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Adaptive Nonlinear Control for Spacecraft Formation Flying with Coupled Translational and Attitude Dynamics

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Abstract

In this paper, we address a tracking control problem for the coupled translational and attitude motion of a follower spacecraft relative to a leader spacecraft. Using the vectrix formalism the translational and attitude dynamics of the leader and follower spacecraft are modeled, where the mutual coupling in each spacecraft's translational and attitude motion induced by their gravitational interaction is duly accounted. Using a Lyapunov framework, nonlinear control and adaptation laws are designed that ensure the global asymptotic convergence of the relative translational and attitude position tracking errors, despite the presence of unknown mass and inertia parameters of the leader and follower spacecraft.

1. Introduction

NASA and the U.S. Air Force are planning several space missions to perform reconnaissance, optical interferometry, passive radiometry, virtual co-observing, stereo-imaging, and terrain mapping using versatile, affordable, and highly capable distributed spacecraft. The distributed spacecraft architecture necessitates interacting spacecraft with system-wide common capabilities (communication, sensing, navigation, control, etc.), operating collectively to accomplish shared mission objectives [1]. Thus, recent years have witnessed a tremendous research interest in distributed spacecraft formation control. For example, linear and nonlinear formation dynamic models have been developed for formation maintenance and a variety of control designs have been proposed to guarantee the desired formation performance (see [2–4] and the numerous references therein). However, a majority of these papers assume the spacecraft to be a point mass, which ignores the influence of angular motion of spacecraft body relative to the earth and other spacecraft in formation. The attitude control of rigid spacecraft has also been a topic of considerable interest over the past three decades. A variety of linear and nonlinear control designs have been developed in the literature for spacecraft attitude control (see, e.g., [5,6] and the numerous references therein).

Note that distributed spacecraft missions such as synthetic aperture radars, optical interferometers, tracking of and orbital rendezvous with comets, asteroids, and orbital debris, etc., necessitate highly maneuverable spacecraft which rely on a systematic control design framework for six degrees-of-freedom (DOF) coupled translational and attitude motion. The control design problem for distributed spacecraft is further exacerbated by the fact that most spacecraft consist of several semi-rigid constituent bodies, thus leading to extreme difficulty in the characterization of their physical parameters, e.g., the inertia matrix. Unfortunately, the distributed spacecraft formation control problem with simultaneous translational and attitude motion control of spacecraft in the presence of inherently coupled system dynamics has not yet received significant attention, not to mention the control design problem with unknown spacecraft mass and inertia parameters.

In this paper, we address a tracking control problem for the coupled translational and attitude motion of a follower spacecraft relative to a leader spacecraft. Using a Lyapunov framework, nonlinear control and adaptation laws are designed that ensure the global asymptotic convergence of the translational and attitude position tracking errors, despite the presence of unknown mass and inertia parameters of the leader and follower spacecraft. Throughout this paper, the leader spacecraft is assumed to track a given desired translational and attitude trajectory using, e.g., the control design of [7].

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2. Mathematical Preliminaries

Throughout this paper several reference frames are employed to characterize the translational and attitude dynamics of spacecraft. Each reference frame used in this paper is assumed to consist of three basis vectors which are right-handed, mutually perpendicular, and of unit length. Let $\mathcal{F}$ denote a reference frame and let $\vec{i}$, $\vec{j}$, and $\vec{k}$ denote the three basis vectors of $\mathcal{F}$. Then $\vec{F}$ denotes a vectrix for the reference frame $\mathcal{F}$, which is defined as [8] $\vec{F} \triangleq \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$. A vector $\vec{A}$ can be expressed in the reference frame $\mathcal{F}$ as $\vec{A} \triangleq a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, where a_1, a_2, and a_3 denote the components of $\vec{A}$ along $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively. Frequently, we will assemble these components as $A \triangleq [a_1\ a_2\ a_3]^T$. Using the above vectrix formalism and the usual vector inner product, a vector $\vec{A}$ can be expressed in the reference frame $\mathcal{F}$ as $\vec{A} = A^T \vec{F} = \vec{F}^T A$.

The vectrix $\vec{F}$ has the following two properties: *i)* $\vec{F} \cdot \vec{F}^T = I_3$, where "$\cdot$" denotes the usual vector dot product and I_n denotes an n dimensional identity matrix and *ii)* $\vec{F} \times \vec{F}^T = \begin{bmatrix} \vec{0} & \vec{k} & -\vec{j} \\ -\vec{k} & \vec{0} & \vec{i} \\ \vec{j} & -\vec{i} & \vec{0} \end{bmatrix}$, where "$\times$" denotes the usual vector cross product. Thus, we can write A in terms of the vector $\vec{A}$ and the vectrix $\vec{F}$ as $A = \vec{F} \cdot \vec{A} = \vec{A} \cdot \vec{F}$. Next, let $\vec{B}$ be a vector, which is expressed in $\mathcal{F}$ by $\vec{B} \triangleq b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, and let $B \triangleq [b_1\ b_2\ b_3]^T$. Then, it follows from the above that $\vec{A} \cdot \vec{B} = A^T B = B^T A$ and $\vec{A} \times \vec{B} = \vec{F}^T A^\times B$, where $A^\times \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$.

Throughout this paper, we will frequently need to express various vectors in two or more different reference frames. In such cases, a rotation matrix will facilitate the transformation of the components of a given vector expressed in one reference frame to the components of the same vector expressed in another reference frame. To illustrate this concept, let us begin by assuming that a vector $\vec{A}$ expressed in the reference frame $\mathcal{F}_{\rm u}$ has components $A_{\rm u}$, i.e., $\vec{A}=\vec{F}_{\rm u}^T A_{\rm u}$, where $\vec{F}_{\rm u}$ denotes the vectrix of the reference frame $\mathcal{F}_{\rm u}$. Similarly, let us assume that the vector $\vec{A}$ expressed in the reference frame $\mathcal{F}_{\rm v}$ has components $A_{\rm v}$, i.e., $\vec{A}=\vec{F}_{\rm v}^T A_{\rm v}$, where $\vec{F}_{\rm v}$ denotes the vectrix of the reference frame $\mathcal{F}_{\rm v}$. It now follows that $A_{\rm u} = \vec{F}_{\rm u} \cdot \vec{A} = \vec{A} \cdot \vec{F}_{\rm u}$ and $A_{\rm v} = \vec{F}_{\rm v} \cdot \vec{A} = \vec{A} \cdot \vec{F}_{\rm v}$. Thus, we now obtain $A_{\rm v} = \vec{F}_{\rm v} \cdot \vec{F}_{\rm u}^T A_{\rm u} = C_{\rm v}^{\rm u} A_{\rm u}$, where $C_{\rm v}^{\rm u} \triangleq \vec{F}_{\rm v} \cdot \vec{F}_{\rm u}^T \in SO(3)$ denotes the rotation matrix that transforms the components of a vector expressed in $\mathcal{F}_{\rm u}$ (*viz.*, $A_{\rm u}$) to the components of the same vector expressed in $\mathcal{F}_{\rm v}$ (*viz.*, $A_{\rm v}$). Here the notation $SO(3)$ represents the set of all 3×3 rotation matrices.

Let us assume that the vector $\vec{A}$ expressed in another reference frame $\mathcal{F}_{\rm w}$ has components $A_{\rm w}$, i.e., $\vec{A}=\vec{F}_{\rm w}^T A_{\rm w}$, where $\vec{F}_{\rm w}$ denotes the vectrix of the reference frame $\mathcal{F}_{\rm w}$. Then, as above, we obtain $A_{\rm u} = \vec{F}_{\rm u} \cdot \vec{F}_{\rm w}^T A_{\rm w} = C_{\rm u}^{\rm w} A_{\rm w}$ and $A_{\rm v} = \vec{F}_{\rm v} \cdot \vec{F}_{\rm w}^T A_{\rm w} = C_{\rm v}^{\rm w} A_{\rm w}$, where $C_{\rm u}^{\rm w} \triangleq \vec{F}_{\rm u} \cdot \vec{F}_{\rm w}^T \in SO(3)$ denotes the rotation matrix that transforms the components of a vector expressed in $\mathcal{F}_{\rm w}$ (*viz.*, $A_{\rm w}$) to the components of the same vector expressed in $\mathcal{F}_{\rm u}$ (*viz.*, $A_{\rm u}$) and $C_{\rm v}^{\rm w} \triangleq \vec{F}_{\rm v} \cdot \vec{F}_{\rm w}^T \in SO(3)$ denotes the rotation matrix that transforms the components of a vector expressed in $\mathcal{F}_{\rm w}$ (*viz.*, $A_{\rm w}$) to the components of the same vector expressed in $\mathcal{F}_{\rm v}$ (*viz.*, $A_{\rm v}$). Finally, it follows from the above that $A_{\rm v} = C_{\rm v}^{\rm u} C_{\rm u}^{\rm w} A_{\rm w}$ and $C_{\rm v}^{\rm w} = C_{\rm v}^{\rm u} C_{\rm u}^{\rm w}$.

3. Spacecraft Relative Dynamics Modeling

In this section, we develop the translational and attitude dynamics of a spacecraft, termed as the follower spacecraft, relative to the leader spacecraft. Each spacecraft is modeled as a rigid body with actuators that provide body-fixed forces and torques about three mutually perpendicular axes that define a body-fixed reference frame (i.e., $\mathcal{F}_{{\rm b}\ell}$ and $\mathcal{F}_{\rm bf}$ located at the mass center of the leader and follower spacecraft, respectively). We fully account for the mutual coupling between the translational and attitude motion of each spacecraft. For the given desired translational and attitude motion trajectories of the follower spacecraft relative to the leader spacecraft, we develop the relative translational and attitude error dynamics. Finally, we state our control objective for the translational and attitude motion of the follower spacecraft relative to the leader spacecraft.

Follower Spacecraft Relative Translational Dynamics. Let $\mathcal{F}_{\rm i}$ be an inertial reference frame fixed at the center of the earth and let $\vec{A}$ be an arbitrary vector measured with respect to the origin of $\mathcal{F}_{\rm i}$. Then, in this paper, $\dot{\vec{A}}(t)$ denotes the time derivative of $\vec{A}(t)$ measured in $\mathcal{F}_{\rm i}$. Using this notation, the translational dynamics of the leader spacecraft is given by [8]

$$\dot{\vec{R}}_\ell = \vec{V}_\ell, \qquad m_\ell \dot{\vec{V}}_\ell = \vec{f}_{{\rm e}\ell} + \vec{f}_{{\rm d}\ell} - \vec{f}_\ell, \qquad (1)$$

where m_ℓ denotes the mass of the leader spacecraft, $\vec{R}_\ell(t)$ and $\vec{V}_\ell(t)$ denote the position and velocity of its mass center, $\vec{f}_{{\rm e}\ell}(t)$ denotes the inverse-square gravitational force that leads to an elliptical orbit [8], $\vec{f}_{{\rm d}\ell}(t)$ denotes the attitude-dependent disturbance force that causes the leader spacecraft trajectory to deviate from an ellipse [8], and $\vec{f}_\ell(t)$ denotes the external control force. The inverse-square gravitational force and the attitude-dependent disturbance force are characterized as $\vec{f}_{{\rm e}\ell} = -\frac{\mu m_\ell}{\|\vec{R}_\ell\|^3}\vec{R}_\ell$ and $\vec{f}_{{\rm d}\ell} = -\frac{3\mu}{2\|\vec{R}_\ell\|^4}\left[\left\{{\rm tr}\left(J_\ell\right)\vec{I} + 2\vec{J}_\ell\right\}\cdot\vec{Z}_\ell - 5\left(\vec{Z}_\ell\cdot\vec{J}_\ell\cdot\vec{Z}_\ell\right)\vec{Z}_\ell\right]$, respectively, where $\mu \triangleq MG$ with M being the mass of the earth and G being the universal gravitational constant, J_ℓ is the constant, positive-definite, symmetric inertia matrix of the leader spacecraft expressed in $\mathcal{F}_{{\rm b}\ell}$, $\vec{J}_\ell \triangleq \vec{F}_{{\rm b}\ell}^T J_\ell \vec{F}_{{\rm b}\ell}$ denotes the central inertia dyadic of the leader spacecraft [8], $\vec{I}$ denotes the dyadic of a 3×3 identity matrix, $\vec{Z}_\ell \triangleq \frac{\vec{R}_\ell}{\|\vec{R}_\ell\|}$ with $\|\vec{R}_\ell\| \triangleq \sqrt{\vec{R}_\ell \cdot \vec{R}_\ell}$, and ${\rm tr}(\cdot)$ denotes the trace of a matrix.

Analogous to the leader spacecraft, the nonlinear translational dynamics of the follower spacecraft is given by

$$\dot{\vec{R}}_{\rm f} = \vec{V}_{\rm f}, \qquad m_{\rm f} \dot{\vec{V}}_{\rm f} = \vec{f}_{\rm ef} + \vec{f}_{\rm df} - \vec{f}_{\rm f}, \qquad (2)$$

where $m_{\rm f}$, $\vec{R}_{\rm f}(t)$, $\vec{V}_{\rm f}(t)$, $\vec{f}_{\rm ef}(t)$, $\vec{f}_{\rm df}(t)$, and $\vec{f}_{\rm f}(t)$ are defined similar to the case of the leader spacecraft.

Next, we develop the translational motion dynamics of the follower spacecraft relative to the leader spacecraft. Before proceeding, for convenience, we introduce the notation

$$\vec{\rho}_{\rm R} \triangleq \vec{R}_{\rm f} - \vec{R}_\ell, \qquad \vec{\rho}_{\rm V} \triangleq \vec{V}_{\rm f} - \vec{V}_\ell\,. \qquad (3)$$

Let $\vec{\omega}_{\rm bf}(t)$ denote the angular velocity of $\mathcal{F}_{\rm bf}$ relative to $\mathcal{F}_{\rm i}$. In this paper, $\mathring{\vec{A}}(t)$ denotes the time derivative of an arbitrary vector $\vec{A}$ measured in $\mathcal{F}_{\rm bf}$. Using this notation, the time derivative of vector $\vec{A}$ measured in $\mathcal{F}_{\rm i}$ is given by

$$\dot{\vec{A}} = \mathring{\vec{A}} + \vec{\omega}_{\rm bf} \times \vec{A}\,. \qquad (4)$$

Following (4) for vectors $\vec{R}_{\rm f}(t), \vec{V}_{\rm f}(t), \vec{\rho}_{\rm R}(t)$, and $\vec{\rho}_{\rm V}(t)$ yields $\dot{\vec{R}}_{\rm f} = \mathring{\vec{R}}_{\rm f} + \vec{\omega}_{\rm bf} \times \vec{R}_{\rm f}$, $\dot{\vec{V}}_{\rm f} = \mathring{\vec{V}}_{\rm f} + \vec{\omega}_{\rm bf} \times \vec{V}_{\rm f}$, $\dot{\vec{\rho}}_{\rm R} = \mathring{\vec{\rho}}_{\rm R} + \vec{\omega}_{\rm bf} \times \vec{\rho}_{\rm R}$, and $\dot{\vec{\rho}}_{\rm V} = \mathring{\vec{\rho}}_{\rm V} + \vec{\omega}_{\rm bf} \times \vec{\rho}_{\rm V}$.

In this paper, we assume that the desired translational position of the follower spacecraft relative to the leader spacecraft denoted by $\vec{\rho}_{\rm R_d}(t)$ is given. In addition, we assume that the time derivative of $\vec{\rho}_{\rm R_d}$ measured in $\mathcal{F}_{\rm i}$ and denoted by $\vec{\rho}_{\rm V_d}(t) \triangleq \dot{\vec{\rho}}_{\rm R_d}$ is given. Note that $\vec{\rho}_{\rm R_d}(t)$ and its first two time derivatives are assumed to be bounded functions of time. Next, following (4) for vectors $\dot{\vec{\rho}}_{\rm R_d}$ and $\dot{\vec{\rho}}_{\rm V_d}$ yields $\dot{\vec{\rho}}_{\rm R_d} = \mathring{\vec{\rho}}_{R_{\rm d}} + \vec{\omega}_{\rm bf} \times \vec{\rho}_{\rm R_d}$ and $\dot{\vec{\rho}}_{\rm V_d} = \mathring{\vec{\rho}}_{V_{\rm d}} + \vec{\omega}_{\rm bf} \times \vec{\rho}_{\rm V_d}$.

Now we develop the error dynamics of the translational motion of the follower spacecraft relative to the leader spacecraft. We begin by introducing the notation

$$\vec{e}_{\rm R_r} \triangleq \vec{\rho}_{\rm R_d} - \vec{\rho}_{\rm R}, \qquad \vec{e}_{\rm V_r} \triangleq \vec{\rho}_{\rm V_d} - \vec{\rho}_{\rm V}\,. \qquad (5)$$

Computing the time derivative of both sides of (5) measured in $\mathcal{F}_{\rm i}$ and performing simple manipulations we get

$$\overset{\bullet}{\vec{e}}_{\rm R_r}=\overset{\circ}{\vec{\rho}}_{R_{\rm d}}-\overset{\circ}{\vec{\rho}}_{\rm R}+\vec{\omega}_{\rm bf}\times\vec{e}_{\rm R_r},\quad \overset{\bullet}{\vec{e}}_{\rm V_r}=\overset{\circ}{\vec{\rho}}_{V_{\rm d}}-\overset{\circ}{\vec{\rho}}_{\rm V}+\vec{\omega}_{\rm bf}\times\vec{e}_{\rm V_r}\,. \tag{6}$$

It follows from (5) that $\overset{\circ}{\vec{e}}_{\rm R_r}=\overset{\circ}{\vec{\rho}}_{R_{\rm d}}-\overset{\circ}{\vec{\rho}}_{\rm R}$ and $\overset{\bullet}{\vec{e}}_{\rm R_r}=\overset{\bullet}{\vec{\rho}}_{\rm R_d}-\overset{\bullet}{\vec{\rho}}_{\rm R}$. Similarly, it follows from (5) that $\overset{\circ}{\vec{e}}_{\rm V_r}=\overset{\circ}{\vec{\rho}}_{V_{\rm d}}-\overset{\circ}{\vec{\rho}}_{\rm V}$ and $\overset{\bullet}{\vec{e}}_{\rm V_r}=\overset{\bullet}{\vec{\rho}}_{\rm V_d}-\overset{\bullet}{\vec{\rho}}_{\rm V}$. Combining (1), (2), and (6), we obtain the translational error dynamics of the follower spacecraft motion relative to the leader spacecraft motion given by

$$\overset{\circ}{\vec{e}}_{\rm R_r}=\vec{e}_{\rm V_r}-\vec{\omega}_{\rm bf}\times\vec{e}_{\rm R_r}, \tag{7}$$

$$\overset{\circ}{\vec{e}}_{\rm V_r}=\overset{\bullet}{\vec{\rho}}_{\rm V_d}-\vec{\omega}_{\rm bf}\times\vec{e}_{\rm V_r}+\frac{1}{m_\ell}\left(\vec{f}_{\rm e\ell}+\vec{f}_{\rm d\ell}\right)-\frac{1}{m_\ell}\vec{f}_\ell -\frac{1}{m_{\rm f}}\left(\vec{f}_{\rm ef}+\vec{f}_{\rm df}\right)+\frac{1}{m_{\rm f}}\vec{f}_{\rm f}\,. \tag{8}$$

Now using the framework of Section 2, various vectors of interest can be expressed in $\mathcal{F}_{\rm bf}$ as follows

$$\Big[R_{\rm f}\,V_{\rm f}\,\rho_{\rm R}\,\dot\rho_{\rm R}\,\rho_{\rm V}\,\dot\rho_{\rm V}\,\rho_{\rm R_d}\,\dot\rho_{\rm R_d}\,\rho_{\rm V_d}\,DV\,e_{\rm R_r}\,\dot e_{\rm R_r}\,e_{\rm V_r}\,\dot e_{\rm V_r}\,\omega_{\rm bf}\,f_{\rm f}\,f_{\rm ef}\,f_{\rm df}\Big]\triangleq\vec{F}_{\rm bf}\cdot\Big[\vec{R}_{\rm f}\,\vec{V}_{\rm f}\,\vec{\rho}_{\rm R}\,\overset{\circ}{\vec{\rho}}_{\rm R}\,\vec{\rho}_{\rm V}\,\overset{\circ}{\vec{\rho}}_{\rm V}\,\vec{\rho}_{\rm R_d}\,\overset{\circ}{\vec{\rho}}_{R_{\rm d}}\,\vec{\rho}_{\rm V_d}\,\overset{\bullet}{\vec{\rho}}_{\rm V_d}\,\vec{e}_{\rm R_r}\,\overset{\circ}{\vec{e}}_{\rm R_r}\,\vec{e}_{\rm V_r}\,\overset{\circ}{\vec{e}}_{\rm V_r}\,\vec{\omega}_{\rm bf}\,\vec{f}_{\rm f}\,\vec{f}_{\rm ef}\,\vec{f}_{\rm df}\Big], \tag{9}$$

where $R_{\rm f}(t), V_{\rm f}(t), \rho_{\rm R}(t), \dot\rho_{\rm R}(t), \rho_{\rm V}(t), \dot\rho_{\rm V}(t), \rho_{\rm R_d}(t), \dot\rho_{\rm R_d}(t), \rho_{\rm V_d}(t), DV(t), e_{\rm R_r}(t), \dot e_{\rm R_r}(t), e_{\rm V_r}(t), \dot e_{\rm V_r}(t), \omega_{\rm bf}(t), f_{\rm f}(t), f_{\rm ef}(t), f_{\rm df}(t)\in\mathbb{R}^3$. Similarly, various vectors of interest can be expressed in $\mathcal{F}_{\rm b\ell}$ as follows $\Big[R_\ell\,V_\ell\,f_\ell\,f_{\rm e\ell}\,f_{\rm d\ell}\Big]\triangleq\vec{F}_{\rm b\ell}\cdot\Big[\vec{R}_\ell\,\vec{V}_\ell\,\vec{f}_\ell\,\vec{f}_{\rm e\ell}\,\vec{f}_{\rm d\ell}\Big]$, where $R_\ell(t), V_\ell(t), f_\ell(t), f_{\rm e\ell}(t), f_{\rm d\ell}(t)\in\mathbb{R}^3$. Next, it follows from Section 2 that various vectors of interest originally expressed in $\mathcal{F}_{\rm b\ell}$ can be expressed in $\mathcal{F}_{\rm bf}$ using the rotation matrix $C_{\rm bf}^{\rm b\ell}$, which is given in the sequel. Thus, using (3), (5) and the vectrix formalism of Section 2, we obtain

$$R_{\rm f}=\rho_{\rm R_d}-e_{\rm R_r}+C_{\rm bf}^{\rm b\ell}R_\ell,\;\; V_{\rm f}=\rho_{\rm V_d}-e_{\rm V_r}+C_{\rm bf}^{\rm b\ell}V_\ell. \tag{10}$$

Finally, an application of the vectrix formalism of Section 2 on (7) and (8) yields

$$\dot e_{\rm R_r}=e_{\rm V_r}-\omega_{\rm bf}^{\times}e_{\rm R_r}, \tag{11}$$

$$\dot e_{\rm V_r}=DV-\omega_{\rm bf}^{\times}e_{\rm V_r}-\tfrac{1}{m_\ell}C_{\rm bf}^{\rm b\ell}\Big[\tfrac{\mu m_\ell}{\|\vec{R}_\ell\|^3}R_\ell+\tfrac{3\mu}{2\|\vec{R}_\ell\|^4}\Big\{{\rm tr}(J_\ell)I_3 +2J_\ell-\tfrac{5R_\ell^TJ_\ell R_\ell}{\|\vec{R}_\ell\|^2}I_3\Big\}\tfrac{R_\ell}{\|\vec{R}_\ell\|}+f_\ell\Big]+\tfrac{1}{m_{\rm f}}\Big[\tfrac{\mu m_{\rm f}}{\|\vec{R}_{\rm f}\|^3}R_{\rm f}+\tfrac{3\mu}{2\|\vec{R}_{\rm f}\|^4} \Big\{{\rm tr}(J_{\rm f})I_3+2J_{\rm f}-\tfrac{5R_{\rm f}^TJ_{\rm f}R_{\rm f}}{\|\vec{R}_{\rm f}\|^2}I_3\Big\}\tfrac{R_{\rm f}}{\|\vec{R}_{\rm f}\|}\Big]+\tfrac{1}{m_{\rm f}}f_{\rm f}. \tag{12}$$

In this paper, the control objective for the translational dynamics of the follower spacecraft relative to the leader spacecraft requires that the mass center of the follower spacecraft relative to the mass center of the leader spacecraft track the desired relative translational motion trajectory i.e., $\vec{\rho}_{\rm R}(t)\to\vec{\rho}_{\rm R_d}(t)$ as $t\to\infty$. In addition, it is required that $\overset{\bullet}{\vec{\rho}}_{\rm R}(t)\to\overset{\bullet}{\vec{\rho}}_{\rm R_d}(t)$ as $t\to\infty$. Taking the time derivative of (3) measured in $\mathcal{F}_{\rm i}$, it follows that $\overset{\bullet}{\vec{\rho}}_{\rm R}=\overset{\bullet}{\vec{R}}_{\rm f}-\overset{\bullet}{\vec{R}}_\ell$, which upon the use of (1), (2), and (3) yields $\overset{\bullet}{\vec{\rho}}_{\rm R}=\vec{\rho}_{\rm V}$. In addition, as noted before, $\overset{\bullet}{\vec{\rho}}_{\rm R_d}=\vec{\rho}_{\rm V_d}$. Using (5) and (9), the follower spacecraft relative translational tracking control objective can be stated as follows

$$\lim_{t\to\infty}e_{\rm R_r}(t),\,e_{\rm V_r}(t)\;=\;0. \tag{13}$$

Remark 3.1. We assume that the desired translational dynamics of the follower spacecraft relative to the leader spacecraft will be typically specified in the earth-fixed inertial reference frame $\mathcal{F}_{\rm i}$. Let $\vec{F}_{\rm i}=\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\end{bmatrix}^T$ denote the vectrix of the inertial reference frame $\mathcal{F}_{\rm i}$. In addition, let $x(t), y(t), z(t)\in\mathbb{R}$ denote the components of $\vec{\rho}_{\rm R_d}$ along $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively. Then, $\vec{\rho}_{\rm R_d}=x\,\vec{i}+y\,\vec{j}+z\,\vec{k}=\vec{F}_{\rm i}^{\,T}D_{\rm r}$, where $D_{\rm r}(t)\in\mathbb{R}^3$, $D_{\rm r}\triangleq[x\;y\;z]^T$. With $\overset{\bullet}{\vec{\rho}}_{\rm R_d}$ and $\overset{\bullet\bullet}{\vec{\rho}}_{\rm R_d}$ denoting the first and second derivatives, respectively, of $\vec{\rho}_{\rm R_d}$ measured in $\mathcal{F}_{\rm i}$, we can express $\overset{\bullet}{\vec{\rho}}_{\rm R_d}$ and $\overset{\bullet\bullet}{\vec{\rho}}_{\rm R_d}$ in $\mathcal{F}_{\rm i}$ as $\dot D_{\rm r}\triangleq\vec{F}_{\rm i}^{T}\cdot\overset{\bullet}{\vec{\rho}}_{\rm R_d}$ and $\ddot D_{\rm r}\triangleq\vec{F}_{\rm i}^{T}\cdot\overset{\bullet\bullet}{\vec{\rho}}_{\rm R_d}$, respectively, where $\dot D_{\rm r}(t), \ddot D_{\rm r}(t)\in\mathbb{R}^3$. Now using the rotation matrix framework of Section 2, we transform $D_{\rm r}$, $\dot D_{\rm r}$, and $\ddot D_{\rm r}$ to the follower spacecraft body-fixed reference frame $\mathcal{F}_{\rm bf}$ as follows $\rho_{\rm R_d}=C_{\rm bf}^{\rm i}D_{\rm r}$, $\rho_{\rm V_d}=C_{\rm bf}^{\rm i}\dot D_{\rm r}$, and $DV=C_{\rm bf}^{\rm i}\ddot D_{\rm r}$, where $C_{\rm bf}^{\rm i}$ is given in the sequel.

Follower Spacecraft Relative Attitude Dynamics. We begin by characterizing the attitude dynamics of leader and follower spacecraft. The attitude dynamics of the leader spacecraft is given by

$$\overset{\bullet}{\vec{h}}_\ell\;=\;\vec{\tau}_{\rm g\ell}+\vec{\tau}_\ell, \tag{14}$$

where $\vec{h}_\ell(t)$ given by $\vec{h}_\ell=\vec{J}_\ell\cdot\vec{\omega}_{\rm b\ell}$ denotes the angular momentum of the leader spacecraft about its mass center, $\vec{\tau}_{\rm g\ell}(t)$ given by $\vec{\tau}_{\rm g\ell}=\frac{3\mu}{\|\vec{R}_\ell\|^3}\,\vec{Z}_\ell\times\vec{J}_\ell\cdot\vec{Z}_\ell$ denotes the gravity gradient torque [8], and $\vec{\tau}_\ell(t)$ denotes the control torque.

Next, let $\vec{\omega}_{\rm b\ell}(t)$ denote the angular velocity of $\mathcal{F}_{\rm b\ell}$ relative to $\mathcal{F}_{\rm i}$ and let $\overset{\circ}{\vec{A}}(t)$ denote the time derivative of an arbitrary vector $\vec{A}$ measured in $\mathcal{F}_{\rm b\ell}$. Then, the time derivative of vector $\vec{h}_\ell$ measured in $\mathcal{F}_{\rm i}$ is given by $\overset{\bullet}{\vec{h}}_\ell=\overset{\circ}{\vec{h}}_\ell+\vec{\omega}_{\rm b\ell}\times\vec{h}_\ell$. Once again, using the framework of Section 2, various vectors of interest can be expressed in the leader spacecraft body-fixed reference frame $\mathcal{F}_{\rm b\ell}$ as $\Big[h_\ell\;\dot\omega_{\rm b\ell}\;\tau_{\rm g\ell}\;\tau_\ell\Big]\triangleq\vec{F}_{\rm b\ell}\cdot\Big[\vec{h}_\ell\;\overset{\circ}{\vec{\omega}}_{\rm b\ell}\;\vec{\tau}_{\rm g\ell}\;\vec{\tau}_\ell\Big]$, where $h_\ell(t), \dot\omega_{\rm b\ell}(t), \tau_{\rm g\ell}(t), \tau_\ell(t)\in\mathbb{R}^3$. Now an application of the vectrix formalism of Section 2 on (14) yields

$$J_\ell\dot\omega_{\rm b\ell}\;=\;-\omega_{\rm b\ell}^{\times}J_\ell\omega_{\rm b\ell}+\tau_{\rm g\ell}+\tau_\ell. \tag{15}$$

The dynamic equation of (15) characterizes the time evolution of the angular velocity vector $\omega_{\rm b\ell}$. Next, we characterize the kinematic equations that relate the time derivative of the leader spacecraft angular orientation to the angular velocity vector, using the nonsingular, four-parameter quaternion representation, as follows [8]

$$\begin{bmatrix} \dot{\varepsilon}_{\mathrm{b}\ell} \\ \dot{\zeta}_{\mathrm{b}\ell} \end{bmatrix} = E(\varepsilon_{\mathrm{b}\ell}, \zeta_{\mathrm{b}\ell})\omega_{\mathrm{b}\ell}, \tag{16}$$

where $E(\varepsilon_{\mathrm{b}\ell}, \zeta_{\mathrm{b}\ell}) \triangleq \frac{1}{2}\begin{bmatrix} \varepsilon^{\times}_{\mathrm{b}\ell} + \zeta_{\mathrm{b}\ell} I_3 \\ -\varepsilon^{T}_{\mathrm{b}\ell} \end{bmatrix}$ and $(\varepsilon_{\mathrm{b}\ell}(t),\, \zeta_{\mathrm{b}\ell}(t)) \in \mathbb{R}^3 \times \mathbb{R}$ represents the quaternion, which characterizes the attitude of $\mathcal{F}_{\mathrm{b}\ell}$ with respect to $\mathcal{F}_{\mathrm{i}}$. By construction, $(\varepsilon_{\mathrm{b}\ell},\, \zeta_{\mathrm{b}\ell})$ must satisfy the unit norm constraint

$$\varepsilon^{T}_{\mathrm{b}\ell}\varepsilon_{\mathrm{b}\ell} + \zeta^{2}_{\mathrm{b}\ell} = 1. \tag{17}$$

Following [8], the rotation matrix $C^{\mathrm{i}}_{\mathrm{b}\ell} \in SO(3)$ that brings the reference frame $\mathcal{F}_{\mathrm{i}}$ onto the reference frame $\mathcal{F}_{\mathrm{b}\ell}$ is given as

$$C^{\mathrm{i}}_{\mathrm{b}\ell} = C(\varepsilon_{\mathrm{b}\ell}, \zeta_{\mathrm{b}\ell}) \triangleq (\zeta^{2}_{\mathrm{b}\ell} - \varepsilon^{T}_{\mathrm{b}\ell}\varepsilon_{\mathrm{b}\ell}) I_3 + 2\varepsilon_{\mathrm{b}\ell}\varepsilon^{T}_{\mathrm{b}\ell} - 2\zeta_{\mathrm{b}\ell}\varepsilon^{\times}_{\mathrm{b}\ell}. \tag{18}$$

The dynamic and kinematic equations of (15), (16) represent the attitude dynamics of the leader spacecraft.

The attitude dynamics of the follower spacecraft is analogously given by (14) with subscript ℓ replaced by f. In addition, various vectors of interest can be expressed in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as $\begin{bmatrix} h_{\mathrm{f}} & \dot{\omega}_{\mathrm{bf}} & \tau_{\mathrm{gf}} & \tau_{\mathrm{f}} \end{bmatrix} \triangleq \vec{F}_{\mathrm{bf}} \cdot \begin{bmatrix} \vec{h}_{\mathrm{f}} & \overset{\circ}{\vec{\omega}}_{\mathrm{bf}} & \vec{\tau}_{\mathrm{gf}} & \vec{\tau}_{\mathrm{f}} \end{bmatrix}$. The attitude dynamics of follower spacecraft is then characterized by the following dynamic and kinematic equations

$$J_{\mathrm{f}}\dot{\omega}_{\mathrm{bf}} = -\omega^{\times}_{\mathrm{bf}} J_{\mathrm{f}}\omega_{\mathrm{bf}} + \tau_{\mathrm{gf}} + \tau_{\mathrm{f}}, \tag{19}$$

$$\begin{bmatrix} \dot{\varepsilon}_{\mathrm{bf}} \\ \dot{\zeta}_{\mathrm{bf}} \end{bmatrix} = E(\varepsilon_{\mathrm{bf}}, \zeta_{\mathrm{bf}})\omega_{\mathrm{bf}}. \tag{20}$$

Finally, following (18), the rotation matrix $C^{\mathrm{i}}_{\mathrm{bf}} \in SO(3)$ that brings the inertial frame $\mathcal{F}_{\mathrm{i}}$ onto the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ is given by $C^{\mathrm{i}}_{\mathrm{bf}} \triangleq C(\varepsilon_{\mathrm{bf}}, \zeta_{\mathrm{bf}})$.

Next, we develop the attitude dynamics of the follower spacecraft relative to the leader spacecraft. Let $(\varepsilon_{\mathrm{r}}(t),\, \zeta_{\mathrm{r}}(t)) \in \mathbb{R}^3 \times \mathbb{R}$ denote the unit quaternion characterizing the mismatch between the orientation of the follower spacecraft $\mathcal{F}_{\mathrm{bf}}$ and the orientation of the leader spacecraft $\mathcal{F}_{\mathrm{b}\ell}$. In addition, $(\varepsilon_{\mathrm{r}},\, \zeta_{\mathrm{r}})$ can be characterized using $(\varepsilon_{\mathrm{bf}},\, \zeta_{\mathrm{bf}})$ and $(\varepsilon_{\mathrm{b}\ell},\, \zeta_{\mathrm{b}\ell})$ as [5,8]

$$\begin{bmatrix} \varepsilon_{\mathrm{r}} \\ \zeta_{\mathrm{r}} \end{bmatrix} = F(\varepsilon_{\mathrm{bf}}, \zeta_{\mathrm{bf}}, \varepsilon_{\mathrm{b}\ell}, \zeta_{\mathrm{b}\ell}) \triangleq \begin{bmatrix} \zeta_{\mathrm{b}\ell}\varepsilon_{\mathrm{bf}} - \zeta_{\mathrm{bf}}\varepsilon_{\mathrm{b}\ell} + \varepsilon^{\times}_{\mathrm{bf}}\varepsilon_{\mathrm{b}\ell} \\ \zeta_{\mathrm{b}\ell}\zeta_{\mathrm{bf}} + \varepsilon^{T}_{\mathrm{b}\ell}\varepsilon_{\mathrm{bf}} \end{bmatrix}. \tag{21}$$

The corresponding rotation matrix $C^{\mathrm{b}\ell}_{\mathrm{bf}} \in SO(3)$ that brings the leader spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{b}\ell}$ onto the follower spacecraft body-fixed frame $\mathcal{F}_{\mathrm{bf}}$ is given by $C^{\mathrm{b}\ell}_{\mathrm{bf}} = C(\varepsilon_{\mathrm{r}}, \zeta_{\mathrm{r}})$ and satisfies $C^{\mathrm{b}\ell}_{\mathrm{bf}} = C^{\mathrm{i}}_{\mathrm{bf}} C^{\mathrm{i}\,T}_{\mathrm{b}\ell}$.

Next, let $\vec{\omega}_{\mathrm{r}}(t)$ denote the angular velocity of $\mathcal{F}_{\mathrm{bf}}$ relative to $\mathcal{F}_{\mathrm{b}\ell}$. Then, it follows that $\vec{\omega}_{\mathrm{r}} = \vec{\omega}_{\mathrm{bf}} - \vec{\omega}_{\mathrm{b}\ell}$. Now using the framework of Section 2, we express $\vec{\omega}_{\mathrm{r}}$ in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as

$$\omega_{\mathrm{r}} \triangleq \vec{F}_{\mathrm{bf}} \cdot \vec{\omega}_{\mathrm{r}}, \tag{22}$$

where $\omega_{\mathrm{r}}(t) \in \mathbb{R}^3$. In addition, we can obtain

$$\omega_{\mathrm{r}} = \omega_{\mathrm{bf}} - C^{\mathrm{b}\ell}_{\mathrm{bf}}\omega_{\mathrm{b}\ell}, \tag{23}$$

$$\overset{\circ}{\vec{\omega}}_{\mathrm{r}} = \overset{\circ}{\vec{\omega}}_{\mathrm{bf}} - \vec{\omega}_{\mathrm{bf}} \times \vec{\omega}_{\mathrm{r}} - \overset{\circ}{\vec{\omega}}_{\mathrm{b}\ell}. \tag{24}$$

Expressing $\overset{\circ}{\vec{\omega}}_{\mathrm{r}}$ in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as $\dot{\omega}_{\mathrm{r}} \triangleq \vec{F}_{\mathrm{bf}} \cdot \overset{\circ}{\vec{\omega}}_{\mathrm{r}}$, $\dot{\omega}_{\mathrm{r}}(t) \in \mathbb{R}^3$, we can now express (24) in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$. Finally, multiplying the resultant expression by J_{f} on both sides, we obtain

$$J_{\mathrm{f}}\dot{\omega}_{\mathrm{r}} = J_{\mathrm{f}}\dot{\omega}_{\mathrm{bf}} - J_{\mathrm{f}}\omega^{\times}_{\mathrm{bf}}\omega_{\mathrm{r}} - J_{\mathrm{f}}C^{\mathrm{b}\ell}_{\mathrm{bf}}\dot{\omega}_{\mathrm{b}\ell}. \tag{25}$$

We now use (15), (19), (23), and (25) to obtain the following attitude dynamics of the follower spacecraft relative to the leader spacecraft

$$J_{\mathrm{f}}\dot{\omega}_{\mathrm{r}} = -\left(\omega_{\mathrm{r}} + C^{\mathrm{b}\ell}_{\mathrm{bf}}\omega_{\mathrm{b}\ell}\right)^{\times} J_{\mathrm{f}}\left(\omega_{\mathrm{r}} + C^{\mathrm{b}\ell}_{\mathrm{bf}}\omega_{\mathrm{b}\ell}\right) - J_{\mathrm{f}}\left(C^{\mathrm{b}\ell}_{\mathrm{bf}}\omega_{\mathrm{b}\ell}\right)^{\times}\omega_{\mathrm{r}} - J_{\mathrm{f}}C^{\mathrm{b}\ell}_{\mathrm{bf}}J^{-1}_{\ell}\left(-\omega^{\times}_{\mathrm{b}\ell}J_{\ell}\omega_{\mathrm{b}\ell} + \tau_{\mathrm{g}\ell} + \tau_{\ell}\right) + \tau_{\mathrm{gf}} + \tau_{\mathrm{f}}. \tag{26}$$

In addition, the attitude kinematics of the follower spacecraft relative to the leader spacecraft is given by

$$\begin{bmatrix} \dot{\varepsilon}_{\mathrm{r}} \\ \dot{\zeta}_{\mathrm{r}} \end{bmatrix} = E(\varepsilon_{\mathrm{r}}, \zeta_{\mathrm{r}})\omega_{\mathrm{r}}. \tag{27}$$

Next, we characterize the desired orientation of the follower spacecraft relative to the leader spacecraft using a desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$. Let $\vec{\omega}_{\mathrm{r_d}}(t)$ denote the desired angular velocity of $\mathcal{F}_{\mathrm{r_d}}$ with respect to $\mathcal{F}_{\mathrm{b}\ell}$ and let $\overset{\odot}{\vec{\omega}}_{\mathrm{r_d}}(t)$ denote the time derivative of $\vec{\omega}_{\mathrm{r_d}}$ measured in $\mathcal{F}_{\mathrm{r_d}}$. Using the framework of Section 2, we express $\vec{\omega}_{\mathrm{r_d}}$ and $\overset{\odot}{\vec{\omega}}_{\mathrm{r_d}}$ in the desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$ as follows $\begin{bmatrix} \omega_{\mathrm{r_d}} & \dot{\omega}_{\mathrm{r_d}} \end{bmatrix} \triangleq \vec{F}_{\mathrm{r_d}} \cdot \begin{bmatrix} \vec{\omega}_{\mathrm{r_d}} & \overset{\odot}{\vec{\omega}}_{\mathrm{r_d}} \end{bmatrix}$, where $\omega_{\mathrm{r_d}}(t), \dot{\omega}_{\mathrm{r_d}}(t) \in \mathbb{R}^3$. The angular orientation of the desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$ with respect to the leader spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{b}\ell}$ is characterized by the desired unit quaternion $(\varepsilon_{\mathrm{r_d}}(t),\, \zeta_{\mathrm{r_d}}(t)) \in \mathbb{R}^3 \times \mathbb{R}$, whose kinematics is governed by

$$\begin{bmatrix} \dot{\varepsilon}_{\mathrm{r_d}} \\ \dot{\zeta}_{\mathrm{r_d}} \end{bmatrix} = E(\varepsilon_{\mathrm{r_d}}, \zeta_{\mathrm{r_d}})\omega_{\mathrm{r_d}}. \tag{28}$$

The corresponding rotation matrix $C^{\mathrm{b}\ell}_{\mathrm{rd}} \in SO(3)$ that brings the leader spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{b}\ell}$ onto the desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$ is given by $C^{\mathrm{b}\ell}_{\mathrm{rd}} = C(\varepsilon_{\mathrm{r_d}}, \zeta_{\mathrm{r_d}})$. Using (28), it follows that $\omega_{\mathrm{r_d}} = 2\left(\zeta_{\mathrm{r_d}}\dot{\varepsilon}_{\mathrm{r_d}} - \dot{\zeta}_{\mathrm{r_d}}\varepsilon_{\mathrm{r_d}}\right) - 2\varepsilon^{\times}_{\mathrm{r_d}}\dot{\varepsilon}_{\mathrm{r_d}}$ and $\dot{\omega}_{\mathrm{r_d}} = 2\left(\zeta_{\mathrm{r_d}}\ddot{\varepsilon}_{\mathrm{r_d}} - \ddot{\zeta}_{\mathrm{r_d}}\varepsilon_{\mathrm{r_d}}\right) - 2\varepsilon^{\times}_{\mathrm{r_d}}\ddot{\varepsilon}_{\mathrm{r_d}}$. In this paper, we will assume that $\varepsilon_{\mathrm{r_d}}$, $\zeta_{\mathrm{r_d}}$, and their first two time derivatives are all bounded functions of time, which along with the above form of $\omega_{\mathrm{r_d}}$ and $\dot{\omega}_{\mathrm{r_d}}$ equations yields boundedness of $\omega_{\mathrm{r_d}}$ and $\dot{\omega}_{\mathrm{r_d}}$, respectively.

Now we develop the error dynamics of the attitude motion of the follower spacecraft relative to the leader spacecraft. Let $(e_{\varepsilon_{\mathrm{r}}}(t),\, e_{\zeta_{\mathrm{r}}}(t)) \in \mathbb{R}^3 \times \mathbb{R}$ denote the unit quaternion characterizing the mismatch between the actual orientation of the follower spacecraft $\mathcal{F}_{\mathrm{bf}}$ relative to the leader spacecraft $\mathcal{F}_{\mathrm{b}\ell}$ and the desired orientation of the follower spacecraft $\mathcal{F}_{\mathrm{r_d}}$ relative to the leader spacecraft $\mathcal{F}_{\mathrm{b}\ell}$. Next, following (21), $(e_{\varepsilon_{\mathrm{r}}},\, e_{\zeta_{\mathrm{r}}})$ can be characterized using $(\varepsilon_{\mathrm{r}},\, \zeta_{\mathrm{r}})$ and $(\varepsilon_{\mathrm{r_d}},\, \zeta_{\mathrm{r_d}})$ as $\left[e^{T}_{\varepsilon_{\mathrm{r}}}\ e_{\zeta_{\mathrm{r}}}\right]^{T} = F(\varepsilon_{\mathrm{r}}, \zeta_{\mathrm{r}}, \varepsilon_{\mathrm{r_d}}, \zeta_{\mathrm{r_d}})$. The corresponding rotation matrix $C^{\mathrm{rd}}_{\mathrm{bf}} \in SO(3)$ that brings the desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$ onto the follower spacecraft body-fixed frame $\mathcal{F}_{\mathrm{bf}}$ is given by [5,8]

$$C^{\mathrm{rd}}_{\mathrm{bf}} = C(e_{\varepsilon_{\mathrm{r}}}, e_{\zeta_{\mathrm{r}}}), \tag{29}$$

and satisfies

$$C_{\rm bf}^{\rm rd} = C_{\rm bf}^{\rm b\ell} C_{\rm rd}^{\rm b\ell\,T}. \tag{30}$$

Next, let $\vec{\omega}_{\rm e_r}(t)$ denote the angular velocity of $\mathcal{F}_{\rm bf}$ with respect to $\mathcal{F}_{\rm r_d}$. Then, it follows that

$$\vec{\omega}_{\rm e_r} = \vec{\omega}_{\rm r} - \vec{\omega}_{\rm r_d}, \tag{31}$$

Now using the framework of Section 2, we express $\vec{\omega}_{\rm e_r}$ in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\rm bf}$ as

$$\omega_{\rm e_r} \triangleq \vec{F}_{\rm bf} \cdot \vec{\omega}_{\rm e_r}, \tag{32}$$

where $\omega_{\rm e_r}(t) \in \mathbb{R}^3$. In addition, we can obtain

$$\omega_{\rm e_r} = \omega_{\rm r} - C_{\rm bf}^{\rm rd}\omega_{\rm r_d}, \tag{33}$$

$$\overset{\circ}{\vec{\omega}}_{\rm e_r} = \overset{\circ}{\vec{\omega}}_{\rm r} - \vec{\omega}_{\rm bf} \times \left(\vec{\omega}_{\rm e_r} - \vec{\omega}_{\rm r}\right) - \overset{\circledcirc}{\vec{\omega}}_{\rm r_d}. \tag{34}$$

Expressing $\overset{\circ}{\vec{\omega}}_{\rm e_r}$ in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\rm bf}$ as $\dot{\omega}_{\rm e_r} \triangleq \vec{F}_{\rm bf} \cdot \overset{\circ}{\vec{\omega}}_{\rm e_r}$, $\dot{\omega}_{\rm e_r}(t) \in \mathbb{R}^3$, noting that $\dot{\omega}_{\rm r} = \vec{F}_{\rm bf} \cdot \overset{\circ}{\vec{\omega}}_{\rm r}$, and using (9), (22), and (32), we can now express (34) in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\rm bf}$. Finally, multiplying the resultant expression by $J_{\rm f}$ on both sides, we obtain

$$J_{\rm f}\dot{\omega}_{\rm e_r} = J_{\rm f}\dot{\omega}_{\rm r} + J_{\rm f}\omega_{\rm bf}^{\times}(\omega_{\rm r} - \omega_{\rm e_r}) - J_{\rm f}C_{\rm bf}^{\rm rd}\dot{\omega}_{\rm r_d}. \tag{35}$$

We now use (26) and (33) to obtain the following open-loop attitude tracking error dynamics of the follower spacecraft relative to the desired attitude reference frame $\mathcal{F}_{\rm r_d}$

$$\begin{aligned} J_{\rm f}\dot{\omega}_{\rm e_r} &= -\left(\omega_{\rm e_r} + C_{\rm bf}^{\rm rd}\omega_{\rm r_d} + C_{\rm bf}^{\rm b\ell}\omega_{\rm b\ell}\right)^{\times} J_{\rm f}\left(\omega_{\rm e_r} + C_{\rm bf}^{\rm rd}\omega_{\rm r_d}\right.\\ &\left.+C_{\rm bf}^{\rm b\ell}\omega_{\rm b\ell}\right) - J_{\rm f}\left(C_{\rm bf}^{\rm b\ell}\omega_{\rm b\ell}\right)^{\times}\left(\omega_{\rm e_r} + C_{\rm bf}^{\rm rd}\omega_{\rm r_d}\right) + J_{\rm f}\left(\omega_{\rm e_r} + C_{\rm bf}^{\rm rd}\omega_{\rm r_d}\right.\\ &\left.+C_{\rm bf}^{\rm b\ell}\omega_{\rm b\ell}\right)^{\times} C_{\rm bf}^{\rm rd}\omega_{\rm r_d} - J_{\rm f}C_{\rm bf}^{\rm rd}\dot{\omega}_{\rm r_d} - J_{\rm f}C_{\rm bf}^{\rm b\ell}J_{\ell}^{-1}\Big(-\omega_{\rm b\ell}^{\times}J_{\ell}\omega_{\rm b\ell}\\ &+\frac{3\mu}{||R_\ell||^5}R_\ell^{\times}J_\ell R_\ell + \tau_\ell\Big) + \frac{3\mu}{||R_{\rm f}||^5}R_{\rm f}^{\times}J_{\rm f}R_{\rm f} + \tau_{\rm f}. \end{aligned} \tag{36}$$

In addition, using (27), (28), (29), and (33), the open-loop attitude tracking error kinematics is given by

$$\begin{bmatrix} \dot{e}_{\varepsilon_r} \\ \dot{e}_{\zeta_r} \end{bmatrix} = E(e_{\varepsilon_r}, e_{\zeta_r})\omega_{\rm e_r}. \tag{37}$$

In this paper, the control objective for the attitude dynamics of the follower spacecraft relative to the leader spacecraft requires that the actual attitude of the follower spacecraft track the desired attitude trajectory, i.e., the rotation matrix $C_{\rm bf}^{\rm b\ell}$ must coincide with the rotation matrix $C_{\rm rd}^{\rm b\ell}$ in steady-state. Using (30), this control objective can be equivalently characterized as $\lim_{t\to\infty} C_{\rm bf}^{\rm rd} = I_3$. Furthermore, it is required that $\vec{\omega}_{\rm r}(t) \to \vec{\omega}_{\rm r_d}(t)$ as $t \to \infty$. With the aid of the unit norm constraint on $(e_{\varepsilon_r}, e_{\zeta_r})$ and (29), (31), and (32), the follower spacecraft attitude tracking control objective can be equivalently stated as follows

$$\lim_{t\to\infty} e_{\varepsilon_r}(t),\ \omega_{\rm e_r}(t) = 0. \tag{38}$$

4. Adaptive Control Law Design for the Follower Spacecraft

In this section, we address the translational and attitude tracking control problem of the follower spacecraft motion relative to the leader spacecraft. The goal of the problem is to design a globally asymptotically convergent controller satisfying (13) and (38) for the open-loop translational and attitude relative error dynamics given by (11), (12), (36), and (37) without the knowledge of the follower and leader spacecraft mass $m_{\rm f}, m_\ell$ and inertia matrices $J_{\rm f}, J_\ell$.

We begin by defining two linear operators L_1, L_2: $\mathbb{R}^3 \to \mathbb{R}^{3\times 6}$ acting on $a = [a_1\ a_2\ a_3]^T$ such that $L_1(a) \triangleq \begin{bmatrix} a_1 & 0 & 0 & 0 & a_3 & a_2 \\ 0 & a_2 & 0 & a_3 & 0 & a_1 \\ 0 & 0 & a_3 & a_2 & a_1 & 0 \end{bmatrix}$, $L_2(a) \triangleq \begin{bmatrix} a_1 & a_1 & a_1 & \\ a_2 & a_2 & a_2 & 0_{3\times 3} \\ a_3 & a_3 & a_3 & \end{bmatrix}$. Next, define $\alpha_\ell \triangleq [\,J_{\ell_{11}}\ J_{\ell_{22}}\ J_{\ell_{33}}\ J_{\ell_{23}}\ J_{\ell_{13}}\ J_{\ell_{12}}]^T$ and $\alpha_{\rm f} \triangleq [J_{\rm f_{11}}\ J_{\rm f_{22}}\ J_{\rm f_{33}}\ J_{\rm f_{23}}\ J_{\rm f_{13}}\ J_{\rm f_{12}}]^T$, where $(\cdot)_{ij}$, $i,j = 1,2,3$, denotes the $ij^{\rm th}$ element of the corresponding matrix. Then, it follows that

$$J_{\rm k}a = L_1(a)\alpha_{\rm k}, \quad {\rm tr}(J_{\rm k})a = L_2(a)\alpha_{\rm k}, \quad {\rm k} \in \{\ell, {\rm f}\}. \tag{39}$$

The design of an adaptive feedback controller for (11), (12), (36), and (37) necessitates parametrization of these equations such that the leader/follower spacecraft mass and inertia matrices appearing in these equations are isolated as an unknown parameter vector. The linear operators L_1, L_2 facilitate the parametrization of most of the terms containing J_ℓ and $J_{\rm f}$ in the error dynamics. Finally, through recursive use of (39) and some judicious matrix transformations it can be shown that $J_{\rm f}C_{\rm bf}^{\rm b\ell}J_\ell^{-1}\omega_{\rm b\ell}^{\times}J_\ell\omega_{\rm b\ell} = A_3\beta_1$, $J_{\rm f}C_{\rm bf}^{\rm b\ell}J_\ell^{-1}R_\ell^{\times}J_\ell R_\ell = \bar{A}_3\beta_1$, and $J_{\rm f}C_{\rm bf}^{\rm b\ell}J_\ell^{-1}\tau_\ell = A_4\beta_2$, where

$$A_1 \triangleq \left[L_1(C_1)\ L_1(C_2)\ L_1(C_3)\right], \quad C_i \triangleq {\rm col}_i(C_{\rm bf}^{\rm b\ell}),$$

$$A_2 \triangleq \omega_{\rm b\ell}^{\times}L_1(\omega_{\rm b\ell}), \quad a_i \triangleq {\rm col}_i(A_2),$$

$$\hat{a} \triangleq \begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix}^T, \quad \hat{A}_2 \triangleq \underbrace{\begin{bmatrix} \hat{a} & \cdots & 0_{1\times 18} \\ \vdots & \ddots & \vdots \\ 0_{1\times 18} & \cdots & \hat{a} \end{bmatrix}}_{18}, \quad A_3 \triangleq A_1\hat{A}_2,$$

$$B_1 \triangleq \begin{bmatrix} \alpha_{\rm f} & 0_{6\times 1} & 0_{6\times 1} \\ 0_{6\times 1} & \alpha_{\rm f} & 0_{6\times 1} \\ 0_{6\times 1} & 0_{6\times 1} & \alpha_{\rm f} \end{bmatrix}, \quad B_2 \triangleq B_1 J_\ell^{-1}, \quad b_i \triangleq {\rm row}_i(B_2),$$

$$\hat{b}_i \triangleq \underbrace{\begin{bmatrix} b_i & \cdots & 0_{1\times 3} \\ \vdots & \ddots & \vdots \\ 0_{1\times 3} & \cdots & b_i \end{bmatrix}}_{6}, \quad \hat{B}_2 \triangleq \begin{bmatrix} \hat{b}_1^T \\ \vdots \\ \hat{b}_{18}^T \end{bmatrix}, \quad \beta_1 \triangleq \hat{B}_2\alpha_\ell,$$

$$\bar{A}_2 \triangleq R_\ell^{\times}L_1(R_\ell), \quad \bar{a}_i \triangleq {\rm col}_i(\bar{A}_2),$$

$$\check{a} \triangleq \begin{bmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_6 \end{bmatrix}^T, \quad \check{A}_2 \triangleq \underbrace{\begin{bmatrix} \check{a} & \cdots & 0_{1\times 18} \\ \vdots & \ddots & \vdots \\ 0_{1\times 18} & \cdots & \check{a} \end{bmatrix}}_{18}, \quad \bar{A}_3 \triangleq A_1\check{A}_2,$$

$$A_4 \triangleq A_1 \underbrace{\begin{bmatrix} \tau_\ell^T & \cdots & 0_{1\times 3} \\ \vdots & \ddots & \vdots \\ 0_{1\times 3} & \cdots & \tau_\ell^T \end{bmatrix}}_{18}, \quad \beta_2 \triangleq \begin{bmatrix} b_1^T \\ \vdots \\ b_{18}^T \end{bmatrix}.$$

Let $\theta_{\rm f}$ denote the unknown parameters as follows $\theta_{\rm f} \triangleq [\ m_{\rm f}\ \ \frac{m_{\rm f}}{m_\ell}\ \ \alpha_{\rm f}^T\ \ \frac{m_{\rm f}}{m_\ell}\alpha_\ell^T\ \ \beta_1^T\ \ \beta_2^T\ ]^T$. To account for the follower and leader spacecraft parameter uncertainty, the control law will contain an adaptation mechanism for on-line estimation of the unknown parameter vector θ_f. Let $\hat{\theta}_f(t) \in \mathbb{R}^{392}$ denote the dynamic estimate of the unknown parameter vector θ_f. The mismatch between the actual and estimated parameters of the follower and leader spacecraft is quantified by the parameter estimation error $\tilde{\theta}_f(t)$ defined as $\tilde{\theta}_f \triangleq \theta_f - \hat{\theta}_f$.

To facilitate the development of tracking control law, we define the filtered translational and attitude relative tracking errors $r_{tr} \triangleq e_{V_r} + \Lambda_{tr} e_{R_r}$ and $r_{ar} \triangleq \omega_{e_r} + \Lambda_{ar} e_{\varepsilon_r}$, respectively, where $r_{tr}(t)$, $r_{ar}(t) \in \mathbb{R}^3$ and Λ_{tr}, $\Lambda_{ar} \in \mathbb{R}^{3\times 3}$ are constant, positive-definite, diagonal, control gain matrices. Defining $r_r \triangleq \left[r_{tr}^T \;\; r_{ar}^T\right]^T$, $r_r(t) \in \mathbb{R}^6$, the combined translational and attitude filtered relative tracking error is given as $r_r = \begin{bmatrix} e_{V_r} + \Lambda_{tr} e_{R_r} \\ \omega_{e_r} + \Lambda_{ar} e_{\varepsilon_r} \end{bmatrix}$.

For notational convenience, we define $\hat{R}_k \triangleq \begin{bmatrix} R_{k_1} & 0 & 0 \\ 0 & R_{k_2} & 0 \\ 0 & 0 & R_{k_3} \end{bmatrix} \begin{bmatrix} R_k{}^T \\ R_k{}^T \\ R_k{}^T \end{bmatrix}$, $M_f \triangleq \begin{bmatrix} m_f I_3 & 0_{3\times 3} \\ 0_{3\times 3} & J_f \end{bmatrix}$, and $u_f \triangleq \begin{bmatrix} f_f \\ \tau_f \end{bmatrix}$, where $k \in \{\ell, f\}$, $R_{k_i} \in \mathbb{R}$, $i = 1, 2, 3$, denotes the i^{th} component of the column vector R_k, and $u_f(t) \in \mathbb{R}^6$. We initiate the control design by rewriting the dynamics of (11), (12), (36), and (37) in terms of the filtered relative tracking error vector r_r. Specifically, differentiating r_r with respect to time, multiplying both sides of the resulting equation by M_f, using (11), (12), (36), (37), $J_f C_{bf}^{b\ell} J_\ell^{-1} \omega_{b\ell}^\times J_\ell \omega_{b\ell} = A_3 \beta_1$, $J_f C_{bf}^{b\ell} J_\ell^{-1} R_\ell^\times J_\ell R_\ell = \bar{A}_3 \beta_1$, and $J_f C_{bf}^{b\ell} J_\ell^{-1} \tau_\ell = A_4 \beta_2$, and rearranging terms yield $M_f \dot{r}_r = W_f \theta_f + u_f$, where

$$W_f \triangleq \begin{bmatrix} W_{f1} & W_{f2} & W_{f3} & W_{f4} & 0_{3\times 324} & 0_{3\times 54} \\ 0_{3\times 1} & 0_{3\times 1} & W_{f5} & 0_{3\times 6} & W_{f6} & W_{f7} \end{bmatrix},$$

$$W_{f1} \triangleq DV + \frac{\mu}{\|R_f\|^3} R_f - C_{bf}^{b\ell} \frac{\mu}{\|R_\ell\|^3} R_\ell - \left(\omega_{e_r} + C_{bf}^{rd} \omega_{r_d} + C_{bf}^{b\ell} \omega_{b\ell}\right)^\times e_{V_r} + \Lambda_{tr}\left(e_{V_r} - \omega_{bf}^\times e_{R_r}\right), \quad W_{f2} \triangleq -C_{bf}^{b\ell} f_\ell,$$

$$W_{f3} \triangleq \frac{3\mu}{2\|R_f\|^5}\left\{L_2(R_f) + 2L_1(R_f) - \frac{5\hat{R}_f L_1(R_f)}{\|R_f\|^2}\right\},$$

$$W_{f4} \triangleq -\frac{3\mu}{2\|R_\ell\|^5} C_{bf}^{b\ell}\left\{L_2(R_\ell) + 2L_1(R_\ell) - \frac{5\hat{R}_\ell L_1(R_\ell)}{\|R_\ell\|^2}\right\},$$

$$W_{f5} \triangleq L_1\left(\tfrac{1}{2}\Lambda_{ar}\left(e_{\varepsilon_r}^\times + e_{\zeta_r} I_3\right)\omega_{e_r}\right) - \left(\omega_{e_r} + C_{bf}^{rd}\omega_{r_d} + C_{bf}^{b\ell}\omega_{b\ell}\right)^\times L_1\left(\omega_{e_r} + C_{bf}^{rd}\omega_{r_d} + C_{bf}^{b\ell}\omega_{b\ell}\right) - L_1\left(C_{bf}^{b\ell}\omega_{b\ell}^\times\left(\omega_{e_r} + C_{bf}^{rd}\omega_{r_d}\right)\right) + L_1\left(\left(\omega_{e_r} + C_{bf}^{rd}\omega_{r_d} + C_{bf}^{b\ell}\omega_{b\ell}\right)^\times C_{bf}^{rd}\omega_{r_d}\right) - L_1\left(C_{bf}^{rd}\dot{\omega}_{r_d}\right) + \frac{3\mu}{\|R_f\|^5} R_f^\times L_1(R_f),$$

$$W_{f6} \triangleq A_3 - \frac{3\mu}{\|R_\ell\|^5}\bar{A}_3, \qquad W_{f7} \triangleq -A_4.$$

Theorem 4.1. Let $K_f \in \mathbb{R}^{6\times 6}$ be a constant, positive-definite, diagonal control gain matrix and let $\Gamma_f \in \mathbb{R}^{392\times 392}$ be a constant, positive-definite, diagonal adaptation gain matrix. Then, the adaptive control law with the control input $u_f(t)$ and the adaptation law for $\hat{\theta}_f(t)$ given by $u_f = -W_f \hat{\theta}_f - K_f r_r - \begin{bmatrix} k_{fR} e_{R_r} \\ k_{f\varepsilon} e_{\varepsilon_r} \end{bmatrix}$ and $\dot{\hat{\theta}}_f = \Gamma_f W_f^T r_r$, respectively, ensures the global asymptotic convergence of the follower spacecraft relative translational position and velocity tracking errors and relative attitude position and velocity tracking errors as delineated by (13) and (38).

Proof. The proof follows by showing that the time derivative of positive-definite function $V \triangleq \frac{1}{2} k_{fR} e_{R_r}{}^T e_{R_r} + \frac{1}{2} r_r{}^T M_f r_r + \frac{1}{2}\tilde{\theta}_f^T \Gamma_f^{-1}\tilde{\theta}_f + k_{f\varepsilon}\left(e_{\varepsilon_r}^T e_{\varepsilon_r} + (e_{\zeta_r} - 1)^2\right)$, where $k_{fR}, k_{f\varepsilon} > 0$, is negative-semidefinite. Next, standard signal chasing arguments are employed to show that all signals in the closed-loop system remain bounded. Finally, Barbalat's Lemma is used to accomplish the result of (13) and (38). □

Remark 4.1. If the leader and follower spacecraft body-fixed reference frame $\mathcal{F}_{b\ell}$ and $\mathcal{F}_{bf}$ are chosen to be aligned with their respective principle axes then J_ℓ and J_f are diagonal [8] and α_k reduces to $\alpha_k \triangleq [J_{k_{11}} \; J_{k_{22}} \; J_{k_{33}}]^T$, $k \in \{\ell, f\}$. Furthermore, the dimensions of the intermediate variables $B_1, B_2, \hat{A}_2, \hat{B}_2, \breve{A}_2, A_3, \bar{A}_3, A_4, \beta_1$, and β_2, the parameter vector θ_f, and the adaptation gain matrix Γ_f are appropriately reduced.

5. Conclusion

In this paper, we addressed a tracking control problem for the coupled translational and attitude motion of a follower spacecraft relative to a leader spacecraft. In particular, the gravity induced mutual coupling in the translational and attitude motion of the leader and follower spacecraft was properly accounted and a Lyapunov based tracking controller was designed. The resulting controller required no knowledge of the mass and inertia parameters of the leader and follower spacecraft. The controller was shown to ensure that the follower spacecraft globally asymptotically tracked the desired translational and angular position and velocity trajectories relative to the leader spacecraft.

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