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A Simulation Model for Passive Attitude Control Systems Used in CubeSats

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CHAPTER 1

Objective

The objective of this document is to describe the components of a simulation model for a nanosatellite's passive attitude control system, derive the related equations and provide a software architecture for the integration of such components.

A simulation of this system must capture the phenomena related to orbital motion, attitude dynamics and the interaction between the geomagnetic field and the magnetic materials used for attitude control, hence the components of such simulation are chosen to be independent models of each phenomena containing some physically motivated interface between them, allowing for the integration of the components to provide the final model.

CHAPTER 2

Geomagnetic Field Model

2.1 Introduction

The International Geomagnetic Reference Field (IGRF) is a standard mathematical description of the Earth's magnetic field widely used in studies of the magnetosphere. The model is developed and maintained by the International Association of Geomagnetism and Aeronomy (IAGA) and it's going to be the model used in this simulation.

The objective of this section is to derive a mathematical expression that defines a magnetic flux density field over a Cartesian space described in spherical coordinates, hence by the end of the derivation it must be possible to assign a vector, representing the local magnetic field, to every point of an orbit around the Earth.

2.2 Model Description

Assuming that there are no sources of magnetic field, that is, free currents, outside the Earth's surface, a scalar potential description of the magnetic field becomes feasible, since the field becomes irrotational and the existence of the potential field can be proved given the existence of a solution to Poisson's equation. Given the linearity of free-space, a well known solution to Poisson's equation can be found by the expansion of Green's function in terms of spherical harmonics, but since the source of magnetic potential, mathematically found by deriving the Poisson equation for the problem, cannot be fully known for the case of the geomagnetic field, a statistical approach is used to find the coefficients of the spherical harmonics expansion, based on measurement data, providing an approximated model for the geomagnetic field. The final equation for the model, described in [1], is:

$$V(r, \theta, \phi, t) = a \sum_{n=1}^{N} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} \left[g_n^m(t)\, cos\,(m\phi) + h_n^m(t)\, sin\,(m\phi)\right] P_n^m(cos\theta) \tag{2.1}$$

Where a is the Earth's radius, g_n^m,h_n^m are named Gauss Coefficients and P_n^m is the Schmidt Quasi-Normalized Associated Legendre Functions of Degree n and Order m, deeply described in [2].

Is this model, the convention used for the spherical coordinate system defines r as the radial distance from the center of the Earth, θ as the co-latitude and ϕ as the east longitude.

Based on the equation 2.1, it is possible to compute the magnetic flux density through the expression:

$$\vec{B} = -\nabla V \tag{2.2}$$

In this equation, the gradient operator is described in spherical coordinates, hence the components of the magnetic flux density vector are given by:

$$\begin{aligned}
B_r &= -\frac{\partial V}{\partial r} \\
B_\theta &= -\frac{1}{r}\frac{\partial V}{\partial \theta} \\
B_\phi &= -\frac{1}{rsin\theta}\frac{\partial V}{\partial \phi}
\end{aligned}$$

An explicit representation for the magnetic flux density vector with regard to it's basis vectors is given by:

$$\vec{B} = B_r\hat{r} + B_\theta\hat{\theta} + B_\phi\hat{\phi} \tag{2.3}$$

Where $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are unit vectors forming a basis for a curvilinear space.

Since vectors are invariant to coordinate transformations (see [3] for a detailed explanation), a direct consequence of the fact that physical quantities must not change when it's mathematical representation changes, the use of the spherical coordinates representation of the magnetic flux density field should not be a problem, as long as the basis vectors are properly defined.

A definition for the basis vectors can be constructed based on the relation between Cartesian coordinates and spherical coordinates, given by:

$$\begin{aligned}
x &= rsin\theta cos\phi \\
y &= rsin\theta sin\phi \\
z &= rcos\theta
\end{aligned}$$

To arrive at a basis all it takes is to differentiate the vector (x,y,z) with respect to (r,ϕ,θ), arriving at a Jacobian matrix. The Jacobian matrix represents the amount of deformation caused by a transformation from Cartesian to spherical coordinates, establishing a relation between the systems. By simple differentiation rules, one gets:

$$\begin{aligned}
\vec{R} &= (x, y, z) \\
\frac{\partial R}{\partial r} &= (sin\theta cos\phi, sin\theta sin\phi, cos\theta) \\
\frac{\partial R}{\partial \theta} &= (rcos\theta cos\phi, rcos\theta sin\phi, -rsin\theta) \\
\frac{\partial R}{\partial \phi} &= (-rsin\theta sin\phi, rsin\theta cos\phi, 0)
\end{aligned}$$

After normalization:

$$
\begin{aligned}
\hat{r} &= \frac{\partial R}{\partial r} = (sin\theta cos\phi, sin\theta sin\phi, cos\theta) \\
\hat{\theta} &= \frac{1}{r}\frac{\partial R}{\partial \theta} = (cos\theta cos\phi, cos\theta sin\phi, -sin\theta) \\
\hat{\phi} &= \frac{1}{rsin\theta}\frac{\partial R}{\partial \phi} = (-sin\phi, cos\phi, 0)
\end{aligned}
$$

By using such basis, it is possible to verify the relation:

$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} = B_r\hat{r} + B_\theta\hat{\theta} + B_\phi\hat{\phi} \tag{2.4}$$

And the spherical coordinates representation of vector field can be used directly in the computation of the magnetic flux density at every point of an orbit around the Earth.

CHAPTER 3

Orbital Motion Model

3.1 Introduction

Since most LEO satellite orbits are near-circular, for this analysis a circular orbit will be assumed, based on the orbital motion model used in [4].

The objective of this model is to associate every instant in time to a position contained in a feasible orbit around the Earth.

3.2 Model Description

The mathematical description of a circular orbit is defined by five parameters: three angles that define the orbital plane with respect to the xy plane, through successive rotations; a radius, defining the distance between the Earth's center to the satellite's center of mass; and an angle that defines the angular position of the satellite's center of mass in the orbit at an instant. Note that since the orbit is approximated as a circle, both the radius and the rotation of the orbital plane are constant, implying that only the angular position in the orbit changes with time.

The construction of the proposed representation of orbital motion can be realized by successive rotations of a circular orbit in the xy plane by a set of angles α, β, γ. Hence any point P contained in the orbit can be computed using the relation:

$$P = R_{\alpha\beta\gamma} P_0 \tag{3.1}$$

Where:

$$R_{\alpha\beta\gamma} = R_\alpha R_\beta R_\gamma \tag{3.2}$$

$$R_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & -sin(\alpha) \\ 0 & sin(\alpha) & cos(\alpha) \end{bmatrix} \tag{3.3}$$

$$R_\beta = \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & & cos(\beta) \end{bmatrix} \tag{3.4}$$

$$R_\gamma = \begin{bmatrix} cos(\gamma) & -sin(\gamma & 0 \\ sin(\gamma) & cos(\gamma) &) \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

$$\tag{3.6}$$

Note that matrix multiplication is not a commutative operation, hence the order of rotations, α-β-γ, must be respected.

Since $P_0(\theta)$ is a point contained in a circular orbit defined in the xy plane, a convenient parameterization for the curve is:

$$P_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rcos\theta \\ rsin\theta \\ 0 \end{bmatrix} \tag{3.7}$$

Where r is the constant radius, and theta is the angular position parameter.

Finally, for the link between the orbital position and time to be established, an expression for the angular position as a function of time must be defined. Assuming that the there is no energy dissipation in the orbital motion, implying that the kinetic energy of the satellite's center of mass is constant, then the angular velocity of such motion is constant and the angular position evolves linearly in time. Mathematically:

$$\theta(t) = \omega_o t = \sqrt{\frac{Gm}{(R_E + h)^3}}\, t \tag{3.8}$$

Where G is the gravitational constant, m is Earth's mass, R_E is Earth's surface average radius and h is the altitude, using the Earth's surface average radius as reference, as established in [4].

CHAPTER 4

Attitude Representation

4.1 Introduction

Before building a concise attitude representation, there are two main topics that must discussed: the frames of reference that are going to be used in the computations and the chosen parameterization for spatial rotations.

Specifying an orientation in space only makes sense in a relative perspective, hence the objetive of this section is to define a procedure through which the numbers resulting from attitude computation can be traced back to it's physical interpretation.

4.2 Reference Frames

To mathematically describe the satellite's attitude in space, it is convenient to set an inertial reference frame, that is, a reference frame that does not, or at least approximatly not, experience acceleration, mostly because it helps at keeping the mathematics simple and clear.

In this work, based on what has been presented on [5], the chosen inertial reference frame is the Earth Centered Inertial Reference Frame. In this reference frame, the x-axis is defined by a unit vector pointing in the direction of the vernal equinox, the point of intersection between the equatorial plane and the eclipitc in which the Sun's trajectory crosses the Equator moving from the Southern Hemisphere to the Northern Hemisphere, as viewed by an observert at Earth. The z-axis is given by a unit vector in the direction of the geographic north pole axis, and, finally, the y-axis unit vector is the result of the cross-product between the x-axis unit vector and the z-axis unit vector. All equations describing attitude dynamics will be written with respect to this reference frame.

Next, an idea of how to the describe attitude based on the definition of an inertial frame will be developed. First, it is relevant to point out that the geometry of the satellite is understood mathematically as a collection of points that move with a moving reference frame attached to the satellite, that is, those points are stationary when observed from such a moving reference frame. This is intuitive. In the inside of a car, if a person is holding a cellphone it seems's like the cellphone is static from the point of view of another passenger, but it is clear that the cellphone's position is changing according to the car's motion, from the perspective of an observer standing outside of the car. It is tempting to the describe, then, a point contained in the satellite's set of points, by:

$$r_I^P = r_I^B + r_B^P \tag{4.1}$$

This equation reads: "the position of the point P with respect to the inertial reference frame, I, equals the position of the origin of the B reference frame with respect to the inertial reference frame, I, plus the position of the point P with respect to the B reference frame". Where B is a reference frame attached to the satellite. To clearify what this expression means, an intermediate reference frame, B', is defined as a reference frame whose origin coincides with the origin of the B reference frame, but has all of it's axis parallel to the inertial frame's axis, that is, it only differs from the inertial reference frame by a translation operation. This means that the relation between the reference frames B and B' is a rotation operation. Mathematically:

$$r_I^P = r_I^B + R_B^{B'} r_{B'}^P \tag{4.2}$$

The capital letter R is a rotation matrix that transforms the basis vectors of the B' reference frame into the basis vectors of the B reference frame and is the proposed definition of attitude. Attitude is the orientation of a body with respect to some reference placement in space that can be described by a the rotation that must take place in order to make the current placement of the body in space equal to the reference placement. In this case, the reference placement is the inertial reference frame, after the translation required by the position description, and the current placement is the so called Body Reference Frame, a reference frame that moves with the satellite and has two additional special properties. The first special property of the body reference frame is that it's origin is at the satellite's center of mass, making it invariant to rotational movements of the satellite. The second special property of the body reference frame is that in this reference frame the Inertia Tensor is constant and diagonal, which means that the basis vectors for this frame of reference are the normalized eigenvectors of the inertia tensor, that is, the body's principal axes of rotation.

4.3 Attitude Parameterization

Now that the question "what is attitude?" has been answered, another question must be raised: "how to describe attitude mathematically?".

The straightforward method to construct an attitude representation is to make use of sequential rotations around different orthogonal axes. Using this approach the rotation matrix assumes the form:

$$R_{\phi\psi\ \theta} = R_\phi R_\psi R_\theta \tag{4.3}$$

$$R_{\phi\psi\ \theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix} \begin{bmatrix} cos\psi & 0 & sin\psi \\ 0 & 1 & 0 \\ -sin\psi & 0 & cos\psi \end{bmatrix} \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4}$$

$$R_{\phi\psi\ \theta} = \begin{bmatrix} cos\psi cos\theta & -cos\psi sin\theta & sin\psi \\ cos\phi sin\theta + sin\phi cos\theta sin\phi & cos\phi cos\theta - sin\phi sin\psi sin\theta & -sin\phi cos\psi \\ sin\phi sin\theta - cos\phi sin\psi cos\theta & cos\theta sin\phi + cos\phi sin\psi sin\theta & cos\phi cos\psi \end{bmatrix} \tag{4.5}$$

Note that the choice of rotation matrices and it's sequence lead to different representations for attitude, in this case the chosen representation is the 3-2-1 sequence, that is, first a rotation about the z-axis, then a rotation about the y-axis and finally a rotation about the z-axis. Also note that this matrices are defined in the B' coordinate system. This is not a very practical representation, mostly beause it is hard to build a simple mathematical description of attitude dynamics based on this representation and it is not a convenient representation for digital implementations of control algorithms, since trigonometric function evaluation can be computationallly expensive and there are several redundant parameters in this representation, a statement supported by the fact that the knowledge of the 3 angles ϕ, ψ, θ, uniquely defines the rotation operation, for a given sequence of rotations, but the rotation matrix has 9 elements.

A more elegant descritption of rotations is build upon Euler's Rotation Theorem, for which a proof can be found in [6], stated originally as:

Theorem: *In whatever way a sphere is turned about its centre, it is always possible to assign a diameter, whose direction in the translated state agrees with that of the initial state.*

This means that for every rotation there exists an axis, that is, for every rotation there exists a vector invariant to this rotation. This can be expressed by:

$$Rv = v \tag{4.6}$$

The theorem's proof states that there exists only one eigenvector for each rotation matrix whose corresponding eigenvalue is real and equal to 1. This vector defines the rotation's unique axis associated with its corresponding matrix, hence any such operation can actually be represented by a rotation axis and an angle of rotation. Based on such an ideia, the Rodrigue's Rotation Formula is going to be derived next to show an alternative way to compute the result of a rotation operation acting upon a vector.

Initially, any vector can be decomposed in two components, one parallel to the rotation axis and one perpendicular to it, using the equations:

$$\vec{x} = \vec{x}_{||} + \vec{x}_{\_|\_} \tag{4.7}$$
$$\vec{x}_{||} = (\vec{x} \cdot \vec{e})\vec{e} \tag{4.8}$$
$$\vec{x}_{\_|\_} = \vec{x} - (\vec{x} \cdot \vec{e})\vec{e} \tag{4.9}$$
$$\tag{4.10}$$

Where $\vec{e}$ is the vector representing the rotation axis. The rotation of $\vec{x}$ is, then, the sum of the rotation of its components, due to linearity of rotations, but since the component parallel to the rotation axis inherits the property of invariance to this specific rotation operation, the only component that is actually affected by the operation is the perpendicular component. Note that, once it is given that the rotation only affects vectors perpendicular to the rotation axis, it is also defined that the rotations can be analysed in terms of a basis for the vector space defined by the plane perpendicular to the rotation axis. A convenient basis for such representation is given by the set $\left(\vec{x}_{\_|\_}, \vec{e} \times \vec{x}_{\_|\_}\right)$, constructed using the perpendicular component of the original vector (before rotation) and a second vector that is perpendicular to both the rotation axis, hence belongs to the rotation plane, and to the perpendicular component of the original vector, providing an orthogonal basis for the rotation plane. Now any vector that is the result of a rotation around the rotation axis can be described as:

$$\vec{x}' = \vec{x}_{||} + \vec{x}_{\_|\_} cos\alpha + \vec{e} \times \vec{v} sin\alpha \tag{4.11}$$

The greek letter α being the rotation angle. Now, given a rotation axis and an angle, the resulting rotated vector can be computed without the usage of rotation matrices. For the last part of this analysis, a final form of attitude parameterization will be shown. This form is the result of some mathematical manipulation of the Rodrigue's rotation formula that results in the following experssion:

$$\vec{x'} = (cos\frac{\alpha}{2} - +\vec{e}sin\frac{\alpha}{2})\vec{x}(cos\frac{\alpha}{2} - \vec{e}sin\frac{\alpha}{2}) \tag{4.12}$$

The careful reader must have realized by now that equation 4.12 implicitly defines a multiplication of the vector $\vec{x}$ by the mathematical entity $(cos\frac{\alpha}{2} \pm \vec{e}sin\frac{\alpha}{2})$. This leads to the introduction of a mathematical object known as a quaternion. The quaternion can be understood as an extension of the complex numbers to a four dimensional space. It is defined by a scalar part and a imaginary part containing three components, written as:

$$q = q_0 + iq_1 + jq_2 + kq_3 \tag{4.13}$$

It is clear that the space of quaternions is isomorphic to the space of four dimensional vectors with real components. The relevant aspect of quaternions is that they come with the definition of a multiplication operation, defined by the multiplication of its basis elements as:

$$\begin{aligned}
i^2 = j^2 = k^2 = ijk &= -1 && (4.14)\\
jk &= i && (4.15)\\
ki &= j && (4.16)\\
kj &= -i && (4.17)\\
ik &= -j && (4.18)\\
ij &= k && (4.19)\\
ji &= -k && (4.20)\\
& && (4.21)
\end{aligned}$$

Multiplication between quaternions can be carried out by using the usual distributive rules from elementary algebra.

To proceed with the discussion about the rotation of vectors in three dimensional space based on quaternions, a pure quaternion must be defined. A pure quaternion is a quaternion whose scalar part is equal to zero. Note that the space of pure quaternions is isomorphic to the space of three dimensional vectors with real components, hence any vector in three dimensional space can also be represented by a pure quaternion. To define quaternion based rotations of three dimensional vectors, inspired by the rotation operations in the complex plane based on multiplication, one must ask if there is a general procedure to multiply a pure quaternion by an arbitrary quaternion and obtain a pure quaternion. The answer is: no. But, if one evaluates the multiplication of three quaternions:

$$v' = qvp \tag{4.22}$$

Where v is a pure quaternion, it can be shown that if p equals the conjugate of q, then v' is always a pure quaternion. The conjugate of q is a quaternion with the same scalar part of q but its imaginary part equals the imaginary part of q multiplied by minus one. By comparing equation 4.22 with equation 4.12, the link between quaternion multiplication and the rotation of vectors in thee dimensional space can finally be established. If

$$q = cos\frac{\alpha}{2} + (ie_1 + je_2 + ke_3)sin\frac{\alpha}{2} \tag{4.23}$$

Where e_i is the i'th component of the unit vector that defines the rotation axis and α is the rotation angle, then the quaternion operation defined in equation 4.22 represents a rotation around an axis $\vec{e}$ by an angle α and any unit quaternion respecting the structure defined in equation 4.23 is a valid attitude parameterization.

Quaternions are going to be used to represent attitude throughout this work.

CHAPTER 5

Attitude Kinematics and Dynamics

5.1 Introduction

The objetive of this section is to derive the laws that govern the evolution of attitude and angular velocity over time due to externally applied torques. The procedure adopted for this derivation is based on going through the time derivatives of the attitude quaternion and the angular momentum of the satellite.

5.2 Attitude Kinematics

Essentially, the derivation of an expression for the rate of change of an attitude quaterion with respect to time starts from the basic concepts on differentiation, that is:

$$\dot{q}(t) = lim_{dt \to 0} \frac{q(t + dt) - q(t)}{dt} \tag{5.1}$$

Where $q(t)$ is the quaternion that relates a reference vector $\vec{v}$ to it's rotated version, $\vec{v}'$, through the equation:

$$\vec{v}'(t) = q(t)\vec{v}q(t)^* \tag{5.2}$$

Hence, as $q(t)$ changes with time, $\vec{v}$ is rotated through space with some angular velocity. by making the substitution $t \to t + dt$ in equation 5.2, it is stated that:

$$\vec{v}'(t + dt) = q(t + dt)\vec{v}q(t + dt)^* \tag{5.3}$$

But since it is known *apriori* that $\vec{v}'(t + dt)$ is given by some rotation that takes $\vec{v}'(t)$ to $\vec{v}'(t + dt)$, given that the subject of this analysis is a rigid body under rotational motion, it is possible to write:

$$\vec{v}'(t + dt) = \delta q \vec{v}'(t) \delta q^* \tag{5.4}$$

$$\vec{v}'(t + dt) = \delta q(q(t)\vec{v}q(t)^*)\delta q^* \tag{5.5}$$

Hence, by comparison of equation 5.3 with equation 5.5:

$$q(t + dt) = \delta q q(t) \tag{5.6}$$

Now, δq is defined conceptually as a quaternion that represents a rotation by some angle $d\theta$ with some instantaneous angular velocity $\vec{\omega}$, which means that:

$$\delta q = cos\frac{d\theta}{2} + \hat{\omega} sin\frac{d\theta}{2} \tag{5.7}$$

$$\delta q = cos\frac{|\vec{\omega}|dt}{2} + \hat{\omega} sin\frac{|\vec{\omega}|dt}{2} \tag{5.8}$$

Where $|\vec{\omega}|$ is the magnitude of angular velocity and $\hat{\omega}$ is an unit vector in the direction of the angular velocity vector, which is the vector that specifies the direction of rotation.

Finally:

$$q(t+dt) = \delta q q(t) \tag{5.9}$$

$$q(t+dt) = (cos\frac{|\vec{\omega}|dt}{2} + \hat{\omega} sin\frac{|\vec{\omega}|dt}{2})q(t) \tag{5.10}$$

$$q(t+dt) - q(t) = (cos\frac{|\vec{\omega}|dt}{2} + \hat{\omega} sin\frac{|\vec{\omega}|dt}{2} - 1)q(t) \tag{5.11}$$

$$q(t+dt) - q(t) = (-2sin^2\frac{|\vec{\omega}|dt}{4} + \hat{\omega} sin\frac{|\vec{\omega}|dt}{2})q(t) \tag{5.12}$$

$$\frac{q(t+dt) - q(t)}{dt} = \left(\frac{-2sin^2\frac{|\vec{\omega}|dt}{4}}{dt} + \frac{\hat{\omega} sin\frac{|\vec{\omega}|dt}{2}}{dt}\right)q(t) \tag{5.13}$$

Taking the limit when dt approaches zero for equation 5.13:

$$lim_{dt\to 0}\frac{q(t+dt) - q(t)}{dt} = lim_{dt\to 0}\left(\frac{-2sin^2\frac{|\vec{\omega}|dt}{4}}{dt} + \frac{\hat{\omega} sin\frac{|\vec{\omega}|dt}{2}}{dt}\right)q(t) \tag{5.14}$$

Since the limits can be treated separately:

$$lim_{dt\to 0}\frac{-2sin^2\frac{|\vec{\omega}|dt}{4}}{dt} = lim_{dt\to 0}\frac{-2sin\frac{|\vec{\omega}|dt}{4}}{dt} sin\left(\frac{|\vec{\omega}|dt}{4}\right) \tag{5.15}$$

$$lim_{dt\to 0}\frac{-2sin\frac{|\vec{\omega}|dt}{4}}{dt} = -2\frac{|\vec{\omega}|}{4} \tag{5.16}$$

$$lim_{dt\to 0} sin\left(\frac{|\vec{\omega}|dt}{4}\right) = 0 \tag{5.17}$$

$$lim_{dt\to 0}\frac{-2sin^2\frac{|\vec{\omega}|dt}{4}}{dt} = 0 \tag{5.18}$$

For the second component of the sum:

$$lim_{dt\to 0}\frac{\hat{\omega} sin\frac{|\vec{\omega}|dt}{2}}{dt} = \frac{\hat{\omega}|\vec{\omega}|}{2} \tag{5.19}$$

Finally, collecting the terms:

$$\dot{q}(t) = lim_{dt\to 0}\left(\frac{-2sin^2\frac{|\vec{\omega}|dt}{4}}{dt} + \frac{\hat{\omega} sin\frac{|\vec{\omega}|dt}{2}}{dt}\right)q(t) \tag{5.20}$$

$$\dot{q}(t) = \frac{1}{2}\vec{\omega}(t)q(t) \tag{5.21}$$

This equations defines the evolution of the attitude over time as a function of the body's instantaneous angular velocity.

5.3 Attitude Dynamics

To conclude the set of differential equations that determine the evolution of the attitude over time, it is necessary to find a differential equation that relates the rate of change of angular momentum to the torques applied to the body, with it's roots on Newton's second law. First, let the angular momentum vector, expressed in terms of the inertial reference basis vectors, be described in it's usual form:

$$h_I = J_I \omega_I^B \tag{5.22}$$

h_I is the total angular momentum expressed in the inertial reference frame, J_I is the inertia tensor expressed in the inertial reference frame and ω_I^B is the angular velocity vector of the body reference frame expressed in the inertial reference frame. By taking the time derivative of angular momentum the following expression is derived:

$$\dot{h}_I = \dot{J}_I \omega_I^B + J_I \dot{\omega}_I^B \tag{5.23}$$

Where the first term expresses the fact that, since the body reference frame is rotating, the inertia tensor described in terms of the axis of the inertial reference frame must change over time. Since this is not computationally convenient, it is easier to express the rate of change of the angular momentum vector in the body fixed reference frame, once the inertia tensor is, by definition, always constant in such frame. The angular momentum vector takes the form:

$$h_B = J_B \omega_B^B \tag{5.24}$$

Due to the fact that the basis vectors of the body reference frame are rotating with, the time derivative of the angular momentum expressed in the body reference frame is not so trivially derived. First, it is convenient to write:

$$J_B \omega_B^B = J_{b1b1} \omega_1^B \hat{b}_1 + J_{b2b2} \omega_2^B \hat{b}_2 + J_{b3b3} \omega_3^B \hat{b}_3 \tag{5.25}$$

Where ω_i^B is the ith component of the angular momentum vector in the body reference frame basis, J_{bibi} is the ith diagonal element of the inertia tensor and b_i is the ith basis vector of the body reference frame coordinate system. Hence, the derivative of the angular momentum takes the form:

$$\dot{h}_B = J_{b1b1} \dot{\omega_1^B} \hat{b}_1 + J_{b2b2} \dot{\omega_2^B} \hat{b}_2 + J_{b3b3} \dot{\omega_3^B} \hat{b}_3 + J_{b1b1} \omega_1^B \dot{\hat{b}}_1 + J_{b2b2} \omega_2^B \dot{\hat{b}}_2 + J_{b3b3} \omega_3^B \dot{\hat{b}}_3 \tag{5.26}$$

$$\dot{h}_B = J_B \dot{\omega}_B^B + J_{b1b1} \omega_1^B \vec{\omega_B} \times \hat{b}_1 + J_{b2b2} \omega_2^B \vec{\omega_B} \times \hat{b}_2 + J_{b3b3} \omega_3^B \vec{\omega_B} \times \hat{b}_3 \tag{5.27}$$

$$\dot{h}_B = J_B \dot{\omega}_B^B + \omega_B^B \times J_B \omega_B^B \tag{5.28}$$

Hence, an alternative form of Newton's second law can be written as

$$\dot{h}_B = J_B \dot{\omega}_B^B + \omega_B^B \times h_B = \tau_B \tag{5.29}$$

Where τ_B is the sum of external torques expressed in terms of the body reference frame basis.

The integration of the differential equations governing the time evolution of angular momentum and attitude, given an initial condition and an expression for the external torques, uniquely describes the rotational motion of the satellite.

Bibliography

[1] E. Thébault et al. International geomagnetic reference field: the 12th generation. *Earth, Planets and Space*, 2015.

[2] D. Winch et al. Geomagnetism and schmidt quasi-normalization. *Geophysical Journal International*, 2005.

[3] A. Hazel. *Chapter 1 Describing the Physical World: Vectors & Tensors*.

[4] C. A. Rigo. *Task scheduling for optimal power management and quality-of-service assurance in CubeSats*, 2021.

[5] K. Vinther K. F. Jensen. *Attitude Determination and Control System for AAUSAT3*, 2010.

[6] R. Palais B. Palais. Euler's fixed point theorem: The axis of a rotation. *Journal of Fixed Point Theory and Applications*, 2007.