Non-well-founded trees in categories *

Benno van den Berg †

Federico De Marchi ‡§

Department of Mathematics, University of Utrecht
P.O. Box 80010, 3508 TA Utrecht, The Netherlands

September 2004

Abstract

Non-well-founded trees are used in mathematics and computer science, for modelling non-well-founded sets, as well as non-terminating processes or infinite data-structures. Categorically, they arise as final coalgebras for polynomial endofunctors, which we call M-types. In order to reason about trees, we need the notion of path, which can be formalised in the internal logic of any locally cartesian closed pretopos with a natural number object. In such categories, we derive existence results about M-types, leading to stability of locally cartesian closed pretoposes with a natural number object and M-types under slicing, formation of coalgebras (for a cartesian comonad), and sheaves for an internal site.

1 Introduction

The relevance of non-well-founded trees to mathematics and computer science was made clear by the work of Peter Aczel [3]. He used them in order to extend the set-theoretic universe by non-well-founded sets. Traditionally, the Axiom of Foundation allows sets to be represented only by well-founded trees, but Aczel's Anti-Foundation Axiom extends this possibility to all non-well-founded trees.

In computer science, non-well-foundedness of trees enables one to describe circular (and, more generally, non-terminating) phenomena. For this reason, they have been used in the theory of concurrency and specification, as well as

*Paper submitted for publication.

† *vdberg@math.uu.nl.*

‡ *marchi@math.uu.nl.*

§ Research supported by NWO Grant n. 613.000.222.

in the study of semantics for programming languages with coinductive types [7, 8, 10, 21, 6].

Categorically, non-well-founded trees over a given signature form what we call its M-type, that is the final coalgebra for the polynomial functor determined by the signature itself, much like well-founded trees form its initial algebra, usually called the W-type of the signature.

In this paper, we look at properties of categories with M-types. Specifically, we prove that they are closed under slicing, formation of coalgebras (for a cartesian comonad), and sheaves (for an internal site).

These constructions have proved useful in topos theory, leading to the formulation of various independence results [9, 20]. Our motivation for studying them in this setting, is the potential for application to the theory of non-well-founded sets and to polymorphism of coinductive types.

The core of the proof, in each case, is to show existence of certain final coalgebras. Generally, this is not an easy task. It is obvious that some functors can not have a final coalgebra (for example the powerset functor in any elementary topos, by Cantor's argument). However, playing with the structure of our categories and restricting our attention to particular classes of functors, we can obtain partial results (see for instance [4, 5, 22, 18]).

In the present work, we shall focus on locally cartesian closed pretoposes. The internal logic of these categories forces us to think of trees in a constructive (and predicative) way. For this purpose, we choose to use the language of paths [8]; to this end we shall require the existence of a natural number object. In formalising our arguments, we encountered a category of objects, which we call proto-coalgebras. We have no notice of them being ever introduced before.

As a byproduct of our work with proto-coalgebras and paths, we could reformulate in our setting a result by Santocanale [19].

We are grateful to Jaap van Oosten and Ieke Moerdijk for their useful comments on an earlier version of this paper.

2 Setting the scene

Throughout the paper, C will denote a locally cartesian closed pretopos with a natural number object. To any morphism $f\colon B \longrightarrow A$ in C, we can then associate a *polynomial functor* $P_f\colon \mathsf{C} \longrightarrow \mathsf{C}$, defined as

$$P_f(X) = \Sigma_{a \in A} X^{B_a}$$

or, more formally, as

$$P_f(X) = \Sigma_A (A \times X \xrightarrow{p_1} A)^{(B \xrightarrow{f} A)},$$

where the exponential is taken in the slice category C/A. In the internal language of C, an element of $P_f(X)$ is a pair (a,t), where $a \in A$ and $t: f^{-1}(a) \longrightarrow X$ assigns to each element in the fibre over a an element in X. Notice in particular that there is an obvious map $\rho: P_f(X) \longrightarrow A$ mapping the pair (a,t) to a. This makes $P_f(X)$ into an object over A, namely the exponential of p_1 and f. Notice also that the action of P_f on maps lives over A, i.e. for a map $\phi: X \longrightarrow Y$ in C the following commutes:

$$\begin{tikzcd} P_f(X) \arrow[rr, "P_f\phi"] \arrow[dr, "\rho"'] & & P_f(Y) \arrow[dl, "\rho"] \\ & A. & \end{tikzcd} \tag{1}$$

The *W-type* associated to f is defined to be the initial algebra for the functor P_f. In Set, this is the set of all well-founded trees built over the signature determined by f, having one term constructor of arity $f^{-1}(a)$ for each element $a \in A$. Initiality allows for definition of maps on trees by structural induction. In this same context, the final P_f-coalgebra turns out to be the set of all trees (including the non-well-founded ones) built over the same signature. We shall call a final P_f-coalgebra the *M-type* associated to f, and we shall denote it by

$$\tau_f: M_f \longrightarrow P_f(M_f).$$

We shall often denote the inverse of τ_f by $\sup_f$ (or just sup, when f is understood). This will map the pair (a,t) to the tree $\sup_a t$, consisting of a root labelled by a and having children indexed by the elements of the fibre of f over a, in such a way that the subtree appended to the branch labelled by $b \in f^{-1}(a)$ is $t(b)$. Mediating arrows to final coalgebras correspond to functions to trees defined by coinduction (for a clear description, look at [10]). We say that C *has M-types*, if terminal coalgebras exist for every polynomial functor. A *coinductive pretopos* will be a locally cartesian closed pretopos with a natural number object which has M-types.

It is the purpose of this paper to prove the closure of coinductive pretoposes under slicing, formation of coalgebras for a cartesian comonad and formation of sheaves over an internal site. But, before proceeding, we state some general properties of M-types which will be needed later.

2.1 Covariant character of M-types

First of all let's note that, by Lambek's lemma [13], the coalgebra map τ_f is an isomorphism. In particular, this determines a *root* map

$$M_f \xrightarrow{\tau_f} P_f(M_f) \xrightarrow{\rho} A,$$

which, by an abuse of notation, will be denoted again by ρ. We are confident that this will generate no confusion.

Given a pullback diagram in C

$$\begin{tikzcd} B' \arrow[r, "\beta"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ A' \arrow[r, "\alpha"'] & A, \end{tikzcd}$$

we can think of α as a morphism of signatures, since the fibre over each $a' \in A'$ is isomorphic to the fibre over $\alpha(a') \in A$. It is therefore reasonable to expect, in such a situation, an induced morphism between $M_{f'}$ and M_f, when these exist.

In fact, as already pointed out in [16], such a pullback square induces a natural transformation $\widetilde{\alpha}\colon P_{f'} \longrightarrow P_f$ such that

$$\rho\widetilde{\alpha} = \alpha\rho. \tag{2}$$

Post-composition with $\widetilde{\alpha}$ turns any $P_{f'}$-coalgebra into one for P_f. In particular, this happens for $M_{f'}$, thus inducing a unique coalgebra homomorphism as in

$$\begin{tikzcd} M_{f'} \arrow[r, "\alpha_!"] \arrow[d, "\tau_{f'}"'] & M_f \arrow[dd, "\tau_f"] \\ P_{f'}(M_{f'}) \arrow[d, "\widetilde{\alpha}"'] & \\ P_f(M_{f'}) \arrow[r, "P_f(\alpha_!)"'] & P_f(M_f). \end{tikzcd} \tag{3}$$

Notice that, by (1) and (2), the morphism $\alpha_!$ preserves the root maps.

2.2 Working with paths

The reader who is familiar with the language of trees will not find it surprising that, in order to carry out some calculations, we introduce the notion of *path* [8]. The reason for doing so is that paths allow us to identify properties of trees in a predicative way. Making an essential use of the internal logic of a locally cartesian closed pretopos C with a natural number object $\mathbb{N}$, we can define paths not just for the W-type and the M-type of a polynomial functor, but for *any* coalgebra.

Assume we are given a P_f-coalgebra

$$X \xrightarrow{\gamma} P_f X.$$

A finite sequence of odd length $\langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ is called a *(finite) path* in (X, γ), if every x_i is in X, every b_i is in B and for every $i < n$ we have

$$x_{i+1} = \gamma(x_i)(b_i). \tag{4}$$

More precisely, if $\gamma(x_i) = (a_i, t_i)$, then we are asking that $x_{i+1} = t_i(b_i)$. In the particular case when X is the final coalgebra M_f, a path $\langle m_0, b_0, \ldots, m_n \rangle$ in this sense coincides precisely with a path in the usual sense in the non-well-founded tree m_0. We will therefore say that such a path *lies* in m_0, and by extension, a path $\langle x_0, b_0, \ldots, x_n \rangle$ lies in $x_0 \in X$ for any coalgebra (X, γ). All paths in a coalgebra (X, γ) are collected in the subobject

$$Paths(\gamma) \rightarrowtail (X + B + 1)^{\mathbb{N}}.$$

Any morphism of coalgebras $\alpha\colon (X, \gamma) \longrightarrow (Y, \delta)$ induces a morphism

$$\alpha^*\colon Paths(\gamma) \longrightarrow Paths(\delta) \tag{5}$$

between the respective objects of paths. A path $\langle x_0, b_0, \ldots, x_n \rangle$ is sent by α^* to $\langle \alpha(x_0), b_0, \ldots, \alpha(x_n) \rangle$. Furthermore, given a path $\tau = \langle y_0, b_0, \ldots, y_n \rangle$ in Y and an x_0 such that $\alpha(x_0) = y_0$, there is a unique path σ starting with x_0 such that $\alpha^*(\sigma) = \tau$. (Proof: define x_{i+1} inductively for every $i < n$ using (4) and put $\sigma = \langle x_0, b_0, \ldots, x_n \rangle$.)

In fact, in order to introduce the concept of path, we need even less than a coalgebra: it is sufficient to have a common environment in which to read equation (4). Given a map $f\colon B \longrightarrow A$ in C, consider the category $P_f - \mathsf{prtclg}$ of P_f*-proto-coalgebras*. Its objects are pairs of maps

$$(\gamma, m) = X \xrightarrow{\gamma} Y \xleftarrow{m} P_f(X), \tag{6}$$

where m is monic. An arrow between (γ, m) and (γ', m') is a pair of maps (α, β) making the following commute:

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"'] & P_f(X) \arrow[l, tail, "m"'] \arrow[d, "P_f(\alpha)"] \\ X' \arrow[r, "\gamma'"'] & Y' & P_f(X'). \arrow[l, tail, "m'"] \end{tikzcd}$$

Notice that there is an obvious inclusion functor

$$I\colon P_f - \mathsf{coalg} \longrightarrow P_f - \mathsf{prtclg}, \tag{7}$$

mapping a coalgebra $\gamma\colon X \longrightarrow P_f(X)$ to the pair $(\gamma, \mathsf{id}_{P_f X})$.

For a proto-coalgebra as in (6), one can introduce the notion of a path in the following way. We shall call an element $x \in X$ *branching* if $\gamma(x)$ lies in the image of m. Then, we call a sequence of odd length $\sigma = \langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ a *path* if it satisfies the properties:

1. $x_i \in X$ is branching for all $i < n$;

2. $b_i \in B_{a_i}$ for all $i < n$;

3. $t_i(b_i) = x_{i+1}$ for all $i < n$;

where (a_i, t_i) is the (unique) element in $P_f X$ such that $\gamma(x_i) = m(a_i, t_i)$. An element $x \in X$ is called *coherent*, if all paths starting with x end with a branching element. So, all coherent elements are automatically branching, and their children, identified through m, are themselves coherent.

The internal language of C makes it possible to identify the object of coherent elements in any proto-coalgebra. This can be shown to be the intersection of the chain X_n of subobjects of X defined inductively as follows: $X_0 = X$; X_1 is the pullback

$$\begin{tikzcd} X_1 \arrow[r, tail, "m_0"] \arrow[d] & X \arrow[d, "\gamma"] \\ P_f(X) \arrow[r, tail, "m"'] & Y; \end{tikzcd}$$

given a subobject $m_{n-1}\colon X_n \rightarrowtail X_{n-1}$, X_{n+1} is the pullback

$$\begin{tikzcd} X_{n+1} \arrow[r, tail, "m_n"] \arrow[d] & X_n \arrow[d, "\gamma m_0 \ldots m_{n-1}"] \\ P_f(X_n) \arrow[r, tail, "m P_f(m_0) \ldots P_f(m_{n-1})"'] & Y. \end{tikzcd}$$

Intuitively, a branching element is one whose image under γ is in $P_f(X)$, hence it has some *branches* determined by the fibres of f. However, nothing ensures that the children nodes will be themselves branching. The subobject X_n above is the object of those elements in X for which we can follow a path for at least n-many steps, that is, we can find branching children up to n consecutive generations. The intersection $Coh(\gamma)$ of the X_n is then the object of coherent elements, for which we know that all their children nodes are still coherent. Hence, we can define on $Coh(\gamma)$ a P_f-coalgebra structure. In fact, this is the biggest coalgebra which we can embed in (γ, m), i.e. a coreflection of the latter for the inclusion functor I of (7).

Proposition 2.1 *The assignment $(\gamma, m) \longmapsto Coh(\gamma)$ mapping any P_f-proto-coalgebra to the object of coherent elements in it, determines a right adjoint Coh to the functor $I\colon P_f - \mathsf{coalg} \longrightarrow P_f - \mathsf{prtclg}$.*

Proof. Consider a proto-coalgebra

$$X \xrightarrow{\gamma} Y \xleftarrow{m} P_f(X),$$

and build the object $Coh(\gamma)$ of coherent elements in X. Because any coherent element $x \in Coh(\gamma)$ is also branching, we can find a (necessarily unique) pair (a,t) such that $\gamma(x) = m(a,t)$. By defining $\chi(x) = (a,t)$ we equip $Coh(\gamma)$ with a P_f-coalgebra structure (notice that, x being coherent, so are the elements in the image of t). The coalgebra $(Coh(\gamma), \chi)$ clearly fits in a commutative diagram

$$\begin{tikzcd}
Coh(\gamma) \arrow[rr, tail, "i"] \arrow[d, "\chi"'] & & X \arrow[d, "\gamma"] \\
P_f(Coh(\gamma)) \arrow[r, tail, "P_f i"'] & P_f(X) \arrow[r, tail, "m"'] & Y.
\end{tikzcd}$$

Let now (X', χ') be any other P_f-coalgebra. Then, given a coalgebra morphism

$$\begin{tikzcd}
X' \arrow[r, "\phi"] \arrow[d, "\chi'"'] & Coh(\gamma) \arrow[d, "\chi"] \\
P_f(X') \arrow[r, "P_f\phi"'] & P_f(Coh(\gamma)),
\end{tikzcd}$$

the pair $(i\phi, mP_f(i\phi))$ clearly determines a proto-coalgebra morphism from $I(X', \chi')$ to (γ, m). Conversely, any proto-coalgebra morphism

$$\begin{tikzcd}
X' \arrow[r, "\chi'"] \arrow[d, "\alpha"'] & P_f(X') \arrow[r, equal] \arrow[d, "\beta"] & P_f(X') \arrow[d, "P_f(\alpha)"] \\
X \arrow[r, "\gamma"'] & Y & P_f(X) \arrow[l, tail, "m"']
\end{tikzcd}$$

has the property that $\alpha(x')$ is branching for any $x' \in X'$. Using an opportune extension to proto-coalgebras of the morphism α^* described in (5) above, one can then easily check that elements in the image of α are coherent. Hence, α factors through the object $Coh(\gamma)$, inducing a coalgebra morphism from (X', χ') to $(Coh(\gamma), \chi)$.

It is now easy to check that the two constructions are mutually inverse, thereby describing the desired adjunction. $\square$

A particular subcategory of proto-coalgebras arises when we have another endofunctor F on C and an injective natural transformation $m: P_f \rightarrowtail F$. In this case, any F-coalgebra $\chi: X \longrightarrow FX$ has an obvious P_f-proto-coalgebra associated to it, namely (χ, m_X). The assignment $(X, \chi) \mapsto (\chi, m_X)$ determines a functor $\widehat{m}: F-\mathsf{coalg} \longrightarrow P_f - \mathsf{prtclg}$, which is clearly faithful.

Proposition 2.2 *The adjunction $I \dashv Coh$ of Proposition 2.1 restricts to an adjunction $m_* \dashv Coh\ \widehat{m}$, where $m_* \colon P_f-\mathsf{coalg} \longrightarrow F-\mathsf{coalg}$ takes $\chi \colon X \longrightarrow P_f X$ to $(X, m_X \chi)$.*

Proof. Consider a P_f-coalgebra (Z, γ) and an F-coalgebra (X, χ). Then, a simple diagram chase, using the naturality of m, shows that F-coalgebra morphisms from $m_*(Z, \gamma)$ to (X, χ) correspond bijectively to morphisms of proto-coalgebras from $I(Z, \gamma)$ to $\widehat{m}(X, \chi)$, hence to P_f-coalgebra homomorphisms from (Z, γ) to $Coh(\widehat{m}(X, \chi))$, by Proposition 2.1. □

2.3 Existence of M-types

The crucial point of the proof that coinductive pretoposes are closed under the various constructions we are going to consider, will always be that of showing existence of M-types. The machinery to do so will be set up in this section.

Traditionally, one can recover non-well-founded trees from well-founded ones, whenever the signature has one specified constant. In fact, the constant allows for the definition of truncation functions, which cut a tree at a certain depth and replace all the term constructors at that level by that specified constant. The way to recover non-well-founded trees is then to consider sequences of trees $(t_n)_{n>0}$ such that each t_n is the truncation at depth n of t_m for all $m > n$. Each such sequence is viewed as the sequence of approximations of a given tree.

In our context, we call a map $f \colon B \longrightarrow A$ *pointed*, when the signature it represents has a specified constant symbol, i.e. if there exists a global element $\perp \colon 1 \longrightarrow A$ such that the following is a pullback:

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \downarrow & & \downarrow f \\ 1 & \xrightarrow[\perp]{} & A. \end{array}$$

The next two statements make clear that, instead of starting with well-founded trees, i.e. with the W-type for f, we can build these approximations from any fixed point of P_f.

Lemma 2.3 *If for some pointed f in C, P_f has a fixed point (that is, a (co)algebra for which the structure map is an isomorphism), then it also has a final coalgebra.*

Proof. Assume X is an algebra whose structure map $\sup \colon P_f X \longrightarrow X$ is an isomorphism. Observe, first of all, that X has a global element

$$\perp \colon 1 \longrightarrow X, \tag{8}$$

namely $\sup_{\perp}(t)$, where $\perp$ is the point of f and t is the unique map from $B_{\perp} = 0$ to X.

Define, by induction, the following truncation functions $tr_n\colon X \longrightarrow X$:

$$\begin{aligned} tr_0 &= \perp \\ tr_{n+1} &= \sup \circ P_f(tr_n) \circ \sup^{-1} \end{aligned}$$

Using these maps, we can define the object M, consisting of all sequences $(\alpha_n \in X)_{n>0}$ with the property:

$$\alpha_n = tr_n(\alpha_m) \text{ for all } n < m.$$

On M, we define a P_f-coalgebra structure $\tau\colon M \longrightarrow P_f M$ as follows. Given a sequence $\alpha = (\alpha_n) \in M$, observe that $\rho(\alpha_n)$ is independent of n and is some element $a \in A$. Hence, each α_n is of the form $\sup_a(t_n)$ for some $t_n\colon B_a \longrightarrow X$, and we define $t\colon B_a \longrightarrow M$ by putting $t(b)_n = t_{n+1}(b)$ for every $b \in B_a$; then $\tau(\alpha) = (a, t)$. Thus, M has the structure of a P_f-coalgebra, and we claim it is the terminal one.

To show this, given another coalgebra $\chi\colon Y \longrightarrow P_f Y$, we wish to define a map of coalgebras $\widehat{p}\colon Y \longrightarrow M$. This means defining maps $\widehat{p}_n\colon Y \longrightarrow X$ for every $n > 0$, with the property that $\widehat{p}_n = tr_n \widehat{p}_m$ for all $n < m$. Intuitively, $\widehat{p}_n$ maps a state of Y to its "unfolding up to level n", which we can mimic in X. Formally, they are defined inductively by

$$\begin{aligned} \widehat{p}_0 &= \perp \\ \widehat{p}_{n+1} &= \sup \circ P_f(\widehat{p}_n) \circ \chi. \end{aligned}$$

It is now easy to show, by induction on n, that $\widehat{p}_n = tr_n \widehat{p}_m$ for all $m > n$. For $n = 0$, both sides of the equation become the constant map $\perp$. Supposing the equation holds for a fixed n and any $m > n$, then for $n + 1$ and any $m > n$ we have

$$\begin{aligned} \widehat{p}_{n+1} &= \sup\ P_f(\widehat{p}_n)\, \chi \\ &= \sup\ P_f(tr_n\, \widehat{p}_m)\, \chi \\ &= \sup P_f(tr_n) \sup^{-1}\ \sup\ P_f(\widehat{p}_m)\, \chi \\ &= tr_{n+1}\, \widehat{p}_{m+1}. \end{aligned}$$

We leave to the reader the verification that $\widehat{p}$ is the unique P_f-coalgebra morphism from X to M. $\square$

Theorem 2.4 *If fixed points exist in* C *for all* P_f *(with* f *pointed), then* C *has M-types .*

Proof. Let $f\colon B \longrightarrow A$ be a map. We freely add a point to the signature represented by f, by considering the composite

$$f_\perp\colon B \xrightarrow{f} A \rightarrowtail^{i} A+1 \tag{9}$$

(with the point $j = \perp\colon 1 \longrightarrow A+1$). Notice that the obvious pullback

$$\begin{tikzcd} B \arrow[r, "\mathsf{id}"] \arrow[d, "f"'] & B \arrow[d, "f_\perp"] \\ A \arrow[r, tail, "i"'] & A+1 \end{tikzcd}$$

determines, by (3), a (monic) natural transformation $i_!\colon P_f \longrightarrow P_{f_\perp}$; hence, by Proposition 2.2, the functor $(i_!)_*\colon P_f-\mathsf{coalg} \longrightarrow P_{f_\perp}-\mathsf{coalg}$ has a right adjoint. Now observe that $P_{f_\perp}$ has a fixed point, by assumption, hence a final coalgebra by Lemma 2.3. This will be preserved by the right adjoint of $(i_!)_*$, hence P_f has a final coalgebra. $\square$

This proof gives a categorical counterpart of the standard set-theoretic construction: add a dummy constant to the signature, build infinite trees by sequences of approximations, then select the actual M-type by taking those infinite trees which involve only term constructors from the original signature. This last passage is performed by the coreflection functor of Proposition 2.2, since branching elements are trees in the M-type of $f_\perp$ whose root is not $\perp$, and coherent ones are trees with no occurrence of $\perp$ at any point.

From this last theorem, we readily deduce the following result, first pointed out to us by Abbott, Altenkirch and Ghani (this was, in fact, one of the motivations for looking at this topic). Here, we call *predicative topos* a locally cartesian closed pretopos with W-types.

Corollary 2.5 *[1] Every predicative topos is a coinductive pretopos.*

Proof. Every predicative topos C has a natural number object, namely the W-type associated to the left inclusion $\mathsf{inl}\colon 1 \longrightarrow 1+1$. Since the W-type associated to a (pointed) map f is a fixed point for P_f, C also has all M-types by the previous theorem. $\square$

Because, as pointed out in [16], an elementary topos with a natural number object has all W-types, so is a predicative topos, this also establishes (in a very roundabout fashion):

Corollary 2.6 *[11] Any elementary topos with a natural number object has M-types.*

Together, these corollaries provide us with a substantial class of examples of coinductive pretoposes. It is still an open question whether there is any relevant examples of a coinductive pretopos that is not a predicative topos.

Although Theorem 2.4 is clearly helpful in proving that certain categories have M-types, it is even more so, when combined with the following observation. Given any monic morphism $\alpha\colon P_f X \longrightarrow X$, we can think of it as identifying branching elements in X, giving at the same time an explicit description of their children. We can therefore form the chain $(X_n)_{n>0}$ of "elements in X with n generations of children". The intersection of the X_n (the object of coherent elements in X) will then be a P_f-coalgebra; in fact, a fixed point for P_f.

Proposition 2.7 *Let C be a locally cartesian closed pretopos with a natural number object, and $f\colon B \longrightarrow A$ a map in it. Then, any prefixed point $\gamma\colon P_f X \longrightarrow X$ (i.e. an algebra whose structure map is monic) has a subalgebra that is a fixed point.*

Proof. Any prefixed point $\alpha\colon P_f X \longrightarrow X$ can be seen as a P_f-proto-coalgebra

$$X \xrightarrow{\mathsf{id}} X \xleftarrow{\alpha} P_f X.$$

Its coreflection $Coh(\mathsf{id}, \alpha)$, as defined in Proposition 2.1, is a P_f-coalgebra $\gamma\colon Y \longrightarrow P_f Y$ (in fact, the largest) fitting in the following commutative square:

$$\begin{tikzcd} Y \arrow[r, tail, "i"] \arrow[d, "\gamma"'] & X \\ P_f Y \arrow[r, tail, "P_f i"'] & P_f X. \arrow[u, tail, "\alpha"'] \end{tikzcd}$$

Now, consider the image under $I\colon P_f-\mathsf{coalg} \longrightarrow P_f - \mathsf{prtclg}$ of the coalgebra $P_f(\gamma)\colon P_f Y \longrightarrow P_f^2 Y$. The morphism of proto-coalgebras

$$\begin{tikzcd} P_f Y \arrow[r, "P_f \gamma"] \arrow[d, "\alpha P_f i"'] & P_f^2 Y \arrow[d, "\alpha P_f(\alpha) P_f^2 i"] & P_f^2 Y \arrow[l, tail, "\mathsf{id}"'] \arrow[d, "P_f(\alpha) P_f^2 i"] \\ X \arrow[r, "\mathsf{id}"'] & X & P_f X \arrow[l, tail, "\alpha"] \end{tikzcd}$$

transposes through $I \dashv Coh$ to a morphism $\phi\colon (P_f Y, P_f \gamma) \longrightarrow (Y, \gamma)$, which is a right inverse of $\gamma\colon (Y, \gamma) \longrightarrow (P_f Y, P_f \gamma)$ by the universal property of (Y, γ). Hence, we have $\gamma\phi = P_f(\phi\gamma) = \mathsf{id}$, proving that γ and ϕ are mutually inverse. $\square$

Putting together Theorem 2.4 and Proposition 2.7, we get at once the following:

Corollary 2.8 *If in a locally cartesian closed pretopos* C *with a natural number object we have prefixed points for every polynomial functor, then* C *has M-types.*

As an application of the techniques in this section, we present the following result, which is to be compared with the one by Santocanale in [19]. An immediate corollary of his Theorem 4.5 is that M-types exist in every locally cartesian closed pretopos with with a natural number object, for maps of the form $f: B \longrightarrow A$ where A is a finite sum of copies of 1. Notice that such an object A has *decidable equality*, i.e. the diagonal $\Delta: A \longrightarrow A \times A$ has a complement in the subobject lattice of $A \times A$. We extend the statement above to *all* maps whose codomain has decidable equality.

Proposition 2.9 *Let* C *be a locally cartesian closed pretopos with a natural number object, and* $f: B \longrightarrow A$ *a morphism in* C. *If* A *has decidable equality, then the M-type for* f *exists.*

Proof. Without loss of generality, we may assume that f is pointed; in fact, if we replace A by $A_\perp = A + 1$ and f by $f_\perp$ as in (9), then $A_\perp$ also has decidable equality, and the existence of an M-type for the composite $f_\perp$ implies that of an M-type for f (see the proof of Theorem 2.4). Then, by Proposition 2.7 and Lemma 2.3, it is enough to show that P_f has a prefixed point.

Let S be the object of all finite sequences of the form

$$\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle$$

where $f(b_i) = a_i$ for all $i < n$. (Like paths in a coalgebra, this object S can be constructed using the internal logic of C.) Now, let V be the object of all decidable subobjects of S (these can be considered as functions $S \longrightarrow 1 + 1$). Define the map $m: P_f V \longrightarrow V$ taking a pair $(a, t: B_a \longrightarrow V)$ to the subobject P of S defined by the following clauses:

1. $\langle a_0 \rangle \in P$ iff $a_0 = a$.
2. $\langle a_0, b_0 \rangle * \sigma \in P$ iff $a_0 = a$ and $\sigma \in t(b_0)$.

(Here, $*$ is the symbol for concatenation.) P is obviously decidable, so m is well-defined. To see that it is monic, suppose $P = m(a,t)$ and $P' = m(a', t')$ are equal. Then,

$$\langle a \rangle \in P \Longrightarrow \langle a \rangle \in P' \Longrightarrow a = a',$$

and, for every $b \in B_a$ and $\sigma \in S$,

$$\begin{aligned} \sigma \in t(b) &\iff \langle a, b \rangle * \sigma \in P \\ &\iff \langle a, b \rangle * \sigma \in P' \\ &\iff \sigma \in t'(b), \end{aligned}$$

so $t = t'$ and m is monic. Hence, (V, m) is a prefixed point for P_f and we are finished. $\square$

Remark 2.10 To obtain the M-type for f from V, one should first deduce a fixed point V' from it, as in Corollary 2.8. This means selecting the coherent elements of V, and these turn out to be those decidable subobjects P of S satisfying the following properties:

1. $\langle a \rangle \in P$ for a unique $a \in A$;
2. if $\langle a_0, b_0, \ldots, a_n \rangle \in P$, then there exists a unique a_{n+1} for any $b_n \in B_{a_n}$ such that $\langle a_0, b_0, \ldots, a_n, b_n, a_{n+1} \rangle \in P$.

Now, we should turn this fixed point into the M-type for f (as in Lemma 2.3), but this step is redundant, since our choice of V is such that V' already is the desired M-type.

3 M-types and slicing

We start by considering preservation of the coinductive pretopos structure under slicing. Let I be an object in a locally cartesian closed pretopos with a natural number object C. Then, it is well-known that the slice category C/I has again the same structure, and the reindexing functor $x^*\colon \mathsf{C}/I \longrightarrow \mathsf{C}/J$ for any map $x\colon J \longrightarrow I$ in C preserves it. So, we can focus on showing the existence of M-types in C/I and their preservation under reindexing.

We shall first concentrate on the existence, proving a "local existence" result, from which we derive a global statement. Then, we shall investigate the action of the reindexing functors.

Let us consider a map

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "\beta"'] & & A \arrow[dl, "\alpha"] \\ & I & \end{tikzcd} \tag{10}$$

in C/I. We shall denote by P_f the polynomial functor determined by f (or, more precisely, by Σf) in C, and by P_f^I the polynomial endofunctor determined in C/I. The functor $P_f\colon \mathsf{C} \longrightarrow \mathsf{C}$ can be extended to a functor $P_f\colon \mathsf{C} \longrightarrow \mathsf{C}/I$; in fact, $P_f X$ lives over A via the root map, and the composite $\alpha\rho\colon P_f X \longrightarrow I$ defines the desired extension.

Lemma 3.1 *There is an injective natural transformation of endofunctors on* C/I

$$i\colon P_f^I \longrightarrow P_f \Sigma_I.$$

Proof. Fix an object $\xi\colon X \longrightarrow I$ in C/I. Then, unfolding the definitions, we have

$$\begin{aligned} \Sigma_I P_f^I(X \xrightarrow{\xi} I) &= \Sigma_I \Sigma_\alpha (A \times_I X \xrightarrow{\alpha^*\xi} A)^{(B \xrightarrow{f} A)} \\ &= \Sigma_A (A \times_I X \xrightarrow{\alpha^*\xi} A)^{(B \xrightarrow{f} A)} \end{aligned}$$

and

$$P_f(\Sigma_I(X \xrightarrow{\xi} I)) = P_f(X) = \Sigma_A(A \times X \xrightarrow{p_1} A)^{(B \xrightarrow{f} A)},$$

where $\alpha\colon A \longrightarrow I$ makes A into an object over I. The pullback $A \times_I X$ fits into the equaliser diagram

$$A \times_I X \xrightarrow{\psi} A \times X \underset{\xi p_2}{\overset{\alpha p_1}{\rightrightarrows}} I,$$

hence ψ is a map over A, since $p_1 \psi = \alpha^* \xi$. Moreover, ψ is monic in C/A, and so is $\Sigma_A \psi^f$, thus giving the desired monomorphism, which is easily seen to live over I. Naturality is also readily checked. $\Box$

Remark 3.2 In the internal language of C, the map $i_{(X,\xi)}$ realises $\Sigma_I P_f^I(X,\xi)$ as the subobject of $P_f(X)$ consisting of those pairs $(a, t\colon f^{-1}(a) \longrightarrow X)$ such that for all $b \in f^{-1}(a)$ it holds $\alpha(a) = \xi t(b)$. In other words, i is the equaliser

$$\begin{tikzcd} \Sigma_I P_f^I(X,\xi) \arrow[r, "i"] & P_f X \arrow[rr, "P_f \xi"] \arrow[dr, "\rho"'] & & P_f I, \\ & & A \arrow[ur, "k"'] & \end{tikzcd}$$

where k maps $a \in A$ to $(a, \lambda b.\alpha(a))$.

If we think of the elements of A as term constructors, this is just saying that $P_f^I(X,\xi)$ applies each a just to elements of X which sit in the same fibre over I.

Using the i of Lemma 3.1, we can build an M-type for f in C/I, whenever M_f exists in C.

Theorem 3.3 *Let* C *be a locally cartesian closed pretopos with a natural number object and* I *an object in* C*. Consider a map* $f\colon B \longrightarrow A$ *over* I*, such that the functor* $P_f\colon \mathsf{C} \longrightarrow \mathsf{C}$ *has a final coalgebra. Then,* f *has an M-type in* C/I*.*

Proof. Let $\tau_f: M_f \longrightarrow P_f M_f$ be the M-type associated to f in C. M_f can be considered as an object over I, by taking the composite μ of the root map $\rho: M_f \longrightarrow A$ with the map $\alpha: A \longrightarrow I$, and (M_f, τ_f) then becomes the final $P_f \Sigma_I$-coalgebra, as one can easily check. The adjunction determined by the natural transformation $i: P_f^I \longrightarrow P_f \Sigma_I$ as in Proposition 2.2 takes the final $P_f \Sigma_I$-coalgebra (M_f, τ_f) to its coreflection M_f^I, and because right adjoints preserve limits, this is the final P_f^I-coalgebra. □

Remark 3.4 The injective natural transformation i of Lemma 3.1 identifies branching elements in $P_f \Sigma_I$ as those obtained by applying a term constructor in A to elements living in its same fibre over I, as discussed in Remark 3.2.

The coreflection process used to build M_f^I out of the M-type (M_f, τ_f), helps understanding which elements of the latter do actually belong to the former. Trees in M_f^I are coherent for the notion of branching determined by P_f^I, hence, not only the children of the root node live in its same fibre over I, but all the children of the children do too, and so on for any node in the tree. In other words, M_f^I consists of those trees in M_f all nodes of which live in the same fibre over I. As such, the object M_f^I can also be described as the equaliser

$$\begin{tikzcd} M_f^I \arrow[r, tail] & M_f \arrow[rr, "\langle \mathsf{id}, \alpha \rangle_!"] \arrow[dr, "\langle \mathsf{id}, \alpha\rho \rangle"'] & & M_{f \times I}, \\ & & M_f \times I \arrow[ur, "\chi"'] & \end{tikzcd}$$

where χ is the map coinductively defined as

$$\chi(\sup_a t, i) = \sup_{(a,i)}(\chi \langle t, i \rangle).$$

As an immediate consequence of Theorem 3.3, we get the following:

Corollary 3.5 *For any given object I of a coinductive pretopos C, the slice category C/I is again a coinductive pretopos.*

Remark 3.6 This last result could have also been proved directly by combining Corollary 2.8 and Lemma 3.1. However, the proof of Theorem 3.3 shows that the construction is actually simpler. More specifically, notice that, in this case, we obtain the M-type for a map f directly after the coreflection, and we do not need to add any dummy variable, nor to build sequences of approximations.

We now look at the reindexing functors. Suppose we are given a morphism $x: J \longrightarrow I$ in C, and consider the induced functor $x^*: \mathsf{C}/I \longrightarrow \mathsf{C}/J$, with left adjoint Σ_x (we shall denote by η the unit of this adjunction). Recall that the reindexing functors, in this case, do always preserve the cartesian closed structure of each slice.

Now, consider a morphism f as in (10), inducing a map $x^*f\colon x^*B \longrightarrow x^*A$ in C/J and a pullback square

$$\begin{array}{ccc} x^*A & \xrightarrow{\alpha^* x} & A \\ {\scriptstyle x^*\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ J & \xrightarrow[x]{} & I. \end{array}$$

By the Beck-Chevalley property for this square, and the fact that x^* preserves exponentials, we then have, for any object (X,ξ) in C/I,

$$\begin{aligned} x^*P^I_f(X,\xi) &= x^*\Sigma_\alpha(X\times_I A\longrightarrow A)^{(B\xrightarrow{f}A)} \\ &= \Sigma_{x^*\alpha}(\alpha^*x)^*(X\times_I A\longrightarrow A)^{(B\xrightarrow{f}A)} \\ &\cong \Sigma_{x^*\alpha}(\alpha^*x)^*(X\times_I A\longrightarrow A)^{(\alpha^*x)^*(B\xrightarrow{f}A)} \\ &= \Sigma_{x^*\alpha}(x^*X\times_J x^*A\longrightarrow x^*A)^{(x^*B\xrightarrow{x^*f}x^*A)} \\ &= P^J_{x^*f}(x^*X, x^*\xi). \end{aligned}$$

Let us write $\theta\colon P^J_{x^*f}x^*X\longrightarrow x^*P^I_f X$ for the isomorphism, and $\widehat{\theta}$ for its transpose under the adjunction $\Sigma_x \dashv x^*$.

Given any coalgebra $\gamma\colon X\longrightarrow P^I_f X$ in C/I, we can map it to a coalgebra

$$x^*X \xrightarrow{x^*\gamma} x^*P^I_f X \xrightarrow{\theta^{-1}} P^J_{x^*f}x^*X.$$

This determines a functor $G\colon P^I_f-\mathsf{coalg}\longrightarrow P^J_{x^*f}-\mathsf{coalg}$.

Now, given a coalgebra $\delta\colon Y\longrightarrow P^J_{x^*f}Y$ in C/J, a morphism $\phi\colon (Y,\delta)\longrightarrow G(X,\gamma)$ determines a commutative square

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[d, "\delta"'] & & x^*X \arrow[d, "x^*\gamma"] \\ P^J_{x^*f}Y \arrow[r, "P^J_{x^*f}\phi"'] & P^J_{x^*f}x^*X \arrow[r, "\theta"'] & x^*P^I_f X, \end{tikzcd}$$

which transposes under $\Sigma_x \dashv x^*$ to the square

$$\begin{tikzcd} \Sigma_x \arrow[rr, "\widehat{\phi}"] \arrow[d, "\Sigma_x\delta"'] & & X \arrow[d, "\gamma"] \\ \Sigma_x P^J_{x^*f}Y \arrow[r, "\Sigma_x P^J_{x^*f}\phi"'] & \Sigma_x P^J_{x^*f}x^*X \arrow[r, "\widehat{\theta}"'] & P^I_f X. \end{tikzcd}$$

By simple diagram chasing, one shows that

$$
\begin{aligned}
\gamma\widehat{\phi} &= \widehat{\theta}\,\Sigma_x P^J_{x^*f}(\phi)\,\Sigma_x(\delta) \\
&= \widehat{\theta}\,\Sigma_x P^J_{x^*f}x^*(\widehat{\phi})\,\Sigma_x P^J_{x^*f}(\eta)\,\Sigma_x(\delta) \\
&= P^I_f(\widehat{\phi})\,\widehat{\theta}\,\Sigma_x P^J_{x^*f}(\eta)\,\Sigma_x(\delta);
\end{aligned}
$$

hence, the functor assigning to (Y, δ) the composite

$$
\Sigma_x Y \xrightarrow{\Sigma_x \delta} \Sigma_x P^J_{x^*f} Y \xrightarrow{\Sigma_x P^J_{x^*f}\eta} \Sigma_x P^J_{x^*f} x^* \Sigma_x Y \xrightarrow{\widehat{\theta}} \Sigma_x Y
$$

is a left adjoint to G.

In particular, this implies that the image of the M-type (M^I_f, τ_f) for f in C/I, along the right adjoint G is the final $P^J_{x^*f}$-coalgebra in C/J, i.e. $G(M^I_f) = x^* M^I_f$ is the M-type for x^*f. This proves the following.

Theorem 3.7 *For any map $x: J \longrightarrow I$ in a coinductive pretopos C, the reindexing functor $x^*: \mathsf{C}/I \longrightarrow \mathsf{C}/J$ preserves the structure of a coinductive pretopos.*

4 M-types and coalgebras

In this section, we turn our attention to the construction of categories of coalgebras for a cartesian comonad (G, ϵ, δ) (by *cartesian* we mean that G preserves finite limits). As for the slicing case, we already know that most of the structure of a coinductive pretopos is preserved by taking coalgebras for G:

Theorem 4.1 *If C is a locally cartesian closed pretopos with natural number object, then so is C_G for a cartesian comonad $G = (G, \epsilon, \delta)$ on C.*

Proof. Theorem 4.2.1 on page 173 of [12] gives us that C_G is cartesian, in fact locally cartesian closed, and that it has a natural number object. The two additional requirements of having finite disjoint sums and being exact are easily verified, since the forgetful functor $U: \mathsf{C}_G \longrightarrow \mathsf{C}$ creates finite limits. □

Given a morphism f of G-coalgebras, this determines a polynomial functor $P_f: \mathsf{C}_G \longrightarrow \mathsf{C}_G$, while its underlying map Uf determines the endofunctor P_{Uf} on C. The two are related as follows:

Proposition 4.2 *Let $f: (B, \beta) \longrightarrow (A, \alpha)$ be a map of G-coalgebras. Then, there* *is an injective natural transformation*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "P_{Uf}"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{i}{\rightarrowtail}"] & \mathsf{C} \arrow[d, "G"] \\ \mathsf{C}_G \arrow[r, "P_f"'] & \mathsf{C}_G, \end{tikzcd}$$

whose mate under the adjunction $U \dashv G$ we shall denote by

$$j\colon UP_f \longrightarrow P_{Uf}U\colon \mathsf{C}_G \longrightarrow \mathsf{C}. \tag{11}$$

In order to prove this result, we first need to introduce a couple of lemmas (the proof of the second can also be found in [12]).

Lemma 4.3 *Let C be a cartesian closed category and G be a cartesian comonad on it. Then, for any object X in C and any coalgebra Y, the exponential $(GX)^Y$ exists and is equal to $G(X^{UY})$.*

Proof. The forgetful functor $U\colon \mathsf{C}_G \longrightarrow \mathsf{C}$ is left adjoint to the cofree coalgebra functor $G\colon \mathsf{C} \longrightarrow \mathsf{C}_G$. Moreover, U preserves products because it creates them; hence, the following bijective correspondence:

$$\begin{array}{rcl} A & \longrightarrow & G(X^{UY}) \\ \hline UA & \longrightarrow & X^{UY} \\ \hline UA \times UY & \longrightarrow & X \\ \hline U(A \times Y) & \longrightarrow & X \\ \hline A \times Y & \longrightarrow & GX. \end{array}$$

$\square$

Lemma 4.4 *Every G-coalgebra (A, α) in C_G, determines a cartesian comonad G' on C/A and an isomorphism between $\mathsf{C}_G/(A, \alpha)$ and $(\mathsf{C}/A)_{G'}$ which respects the forgetful functors $U\colon \mathsf{C}_G/(A, \alpha) \longrightarrow \mathsf{C}/A$ and $U'\colon (\mathsf{C}/A)_{G'} \longrightarrow \mathsf{C}/A$.*

Proof. The comonad G' is computed on an object $t\colon X \longrightarrow A$ in C/A by taking the following pullback:

$$\begin{tikzcd} G'X \arrow[r, tail] \arrow[d, "G't"'] & GX \arrow[d, "Gt"] \\ A \arrow[r, tail, "\alpha"'] & GA. \end{tikzcd} \tag{12}$$

Then, there is a natural one-to-one correspondence between the dotted arrows over A and GA in

$$\begin{tikzcd}
X \arrow[dr, dotted, "r"] \arrow[drr, dotted, "s"] \arrow[ddr, bend right, "t"'] & & \\
& G'X \arrow[r, tail] \arrow[d, "G't"] & GX \arrow[d, "Gt"] \\
& A \arrow[r, tail, "\alpha"'] & GA.
\end{tikzcd}$$

It is now easy to show that if r is a G'-coalgebra, then the corresponding s is a G-coalgebra, and vice versa. Hence, the isomorphism. Moreover,

$$U(t\colon(X,s)\longrightarrow(A,\alpha)) = t = U'(t,r);$$

therefore, the action of the forgetful functors is respected.

Note that both horizontal arrows in the pullback (12) are monic, because ϵ_A is a retraction of the G-coalgebra α. $\square$

Proof of Proposition 4.2. Recall that $P_f(GX)$ is the source of the exponential $(A \times GX \longrightarrow A)^f$ in the category $\mathsf{C}_G/(A,\alpha)$. However, this can be more easily computed in the category of G'-coalgebras. First of all, since G preserves products, the object $A \times GX \longrightarrow A$ corresponds, through the isomorphism of Lemma 4.4, to $G'(A \times X \longrightarrow A)$. Then, using Lemma 4.3, the exponential takes the form $G'((A \times X \longrightarrow A)^{Uf})$, and fits in the following pullback square, which is an instance of (12):

$$\begin{tikzcd}
G'((A \times X \to A)^{Uf}) \arrow[r, tail, "i_X"] \arrow[d] & G((A \times X \to A)^{Uf}) \arrow[d] \\
A \arrow[r, tail, "\alpha"'] & GA.
\end{tikzcd}$$

Now notice that the top-right entry of the diagram is exactly $GP_{Uf}(X)$, hence the map i therein defines the X-th component of the desired natural transformation. $\square$

We are now ready to formulate a local existence result for M-types in categories of coalgebras.

Theorem 4.5 *Let $f\colon(B,\beta)\longrightarrow(A,\alpha)$ be a map of G-coalgebras. If the underlying map Uf has an M-type in C, then the functor $P_f\colon \mathsf{C}_G \longrightarrow \mathsf{C}_G$ has a final coalgebra in C_G.*

Proof. The natural transformation i of Proposition 4.2 allows us to turn any P_{Uf}-coalgebra into a P_f-proto-coalgebra. In particular, for the M-type $\tau\colon M =$

$M_{Uf} \longrightarrow P_{Uf}M$ in C, we get the proto-coalgebra

$$GM \xrightarrow{G\tau} GP_{Uf}M \xleftarrow{i_M} P_f GM,$$

whose coreflection $Coh(M) = Coh(G\tau, i_M)$ is final in P_f–coalg. To see this, consider another coalgebra (X, γ) (therefore, X is a G-coalgebra, and $\gamma\colon X \longrightarrow P_f X$ is a homomorphism of G-coalgebras). To give a morphism of P_f-coalgebras from (X, γ) to $Coh(M)$ is the same, through $I \dashv Coh$, as giving a map $\psi\colon X \longrightarrow GM$ in C_G which is a morphism of P_f-proto-coalgebras, i.e. that makes the following commute:

$$\begin{tikzcd}
X \arrow[rr, "\gamma"] \arrow[d, "\psi"'] & & P_f X \arrow[d, "P_f\psi"] \\
GM \arrow[r, "G\tau"'] & GP_{Uf}M & P_f GM. \arrow[l, "i_M"]
\end{tikzcd}$$

This transposes, through $U \dashv G$, to the following diagram in C, where j is the natural transformation defined in (11):

$$\begin{tikzcd}
UX \arrow[r, "U\gamma"] \arrow[d, "\widehat{\psi}"'] & UP_f X \arrow[r, "j_X"] & P_{Uf}UX \arrow[d, "P_{Uf}\widehat{\psi}"] \\
M \arrow[rr, "\tau"'] & & P_{Uf}M.
\end{tikzcd}$$

But finality of M implies that there is precisely one such $\widehat{\psi}$ for any coalgebra (X, γ), hence finality is proved. $\square$

Corollary 4.6 *If C is a coinductive pretopos and $G = (G, \epsilon, \delta)$ is a cartesian comonad on C, then the category C_G of (Eilenberg-Moore) coalgebras for G is again a coinductive pretopos.*

Remark 4.7 Notice that Corollary 4.6 could also be deduced by Corollary 2.8, in conjunction with Proposition 4.2. However, analogously to what happens in the slicing case, Theorem 4.5 shows that we do not need to perform the whole construction, since the coreflection step gives directly the final coalgebra.

Remark 4.8 In particular, this result shows stability of coinductive pretoposes under the glueing construction, since this is a special case of taking coalgebras for a cartesian comonad (see [23]).

5 M-types and sheaves

In this section, we turn our attention to formation of sheaves for an internal site in a coinductive pretopos. Our aim is to show that the resulting category is again a coinductive pretopos. The structure of the proof mimics the one for W-types in [16], but we now make use of the finality of M-types and, implicitly, of the language of paths.

We start by considering an internal category $\mathbb{C}$ in a coinductive pretopos E, with object of objects $\mathbb{C}_0$ and domain and codomain morphisms indicated by $d_0, d_1 : \mathbb{C}_1 \longrightarrow \mathbb{C}_0$, respectively. We shall assume $\mathbb{C}$ to have pullbacks and to come equipped with a collection T of covering families, satisfying the axioms of a site [15, Def. 2 p.111].

Notice that the category of presheaves $\mathrm{Psh}(\mathbb{C})$ is then the category of coalgebras for a cartesian comonad on the slice category $\mathsf{E}/\mathbb{C}_0$ (see for instance [12, Example A.4.2.4 (b)]). By the results of the previous sections, we get at once

Proposition 5.1 *The presheaf category Psh($\mathbb{C}$) is a coinductive pretopos.*

Before proceeding to the sheaf case, we find it useful to have a closer look at the construction of M-types for categories of presheaves. This will provide a concrete description of the M-types, and at the same time set some notation about polynomial functors in presheaf (and sheaf) categories.

5.1 M-types in presheaves

First of all, we need to introduce the functor $|\cdot| : \mathrm{Psh}(\mathbb{C}) \longrightarrow \mathsf{E}$ which takes a presheaf $\mathcal{A}$ to its "underlying object" $|\mathcal{A}| = \{(a, C) \mid a \in \mathcal{A}(C)\}$. This is just the composite of the forgetful functor $U : \mathrm{Psh}(\mathbb{C}) \longrightarrow \mathsf{E}/\mathbb{C}_0$ with $\Sigma_{\mathbb{C}_0} : \mathsf{E}/\mathbb{C}_0 \longrightarrow \mathsf{E}$.

Let $f : \mathcal{B} \longrightarrow \mathcal{A}$ be a morphism of presheaves. Then, the "fibre" $\mathcal{B}_a$ of f over $a \in \mathcal{A}(C)$ for an object C in $\mathbb{C}$ is a presheaf, whose action on D is described in the internal language of E as

$$\mathcal{B}_a(D) = \{(\beta, b) \mid \beta : D \longrightarrow C, a \cdot \beta = f(b)\}$$

and restriction along a morphism $\delta : D' \longrightarrow D$ is defined as

$$(\beta, b) \cdot \delta = (\beta\delta, b \cdot \delta).$$

The image $P_f(X)$ of a presheaf X is defined on an object C of $\mathbb{C}$ as

$$P_f(X)(C) = \{(a, t) \mid a \in \mathcal{A}(C),\, t : \mathcal{B}_a \longrightarrow X\}, \tag{13}$$

where t is a presheaf morphism. Restriction along a morphism $\alpha : C' \longrightarrow C$ is defined as

$$(a, t) \cdot \alpha = (a \cdot \alpha,\, \alpha^*(t)),$$

where $\alpha^*(t)(\beta, b) = t(\alpha\beta, b)$.

But the presheaf morphism f also induces a map

$$f' \colon \Sigma_{(a,C)\in|\mathcal{A}|}|\mathcal{B}_a| \longrightarrow |\mathcal{A}|$$

whose fibre over (a,C) is precisely $|\mathcal{B}_a|$. This induces a polynomial endofunctor on E, which acts on an object Y as

$$P_{f'}Y = \{((a,C),t) \mid a \in \mathcal{A}(C),\ t \colon |\mathcal{B}_a| \longrightarrow Y\}.$$

Notice that we can always think of $P_{f'}Y$ as the underlying object of a presheaf, whose section over C are those elements of $P_{f'}Y$ of the form $((a,C),t)$. Given a morphism $\alpha \colon C' \longrightarrow C$ for which $a \cdot \alpha = a'$, this induces a map $\widetilde{\alpha} \colon |\mathcal{B}_{a'}| \longrightarrow |\mathcal{B}_a|$. Restriction along α is then defined by $((a,C),t)\cdot\alpha = ((a',C'),t\widetilde{\alpha})$.

In particular, f' induces an endofunctor $F \colon \mathrm{Psh}(\mathbb{C}) \longrightarrow \mathrm{Psh}(\mathbb{C})$, which verifies the equation

$$|FX| = P_{f'}|X| = \{((a,C),t) \mid a \in \mathcal{A}(C),\ t \colon |\mathcal{B}_a| \longrightarrow |X|\} \tag{14}$$

for any presheaf X on $\mathbb{C}$. Because each morphism of presheaves $\mathcal{B}_a \longrightarrow X$ induces a map on the underlying objects, it is clear by (13) and (14) that P_f is a subfunctor of F. We shall denote by m the corresponding injective natural transformation.

By Proposition 2.2, we can then determine an M-type for f by giving a final coalgebra for the functor F and taking its coreflection in $P_f-\mathsf{coalg}$. The latter is readily obtained from the M-type $(M_{f'}, \tau_{f'})$ for f' in E. In fact, as we saw above, the object $P_{f'}M_{f'}$ has a presheaf structure, and we can translate it on $M_{f'}$ via the isomorphism $\tau_{f'}$, which then becomes a presheaf morphism. To see that it is the final F-coalgebra, it is now enough to observe that, given any other F-coalgebra $\gamma \colon X \longrightarrow FX$, finality of $M_{f'}$ determines a unique morphism of $P_{f'}$-coalgebras

$$\begin{tikzcd}
{|X|} \arrow[r, dotted] \arrow[d, "|\gamma|"'] & M_{f'} \arrow[d, "\tau_{f'}"] \\
P_{f'}|X| = |FX| \arrow[r, dotted] & P_{f'}M_{f'},
\end{tikzcd}$$

which is easily seen to be a morphism of presheaves.

We have just given a proof of the following local existence result:

Theorem 5.2 *Consider a map $f \colon \mathcal{B} \longrightarrow \mathcal{A}$ in Psh($\mathbb{C}$). If the induced map f' has an M-type in E, then f has an M-type in Psh($\mathbb{C}$).*

Remark 5.3 The natural transformation m, in this case, selects as branching only those trees $\sup_{(a,C)} t$ in $M_{f'}$ for which the map $t \colon |\mathcal{B}_a| \longrightarrow M_{f'}$ is the underlying map of a presheaf morphism. The reader who is familiar with the language of [16], will recognise that trees coherent for m are those called *natural* there.

5.2 M-types in sheaves

Once again, it is already known that the category of sheaves for an internal site in a locally cartesian closed pretopos with a natural number object has again the same structure, so, as in the previous cases, all we need to show is that Sh($\mathbb{C}$) has M-types. Analogously to what is done in [16], we get the result by showing the following.

Proposition 5.4 *Let $f\colon\mathcal{B}\longrightarrow\mathcal{A}$ be a morphism of sheaves, and $\mathcal{M}$ the corresponding M-type in Psh($\mathbb{C}$). Then, $\mathcal{M}$ is a sheaf.*

Proof. The proof will work as follows. First, we shall define a presheaf $\mathcal{M}^+$, which is the separated presheaf associated to $\mathcal{M}$ (see [15]). Then, we will show that $\mathcal{M}^+$ is a P_f-coalgebra with mediating morphism $\widehat{m}\colon\mathcal{M}^+\longrightarrow\mathcal{M}$; moreover, the unit $\eta\colon\mathcal{M}\longrightarrow\mathcal{M}^+$ is a coalgebra homomorphism. This will imply that $\widehat{m}\eta = \mathsf{id}_{\mathcal{M}}$, hence η is mono, i.e. $\mathcal{M}$ is separated. Existence of the glueing of a compatible family will be shown to be determined by $\widehat{m}$, and this will prove $\mathcal{M}$ to be a sheaf.

The presheaf $\mathcal{M}^+$ is defined on an object C in $\mathbb{C}$ as

$$\mathcal{M}^+(C) = \left\{ (\{C_i \xrightarrow{c_i} C\},\, T_i \in \mathcal{M}(C_i)) \;\middle|\; \begin{array}{l} \{c_i\} \text{ is a covering family and} \\ \text{the } T_i\text{'s are compatible} \end{array} \right\} / \sim$$

where

$$(\{C_i \xrightarrow{c_i} C\}, T_i) \sim (\{C'_j \xrightarrow{c'_j} C\}, S_j)$$

if there is a common refinement of $\{c_i\}$ and $\{c'_j\}$ on which the T_i's and the S_j's agree. More precisely, we require the existence of a covering family $\{c''_k\colon C''_k\longrightarrow C\}$ such that every c''_k factors as $c_i\sigma_{ki}$ and $c'_j\tau_{kj}$ for opportune maps σ_{ki} and τ_{kj}, with $T_i\cdot\sigma_{ki} = S_j\cdot\tau_{kj}$. We shall indicate the equivalence class of a pair $(\{c_i\}, T_i)$ by enclosing it in square brackets.

By the stability axiom for a site, a given morphism $\beta\colon D\longrightarrow C$ in $\mathbb{C}$ induces, via pullback, a restriction of a covering family $\{c_i\}$ over C to a covering family $\{d_i\}$ as below:

$$\begin{tikzcd} D_i \arrow[r, "\beta_i"] \arrow[d, "d_i"'] & C_i \arrow[d, "c_i"] \\ D \arrow[r, "\beta"'] & C. \end{tikzcd} \tag{15}$$

We can then define the restriction morphism of $\mathcal{M}^+$ by letting

$$[\{c_i\}, T_i]\cdot\beta = [\{d_i\}, T_i\cdot\beta_i]. \tag{16}$$

We now need to show that $-\cdot\beta$ is a well-defined map and that it satisfies the functoriality properties, thus making $\mathcal{M}^+$ into a presheaf. We shall go through this calculation in some detail, in order to illustrate the kind of arguments used. Analogous forms of reasoning will occur several times in this proof. All the successive details will then be omitted.

Consider the two related pairs

$$(\{c_i\}, T_i) \sim (\{c'_j\}, S_j).$$

It is clearly enough to consider the case where the covering family $\{c'_j\}$ is a refinement of the family $\{c_i\}$, say via maps σ_{ji}, in such a way that for each i there is a j with $T_i \cdot \sigma_{ji} = S_j$. Pulling back the families $\{c_i\}$ and $\{c'_j\}$ along β, we get covering families $\{d_i\}$ and $\{d'_j\}$ as below:

$$\begin{tikzcd} D_i \arrow[r, "\beta_i"] \arrow[d, "d_i"'] & C_i \arrow[d, "c_i"] \\ D \arrow[r, "\beta"'] & C \end{tikzcd} \qquad \begin{tikzcd} D'_j \arrow[r, "\beta'_j"] \arrow[d, "d'_j"'] & C'_j \arrow[d, "c'_j"] \\ D \arrow[r, "\beta"'] & C. \end{tikzcd}$$

The maps σ_{ji} pull back to maps τ_{ji} such that $d'_j = d_i\tau_{ji}$, hence $\{d'_j\}$ is a refinement of $\{d_i\}$. Moreover, we have

$$(T_i \cdot \beta_i) \cdot \tau_{ji} = T_i \cdot (d_i\tau_{ji}) = T_i \cdot (\sigma_{ji}\beta'_j) = (T_i \cdot \sigma_{ji}) \cdot \beta'_j = S_j \cdot \beta'_j,$$

whence $[\{d_i\}, T_i \cdot \beta_i] = [\{d'_j\}, S_j \cdot \beta'_j]$. This proves that restriction along β is well-defined.

Now we show that this restriction operation defines a presheaf structure on $\mathcal{M}^+$. Preservation of identities is easily verified. As for composition, consider an element $[\{c_i\}, T_i]$ in $\mathcal{M}^+(C)$ and composable maps

$$E \xrightarrow{\delta} D \xrightarrow{\beta} C.$$

Then, we want to show that

$$(\{e_i\}, T_i \cdot \gamma_i) \sim (\{e'_i\}, T_i \cdot \beta_i\delta_i), \tag{17}$$

where the maps e_i, γ_i, e'_i, β_i and δ_i arise as shown by the pullback diagrams below:

$$\begin{tikzcd} E_i \arrow[r, "\gamma_i"] \arrow[d, "e_i"'] & C_i \arrow[d, "c_i"] \\ E \arrow[r, "\beta\delta"'] & C \end{tikzcd} \qquad \begin{tikzcd} E'_i \arrow[r, "\delta_i"] \arrow[d, "e'_i"'] & D_i \arrow[r, "\beta_i"] \arrow[d, "d_i"] & C_i \arrow[d, "c_i"] \\ E \arrow[r, "\delta"'] & D \arrow[r, "\beta"'] & C. \end{tikzcd}$$

By pullback-pasting, there are obvious induced isomorphisms $\sigma_i : E'_i \longrightarrow E_i$ such that $\gamma_i\sigma_i = \beta_i\delta_i$ and $e_i\sigma_i = e'_i$. Therefore, the family $\{e'_i\}$ is a refinement of $\{e_i\}$; moreover,

$$T_i \cdot \beta_i\delta_i = T_i \cdot \gamma_i\sigma_i,$$

which proves (17).

So, $\mathcal{M}^+$ is a presheaf, and there is an obvious presheaf morphism $\eta: \mathcal{M} \longrightarrow \mathcal{M}^+$, whose component on an object C maps a tree $T \in \mathcal{M}(C)$ to $[\{\mathsf{id}_C\}, T]$. Equation (16) immediately implies that η is natural. We shall soon show that it is also a coalgebra morphism, but first we have to equip $\mathcal{M}^+$ with a P_f-coalgebra structure.

For this, we need a presheaf morphism $m: \mathcal{M}^+ \longrightarrow P_f \mathcal{M}^+$. This will associate to an element $F = [\{c_i: C_i \longrightarrow C\}, T_i]$ in $\mathcal{M}^+(C)$ a pair (a, t) with a in $\mathcal{A}(C)$ and $t: \mathcal{B}_a \longrightarrow \mathcal{M}^+$ a presheaf morphism. Suppose for each i that $T_i = \sup_{(a_i, C_i)} t_i$, with $t_i: \mathcal{B}_{a_i} \longrightarrow \mathcal{M}$. Then, since the T_i's form a compatible family, so do the a_i's, thus determining a unique amalgamation a such that $a \cdot c_i = a_i$, because $\mathcal{A}$ is a sheaf. In order to define the action of t, consider an element (β, b) in $\mathcal{B}_a(D)$ (so that $\beta: D \longrightarrow C$ and $a \cdot \beta = f(b)$). The compatible family $\{c_i\}$ restricts along β to a compatible family $\{d_i\}$ as in (15), and we define

$$t(\beta, b) = [\{d_i\}, t_i(\beta_i, b \cdot d_i)].$$

By a standard calculation (of the kind shown above), it is easy to show that t defines a presheaf morphism and that the definition of both a and t does not depend on the choice of the representative for F. Hence, the morphism m is well-defined, and again by similar arguments, we can then show that it defines a presheaf morphism.

So, we now have a coalgebra m on the presheaf $\mathcal{M}^+$, for which it is easily checked that the presheaf morphism η defined above is in fact a coalgebra homomorphism. Moreover, there is a unique map of coalgebras $\widehat{m}: \mathcal{M}^+ \longrightarrow \mathcal{M}$, determined by the finality of $\mathcal{M}$. The composite $\widehat{m}\eta$ is then a coalgebra morphism from $\mathcal{M}$ to itself, therefore it must be the identity. In particular, this implies that η is monic, i.e. $\mathcal{M}$ is separated. To complete the proof of the statement, we now need to show that every compatible family has an amalgamation. In fact, it turns out that the glueing of a family F is determined by its image under $\widehat{m}$.

More precisely, given a covering family $\{c_i: C_i \longrightarrow C\}$ over C and a matching family of trees $T_i \in \mathcal{M}(C_i)$, we want to show that $T = \widehat{m}([\{c_i\}, T_i])$ is the amalgamation of the T_i's; that is, $T \cdot c_i = T_i$ for all i. To this end, fix an index i_0, and restrict the covering family $\{c_i\}$ along the map c_{i_0} as below:

$$\begin{tikzcd} D_i \arrow[r, "c_i'"] \arrow[d, "d_i"'] & C_i \arrow[d, "c_i"] \\ C_{i_0} \arrow[r, "c_{i_0}"'] & C. \end{tikzcd} \tag{18}$$

We then have that

$$\begin{aligned} \widehat{m}([\{c_i\}, T_i]) \cdot c_{i_0} &= \widehat{m}([\{c_i\}, T_i] \cdot c_{i_0}) \\ &= \widehat{m}([\{d_i\}, T_i \cdot c_i']) \end{aligned}$$

But clearly, the covering family $\{d_i\}$ also forms a refinement of $\{\mathsf{id}_{C_{i_0}}\}$, and since the T_i's are a matching family (hence, in particular, for the pullbacks in (18) we have that $T_{i_0} \cdot d_i = T_i \cdot c_i'$), it follows at once that

$$(\{d_i\}, T_i \cdot c_i') \sim (\{\mathsf{id}_{C_{i_0}}\}, T_{i_0}),$$

whence, because $\widehat{m}\eta = \mathsf{id}$,

$$T \cdot c_{i_0} = \widehat{m}([\{c_i\}, T_i]) \cdot c_{i_0} = \widehat{m}([\{\mathsf{id}_{C_{i_0}}\}, T_{i_0}]) = \widehat{m}\eta(T_{i_0}) = T_{i_0}.$$

$\square$

Remark 5.5 Notice that this is the only proof we have given where we make use of the exactness of coinductive pretoposes. In any other argument regularity would have been enough.

Piecing this result together with the remarks at the beginning of this section, we get at once the following:

Theorem 5.6 *Let* E *be a coinductive pretopos and* $(\mathbb{C}, T)$ *an internal site in* E*. Then, the category Sh(*$\mathbb{C}, T$*) of sheaves over* $\mathbb{C}$ *in* E *is again a coinductive pretopos.*

6 Concluding remarks

The original motivation for writing this paper was to set the categorical framework in which to work with non-well-founded trees. The natural continuation of this research is two-folded, with applications to set theory and type theory.

The reason to expect such relationships to exist is the well-established correspondence between categories with W-types, Martin-Löf type theory, and Aczel's constructive set theory (the formal system CZF) (see for example [16, 17, 2]). We suggest that this correspondence should have a counterpart in the non-well-founded world, between coinductive pretoposes, Martin-Löf type theory with coinductive types, and Aczel's non-well-founded set theory ($CZF^- + AFA$). This would provide the ground for successive research in various directions.

Corollary 2.5 reveals that there are links between the well-founded and the non-well-founded world. In particular, it shows that within Martin-Löf type theory we can use well-founded types to model certain coinductive types. An analogous connection arises from the work of Lindström [14], where it is shown how to construct non-well-founded sets in Martin-Löf type theory. Making the correspondence above clear, should bring the two approaches together.

Ultimately, we expect our result on sheaves to lead to independence results in non-well-founded set theory, and to Martin-Löf type theory with coinductive types.

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