i$ and $\tilde{g}_{ij} - g_{ij} = h_j - h_i$.

This means that the space of prequantizations of (X, ω) is similar to an *orbifold*: it has points which are connected by gauge equivalences: there is a *groupoid* of pre-quantum structures on a manifold X. Otherwise this space of prequantizations is similar to the spaces $\mathbf{\Omega}^2_{\text{cl}}$ of differential forms, in that for each smooth manifold there is a collection of smooth such data and it may consistently be pullback back along smooth functions of smooth manifolds.

As before for the presymplectic differential forms in 2.2 it will be useful to find a moduli space for such prequantum structures. This certainly cannot exist as a smooth manifold, but due to the gauge transformations between prequantizations it can also not exist as a more general smooth space. However, it does exist as a *smooth groupoid*, def. 2.64.

Definition 2.66. For $X = \mathbb{R}^n$ a Cartesian space, wrrite $\Omega^1(X)$ for the set of smooth differential 1-forms on X and write $C^\infty(X, U(1))$ for the set of smooth circle-group valued function on X. There is an action

$$\rho : C^\infty(X, U(1)) \times \Omega^1(\mathbb{R}^n) \longrightarrow \Omega^1(X, U(1))$$

of functions on 1-forms A by gauge transformation g, given by the formula

$$\rho(g)(A) := A + \mathbf{d}\mathrm{log} g \,.$$

Hence if $g = \exp(i\kappa)$ is given by the exponential of a smooth real valued function (which is always the case on $\mathbb{R}^n$) then this is

$$\rho(g)(A) := A + \mathbf{d}\kappa \,.$$

Definition 2.67. Write

$$\mathbf{B}U(1)_{\mathrm{conn}} \in \mathbf{H} \,,$$

for the smooth groupoid, def. 2.60, which for Cartesian space $\mathbb{R}^n$ has as groupoid of coordinate charts the homotopy quotient, def. 2.56, of the smooth functions on the coordinate chart acting on the smooth 1-forms on the coordinate chart.

$$\mathbf{B}U(1)_{\mathrm{conn}} \;:\; \mathbb{R}^n \;\mapsto\; \Omega^1(\mathbb{R})//\mathbb{C}^\infty(\mathbb{R}^n, U(1)) \,.$$

Equivalently this is the smooth homotopy quotient, def. 2.61, of the smooth group $U(1) \in \mathbf{H}$ acting on the universal smooth moduli space $\mathbf{\Omega}^1$ of smooth differential 1-forms:

$$\mathbf{B}U(1)_{\mathrm{conn}} \simeq \mathbf{\Omega}^1//U(1) \,.$$

We call this the *universal moduli stack of prequantizations* or *universal moduli stack of $U(1)$-principal connections.*

Remark 2.68. This smooth groupoid $\mathbf{B}U(1)_{\mathrm{conn}} \simeq \mathbf{\Omega}^1//U(1)$ is equivalently characterized by the following properties.

1. For X any smooth manifold, smooth functions

$$X \longrightarrow \mathbf{B}U(1)_{\mathrm{conn}}$$

are equivalent to prequantum structures $(\{\theta_i\}, \{g_{ij}\})$ on X,

2. a homotopy

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbf{B}U(1)_{\mathrm{conn}} \end{tikzcd}$$

between two such maps is equivalently a gauge transformation $(\{h_i\})$ between these prequantizations.

Proposition 2.69. *There is then in* $\mathbf{H}$ *a morphism*

$$F:\; \mathbf{B}U(1)_{\mathrm{conn}} \longrightarrow \mathbf{\Omega}^2_{\mathrm{cl}}$$

from this universal moduli stack of prequantizations back to the universal smooth moduli space of closed differential 2-form. This is the universal curvature *map in that for $\nabla : X \longrightarrow \mathbf{B}U(1)_{\mathrm{conn}}$ a prequantization datum $(\{\theta_i\}, \{g_{ij}\})$, the composite*

$$F_{(-)} :\; X \xrightarrow{\nabla} \mathbf{B}U(1)_{\mathrm{conn}} \xrightarrow{F_{(-)}} \mathbf{\Omega}^2_{\mathrm{cl}}$$

is the closed differential 2-form on X characterized by $\omega|_{U_i} = \mathbf{d}\theta_i$, for every patch U_i. Again, this property characterizes the map $F_{(-)}$ and may be taken as its definition.

Using this language of the $(2,1)$-topos $\mathbf{H}$ of smooth groupoids, we may then formally capture the above discussion of prequantization as follows:

Definition 2.70. Given a symplectic manifold (X, ω), regarded by prop. 2.16 as an object $(X \xrightarrow{\omega} \mathbf{\Omega}^2_{\mathrm{c}}) \in \mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}$, then a *prequantization* of (X, ω) is a lift ∇ in the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\nabla"] \arrow[dr, "\omega"'] & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\ & \Omega^2_{\mathrm{cl}} \end{tikzcd}$$

in $\mathbf{H}$, hence is a lift of (X, ω) through the *base change* functor (see prop. A.2 for this terminology) or *dependent sum* functor (see def. A.3)

$$\sum_{F_{(-)}} : \mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}} \longrightarrow \mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}$$

that goes from the slice over the universal moduli stack of prequantizations to the slice over the universal smooth moduli space of closed differential 2-forms.

Moreover, in this language of geometric homotopy theory we then also find a conceptual re-statement of the descent of the (pre-)symplectic potential to the reduced phase space, from 2.7:

Proposition 2.71. *Given a covariant phase space X with (pre-)symplectic potential θ and gauge group action $\rho : G \times X \longrightarrow X$, a G-equivariant structure on θ, def. 2.50, is equivalently an extension ∇_{red} of θ along the map to the smooth homotopy quotient $X//G$ as a $(\mathbb{R}/\Gamma)$-principal connection, hence a diagram in $\mathbf{H}$ of the form*

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[d] & \mathbf{B}U(1)_{\mathrm{conn}} \\ X//G \arrow[ur, dashed, "\nabla_{\mathrm{red}}"'] & \end{tikzcd} .$$

2.10 The classical action, the Legendre transform and Hamiltonian flows

The reason to consider Hamiltonian symplectomorphisms, prop. 2.40 instead of general symplectomorphisms, is really because these give homomorphisms not just between plain symplectic manifolds, but between their prequantizations, def. 2.70. To these we turn now.

Consider a morphism

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\nabla"'] & \arrow[d, phantom, "\swarrow"] & X \arrow[dl, "\nabla"] \\ & \mathbf{B}U(1)_{\mathrm{conn}} & \end{tikzcd} ,$$

hence a morphism in the slice topos $\mathbf{H}_{/\mathbf{B}U(1)_{conn}}$. This has been discussed in detail in [15].

One finds that infinitesimally such morphisms are given by a Hamiltonian and its Legendre transform.

Proposition 2.72. *Consider the phase space $(\mathbb{R}^2,\ \omega = \mathbf{d}q \wedge \mathbf{d}p)$ of example 2.1 equipped with its canonical prequantization by $\theta = p\mathbf{d}q$ from example 2.65. Then for $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ a Hamiltonian, and for $t \in \mathbb{R}$ a parameter ("time"), a lift of the Hamiltonian symplectomorphism $\exp(t\{H, -\})$ from $\mathbf{H}$ to the slice topos $\mathbf{H}_{/\mathbf{B}U(1)_{conn}}$ is given by*

$$\begin{tikzcd} X \arrow[rr, "\exp(t\{H,-\})"] \arrow[dr, "\theta"'] & \arrow[d, phantom, "\swarrow \exp(iS_t)"] & X \arrow[dl, "\theta"] \\ & \mathbf{B}U(1)_{\mathrm{conn}} & \end{tikzcd} ,$$

where

- *$S_t : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the action functional of the classical trajectories induced by H,*
- *which is the integral $S_t = \int_0^t L\, dt$ of the Lagrangian $L\, dt$ induced by H,*
- *which is the Legendre transform*

$$L := p\frac{\partial H}{\partial p} - H \,:\, \mathbb{R}^2 \longrightarrow \mathbb{R}\,.$$

In particular, this induces a functor

$$\exp(iS) \,:\, \mathrm{Bord}_1^{\mathrm{Riem}} \longrightarrow \mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}\,.$$

Conversely, a symplectomorphism, being a morphism in $\mathbf{H}_{/\mathbf{\Omega}^2_{cl}}$ is a Hamiltonian symplectomorphism precisely if it admits such a lift to $\mathbf{H}_{/\mathbf{B}U(1)_{conn}}$.

This is a special case of the discussion in [15]. Proof. The canonical prequantization of $(\mathbb{R}^2, \mathbf{d}q \wedge \mathbf{d}p)$ is the globally defined connection on a bundle—connection 1-form

$$\theta := p\,\mathbf{d}q\,.$$

We have to check that on *graph*$(\exp(t\{H,-\}))$ we have the equation

$$p_2 \wedge \mathbf{d}q_2 = p_1 \wedge \mathbf{d}q_1 + \mathbf{d}S\,.$$

Or rather, given the setup, it is more natural to change notation to

$$p_t \wedge \mathbf{d}q_t = p \wedge \mathbf{d}q + \mathbf{d}S\,.$$

Notice here that by the nature of $\mathrm{graph}(\exp(t\{H,-\}))$ we can identify

$$\mathrm{graph}(\exp(t\{H,-\})) \simeq \mathbb{R}^2$$

and under this identification

$$q_t = \exp(t\{H,-\})q$$

and

$$p_t = \exp(t\{H,-\})p\,.$$

It is sufficient to check the claim infinitesimal object—infinitesimally. So let $t = \epsilon$ be an infinitesimal, hence such that $\epsilon^2 = 0$. Then the above is Hamilton's equations and reads equivalently

$$q_\epsilon = q + \frac{\partial H}{\partial p}\epsilon$$

and

$$p_\epsilon = p - \frac{\partial H}{\partial q}\epsilon\,.$$

Using this we compute

$$\begin{aligned}
\theta_\epsilon - \theta &= p_\epsilon \wedge \mathbf{d}q\epsilon - p \wedge \mathbf{d}q \\
&= \left(p - \frac{\partial H}{\partial q}\epsilon\right) \wedge \mathbf{d}\left(q + \frac{\partial H}{\partial p}\epsilon\right) - p \wedge \mathbf{d}q \\
&= \epsilon\left(p \wedge \mathbf{d}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q} \wedge \mathbf{d}q\right) \\
&= \epsilon\left(\mathbf{d}\left(p\frac{\partial H}{\partial p}\right) - \frac{\partial H}{\partial p} \wedge \mathbf{d}p - \frac{\partial H}{\partial q} \wedge \mathbf{d}q\right) \\
&= \epsilon\mathbf{d}\left(p\frac{\partial H}{\partial p} - H\right)
\end{aligned}\,.$$

□

Remark 2.73. When one speaks of symplectomorphisms as "canonical transformations" (see e.g. [1], p. 206), then the function S in prop. 2.72 is also known as the "generating function of the canonical transformation", see [1], chapter 48.

Remark 2.74. Proposition 2.72 says that the slice topos $\mathbf{H}_{/\mathbf{B}U(1)_{conn}}$ unifies classical mechanics in its two incarnations as Hamiltonian mechanics and as Lagrangian mechanics. A morphism here is a diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \mathbf{B}U(1)_{\mathrm{conn}} & \end{tikzcd}$$

and which may be regarded as having two components: the top horizontal 1-morphism as well as the homotopy/2-morphism filling the slice. Given a smooth flow of these, the horizontal morphism is the flow of a Hamiltonian vector field for some Hamiltonian function H, and the 2-morphism is a $U(1)$-gauge transformation given (locally) by a $U(1)$-valued function which is the exponentiated action functional that is the integral of the Lagrangian L which is the Legendre transform of H.

So in a sense the prequantization lift through the base change/dependent sum along the universal curvature map

$$\sum_{F_{(-)}} : \mathbf{H}_{/\mathbf{B}U(1)_{conn}} \longrightarrow \mathbf{H}_{/\mathbf{\Omega}^2_{cl}}$$

is the Legendre transform which connects Hamiltonian mechanics with Lagrangian mechanics.

2.11 The classical action functional pre-quantizes Lagrangian correspondences

We may sum up these observations as follows.

Definition 2.75. Given a Lagrangian correspondence

$$\begin{tikzcd} & \mathrm{graph}(\phi) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X_1 \arrow[dr, "\omega_1"'] \arrow[rr, phantom, "="] & & X_2 \arrow[dl, "\omega_2"] \\ & \Omega^2_{\mathrm{cl}} & \end{tikzcd}$$

as in prop. 2.22, a *prequantization* of it is a lift of this diagram in $\mathbf{H}$ to a diagram of the form

$$\begin{tikzcd} & \mathrm{graph}(\phi) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X_1 \arrow[dr, "\nabla_1"] \arrow[ddr, "\omega_1"'] \arrow[rr, phantom, "\Swarrow"] & & X_2 \arrow[dl, "\nabla_2"'] \arrow[ddl, "\omega_2"] \\ & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] & \\ & \Omega^2_{\mathrm{cl}} & \end{tikzcd}$$

Remark 2.76. This means that a prequantization of a Lagrangian correspondence is a prequantization of the source and target symplectic manifolds by prequantum circle bundles as in def. 2.70, together with a choice of (gauge) equivalence between thes respective pullback of these two bundles to the correspondence space. More abstractly, such a prequantization is a lift through the base change/dependent sum map along the universal curvature morphism

$$\mathrm{Corr}\left(\sum_{F_{(-)}}\right) : \mathrm{Corr}\left(\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}\right) \longrightarrow \mathrm{Corr}\left(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}\right) .$$

From prop. 2.72 and under the equivalence of example 2.21 it follows that smooth 1-parameter groups of prequantized Lagrangian correspondences are equivalently Hamiltonian flows, and that the prequantizaton of the underlying Hamiltonian correspondences is given by the classical action funtional.

In summary, the description of classical mechanics here identifies prequantized Lagrangian correspondences schematically as follows:

$$\begin{tikzcd}
 & \mathrm{graph}\left(\exp\left(t\{H,-\}\right)\right) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\Swarrow\ \exp\left(\frac{i}{\hbar}S_t\right)=\exp\left(\frac{i}{\hbar}\int_0^t L dt\right)"] & \\
X \arrow[dr, "\nabla_{\mathrm{in}}"'] & & X \arrow[dl, "\nabla_{\mathrm{out}}"] \\
 & \mathbf{B}U(1)_{\mathrm{conn}} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{space of trajectories} \arrow[dl, "\text{initial values}"'] \arrow[dr, "\text{Hamiltonian evolution}"] \arrow[dd, phantom, "\Swarrow\ \text{action functional}"] & \\
\text{incoming configurations} \arrow[dr, "\text{prequantum bundle}"'] & & \text{outgoing configurations} \arrow[dl, "\text{prequantum bundle}"] \\
 & \text{2-group of phases} &
\end{tikzcd}$$

This picture of classical mechanics as the theory of correspondences in higher slices topos is what allows a seamless generalization to a local discussion of prequantum field theory in [54].

2.12 Quantization, the Heisenberg group, and slice automorphism groups

While we do not discussion genuine quantization here (in a way adapted to the perspective here this is discussed in [43]) it is worthwhile to notice that the perspective of classical mechanics by correspondences in slice toposes seamlessly leads over to *quantization* by recognizing that the slice automorphism groups of the prequantized phase spaces are nothing but the "quantomorphisms groups" containing the famous Heisenberg groups of quantum operators. This has been developed for higher prequantum field theory in [15], see D.3 below. Here we give an exposition, which re-amplifies some of the structures already found above.

Quantization of course was and is motivated by experiment, hence by observation of the observable universe: it just so happens that quantum mechanics and quantum field theory correctly account for experimental observations where classical mechanics and classical field theory gives no answer or incorrect answers (see for instance [13]). A historically important example is the phenomenon called the "ultraviolet catastrophe", a paradox predicted by classical statistical mechanics which is *not* observed in nature, and which is corrected by quantum mechanics.

But one may also ask, independently of experimental input, if there are good formal mathematical reasons and motivations to pass from classical mechanics to quantum mechanics. Could one have been led to quantum mechanics by just pondering the mathematical formalism of classical mechanics? (Hence more precisely: is there a natural "Synthetic quantum field theory" [53]).

The following spells out an argument to this effect.

So to briefly recall, a system of classical mechanics/prequantum field theory—prequantum mechanics is a phase space, formalized as a symplectic manifold (X, ω). A symplectic manifold is in particular a

Poisson manifold, which means that the algebra of functions on phase space X, hence the algebra of *classical observables*, is canonically equipped with a compatible Lie bracket: the *Poisson bracket*. This Lie bracket is what controls dynamics in classical mechanics. For instance if $H \in C^\infty(X)$ is the function on phase space which is interpreted as assigning to each configuration of the system its energy – the Hamiltonian function – then the Poisson bracket with H yields the infinitesimal object—infinitesimal time evolution of the system: the differential equation famous as Hamilton's equations.

Something to take notice of here is the *infinitesimal* nature of the Poisson bracket. Generally, whenever one has a Lie algebra $\mathfrak{g}$, then it is to be regarded as the infinitesimal object—infinitesimal approximation to a globally defined object, the corresponding Lie group (or generally smooth group) G. One also says that G is a *Lie integration* of $\mathfrak{g}$ and that $\mathfrak{g}$ is the Lie differentiation of G.

Therefore a natural question to ask is: *Since the observables in classical mechanics form a Lie algebra under Poisson bracket, what then is the corresponding Lie group?*

The answer to this is of course "well known" in the literature, in the sense that there are relevant monographs which state the answer. But, maybe surprisingly, the answer to this question is not (at time of this writing) a widely advertized fact that has found its way into the basic educational textbooks. The answer is that this Lie group which integrates the Poisson bracket is the "quantomorphism group", an object that seamlessly leads to the quantum mechanics of the system.

Before we spell this out in more detail, we need a brief technical aside: of course Lie integration is not quite unique. There may be different global Lie group objects with the same Lie algebra.

The simplest example of this is already one of central importance for the issue of quantization, namely, the Lie integration of the abelian line Lie algebra $\mathbb{R}$. This has essentially two different Lie groups associated with it: the simply connected topological space—simply connected translation group, which is just $\mathbb{R}$ itself again, equipped with its canonical additive abelian group structure, and the discrete space—discrete quotient of this by the group of integers, which is the circle group

$$U(1) = \mathbb{R}/\mathbb{Z}\,.$$

Notice that it is the discrete and hence "quantized" nature of the integers that makes the real line become a circle here. This is not entirely a coincidence of terminology, but can be traced back to the heart of what is "quantized" about quantum mechanics.

Namely, one finds that the Poisson bracket Lie algebra $\mathfrak{poiss}(X,\omega)$ of the classical observables on phase space is (for X a connected topological space—connected manifold) a Lie algebra extension of the Lie algebra $\mathfrak{ham}(X)$ of Hamiltonian vector fields on X by the line Lie algebra:

$$\mathbb{R} \longrightarrow \mathfrak{poiss}(X,\omega) \longrightarrow \mathfrak{ham}(X)\,.$$

This means that under Lie integration the Poisson bracket turns into an central extension of the group of Hamiltonian symplectomorphisms of (X,ω). And either it is the fairly trivial non-compact extension by $\mathbb{R}$, or it is the interesting central extension by the circle group $U(1)$. For this non-trivial Lie integration to exist, (X,ω) needs to satisfy a quantization condition which says that it admits a prequantum line bundle. If so, then this $U(1)$-central extension of the group $Ham(X,\omega)$ of Hamiltonian symplectomorphisms exists and is called... the "quantomorphism group" $\mathrm{QuantMorph}(X,\omega)$:

$$U(1) \longrightarrow \mathrm{QuantMorph}(X,\omega) \longrightarrow \mathrm{HamSympl}(X,\omega)\,.$$

More precisely, this group is just the slice automorphism group:

Proposition 2.77. *Let (X,ω) be a symplectic manifold with prequantization $\nabla : X \longrightarrow \mathbf{B}U(1)_{\mathrm{conn}}$, according to def. 2.70, then the smooth automorphism group of ∇ regarded as an object in the higher slice topos*

$\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}$ *is the quantomorphism group* $\mathrm{QuantMorph}(X, \omega)$

$$\begin{aligned}
\mathrm{QuantMorph}(X,\omega) &\simeq \mathbf{Aut}_{\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}}(\nabla) \\
&\simeq \mathbf{Aut}_{\mathrm{Corr}\left(\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}\right)}(\nabla) \\
&\simeq \left\{ \begin{tikzcd} X \arrow[rr, "\phi", "\simeq"'] \arrow[dr, "\nabla"'] & \arrow[d, phantom, "\overset{\Longleftarrow}{\simeq}"] & X \arrow[dl] \\ & \mathbf{B}U(1)_{\mathrm{conn}} & \end{tikzcd} \right\}
\end{aligned}$$

in that

1. *The Lie algebra of* $\mathrm{QuantMorph}(X, \omega)$ *is the Poisson bracket Lie algebra of* (X, ω)*;*
2. *This group constitutes a* $U(1)$*-central extension of the group of Hamiltonian symplectomorphisms.*

While important, for some reason this group is not very well known, which is striking because it contains a small subgroup which is famous in quantum mechanics: the *Heisenberg group*.

More precisely, whenever (X, ω) itself has a Hamiltonian action—compatible group structure, notably if (X, ω) is just a symplectic vector space (regarded as a group under addition of vectors), then we may ask for the subgroup of the quantomorphism group which covers the (left) action of phase space (X, ω) on itself. This is the corresponding Heisenberg group $\mathrm{Heis}(X, \omega)$, which in turn is a $U(1)$-central extension of the group X itself:

$$U(1) \longrightarrow \mathrm{Heis}(X, \omega) \longrightarrow X\,.$$

Proposition 2.78. *If* (X, ω) *is a symplectic manifold that at the same time is a group which acts on itself by Hamiltonian diffeomorphisms, then the Heisenberg group of* (X, ω) *is the pullback* $\mathrm{Heis}(X, \omega)$ *of smooth groups in the following diagram in* $\mathbf{H}$

$$\begin{tikzcd} \mathrm{Heis}(X,\omega) \arrow[r] \arrow[d] & \mathrm{QuantMorph}(X,\omega) \arrow[d] \\ X \arrow[r] & \mathrm{HamSympl}(X,\omega) \end{tikzcd}\,.$$

Remark 2.79. In other words this exhibits $\mathrm{QuantMorph}(X, \omega)$ as a universal $U(1)$-central extension characteristic of quantum mechanics from which various other $U(1)$-extension in QM are obtained by pullback/restriction. In particular all *classical anomalies* arise this way, discussed below in 2.14.

At this point it is worth pausing for a second to note how the hallmark of quantum mechanics has appeared as if out of nowhere simply by applying Lie integration to the Lie algebra—Lie algebraic structures in classical mechanics:

if we think of Lie integration—Lie integrating $\mathbb{R}$ to the interesting circle group $U(1)$ instead of to the uninteresting translation group $\mathbb{R}$, then the name of its canonical basis element $1 \in \mathbb{R}$ is canonically "i", the imaginary unit. Therefore one often writes the above central extension instead as follows:

$$i\mathbb{R} \longrightarrow \mathfrak{poiss}(X, \omega) \longrightarrow \mathfrak{ham}(X, \omega)$$

in order to amplify this. But now consider the simple special case where $(X, \omega) = (\mathbb{R}^2, dp \wedge dq)$ is the 2-dimensional symplectic vector space which is for instance the phase space of the particle propagating on the line. Then a canonical set of generators for the corresponding Poisson bracket Lie algebra consists of the linear functions p and q of classical mechanics textbook fame, together with the *constant* function. Under the above Lie theoretic identification, this constant function is the canonical basis element of $i\mathbb{R}$, hence purely Lie theoretically it is to be called "i".

With this notation then the Poisson bracket, written in the form that makes its Lie integration manifest, indeed reads

$$[q, p] = i\,.$$

Since the choice of basis element of $i\mathbb{R}$ is arbitrary, we may rescale here the i by any non-vanishing real number without changing this statement. If we write "$\hbar$" for this element, then the Poisson bracket instead reads

$$[q, p] = i\hbar\,.$$

This is of course the hallmark equation for quantum physics, if we interpret $\hbar$ here indeed as Planck's constant. We see it arises here merely by considering the non-trivial (the interesting, the non-simply connected) Lie integration of the Poisson bracket.

This is only the beginning of the story of quantization, naturally understood and indeed "derived" from applying Lie theory to classical mechanics. From here the story continues. It is called the story of *geometric quantization*. We close this motivation section here by some brief outlook.

The quantomorphism group which is the non-trivial Lie integration of the Poisson bracket is naturally constructed as follows: given the symplectic form ω, it is natural to ask if it is the curvature 2-form of a $U(1)$-principal connection ∇ on complex line bundle L over X (this is directly analogous to Dirac charge quantization when instead of a symplectic form on phase space we consider the the field strength 2-form of electromagnetism on spacetime). If so, such a connection (L, ∇) is called a *prequantum line bundle* of the phase space (X, ω). The quantomorphism group is simply the automorphism group of the prequantum line bundle, covering diffeomorphisms of the phase space (the Hamiltonian symplectomorphisms mentioned above).

As such, the quantomorphism group naturally acts on the space of sections of L. Such a section is like a wavefunction, except that it depends on all of phase space, instead of just on the "canonical coordinates". For purely abstract mathematical reasons (which we won't discuss here, but see at *motivic quantization* for more) it is indeed natural to choose a "polarization" of phase space into canonical coordinates and canonical momenta and consider only those sections of the prequantum line bundle which depend only on the former. These are the actual *wavefunctions* of quantum mechanics, hence the *quantum states*. And the subgroup of the quantomorphism group which preserves these polarized sections is the group of exponentiated quantum observables. For instance in the simple case mentioned before where (X, ω) is the 2-dimensional symplectic vector space, this is the Heisenberg group with its famous action by multiplication and differentiation operators on the space of complex-valued functions on the real line.

2.13 Integrable systems, moment maps and homomorphism into the Poisson bracket Lie algebra

Remark 2.80. Given a phase space (pre-)symplectic manifold (X, ω), and given $n \in \mathbb{N}$, then Lie algebra homomorphisms

$$\mathbb{R}^n \longrightarrow \mathfrak{pois}(X, \omega)$$

from the abelian Lie algebra on n generators into the Poisson bracket Lie algebra, def. 2.34 are equivalently choices of n-tuples of Hamiltonians $\{H_i\}_{i=1}^n$ (and corresponding Hamiltonian vector fields v_i) that pairwise commute with each other under the Poisson bracket, $\forall_{i,j}\{H_i, H_j\} = 0$. If the set $\{H_i\}_i$ is maximal with this property and one of the H_i is regarded the time evolution Hamiltonian of a physical system, then one calls this system *integrable*.

By the discussion in 2.12, the Lie integration of the Lie algebra homomorphism $\mathbb{R}^n \longrightarrow \mathfrak{pois}(X, \omega)$ is a morphism of smooth groupoids

$$\mathbf{B}(\mathbb{R}^n) \longrightarrow \mathbf{BAut}_{/\mathbf{B}U(1)_{\mathrm{conn}}}(\nabla) \hookrightarrow \mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}$$

from the smooth delooping groupoid (def. 2.55) of $\mathbb{R}^n$, now regarded as the translation group of n-dimensional Euclidean space, to the automorphism group of any pre-quantization of the phase space (its quantomoprhism group).

Remark 2.81. Below in 3.3 we re-encounter this situation, but in a more refined context. There we find that n-dimensional classical field theory is encoded by a homomorphism of the form

$$\mathbb{R}^n \longrightarrow \mathfrak{pois}(X, \omega)\,,$$

where however now ω is a closed differential form of degree $(n+1)$ and where $\mathfrak{pois}(X, \omega)$ is a homotopy-theoretic refinement of the Poisson bracket Lie algebra (a *Lie n-algebra* or $(n-1)$-type in homotopy Lie algebras). In that context such a homomorphism does not encode a set of strictly Poisson-commuting Hamiltonians, but a of Hamiltonian flows in the n spacetime directions of the field theory which commute under an n-ary higher bracket only *up to* a specified homotopy. That specified homotopy is the de Donder-Weyl-Hamiltonian of classical field theory.

Remark 2.82. For $\mathfrak{g}$ any Lie algebra and (X, ω) a (pre-)symplectic manifold, a Lie algebra homomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{pois}(X, \omega)$$

is called a *moment map*. Equivalently this is an actin of $\mathfrak{g}$ by Hamiltonian vector fields *with* chosen Hamiltonians. The Lie integration of this is a homomorphism of smooth groups

$$G \longrightarrow \mathbf{Aut}_{/\mathbf{B}U(1)_{\mathrm{conn}}} \simeq \mathrm{QuantMorph}(X, \omega)$$

from a Lie group integrating $\mathfrak{g}$ to the quantomorphism group. This is called a *Hamiltonian G-action.*

2.14 Classical anomalies and projective symplectic reduction

Above in 2.7 we saw that for a gauge symmetry to act consistently on a phase space, it needs to act by *Hamiltonian diffeomorphisms*, because this is the data necessary to put a gauge-equivariant structure on the symplectic potential (hence on the pre-quantization of the phase space).

Under mild conditions every single infinitesimal gauge transformation comes from a Hamiltonian. But these Hamiltonians may not combine to a genuine Hamiltonian action, remark 2.82, but may be specified only up to addition of a locally constant function, and it may happen that these locally constant "gauges" may not be chose globally for the whole gauge group such as to make the whole gauge group act by Hamiltonians. This is the lifting problem of pre-quantization discussed above in 2.9.

But if the failure of the local Hamiltonians to combine to a global Hamiltonian is sufficiently coherent in that it is given by a *group 2-cocycle*, then one can at least find a Hamiltonian action by a central extension of the gauge group. This phenomenon is known as a *classical anomaly* in field theory:

Definition 2.83. Let (X, ω) be a phase space symplectic manifold and let $\rho : G \times X \longrightarrow X$ be a smooth action of a Lie group G on the underlying smooth manifold by Hamiltonian symplectomorphisms, hence a group homomorphism

$$G \longrightarrow \mathrm{HamSympl}(X, \omega)\ .$$

Then we say this system has a *classical anomaly* if this morphism lifts to the quantomorphism group, prop. 2.77, only up to a central extension $\widehat{G} \longrightarrow G$, hence if it fits into the following diagram of smooth group, without the dashed diagonal morphism existing:

$$\begin{tikzcd}
\widehat{G} \arrow[r] \arrow[d] & \mathrm{QuantMorph}(X, \omega) \arrow[d] \\
G \arrow[r, "\rho"'] \arrow[ur, dashed] & \mathrm{HamSympl}(X, \omega)
\end{tikzcd}\ .$$

This is the Lie-integrated version of the Lie-algebraic definition in appendix 5 of [1]. For a list of examples of classical anomalies in field theories see [57].

Remark 2.84. Comparison with prop. 2.78 above shows that for (X, ω) a symplectic group acting on itself by Hamiltonian symplectomorphism, then its Heisenberg group is the "universal classical anomaly".

3 De Donder-Weyl field theory via higher correspondences

We now turn attention from just classical *mechanics* (hence of dynamics along a single parameter, such as the Hamiltonian time parameter in 2.6 above) to, more generally, classical *field theory*, which is dynamics parameterized by higher dimensional manifolds ("spacetimes" or "worldvolumes"). Or rather, we turn attention to the *local* description of classical field theory.

Namely, the situation of example 2.2 above, where a trajectory of a physical system is given by a 1-dimensional curve $[0,1] \longrightarrow Y$ in a space Y of fields can – and traditionally is – also be applied to field theory, if only we allow Y to be a smooth space more general than a finite-dimensional manifold. Specifically, for a field theory on a parameter manifold Σ_n of some dimension n (to be thought of as spacetime or as the "worldvolume of a brane"), and for **Fields** a smooth moduli space of of fields, a *local* field configuration is a map

$$\phi : \Sigma_n \longrightarrow \mathbf{Fields}\,.$$

If however $\Sigma_d \simeq \Sigma_{d-1} \times \Sigma_1$ is a cylinder with $\Sigma_1 = [0,1]$ over a base manifold Σ_{d-1} (a *Cauchy surface* if we think of Σ as spacetime), then such a map is equivalently a map out of the interval into the mapping space of Σ_{d-1} into **Fields**:

$$\phi_{\Sigma_{d-1}} : \Sigma_1 \longrightarrow [\Sigma_{d-1}, \mathbf{Fields}]\,.$$

This brings the field theory into the form of example 2.2, but at the cost of making it "spatially non-local": for instance the energy of the system, as discussed in 2.6, would at each point of Σ_1 be the energy contained in the fields over all of Σ_{d-1}, while the information that this energy arises from integrating contributions localized along Σ_{d-1} is lost.

In more mathematical terms this means that by *transgression to codimension 1* classical field theory takes the form of classical mechanics as discussed above in 2.6. To "localize" the field theory again (make it "extended" or "multi-tiered") we have to undo this process and "de-transgress" classical mechanics to full codimension.

At the level of Hamilton's differential equations, def. 2.29, such a localization is "well known", but much less famous than Hamilton's equations: it is the multivariable variational calculus of Carathéodory, de Donder, and Weyl, as reviewed for instance in section 2 of [23]. Below in 3.3 we show that the de Donder-Weyl equation secretly describes the Lie integration of a higher Poisson bracket Lie algebra in direct but higher analogy to how in 2.12 we saw that the ordinary Hamilton equations exhibit the Lie integration of the ordinary Poisson bracket Lie algebra.

From this one finds that an n-dimensional *local* classical field theory is described not by a symplectic 2-form as a system of classical mechanics is, but by a differential $(n+1)$-form which transgresses to the 2-form after passing to mapping spaces. This point of view has been explored under the name of "covariant mechanics" or "multisymplectic geometry" (see [21] for a review) and "n-plectic geometry" ([16]). Here we show, based on the results in [15], how both of these approaches are unified and "pre-quantized" to a global description of local classical field theory by systems of higher correspondences in higher slices toposes, in higher generalization to the picture which we found in 2.11 for classical mechanics.

3.1 Local field theory Lagrangians and n-plectic smooth spaces

Traditionally, a classical field over a *spacetime* Σ is encoded by a fiber bundle $E \to X$, the *field bundle*. The fields on X are the sections of E.

Example 3.1. Let $d \in \mathbb{N}$ and let $\Sigma = \mathbb{R}^{d-1,1}$ be the d-dimensional real vector space, regarded as a pseudo-Riemannian manifold with the Minkowski metric η (*Minkowski spacetime*). Let moreover F be a finite dimensional real vector space – the *field fiber* – eqipped with a positive definite bilinear form k. Consider the bundle $\Sigma \times F \to \Sigma$, to be called the *field bundle*, and write

$$(X \to \Sigma) := \left(j^1_\infty (\Sigma \times F) \to \Sigma\right)$$

for its first jet bundle.

If we denote the canonical coordinates of Σ by $\sigma^i : \Sigma \to \mathbb{R}$ for $i \in \{0, \cdots, n-1\}$, and choose a dual basis

$$\phi^a : F \to \mathbb{R}$$

of F (hence with $a \in \{1, \cdots, \dim(V)\}$) then X is the vector space with canonical dual basis elements labeled by

$$\{\sigma^i\}, \{\phi^a\}, \{\phi^a_{,i}\}$$

and equipped with bilinear form $(\eta \oplus k \oplus (\eta \otimes k))$. While all of these are coordinates on X, traditionally one says that

1. the functions
$$\sigma^i : X \longrightarrow \mathbb{R}$$
are the *spacetime coordinates*;
2. the functions
$$\phi^a : X \longrightarrow \mathbb{R}$$
are are the *canonical coordinates* of the F-field
3. the functions
$$p^i_a := \eta^{ij} k_{ab} \phi^b_{,j} : X \longrightarrow \mathbb{R}$$
are the *canonical momenta* of the *free* F-field.

Definition 3.2. Given a field bundle $X = j^1_\infty(\Sigma \times F) \to \Sigma$ as in example 3.1, the *free field theory local kinetic Lagrangian* is the horizontal differential n-form

$$L^{\mathrm{loc}}_{\mathrm{kin}} \in \Omega^{n,0}(X)$$

given by

$$\begin{aligned} L^{\mathrm{loc}}_{\mathrm{kin}} &:= \langle \nabla\phi, \nabla\phi \rangle \wedge \mathrm{vol}_\Sigma \\ &:= \left(\tfrac{1}{2} k_{ab} \eta^{ij} \phi^a_{,i} \phi^b_{,j}\right) \wedge \mathbf{d}\sigma^1 \wedge \cdots \wedge \mathbf{d}\sigma^d \end{aligned}$$

(where a sum over repeated indices is understood). Here we regard the volume form of Σ canonically as a horizontal differential form on the first jet bundle

$$\mathrm{vol}_\Sigma := \mathbf{d}\sigma^1 \wedge \cdots \wedge \mathbf{d}\sigma^a \quad \in \quad \Omega^{d,0}_\Sigma(X)\,.$$

The localized analog of example 2.2 is now the following.

Definition 3.3. Given a free field bundle as in example 3.1 and given a horizontal n-form

$$L^{\mathrm{loc}} \in \Omega^{n,0}(X)$$

on its first jet bundle, regarded as a local Lagrangian as in def. 3.2, then the associated *Lagrangian current* is the n-form

$$\theta^{\mathrm{loc}} \in \Omega^{n-1,1}(X)$$

given by the formula

$$\theta^{\mathrm{loc}} := \iota_{\partial_i}\left(\frac{\partial}{\partial \phi^a_{,i}} L^{\mathrm{loc}}\right) \wedge \mathbf{d}\phi^a$$

(where again a sum over repeated indices is understood). We say that the corresponding *pre-symplectic current* or *pre-n-plectic form* is

$$\omega^{\mathrm{loc}} := \mathbf{d}\theta^{\mathrm{loc}}\,.$$

Remark 3.4. The formula in def. 3.3 is effectively that for the pre-symplectic current as it arises in the discussion of *covariant phase spaces* in [64, 12]. In the coordinates of example 3.1 the Lagrangian current reads

$$\theta^{\mathrm{loc}} = p^i_a \wedge \mathbf{d}\phi^a \wedge \iota_{\partial_i}\mathrm{vol}_\Sigma$$

and hence the pre-symplectic current reads

$$\omega^{\mathrm{loc}} = \mathbf{d}p^i_a \wedge \mathbf{d}\phi^a \wedge \iota_{\partial_i}\mathrm{vol}_\Sigma$$

In this form this is manifestly the $(n-1,1)$-component of the canonical "multisymplectic form" that is considered in multisymplectic geometry, see for instance section 2 of [23].

This direct relation between the covariant phase space formulation and the multisymplectic description of local classical field theory seems not to have been highlighted much in the literature. It essentially appears in section 3.2 of [21] and in section 2.1 of [47].

Example 3.5. Consider the simple case $d = 1$ hence $\Sigma = \mathbb{R}$, and $F = \mathbb{R}$, both equipped with the canonical bilinear form on $\mathbb{R}$ (given by multiplication). Jet prolongation followed by evaluation yields the smooth function

$$\mathrm{ev}_\infty : [\Sigma, F] \times \Sigma \stackrel{(j_\infty, \mathrm{id})}{\longrightarrow} \Gamma_\Sigma(X) \times \Sigma \stackrel{\mathrm{ev}}{\longrightarrow} X\,.$$

Then the pullback of the local free field Lagrangian of def. 3.2 along this map is the kinetic Lagrangian of example 2.3:

$$L_{\mathrm{kin}} = \mathrm{ev}^*_\infty L^{\mathrm{loc}}_{\mathrm{kin}}\,.$$

The pullback of the corresponding Lagrangian current according to def. 3.3 is the pre-symplectic potential θ in example 2.2

$$\theta = \mathrm{ev}^*_\infty \theta^{\mathrm{loc}}\,.$$

Definition 3.6. For $d \in \mathbb{N}$, write $\Sigma = \Sigma_1 \times \Sigma_{d-1}$ for the decomposition of Minkowski spacetime into a time axis Σ_1 and a spatial slice Σ_{d-1}, hence with $\Sigma_1 = \mathbb{R}$ the real line. Restrict attention to sections of the field bundle which are periodic in all spatial directions, hence pass to the $(d-1)$-torus $\Sigma_{d-1} := \mathbb{R}^d/\mathbb{Z}^d$ (in order to have a compact spatial slice). Then given a free field local Lagrangian as in def. 3.2, say that its *transgression to codimension 1* is the pullback of the local Lagrangian n-form along

$$\mathrm{ev}_\infty : [\Sigma_1, [\Sigma_{d-1}, F \times \Sigma_1 \times \Sigma_{d-1} \stackrel{\simeq}{\longrightarrow} [\Sigma, F] \times \Sigma \stackrel{(j_\infty, \mathrm{id})}{\longrightarrow} \Gamma_\Sigma(X) \times \Sigma \stackrel{\mathrm{ev}}{\longrightarrow} X$$

followed by fiber integration $\int_{\Sigma_{d-1}}$ over space Σ_{d-1}, to be denoted

$$L_{\mathrm{kin}} := \int_{\Sigma_{d-1}} \mathrm{ev}^*_\infty L^{\mathrm{loc}}_{\mathrm{kin}}\,.$$

Similarly the transgression to codimension 1 of the Lagrangian current, def. 3.3 is

$$\theta := \int_{\Sigma_{d-1}} \mathrm{ev}^*_\infty \theta^{\mathrm{loc}}\,.$$

Remark 3.7. This is the standard way in which the kinetic Lagrangians in example 2.2 arise by transgression of local data.

It is useful to combine this data as follows.

Definition 3.8. Given a first jet bundle $j^1_\infty(\Sigma \times X)$ as in example 3.1, we write

1. $j^1_\infty(\Sigma \times X)^* \to \Sigma \times X$ for its fiberwise linear densitized dual, as a bundle over the field bundle, to be called the *dual first jet bundle*;
2. $j^1_\infty(\Sigma \times X)^\vee \to X\Sigma \times X$ for the fiberwise affine densitized dual, to be called the *affine dual first jet bundle.*

Remark 3.9. With respect to the canonical coordinates in example 3.1, the canonical coordinates of the dual first jet bundle are $\{\sigma^i, \phi^a, p^i_a\}$ (spacetime coordinates, fields and canonical field momenta) and the canonical coordinates of the affine dual first jet bundle are $\{\sigma^i, \phi^a, p^i_a, e\}$ with one more coordinate e.

Definition 3.10. 1. The *canonical pre-n-plectic form* on the affine dual first jet bundle, def. 3.8, is

$$\omega_e := \mathbf{d}\phi^a \wedge \mathbf{d}p^i_a \wedge \iota_{\partial_{\sigma^i}} \mathrm{vol}_\Sigma + \mathbf{d}e \wedge \mathrm{vol}_\Sigma \quad \in \Omega^{n+1}(j^1(\Sigma \times X)^\vee)\,.$$

2. Given a function $H \in C^\infty(j^1(\Sigma \times X)^*)$ on the linear dual first jet bundle, def. 3.8, then the corresponding *DWH pre-n-plectic form* is

$$\omega_H := \mathbf{d}\phi^a \wedge \mathbf{d}p^i_a \wedge \iota_{\partial_{\sigma^i}} \mathrm{vol}_\Sigma + \mathbf{d}H \wedge \mathrm{vol}_\Sigma \quad \in \Omega^{n+1}(j^1(\Sigma \times X)^*)\,.$$

Definition 3.11 (local Legendre transform)**.** Given a local Lagrangian as in def. 3.2, hende a horzontal n-form $L^{\mathrm{loc}} \in \Omega^{(n,0)}(J^1(E))$ on the jets of the field bundle, its *local Legendre transform* is the function

$$\mathbb{F}L^{\mathrm{loc}} : J^1(X) \longrightarrow (J^1(X))^\vee$$

from jets to the affine dual jet bundle, def. 3.8 which is the first order Taylor series of L^{loc}.

This definition was suggested in section 2.5 of [21]. It conceptualizes the traditional notion of local Legendre transform:

Example 3.12. In the local coordinates of example 3.1, the Legendre transform of a local Lagrangian L^{loc}, def. 3.11 has affine dual jet bundle coordinates given by

$$p^i_a = \frac{\partial L^{\mathrm{loc}}}{\partial \phi^a_{,i}}$$

and

$$e = L^{\mathrm{loc}} - \frac{\partial L^{\mathrm{loc}}}{\partial \phi^a_{,i}} \phi^a_{,i}\,.$$

The latter expression is what is traditionally taken to be the local Legendre transform of L^{loc}.

The following observation relates the canonical pre-n-plectic form ω_e on the affine dual jet bundle to the central ingredients of the covariant phase space formalism.

Proposition 3.13. *Given a local Lagrangian $L^{\mathrm{loc}} \in \Omega^{(n,0)}(J^1(E))$, then the pullback of the canonical pre-n-plectic form ω_e, def. 3.10, along the local Legendre transform $\mathbb{F}L^{\mathrm{loc}}$ of def. 3.11 is the sum of the Euler-Lagrange equation term $\mathrm{EL}_{L^{\mathrm{loc}}} \in \Omega^{(n,1)}(J^1(X))$ and of the canonical pre-n-plectic current $\mathbf{d}_v\theta_{L^{\mathrm{loc}}} \in \Omega^{(n-1,2)}(J^1(X))$ of def. 3.3:*

$$\begin{aligned} \omega_{L^{\mathrm{loc}}} &:= (\mathbb{F}L^{\mathrm{loc}})^*\omega_e \\ &= \mathrm{EL}_{L^{\mathrm{loc}}} + \mathbf{d}_v\theta_{\mathrm{L^{loc}}} \end{aligned}\,.$$

This follows with equation (54) and theorem 1 of [21].[1] In 3.3 below we see how using this the equations of motion of the field theory are naturally expressed.

In conclusion, we find that where phase spaces in classical mechanics are given by smooth spaces equipped with a closed 2-form, phase spaces in "de-transgressed" or "covariant" or "localized" classical field theory of dimension n are given by smooth spaces equipped with a closed $(n+1)$-form. To give this a name we say [15]:

Definition 3.14. For $n \in \mathbb{N}$, a *pre-n-plectic smooth space* is a smooth space X and a smooth closed $(n+1)$-form

$$\omega : X \longrightarrow \mathbf{\Omega}^{n+1}_{\mathrm{cl}} ,$$

hence an object of the slice topos

$$(X, \omega) \in \mathbf{H}_{/\mathbf{\Omega}^{n+1}_{\mathrm{cl}}} .$$

3.2 Local observables, conserved currents and higher Poisson bracket homotopy Lie algebras

Above in 2.5 we discussed how functions on a phase space are interpreted as observables of states of the mechanical system, for instance the energy of the system. Now in 3.1 above we saw that that notably the energy of an n-dimensional field theory at some moment in time (over some spatial hyperslice of spacetime) is really the integral over $(d-1)$-dimensional space Σ_{d-1} of an *energy density* $(d-1)$-form H^{loc}, hence by def. 3.6 the transgression of an $(n-1)$-form on the localized n-plectic phase space:

$$H = \int_{\Sigma_{n-1}} \mathrm{ev}_\infty^* H^{\mathrm{loc}} .$$

Therefore in analogy with the notion of observables on a symplectic manifold, given an n-plectic manifold, def. 3.14, its degree-$(n-1)$ differential forms may be called the *local observables* of the system. To motivate from physics how exactly to formalize such local observables (which we do below in def. 3.16), we first survey how such local observables appear in the physics literature:

Example 3.15 (currents in physics as local observables)**.** In the situation of example 3.1, consider a vector field $j \in \Gamma(T\Sigma_d)$ on the d-dimensional Minkowski spacetime $\Sigma_d = \mathbb{R}^{d-1,1}$. In physics this represents a quantity which – for an inertial observer characterized by the coordinates chosen in example 3.1 – has local density j^0 at each point in space and time, of a quantity that flows through space as given by the vector $(j^1, \cdots, j^{d-1})$.

For instance in the description of electric sources distributed in spacetime, the component j^0 would be an *electric charge density* and the vector $(j^1, \cdots, j^{d-1})$ would be the *electric current density*. To emphasize that therefore j combines the information of a spatial current with the density of the substance that flows, traditional physics textbooks call j a "d-current" – usually a "4-current" when identifying d with the number of macroscopic spacetime dimensions of the observable universe. But once the spacetime context is understood, one just speaks of j as a *current*.

The currents of interest in physics are those which satisfy a *conservation law*, a law which states that the change in coordinate time σ^0 of the density j^0 is equal to the negative of the divergence of the spatial current, hence that the spacetime divergence of j vanishes:

$$\mathrm{div}(j) = \frac{\partial j^0}{\partial \sigma^0} + \sum_{i=1}^{n-1} \frac{\partial j^i}{\partial \sigma_i} = 0 .$$

[1] This statement and its formulation in terms of notions in the variational bicomplex as given here has kindly been amplified to us by Igor Khavkine.

If this is the case, one calls the current j a *conserved current*. (Beware that the "conserved" is so important in applications that it is often taken to be implicit and notationally suppressed.)

In order to formulate the notion of divergence of a vector field intrinsically (as opposed with respect to a chosen coordinate system as above), one needs a specified volume form $\mathrm{vol}_\Sigma \in \Omega^d(\Sigma_d)$ of spacetime. With that given, the divergence $\mathrm{div}(j) \in C^\infty(\Sigma_d)$ of the vector field is defined by the equation

$$\mathrm{div}(j) \wedge \mathrm{vol}_{\Sigma_d} := \mathcal{L}_j \mathrm{vol}_{\Sigma_d} = \mathbf{d}\left(\iota_j \mathrm{vol}_\Sigma\right) .$$

In particular, a current j is a conserved current precisely if the degree-$(n-1)$ differential form

$$J := \iota_j \mathrm{vol}_{\Sigma_d}$$

is a closed differential form

$$(j \in \Gamma(T\Sigma_d) \text{ is a conserved current}) \quad \Leftrightarrow \quad (\mathbf{d}J = 0) .$$

Due to this and related relations, one finds eventually that the degree-$(d-1)$ differential form J itself is the more fundamental mathematical reflection of the physical current. But by the above introduction, this is in turn the same as saying that a current is a local observable. Accordingly, we will often use the terms "current" and "local observable" interchangeably.

If currents are local observables, then by the above discussion their integral over a spatial hyperslice of spacetime is to be the corresponding global observable. In the special case of the electromagnetic current J_{el}, the laws of electromagnetism in the form of *Maxwell's equation*

$$J_{\mathrm{el}} = \mathbf{d} \star F_{\mathrm{em}}$$

say that this integral – assuming now that J_{el} is spatially compactly supported – is the integral of the Hodge dual electromagnetic field strength F_{em} over the boundary of a 3-ball $D^3 \hookrightarrow \Sigma_{d-1}$ enclosing the support of the electromagnetic current. This is the *total electric charge* Q_{el} in space:

$$Q_{\mathrm{el}} = \int_{S^2} *F_{\mathrm{em}} = \int_{D^3} J_{\mathrm{el}} = \int_{\Sigma_{d-1}} J_{\mathrm{el}} .$$

Based on this example, in physics one generally speaks of the integral of a spacetime current over space as a *charge*. So charges are the global observables of the local observables, which are currents.

Notice that for a *conserved* current the corresponding charge is also conserved in that it does not change with time or in fact under any isotopy of Σ_{d-1} inside Σ_d, due to Stokes' theorem:

$$\begin{aligned} \mathbf{d}_{\Sigma_1} Q &= \mathbf{d}_{\Sigma_1} \int_{\Sigma_{d-1}} J \\ &= \int_{\Sigma_{d-1}} \mathbf{d}_{\Sigma_d} J \\ &= 0 \end{aligned} .$$

Therefore currents in physics are necessarily subject of *higher gauge equivalences*: if J is a conserved current $(d-1)$-form, then for any $(d-2)$-form α the sum $J + \mathbf{d}\alpha$ is also a conserved current, which, by Stokes' theorem, has the same total charge as J in any $(d-1)$-ball in space, and has the same flux as J through the boundary of that $(d-1)$-ball. This means that the conserved currents J and $J + \mathbf{d}\alpha$ are physically equivalent, while nominally different, hence that α exhibits a *gauge equivalence transformation* between currents

$$\alpha : J \xrightarrow{\simeq} (J' = J + \mathbf{d}\alpha) .$$

The analogous consideration holds for α itself: for any $(d-3)$-form β also $\alpha + \mathbf{d}\beta$ exhibits a gauge transformation between the currents J and J' above. One says this is a *gauge of gauge*-transformation or a *higher gauge transformation* of second order. This phenomenon continues up to the 0-forms (the smooth functions), which therefore are $(d-1)$-fold higher gauge transformations between consderved currents on a d-dimensional spacetime.

Finally notice that in a typical application to physics, a current form J is naturally defined also "off shell", hence for all field configurations (say of the electromagnetic field), but its conservation law only holds "on shell", hence when these field configurations satisfy their equations of motion (to which we come below in 3.3). Since the n-plectic localized phase spaces in the discussion in 3.1 above a priori contain all field configurations, we are not to expect that a local observable $(d-1)$-form J is a conserved current only if its differential strictly vanishes, but already if its differential vanishes at least on those d-tuples of vector fields $v_1 \vee \cdots \vee v_d$ which are tangent to jets of those sections of the field bundle that satisfy their equations of motion:

$$(J \text{ is conserved current}) \Leftrightarrow ((v_1 \vee \cdots \vee v_d \text{ satisfies field equations of motion}) \Rightarrow \iota_{v_1 \vee \cdots \vee v_n} \mathbf{d} J = 0) \ .$$

This we formalize below by the "n-plectic Noether theorem", prop. 3.24. There we will see how such conserved current $(d-1)$-forms arise from vector fields v that consitute infinitesimal symmetries of a Hamiltonian function, by the evident higher degree generalizatin of Hamilton's equations, namely $\mathbf{d}J = \iota_v \omega$.

In summary, example 3.15 motivates the following definition (first proposed in [46] and then interpreted in homotopy topos theory in [15, 16]) of the localized/higher analog of the Poisson bracket Lie algebra of observables, defs. 2.32, 2.34, as we pass from global observable on (pre-)symplectic manifolds to local observables on (pre-)n-plectic manifolds. The general abstract characterization of the following definition we give below in 3.4.

Definition 3.16 (higher Poisson bracket of local observables)**.** Given a pre-n-plectic manifold (X, ω), its vector space of *local Hamiltonian observables* is

$$\Omega^{n-1}_\omega(X) := \left\{ (v, J) \in \Gamma(TX) \oplus \Omega^{n-1}(X) \mid \iota_v \omega = -\mathbf{d} J \right\} \ .$$

We say that the de Rham complex ending in these Hamiltonian observables is the *complex of local observables* of (X, ω), denoted

$$\Omega^\bullet_\omega(X) := \left(C^\infty(X) \xrightarrow{\mathbf{d}} \Omega^1(X) \xrightarrow{\mathbf{d}} \cdots \xrightarrow{\mathbf{d}} \Omega^{n-2}(X) \xrightarrow{(0,\mathbf{d})} \Omega^{n-1}_\omega(X) \right) \ .$$

The *binary higher Poisson bracket* on local Hamiltonian observables is the linear map

$$\{-,-\} \ : \ \Omega^{n-1}_\omega(X) \otimes \Omega^{n-1}_\omega(X) \longrightarrow \Omega^{n-1}_\omega(X)$$

given by the fomula

$$[(v_1, J_1) \, , (v_2, J_1)] := [([v_1, v_2] \, , \, \iota_{v_1 \vee v_2} \omega)] \ ;$$

and for $k \geq 3$ the *k-ary higher Poisson bracket* is the linear map

$$\{-, \cdots, -\} \ : \ \left(\Omega^{n-1}_\omega(X) \right)^{\otimes^k} \longrightarrow \Omega^{n+1-k}(X)$$

given by the formula

$$[(v_1, J_1) \, , \cdots , (v_k, J_k)] := (-1)^{\lfloor \frac{k-1}{2} \rfloor} \iota_{v_1 \vee \cdots \vee v_k} \omega \, .$$

The chain complex of local observables equipped with these linear maps for all k we call the *higher Poisson bracket homotopy Lie algebra* of (X, ω), denoted

$$\mathfrak{pois}(X, \omega) := (\Omega^\bullet_\omega(X), \{-,-\} \, , \{-,-,-\} \, , \cdots) \ .$$

Remark 3.17. What we call a *homotopy Lie algebra* in def. 3.16 is what originally was called a *strong homotopy Lie algebra* and what these days is mostly called an L_∞*-algebra* or, since the above chain complex is concentrated in the lowest n degrees, a *Lie n-algebra*. These are the structures that are to group-like smooth homotopy types as Lie algebras are to smooth groups. The reader can find all further details which we need not dwell on here as well as pointers to the standard literature in [16].

Remark 3.18. For $n = 2$ definition 3.16 indeed reproduces the definition of the ordinary Poisson bracket Lie algebra, def. 2.34.

3.3 Field equations of motion, higher Maurer-Cartan elements, and higher Lie integration

Where in classical mechanics the equations of motion that determine the physically realized trajectories are Hamilton's equations, def. 2.29, in field theory the equations of motion are typically *wave equations* on spacetime. But as we localize from (pre-)symplectic phase spaces to (pre-)n-plectic phase spaces as in 3.1 above, Hamilton's equations also receive a localization to the *Hamilton-de Donder-Weyl* equation. This indeed coincides with the field-theoretic equations of motion. We briefly review the classical idea of de Donder-Weyl formalism and then show how it naturally follows from a higher geometric version of Hamilton's equations in n-plectic geometry.

Definition 3.19. Let (X, ω) be a pre-n-plectic smooth manifold, and let $H \in C^\infty(X)$ be a smooth function, to be called the *de Donder-Weyl Hamiltonian*. Then for $v_i \in \Gamma(TX)$ with $i \in \{1, \cdots, n\}$ an n-tuple of vector fields, the *Hamilton-de Donder-Weyl* equation is

$$(\iota_{v_n} \cdots \iota_{v_1})\omega = \mathbf{d}H\,.$$

Generally, for $J \in \Omega^{n-k}(X)$ a smooth differential form for $1 \leq k \leq n$, and for $\{v_i\}$ a k-tuple of vector fields, the *extended Hamilton-deDonder-Weyl equation* is

$$\iota_{v_k} \cdots \iota_{v_1}\omega = \mathbf{d}J\,.$$

We now first show how this describes equations of motion of field theories. Then we discuss how this de Donder-Weyl-Hamilton equation is naturally found in higher differential geometry. For simplicity of exposition we stick with the simple local situation of example 3.1. The ambitious reader can readily generalize all of the following discussion to non-trivial and non-linear field bundles.

Definition 3.20. Let $\Sigma \times X \to \Sigma$ be a field bundle as in example 3.1. For $\Phi := (\phi^i, p^a_i) : \Sigma \to j^1(\Sigma \times X)^*$ a section of the linear dual jet bundle write

$$v_i^\Phi = \frac{\partial}{\partial \sigma^i} + \frac{\partial \phi^a}{\partial \sigma^i}\frac{\partial}{\partial \phi^a} + \frac{\partial p_a^j}{\partial \sigma^i}\frac{\partial}{\partial p_a^j}$$

for its canonical basis of tangent vector fields. Similarly for $\Phi := (\phi^i, p^a_i, e) : \Sigma \to j^1(\Sigma \times X)^\vee$ a section of the affine dual jet bundle write

$$v_i^\Phi = \frac{\partial}{\partial \sigma^i} + \frac{\partial \phi^a}{\partial \sigma^i}\frac{\partial}{\partial \phi^a} + \frac{\partial p_a^j}{\partial \sigma^i}\frac{\partial}{\partial p_a^j} + \frac{\partial e}{\partial \sigma^i}\frac{\partial}{\partial e}$$

for its canonical basis of tangent vector fields.

Proposition 3.21. *For $(\Sigma \times X) \to \Sigma$ a field bundle as in example 3.1, let $H \in C^\infty(j^1(\Sigma \times X)^*)$ be a function on the linear dual (and hence on the affine dual) first jet bundle. Then for a section Φ of the* *linear dual field bundle the homogeneous ("relativistic") de Donder-Weyl-Hamilton equation, def. 3.19, of the Hamiltonian pre-n-plectic form, def. 3.10,*

$$\left(\iota_n^\Phi \cdots \iota_1^\Phi\right) \omega_H = 0$$

has a unique lift, up to an addive constant, to a solution of the DWH equation on the affine dual field bundle of the form

$$\left(\iota_n^\Phi \cdots \iota_1^\Phi\right) \omega_e = \mathbf{d}(H + e)\,.$$

Moreover, both these equations are equivalent to the following system of differential equations

$$\partial_i \phi^a = \frac{\partial H}{\partial p_a^i} \quad ; \quad \partial_i p_a^i = -\frac{\partial H}{\partial \phi^a}\,.$$

The last system of differential equations is the form in which the de Donder-Weyl-Hamilton equation is traditionally displayed, see for instance theorem 2 [47]. The inhomogeneous version on the affine dual first jet bundle above has been highlighted in [23], around equation (4) there.

Example 3.22. For a field bundle as in example 3.1, the standard form of an energy density function for a field theory on Σ is

$$H \mathrm{vol}_\Sigma = L_{\mathrm{kin}} + V(\{\phi^a\}) \mathrm{vol}_\Sigma\,,$$

where the first summand is the kinetic energy density from example 3.2 and where the second is any potential term as in example 2.3. More explicitly this means that

$$H = \langle \nabla\phi, \nabla\phi \rangle + V(\{\phi^a\}) = k^{ab} \eta_{ij} p_a^i p_b^j + V(\{\phi^a\})\,.$$

For this case the first component of the Hamilton-de Donder-Weyl equation in the form of prop. 3.21 is the equation

$$\partial_i \phi^a = k^{ab} \eta_{ij} p_b^j\,.$$

This identifies the canonical momentum with the actual momentum. More formally, this first equation *enforces the jet prolongation* in that it forces the section of the dual first jet bundle to the field bundle to be the actual dual jet of an actual section of the field bundle.

Using this, the second component of the DWH equation in the form of prop. 3.21 is equivalently the *wave equation*

$$\eta^{ij} \partial_i \partial_j \phi^a = -\frac{\partial V}{\partial \phi^a}$$

with inhomogeneity given by the gradient of the potential. These equations are the hallmark of classical field theory.

In full generality we can express the Euler-Lagrange equations of motion of a local Lagrangian in Hamilton-de Donder-Weyl form by prop. 3.13.

In order for the Hamilton-de Donder-Weyl equation to qualify as a good "localization" or "de-transgression" of non-covariant classical field theory as in example 2.2 it should be true that it reduces to this under transgression. This is indeed the case[2]

Proposition 3.23. *With $\omega_{L^{\mathrm{loc}}}$ as in prop. 3.13, we have that for any Cauchy surface Σ_{n-1} that transgression of $\omega_{L^{\mathrm{loc}}}$ yields the covariant phase space pre-symplectic form of example 2.2.*

Using the n-plectic formulation of the Hamilton-de Donder-Weyl equation, we naturally obtain now the following n-plectic formulation of the refinement of the "symplectic Noether theorem", def. 2.37, form mechanics to field theory:

[2] Again thanks go to Igor Khavkine for discussion of this point.

Proposition 3.24 (n-plectic Noether theorem)**.** *Let (X,ω) be a pre-n-plectic manifold equipped with a function $H \in C^\infty(X)$, to be regarded as a de Donder-Weyl Hamiltonian. If a vector field $v \in \Gamma(TX)$ is a symmetry of H in that*

$$\iota_v \mathbf{d}H = 0\,,$$

then along any n-vector field $v_1 \vee \cdots \vee v_n$ which solves the Hamilton-de Donder-Weyl equation, def. 3.19, the corresponding current $\mathbf{J}_v := \iota_v\omega$ is conserved, in that

$$\iota_{(v_1,\cdots,v_n)}\mathbf{d}J_v = 0\,.$$

Conversely, if a current is conserved on solutions to the Hamilton-de Donder-Weyl equations of motion this way, then it generates a symmetry of the de Donder-Weyl Hamiltonian.

Proof. By the various definitions and assumptions we have

$$\begin{aligned}\iota_{v_1\vee\cdots\vee v_n}\mathbf{d}J_{n+1} &= \iota_{v_1\vee\cdots\vee v_n}\iota_v\omega \\ &= (-)^n\iota_v\iota_{v_1\vee\cdots\vee v_n}\omega \\ &= \iota_v\mathbf{d}H \\ &= 0\end{aligned}\,.$$

□

This shows how the multisymplectic/n-plectic analog of the symplectic formulation of Hamilton's equations, def. 2.29, serves to encode the equations of motion, the symmetries and the conserved currents of classical field theory. But in 2.10 and 2.12 above we had seen that the symplectic formulation of Hamilton's equations in turn is equivalently just an infinitesimal characterization of the automorphisms of a pre-quantized phase space $X \xrightarrow{\nabla} \mathbf{B}U(1)_{\mathrm{conn}}$ in the higher slice topos $\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}$. This suggests that n-dimensional Hamilton-de Donder-Weyl flows should characterize n-fold homotopies in the higher automorphism group of a higher prequantization, regarded as an object in a higher slice topos to be denoted $\mathbf{H}_{/\mathbf{B}^nU(1)_{\mathrm{conn}}}$. This we come to below in 3.4.

Here we now first consider the infinitesimal aspect this statement. To see what this will look like, observe that the statement for $n=1$ is that the Lie algebra of slice automorphisms of ∇ is the Poisson bracket Lie algebra $\mathfrak{pois}(X,\omega)$ whose elements, by def. 2.34, are precisely the pairs (v,H) that satisfy Hamilton's equation $\iota_v\omega = H$. To say this more invariantly: Hamilton's equations on (X,ω) precisely characterize the Lie algebra homomorphisms of the form

$$\mathbb{R} \longrightarrow \mathfrak{pois}(X,\omega)\,,$$

where on the left we have the abelian Lie algebra on a single generator. This suggests that for a (pre-)n-plectic manifold, we consider homotopy Lie algebra homomorphism of the form

$$\mathbb{R}^n \longrightarrow \mathfrak{pois}(X,\omega)\,,$$

where now on the left we have the abelian Lie algebra on n generators, regarded canonically as a homotopy Lie algebra. In comparison with prop. 2.45, this may be thought of as characterizing the infinitesimal approximation to an evolution n-functor from Riemannian n-dimensional cobordisms into the (delooping of) the higher Lie integration of $\mathfrak{pois}(X,\omega)$ (recall remark 2.80 above).

Such homomorphisms of homotopy Lie algebras are computed as follows.

Definition 3.25. Given a pre-n-plectic smooth space (X,ω), write

$$\mathfrak{pois}(X,\omega)^{(\square^n)} := (\wedge^\bullet\mathbb{R}^n)\otimes\mathfrak{pois}(X,\omega)$$

for the homotopy Lie algebra obtained from the Poisson bracket Lie n-algebra of def. 3.16 by tensoring with the Grassmann algebra on n generators, hence the graded-symmetric algebra on n generators in degree 1.

Remark 3.26. A basic fact of homotopy Lie algebra theory implies that homomorphisms of the form $\mathbb{R}^n \longrightarrow \mathfrak{pois}(X,\omega)$ are equivalent to elements $\mathcal{J} \in \mathfrak{pois}(X,\omega)^{\Delta^n}$ of degree 1, which satisfy the *homotopy Maurer-Cartan equation*

$$d\mathcal{J} + \tfrac{1}{2}\{\mathcal{J},\mathcal{J}\} + \tfrac{1}{6}\{\mathcal{J},\mathcal{J},\mathcal{J}\} + \cdots = 0$$

Example 3.27. Write $\{\mathbf{d}\sigma^i\}_{i=1}^n$ for the generators of $\wedge^\bullet\mathbb{R}^n$. Then a general element of degree 1 in $\mathfrak{pois}(X,\omega)^{(\square^n)}$ is of the form

$$\mathcal{J} = \mathbf{d}\sigma^i \otimes (v_i, J_i) + \mathbf{d}\sigma^i \wedge \mathbf{d}\sigma^j \otimes J_{ij} + \mathbf{d}\sigma^i \wedge \mathbf{d}\sigma^j \wedge \mathbf{d}\sigma^k \otimes J_{ijk} + \cdots + (\mathbf{d}\sigma^1 \wedge \cdots \wedge \mathbf{d}\sigma^n) \otimes H\,,$$

where

1. $v_i \in \Gamma(TX)$ is a vector field and $J_i \in \Omega^n(X)$ is a differential n-foms such that $\iota_{v_i}\omega = \mathbf{d}J_i$
2. $J_{i_1\cdots i_k} \in \Omega^{n+1-k}(X)$;
3. $H \in C^\infty(X)$.

From this we deduce the following.

Proposition 3.28. *Given a pre-n-plectic smooth space (X,ω), the extended Hamilton-de Donder-Weyl equations, def. 3.19, characterize, under the identification of example 3.27, the homomorphims of homotopy Lie algebras from $\mathbb{R}^n$ into the higher Poisson bracket Lie n-algebra of def. 3.16:*

$$(\mathcal{J} : \mathbb{R}^n \longrightarrow \mathfrak{pois}(X,\omega)) \quad\Leftrightarrow\quad \left\{ \begin{array}{l} \iota_{v_n}\cdots\iota_{v_1}\omega = \mathbf{d}H \\ \iota_{v_{ik}}\cdots\iota_{v_{i_2}}\iota_{v_{i_1}}\omega = \mathbf{d}J_{i_1 i_2\cdots i_k} \quad \forall_k \forall_{i_1,\cdots,i_k} \end{array} \right.$$

Remark 3.29. The Lie integration of the Lie n-algebra $\mathfrak{pois}(X,\omega)$ is the smooth n-groupoid whose n-cells are Maurer-Cartan elements in

$$\Omega^\bullet_{\mathrm{si}}(\Delta^n) \otimes \mathfrak{pois}(X,\omega)\,,$$

see [19] for details. The construction in def. 3.25 is a locally constant approximation to that. In general there are further σ-dependent terms.

Due to [15, 16] we have that the Lie integration of $\mathfrak{pois}(X,\omega)$ is the automorphism n-group $\mathbf{Aut}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}(\nabla)$ of any pre-quantization ∇ of (X,ω), see D. This means that the above maps

$$\mathbb{R}^n \longrightarrow \mathfrak{pois}(X,\omega)$$

are infinitesimal approximations to something lie n-functors of the form

$$\text{“ } \mathrm{Bord}_n^{\mathrm{Riem}} \longrightarrow \mathbf{H}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}} \text{”}$$

in higher dimensional analogy of prop. 2.45. This we come to below.

3.4 Higher gauge theory, smooth ∞-groupoids and homotopy toposes

(...) C, D (...)

3.5 Higher Chern-Simons-type boundary field theory and higher Chern-Weil theory

(...) [15]

(...explain how Wilson loops in an ambient gauge theory are 1d topological prequantum systems that mathematically are the content of Kirillov's orbit method... explain how the symplectic groupoid $\mathfrak{g}^*//G$ provides the "universal Wilson loop" system as an "internal degrees of freedom" analog to the off-shell Poisson bracket...) from [18]

3.6 Source terms, the off-shell Poisson bracket and holography with symplectic groupoids

We connect now the discussion of mechanics in 2 to that of higher Chern-Simons field theory in 3.5 by showing that the space of all trajectories of a mechanical system naturally carries a Poisson brakcet structure which is foliated by symplectic leafs that are labled by source terms.[3] The corresponding leaf space is naturally refined to the symplectic groupoid that is the moduli stack of fields of the non-perturbative 2s Poisson-Chern-Simons theory. This yields a precise implementation of the "holographic principle" where the 2d Poisson-Chern-Simons theory in the bulk carries on its boundary a 1d field theory (mechanical system) such that fields in the bulk correspond to sources on the boundary.

Let (X, ω) be a symplectic manifold. We write

$$\{-,-\} : C^\infty(X) \otimes C^\infty(X) \longrightarrow C^\infty(X)$$

for the Poisson bracket induced by the symplectic form ω, hence by the Poisson bivector $\pi := \omega^{-1}$.

For notational simplicity we will restrict attention to the special case that

$$X = \mathbb{R}^2 \simeq T^*\mathbb{R}$$

with canonical coordinates

$$q, p : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

and symplectic form

$$\omega = \mathbf{d}q \wedge \mathbf{d}p\,.$$

The general case of the following discussion is a straightforward generalization of this, which is just notationally more inconvenient.

Write $I := [0,1]$ for the standard interval regarded as a smooth manifold manifold with boundary—with boundary. The mapping space

$$PX := [I, X]$$

canonically exists as a smooth space, but since I is compact topological space—compact this structure canonically refines to that of a Frchet manifold. This implies that there is a good notion of tangent space TPX. The task now is to construct a certain Poisson bivector as a section $\pi \in \Gamma^{\wedge 2}(TPX)$.

Among the smooth functions on PX are the evaluation maps

$$ev : PX \times I = [I, X] \times I \longrightarrow X$$

whose components we denote, as usual, for $t \in I$ by

$$q(t) := q \circ ev_t : PX \longrightarrow \mathbb{R}$$

and

$$p(t) := p \circ ev_t : PX \longrightarrow \mathbb{R}\,.$$

Generally for $f \colon X \to \mathbb{R}$ any smooth function, we write $f(t) := f \circ ev_t \in C^\infty(PX)$. This defines an embedding

$$C^\infty(X) \times I \hookrightarrow C^\infty(PX)\,.$$

Similarly we have

$$\dot q(t) : PX \longrightarrow \mathbb{R}$$

[3]This phenomenon was kindly pointed out to us by Igor Khavkine.

and

$$\dot{q}(t) \,:\, PX \longrightarrow \mathbb{R}$$

obtained by differentiation of $t \mapsto q(t)$ and $t \mapsto p(t)$.

Let now

$$H \,:\, X \times I \longrightarrow \mathbb{R}$$

be a smooth function, to be regarded as a time-dependent Hamiltonian. This induces a time-dependent function on trajectory space, which we denote by the same symbol

$$H \,:\, PX \times I \overset{(ev,id)}{\longrightarrow} X \times X \overset{H}{\longrightarrow} \mathbb{R}\,.$$

Hence for $t \in I$ we write

$$H(t) \,:\, PX \times \{t\} \overset{(ev,id)}{\longrightarrow} X \times \{t\} \overset{H}{\longrightarrow} \mathbb{R}$$

for the function that assigns to a trajectors $(q(-), p(-)) \colon I \longrightarrow X$ its energy at (time) parameter value t. Define then the Euler-Lagrange equation—Euler-Lagrange density induced by H to be the functions

$$\mathrm{EL}(t) \,:\, PX \longrightarrow \mathbb{R}^2$$

with components

$$\mathrm{EL}(t) = \left(\begin{array}{c} \dot{q}(t) - \frac{\partial H}{\partial p}(t) \\ \dot{p}(t) + \frac{\partial H}{\partial p}(t) \end{array} \right) .$$

The trajectories $\gamma \colon I \to X$ on which $EL(t)$ vanishes for all $t \in I$ are equivalently those

- for which the tangent vector $\dot{\gamma} \in T_\gamma X$ is a Hamiltonian vector field—Hamiltonian vector for H;
- which satisfy Hamilton's equations equations of motion—of motion for H.

Since the differential equations $\mathrm{EL} = 0$ have a unique solution for given initial data $(q(0), p(0))$, the evaluation map

$$\{\gamma \in PX | \forall_{t \in I} \; EL_\gamma(t) = 0\} \overset{\gamma \mapsto \gamma(0)}{\longrightarrow} X$$

is an equivalence (an isomorphism of smooth spaces).

Write

$$\mathrm{Poly}(PX) \hookrightarrow C^\infty(PX)$$

for the subalgebra of smooth functions on path space which are polynomials of integrals over I, of the smooth functions in the image of $C^\infty(X) \times I \hookrightarrow C^\infty(PX)$ and all their derivatives along I.

Define a bilinear function

$$\{-,-\} \,:\, \mathrm{Poly}(PX) \otimes \mathrm{Poly}(PX) \longrightarrow \mathrm{Poly}(PX)$$

as the unique function which is a derivation in both arguments and moreover is a solution to the differential equations

$$\frac{\partial}{\partial t_2}\{f(t_1), q(t_2)\} = \left\{ f(t_1), \frac{\partial H}{\partial p}(t_2) \right\}$$

$$\frac{\partial}{\partial t_2}\{f(t_1), p(t_2)\} = -\left\{ f(t_1), \frac{\partial H}{\partial q}(t_2) \right\}$$

subject to the initial conditions

$$\{f(t), q(t)\} = \{f, q\}$$

$$\{f(t), p(t)\} = \{f, p\}$$

for all $t \in I$, where on the right we have the original Poisson bracket on X.

This bracket directly inherits skew-symmetry and the Jacobi identity from the Poisson bracket of (X, ω), hence equips the vector space $Poly(PX)$ with the structure of a Lie bracket. Since it is by construction also a derivation of $Poly(PX)$ as an associative algebra, we have that

$$(\mathrm{Poly}\,(PX)\,,\ \{-,-\}) \quad \in P_1 Alg$$

is a Poisson algebra. This is the "off-shell Poisson algebra" on the space of trajectories in (X, ω).

Observe that by construction of the off-shell Poisson bracket, specifically by the differential equations defining it, the Euler-Lagrange equation—Euler-Lagrange function EL generate a Poisson reduction—Poisson ideal.

For instance

$$\left(\begin{array}{ccc} \frac{\partial}{\partial t_2} \{f(t_1), q(t_2)\} & = & \left\{ f(t_1), \frac{\partial H}{\partial p}(t_2) \right\} \\ \frac{\partial}{\partial t_2} \{f(t_1), p(t_2)\} & = & -\left\{ f(t_1), \frac{\partial H}{\partial q}(t_2) \right\} \end{array} \right) \quad \Leftrightarrow \quad (\{f(t_1),\ EL(t)\} = 0)\ .$$

Moreover, since $\{EL(t) = 0\}$ are equations of motion the Poisson reduction defined by this Poisson idea is the subspace of those trajectories which are solutions of Hamilton's equations, hence the "on-shell trajectories".

As remarked above, the initial value map canonically identifies this on-shell trajectory space with the original phase space manifold X. Moreover, by the very construction of the off-shell Poisson bracket as being the original Poisson bracket at equal times, hence in particular at time $t = 0$, it follows that restricted to the zero locus $EL = 0$ the off-shell Poisson bracket becomes symplectic manifold—symplectic.

All this clearly remains true with the function EL replaced by the function $EL - J$, for $J \in C^\infty(I)$ any function of the (time) parameter (since $\{J, -\} = 0$). Any such choice of J hence defines a symplectic subspace

$$\{\gamma \in PX \mid \forall_{t \in I}\ EL_\gamma(t) = J\}$$

of the off-shell Poisson structure on trajectory space. Hence $(OX, \{-,-\})$ has a foliation by symplectic leaves with the leaf space being the smooth space $C^\infty(I)$ of smooth functions on the interval.

Notice that changing $\mathrm{EL} \mapsto \mathrm{EL} - J$ corresponds changing the time-dependent Hamiltonian H as

$$H \mapsto H - Jq\,.$$

Such a term linear in the canonical coordinates (the field (physics)—fields) is a *source term*. (The action functionals with such source terms added serve as integrands of generating functions for correlators in statistical mechanics and in quantum mechanics.)

Hence in conclusion we find the following statement:

The trajectory space (history space) of a mechanical system carries a natural Poisson manifold—Poisson structure whose symplectic leaves are the subspaces of those trajectories which satisfy the equations of motion with a fixed source term and hence whose symplectic leaf space is the space of possible sources.

Notice what becomes of this statement as we consider the the 2d Chern-Simons theory induced by the off-shell Poisson bracket (the non-perturbative field theory—non-pertrbative Poisson sigma-model) whose moduli stack of field (physics)—fields is the symplectic groupoid $SG\,(PX, \{-,-\})$ induced by the Poisson structure.

By the discussion at ... the Poisson space $(PX, \{-,-\})$ defines a boundary field theory (in the sense of local prequantum field theory) for this 2d Chern-Simons theory, exhibited by a boundary correspondence of the form

$$
\begin{tikzcd}
 & PX \arrow[dl] \arrow[dr] & \\
* \arrow[dr] & & \mathrm{SG}\left(PX, \{-,-\}\right) \arrow[dl] \arrow[ll, phantom, "\swarrow"] \\
 & \mathbf{B}^2 U(1) \arrow[d] & \\
 & \mathrm{KUMod} &
\end{tikzcd}
.
$$

Notice that the symplectic groupoid is a version of the symplectic leaf—symplectic leaf space of the given Poisson manifold (its 0-truncation is exactly the leaf space). Hence in the case of the off-shell Poisson bracket, the symplectic groupoid is the space of *sources* of a mechanical system. At the same time it is the moduli space of field (physics)—fields of the 2d Chern-Simons theory of which the mechanical system is the boundary field theory. Hence the field (physics)—fields of the bulk field theory are identified with the sources of the boundary field theory. Hence conceptually the above boundary correspondence diagram is of the following form

$$
\begin{tikzcd}
 & \text{Sources} \arrow[dl] \arrow[dr] & \\
* \arrow[dr] & & \text{Fields} \arrow[dl] \arrow[ll, phantom, "\swarrow"] \\
 & \text{Phases} &
\end{tikzcd}
.
$$

A Homotopy toposes and Cohesive homotopy types

For reference, here we very briefly collect some pointers to basics on homotopy toposes [31], homotopy type theory [59], cohesive homotopy type theory [50] and the the categorical semantics relating them. The sections to follow discuss the aspects of this that are relevant in the main text in detail.

For the most basic notions of category theory see the first pages of [38] or A.1 in [31].

Definition A.1. A category $\mathcal{C}$ is called *cartesian closed* if it has Cartesian products $X \times Y$ of all objects $X, Y \in \mathcal{C}$ and if there is for each $X \in \mathcal{C}$ a mapping space functor $[X, -] : \mathcal{C} \longrightarrow \mathcal{C}$, characterized by the fact that there is a bijection of hom-sets

$$\mathcal{C}(X \times A, Y) \simeq \mathcal{C}(A, [X, Z])$$

natural in the objects $A, X, Y \in \mathcal{C}$. A category $\mathcal{C}$ is called *locally cartesian closed* if for each object $X \in \mathcal{C}$ the slice category $\mathcal{C}_{/X}$ is a cartesian closed category.

The main example of locally cartesian closed categories of interest here are toposes, to which we come below in def. A.9. It is useful to equivalently re-express local cartesian closure in terms of *base change*:

Proposition A.2. *If $\mathcal{C}$ is a locally cartesian closed category, def. A.1, then for $f : X \longrightarrow Y$ any morphism in $\mathcal{C}$ there exists an adjoint triple of functors between the slice categories over X and Y (called* base change *functors)*

$$\mathcal{C}_{/\Gamma_1} \underset{\xrightarrow[f_*]{}}{\overset{\xrightarrow{f_!}}{\xleftarrow{f^*}}} \mathcal{C}_{/\Gamma_2} \ ,$$

where f^* is given by pullback along f, $f_!$ is its left adjoint and f_* its right adjoint. Conversely, if a category $\mathcal{C}$ has pullbacks and has for every morphism f a left and right adjoint $f_!$ and f_* to the pullback functor f^*, then it is locally cartesian closed.

It turns out that base change may usefully be captured syntactically such as to constitute a flavor of formal logic called *constructive set theory* or *type theory* [37]:

Definition A.3. Given a locally cartesian closed category $\mathcal{C}$, one says equivalently that

- its internal logic is a *dependent type theory*;
- it provides *categorical semantics* for dependent type theory

as follows:

- the objects of $\mathcal{C}$ are called the *types*;
- the objects in a slice $\mathcal{C}_{/\Gamma}$ are called the types *in context* Γ or *dependent on* Γ, denoted

$$\Gamma \ \vdash \ X : \mathrm{Type}$$

- a morphism $* \to X$ (from the terminal object into any object X) in a slice $\mathcal{C}_\Gamma$ is called *a term of type* X in context Γ, and denoted

$$\Gamma \ \vdash \ x : X$$

  or more explicitly

$$a : \Gamma \ \vdash \ x(a) : X(a);$$

- given a morphism $f : \Gamma_1 \longrightarrow \Gamma_2$ in $\mathcal{C}$ with its induced base change adjoint triple of functors between slice categories from prop. A.2

$$\mathcal{C}_{/\Gamma_1} \underset{\xrightarrow[f_*]{}}{\overset{\xrightarrow{f_!}}{\xleftarrow{f^*}}} \mathcal{C}_{/\Gamma_2}$$

  then

– given a morphism $(* \to X)$ in $\mathcal{C}_{/\Gamma_2}$, hence a term $\Gamma_2 \vdash x : X$, then its pullback by f^* is denoted by *substitution* of variables
$$a : \Gamma_1 \vdash x(f(a)) : X(f(a)),$$

– given an object $X \in \mathcal{C}_{\Gamma_1}$ its image $f_!(X) \in \mathcal{C}_{/\Gamma_2}$ is called the *dependent sum* of X along f and is denoted as
$$\Gamma_2 \vdash \sum_f X : \mathrm{Type},$$

– given an object $X \in \mathcal{C}_{\Gamma_1}$ its image $f_*(X) \in \mathcal{C}_{/\Gamma_2}$ is called the *dependent product* of X along f and is denoted as
$$\Gamma_2 \vdash \prod_f X : \mathrm{Type},$$

- the universal property of the adjoints $(f_! \dashv f^* \dashv f_*)$ translates to evident rules for introducing and for transforming terms of these dependent sum/product types, called *term introduction* and *term elimination* rules.

When this syntactic translation is properly formalized, it yields an equivalent description of locally cartesian closed categories:

Proposition A.4 ([56, 10])**.** *There is an equivalence of* 2*-categories between locally cartesian closed categories and dependent type theories.*

Remark A.5. Given any object $X \in \mathcal{C}_{/\Gamma}$, its diagonal $X \longrightarrow X \times X$ regarded as an object of $\mathcal{C}_{/(\Gamma \times X \times X)}$ serves as the *identity type* of X, denoted

$$\Gamma, (x_1, x_2) : X \times X \vdash (x_1 = x_2) : \mathrm{Type}.$$

Namely given two terms $x_1, x_2 : X$, then a term $\Gamma \vdash p : (x_1 = x_2)$ is as a morphism in $\mathcal{C}$ an element on the diagonal of $X \times X$ and in the type theory is a *proof of equality* of x_1 and x_2. If there is such a proof of equality then it is unique, since the diagonal is always a monomorphism.

But consider now the case that $\mathcal{C}$ in addition carries the structure of a *model category* (see A.2 in [31] for a review). Then there is for each X a path space object $X^I \longrightarrow X \times X$. Using this as the categorical semantics of identity types, instead of the plain diagonal $X \longrightarrow X \times X$, means to make identity behave instead like *higher gauge equivalence* in physics: there are then possibly many equivalences between two terms of a given type, and many equivalences between equivalences, and so on. If $\mathcal{C}$ is moreover right proper as a model category and such that its cofibrations are precisely its monomorphisms, then there exists a variant of the dependent type theory of remark A.3 reflecting these homotopy-theoretic identity types. This is called dependent type theory *with intensional identity types* or, more recently, *homotopy type theory* (without, necessarily, univalence). At the same time, such a model category is a presentation for the homotopy-theoretic analogy of a locally cartesian closed category: a *locally cartesian closed* $(\infty, 1)$*-category* (see A.3 of [31]).

The following was maybe first explicitly suggested by [27]. A proof of the technical details involved appeared in [9].

Proposition A.6. *Up to equivalence, the internal type theory of a locally Cartesian closed* $(\infty, 1)$*-category is homotopy type theory (without necessarily univalence) and conversely homotopy type theory (without necessarily univalence) has categorical semantics in locally cartesian closed* $(\infty, 1)$*-categories.*

We are interested in those locally cartesian closed $(\infty, 1)$-categories which are also *toposes*. There are several equivalent definitions of these, the one we refer to in the main text above is the following.

Definition A.7. A *coverage* (or *Grothendieck pre-topology*) on a small category $\mathcal{S}$ is an assignment to each object $U \in \mathcal{S}$ of a set of famillies $\{U_i \to U\}_i$ of morphisms into U – to be called the *covering families* – such that for any such covering family and for any other morphism $V \longrightarrow U$ into U , there exists a covering family $\{V_j \to V\}_j$ such that for all its elements V_i there exist dashed lifts in a diagram of the form

$$\begin{tikzcd} V_j \arrow[r, dashed] \arrow[d] & U_{i(j)} \arrow[d] \\ V \arrow[r] & U \end{tikzcd} .$$

Definition A.8. Given a small category $\mathcal{S}$, write $\mathrm{PSh}(\mathcal{S}) = \mathrm{Func}(\mathcal{S}^{\mathrm{op}}, \mathrm{Set})$ for the category of presheaves over it. Every object $U \in \mathcal{S}$ represents the presheaf given by $\mathcal{S}(-, U)$, which we usually just denote by the symbol U, too. Given a covering family $\{U_i \to U\}_i$, def. A.7, its *Čech nerve* $C(\{U_i\}) \in \mathrm{PSh}(\mathcal{S})$ is the universal construction

$$C(\{U_i\}) := \underset{\longrightarrow}{\lim} \left(\coprod_{i,j} U_i \underset{U}{\times} U_j \rightrightarrows \coprod_i U_i \right)$$

in $\mathrm{PSh}(\mathcal{S})$. The universal property of this construction induces a canonical morphism

$$C(\{U_i\}) \longrightarrow U \quad \in \mathcal{S}$$

to be called the *covering morphism* of the given covering family.

Definition A.9. A category $\mathbf{H}$ is a *sheaf topos* if there exists a small category $\mathcal{S}$ equipped with a coverage, def. A.7, such that $\mathbf{H}$ is equivalent to the category given by universally turning the covering morphisms in $\mathrm{PSh}(\mathcal{S})$, def. A.8, into isomrophisms (the *localization* of the presheaf category at the covering morphisms):

$$\mathbf{H} \simeq \mathrm{PSh}(\mathcal{S})[\{\text{covering morphisms}^{-1}\}] .$$

For working with a topos $\mathbf{H}$ presented as a localization of a category of presheaves, one uses two equivalent characterizations.

Proposition A.10. *Let* $\mathbf{H}$ *be a sheaf topos induced by a small category* $\mathcal{S}$ *with coverage according to def. A.9. Then:*

- $\mathbf{H}$ *is equivalent to the full subcategory of* $\mathrm{PSh}(\mathcal{S})$ *of those objects* $X \in \mathrm{PSh}(\mathcal{S})$ *which are* local objects *in that for every covering morphism* $C(\{U_i\}) \longrightarrow U$ *the induced function*

  $$\mathrm{Hom}(U, X) \longrightarrow \mathrm{Hom}(C(\{U_i\}), X)$$

  is a bijection. This is the sheaf condition *and presheafs satisfying it are called* sheaves.

- $\mathbf{H}$ *is* also *equivalent to the category whose objects are* all *the objects of* $\mathrm{PSh}(C)$, *but whose morphisms* $X \longrightarrow Y$ *are equivalence classes of correspondences in* $\mathrm{PSh}(C)$ *of the form*

  $$X \xleftarrow{\simeq} \widehat{X} \longrightarrow Y ,$$

  where the left leg is a local morphism *in that for each local object* A *(as above) the induced function*

  $$\mathrm{Hom}(Y, A) \longrightarrow \mathrm{Hom}(X, A)$$

  is a bijection.

All this has the following fairly straightforward refinement to homotopy theory.

Definition A.11. Given a small category $\mathcal{S}$, write

$$\mathrm{sPSh}(\mathcal{S}) := \mathrm{Func}(\mathcal{S}^{\mathrm{op}}, \mathrm{sSet})$$

for the category of *simplcial presheaves* on $\mathcal{S}$, functors from $\mathcal{S}^{\mathrm{op}}$ to the category of simplicial sets. A *weak equivalence* in this category is a morphism of presheaves which over each object of $\mathcal{S}$ is a weak homotopy equivalence of simplicial sets. The *projective model category structure* on $\mathrm{PSh}(\mathcal{S})$ in addition has fibrantions given by the objectwise Kan fibrations, whereas the *injective model category structure* on $\mathrm{PSh}(\mathcal{S})$ has cofibrations given by the objectwise monomorphisms.

See for instance A.2.8 of [31] for a review.

Definition A.12. For $\{U_i \to U\}_i$ a covering family in $\mathcal{S}$, def. A.7, its *simplicial Čech nerve* is the simplcial presheaf $C(\{U_i\}) \in \mathrm{sPSh}(\mathcal{S})$ given by the simplicial diagram

$$C(\{U_i\}) := \left(\cdots\cdots \coprod_{i,j,k} U_i \underset{U}{\times} U_j \underset{U}{\times} U_k \rightrightarrows\!\!\!\!\!\!\to \coprod_{i,j} U_i \underset{U}{\times} U_j \rightrightarrows \coprod_i U_u \right) .$$

The component maps induce a canonical morphism

$$C(\{U_i\}) \longrightarrow U \quad \in \mathrm{sPSh}(\mathcal{S})$$

for each covering family, to be called the corresponding (simplicial) *covering morphism*.

Definition A.13. A (1-localic) *homotopy topos* or $(\infty,1)$*-topos* $\mathbf{H}$ is an $(\infty,1)$-category which is presented by the left Bousfield localization (see A.3.7 of [31] for a review) of either the projective or the injective model stucture on the category of simplicial presheaves over some small category $\mathcal{S}$, def. A.11, at the simplicial covering morphisms, def. A.12 of some coverage on $\mathcal{S}$, def. A.7: $\mathrm{sPSh}(\mathcal{S})$:

$$\mathbf{H} \simeq \mathrm{sPSh}(\mathcal{S})[\{\text{covering morphisms}\}^{-1}] .$$

This statement is the result of a long development, involving results by, among others, Joyal, Jardine, Dugger, Toën, Vezzosi, Rezk and Lurie. A detailed discussion of the statement in the above form and in view of the class of examples of relevance in the main text above is in [42, 50].

An object in a (1-localic) $(\infty,1)$-topos $\mathbf{H}$ may be thought of as a homotopy type that is equipped with geometric structure which is modeled on the site $\mathcal{S}$, regarding the objects of $\mathcal{S}$ as the basic geometric spaces of the given notion of geometry. Therefore we may speak of objects in $\mathbf{H}$ as being *geometric homotopy types*.

This notion of "geometric" is very general. We may ask how to characterize those geometric homotopy theories which encode geometries that are "differential" in that basic constructions known from traditional differential geometry have sensible analogs in these contexts. It turns out [50] that the following simple axioms on $\mathbf{H}$ ensure that $\mathbf{H}$ is a decent differential geometric homotopy theory. We first say this in the language of ∞-toposes and then after that in their internal homotopy type theory.

Definition A.14 ([50])**.** An $(\infty,1)$-topos $\mathbf{H}$ is called *cohesive* with respect to the base ∞-topos $\infty\mathrm{Grpd}$ if there is given adjoint quadruple of ∞-functors

$$(\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}) \;:\; \mathbf{H} \underset{\underset{\mathrm{coDisc}}{\longleftarrow}}{\overset{\overset{\Pi}{\longrightarrow}}{\underset{\Gamma}{\overset{\mathrm{Disc}}{\leftrightarrows}}}} \infty\mathrm{Grpd} \;,$$

where Disc and coDisc are fully faithful and where Π preserves finite products. Moreover, a cohesive ∞-topos is *differentially cohesive* with respect to another cohesive ∞-topos $\mathbf{H}_{\mathrm{red}}$ if there is given an adjoint quadruple of adjoint ∞-functors

$$(i_! \dashv i^* \dashv i_* \dashv i^!) \;:\; \mathbf{H}_{\mathrm{red}} \underset{\underset{i^!}{\longleftarrow}}{\overset{\overset{i_!}{\longrightarrow}}{\underset{i_*}{\overset{i^*}{\leftrightarrows}}}} \mathbf{H} \;,$$

such that the composite $\Gamma \circ i_*$ exhibits the cohesion of $\mathbf{H}_{\mathrm{red}}$ over $\infty\mathrm{Grpd}$.

We explain below in D what these axioms *mean* for geometric homotopy theory. Restricted to ordinary toposes the axioms of cohesion have been promoted by William Lawvere, informally in [28] and formally in [30], as an axiomatization of those toposes that serve as a good context for geometry. The axioms of differential cohesion turn out [50] to abstractly capture central aspects of another proposal by Lawvere, namely the axiomatization of toposes for "synthetic differential geometry", which was explictly developed with the goal of modeling classical continuum physics [29].

Remark A.15. In terms of the internal homotopy type theory [59] of ∞-toposes, the axiom of cohesion says that there exists an adjoint triple of *higher modalities* [55] on the type system, namely the idempotent ∞-(co-)monads

$$\int \dashv \flat \dashv \sharp$$

on $\mathbf{H}$ which are induced as the sequential composites of the above adjoint functors. We call this the *shape modality* $\int$, the *flat modality* $\flat$ and the *sharp modality* $\sharp$. They are discussed in more detail below in D.

Similarly the axioms of differential cohesion in def. A.14 meain in the internal homotopy type theory that there is another adjoint triple of higher modalities, which we denote

$$\mathrm{Red} \dashv \int_{\mathrm{inf}} \dashv \flat_{\mathrm{inf}}$$

and call the *reduction modality*, the *infinitesimal shape modality* and the *infinitesimal flat modality*, respectively. Notice that the central difference between the two adjoint triples of higher modalities is that the first is of the form "∞-monad $\dashv$ ∞-co-monad $\dashv$ ∞-monad", while the second is of the complementary form "∞-comonad $\dashv$ ∞-monad $\dashv$ ∞-co-monad".

This ends our lightning review of cohesive homotopy type theory. The sections below expand on details of relevance in the discussion in the main text above. For more see [50].

- In B *Differential geometry via Cohesive 0-types* we develop the very basics of differential geometry in the topos of smooth 0-types. For type theorist readers absolutely unfamiliar with differential geometry this may help with the basic notions needed in the main text. Readers familiar with traditional differential geometry may find it helpful to see how this is developed in the topos-theoretic refinement which we make use of in the main text.
- In C *Higher geometry via Geometric homotopy types* we review in a little more detail the notion of homotopy toposes (∞-toposes) from the point of view of doing higher geometry inside them. In particular we briefly review the theory of homotopy fiber bundles from [41].
- In D *Differential cohomology via Cohesive homotopy types* we review the axioms of *cohesion* on a homotopy topos, how they induce moduli for differential cohomology and higher principal connections [50], and how the slice automorphisms of these yield the higher quantomorphism groups and higher Heisenberg groups.

B Differential geometry via Smooth 0-types

We introduce here basics of *differential geometry* naturally formulated via the topos of smooth 0-types. Type theorists may regard this as an introduction to differential geometry and differential geometers may regard this as an introduction to useful topos- and type-theoretic reasoning in differential geometry.

Below in D we axiomatize the properties of a (higher) topos such as to support a good differential geometry and differential cohomology theory, namely that it be *cohesive* and that is objects have the geometric interpretation of being *cohesive types*.

B.1 Coordinate systems

Every kind of geometry is modeled on a collection of archetypical basic spaces and geometric homomorphisms between them. In differential geometry the archetypical spaces are the abstract standard Cartesian coordinate systems, denoted $\mathbb{R}^n$, in every dimension $n \in \mathbb{N}$, and the geometric homomorphism between them are smooth functions $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, hence smooth (and possibly degenerate) coordinate transformations.

Here we discuss the central aspects of the nature of such abstract coordinate systems in themselves. At this point these are not yet coordinate systems on some other space. That is instead the topic of the next section Smooth spaces.

The continuum real (world-)line

The fundamental premise of differential geometry as a model of geometry in physics is the following.

bf Premise. *The abstract worldline of any particle is modeled by the continuum real line* $\mathbb{R}$.

This comes down to the following sequence of premises.

1. There is a linear ordering of the points on a worldline: in particular if we pick points at some intervals on the worldline we may label these in an order-preserving way by integers

$$\mathbb{Z}\,.$$

2. These intervals may each be subdivided into n smaller intervals, for each natural number n. Hence we may label points on the worldline in an order-preserving way by the rational numbers

$$\mathbb{Q}\,.$$

3. This labeling is dense: every point on the worldline is the supremum of an inhabited bounded subset of such labels. This means that a worldline is the *real line*, the continuum of real numbers

$$\mathbb{R}\,.$$

The adjective"real" in "real number" is a historical shadow of the old idea that real numbers are related to observed reality, hence to physics in this way. The experimental success of this assumption shows that it is valid at least to very good approximation.

Speculations are common that in a fully exact theory of quantum gravity, currently unavailable, this assumption needs to be refined. For instance in p-adic physics one explores the hypothesis that the relevant completion of the rational numbers as above is not the reals, but p-adic numbers $\mathbb{Q}_p$ for some prime number $p \in \mathbb{N}$. Or for example in the study of QFT on non-commutative spacetime one explore the idea that at small scales the smooth continuum is to be replaced by an object in noncommutative geometry. Combining these two ideas leads to the notion of non-commutative analytic space as a potential model for space in physics. And so forth.

For the time being all this remains speculation and differential geometry based on the continuum real line remains the context of all fundamental model building in physics related to observed phenomenology. Often it is argued that these speculations are necessitated by the very nature of quantum theory applied to gravity. But, at least so far, such statements are not actually supported by the standard theory of quantization: we discuss below in Geometric quantization how not just classical physics but also quantum theory, in the best modern version available, is entirely rooted in differential geometry based on the continuum real line.

This is the motivation for studying models of physics in geometry modeled on the continuum real line. On the other hand, in all of what follows our discussion is set up such as to be maximally independent of this specific choice (this is what *topos theory* accomplishes for us). If we do desire to consider another choice of archetypical spaces for the geometry of physics we can simply "change the site", as discussed below and many of the constructions, propositions and theorems in the following will continue to hold. This is notably what we do below in Supergeometric coordinate systems when we generalize the present discussion to a flavor of differential geometry that also formalizes the notion of fermion particles: "differential supergeometry".

Cartesian spaces and smooth functions

Definition B.1. A function of sets $f : \mathbb{R} \to \mathbb{R}$ is called a *smooth function* if, coinductively:

1. the derivative $\frac{df}{dx} : \mathbb{R} \to \mathbb{R}$ exists;
2. and is itself a smooth function.

Definition B.2. For $n \in \mathbb{N}$, the *Cartesian space* $\mathbb{R}^n$ is the set

$$\mathbb{R}^n = \{(x^1, \cdots, x^n) | x^i \in \mathbb{R}\}$$

of n-tuples of real numbers. For $1 \leq k \leq n$ write

$$i^k : \mathbb{R} \to \mathbb{R}^n$$

for the function such that $i^k(x) = (0, \cdots, 0, x, 0, \cdots, 0)$ is the tuple whose kth entry is x and all whose other entries are $0 \in \mathbb{R}$; and write

$$\shortmid^k : \mathbb{R}^n \to \mathbb{R}$$

for the function such that $p^k(x^1, \cdots, x^n) = x^k$.

A *homomorphism* of Cartesian spaces is a smooth function

$$f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2} ,$$

hence a function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ such that all partial derivatives exist and are continuous.

Example B.3. Regarding $\mathbb{R}^n$ as an $\mathbb{R}$-vector space, every linear function $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is in particular a smooth function.

Remark B.4. But a homomorphism of Cartesian spaces in def. B.2 is *not* required to be a linear map. We do *not* regard the Cartesian spaces here as vector spaces.

Definition B.5. A smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is called a *diffeomorphism* if there exists another smooth function $\mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ such that the underlying functions of sets are inverse to each other

$$f \circ g = \mathrm{id}$$

and

$$g \circ f = \mathrm{id} .$$

Proposition B.6. *There exists a diffeomorphism* $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ *precisely if* $n_1 = n_2$.

Definition B.7. We will also say equivalently that

1. a Cartesian space $\mathbb{R}^n$ is an *abstract coordinate system*;
2. a smooth function $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is an *abstract coordinate transformation*;
3. the function $p^k : \mathbb{R}^n \to \mathbb{R}$ is the kth *coordinate* of the coordinate system $\mathbb{R}^n$. We will also write this function as $x^k : \mathbb{R}^n \to \mathbb{R}$.
4. for $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ a smooth function, and $1 \leq k \leq n_2$ we write
   (a) $f^k := p^k \circ f$
   (b) $(f^1, \cdots, f^n) := f$.

Remark B.8. It follows with this notation that

$$\mathrm{id}_{\mathbb{R}^n} = (x^1, \cdots, x^n) : \mathbb{R}^n \to \mathbb{R}^n \,.$$

Hence an abstract coordinate transformation

$$f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$$

may equivalently be written as the tuple

$$\left(f^1\left(x^1, \cdots, x^{n_1}\right), \cdots, f^{n_2}\left(x^1, \cdots, x^{n_1}\right)\right) \,.$$

Proposition B.9. *Abstract coordinate systems form a* category *– to be denoted* CartSp *– whose*

- *objects are the abstract coordinate systems* $\mathbb{R}^n$ *(the class of objects is the set* $\mathbb{N}$ *of natural numbers* n*);*
- *morphisms* $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ *are the abstract coordinate transformations = smooth functions.*

Composition of morphisms is given by composition of functions.

We have that

1. *The identity morphisms are precisely the identity functions.*
2. *The isomorphisms are precisely the diffeomorphisms.*

Definition B.10. Write $\mathrm{CartSp}^{\mathrm{op}}$ for the opposite category of CartSp.

This is the category with the same objects as CartSp, but where a morphism $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ in CartSp^{op} is given by a morphism $\mathbb{R}^{n_1} \leftarrow \mathbb{R}^{n_2}$ in CartSp.

We will be discussing below the idea of exploring smooth spaces by laying out abstract coordinate systems in them in all possible ways. The reader should begin to think of the sets that appear in the following definition as the *set of ways* of laying out a given abstract coordinate systems in a given space.

Definition B.11. A functor $X : \mathrm{CartSp}^{op} \to Set$ (a "presheaf") is

1. for each abstract coordinate system U a set $X(U)$
2. for each coordinate transformation $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ a function $X(f) : X(\mathbb{R}^{n_1}) \to X(\mathbb{R}^{n_2})$

such that

1. identity is respected $X(id_{\mathbb{R}^n}) = id_{X(\mathbb{R}^n)}$;
2. composition is respected $X(f_2) \circ X(f_1) = X(f_2 \circ f_1)$

The fundamental theorems about smooth functions

The special properties smooth functions that make them play an important role different from other classes of functions are the following.

1. existence of bump functions and partitions of unity
2. the Hadamard lemma and Borel's theorem

Or maybe better put: what makes smooth functions special is that the first of these properties holds, while the second is still retained.

B.2 Smooth 0-types

We now discuss concretely the definition of smooth sets/smooth spaces and of homomorphisms between them, together with basic examples and properties.

Plots of smooth spaces and their gluing

The general kind of "smooth space" that we want to consider is something that can be *probed* by laying out coordinate systems as in def. B.1 inside it, and that can be obtained by *gluing* all the possible coordinate systems in it together.

At this point we want to impose no further conditions on a "space" than this. In particular we do not assume that we know beforehand a set of points underlying X. Instead, we define smooth spaces X entirely *operationally* as something about which we can ask "Which ways are there to lay out $\mathbb{R}^n$ inside X?" and such that there is a self-consistent answer to this question. The following definitions make precise what we mean by this.

For brevity we will refer "a way to lay out a coordinate system in X" as a *plot* of X. The first set of consistency conditions on plots of a space is that they respect *coordinate transformations*. This is what the following definition formalizes.

Definition B.12. A *smooth pre-space* X is

1. a collection of sets: for each Cartesian space $\mathbb{R}^n$ (hence for each natural number n) a set

   $$X(\mathbb{R}^n) \in \mathrm{Set}$$

   – to be thought of as the *set of ways of laying out* $\mathbb{R}^n$ *inside* X;

2. for each abstract coordinate transformation, hence for each smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ a function between the corresponding sets

   $$X(f) : X(\mathbb{R}^{n_2}) \to X(\mathbb{R}^{n_1})$$

   – to be thought of as the function that sends a *plot* of X by $\mathbb{R}^{n_2}$ to the correspondingly transformed plot by $\mathbb{R}^{n_1}$ induced by laying out $\mathbb{R}^{n_1}$ inside $\mathbb{R}^{n_2}$.

such that this is compatible with coordinate transformations:

1. the identity coordinate transformation does not change the plots:

   $$X(id_{\mathbb{R}^n}) = id_{X(\mathbb{R}^n)}\,,$$

2. changing plots along two consecutive coordinate transformations $f_1 \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $f_2 \colon \mathbb{R}^{n_2} \to \mathbb{R}^{n_3}$ is the same as changing them along the composite coordinate transformation $f_2 \circ f_1$:

   $$X(f_1) \circ X(f_2) = X(f_2 \circ f_1)\,.$$

But there is one more consistency condition for a collection of plots to really be probes of some space: it must be true that if we glue small coordinate systems to larger ones, then the plots by the larger ones are the same as the plots by the collection of smaller ones that agree where they overlap. We first formalize this idea of "plots that agree where their coordinate systems overlap".

Definition B.13. Let X be a smooth pre-space, def. B.12. For $\{U_i \to \mathbb{R}^n\}_{i\in I}$ a good open cover, let

$$\mathrm{GluedPlots}(\{U_i \to \mathbb{R}^n\}, X) \in \mathrm{Set}$$

be the set of I-tuples of U_i-plots of X which coincide on all double intersections

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "\iota_i"'] \arrow[dr, "\iota_j"] & \\ U_i \arrow[dr] & & U_J \arrow[dl] \\ & \mathbb{R}^n & \end{tikzcd}$$

(also called the *matching families* of X over the given cover):

$$\mathrm{GluedPlots}(\{U_i \to \mathbb{R}^n\}, X) \;:=\; \left\{ \, (p_i \in X(U_i))_{i \in I} \;|\; \forall_{i,j \in I} \;:\; X(\iota_i)(p_i) = X(\iota_j)(p_j) \, \right\} .$$

Remark B.14. In def. B.13 the equation

$$X(\iota_i)(p_i) = X(\iota_j)(p_j)$$

says in words:

"The plot p_i of X by the coordinate system U_i inside the bigger coordinate system $\mathbb{R}^n$ coincides with the plot p_j of X by the other coordinate system U_j inside X when both are restricted to the intersection $U_i \cap U_j$ of U_i with U_j inside $\mathbb{R}^n$."

Remark B.15. For each differentially good open cover $\{U_i \to X\}_{i \in I}$ and each smooth pre-space X, def. B.12, there is a canonical function

$$X(\mathbb{R}^n) \to \mathrm{GluedPlots}(\{U_i \to \mathbb{R}^n\}, X)$$

from the set of $\mathbb{R}^n$-plots of X to the set of tuples of glued plots, which sends a plot $p \in X(\mathbb{R}^n)$ to its restriction to all the $\phi_i \colon U_i \hookrightarrow \mathbb{R}^n$:

$$p \mapsto (X(\phi_i)(p))_{i \in I} .$$

If X is supposed to be consistently probable by coordinate systems, then it must be true that the set of ways of laying out a coordinate system $\mathbb{R}^n$ inside it coincides with the set of ways of laying out tuples of glued coordinate systems inside it, for each good cover $\{U_i \to \mathbb{R}^n\}$ as above. Therefore:

Definition B.16. A smooth pre-space X, def. B.12 is a *smooth space* if for all differentially good open covers $\{U_i \to \mathbb{R}^n\}$, the canonical function of remark B.15 from plots to glued plots is a bijection

$$X(\mathbb{R}^n) \stackrel{\simeq}{\to} \mathrm{GluedPlots}(\{U_i \to \mathbb{R}^n\}, X) .$$

Remark B.17. We may think of a smooth space as being a kind of space whose *local models* (in the general sense discussed at *geometry*) are Cartesian spaces:

while definition B.16 explicitly says that a smooth space is something that is *consistently probeable* by such local models; by a general abstract fact that is sometimes called the *co-Yoneda lemma*, it follows in fact that smooth spaces are precisely the objects that are obtained by *gluing coordinate systems* together.

For instance we will see that two open 2-balls $\mathbb{R}^2 \simeq D^2$ along a common rim yields the smooth space version of the sphere S^2, a basic example of a smooth manifold. But before we examine such explicit constructions, we discuss here for the moment more general properties of smooth spaces.

Example B.18. For $n \in \mathbb{R}^n$, there is a smooth space, def. B.16, whose set of plots over the abstract coordinate systems $\mathbb{R}^k$ is the set

$$\mathrm{CartSp}(\mathbb{R}^k, \mathbb{R}^n) \in \mathrm{Set}$$

of smooth functions from $\mathbb{R}^k$ to $\mathbb{R}^n$.

Clearly this is the rule for plots that characterize $\mathbb{R}^n$ itself as a smooth space, and so we will just denote this smooth space by the same symbols "$\mathbb{R}^n$":

$$\mathbb{R}^n \colon \mathbb{R}^k \mapsto \mathrm{CartSp}(\mathbb{R}^k, \mathbb{R}^n)\,.$$

In particular the real line $\mathbb{R}$ is this way itself a smooth space.

In a moment we find a formal justification for this slight abuse of notation.

Another basic class of examples of smooth spaces are the discrete smooth spaces:

Definition B.19. For $S \in \mathrm{Set}$ a set, write

$$\mathrm{Disc}S \in \mathrm{Smooth0Type}$$

for the smooth space whose set of U-plots for every $U \in \mathrm{CartSp}$ is always S.

$$\mathrm{Disc}S \colon U \mapsto S$$

and which sends every coordinate transformation $f \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ to the identity function on S.

A smooth space of this form we call a *discrete smooth space*.

More examples of smooth spaces can be built notably by intersecting images of two smooth spaces inside a bigger one. In order to say this we first need a formalization of homomorphism of smooth spaces. This we turn to now.

Homomorphisms of smooth spaces

We discuss "functions" or "maps" between smooth spaces, def. B.16, which preserve the smooth space structure in a suitable sense. As with any notion of function that preserves structure, we refer to them as *homomorphisms*.

The idea of the following definition is to say that whatever a homomorphism $f : X \to Y$ between two smooth spaces is, it has to take the plots of X by $\mathbb{R}^n$ to a corresponding plot of Y, such that this respects coordinate transformations.

Definition B.20. Let X and Y be two smooth spaces, def. B.16. Then a homomorphism $f \colon X \to Y$ is

- for each abstract coordinate system $\mathbb{R}^n$ (hence for each $n \in \mathbb{N}$) a function $f_{\mathbb{R}^n} : X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ that sends $\mathbb{R}^n$-plots of X to $\mathbb{R}^n$-plots of Y

such that

- for each smooth function $\phi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ we have

  $$Y(\phi) \circ f_{\mathbb{R}^{n_1}} = f_{\mathbb{R}^{n_2}} \circ X(\phi)\,,$$

  hence a commuting diagram

  $$\begin{tikzcd} X(\mathbb{R}^{n_1}) \arrow[r, "f_{\mathbb{R}^{n_1}}"] \arrow[d, "X(\phi)"] & Y(\mathbb{R}^{n_1}) \arrow[d, "Y(\phi)"] \\ X(\mathbb{R}^{n_2}) \arrow[r, "f_{\mathbb{R}^{n_2}}"] & Y(\mathbb{R}^{n_1}) \end{tikzcd} .$$

For $f_1 : X \to Y$ and $f_2 : X \to Y$ two homomorphisms of smooth spaces, their composition $f_2 \circ f_1 \colon X \to Y$ is defined to be the homomorphism whose component over $\mathbb{R}^n$ is the composite of functions of the components of f_1 and f_2:

$$(f_2 \circ f_1)_{\mathbb{R}^n} := f_{2\mathbb{R}^n} \circ f_{1\mathbb{R}^n}\,.$$

Definition B.21. Write Smooth0Type for the category whose objects are smooth spaces, def. B.16, and whose morphisms are homomorphisms of smooth spaces, def. B.20.

At this point it may seem that we have now *two different* notions for how to lay out a coordinate system in a smooth space X: on the hand, X comes by definition with a rule for what the set $X(\mathbb{R}^n)$ of its $\mathbb{R}^n$-plots is. On the other hand, we can now regard the abstract coordinate system $\mathbb{R}^n$ itself as a smooth space, by example B.18, and then say that an $\mathbb{R}^n$-plot of X should be a homomorphism of smooth spaces of the form $\mathbb{R}^n \to X$.

The following proposition says that these two superficially different notions actually naturally coincide.

Proposition B.22. *Let X be any smooth space, def. B.16, and regard the abstract coordinate system $\mathbb{R}^n$ as a smooth space, by example B.18. There is a natural bijection*

$$X(\mathbb{R}^n) \simeq Hom_{\mathrm{Smooth0Type}}(\mathbb{R}^n, X)$$

between the postulated *$\mathbb{R}^n$-plots of X and the* actual *$\mathbb{R}^n$-plots given by homomorphism of smooth spaces $\mathbb{R}^n \to X$.*

Proof. This is a special case of the *Yoneda lemma*. The reader unfamiliar with this should write out the simple proof explicitly: use the defining commuting diagrams in def. B.20 to deduce that a homomorphism $f : \mathbb{R}^n \to X$ is uniquely fixed by the image of the identity element in $\mathbb{R}^n(\mathbb{R}^n) := \mathrm{CartSp}(\mathbb{R}^n, \mathbb{R}^n)$ under the component function $f_{\mathbb{R}^n} : \mathbb{R}^n(\mathbb{R}^n) \to X(\mathbb{R}^n)$. □

Example B.23. Let $\mathbb{R} \in \mathrm{Smooth0Type}$ denote the real line, regarded as a smooth space by def. B.18. Then for $X \in \mathrm{Smooth0Type}$ any smooth space, a homomorphism of smooth spaces

$$f \colon X \to \mathbb{R}$$

is a *smooth function on X* Prop. B.22 says here that when X happens to be an abstract coordinate system regarded as a smooth space by def. B.18, then this general notion of smooth functions between smooth spaces reproduces the basic notion of def, B.2.

Definition B.24. The 0-dimensional abstract coordinate system $\mathbb{R}^0$ we also call the *point* and regarded as a smooth space we will often write it as

$$* \in \mathrm{Smooth0Type}\,.$$

For any $X \in \mathrm{Smooth0Type}$, we say that a homomorphism

$$x \colon * \to X$$

is a *point of X*.

Remark B.25. By prop. B.22 the points of a smooth space X are naturally identified with its 0-dimensional plots, hence with the "ways of laying out a 0-dimensional coordinate system" in X:

$$Hom(*, X) \simeq X(\mathbb{R}^0)\,.$$

Products and fiber products of smooth spaces

Definition B.26. Let $X, Y \in \mathrm{Smooth0Type}$ by two smooth spaces. Their *product* is the smooth space $X \times Y \in \mathrm{Smooth0Type}$ whose plots are pairs of plots of X and Y:

$$X \times Y(\mathbb{R}^n) := X(\mathbb{R}^n) \times Y(\mathbb{R}^n) \quad \in \mathrm{Set}\,.$$

The *projection on the first factor* is the homomorphism

$$p_1 \colon X \times Y \to X$$

which sends $\mathbb{R}^n$-plots of $X \times Y$ to those of X by forming the projection of the cartesian product of sets:

$$p_{1\mathbb{R}^n} \colon X(\mathbb{R}^n) \times Y(\mathbb{R}^n) \xrightarrow{p_1} X(\mathbb{R}^n)\,.$$

Analogously for the *projection to the second factor*

$$p_2 \colon X \times Y \to Y\,.$$

Proposition B.27. *Let* $* = \mathbb{R}^0$ *be the point, regarded as a smooth space, def. B.24. Then for* $X \in \mathrm{Smooth0Type}$ *any smooth space the canonical projection homomorphism*

$$X \times * \to X$$

is an isomorphism.

Definition B.28. Let $f \colon X \to Z$ and $g \colon Y \to Z$ be two homomorphisms of smooth spaces, def. B.20. There is then a new smooth space to be denoted

$$X \times_Z Y \in \mathrm{Smooth0Type}$$

(with f and g understood), called the *fiber product* of X and Y along f and g, and defined as follows:
the set of $\mathbb{R}^n$-plots of $X \times_Z Y$ is the set of pairs of plots of X and Y which become the same plot of Z under f and g, respectively:

$$(X \times_Z Y)(\mathbb{R}^n) = \{(p_X \in X(\mathbb{R}^n), p_Y \in Y(\mathbb{R}^n)) \mid f_{\mathbb{R}^n}(p_X) = g_{\mathbb{R}^n}(p_Y)\}\,.$$

Smooth mapping spaces and smooth moduli spaces

Definition B.29. Let $\Sigma, X \in \mathrm{Smooth0Type}$ be two smooth spaces, def. B.16. Then the *smooth mapping space*

$$[\Sigma, X] \in \mathrm{Smooth0Type}$$

is the smooth space defined by saying that its set of $\mathbb{R}^n$-plots is

$$[\Sigma, X](\mathbb{R}^n) := \mathrm{Hom}(\Sigma \times \mathbb{R}^n, X)\,.$$

Here in $\Sigma \times \mathbb{R}^n$ we first regard the abstract coordinate system $\mathbb{R}^n$ as a smooth space by example B.18 and then we form the product smooth space by def. B.26.

Remark B.30. This means in words that a $\mathbb{R}^n$-plot of the mapping space $[\Sigma, X]$ is a smooth $\mathbb{R}^n$-parameterized collection of homomorphisms $\Sigma \to X$.

Proposition B.31. *There is a natural bijection*

$$\mathrm{Hom}(K, [\Sigma, X]) \simeq \mathrm{Hom}(K \times \Sigma, X)$$

for every smooth space K.

Proof. With a bit of work this is straightforward to check explicitly by unwinding the definitions. It follows however from general abstract results once we realize that $[-,-]$ is of course the *internal hom* of smooth spaces. $\square$

Remark B.32. This says in words that a smooth function from any K into the mapping space $[\Sigma, X]$ is equivalently a smooth function from $K \times \Sigma$ to X. The latter we may regard as a *K-parameterized smooth collections* of smooth functions $\Sigma \to X$. Therefore in view of the previous remark B.30 this says that smooth mapping spaces have a universal property not just over abstract coordinate systems, but over all smooth spaces.

We will therefore also say that $[\Sigma, X]$ is the *smooth moduli space* of smooth functions from $\Sigma \to X$, because it is such that smooth maps $K \to [\Sigma, X]$ into it *modulate*, as we move around on K, a family of smooth functions $\Sigma \to X$, depending on K.

Proposition B.33. *The set of points, def. B.24, of a smooth mapping space $[\Sigma, X]$ is the bare set of homomorphism $\Sigma \to X$: there is a natural isomorphism*

$$\mathrm{Hom}(*, [\Sigma, X]) \simeq \mathrm{Hom}(\Sigma, X)\,.$$

Proof. Combine prop. B.31 with prop. B.27. □

Example B.34. Given a smooth space $X \in \mathrm{Smooth0Type}$, its smooth *path space* is the smooth mapping space

$$\mathbf{P}X := [\mathbb{R}^1, X]\,.$$

By prop. B.33 the points of PX are indeed precisely the smooth trajectories $\mathbb{R}^1 \to X$. But PX also knows how to *smoothly vary* such smooth trajectories.

This is central for variational calculus which determines equations of motion in physics.

Remark B.35. In physics, if X is a model for spacetime, then PX may notably be interpreted as the smooth space of worldlines *in* X, hence the smooth space of paths or *trajectories* of a particle in X.

Example B.36. If in the above example B.34 the path is constraind to be a loop in X, one obtains the *smooth loop space*

$$\mathbf{L}X := [S^1, X]\,.$$

The smooth moduli space of smooth functions

In example B.23 we saw that a smooth function on a general smooth space X is a homomorphism of smooth spaces, def. B.20

$$f\colon X \to \mathbb{R}\,.$$

The collection of these forms the hom-set $\mathrm{Hom}_{\mathrm{Smooth0Type}}(X, \mathbb{R})$. But by the discussion in B.2 such hom-sets are naturally refined to smooth spaces themselves.

Definition B.37. For $X \in \mathrm{Smooth0Type}$ a smooth space, we say that the *moduli space of smooth functions* on X is the smooth mapping space (def. B.29), from X into the standard real line $\mathbb{R}$

$$[X, \mathbb{R}] \in \mathrm{Smooth0Type}\,.$$

We will also denote this by

$$\mathbf{C}^\infty(X) := [X, \mathbb{R}]\,,$$

since in the special case that X is a Cartesian space this is the smooth refinement of the set $C^\infty(X)$ of smooth functions, def. B.2, on X.

Remark B.38. We call this a *moduli space* because by prop. B.31 above and in the sense of remark B.32 it is such that smooth functions into it *modulate* smooth functions $X \to \mathbb{R}$.

By prop. B.33 a point $* \to [X, \mathbb{R}^1]$ of the moduli space is equivalently a smooth function $X \to \mathbb{R}^1$.

Outlook

Later we define/see the following:

- A *smooth manifold* is a smooth space that is *locally equivalent* to a coordinate system;
- A *diffeological space* is a smooth space such that every coordinate labels a point in the space. In other words, a diffeological space is a smooth space that has an underlying set $X_s \in Set$ of points such that the set of $\mathbb{R}^n$-plots is a subset of the set of all functions:

$$X(\mathbb{R}^n) \hookrightarrow \mathrm{Functions}(\mathbb{R}^n, S_s)\,.$$

We discuss below a long sequence of faithful inclusions

$\{$coordinate systems $\} \hookrightarrow \{$smooth manifolds$\} \hookrightarrow \{$diffeological spaces$\} \hookrightarrow \{$smooth spaces$\} \hookrightarrow \{$smooth groupoids$\} \hookrightarrow \cdots$

B.3 Differential forms

A fundamental concept in differential geometry is that of *differential forms.* We here introduce this in the spirit of the topos of smooth spaces.

Differential forms on abstract coordinate systems

We introduce the basic concept of a *smooth differential form* on a Cartesian space $\mathbb{R}^n$. Below in B.65 we use this to define differential forms on any smooth space.

Definition B.39. For $n \in \mathbb{N}$ a *smooth differential 1-form* ω on the Cartesian space $\mathbb{R}^n$ is an n-tuple

$$(\omega_i \in \mathrm{CartSp}\,(\mathbb{R}^n, \mathbb{R}))_{i=1}^n$$

of smooth functions, which we think of equivalently as the coefficients of a formal linear combination

$$\omega = \sum_{i=1}^n f_i \mathbf{d}x^i$$

on a set $\{\mathbf{d}x^1, \mathbf{d}x^2, \cdots, \mathbf{d}x^n\}$ of cardinality n.

Write

$$\Omega^1(\mathbb{R}^k) \simeq \mathrm{CartSp}(\mathbb{R}^k, \mathbb{R})^{\times k} \in \mathrm{Set}$$

for the set of smooth differential 1-forms on $\mathbb{R}^k$.

Remark B.40. We think of $\mathbf{d}x^i$ as a measure for infinitesimal displacements along the x^i-coordinate of a Cartesian space. This idea is made precise by the notion of *parallel transport.*

If we have a measure of infintesimal displacement on some $\mathbb{R}^n$ and a smooth function $f\colon \mathbb{R}^{\tilde{n}} \to \mathbb{R}^n$, then this induces a measure for infinitesimal displacement on $\mathbb{R}^{\tilde{n}}$ by sending whatever happens there first with f to $\mathbb{R}^n$ and then applying the given measure there. This is captured by the following definition.

Definition B.41. For $\phi\colon \mathbb{R}^{\tilde{k}} \to \mathbb{R}^k$ a smooth function, the *pullback of differential 1-forms* along ϕ is the function

$$\phi^* \colon \Omega^1(\mathbb{R}^k) \to \Omega^1(\mathbb{R}^{\tilde{k}})$$

between sets of differential 1-forms, def. B.39, which is defined on basis-elements by

$$\phi^* \mathbf{d}x^i := \sum_{j=1}^{\tilde{k}} \frac{\partial \phi^i}{\partial \tilde{x}^j} \mathbf{d}\tilde{x}^j$$

and then extended linearly by

$$
\begin{aligned}
\phi^*\omega &= \phi^*\left(\sum_i \omega_i \mathbf{d}x^i\right) \\
&:= \sum_{i=1}^{k} (\phi^*\omega)_i \sum_{j=1}^{\tilde{k}} \frac{\partial \phi^i}{\partial \tilde{x}^j} \mathbf{d}\tilde{x}^j \quad . \\
&= \sum_{i=1}^{k} \sum_{j=1}^{\tilde{k}} (\omega_i \circ \phi) \cdot \frac{\partial \phi^i}{\partial \tilde{x}^j} \mathbf{d}\tilde{x}^j
\end{aligned}
$$

Remark B.42. The term "pullback" in *pullback of differential forms* is not really related, certainly not historically, to the term *pullback* in category theory. One can relate the pullback of differential forms to categorical pullbacks, but this is not really essential here. The most immediate property that both concepts share is that they take a morphism going in one direction to a map between structures over domain and codomain of that morphism which goes in the other direction, and in this sense one is "pulling back structure along a morphism" in both cases.

Even if in the above definition we speak only about the set $\Omega^1(\mathbb{R}^k)$ of differential 1-forms, this set naturally carries further structure.

Definition B.43. The set $\Omega^1(\mathbb{R}^k)$ is naturally an abelian group with addition given by componentwise addition

$$
\begin{aligned}
\omega + \lambda &= \sum_{i=1}^{k} \omega_i \mathbf{d}x^i + \sum_{j=1}^{k} \lambda_j \mathbf{d}x^j \\
&= \sum_{i=1}^{k} (\omega_i + \lambda_i) \mathbf{d}x^j
\end{aligned} \quad ,
$$

Moreover, the abelian group $\Omega^1(\mathbb{R}^k)$ is naturally equipped with the structure of a module over the ring $C^\infty(\mathbb{R}^k, \mathbb{R}) = \mathrm{CartSp}(\mathbb{R}^k, \mathbb{R})$ of smooth functions, where the action $C^\infty(\mathbb{R}^k, \mathbb{R}) \times \Omega^1(\mathbb{R}^k) \to \Omega^1(\mathbb{R}^k)$ is given by componentwise multiplication

$$
f \cdot \omega = \sum_{i=1}^{k} (f \cdot \omega_i) \mathbf{d}x^i \, .
$$

Remark B.44. More abstractly, this just says that $\Omega^1(\mathbb{R}^k)$ is the free module over $C^\infty(\mathbb{R}^k)$ on the set $\{\mathbf{d}x^i\}_{i=1}^k$.

The following definition captures the idea that if $\mathbf{d}x^i$ is a measure for displacement along the x^i-coordinate, and $\mathbf{d}x^j$ a measure for displacement along the x^j coordinate, then there should be a way te get a measure, to be called $\mathbf{d}x^i \wedge \mathbf{d}x^j$, for infinitesimal *surfaces* (squares) in the x^i-x^j-plane. And this should keep track of the orientation of these squares, whith

$$
\mathbf{d}x^j \wedge \mathbf{d}x^i = -\mathbf{d}x^i \wedge \mathbf{d}x^j
$$

being the same infinitesimal measure with orientation reversed.

Definition B.45. For $k, n \in \mathbb{N}$, the *smooth differential forms* on $\mathbb{R}^k$ is the exterior algebra

$$
\Omega^\bullet(\mathbb{R}^k) := \wedge^\bullet_{C^\infty(\mathbb{R}^k)} \Omega^1(\mathbb{R}^k)
$$

over the ring $C^\infty(\mathbb{R}^k)$ of smooth functions of the module $\Omega^1(\mathbb{R}^k)$ of smooth 1-forms, prop. B.43.

We write $\Omega^n(\mathbb{R}^k)$ for the sub-module of degree n and call its elements the *smooth differential n-forms*.

Remark B.46. Explicitly this means that a differential n-form $\omega \in \Omega^n(\mathbb{R}^k)$ on $\mathbb{R}^k$ is a formal linear combination over $C^\infty(\mathbb{R}^k)$ of basis elements of the form $\mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_n}$ for $i_1 < i_2 < \cdots < i_n$:

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_n < k} \omega_{i_1, \cdots, i_n} \mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_n} \, .$$

Remark B.47. The pullback of differential 1-forms of def. B.39 extends as an $C^\infty(\mathbb{R}^k)$-algebra homomorphism to $\Omega^n(-)$, given for a smooth function $f \colon \mathbb{R}^{\tilde{k}} \to \mathbb{R}^k$ on basis elements by

$$f^* \left(\mathbf{d}x^{i_1} \wedge \cdots \wedge \mathbf{d}x^{i_n} \right) = \left(f^* \mathbf{d}x^{i_1} \wedge \cdots \wedge f^* \mathbf{d}x^{i_n} \right) \, .$$

Differential forms on smooth spaces

Above we have defined differential n-form on abstract coordinate systems. Here we extend this definition to one of differential n-forms on arbitrary smooth spaces. We start by observing that the space of *all*3 differential n-forms on cordinate systems themselves naturally is a smooth space.

Proposition B.48. *The assignment of differential n-forms*

$$\Omega^n(-) \colon \mathbb{R}^k \mapsto \Omega^n(\mathbb{R}^k)$$

of def. B.45 together with the pullback of differential forms-functions of def. B.47

$$\begin{tikzcd} \mathbb{R}^{k_1} \arrow[r, mapsto] & \Omega^1(\mathbb{R}^{k_1}) \arrow[d, "f^*"] \\ \mathbb{R}^{k_2} \arrow[r, mapsto] \arrow[u, "f"] & \Omega^1(\mathbb{R}^{k_2}) \end{tikzcd}$$

defines a smooth space in the sense of def. B.16:

$$\Omega^n(-) \in \mathrm{Smooth0Type} \, .$$

Definition B.49. We call this

$$\Omega^n \colon \mathrm{Smooth0Type}$$

the *universal smooth moduli space* of differential n-forms.

The reason for this terminology is that homomorphisms of smooth spaces into Ω^1 *modulate* differential n-forms on their domain, by prop. B.22 (and hence by the Yoneda lemma):

Example B.50. For the Cartesian space $\mathbb{R}^k$ regarded as a smooth space by example B.18, there is a natural bijection

$$\Omega^n(\mathbb{R}^k) \simeq Hom(\mathbb{R}^k, \Omega^1)$$

between the set of smooth n-forms on $\mathbb{R}^n$ according to def. B.39 and the set of homomorphism of smooth spaces, $\mathbb{R}^k \to \Omega^1$, according to def. B.20.

In view of this we have the following elegant definition of smooth n-forms on an arbitrary smooth space.

Definition B.51. For $X \in \mathrm{Smooth0Type}$ a smooth space, def. B.16, a *differential n-form* on X is a homomorphism of smooth spaces of the form

$$\omega \colon X \to \Omega^n(-) \, .$$

Accordingly we write

$$\Omega^n(X) := \mathrm{Smooth0Type}(X, \Omega^n)$$

for the set of smooth n-forms on X.

We may unwind this definition to a very explicit description of differential forms on smooth spaces. This we do in a moment in remark B.55.

Notice the following

Proposition B.52. *Differential 0-forms are equivalently smooth* $\mathbb{R}$*-valued functions:*

$$\Omega^0 \simeq \mathbb{R}\,.$$

Definition B.53. For $f\colon X \to Y$ a homomorphism of smooth spaces, def. B.20, the *pullback of differential forms* along f is the function

$$f^*\colon \Omega^n(Y) \to \Omega^n(X)$$

given by the hom-functor into the smooth space Ω^n of def. B.49:

$$f^* := Hom(-, \Omega^n)\,.$$

This means that it sends an n-form $\omega \in \Omega^n(Y)$ which is modulated by a homomorphism $Y \to \Omega^n$ to the n-form $f^*\omega \in \Omega^n(X)$ which is modulated by the composition—composite $X \xrightarrow{f} Y \to \Omega^n$.

By the Yoneda lemma we find:

Proposition B.54. *For* $X = \mathbb{R}^{\tilde{k}}$ *and* $Y = \mathbb{R}^k$ *definition B.53 reproduces def. B.47.*

Remark B.55. Using def. B.53 for unwinding def. B.51 yields the following explicit description:

a differential n-form $\omega \in \Omega^n(X)$ on a smooth space X is

1. for each way $\phi\colon \mathbb{R}^k \to X$ of laying out a coordinate system $\mathbb{R}^k$ in X a differential n-form

$$\phi^*\omega \in \Omega^n(\mathbb{R}^k)$$

   on the abstract coordinate system, as given by def. B.45;

2. for each abstract coordinate transformation $f\colon \mathbb{R}^{k_2} \to \mathbb{R}^{k_1}$ a corresponding compatibility condition between local differential forms $\phi_1\colon \mathbb{R}^{k_1} \to X$ and $\phi_2\colon \mathbb{R}^{k_2} \to X$ of the form

$$f^*\phi_1^*\omega = \phi_2^*\omega\,.$$

Hence a differential form on a smooth space is simply a collection of differential forms on all its coordinate systems such that these glue along all possible coordinate transformations.

The following adds further explanation to the role of $\Omega^n \in \mathrm{Smooth0Type}$ as a *moduli space*. Notice that since Ω^n is itself a smooth space, we may speak about differential n-forms on Ω^n itsefl.

Definition B.56. The *universal differential* n*-forms* is the differential n-form

$$\omega^n_{univ} \in \Omega^n(\Omega^n)$$

which is modulated by the identity homomorphism $id\colon \Omega^n \to \Omega^n$.

With this definition we have:

Proposition B.57. *For* $X \in \mathrm{Smooth0Type}$ *any smooth space, every differential* n*-form on* X*,* $\omega \in \Omega^n(X)$ *is the pullback of differential forms, def. B.53, of the universal differential* n*-form, def. B.56, along a homomorphism* f *from* X *into the moduli space* Ω^n *of differential* n*-forms:*

$$\omega = f^*\omega^n_{\mathrm{univ}}\,.$$

Remark B.58. This statement is of course in a way a big tautology. Nevertheless it is a very useful tautology to make explicit. The whole concept of differential forms on smooth spaces here may be thought of as simply a variation of the theme of the Yoneda lemma.

Concrete smooth spaces

The smooth universal moduli space of differential forms $\Omega^n(-)$ from def. B.49 is noteworthy in that it has a property not shared by many smooth spaces that one might think of more naively: while evidently being "large" (the space of all differential forms!) it has "very few points" and"very few k-dimensional subspaces" for low k. In fact

Proposition B.59. *For $k < n$ the smooth space Ω^n admits only a unique probe by $\mathbb{R}^k$:*

$$\mathrm{Hom}(\mathbb{R}^k, \mathbf{\Omega}^n) \simeq \Omega^n(\mathbb{R}^k) = \{0\}\,.$$

So while $\mathbf{\Omega}^n$ is a large smooth space, it is "not supported on probes" in low dimensions in as much as one might expect, from more naive notions of smooth spaces.

We now formalize this. The formal notion of an smooth space which is *supported on its probes* is that of a *concrete object*. There is a univeral map that sends any smooth space to its *concretification*. The universal moduli spaces of differential forms turn out to be *non-concrete* in that their concetrification is the point.

Definition B.60. Let $\mathbf{H}$ be a local topos. Write $\sharp\colon \mathbf{H} \to \mathbf{H}$ for the corresponding sharp modality, def. A.14. Then.

1. An object $X \in \mathbf{H}$ is called a *concrete object* if

   $$\mathrm{DeCoh}_X\colon X \to \sharp X$$

   is a monomorphism.

2. For $X \in \mathbf{H}$ any object, its *concretification* $\mathrm{Conc}(X) \in \mathbf{H}$ is the image factorization of DeCoh_X, hence the factorization into an epimorphism followed by a monomorphism

   $$\mathrm{DeCoh}_X : X \to \mathrm{Conc}(X) \hookrightarrow \sharp X\,.$$

Remark B.61. Hence the concretification $\mathrm{Conc}(X)$ of an object X is itself a concrete object and it is universal property—universal with this property.

Proposition B.62. *Let C be a site of definition for the local topos $\mathbf{H}$, with terminal object $*$. Then for $X\colon C^{op} \to Set$ a sheaf, $DeCoh_X$ is given over $U \in C$ by*

$$X(U) \xrightarrow{\simeq} \mathbf{H}(U,X) \overset{\Gamma_{U,X}}{\to} Set(\Gamma(U),\Gamma(X))\,.$$

Proposition B.63. *For $n \geq 1$ we have*

$$\mathrm{Conc}(\Omega^n) \simeq *\,.$$

In this sense the smooth moduli space of differential n-forms is *maximally non-concrete.*

Smooth moduli spaces of differential forms on a smooth space

We discuss the smooth space of differential forms *on a fixed smooth space X*.

Remark B.64. For X a smooth space, the smooth mapping space $[X, \Omega^n] \in \mathrm{Smooth0Type}$ is the smooth space whose $\mathbb{R}^k$-plots are differential n-forms on the product $X \times \mathbb{R}^k$

$$[X,\Omega^n]\colon \mathbb{R}^k \mapsto \Omega^n(X \times \mathbb{R}^k)\,.$$

This is not *quite* what one usually wants to regard as an $\mathbb{R}^k$-parameterized of differential forms on X. That is instead usually meant to be a differential form ω on $X \times \mathbb{R}^k$ which has "no leg along $\mathbb{R}^k$". Another way to say this is that the family of forms on X that is represented by some ω on $X \times \mathbb{R}^k$ is that which over a point $v\colon * \to \mathbb{R}\mathbb{R}^k$ has the value $(id_X, v)^*\omega$. Under this pullback of differential forms any components of ω with "legs along $\mathbb{R}^k$" are identified with the 0 differential form

This is captured by the following definition.

Definition B.65. For $X \in \mathrm{Smooth0Type}$ and $n \in \mathbb{N}$, the *smooth space of differential n-forms* $\mathbf{\Omega}^n(X)$ on X is the concretification, def. B.60, of the smooth mapping space $[X, \Omega^n]$, def. B.29, into the smooth moduli space of differential n-forms, def. B.49:

$$\mathbf{\Omega}^n(X) := \mathrm{Conc}([X, \Omega^n])\,.$$

Proposition B.66. *The $\mathbb{R}^k$-plots of $\mathbf{\Omega}^n(\mathbb{R}^k)$ are indeed smooth differential n-forms on $X \times \mathbb{R}^k$ which are such that their evaluation on vector fields tangent to $\mathbb{R}^k$ vanish.*

Proof. By def. **??**, def. B.60 and prop. B.62 the set of plots of $\mathbf{\Omega}^n(X)$ over $\mathbb{R}^k$ is the image of the function

$$\Omega^n(X \times \mathbb{R}^k) \simeq \mathrm{Hom}_{\mathrm{Smooth0Type}}(\mathbb{R}^k, [X, \Omega^n]) \overset{\Gamma_{\mathbb{R}^k,[X,\Omega^n]}}{\to} \mathrm{Hom}_{\mathrm{Set}}(\Gamma(\mathbb{R}^k), \Gamma[X, \Omega^n]) \simeq \mathrm{Hom}_{\mathrm{Set}}(\mathbb{R}^k_s, \Omega^n(X))\,,$$

where on the right $\mathbb{R}^k_s$ denotes, just for emphasis, the underlying set of $\mathbb{R}^k_s$. This function manifestly sends a smooth differential form $\omega \in \Omega^n(X \times \mathbb{R}^k)$ to the function from points v of $\mathbb{R}^k$ to differential forms on X given by

$$\omega \mapsto (v \mapsto (id_X, v)^* \omega)\ .$$

Under this function all components of differential forms with a "leg along" $\mathbb{R}^k$ are sent to the 0-form. Hence the image of this function is the collection of smooth forms on $X \times \mathbb{R}^k$ with "no leg along $\mathbb{R}^k$". □

Remark B.67. For $n = 0$ we have (for any $X \in \mathrm{Smooth0Type}$)

$$\begin{aligned}\mathbf{\Omega}^0(X) &:= Conc[X, \Omega^1] \\ &\simeq Conc[X, \mathbb{R}] \quad , \\ &\simeq [X, \mathbb{R}]\end{aligned}$$

by prop. B.67.

C Higher geometry via Geometric homotopy types

(This section essentially coincides with sections 2.1, 2.2 in [15].)

Above we indicated how traditional geometric prequantum theory has a natural formulation in terms of stacks of groupoids over the site of smooth manifolds. Accordingly, higher geometric prequantum theory has a natural formulation in terms of stacks of *higher groupoids* (homotopy types) on a site of geometric test spaces. Conversely, since a collection of higher stacks forms a context called an ∞-*topos*, and since these are particularly well-behaved contexts for formulating geometric theories, our formulation of higher prequantum geometry is guided by notions that are natural in higher topos theory. Such an axiomatic approach guarantees robust general notions: everything that we discuss here makes sense and holds in *every* ∞-topos whatsoever, be it one that models higher/derived differential geometry, complex geometry, analytic geometry, supergeometry etc.

The only constraining assumption that we need later on arises in D below, when we turn from plain geometric cohomology to *differential cohomology*. For that to make sense we need to impose a minimum of axioms that guarantees that the ambient ∞-topos supports not only a good notion of fiber/principal ∞-bundles, as every ∞-topos does, but also of connections on such bundles.

This section is a brief commented list of some basic constructions and facts in higher geometry/higher topos theory which we need below; the foundational aspects in C.1 taken from [31], and the fiber bundle and representation theory in C.2 taken from [41, 42].

C.1 Homotopy toposes of geometric ∞-groupoids

The notion of ∞-*topos* [31] combines geometry with homotopy theory, hence with higher gauge symmetry: given a category C of geometric test spaces (hence equipped with a Grothendieck topology), the ∞-*topos of* ∞-*stacks* over it, denoted $\mathrm{Sh}_\infty(C)$, is the homotopy theory obtained by taking the category $[C^{\mathrm{op}}, \mathrm{KanCplx}]$ of Kan-complex valued presheaves on C and then universally turning *local* homotopy equivalence between such presheaves (local as seen by the Grothendieck topology) into global homotopy equivalences. This process is called *simplicial localization* (see [42] for a review and further details), denoted by the right hand side of

$$\mathrm{Sh}_\infty(C) \simeq L_{\mathrm{lhe}}[C^{\mathrm{op}}, \mathrm{KanCplx}]\,.$$

More generally, for $\mathcal{C}$ any category and $W \subset \mathrm{Mor}(\mathcal{C})$ a collection of morphisms, there is the homotopy theory of the simplicial localization $L_W\mathcal{C}$ obtained by universally turning the morphisms in W into homotopy equivalences. This is the ∞-*category* induced by $(\mathcal{C}, W)$. A homotopy-theoretic functor $L_{W_1}\mathcal{C}_1 \to L_{W_2}\mathcal{C}_2$ between such homotopy-theoretic categories is an ∞-*functor*. If this is induced from an ordinary functor $\mathcal{C}_1 \to \mathcal{C}_2$ it is also called a (total) *derived functor*. The example of this that we use prominently is the Dold-Kan functor below in remark C.4.

Example C.1. The basic example is the ∞-topos $\infty\mathrm{Grpd}$ of ∞-groupoids (hence of geometrically discrete ∞-groupoids!). This is presented equivalently by the simplicial localization of the category KanCplx of Kan complexes at the homotopy equivalences, or of the category Top of (compactly generated weakly Hausdorff) topological spaces at the weak homotopy equivalences:

$$\infty\mathrm{Grpd} \simeq L_{\mathrm{he}}\mathrm{KanCplx} \simeq L_{\mathrm{whe}}\mathrm{Top}\,.$$

Hence this is just traditional homotopy theory thought of as the ∞-topos of geometrically discrete ∞-groupoids.

Example C.2. The most immediate choice of ∞-topos which subsumes traditional differential geometry, foliation/orbifold theory and Lie groupoid/differentiable stack theory is that of ∞-stacks over the site of smooth manifolds with its standard Grothendieck topology of open covers. We write

$$\mathrm{Smooth}\infty\mathrm{Grpd} \simeq \mathrm{Sh}_\infty(\mathrm{SmthMfd}) \simeq L_{\mathrm{lhe}}[\mathrm{SmoothMfd}^{\mathrm{op}}, \mathrm{KanCplx}]$$

for this ∞-topos. The ordinary category of smooth manifolds is faithfully embedded into this ∞-topos, as is the collection of Lie groupoids with generalized/Morita-morphisms between them ("differentiable stacks"). More in detail, a Lie groupoid $\mathcal{G} = \left(\mathcal{G}_1 \underset{s}{\overset{t}{\rightrightarrows}} \mathcal{G}_0 \right)$ is identified, up to equivalence, with the presheaf of Kan complexes given by

$$\underline{\mathcal{G}} : U \mapsto N\left(C^\infty(U, \mathcal{G}_1) \underset{C^\infty(U,s)}{\overset{C^\infty(U,t)}{\rightrightarrows}} C^\infty(U, \mathcal{G}_0\ ) \right)$$

for every smooth manifold U, where $N : \mathrm{Grpd} \to \mathrm{KanCplx}$ is the nerve functor. See [39] for how orbifolds and foliations are special cases of Lie groupoids and hence are similarly embedded into $\mathrm{Smooth}\infty\mathrm{Grpd}$. Basic tools for explicit computations with objects in $\mathrm{Smooth}\infty\mathrm{Grpd}$ and similar contexts of higher geometry are discussed in [19, 42, 50].

Remark C.3. As in traditional homotopy theory, when we draw a commuting diagram of morphisms in an ∞-category, it is always understood that they commute up to a specified homotopy. We will often notationally suppress these homotopies that fill diagrams, except if we want to give them explicit labels. For instance, in the figure below, the diagram of morphisms in an ∞-category on the left hand side always means the more explicit diagram displayed on the right hand side:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & A & \end{tikzcd} \quad := \quad \begin{tikzcd} X \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\swArrow"] & Y \arrow[dl] \\ & A & \end{tikzcd} .$$

In the same spirit all the universal constructions that we mention in the following refer to their homotopy-correct version. Notably fiber products in the following always are homotopy fiber products. With homotopies thus understood, most of the familiar basic facts of category theory generalize verbatim to ∞-category theory. For instance a basic fact that we make repeated use of is the pasting law for homotopy pullbacks: if we have two adjacent square diagrams and the right square is a homotopy pullback, then the left square is also such a pullback if and only if the total rectangle is.

Remark C.4 (generalized nonabelian sheaf cohomology)**.** For $X, A \in \mathbf{H}$ any two objects in an ∞-topos, we have an ∞-groupoid $\mathbf{H}(X, A) \in \infty\mathrm{Grpd}$ consisting of morphisms from X to A, homotopies between such morphisms and higher homotopies between these, etc. We may think of this as the ∞-groupoid of cocycles, coboundaries and higher coboundaries on X with coefficients in A. The set of connected components of this ∞-groupoid

$$H(X, A) := \pi_0 \mathbf{H}(X, A)$$

is the *cohomology* of X with coefficients in A. This notion of cohomology in an ∞-topos unifies abelian sheaf cohomology with the generalized cohomology theories of algebraic topology and generalizes both to nonabelian cohomology that classifies higher principal bundles in $\mathbf{H}$ (this we come to below in C.2). Hence the *homotopy category* H of an ∞-topos $\mathbf{H}$ may be thought of as a *generalized nonabelian sheaf cohomology theory*: the fact that it is a sheaf cohomology theory means that it encodes "geometric cohomology", for instance "smooth cohomology" in example C.2. Ordinary abelian sheaf cohomology is reproduced as the special case where the coefficient object $A \in L_{\mathrm{he}}[C^{\mathrm{op}}, \mathrm{KanCplx}]$ is in the essential image of the Dold-Kan correspondence

$$\mathrm{DK}: \ \mathrm{Ch}_{\bullet \geq 0}(\mathrm{Ab}) \xrightarrow{\simeq} \mathrm{Ab}^{(\Delta^{\mathrm{op}})} \xrightarrow{\mathrm{forget}} \mathrm{KanCplx} \ ,$$

which regards a sheaf of chain complexes of abelian groups equivalently as a sheaf of simplicial abelian groups (whose normalized chain complex is the original complex), hence in particular as a sheaf of Kan complexes.

A crucial point of ∞-toposes is that they share the general abstract properties of classical homotopy theory in $\infty\mathrm{Grpd} \simeq L_{\mathrm{whe}}\mathrm{Top}$ (example C.1). In our discussion of higher prequantum geometry we need specifically the following three technical aspects of homotopy theory in ∞-toposes:

1. Moore-Postnikov-Whitehead-theory;
2. relative theory over a base and base change;
3. looping and delooping.

In the remainder of this section we state the corresponding definitions and results that are used later on. The reader not interested in this level of technical detail should maybe skip ahead and come back here as need be.

There is a notion of *homotopy groups* π_n of objects in $\mathbf{H}$, however these are not groups in Set but group objects in the 1-topos (sheaf topos) of 0-truncated objects of $\mathbf{H}$. With respect to these homotopy sheaves there is Moore-Postnikov-Whitehead theory:

Remark C.5. An object $A \in \mathbf{H}$ is called *n-truncated* if for all $X \in \mathbf{H}$ the ∞-groupoid $\mathbf{H}(X, A)$ is a homotopy n-type. The n-truncated objects in $\mathbf{H} = L_{\mathrm{lwe}}[C^{\mathrm{op}}, \mathrm{KanCplx}]$ are the *stacks of n-groupoids* on C. For $n = 1$ these are ordinary stacks and for $n = 0$ these are ordinary sheaves on C.

Proposition C.6. *The full sub-∞-category of n-truncated objects $\tau_{\leq n}\mathbf{H} \hookrightarrow \mathbf{H}$ in an ∞-topos is reflectively embedded, which means that there is an idempotent truncation projection $\tau_n : \mathbf{H} \to \mathbf{H}$ which sends an arbitrary ∞-stack X to its universal approximation by an n-truncated object $\tau_n X$, the nth* Postnikov stage *of X as seen in $\mathbf{H}$.*

More generally given a morphism $f : X \to Y$ in $\mathbf{H}$, there is a tower of factorizations

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted, no head] & \\
 & \mathrm{im}_3(f) \arrow[d] \arrow[ddr] & \\
 & \mathrm{im}_2(f) \arrow[d] \arrow[dr] & \\
X \arrow[uur] \arrow[ur] \arrow[r, two heads] \arrow[rr, bend right, "f"'] & \mathrm{im}_1(f) \arrow[r, hook] & Y
\end{tikzcd}$$

with the property that for all $n \in \mathbb{N}$ the morphism $X \longrightarrow \mathrm{im}_n(f)$ is an epimorphism on π_0, an isomorphism on $\pi_{<n-1}$, and that $\mathrm{im}_n(f) \longrightarrow Y$ is an injection on π_{n-1} and an isomorphism on all $\pi_{\geq n}$.

This is part of [31, section 5.5.6 and 6.5].

Definition C.7. We call the objects $\mathrm{im}_n(f)$ in prop. C.6 the *n-image* of f and say that morphisms of the form $X \longrightarrow \mathrm{im}_n(f)$ are *n-epimorphisms* and that morphisms of the form $\mathrm{im}_n(f) \longrightarrow Y$ are *n-monomorphisms* (in [31] these are called $(n-1)$-*connective* and $(n-2)$-*truncated* morphisms, respectively).

For $n = 1$ the n-image factorization has a useful more explicit characterization:

Proposition C.8. *For $f : X \to Y$ a morphism in an ∞-topos $\mathbf{H}$, consider the homotopy-colimiting cocone under its Čech nerve simplicial diagram as indicated in the top row of the following diagram*

$$\begin{tikzcd}
\cdots \; X \underset{Y}{\times} X \underset{Y}{\times} X \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X \underset{Y}{\times} X \arrow[r, shift left] \arrow[r, shift right] \arrow[l, shift left=2] \arrow[l, shift right=2] & X \arrow[r, two heads, "p"] \arrow[dr, "f"'] \arrow[l] & \left(\varinjlim_n X^{\times_Y^{n+1}}\right) \simeq \mathrm{im}_1(f) \arrow[d, hook, dashed, "i"] \\
 & & & Y
\end{tikzcd}$$

Since $f : X \to Y$ canonically extends to a homotopy cocone under its own Čech nerve, the universal property of the ∞-colimit induces a vertical dashed map i, is indicated. The resulting factorization of f is its 1-image factorization, as shown.

Another important aspect of ∞-toposes which is familiar both from traditional geometry as well as from traditional homotopy theory is the possibility of working relatively over a base object, the construction which we amplified above in 2.12: given an object $X \in \mathbf{H}$, an *object over* X is just a map $E \to X$ into X, and the collection of all of these with maps between them that fix X is written $\mathbf{H}_{/X}$ and called the *slice ∞-topos* over X.

Proposition C.9. *For all $X \in \mathbf{X}$ the slice $\mathbf{H}_{/X}$ is again an ∞-topos and the ∞-functor $\sum\limits_X : (E \to X) \mapsto E$*

is the left part of an adjoint triple of base change ∞*-functors:*

$$\left(\sum_X \dashv X \times (-) \dashv \prod_X\right) : \mathbf{H}_{/X} \underset{\prod_X}{\overset{\sum_X}{\underset{\longleftarrow X\times(-)\longrightarrow}{\longrightarrow}}} \mathbf{H} \;.$$

This is [31, prop. 6.3.5.1].

Remark C.10. Using the right adjointness of $\prod\limits_X$ in prop. C.9 one finds that it sends a bundle $E \to X$ to its space of sections, regarded naturally as a geometric ∞-groupoid itself, hence as an object of $\mathbf{H}$:

$$\mathbf{\Gamma}_X(E) \simeq \prod_X E \quad \in \mathbf{H}\,.$$

The underlying discrete ∞-groupoid of sections (forgetting the geometric structure) is as usual given by further evaluation on the point

$$\Gamma_X(E) \simeq \Gamma\left(\mathbf{\Gamma}_X(E)\right) \in \infty\mathrm{Grpd}\,.$$

This is the ∞-groupoid whose objects are naturally identified with sections σ in

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[ur, dashed, "\sigma"] \arrow[r, equal] & X \end{tikzcd} ,$$

whose morphisms are homotopies of such sections, etc. The geometric ∞-groupoid of sections $\mathbf{\Gamma}_X(E) \in \mathbf{H}$ plays a central role in the discussion of genuine higher prequantum geometry starting below in C.4.

Remark C.11. Given $A \in \mathbf{H}$, the intrinsic cohomology, as in remark C.4, of the slice ∞-topos $\mathbf{H}_{/A}$ as in prop. C.9 is equivalently *twisted cohomology* in $\mathbf{H}$ with twist coefficients A: a domain object $\mathcal{X} = (X \stackrel{\phi}{\to} A) \in \mathbf{H}_{/A}$ is equivalently an object $X \in \mathbf{H}$ equipped with a twisting cocycle ϕ, and a codomain object $\mathcal{A} = (E \to A)$ is equivalently a *local coefficient bundle*. See [41, section 4.3] for a general abstract account of this and [18, 51] for examples relevant to higher prequantum theory. This observation, combined with our discussion in D.4, implies that higher prequantum geometry is, equivalently, the geometry of spaces equipped with a (differential) cohomological twist. An archetypical example of this general phenomenon is the identification of the (pre-)quantum 2-states of the higher prequantized WZW model with cocycles in twisted K-theory.

A *group object* in $\mathbf{H}$ – an ∞*-group* equipped with geometric structure as encoded by $\mathbf{H}$ – may be defined to be an object equipped with a coherently homotopy-associative- hence A_∞-multiplication, such that its 0-truncation (to a sheaf) is an ordinary group. The standard example for this is the *loop space object* $\Omega_x X$ of any object X which is equipped with a global point, $x : * \to X$. This is the homotopy fiber product of the point with itself, $\Omega_x X := * \underset{X}{\times} *$. Here we can assume without restriction that X only has that single global point, up to equivalence, hence that X is *pointed connected*.

Remark C.12. For $f : B \to C$ any morphism of pointed objects in $\mathbf{H}$, forming successive homotopy fibers yields, due to the pasting law, a long homotopy fiber sequence in $\mathbf{H}$ of the form

$$\cdots \longrightarrow \Omega B \xrightarrow{\Omega f} \Omega C \longrightarrow A \longrightarrow B \longrightarrow C \;,$$

which repeats to the left with successive loopings of the original morphism. Given an object of the form $\mathbf{B}^n G \in \mathbf{H}$ we write $H^n(X, G) := H(X, \mathbf{B}^n G)$ for the degree-n cohomology of X with coefficients in G. Since $\mathbf{H}(X, -)$ preserves homotopy fibers every morphism induces the expected long exact sequence in generalized nonabelian sheaf cohomology.

Another incarnaton of this lifting of long sequences of homotopy groups to long sequences of homotopy types is the following.[4]

Proposition C.13. *For f a morphism of pointed objects, there is for each $n \in \mathbb{N}$ a natural equivalence*

$$\mathrm{im}_n \circ \Omega(f) \simeq \Omega \circ \mathrm{im}_{n+1}(f)$$

between the n-image of the looping of f and the looping of the $(n+1)$-image of f.

A fact that we make constant use of is that up to equivalence every ∞-group in an ∞-topos arises as the loop space object of another object, and essentially uniquely so:

Proposition C.14. *The ∞-functor Ω that forms loop space objects is an equivalence of ∞-categories*

$$\mathrm{Grp}(\mathbf{H}) \underset{\mathbf{B}}{\overset{\Omega}{\underset{\simeq}{\rightleftarrows}}} \mathbf{H}^{*/}_{\geq 1}$$

from pointed connected objects in $\mathbf{H}$ to group objects in $\mathbf{H}$.

In $\mathbf{H} = \infty\mathrm{Grpd}$ (example C.1) this is a classical fact of homotopy theory due to people like Kan, Milnor, May and Segal. In an arbitrary ∞-topos this is [35, theorem 5.1.3.6].

Remark C.15. The inverse ∞-functor $\mathbf{B}$ to looping usually called the *delooping* functor. The boldface here is to serve as a reminder that this is the delooping in $\mathbf{H}$, hence in general a *geometric* delooping which is richer (when both are comparable via a notion of geometric realization, which we come to below in D) than the familiar delooping in ∞Grpd, example C.1, which is traditionally denoted by "B".

Example C.16. In $\mathbf{H} = \infty\mathrm{Grpd}$, example C.1, every simplicial group – a Kan complex equipped with an ordinary group structure – presents a group object, hence a (geometrically discrete) ∞-group, and up to equivalence all ∞-groups arise this way.

Example C.17. In $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$, example C.2, every Lie group is canonically a smooth ∞-group

$$G \in \mathrm{LieGrp} = \mathrm{Grp}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Grp}(\mathrm{Sh}_\infty(\mathrm{SmthMfd})) \simeq \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})\,.$$

Accordingly a simplicial Lie group represents a smooth ∞-group. Generally, simplicial sheaves of groups represent all smooth ∞-groups, up to equivalence.

Specific examples of smooth ∞-groups that we encounter (e.g. [49, 51]) are the *smooth string 2-group* and the *smooth fivebrane 6-group*

$$\mathrm{String}, \mathrm{Fivebrane} \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})\,.$$

These participate in a smooth refinement of the Whitenead tower of BO from ∞Grpd to Smooth∞Grpd,

[4] U.S. thanks Egbert Rijke for discussion of this point.

exhibited by a diagram in Smooth∞Grpd of the form

$$
\begin{tikzcd}
\vdots \arrow[d, dotted, no head] & \vdots \arrow[d, dotted, no head] \\
\mathbf{B}\mathrm{Fivebrane} \arrow[r] \arrow[d] & BO\langle 8\rangle \arrow[d] \\
\mathbf{B}\mathrm{String} \arrow[r] \arrow[d] & BO\langle 4\rangle \arrow[d] \\
\mathbf{B}\mathrm{Spin} \arrow[r] \arrow[d] & B\mathrm{Spin} \arrow[d] \\
\mathbf{B}\mathrm{SO} \arrow[r] \arrow[d] & B\mathrm{SO} \arrow[d] \\
\mathbf{B}O \arrow[r, "\int"] & BO
\end{tikzcd}
$$

(Here the horizontal maps denote geometric realization, discussed below in D, example D.1.) A review of the smooth String 2-group (and of its ∞-Lie algebra, the $\mathfrak{string}$ Lie 2-algebra) in a context of higher prequantum theory and string geometry is in the appendix of [17]; here we encounter this below in **??**. The smooth Fivebrane 6-group was constructed in [19], a discussion in the context of higher geometric prequantum theory is in [49]. For more on these matters see [50, section 5].

A group object G may admit and be equipped with further deloopings $\mathbf{B}^k G$, for $k \in \mathbb{N}$. (In terms of $A_\infty \simeq E_1$-structure this is a lift to E_k-structure, where E_k is the little k-cubes ∞-operad.) The higher the value of k here, the *closer to abelian* the ∞-group is:

Definition C.18. Given a group object $G \in \mathrm{Grp}(\mathbf{H})$,

1. it is equipped with the structure of a *braided group object* if equivalently
   - $\mathbf{B}G$ is equipped with a further delooping $\mathbf{B}^2 G$;
   - $\mathbf{B}G$ is itself equipped with the structure of a group object;
2. it is equipped with the structure of a *sylleptic group object* if equivalently
   - $\mathbf{B}G$ is eqipped with two further deloopings $\mathbf{B}^3 G$;
   - $\mathbf{B}G$ is itself equipped with the structure of a braided group object;
3. it is equipped with the structure of an *abelian group object* if it is equipped with ever higher deloopings, hence if it is an *infinite loop space object* in $\mathbf{H}$.

We write

$$\mathrm{Grp}_\infty(\mathbf{H}) \to \cdots \to \mathrm{Grp}_3(\mathbf{H}) \to \mathrm{Grp}_2(\mathbf{H}) \to \mathrm{Grp}_1(\mathbf{H}) := \mathrm{Grp}(\mathbf{H})$$

for the ∞-categories of abelian ∞-groups ... sylleptic ∞-groups, braided ∞-groups and ∞-groups, respectively, with the evident forgetful functors between them.

Example C.19. Given an abelian Lie group such as the circle group

$$U(1) \in \mathrm{Grp}(\mathrm{Smoothmfd}) \hookrightarrow \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grp})$$

it is canonically an abelian ∞-group. For every $n \in \mathbb{N}$ the n-fold geometric delooping

$$\mathbf{B}^n U(1) \in \mathrm{Smooth}\infty\mathrm{Grpd}$$

canonically exists and is presented under the Dold-Kan correspondence, remark C.4, by the chain complex of sheaves of abelian groups concentrated on $\underline{U}(1) = C^\infty(-, U(1))$ in degree n:

$$\mathbf{B}^n U(1) \simeq \mathrm{DK}\left(\underline{U}(1)[n]\right) \in L_{\mathrm{lwhe}}[\mathrm{SmthMfd}^{\mathrm{op}}, \mathrm{KanCplx}] \simeq \mathrm{Smooth}\infty\mathrm{Grpd}\,.$$

The analogous statements holds for the multiplicative Lie group $\mathbb{C}^\times$ of invertible complex numbers. While the canoncial inclusion

$$\mathbf{B}^n U(1) \hookrightarrow \mathbf{B}\mathbb{C}^\times$$

is not an equivalence in Smooth∞Grpd, it becomes an equivalence under geometric realization $\int$: Smooth∞Grpd $\to$ ∞Grpd $\simeq L_{\mathrm{whe}}$Top (see D below), which maps both to

$$\int (\mathbf{B}^n U(1)) \simeq \int \left(\mathbf{B}^n \mathbb{C}^\times\right) \simeq K(\mathbb{Z}, n+1)\,.$$

Although we happen to talk about $U(1)$-principal (higher) bundles throughout, using this relation all of our discussion is directly adapted to $\mathbb{C}^\times$-principal (higher) bundles, which is the default in some part of the literature.

Remark C.20. For $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group, the geometric cohomology, remark C.4, of the delooping object $\mathbf{B}G \in \mathbf{H}$ of prop. C.14 is the *∞-group cohomology* of G:

$$H_{\mathrm{grp}}(G, A) := H(\mathbf{B}G, A) := \pi_0 \mathbf{H}(\mathbf{B}G, A)\,.$$

Example C.21. For $G \in \mathrm{Grp}(\mathrm{SmoothMfd}) \hookrightarrow \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ a Lie group regarded as a smooth ∞-group as in example C.17, and for $A = \mathbb{R}$ or $A = \mathbb{Z}$ or $A = U(1)$, the intrinsic group cohomology of G in Smooth∞Grpd according to remark C.20 with coefficients in $\mathbf{B}^n A$ coincides with Segal-Brylinski Lie group cohomology in degree n with these coefficients. In particular for G a compact Lie group we have

$$H^n_{\mathrm{grp}}(G, U(1)) := \pi_0 \mathrm{Smooth}\infty\mathrm{Grpd}(\mathbf{B}G, \mathbf{B}^n U(1)) \simeq H^{n+1}(BG, \mathbb{Z})\,,$$

where on the far right we have the traditonal (for instance singular) cohomology of the classifying space BG. This is a first class of examples of geometric refinement to ∞-toposes: it says that every traditional universal characteristic class $[c] \in H^{n+1}(BG, \mathbb{Z})$ is represented by a smooth cocycle $\nabla^0 : \mathbf{B}G \to \mathbf{B}^n U(1)$ – or equivalently, as we discuss below in C.2, by a smooth $(\mathbf{B}^{n-1}U(1))$-principal bundle on $\mathbf{B}G$.

This is discussed in [50]. Below in **??** these smooth refinements of universal characteristic classes are seen to be the higher prequantum bundles of n-dimensional Chern-Simons type field theories.

Remark C.22. While every object of an ∞-topos may be thought of as a higher groupoid equipped a some type of geometric structure, there is a subtlety to take note of when comparing to groupoids as traditionally used in geometry: a Lie groupoid or "differentiable stack" $\mathcal{G}$, as in example C.2, is usually (often implicitly) regarded as a groupoid *equipped with an atlas*, namely with the canonical map $\mathcal{G}_0 \twoheadrightarrow \mathcal{G}$ from the space of objects, regarded as a groupoid with only identity morphisms. This map is a 1-epimorphism, def. C.7, hence a cover or atlas of $\mathcal{G}$ by $\mathcal{G}_0$, as seen in Smooth∞Grpd.

Example C.23. For every group object G there is by prop. C.14 an essentialy unique morphism $* \twoheadrightarrow \mathbf{B}G$. This is a 1-epimorphism, def. C.7, hence exhibits the point as an *atlas* of $\mathbf{B}G$. The Čech nerve of this point inclusion is a simplicial object $(\mathbf{B}G)_n = G^{\times^n} \in \mathbf{H}^{\Delta^{\mathrm{op}}}$ in $\mathbf{H}$, generalizing the familiar bar construction on a group.

More generally, we say :

Definition C.24. A simplicial object $X_\bullet \in \mathbf{H}^{(\Delta^{\mathrm{op}})}$ in an ∞-topos $\mathbf{H}$ is a *pre-category* object if for all $n \in \mathbb{N}$ the canonical projection maps

$$p_n : \; X_n \xrightarrow{\simeq} X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1$$

(with n factors on the right) are equivalences, as indicated (the *Segal conditions*). These conditions imply a coherently associative partial composition operation on X_1 over X_0 given by

$$\circ : \; X_1 \underset{X_0}{\times} X_1 \underset{\simeq}{\xrightarrow{p_2^{-1}}} X_2 \xrightarrow{d_1} X_1 \; .$$

If in a pre-category object all of X_1 is invertible (up to homotopies in X_2) under this composition operation, then it is called a *groupoid object.* We write $\mathrm{Grpd}(\mathbf{H}) \hookrightarrow \mathbf{H}^{\Delta^{\mathrm{op}}}$ for the full sub-∞-category of the simplicial objects in $\mathbf{H}$ on the groupoid objects.

This is [31, def. 6.1.2.7], here stated as in [33, section 1.1]. The following asserts that groupoid objects in this sense are indeed equivalently just objects of $\mathbf{H}$, but equipped with an atlas:

Proposition C.25 ($\frac{1}{3}$-Giraud-Rezk-Lurie axioms)**.** *Sending 1-epimorphisms in* $\mathbf{H}$*, def. C.7, to their Čech nerve simplicial objects is an equivalence of ∞-categories onto the groupoid objects in* $\mathbf{H}$*:*

$$(\mathbf{H}^{(\Delta^1)})_{1\mathrm{epi}} \xrightarrow{\simeq} \mathrm{Grpd}(\mathbf{H}) \; .$$

This is in [31, theorem 6.1.0.6, below cor. 6.2.3.5].

In order to reflect this state of affairs notationally, we stick here to the following convention on notation and terminology:

- An object $X \in \mathbf{H} = L_{\mathrm{lhe}}[C^{\mathrm{op}}, \mathrm{KanCplx}]$ we call an *∞-groupoid* (*parameterized over C* or *with C-geometric structure*);
- a 1-epimorphism $X_0 \twoheadrightarrow X$ we call an *atlas for the ∞-groupoid X*;
- an object $X_\bullet \in \mathrm{Grpd}(\mathbf{H}) \hookrightarrow \mathbf{H}^{(\Delta^{\mathrm{op}})}$ we call a *(higher) groupoid object* in $\mathbf{H}$;
- the homotopy colimit over the simplicial diagram underlying a higher groupoid object, hence its *realization* as an ∞-groupoid, we indicate with the same symbol, but omitting the subscript decoration:

$$X := \varinjlim X_\bullet := \varinjlim_n X_n \, ;$$

- hence given a higher groupoid object denoted $X_\bullet \in \mathrm{Grpd}(\mathbf{H})$, the ∞-groupoid with atlas that corresponds to it under prop. C.25 we denote by $(\, X_0 \twoheadrightarrow X \,) \in (\mathbf{H}^{(\Delta^1)})_{1\mathrm{epi}}$.

C.2 Higher geometric fiber bundles

If we think of a group object $G \in \mathrm{Grp}(\mathbf{H})$ as an A_∞-algebra object in $\mathbf{H}$, then there is an evident notion of *A_∞-actions* of G on any object in $\mathbf{H}$. This defines an ∞-category $G\mathrm{Act}(\mathbf{H})$ of G-∞-actions and G-equivariant maps between these. In traditional geometry, one constructs from a G-space V a universal associated bundle $EG \times_G V \to BG$. Analogously, in higher geometry we have a useful equivalent reformulation of G-∞-actions:

Proposition C.26. *For $G \in \mathrm{Grp}(\mathbf{H})$ there is an equivalence*

$$(\mathbf{E}G) \times_G (-) : \; G\mathrm{Act}(\mathbf{H}) \xrightarrow{\simeq} \mathbf{H}_{/\mathbf{B}G}$$

between the ∞-category of ∞-actions of G and the slice ∞-topos over the delooping $\mathbf{B}G$.

This is [41, theorem 3.19, section 4.1]. In prop. C.32 below we will see that this equivalence is exhibited by sending an ∞-action (V, ρ) to the corresponding *universal ρ-associated V-fiber ∞-bundle* over $\mathbf{B}G$. This explains our choice of notation for the ∞-functor $(\mathbf{E}G) \times_G (-)$.

Definition C.27. For $G \in \mathrm{Grp}(\mathbf{H})$, a *$G$-principal ∞-bundle* in $\mathbf{H}$ is a map $P \to X$ equipped with an action of G on P over X such that the map is the ∞-quotient projection $P \to X \simeq P//G$.

Write $G\mathrm{Bund}_X(\mathbf{H})$ for the ∞-category of G-principal ∞-bundles and G-equivariant maps between them fixing the base.

See [41, section 3.1] for some background discussion on the higher geometry of G-principal ∞-bundles.

Proposition C.28. *For $G \in \mathrm{Grp}(\mathbf{H})$, the map that sends a morphism in $\mathbf{H}$ of the form $X \to \mathbf{B}G$ to its homotopy fiber over the essentially unique point of $\mathbf{B}G$ exhibits an equivalence with the ∞-groupoid of G-principal bundles over X:*

$$\mathrm{fib} : \; \mathbf{H}(X, \mathbf{B}G) \xrightarrow{\simeq} G\mathrm{Bund}_X(\mathbf{H}) \; .$$

This is [41, theorem 3.19].

Remark C.29. Prop. C.28 says that the delooping $\mathbf{B}G \in \mathbf{H}$ is the *moduli ∞-stack* of *G-principal ∞-bundles*. This means that for any object $X \in \mathbf{H}$, maps $\nabla^0 : X \to \mathbf{B}G$ correspond to G-principal ∞-bundles $P \to X$ over X, and (higher) homotopies of ∇^0 correspond to (higher) gauge transformations of $P \to X$. Moreover, prop. C.28 says that the point inclusion into $\mathbf{B}G$ is equivalently the *universal G-principal ∞-bundle* $\mathbf{E}G \to \mathbf{B}G$ over the moduli ∞-stack. Here and in the following we tend to denote modulating maps of G-principal bundles by "∇^0", because below in D.4 we find that in higher prequantum geometry it is natural to regard these maps as the leftmost stage in a sequence of analogous but richer maps whose rightmost stage is a G-principal connection.

Remark C.30. Higher geometry conceptually simplifies and strenghtens higher principal bundle theory, thereby avoiding certain difficulties which arise in ordinary principal bundle theory. For example we observe that prop. C.28 has a stronger formulation, which says that, conversely, for *every* G-∞-action on some object $V \in \mathbf{H}$, the ∞-quotient map $V \to V//G$ is a G-principal ∞-bundle, and that all G-principal ∞-bundles arise this way. This is a statement wildly false in ordinary geometry! It becomes true in higher geometry because homotopy colimits "correct" the quotients by non-free actions.

Example C.31. For $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group and $A \in \mathrm{Grp}_{n+2}(\mathbf{H})$ a sufficiently deloopable ∞-group, a map of the form $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^{n+2}A$ is, by remark C.20, equivalently a cocycle representing a class $c \in H^{n+2}_{\mathrm{grp}}(G, A)$ in the degree-n ∞-group cohomology of G with coefficients in A. The $\mathbf{B}^{n+1}A$-principal ∞-bundle

$$\begin{tikzcd} \mathbf{B}^{n+1}A \arrow[r] & \mathbf{B}\widehat{G} \arrow[d, "\mathrm{fib}(\mathbf{c})"] \\ & \mathbf{B}G \end{tikzcd}$$

which is classified by $\mathbf{c}$ according to prop. C.28 is the delooping of the *∞-group extension*

$$\begin{tikzcd} \mathbf{B}^{n}A \arrow[r] & \widehat{G} \arrow[d, "\Omega\mathrm{fib}(\mathbf{c})"] \\ & G \end{tikzcd}$$

which is classified by c. Below in **??** we discuss how in higher prequantum geometry the ∞-bundles of the form $\mathrm{fib}(\mathbf{c})$ appear as higher prequantum bundles of higher Chern-Simons-type field theories, while $\Omega\mathrm{fib}(\mathbf{c})$ are those of the corresponding higher Wess-Zumino-Witten type field theories.

For $P \to X$ a G-principal ∞-bundle and given an ∞-action ρ of G on V there is the corresponding *associated* V-fiber ∞-bundle $P \times_G V \to X$ obtained by forming the ∞-quotient of the diagonal ∞-action of G on $P \times V$. The equivalence of prop C.26 may be understood as sending (V, ρ) to the map $V//G \to \mathbf{B}G$ which is the *universal ρ-associated V-fiber ∞-bundle*:

Proposition C.32. *For $P \to X$ a G-principal ∞-bundle in $\mathbf{H}$ and for (V, ρ) a G-action, the ρ-associated V-fiber ∞-bundle $P \times_G V \to X$ fits into a homotopy pullback square of the form*

$$\begin{tikzcd} P \times_G V \arrow[r] \arrow[d] & V//G \arrow[d, "\rho"] \\ X \arrow[r, "\nabla^0"] & \mathbf{B}G \end{tikzcd} ,$$

where the bottom map modulates the given G-principal bundle by prop. C.28 and where the right map is the incarnation of ρ under the equivalence of prop. C.26.

This is [41, prop. 4.6].

Remark C.33. The internal hom (mapping stack) $[(V_1, \rho_1), (V_2, \rho_2)] \in G\mathrm{Act}(\mathbf{H})$ between two G-actions ρ_1 and ρ_2 on objects $V_1, V_2 \in \mathbf{H}$ is, by prop. C.26, the internal hom $[V_1, V_2] \in \mathbf{H}$ of these two objects equipped with the induced *conjugation action* ρ_{conj} of G:

$$[(V_1, \rho_1), (V_2, \rho_2)] \simeq ([V_1, V_2], \rho_{\mathrm{conj}}) .$$

Therefore the *G-homomorphisms* $V_1 \to V_2$ are those elements of $[V_1, V_2]$ which are invariant under this conjugation action, hence the homotopy fixed points of the conjugation-∞-action. With prop. C.26 and prop. C.32 these are equivalently the sections of the universal ρ_{conj}-associated $[V_1, V_2]$-fiber ∞-bundle. Therefore by remark C.10 the geometric ∞-groupoid/$\mathbf{H}$-object of G-equivariant maps $V_1 \to V_2$ is

$$[(V_1, \rho_1), (V_2, \rho_2)]_{\mathbf{H}} := \prod_{\mathbf{B}G} [(V_1, \rho_1), (V_2, \rho_2)] .$$

We will usually write just ρ for (ρ, V) if the space V that the action is defined on is understood. Notably with prop. C.32 it follows that:

Proposition C.34. *The space of sections of a V-fiber ∞-bundle which is ρ-associated to a principal bundle modulated by ∇^0, is naturally equivalent to the space of maps $\nabla^0 \to \rho$ in the slice over $\mathbf{B}G$:*

$$\mathbf{\Gamma}_X (P \times_G V) \simeq \prod_{\mathbf{B}G} [\nabla^0, \rho] .$$

Example C.35. For $G \in \mathrm{Grp}(\mathrm{SmoothMfd}) \hookrightarrow \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ a Lie group as in example C.17 and for $\rho : V \times G \to V$ an ordinary representation of G on a (vector) space V, the corresponding map $V//G \to \mathbf{B}G$ in $\mathrm{Smooth}\infty\mathrm{Grpd}$ given by prop. C.26 has as its domain the object which is presented by the traditional action Lie groupoid (also called "translation Lie groupoid" etc.)

$$V//G := \left(\begin{tikzcd} V \times G \arrow[r, shift left=2, "\rho"] \arrow[r, shift right=2, "p_1"'] & V \arrow[l] \end{tikzcd} \right) .$$

The map itself is presented by the evident functor which forgets the V-factor. Let then $\mathcal{U} = \{U_\alpha \to X\}_\alpha$ be a good open cover of a smooth manifold X, so that its Čech nerve groupoid $C(\mathcal{U})$ is an equivalent resolution of X. Then a modulating map $\nabla^0 : X \to \mathbf{B}G$ is equivalently a zig-zag

$$X \xleftarrow{\simeq} C(\mathcal{U}) \xrightarrow{g} \mathrm{B}(*, G, *)$$

hence a Čech cocycle $\{g_{\alpha,\beta} \in C^\infty(U_{\alpha\beta}, G) | g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}\}$. (An introduction to these kinds of arguments is in [19].) So a dashed lift in

$$\begin{tikzcd} & \mathrm{B}(V,G,*) \arrow[d] \\ C(\mathcal{U}) \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] \arrow[d, "\simeq"] & \mathrm{B}(*,G,*) \\ X \end{tikzcd} \quad \simeq \quad \begin{tikzcd} & V//G \arrow[d] \\ X \arrow[ur, dashed, "\sigma"] \arrow[r, "\nabla^0"] & \mathbf{B}G \end{tikzcd}$$

is a choice of smooth functions of the form $\{\sigma_\alpha \in C^\infty(U_\alpha, V) | \rho(\sigma_\alpha, g_{\alpha\beta}) = \sigma_\beta\}$. This is the traditional description terms of local data of a section of the associated V-fiber bundle $P \times_G V$.

Example C.36. Every ordinary central extension of Lie groups, such as $U(1) \to U(n) \to PU(n)$, deloops in Smooth∞Grpd to a long homotopy fiber sequence of the form

$$\begin{tikzcd} \mathbf{B}U(1) \arrow[r] & \mathbf{B}U(n) \arrow[r] & \mathbf{B}PU(n) \arrow[d, "\mathbf{dd}_n"] \\ & & \mathbf{B}^2U(1) \end{tikzcd},$$

where $\mathbf{B}^2U(1)$ is as in example C.19, and where the vertical map is a smooth refinement of the group 2-cocycle which classifies the extension. In this particular example, the vertical map is the universal *Dixmier-Douady class* on smooth $PU(n)$-principal bundles. Under prop. C.26 the right part of this fiber sequence exibits an ∞-action of the smooth circle 2-group $\mathbf{B}U(1)$ on the moduli stack $\mathbf{B}U(n)$. Accordingly, for a map $\nabla^0 : X \to \mathbf{B}^2U(1)$ modulating a $(\mathbf{B}U(1))$-principal 2-bundle, a section σ of the associated $\mathbf{B}U(n)$-fiber 2-bundle[5] over X is a dashed lift in

$$\begin{tikzcd} & \mathbf{B}PU(n) \arrow[d, "\mathbf{dd}_n"] & \simeq (\mathbf{B}U(n)//\mathbf{B}U(1)) \\ X \arrow[ur, dashed, "\sigma"] \arrow[r, "\nabla^0"] & \mathbf{B}^2U(1) \end{tikzcd}.$$

The map ∇^0 here is equivalent to what is commonly called a *bundle gerbe* over X, and lifts σ as shown here are equivalent to what are called *bundle gerbe modules* or rank-n *twisted unitary bundles* (see for instance [58][5]). Hence twisted bundles are sections of 2-bundles, in accord with the general remark C.11.

As before for the case of ordinary sections in 2.12, the universal associated $\mathbf{B}U(n)$-principal bundle over $\mathbf{B}^2U(1)$ has a differential refinement to a bundle over $\mathbf{B}^2U(1)_{\mathrm{conn}}$ such that dashed lifts in

$$\begin{tikzcd} & (\mathbf{B}U(n)//\mathbf{B}U(1))_{\mathrm{conn}} \arrow[d, "(\mathbf{dd}_n)_{\mathrm{conn}}"] \\ X \arrow[ur, dashed, "\sigma_{\mathrm{conn}}"] \arrow[r, "\nabla"] & \mathbf{B}^2U(1)_{\mathrm{conn}} \end{tikzcd}$$

are equivalently twisted bundles with connection.

[5]This is a *Giraud $U(n)$-gerbe* over X, see [41, section 4.4].

C.3 Bisections of higher groupoids

Traditionally, for $\mathcal{G}_\bullet = \left(\mathcal{G}_1 \underset{s}{\overset{t}{\rightrightarrows}} \mathcal{G}_0 \right)$ a Lie groupoid, a *bisection* is defined to be a smooth function $\sigma : \mathcal{G}_0 \to \mathcal{G}_1$ such that $s \circ \sigma = \mathrm{id}_{\mathcal{G}_0}$ and such that $\phi := t \circ \sigma : \mathcal{G}_0 \to \mathcal{G}_0$ is a diffeomorphism. Just like we observed for similar cases mentioned in 2.12, a moment of reflection reveals that the group of these bisections is eqivalently the following group of triangular diagrams under pasting composition

$$\mathbf{BiSect}(\mathcal{G}_\bullet) = \left\{ \begin{tikzcd} \mathcal{G}_0 \arrow[rr, "\phi", "\simeq"'] \arrow[dr, "p_{\mathcal{G}}"'] & \arrow[d, phantom, "\swarrow \sigma^{-1}"] & \mathcal{G}_0 \arrow[dl, "p_{\mathcal{G}}"] \\ & \mathcal{G} & \end{tikzcd} \right\} .$$

Hence $\mathbf{BiSect}(\mathcal{G}_\bullet)$ is the group of automorphisms of the canonical atlas $p_{\mathcal{G}}$ of a Lie groupoid, computed in the slice over the Lie groupoid itself.

In view of prop. C.25 and remark C.33 we have the following natural generalization of the notion of groupoid bisections to higher geometry.

Definition C.37. 1. For $G \in \mathrm{Grp}(\mathbf{H})$, the $\mathbf{H}$*-valued automorphism group* of a G-action ρ is

$$\mathbf{Aut}_{\mathbf{H}}(\rho) := \prod_{\mathbf{B}G} \mathbf{Aut}(\rho) .$$

2. For $\mathcal{G}_\bullet \in \mathrm{Grpd}(\mathbf{H})$ a groupoid object, def. C.24, its ∞*-group of bisections* is

$$\mathbf{BiSect}(\mathcal{G}_\bullet) := \prod_{\mathcal{G}} \mathbf{Aut}(p_{\mathcal{G}}) ,$$

where $p_{\mathcal{G}} : \mathcal{G}_0 \twoheadrightarrow \mathcal{G}$ is the atlas of ∞-groupoids which corresponds to $\mathcal{G}_\bullet$ under the equivalence of prop. C.28.

The following proposition reveals a fundamental property of $\mathbf{H}$-valued automorphism ∞-groups in slices. As we show below, it is via this property that such ∞-groups control higher prequantum geometry.

Proposition C.38. *For* $(\sum_X E \xrightarrow{E} X) \in \mathbf{H}_{/X}$ *an object in the slice* ∞*-topos of* $\mathbf{H}$ *over some* $X \in \mathbf{H}$*, there is an* ∞*-fiber sequence in* $\mathbf{H}$ *of the form*

$$\Omega_E \left[\sum_X E, X \right] \longrightarrow \mathbf{Aut}_{\mathbf{H}}(E) \longrightarrow \mathbf{Aut}\left(\sum_X E \right) \xrightarrow{E \circ (-)} \left[\sum_X E, X \right] .$$

Here the object on the far right is regarded as pointed by E and the object on the far left is its loop space object as in prop. C.14.

The above is the general abstract formalization of the basic idea schematically indicated in 2.12. This class of extensions is the archetype of all the ∞-group extensions in higher prequantum theory that we find, namely the integrated ∞-Atiyah sequence in C.4 and the quantomorphism ∞-group extension in D.3.

C.4 Higher Atiyah groupoids

A fundamental construction in the traditional theory of G-principal bundles $P \to X$ is that of the corresponding *Atiyah Lie algebroid* [2] and the Lie groupoid which integrates it. This Lie groupoid is usually called the *gauge groupoid* of P. However, we see in D.4 that in higher geometry there is a whole tower of higher groupoids that could go by this name. So for definiteness we stick here with the tradition of naming Lie groupoids and Lie algebroids alike and speak of the *Atiyah groupoid* $\mathrm{At}(P)_\bullet$.

For G an ordinary Lie group and $P \to X$ an ordinary G-principal bundle, the corresponding Atiyah groupoid $\mathrm{At}(P)_\bullet$ is the Lie groupoid whose manifold of objects is X, and whose morphisms between two points are the G-equivariant maps between the fibers of P over these points. Since a G-equivariant map between two G-torsors over the point is fixed by its image on any one point, $\mathrm{At}(P)_\bullet$ is usually written as on the left-hand side of

$$\begin{array}{ccc} \mathrm{At}(P)_\bullet & \to & \mathrm{Pair}(X)_\bullet \\ = & & = \\ \left(\begin{array}{c} (P \times P)/_{\mathrm{diag}} G \\ \downarrow\uparrow\downarrow \\ X \end{array} \right) & & \left(\begin{array}{c} X \times X \\ \downarrow\uparrow\downarrow \\ X \end{array} \right) \end{array},$$

where on the right-hand we display the *pair groupoid* of X. As previously discussed in 2.12, there is a conceptual simplification to this construction after the embedding into the ∞-topos $\mathrm{Smooth}\infty\mathrm{Grpd}$, example C.2. Within this context the construction can be expressed in terms of the moduli stack $\mathbf{B}G$ of G-principal bundles of prop. C.29. Namely, if $\nabla^0 : X \to \mathbf{B}G$ is the map which modulates $P \to X$, then:

Proposition C.39. *The object of morphisms of* $\mathrm{At}(P)_\bullet$ *is naturally identified with the homotopy fiber product of* ∇^0 *with itself:*

$$\mathrm{At}(P)_1 := (P \times P)/_{\mathrm{diag}} G \simeq X \underset{\mathbf{B}G}{\times} X\,.$$

Moreover, the canonical atlas *of the Atiyah groupoid, given by the canonical inclusion* $p_{\mathrm{At}(P)} : X \longrightarrow\!\!\!\!\!\rightarrow \mathrm{At}(P)$, *is equivalently the homotopy-colimiting cocone under the full Čech nerve of the classifying map* ∇^0*:*

$$\cdots\cdots\; X \underset{\mathbf{B}G}{\times} X \underset{\mathbf{B}G}{\times} X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X \underset{\mathbf{B}G}{\times} X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X \xrightarrow{\;p_{\mathrm{At}(P)}\;}\!\!\!\!\rightarrow \left(\underset{\longrightarrow n}{\lim} X^{\times^{n+1}_{\mathbf{B}G}} \right) \simeq \mathrm{At}(P)\;\;.$$

The full impact of this reformulation in the present context of automorphism groups in slices is seen by looking at the group of bisections, def. C.37, of the Atiyah groupoid. In these terms, the above proposition C.39 becomes:

Proposition C.40. *For G a Lie group, the Atiyah groupoid* $\mathrm{At}(P)_\bullet$ *of a G-principal bundle $P \to X$ over a smooth manifold X is the Lie groupoid which is universal with the property that its group of bisections is naturally equivalent to the group of automorphisms of the modulating map ∇^0 of $P \to X$ (according to prop. C.28) in the slice:*

$$\begin{array}{ccc} \mathbf{BiSect}(\mathrm{At}(P)_\bullet) & \simeq & \mathbf{Aut}_{\mathbf{H}}(\nabla^0) \\ = & & = \\ \left\{ \begin{tikzcd} X \arrow[rr, "\phi", "\simeq"'] \arrow[dr, "p_{\mathrm{At}(P)}"'] & \arrow[d, phantom, "\swArrow"] & X \arrow[dl, "p_{\mathrm{At}(P)}"] \\ & \mathrm{At}(P) & \end{tikzcd} \right\} & & \left\{ \begin{tikzcd} X \arrow[rr, "\phi", "\simeq"'] \arrow[dr, "\nabla^0"'] & \arrow[d, phantom, "\swArrow"] & X \arrow[dl, "\nabla^0"] \\ & \mathbf{B}G & \end{tikzcd} \right\} \end{array}.$$

Therefore, even though we have not yet introduced differential cohomology into the picture (this we turn to below in D), comparison with the discussion in 2.12 shows why Atiyah groupoids are central to prequantum geometry: prequantum geometry is about automorphisms of modulating maps in slices, and the Atiyah groupoid is the universal solution to making that a group of bisections, hence making this the automorphisms of an atlas of a Lie groupoid in the slice over that Lie groupoid.

Remark C.41. There is a more abstract formulation of this statement, which is useful in generalizing it: prop. C.39 together with prop. C.8 implies that after the canonical embedding of Lie groupoids into the ∞-topos Smooth∞Grpd of example C.2, the Atiyah Lie groupoid is the 1-image, in the sense of def. C.7, of the modulating map ∇^0 of $P \to X$ and its canonical atlas is the corresponding 1-image projection, hence the *first relative Postnikov stage* of ∇^0:

$$\nabla^0 : X \xrightarrow{p_{\mathrm{At}(P)}} \mathrm{At}(P) \hookrightarrow \mathbf{B}G \ .$$

In particular we have a canonical factorizing map from $\mathrm{At}(P)$ to $\mathbf{B}G$ which is a 1-monomorphism, and this implies that the components of any natural transformation from ∇^0 to itself factor through this *fully faithful* inclusion:

$$\left\{ \begin{tikzcd} X \arrow[rr, "\phi", "\simeq"'] \arrow[dr, "\nabla^0"'] & \arrow[d, phantom, "\swarrow"] & X \arrow[dl, "\nabla^0"] \\ & \mathbf{B}G & \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} X \arrow[rr, "\phi", "\simeq"'] \arrow[dr, "p"] \arrow[ddr, "\nabla^0"', bend right] & \arrow[d, phantom, "\swarrow"] & X \arrow[dl, "p"'] \arrow[ddl, "\nabla^0", bend left] \\ & \mathrm{At}(P) \arrow[d, hook] & \\ & \mathbf{B}G & \end{tikzcd} \right\} .$$

This relation translates to a proof of prop. C.40.

In view of these observations, it is then clear what the general definition of higher Atiyah groupoids should be: Let $\mathbf{H}$ be an ∞-topos, let $G \in \mathrm{Grp}(\mathbf{H})$ be an ∞-group and let $P \to X$ be a G-principal ∞-bundle in $\mathbf{H}$, as discussed above in C.

Definition C.42. The *higher Atiyah groupoid* $\mathrm{At}(P)_\bullet \in \mathrm{Grpd}(\mathbf{H})$ of P is the groupoid object, def. C.24, which under prop. C.25 corresponds to the 1-image projection $p_{\mathrm{At}(P)}$

$$\nabla^0 : X \xrightarrow{p_{\mathrm{At}(P)}} \mathrm{At}(P) \hookrightarrow \mathbf{B}G$$

of the map ∇^0 which modulates $P \to X$ via prop. C.28.

As an illustration for the use of higher Atiyah groupoids in higher geometry, notice the following immediate rederivation and refinement to higher geometry of the classical statement in Lie groupoid theory, which says that every principal bundle arises as the source fiber of its Atiyah groupoid:

Proposition C.43. *For $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group, every G-principal ∞-bundle $P \to X$ in $\mathbf{H}$ over an inhabited (= (-1)-connected) object X is equivalently the source-fiber of a transitive higher groupoid $\mathcal{G}_\bullet \in \mathrm{Grpd}(\mathbf{H})$ with vertex ∞-group G (automorphism ∞-group of any point). Here in particular we can set $\mathcal{G}_\bullet = \mathrm{At}(P)_\bullet$.*

Proof. The outer rectangle of

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d] & * \arrow[r, "\simeq"] \arrow[d, "x"] & * \arrow[d] \\ X \arrow[r, two heads] \arrow[rr, bend right, "\nabla^0"'] & \mathrm{At}(P) \arrow[r, hook] & \mathbf{B}G \end{tikzcd}$$

is an ∞-pullback by prop. C.28. Also the right sub-square is an ∞-pullback (for any global point $x \in X$) because by ∞-pullback stability of 1-epimorphisms and 1-monomorphisms the top right morphism is a 1-monomorphism from an inhabited object to the terminal object and hence is an equivalence. Now by the pasting law for ∞-pullbacks also the left sub-square is an ∞-pullback and this exhibits P as the source fiber of $\mathrm{At}(P)$ over $x \in X$. $\square$

Here we are interested in the following generalization to higher Atiyah groupoids of the classical facts reviewed at the beginning of this section. While this is a fairly elementary result in higher topos theory, we highlight it as a theorem since it serves as the blueprint for the differential refinement in theorem D.35 below.

Theorem C.44. *In the situation of def. C.42, there is a canonical equivalence*

$$\mathbf{BiSect}(\mathrm{At}(P)_\bullet) \simeq \mathbf{Aut}_{\mathbf{H}}(\nabla^0)$$

between the ∞-group of bisections, def. C.37, of the higher Atiyah groupoid of a G-principal ∞-bundle P and the $\mathbf{H}$-valued automorphism ∞-group of its modulating map ∇^0, according to prop. C.28. Moreover, the ∞-group of bisections of the higher Atiyah ∞-groupoid is an ∞-group extension, example C.31, of the form

$$\begin{tikzcd}
\Omega_{\nabla^0}[X, \mathbf{B}G] \arrow[r] & \mathbf{BiSect}(\mathbf{At}(P)_\bullet) \arrow[r] \arrow[d, phantom, "\simeq"] & \mathbf{Aut}(X) \ , \\
& \mathbf{Aut}_{\mathbf{H}}(\nabla^0) &
\end{tikzcd}$$

where on the right we have the canonical forgetful map.

Proof. By the defining property of 1-monomorphisms and by prop. C.38. $\square$

Remark C.45. Together with prop. C.26, this theorem says that higher Atiyah groupoids are related to G-equivariant maps between the fibers of their principal ∞-bundles in just the way that one expects from the traditional situation.

Also notice that this theorem together with prop. C.34 implies:

Corollary C.46. *There is a canonical ∞-action of bisections of* $\mathrm{At}(P)_\bullet$ *on the space of sections of any associated V-fiber ∞-bundle:*

$$\mathbf{BiSect}(\mathrm{At}(P)_\bullet) \times \mathbf{\Gamma}_X(P \times_G V) \to \mathbf{\Gamma}_X(P \times_G V)\,.$$

In view of theorem C.44, and example C.31 we may ask for a cocycle that classifies the higher Atiyah extension. This can not exist on all of $\mathbf{Aut}(X)$, in general, but just on the part that is in the 1-image of the projection from bisections:

Definition C.47. For $P \to X$ a G-principal ∞-bundle, write $\mathbf{Aut}_P(X) \in \mathrm{Grp}(\mathbf{H})$ for the full sub-∞-group of the automorphism ∞-group of X on those elements that have a lift to an autoequivalence of P, hence the 1-image of the right map in prop. C.44:

$$\mathbf{BiSect}(\mathrm{At}(P)_\bullet) \twoheadrightarrow \mathbf{Aut}_P(X) \hookrightarrow \mathbf{Aut}(X)\ .$$

Theorem C.48. *The fiber sequence of theorem C.44 extends to a long homotopy fiber sequence in $\mathbf{H}$ of the form*

$$\Omega_{\nabla^0}[X, \mathbf{B}G] \longrightarrow \mathbf{BiSect}(\mathrm{At}(P)_\bullet) \twoheadrightarrow \mathbf{Aut}_P(X) \xrightarrow{\nabla^0 \circ (-)} \mathbf{B}\left(\Omega_{\nabla^0}[X, \mathbf{B}G]\right)\ .$$

Moreover, if G is a sylleptic ∞-group, def. C.18, so that the rightmost object is itself naturally an ∞-group, then this naturally lifts to a long homotopy fiber sequence in $\mathrm{Grp}(\mathbf{H})$*. In this case the delooping $\mathbf{B}(\nabla^0 \circ (-))$ is the ∞-group cocycle (example C.31) that classifies* $\mathbf{BiSect}(\mathrm{At}(P)_\bullet)$ *as an $\Omega_{\nabla^0}[X, \mathbf{B}G]$-extension of the ∞-group $\mathbf{Aut}_P(X)$.*

Proof. First consider the underlying morphisms in $\mathbf{H}$. By theorem C.44 and by general properties of automorphisms in slices, the outer rectangle in the diagram

$$\begin{tikzcd} \mathbf{BiSect}(\mathrm{At}(P)_\bullet) \arrow[r, two heads] \arrow[d] & \mathbf{Aut}_P(X) \arrow[r, hook] \arrow[d] & \mathbf{Aut}(X) \arrow[d, "\nabla^0 \circ (-)"] \\ * \arrow[r, two heads] \arrow[rr, bend right, "\vdash \nabla^0"'] & \mathbf{B}\left(\Omega_{\nabla^0}[X, \mathbf{B}G]\right) \arrow[r, hook] & {[X, \mathbf{B}G]} \end{tikzcd}$$

is a homotopy pullback. We form the 1-image factorization of the bottom map as indicated and observe that by homotopy pullback stability of 1-monomorphisms and 1-epimorphisms in an ∞-topos also the right and in particular also the left sub-square are then homotopy pullbacks.

Now if G is equipped with the structure of a sylleptic ∞-group, it remains to see that the vertical map in the middle lifts to a homomorphism of ∞-groups such that the left square is also a homotopy pullback in $\mathrm{Grp}(\mathbf{H})$.

To that end, first regard the point in the bottom left as the trivial ∞-group, and hence the bottom horizontal map uniquely as an ∞-group homomorphism. This way, by theorem C.14 and by prop. C.13, the top and bottom horizontal factorizations naturally lift to $\mathrm{Grp}(\mathbf{H})$ as the looping of the 2-image factorization of the delooped horizontal morphisms. Therefore the left part of the diagram naturally lifts to a diagram of simplicial objects as shown by the solid arrows in

$$\begin{tikzcd} \mathbf{BiSect}(\mathrm{At}(P)_\bullet)^{\times^{\bullet+1}} \arrow[r] \arrow[d] & \mathbf{Aut}_P(X)^{\times^{\bullet+1}} \arrow[d, dashed, "(\nabla \circ (-))_\bullet"] \\ *^{\times^{\bullet+1}} \arrow[r] & (\mathbf{B}(\Omega_{\nabla^0}[X, \mathbf{B}G]))_\bullet \end{tikzcd},$$

and we have to produce the dashed morphism on the right as a simplicial morphism lifting $\nabla^0 \circ (-) = (\nabla^0 \circ (-))_0$. Observe that each degree of the horizontal simplicial maps here is a 1-epimorphism in $\mathbf{H}$, because a finite product of 1-epimorphisms is still a 1-epimorphism (this follows for instance with the characterization of 1-epimorphism in prop. C.8 together with the fact that Δ^{op} is a sifted ∞-category [31, prop. 5.3.1.20], so that homotopy colimits over it preserve finite products [31, lemma 5.5.8.11]). But, again by prop. C.8, this induces naturally and essentially uniquely in each degree the dashed vertical morphism as the unique map between homotopy colimiting cocones under the Čech nerves of the vertical maps in this degree. Notice that here $(\nabla \circ (-))_k \simeq (\nabla \circ (-))^{\times^{k+1}}$, necessarily, the point being that naturally implies that these components constitute a morphism of simplicial objects. Hence this diagram is degreewise a homtopy pullback in $\mathbf{H}$, hence is a homotopy pullback in $\mathbf{H}^{\Delta^{\mathrm{op}}}$ and therefore finally also in $\mathrm{Grp}(\mathbf{H})$. □

This class of ∞-group extensions introduced in theorem C.44 and theorem C.48 is the source of all extensions that we consider here, and hence the source of all the fundamental extensions in traditional and in higher prequantum geometry.

For instance, a slight variation of theorem C.48 adapts it to the context of differential moduli discussed below in D. There it yields the central statement about the quantomorphism ∞-group extension in theorem D.35. Also *higher Courant groupoids* are of this form, discussed in D.4 below: they are intermediate between higher Atiyah groupoids and higher quantomorphism groupoids.

This fundamental unification of higher prequantum geometry via the theory of higher Atiyah groupoids is even stronger when we shift emphasis away from ∞-groups of bisections of a higher groupoid to the higher groupoid itself. Clearly, the group of bisections of a groupoid, being really the group of *global* bisections, is a global incarnation of that groupoid, and hence forgets some of its local structure. Looking back through the discussion in 2.12, we see that the main reason why one passes to groups of bisections is because these canonically *act*. For instance we saw that a prequantum operator is a tangent to a global bisection of the quantomorphism groupoid, and its action on prequantum states is inherited from the canonical action of that group of bisections.

But in fact there is a natural notion of actions of higher groupoids themselves, which refines the notion of action of their ∞-groups of bisections:

Definition C.49. For $\mathcal{G}_\bullet \in \mathrm{Grpd}(\mathbf{H})$ a groupoid object and for $p : E \to \mathcal{G}_0$ an object over $\mathcal{G}_0$, a *groupoid action* of $\mathcal{G}_\bullet$ on (the space of sections of) E is another groupoid object

$$(E//\mathcal{G})_\bullet \in \mathrm{Grpd}(\mathbf{H})$$

corresponding to an ∞-groupoid with atlas $E \twoheadrightarrow E//\mathcal{G}$ and an ∞-pullback diagram of atlases of the form

$$\begin{tikzcd} E \arrow[r, two heads] \arrow[d, "p"] & E//\mathcal{G} \arrow[d] \\ \mathcal{G}_0 \arrow[r, two heads] & \mathcal{G} \end{tikzcd} .$$

To see heuristically how such a definition indeed encodes an action, it is helpful to think of path lifting: For an element $e \in E$ and a morphism $(p(e) \xrightarrow{g} y) \in \mathcal{G}^{(\Delta^1)}$ in $\mathcal{G}_\bullet$, the $\mathcal{G}$-action of g on e corresponds to a lift of g to a morphism $(e \xrightarrow{\hat{g}} \tilde{e}) \in (E//\mathcal{G})^{\Delta^1}$ in the action groupoid, which takes e to a morphism $\tilde{e}$ sitting over y. Notice that for $\mathcal{G} \simeq \mathbf{B}G$ the delooping groupoid of an ∞-group, this reduces to the definition of actions of ∞-groups discussed around prop. C.26.

With this it is straightforward to see the canonical action of a higher Atiyah groupoid on sections of any bundle associated to its correpsonding principal bundle without passing to global bisections:

Example C.50. Given a G-principal ∞-bundle $P \to X$ modulated by a map $\nabla^0 : X \to \mathbf{B}G$ (prop. C.28), and given a G-∞-action (V, ρ) exhibited by the universal V-bundle $V//G \to \mathbf{B}G$ (prop. C.26), recall that there is a ρ-associated V-fiber $P \times_G V$ which fits into the homotopy pullback square described in prop. C.32. The canonical ∞-action of the higher Atiyah groupoid $\mathrm{At}(P)_\bullet$ (def. C.42) on the sections of $P \times_X V$ is exhibited by the left square in the following pasting diagram of homotopy pullback squares:

$$\begin{tikzcd} P \times_G V \arrow[r] \arrow[d] & (P \times_G V)//\mathrm{At}(P) \arrow[r] \arrow[d] & V//G \arrow[d, "\rho"] \\ X \arrow[r, two heads] \arrow[rr, bend right, "\nabla^0"'] & \mathrm{At}(P) \arrow[r, hook, two heads] & \mathbf{B}G \end{tikzcd} .$$

D Differential cohomology via Cohesive homotopy types

(This section essentially coincides with sections 2.3, 2.4 in [15].)

The discussion of higher gauge groupoids in C.4 makes sense in any ∞-topos and hence provides a general robust theory of higher Atiyah groupoids, C.4, in all kinds of notions of geometry. However, our discussion in D.4 below, involving the higher Heisenberg/quantomorphism groupoids and higher Courant groupoids necessary for genuine higher prequantum geometry, requires that in the ambient ∞-topos one can give meaning to refining a G-principal bundle to a *G-principal connection*. Hence it requires to be able to refine plain (albeit geometric) cohomology described in remark C.4 to *differential cohomology*. In the same fashion as indicated at the beginning of C, we want to incorporate this in a flexible but robust way that allows all constructions and results to be interpreted as much as possible in various flavors of geometry such as higher/derived differential geometry, analytic geometry, supergeometry, etc. In order to achieve this, we now impose a minimum set of axioms on our ambient ∞-topos $\mathbf{H}$, called *cohesion* [50], that guarantees the existence of a consistent notion of differential cohomology in $\mathbf{H}$. Then we briefly indicate some examples and give a list of those basic constructions and results available in such a context which we use in the following chapters D.4 and D.3 for the formulation and study of higher prequantum geometry.

The most basic ingredient of any theory of differential cohomology is that for any coefficient object $\mathbf{B}G \in \mathbf{H}$ there is the corresponding object $\mathbf{B}G_{\mathrm{disc}}$ of *discrete* coefficients equipped with a map $u_{\mathbf{B}G} : \mathbf{B}G_{\mathrm{disc}} \to \mathbf{B}G$, such that a lift through this map is equivalently a *flat* G-principal connection:

$$\begin{tikzcd} & \mathbf{B}G_{\mathrm{disc}} \arrow[d, "u_{\mathbf{B}G}"] \\ X \arrow[ur, dashed, "\nabla_{\mathrm{flat}}"] \arrow[r, "\nabla^0"'] & \mathbf{B}G \end{tikzcd} .$$

(A simple familiar example captured by this formalization is the classification of $U(1)$-principal bundles by degree-1 Čech cohomology with coefficients in the sheaf of $U(1)$-valued functions as compared to the classification of flat $U(1)$-principal connections by singular cohomology with coefficients in the discrete group underlying $U(1)$.)

Something close to this already exists in every ∞-topos $\mathbf{H}$: if we let

$$\flat := \mathrm{LConst} \circ \Gamma \ : \ \mathbf{H} \to \mathbf{H}$$

be the composite of the ∞-functor Γ which forms global sections of ∞-stacks, with its left adjoint LConst, the ∞-functor which forms locally constant ∞-stacks, then we set

$$\mathbf{B}G_{\mathrm{disc}} := \flat(\mathbf{B}G) \simeq \mathbf{B}(\flat G)\,.$$

(The symbol "$\flat$" is pronounced "flat", alluding to the relation of discrete coefficients to flat principal ∞-connections.) The counit of the adjunction ($\mathrm{LConst} \dashv \Gamma$) gives the map $u_{\mathbf{B}G}$ described above. In order to have a consistent interpretation of $\flat G$ as the geometrically discrete version of G, it must be true that universally turning an already discrete object again into a discrete object does not change it, hence that $u_{\flat(-)}$ is an equivalence $\flat(\flat(-)) \xrightarrow{\simeq} \flat(-)$. This is the *first axiom of cohesion*.

Notice that with this first axiom we may think of the image of $\flat$ as constituting a canonical inclusion of $\infty\mathrm{Grpd}$ into $\mathbf{H}$ as the geometrically discrete ∞-groupoids. In the following we freely make use of this and speak of traditional objects of homotopy theory, such as Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$, as objects of $\mathbf{H}$.

Moreover, cohomology with discrete coefficients should have a consistent interpretation in terms of flat principal ∞-connections (often called *local systems of coefficients*, but better called *flat local systems of coefficients* as there are also non-flat bundles of local coefficients, see remark C.11 above) and these should have a notion of (higher) *parallel transport*. In order to satisfy such design criteria, there must exist for every space $X \in \mathbf{H}$ there exists its *fundamental ∞-groupoid* (also called *Poincaré groupoid*) $\int X$ such that

$$\text{maps} \qquad \underbrace{X \longrightarrow \flat\mathbf{B}G}_{\text{cocycle with discrete coefficients}} \qquad \text{are naturally equivalent to maps} \qquad \underbrace{\textstyle\int X \longrightarrow \mathbf{B}G}_{\text{flat parallel transport}} \quad .$$

Technically this means that $\flat$ has a left adjoint, or equivalently that it preserves all homotopy limits. This is the *second axiom of cohesion*.

It follows that with $\mathbf{c} : X \to \mathbf{B}G$ a map in $\mathbf{H}$, its image $\int \mathbf{c} : \int X \to \int \mathbf{B}G$ can be identified with a map of bare homotopy types in $\infty\mathrm{Grpd} \simeq L_{\mathrm{whe}}\mathrm{Top}$, hence that $\int$ behaves like *geometric realization* of ∞-stacks. This allows us to say what it means in $\mathbf{H}$ to geometrically refine a cohomology class. For instance the geometric refinement $\mathbf{c}$ in $\mathbf{H}$ of a universal integral characteristic class c is a diagram of the form

$$\begin{tikzcd} \mathbf{B}G \arrow[r, "\mathbf{c}"] \arrow[d, "\int"] & \mathbf{B}^n U(1) \arrow[d, "\int"] \\ BG \arrow[r, "c"] & K(\mathbb{Z}, n+1) \end{tikzcd} .$$

(We see several examples of this below in **??**.) For this interpretation to be consistent it must be true that the geometric realization of the point is contractible, and that the realization of a product is the product of the realizations. This is the *third axiom of cohesion*.

In the presence of these axioms there is a notion of non-flat principal ∞-connections, hence there is a notion of *differential cohomology* in $\mathbf{H}$, def. D.8 below, whose coefficients are differentially refined moduli ∞-stacks which we denote by $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$. A special aspect of differential coefficients, discussed in detail below, is that for any object $X \in \mathbf{H}$, the internal hom $[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}]$ (the *mapping stack*) is not in general the correct moduli stack $\mathbb{G}\mathbf{Conn}(X)$ of $\mathbb{G}$-principal connections on X: it has the correct global points, but not in general the expected geometric structure. One of the results presented below is that the correct differential moduli stack exists if $\mathbf{H}$ satisfies one more condition: The ∞-functor $\flat$ also has a right adjoint operator, to be denoted $\sharp$. This is the *fourth axiom of cohesion*.

Example D.1. Our running example C.2, $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$, is cohesive. Here $\int$ sends manifolds $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ to their standard fundamental ∞-groupoid, the singular simplicial complex $\mathrm{Sing}(X) \in L_{\mathrm{whe}}\mathrm{sSet} \overset{\mathrm{LConst}}{\hookrightarrow} \mathrm{Smooth}\infty\mathrm{Grpd}$, and sends moduli stacks $\mathbf{B}G$ of Lie groups and, more generally of simplicial Lie groups, to their traditional classifying spaces $BG \in L_{\mathrm{whe}}\mathrm{Top} \overset{\mathrm{LConst}}{\hookrightarrow} \mathrm{Smooth}\infty\mathrm{Grpd}$. Generally, $\int$ sends an ∞-stack, regarded as an ∞-functor $\mathrm{SmoothMfd}^{\mathrm{op}} \to \infty\mathrm{Grpd}$ to its homotopy colimit.

Moreover, the operator $\sharp$ on $\mathrm{Smooth}\infty\mathrm{Grpd}$ characterizes *concrete sheaves* on the site of smooth manifolds: the 0-truncated objects $X \in \mathrm{Sh}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ such that the unit $X \to \sharp X$ is a 1-monomorphism. These are equivalently the *diffeological spaces*.

(If one further enhances the axioms of cohesion to those for *differential cohesion* then one can intrinsically characterize also the smooth manifolds, the orbifolds and generally the étale ∞-groupoids. This is discussed in [50, 3.5 and 3.10].)

The following table summarizes these four axioms of cohesion and their immediate interpretation in $\mathbf{H}$:

Axiom	$\mathbf{H}$ has a notion of...
$\flat := \mathrm{LConst} \circ \Gamma$ is idempotent.	discrete coefficients;
There is a left adjoint $\int$ to $\flat$.	flat principal ∞-connections / flat local systems of coefficients;
$\int$ preserves finite products.	geometric realization;
There is a right adjoint $\sharp$ to $\flat$.	differential moduli stacks.

But the point of these axioms is that they naturally imply more constructions of a differential geometric and differential cohomological nature. Some of these we turn to now.

D.1 Differential coefficients

The crucial ingredient for defining differential cohomology is the existence of universal *curvature characteristics*. Given these, differential cohomology is simply, as we see below, *curvature-twisted cohomology*, in the general sense of twisted cohomology as described in remark C.11. We find universal curvature characteristics encoded in the higher homotopy fibers of the counit of the $\flat$-operator given by the axioms of cohesion:

Definition D.2. For $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group in a cohesive ∞-topos, with delooping $\mathbf{B}G$ (prop. C.14) consider the long homotopy fiber sequence, remark C.12, induced by the map $u_{\mathbf{B}G}$, hence consider the following pasting diagram of homotopy pullbacks:

$$
\begin{tikzcd}
\flat G \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\theta"] & * \arrow[d] \\
* \arrow[r] & \flat_{\mathrm{dR}}\mathbf{B}G \arrow[r] \arrow[d] & \flat\mathbf{B}G \arrow[d, "u_{\mathbf{B}G}"] \\
& * \arrow[r] & \mathbf{B}G
\end{tikzcd}
\quad .
$$

Here we say that

- $\flat_{\mathrm{dR}}\mathbf{B}G$ is the *de Rham coefficient object* of $\mathbf{B}G$;
- θ is the *Maurer-Cartan form* on G.

Moreover, if $\mathbb{G} \in \mathrm{Grp}_2(\mathbf{H})$ is a braided ∞-group, def. C.18, then we say that the Maurer-Cartan form of its delooping group is the *universal curvature characteristic* of $\mathbb{G}$, denoted

$$
\mathrm{curv}_{\mathbb{G}} := \theta_{\mathbf{B}\mathbb{G}} \simeq \mathbf{B}\theta_{\mathbb{G}} : \ \mathbf{B}\mathbb{G} \longrightarrow \mathbf{B}\flat_{\mathrm{dR}}\mathbf{B}\mathbb{G} \simeq \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \ .
$$

Example D.3. For $G \in \mathrm{Grp}(\mathrm{SmthMf}) \hookrightarrow \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ a Lie group regarded as a smooth ∞-group as in example C.17,

$$
\flat_{\mathrm{dR}}\mathbf{B}G = \Omega^1_{\mathrm{flat}}(-, \mathfrak{g})
$$

is given by the traditional sheaf of flat Lie-algebra valued forms and $\theta : G \to \flat_{\mathrm{dR}}\mathbf{B}G$ is, under the Yoneda embedding, the traditional Maurer-Cartan form $\theta \in \Omega^1_{\mathrm{flat}}(G)$.

Example D.4. For $n \geq 1$ and $\mathbb{G} = \mathbf{B}^{n-1}U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ the smooth circle n-group as in example C.19, we have that

$$
\flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) \simeq \mathbf{B}^n\flat_{\mathrm{dR}}\mathbf{B}U(1) \simeq \mathrm{DK}(\Omega^1_{\mathrm{cl}}[n]) \simeq \mathrm{DK}\left(\Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} \Omega^{n+1}_{\mathrm{cl}} \right)
$$

is presented under the Dold-Kan correspondence, remark C.4, by the trunacated and shifted de Rham complex. Moreover, the universal curvature characteristic $\mathrm{curv}_{\mathbf{B}^{n-1}U(1)} = \mathbf{B}^n\theta_{U(1)}$ is presented by the map which equips a $(\mathbf{B}^{n-1}U(1))$-principal n-bundle with a *pseudo-connection* and then sends that to the corresponding curvature in de Rham hypercohomology.

This is discussed in [19, prop. 3.2.26] and [50, section 4.4.16].

The following is a direct consequence of the axioms, but central for their interpretation in differential cohomology.

Proposition D.5. *The universal curvature characteristic of def. D.2 is the obstruction to lifting $\mathbb{G}$-principal ∞-bundles $\nabla^0 : X \to \mathbf{B}\mathbb{G}$ to flat ∞-connections $\nabla_{\mathrm{flat}} : X \to \flat\mathbf{B}\mathbb{G}$.*

Therefore:

Definition D.6. Given a braided ∞-group $\mathbb{G} \in \mathrm{Grp}_2(\mathbf{H})$, *differential $\mathbb{G}$-cohomology* is $\mathrm{curv}_{\mathbb{G}}$*-twisted cohomology*, according to remark C.11.

Usually, as in our applications below in D.4 and D.3, one chooses an object that represents a certain class of curvature twists and then restricts attention to differential cohomology obtained from just these twists. This we turn to now.

If $\mathbf{H}$ comes equipped with differential cohesion, then it is typically desireable to consider a 0-truncated object to be denoted $\Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \in \mathbf{H}$ which is equipped with a map

$$F_{\mathbb{G}} : \; \Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \longrightarrow \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G}$$

such that for all manifolds Σ the map $[\Sigma, F_{\mathbb{G}}]$ is a 1-epimorphism, def. C.7, and such that $\Omega^2_{\mathrm{cl}}(-,\mathbb{G})$ is minimal with this property.

Example D.7. For $\mathbb{G} = \mathbf{B}^{n-1}U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ the smooth circle n-group as in example D.4, the standard choice is to take

$$\Omega^2_{\mathrm{cl}}(-,\mathbf{B}^{n-1}U(1)) := \Omega^{n+1}_{\mathrm{cl}} \in \mathrm{Sh}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$$

to be the ordinary sheaf of closed $(n+1)$-forms under the canonical inclusion into the de Rham hypercomplex presentation for $\flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$ from example D.4. That this is a 1-epimorphism over manifolds is then equivalently the statement that every de Rham hypercohomology class on a smooth manifold has a representative by a globally defined closed differential form.

Definition D.8. For $\Omega^2_{\mathrm{cl}}(-,\mathbb{G})$ a choice of curvature twists as above, we write $\mathbf{B}\mathbb{G}_{\mathrm{conn}} \in \mathbf{H}$ for the homotopy pullback in

$$\begin{tikzcd} \mathbf{B}\mathbb{G}_{\mathrm{conn}} \arrow[r, "F_{(-)}"] \arrow[d, "u_{\mathbf{B}\mathbb{G}}"] & \Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \arrow[d, "F_{\mathbb{G}}"] \\ \mathbf{B}\mathbb{G} \arrow[r, "\mathrm{curv}_{\mathbb{G}}"] & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \end{tikzcd} .$$

We say that lifts ∇ in

$$\begin{tikzcd} & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \arrow[d] \\ X \arrow[ur, dashed, "\nabla"] \arrow[r, "\nabla^0"] & \mathbf{B}\mathbb{G} \end{tikzcd}$$

correspond to equipping the $\mathbb{G}$-principal bundle modulated by ∇^0 with a $\mathbb{G}$-*princial connection*. We say that lifts ∇ in

$$\begin{tikzcd} & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\ X \arrow[ur, dashed, "\nabla"] \arrow[r, "\omega"] & \Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \end{tikzcd}$$

are $\mathbb{G}$-*prequantizations* of that datum $\omega : X \to \Omega^2_{\mathrm{cl}}(-,\mathbb{G})$.

Remark D.9. A general cocycle in twisted cohomology, def. C.11, with respect to the universal curvature characteristic $\mathrm{curv}_{\mathbb{G}}$ of def. D.2 and for some twist $\tilde{\omega}$ is given by a diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} X \arrow[rr, "\nabla^0"] \arrow[dr, "\tilde{\omega}"'] \arrow[rr, phantom, shift right=4ex, "\swarrow\nabla"] & & \mathbf{B}\mathbb{G} \arrow[dl] \\ & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} & \end{tikzcd} .$$

This is a cocycle with coefficients in $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ of def. D.8 if its curvature twist ω factors through the prescribed curvature coefficients $F_{\mathbb{G}}$ as a form datum $\omega \in \Omega^2_{\mathrm{cl}}(-,\mathbb{G})$. Because in that case the universal property of the homotopy pullback identifies ∇ with the dashed morphism in the following diagram

$$\begin{tikzcd}
X \arrow[rr, dashed, "\nabla"] \arrow[rrr, bend left, "\nabla^0"] \arrow[dr, "\omega"'] \arrow[ddrr, bend right, "\tilde{\omega}"'] & & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \arrow[r] \arrow[dl, "F_{(-)}"] & \mathbf{B}\mathbb{G} \arrow[ddl, "\mathrm{curv}_{\mathbb{G}}"] \\
& \Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \arrow[dr, "F_{\mathbb{G}}"] & & \\
& & \flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} &
\end{tikzcd}$$

Example D.10. For $\mathbb{G} = \mathbf{B}^{n-1}U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpds})$ the smooth circle n-group with the standard choice of curvature twists as in example D.7, the differential coefficient object $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is presented under the Dold-Kan correspondence by the Deligne complex

$$\mathbf{B}^n U(1)_{\mathrm{conn}} \simeq \mathrm{DK}\left(\Omega^{\bullet \leq n}(-,U(1))[n]\right) = \mathrm{DK}\left(\underline{U}(1) \xrightarrow{d\log} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \right)$$

and the universal curvature form $F_{(-)} : \mathbf{B}^n U(1)_{\mathrm{con}} \to \Omega^{n+1}_{\mathrm{cl}}$ is presented under $\mathrm{DK}(-)$ by the standard Deligne curvature chain map

$$\begin{tikzcd}
\underline{U}(1) \arrow[r, "d\log"] \arrow[d] & \Omega^1 \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & \Omega^n \arrow[d, "d"] \\
0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r, "0"] & \Omega^{n+1}_{\mathrm{cl}}
\end{tikzcd} .$$

In particular for X a smooth manifold (paracompact) and for $\mathcal{U} = \{U_\alpha \to X\}_\alpha$ a good open cover, the chain complex

$$\mathrm{Tot}(\mathcal{U}, \Omega^{\bullet \leq n}(-,U(1))[n])^n \xrightarrow{d_{\mathrm{tot}}} \cdots \xrightarrow{d_{\mathrm{tot}}} \mathrm{Tot}(\mathcal{U}, \Omega^{\bullet \leq n}(-,U(1))[n])^1 \xrightarrow{d_{\mathrm{tot}}} \mathrm{Tot}(\mathcal{U}, \Omega^{\bullet \leq n}(-,U(1))[n])^0_{\mathrm{cl}}$$

is under the Dold-Kan correspondence a presentation of $\mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}}) \in \infty\mathrm{Grpd}$. Hence with respect to traditional terminology we have:

- A $U(1)$-principal connection in the above sense is, over a manifold $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$, equivalently a $U(1)$-principal connection in the traditional sense. Over a quotient groupoid $X//G$ it is a G-equivariant connection.
- A $(\mathbf{B}U(1))$-principal connection in the above sense is, over a manifold, equivalently a $U(1)$-*bundle gerbe with connection and curving*;
- A $(\mathbf{B}^2 U(1))$-principal connection in the above sense is, over a manifold, equivalently a $U(1)$-*bundle 2-gerbe with connection and curving and 3-form connection.*

This is discussed in [19].

For the differential gauge groupoids in D.4 below, we also need the differential coefficients which are intermediate betweem genuine principal ∞-connections and plain principal ∞-bundles:

Remark D.11. If $\mathbb{G}$ is braided, hence equipped with a further delooping, then we usually demand that the choice $F_{\mathbb{G}}$ of curvature twists is compatible with the delooping in that we have a factorization

$$\begin{tikzcd}
\Omega^2_{\mathrm{cl}}(-,\mathbf{B}\mathbb{G}) \arrow[r] \arrow[rrr, bend right, "F_{\mathbf{B}\mathbb{G}}"'] & \mathbf{B}\Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \arrow[r, "\mathbf{B}F_{\mathbb{G}}"] & \mathbf{B}\flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \arrow[r, "\simeq"] & \flat_{\mathrm{dR}}\mathbf{B}^3\mathbb{G}
\end{tikzcd} .$$

Proposition D.12. *For $\mathbb{G} \in \mathrm{Grp}_3(\mathbf{H})$ a sylleptic ∞-group, def. C.18, and given a factorization of curvature twists as in remark D.11, there is canonically induced a factorization*

$$\begin{tikzcd} \mathbf{B}^2\mathbb{G}_{\mathrm{conn}} \arrow[r] \arrow[rr, bend right, "u_{\mathbf{B}^2\mathbb{G}}"'] & \mathbf{B}\left(\mathbf{B}\mathbb{G}_{\mathrm{conn}}\right) \arrow[r, "\mathbf{B}u_{\mathbf{B}\mathbb{G}}"] & \mathbf{B}^2\mathbb{G} \end{tikzcd}$$

of the forgetful map from $\mathbf{B}\mathbb{G}$-principal connections to the underlying $\mathbf{B}\mathbb{G}$-principal bundles through the delooping of $\mathbb{G}$-principal connection.

Proof. We have a pasting diagram of ∞-pullbacks of the form

$$\begin{tikzcd} \mathbf{B}^2\mathbb{G}_{\mathrm{conn}} \arrow[r] \arrow[d] \arrow[ddd, bend right=60, "u_{\mathbf{B}^2\mathbb{G}}"'] & \Omega^2_{\mathrm{cl}}(-,\mathbf{B}\mathbb{G}) \arrow[d] \arrow[ddd, bend left=60, "F_{\mathbf{B}\mathbb{G}}"] \\ \mathbf{B}(\mathbf{B}\mathbb{G}_{\mathrm{conn}}) \arrow[r] \arrow[d, "\mathbf{B}u_{\mathbf{B}\mathbb{G}}"] & \mathbf{B}\Omega^2_{\mathrm{cl}}(-,\mathbb{G}) \arrow[d] \\ \mathbf{B}\mathbf{B}\mathbb{G} \arrow[r, "\mathbf{B}\mathrm{curv}_{\mathbb{G}}"] \arrow[d, "\simeq"] & \mathbf{B}\flat_{\mathrm{dR}}\mathbf{B}^2\mathbb{G} \arrow[d, "\simeq"] \\ \mathbf{B}^2\mathbb{G} \arrow[r, "\mathrm{curv}_{\mathbf{B}\mathbb{G}}"] & \flat_{\mathrm{dR}}\mathbf{B}^3\mathbb{G} \end{tikzcd}$$

□

Definition D.13. In the situation of prop. D.12 and with $n \in \mathbb{N}$ given such that $\mathbb{G} \in \mathbf{H}$ is n-truncated, we write $\nabla^n := \nabla$, ∇^{n-1} and ∇^0 for the three degrees of notions of $\mathbb{G}$-principal connections as in the diagram

$$\begin{tikzcd} & \mathbf{B}^2\mathbb{G}_{\mathrm{conn}} \arrow[d] & \mathbf{B}\mathbb{G}\text{-principal connection} \\ & \mathbf{B}\left(\mathbf{B}\mathbb{G}_{\mathrm{conn}}\right) \arrow[d] & \begin{array}{c}\mathbf{B}\mathbb{G}\text{-principal connection} \\ \text{without top-degree connection forms}\end{array} \\ X \arrow[ruu, "\nabla"] \arrow[ru, "\nabla^1"] \arrow[r, "\nabla^0"'] & \mathbf{B}^2\mathbb{G} & \mathbf{B}\mathbb{G}\text{-principal } \infty\text{-bundle} \end{tikzcd}$$

.

Example D.14. If $\mathbb{G} = U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ in example D.10 then

$$\mathbf{B}(\mathbf{B}U(1)_{\mathrm{conn}}) \simeq \mathrm{DK}\left(\underline{U}(1) \xrightarrow{d\log} \Omega^1 \longrightarrow 0 \right)$$

is the moduli 2-stack for what in the literature are traditionally known as $U(1)$-*bundle gerbes with connective structure but without curving.*

Remark D.15. More generally, for sufficiently highly deloopable $\mathbb{G}$ and compatibly chosen curvature twists in each degree, there are towers of factorizations of principal connection data

$$
\begin{tikzcd}
 & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \arrow[d] \\
 & \mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}} \arrow[d] \\
 & \vdots \arrow[d] \\
X \arrow[uuur, "\nabla"] \arrow[uur, "\nabla^{n-1}"] \arrow[r, "\nabla^0"] & \mathbf{B}\mathbb{G}
\end{tikzcd}
\,.
$$

D.2 Differential moduli

While differential coefficients $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ as discussed in D.1 are the basis for any discussion of principal connections and differential cocycles, for the discussion of quantomorphism groups and (higher) Courant groupoids in (higher) prequantum geoemtry below in D.4 it is crucial that we refine the construction to "concretified" differential moduli stacks. The issue here is illustrated by the following

Example D.16. For $n \geq 1$ let $\Omega^n \in \mathrm{Sh}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ be the ordinary sheaf of smooth differential n-forms and let $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ be a smooth manifold. By the ∞-Yoneda lemma, the *external* hom from X to Ω^n is the set of smooth differential n-forms on X:

$$\mathbf{H}(X, \Omega^n) \simeq \Omega^n(X) \in \mathrm{Set} \hookrightarrow \infty\mathrm{Grpd}\,.$$

However, for various applications in differential geometry we want not just this set, but the canonical structure of a *smooth space* in particular of a *diffeological space* on this set. For instance we might have a functional on the space of n-forms on X (an action functional of a form field theory, a Hitchin function or similar) which depends smoothly on its arguments, and of which we want to form the (variational) derivative. An immediate candidate for the smooth space of n-forms on X is the *internal hom* (the *mapping stack*)

$$[X, \Omega^n] \in \mathrm{Smooth}\infty\mathrm{Grpd}\,.$$

This is an object with natural and nontrivial smooth structure (if X is not discrete) and its global points $* \to [X, \Omega^n]$ are indeed equivalently differential n-forms on X. However, $[X, \Omega^n]$ does not have the smooth structure which is expected of the smooth space of n-forms on X: for U a smooth test manifold, a map $U \to [X, \Omega^n]$ is equivalently a map $U \times X \to \Omega^n$, which by the Yoneda lemma is equivalently a differential n-form on the product manifold $U \times X$. This is too much: a smoothly U-parameterized collection of differential n-forms on U should be just a *vertical* differential n-form on the bundle, $U \times X \to U$, hence a form on $X \times U$ with "no legs along U". Notice that this issue disappears for $n = 0$, hence when we are dealing not with differential forms but with smooth functions: this issue is one genuine to *differential* cocycles.

Definition D.17. Write

$$\mathbf{\Omega}^n(X) \in \mathrm{Sh}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$$

for the sheaf of such vertical n-forms.

This is the correct *moduli stack of differential n-forms on X*. In this example it is easy enough to just define this by hand. But sticking to our goal of providing a flexible but robust general theory that applies broadly to higher/derived geometry and to different flavors of geometry, we observe the following abstract characterization:

Proposition D.18. *The moduli object of differential forms, def. D.17, is the 1-image, def. C.6, of the unit of the $\sharp$-monad of smooth cohesion, example D.1, applied to the internal hom of example D.16:*

$$[X, \Omega^n] \twoheadrightarrow \mathbf{\Omega}^n(X) \hookrightarrow \sharp[X, \Omega^n] \ .$$

Generally we say:

Definition D.19. Given a cohesive ∞-topos $\mathbf{H}$ (or just a *local* ∞*-topos*, equipped with a $\sharp$-operator), we say that a 0-truncated object $X \in \tau_{\leq 0}\mathbf{H} \hookrightarrow \mathbf{H}$ is *concrete* if $X \to \sharp X$ is a 1-monomorphism. Moreover we say that the 1-image projection of this map is the *concretification* of X.

Hence the moduli stack $\mathbf{\Omega}^n(X)$ of differential forms on X is the concretification of the mapping stack $[X, \Omega^n]$ of maps into the "differential coefficient object" Ω^n. As we pass to differential coefficient objects that are not 0-truncated, we have to concretify the moduli stack degreewise, as shown by the following example.

Example D.20. For G a Lie group such as $G = U(1)$, let $\mathbf{B}G_{\mathrm{conn}} \in \mathrm{Smooth}\infty\mathrm{Grpd}$ be the universal moduli stack for G-principal connections as in example D.10. For $X \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ a smooth manifold, write

$$[X, \mathbf{B}G_{\mathrm{conn}}] \twoheadrightarrow \sharp_1[X, \mathbf{B}G_{\mathrm{conn}}]$$

for the 1-image projection of its $\sharp$-unit. Over a test manifold U we have that the groupoid of U-plots $U \to \sharp_1[X, \mathbf{B}G_{\mathrm{conn}}]$ is equivalently that which has as objects the smoothly U-parameterized collections of G-principal connections on X, and has as morphisms the *discretely* U-parameterized collections of gauge transformations between these. Hence the 1-image factorization here has correctly concretified the collections of objects, but has forgotten all geometric structure on the collection of morphisms. On the other hand, a morphism of G-principal connections is just a morphism of the underlying G-principal bundles (satisfying the condition that it respects the connections) and the mapping stack $[X, \mathbf{B}G]$ into the universal moduli for plain G-principal connections does correctly encode the geometric structure on collections of these. Therefore we can form the homotopy fiber product $G\mathbf{Conn}(X)$ in the following diagram

$$\begin{tikzcd}
 & {[X, \mathbf{B}G_{\mathrm{conn}}]} \arrow[d, "\mathrm{conc}"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & G\mathbf{Conn}(X) \arrow[dl] \arrow[dr] & \\
\sharp_1[X, \mathbf{B}G_{\mathrm{conn}}] \arrow[dr, "{\sharp_1[X, u_{\mathbf{B}G}]}"'] & & {[X, \mathbf{B}G]} \arrow[dl] \\
 & \sharp_1[X, \mathbf{B}G] &
\end{tikzcd}$$

which is the correct moduli stack $G\mathbf{Conn}(X)$ of G-principal connections on X. Its U-plots $U \to G\mathbf{Conn}(X)$ form the groupoid of smoothly U-parameterized collections of G-principal connections on X and of smoothly U-parameterized collections of gauge transformations between these. The vertical morphism labeled conc above is the one induced by the naturality of the $\sharp_1$-unit and the universailty of the homotopy pullbacl. This we may call the *differential concreteification* map in this case.

Remark D.21. The Chevalley-Eilenberg algebra of function on the Lie algebroid of $G\mathbf{Conn}(X)$ is what in the literature is known as the (off-shell) *BRST complex* of G-gauge theory: the functions on the cotangents to the unit morphisms in the gorupoid $G\mathrm{Conn}(X)$ are what are called the *ghosts* in the BRST complex.

Remark D.22. For $X = *$ the point the differential concretification map here is the forgetful map from the universal modili stack of G-principal connections to that of G-principal bundles

$$\begin{tikzcd} {[*, \mathbf{B}G_{\mathrm{conn}}]} \arrow[d, "\mathrm{conc}"] & \simeq & \mathbf{B}G_{\mathrm{conn}} \arrow[d, "u_{\mathbf{B}G}"] \\ G\mathbf{Conn}(*) & \simeq & \mathbf{B}G \end{tikzcd} .$$

Therefore we make the following general

Definition D.23. Let $\mathbb{G} \in \mathrm{Grp}_2(\mathbf{H})$ a braided ∞-group equipped with a tower of curvature twists and an induced tower of differential coefficients $\mathbf{B}\mathbb{G}_{\mathrm{conn}^\bullet}$ as in remark D.15. Then for $X \in \mathbf{H}$ any object, the *moduli ∞-stack of $\mathbb{G}$-connections on X* is the iterated homotopy fiber product

$$\mathbb{G}\mathbf{Conn}(X) := \left(\sharp_1[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \underset{\sharp_1[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^1}]}{\times} \sharp_2[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}^1}] \underset{\sharp_2[X,\mathbf{B}\mathbb{G}_{\mathrm{conn}^2}]}{\times} \cdots \underset{\sharp_{n-1}[X,\mathbf{B}^n\mathbb{G}]}{\times} [X, \mathbf{B}\mathbb{G}] \right) ,$$

where $\sharp_k(-)$ denotes the k-image factorization, def. C.7 of the unit of the $\sharp$-operator.

We check that this indeed has the correct ouput in our running example:

Proposition D.24. *For $\mathbb{G} = \mathbf{B}^{n-1}U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ the smooth circle n-group as in example D.10 and for $X \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ a smooth manifold, the object*

$$(\mathbf{B}^{n-1}U(1))\mathbf{Conn}(X) \in \mathrm{Smooth}\infty\mathrm{Grpd}$$

of def. D.23 is presented by the presheaf of n-groupoids which to $U \in \mathrm{SmoothMfd}$ assigns the n-groupoid of smoothly U-parameterized collections of Deligne cocycles on X, of smoothly U-paramatererized collections of gauge transformations between these, and so on.

Proof. This follows by an argument generalizing the discussion in example D.20. Details are in [43]. □

Remark D.25. By construction and by the universal property of homotopy limits, there is a canonical projection map

$$[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \longrightarrow \mathbb{G}\mathbf{Conn}(X)$$

from the mapping stack from X into the universal moduli stack of $\mathbb{G}$-principal connections of def. D.8 to the concretified moduli stack of $\mathbb{G}$-principal connections of def. D.23. We call this the *differential concretification* map. This is the higher variant of the concretification map of 0-truncated objects of def. D.19.

For the discussion of the quantomorphism ∞-group extension in D.3 we need the relation of the differential moduli of def. D.23 to their restriction to *flat* differential cocycles:

Definition D.26. In the situation of def. D.23 we say that

$$\begin{aligned} &\mathbb{G}\mathbf{FlatConn}(X) \\ &:= \sharp[X, \flat\mathbf{B}\mathbb{G}] \underset{\sharp[X,\Omega(\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}}])}{\times} \sharp_1[X, \Omega(\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}})] \underset{\sharp_1[X,\Omega(\mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-2}})]}{\times} \cdots \underset{\sharp_n[X,\Omega(\mathbf{B}\mathbb{G}_{\mathrm{conn}^0})]}{\times} [X, \mathbb{G}] \end{aligned}$$

is the *moduli object for flat $\mathbb{G}$-connections on X.*

Example D.27. In the context of prop. D.24 one checks that this reproduces the moduli of flat Deligne cocycles.

The crucial general abstract relation between differential moduli and flat differential moduli is now the following statement, which says that the loop space objects of the differential moduli objects are the flat differential moduli objects.

Proposition D.28. *1. If $\mathbb{G}$ is an abelian 0-truncated group object and if $\int X$ is connected, then for every $\nabla : X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$*

$$\Omega_\nabla\left(\mathbb{G}\mathbf{Conn}\left(X\right)\right) \simeq \mathbb{G}$$

2. If $\mathbb{G}$ is not 0-truncated then

$$\Omega_*\left(\mathbb{G}\mathbf{Conn}(X)\right) \simeq (\Omega\mathbb{G})\mathbf{FlatConn}(X)\,.$$

3. If moreover $\mathbb{G}$ is a sylleptic ∞-group, then for ever $\nabla \in \mathbb{G}\mathbf{Conn}(X)$ we have

$$\Omega_\nabla\left(\mathbb{G}\mathbf{Conn}(X)\right) \simeq (\Omega\mathbb{G})\,\mathbf{FlatConn}(X)\,.$$

Proof. For the second statement observe that forming loop space objects distributes over homotopy fiber products, respects the internal hom in the second argument, commutes with the $\sharp$-operator (since that is right adjoint) and intertwines n-images as

$$\Omega \circ \mathrm{im}_n \simeq \mathrm{im}_{n-1}\Omega\,,$$

by prop. C.13. The first statement follows in the same way. For the third we use that if $\mathbb{G}$ is sylleptic then $\mathbb{G}\mathbf{Conn}(X)$ itself is canonically a group and then the group product canonically identifies the loop space at any given point with that at the neutral element. □

Remark D.29. The analogous construction for the not-concretetified moduli stacks produces only the discrete underlying ∞-groupoid of flat higher connections, but not its cohesive structure:

$$\Omega_*[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \simeq [X, \flat\mathbf{B}(\Omega\mathbb{G})] \simeq \flat\left((\Omega\mathbb{G})\,\mathbf{FlatConn}(X)\right)\,.$$

D.3 Higher quantomorphism- and Heisenberg-groupoids

As in the discussion in D, let $\mathbf{H}$ be a cohesive ∞-topos (such as that of smooth ∞-groupoids, example D.1), let $\mathbb{G} \in \mathrm{Grp}_2(\mathbf{H})$ be a braided ∞-group in $\mathbf{H}$, let $\mathbf{B}\mathbb{G}_{\mathrm{conn}}$ be the universal moduli ∞-stack of $\mathbb{G}$-principal connections.

Definition D.30. For $\nabla : X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ the map modulating a $\mathbb{G}$-principal connection, the corresponding *higher quantomorphism groupoid* $\mathrm{At}(\nabla)_\bullet \in \mathrm{Grpd}(\mathbf{H})$ or *higher contactomorphism groupoid* induced by ∇ is the corresponding higher Atiyah-groupoid according to def. C.42, hence under the equivalence of prop. C.25 is the ∞-groupoid with atlas which is the 1-image projection

$$X \twoheadrightarrow \mathrm{At}(\nabla) := \mathrm{im}_1(\nabla)$$

of ∇.

Remark D.31. By prop. C.38 the unconcretified ∞-group of bisections of the higher quantomorphism groupoid $\mathrm{At}(\nabla)_\bullet$ of def. D.30 sits in a homotopy fiber sequence of the form

$$\mathbf{BiSect}(\mathrm{At}(\nabla)_\bullet) \longrightarrow \mathbf{Aut}(X) \xrightarrow{\nabla\circ(-)} [X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}]\ ,$$

with the object on the right taken to be pointed by ∇. But now that we are considering a differential cocycle, not just from a bundle cocycle, the same kind of reasoning as in example D.16 shows that this ∞-group of bisections does have the correct global points, but does not quite have the geometric structure on these that one would typically need in applications (such as in the theorems below in D.3). Instead, one wants the *differentially concretified* version of $\mathbf{BiSect}(\mathrm{At}(\nabla)_\bullet)$, along the lines of the above discussion around def. D.23.

But in view of the above fiber sequence, there is a natural candidate of such differential concretification:

Definition D.32. The *quantomorphism ∞-group* of a $\mathbb{G}$-principal connection ∇ is the homotopy fiber $\mathbf{QuantMorph}(\nabla) \in \mathrm{Grp}(\mathbf{H})$ in

$$\mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{Aut}(X) \xrightarrow{\nabla\circ(-)} \mathbb{G}\mathbf{Conn}(X)$$

,

where the right morphism is the composite of $\nabla \circ (-)$ with the differential concretification projection $[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \longrightarrow \mathbb{G}\mathbf{Conn}(X)$ of remark D.25.

Remark D.33. The canonical forgetful map $u_{\mathbf{B}\mathbb{G}} : \mathbf{B}\mathbb{G}_{\mathrm{conn}} \to \mathbf{B}\mathbb{G}$ induces a morphism from the higher quantomorphism groupoid to the Atiyah groupoid of the underlying $\mathbb{G}$-principal bundle

$$\mathrm{At}(\nabla)_\bullet \longrightarrow \mathrm{At}(\nabla^0)_\bullet$$

which is the identity on objects. This in turn induces a canonical homomorphism

$$u_{\mathbf{B}\mathbb{G}} \circ (-) : \ \mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{BiSect}(\mathrm{At}(P)_\bullet)$$

from the quantomorphism ∞-group, def. D.32, into that of bisections of the Atiyah groupoid, prop. C.40. Thereby, via prop. C.46, the quantomorphism ∞-group acts on the space of sections of any associated V-fiber ∞-bundle to ∇^0. This is the *higher prequantum operator* action. It is the global version of the canonical action of the higher quantomorphism groupoid itself, in the sense of groupoid actions of def. C.49, which is exhibited, in analogy with def. C.50 by the left square in the following pasting diagram of ∞-pullbacks:

$$\begin{tikzcd}
P \times_{\mathbb{G}} V \arrow[r] \arrow[d] & (P \times_{\mathbb{G}} V)//\mathrm{Qu}(\nabla) \arrow[rr] \arrow[d] & & V//G \arrow[d, "\rho"] \\
X \arrow[r, two heads] & \mathrm{Qu}(\nabla) \arrow[r, hook] \arrow[dr] & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \arrow[r] & \mathbf{B}\mathbb{G} \\
& & \mathrm{At}(\nabla^0) \arrow[ur, hook] &
\end{tikzcd}$$

Given all of the above, we now have the following list of evident generalizations of traditional notions in prequantum theory.

Definition D.34. Let $\nabla : X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ be given, regarded as a prequantum ∞-bundle as in def. D.8. Then

1. the *Hamiltonian symplectomorphism group* $\mathbf{HamSympl}(\nabla) \in \mathrm{Grp}(\mathbf{H})$ is the sub-∞-group of the automorphisms of X which is the 1-image, def. C.7, of the quantomorphisms:

$$\mathbf{QuantMorph}(\nabla) \twoheadrightarrow \mathbf{HamSympl}(\nabla) \hookrightarrow \mathbf{Aut}(X)$$

2. for $G \in \mathrm{Grp}(\mathbf{H})$ an ∞-group, a *Hamiltonian G-action* on X is an ∞-group homomorphim

$$G \xrightarrow{\phi} \mathbf{HamSympl}(\nabla) \hookrightarrow \mathbf{Aut}(X) \ ;$$

3. an *integrated G-momentum map* is an action by quantomorphisms

$$G \xrightarrow{\hat{\phi}} \mathbf{QuantMorph}(\nabla) \hookrightarrow \mathbf{Aut}(X) \ ;$$

4. given a Hamiltonian G-action ϕ, the corresponding *Heisenberg ∞-group* $\mathbf{Heis}_\phi(\nabla)$ is the homotopy fiber product in

$$\begin{tikzcd} \mathbf{Heis}_\phi(\nabla) \arrow[r] \arrow[d] & \mathbf{QuantMorph}(\nabla) \arrow[d] \\ G \arrow[r, "\phi"] & \mathbf{HamSympl}(\nabla) \end{tikzcd} .$$

As in the disucssion in D.1, let $\mathbf{H}$ be a cohesive ∞-topos (such as Smooth∞Grpd of example D.1), let $\mathbb{G} \in \mathrm{Grp}_2(\mathbf{H})$ a braided ∞-group, def. C.18, let $X \in \mathbf{H}$ any object, let $\omega : X \to \Omega^2_{\mathrm{cl}}(-, \mathbb{G})$ be a flat differential form datum and let $\nabla : X \to \mathbf{B}\mathbb{G}_{\mathrm{conn}}$ a $\mathbb{G}$-prequantization of it. Then we have the following characterization of the corresponding quantomorphism ∞-group of def. D.32.

Theorem D.35. *There is a long homotopy fiber sequence in* $\mathrm{Grp}(\mathbf{H})$ *of the form*

- *if* $\mathbb{G}$ *is 0-truncated:*

$$\mathbb{G} \longrightarrow \mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{HamSympl}(\nabla) \xrightarrow{\nabla\circ(-)} \mathbf{B}\left(\mathbb{G}\mathbf{ConstFunct}(X)\right)$$

- *otherwise:*

$$(\Omega\mathbb{G})\mathbf{FlatConn}(X) \longrightarrow \mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{HamSympl}(\nabla) \xrightarrow{\nabla\circ(-)} \mathbf{B}\left((\Omega\mathbb{G})\,\mathbf{FlatConn}(\nabla)\right) ,$$

which hence exhibits the quantomorphism group $\mathbf{QuantMorph}(\nabla) \in \mathrm{Grp}(\mathbf{H})$ *as an ∞-group extension, example C.31, of the ∞-group of Hamiltonian symplectomorphisms, def. D.34, by the differential moduli of flat $\Omega\mathbb{G}$-principal connections on X, def. D.26, classified by an ∞-group cocycle which is given by postcomposition with ∇ itself.*

Proof. This is an immediate variant, under the differential concretification of def. D.23, of the higher Atiyah sequence of theorem C.48: Consider the natural 1-image factorization of the horizontal maps in the defining ∞-pullback of def. D.32:

$$\begin{tikzcd} \mathbf{QuanMorph}(\nabla) \arrow[r, two heads] \arrow[d] & \mathbf{HamSympl}(\nabla) \arrow[r, hook] \arrow[d, "\nabla\circ(-)"] & \mathbf{Aut}(X) \arrow[d, "\nabla\circ(-)"] \\ * \arrow[r, two heads] \arrow[rr, bend right, "\vdash\nabla"'] & \mathbf{B}\left(\Omega_\nabla\left(\mathbb{G}\mathbf{Conn}(X)\right)\right) \arrow[r, hook] & \mathbb{G}\mathbf{Conn}(X) \end{tikzcd} .$$

By homotopy pullback stability of both 1-epimorphisms and 1-monomorphisms and by essential uniqueness of 1-image factorizations this is a pasting diagram of homotopy pullback squares. The claim then follows with prop. D.28 as in the proof of theorem C.48. □

The analogous statement also holds for Heisenberg ∞-groups:

Corollary D.36. *If $\phi : G \to \mathbf{HamSympl}(\nabla) \hookrightarrow \mathbf{Aut}(X)$ is any Hamiltonian G-action, def. D.34, then the corresponding Heisenberg ∞-group sits in the ∞-fiber sequence*

$$(\Omega\mathbb{G})\mathbf{FlatConn}(X) \longrightarrow \mathbf{Heis}_\phi(\nabla) \longrightarrow G \xrightarrow{\nabla\circ(-)} \mathbf{B}\left((\Omega\mathbb{G})\,\mathbf{FlatConn}(\nabla)\right) ,$$

Proof. By the pasting law for homotopy pullbacks. □

Example D.37. For $\mathbb{G} = U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ the smooth circle group as in example D.4 for $n = 1$, and for $X \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ a connected smooth manifold, theorem D.35 reproduces the traditional quantomorphism group as a $U(1)$-extension of the traditional group of Hamiltonian symplectomorphisms, as discussed for instance in [45, 60].

In order to put the higher generalizations of the quantomorphism extensions into this context, we notice the following basic fact.

Proposition D.38. *For $\mathbb{G} = \mathbf{B}U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ the smooth circle 2-group as in example D.4 for $n = 2$, consider $X \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ a connected and simply connected smooth manifold. Then from prop. D.24 and example D.27 one obtains an equivalence of smooth group stacks*

$$U(1)\mathbf{FlatConn}(X) \simeq \mathbf{B}U(1)\,.$$

Generally, for $n \geq 1$ and for $\mathbb{G} = \mathbf{B}^n U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ the smooth circle $(n+1)$-group as in example D.4, there is for X an n-connected smooth manifold an equivalence of smooth ∞-groups

$$(\mathbf{B}^{n-1}U(1))\mathbf{FlatConn}(X) \simeq \mathbf{B}^n U(1)\,.$$

Proof. We use the description of $U(1)\mathbf{FlatConn}(X)$ given by prop. D.24 and example D.27. First notice then that on a simply connected manifold there is up to equivalence just a single flat connection, hence $U(1)\mathbf{FlatConn}(X)$ is pointed connected. Moreover, an auto-gauge transformation from that single flat connection (any one) to itself is a $U(1)$-valued function which is *constant on X*. But therefore by prop. D.24 the U-plots of the first homotopy sheaf of $U(1)\mathbf{FlatConn}(X)$ are smoothly U-parameterized collections of constant $U(1)$-valued functions on X, hence are smoothly U-parameterized collections of elements in $U(1)$, hence are smooth $U(1)$-valued functions on U. These are, by definition, equivalently the U-plots of automorphisms of the point in $\mathbf{B}U(1)$.

The other cases work analogously. $\square$

Remark D.39. Therefore in the situation of prop. D.38 the quantomorphism ∞-group is a smooth 2-group extension by the circle 2-group $\mathbf{B}U(1)$. The archetypical example of $\mathbf{B}U(1)$-extensions is the smooth *String 2-group*, example C.17. Indeed, this occurs as the Heisenberg 2-group extension of the WZW sigma-model regarded as a local 2-dimensional quantum field theory. This we turn to in **??** below.

D.4 Higher Courant groupoids

Given a $\mathbb{G}$-principal ∞-connection

$$\begin{tikzcd} & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \arrow[d, "u_{\mathbf{B}\mathbb{G}}"] \\ X \arrow[ur, "\nabla"] \arrow[r, "\nabla^0"'] & \mathbf{B}\mathbb{G} \end{tikzcd}$$

we have considered in C.4 the corresponding higher Atiyah groupoid $\mathrm{At}(\nabla^0)_\bullet$ and in D.3 the higher quantomorphism groupoid $\mathrm{At}(\nabla)$ equipped with a canonical map $\mathrm{At}(\nabla)_\bullet \longrightarrow \mathrm{At}(\nabla^0)_\bullet$. But in view of the towers of differential coefficients discussed in D.1 this has a natural generalization to towers of higher groupoids interpolating between the higher Atiyah groupoid and the higher quantomorphism groupoid.

In particular, let $\mathbb{G} \in \mathrm{Grp}_3(\mathbf{H})$ a sylleptic ∞-group, def. C.18, with compatibly chosen factorization of differential form coefficients and induced factorization of differential coefficients

$$\mathbf{B}^2\mathbb{G}_{\mathrm{conn}} \longrightarrow \mathbf{B}(\mathbf{B}\mathbb{G}_{\mathrm{conn}}) \longrightarrow \mathbf{B}^2\mathbb{G}$$

by prop. D.12. Then in direct analogy with def. D.30 we set:

Definition D.40. For $\nabla^{n-1} : X \to \mathbf{B}(\mathbf{B}\mathbb{G}_{\mathrm{conn}})$ a $\mathbb{G}$-principal connection without top-degree connection data as in def. D.13, we say that the corresponding *higher Courant groupoid* is the corresponding higher Atiyah groupoid $\mathrm{At}(\nabla^{n-1})_\bullet \in \mathrm{Grpd}(\mathbf{H})$, hence the groupoid object which by prop. C.25 is equivalent to the ∞-groupoid with atlas given by the 1-image factorization of ∇^{n-1}

$$X \twoheadrightarrow \mathrm{At}(\nabla^{n-1}) := \mathrm{im}_1(\nabla^{n-1}) \ .$$

Example D.41. If $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ is the ∞-topos of smooth ∞-groupoids from example D.1 and $\mathbb{G} = \mathbf{B}U(1) \in \mathrm{Grp}_\infty(\mathbf{H})$ is the smooth circle 2-group as in example C.19 and if finally $X \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ is a smooth manifold, then by example D.14 a map $\nabla^1 : X \to \mathbf{B}(\mathbf{B}U(1)_{\mathrm{conn}})$ is equivalently a "$U(1)$-bundle gerbe with connective structure but without curving" on X.

In this case the higher Courant groupoid according to def. D.40 is a smooth 2-groupoid and its ∞-group of bisections $\mathbf{BiSect}(\mathrm{At}(\nabla^1)_\bullet)$ of def. C.37 is a smooth 2-group. The points of this 2-group are equivalently pairs (ϕ, η) consisting of a diffeomorphism $\phi : X \xrightarrow{\simeq} X$ and an equivalence of bundle gerbes with connective structure but without curving of the form $\eta : \phi^*\nabla^{n-1} \xrightarrow{\simeq} \nabla^{n-1}$. A homotopy of bisections between two such pairs $(\phi_1, \eta_1) \to (\phi_2, \eta_2)$ exists if $\phi_1 = \phi_2$ and is then given by a higher gauge equivalence κ $\kappa : \eta_1 \xrightarrow{\simeq} \eta_2$. Moreover, with prop. D.24 the smooth structure on the differentially concretified 2-group of such bisections is the expected one, where U-plots are smooth U-parameterized collections of diffeomorphisms and of bundle gerbe gauge transformations.

Precisely these smooth 2-groups have been studied in [11]. There it was shown that the Lie 2-algebras that correspond to them under Lie differentiation are the Lie 2-algebras of sections of the *Courant Lie 2-algebroid* which is traditionally associated with a bundle gerbe with connective structure. (See the citations in [11] for literature on Courant Lie 2-algebroids.) Therefore the abstractly defined smooth higher Courant groupoid $\mathrm{At}(\nabla^{n-1})$ according to def. D.40 indeed is a Lie integration of the traditional Courant Lie 2-algebroid assigned to ∇^{n-1}, hence is the *smooth Courant 2-groupoid*.

Example D.42. More generally, in the situation of example D.41 consider now for some $n \geq 1$ the smooth circle n-group $\mathbb{G} = \mathbf{B}^{n-1}U(1)$ as in example C.19. Then by example D.10 a map

$$\nabla^{n-1} : X \longrightarrow \mathbf{B}(\mathbf{B}^{n-1}U(1)_{\mathrm{conn}})$$

is equivalently a Deligne cocycle on X in degree $(n+1)$ without n-form data.

To see what the corresponding smooth higher Courant groupoid $\mathrm{At}(\nabla^{n-1})$ is like, consider first the local case in which ∇^{n-1} is trivial. In this case a bisection of $\mathrm{At}(\nabla^{n-1})$ is readily seen to be a pair consisting of a diffeomorphism $\phi \in \mathrm{Diff}(X)$ together with an $(n-1)$-form $H \in \Omega^{n-1}(X)$, satisfying no further compatibility condition. This means that there is an L_∞-algebra representing the Lie differentiation of the higher Courant groupoid $\mathrm{At}(\nabla^{n-1})_\bullet$ which in lowest degree is the space of sections of a bundle on X which is locally the sum $TX \oplus \wedge^{n-1}T^*X$ of the tangent bundle with the $(n-1)$-form bundle. This is precisely what the sections of higher Courant Lie n-algebroids are supposed to be like, see for instance [63].

Finally, if we are given a tower of differential refinements of $\mathbb{G}$-principal bundles as discussed in D.1

$$\begin{tikzcd}
 & & \mathbf{B}\mathbb{G}_{\mathrm{conn}} \arrow[d] \\
 & & \mathbf{B}\mathbb{G}_{\mathrm{conn}^{n-1}} \arrow[d] \\
 & & \vdots \arrow[d] \\
X \arrow[uuurr, "\nabla"] \arrow[uurr, "\nabla^{n-1}"'] \arrow[rr, "\nabla^0"'] & & \mathbf{B}\mathbb{G}
\end{tikzcd}$$

then there is correspondingly a tower of higher gauge groupoids:

$$
\begin{array}{ccccccccc}
\begin{array}{c}\text{higher}\\ \text{Quantomorphism}\\ \text{groupoid}\end{array} & & \begin{array}{c}\text{higher}\\ \text{Courant}\\ \text{groupoid}\end{array} & & \cdots & & \begin{array}{c}\text{intermediate}\\ \text{differential}\\ \text{higher}\\ \text{Atiyah}\\ \text{groupoid}\end{array} & \cdots & \begin{array}{c}\text{higher}\\ \text{Atiyah}\\ \text{groupoid}\end{array} \\
\\
\mathrm{At}(\nabla)_\bullet & \longrightarrow & \mathrm{At}(\nabla^{n-1})_\bullet & \longrightarrow & \cdots & \longrightarrow & \mathrm{At}(\nabla^k) & \longrightarrow \cdots \longrightarrow & \mathrm{At}(\nabla^0)
\end{array}
.
$$

The further intermediate stages appearing here seem not to correspond to anything that has already been given a name in traditional literature. We might call them *intermediate higher differential gauge groupoids*. These structures are an integral part of higher prequantum geometry.

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