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Engineering Notes

Solutions of Tschauner–Hempel Equations

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I. Introduction

THE well-known Tschauner–Hempel equations [1–3] developed in the 1960s provide the means for a direct approach to the linearized relative motion problem near a satellite in Keplerian orbit. The solution of these equations for arbitrary eccentricity, however, was found earlier by Lawden [4] in 1954 to describe the primer vector associated with the unpowered part of a fuel-optimal rocket trajectory in an inverse square law gravitational field. The solution obtained by Lawden [4] involves the use of an integral I. Carter and Humi [5] also studied the Tschauner–Hempel equations and evaluated the integral I in a more elegant way to express the solution for general noncircular Keplerian orbits in 1987. Later, however, Carter [6] found that the integral I is singular when the true anomaly f is a multiple of π. To remove the singularities, Carter [6] instead used a new integral J to construct the solution. However, the solution presented in terms of J becomes singular at $e = 0$. To avoid the troublesome singularity, Carter [7] transformed the integrand in J into another form that involved a new integral K. It is difficult to evaluate the integrals J and K using the true anomaly. Instead of using the true anomaly, Carter [6,7] performed the evaluation of J and K using an eccentric anomaly for elliptical orbits and a hyperbolic anomaly for hyperbolic orbits. As a consequence, the resulting solution was not particularly simple for engineering use. To overcome this problem, Yamanaka and Ankersen [8] proposed a new integral term in 2002 to replace the integrals suggested by Carter [6,7]. The suggested new integral could be easily calculated from the transition time, thus avoiding the eccentric or hyperbolic anomaly. Yamanaka and Ankersen [8] studied only the case of elliptical orbits. Using the Wronskian of Yamanaka and Ankersen [8], this Note shows that the solution obtained there is also valid for the hyperbolic orbits: $e > 1$. Unfortunately, the solution of Yamanaka and Ankersen [8] fails when the orbit is parabolic: $e = 1$.

These two existing solutions, respectively, proposed by Carter [7] and Yamanaka and Ankersen [8] for the Tschauner–Hempel equations have received considerable attention from subsequent researchers. Some have thought that the solution of Yamanaka and Ankersen [8] for elliptic orbits completely differed from the one of Carter [7]. However, this Note proves that the two independent solutions for the case of elliptic and hyperbolic orbits are actually equivalent. This conclusion is achieved using a new integral D presented herein. According to this finding, Carter's solution [7] is modified here for Tschauner–Hempel equations by expressing it in terms of the true anomaly and transition time rather than the eccentric anomaly or hyperbolic anomaly. This new solution is valid for any Keplerian orbit ($e \geq 0$) and is simpler and more convenient than other versions found in the literature.

The remainder of this Note is organized as follows. Section II presents the Tschauner–Hempel equations and the solutions obtained by Carter [7] and Yamanaka and Ankersen [8]. Section III describes our recently found integral and the proof of the equivalent of the two solutions. The modified solutions and state transition matrix of the Tschauner–Hempel equations expressed in terms of the true anomaly and transition time are also presented in Sec. III. Finally, the conclusions can be found in Sec. IV.

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II. Existing Solutions of Tschauner–Hempel Equations

A. Tschauner–Hempel Equations

The local-vertical/local-horizontal (LVLH) frame [9] is adopted to describe the relative motion in this Note. The following ordering of the state variables is therefore different from that used by Carter [7] and Yamanaka and Ankersen [8]. We define $\boldsymbol{X} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$ as the state variable of the deputy in the LVLH frame attached to the chief. Here, the dot indicates the time derivative. By using the following transformation, the actual position is changed into a relatively concise form:

$$\left[\tilde{x}, \tilde{y}, \tilde{z}, \tilde{x}', \tilde{y}', \tilde{z}'\right]^T = A(f) \cdot \left[x, y, z, \dot{x}, \dot{y}, \dot{z}\right]^T \tag{1}$$

where the prime indicates the first derivative with respect to the true anomaly, and the 6×6 real matrix $A(f)$ follows from the chain rule:

$$A(f) = \begin{bmatrix} \frac{\mu(1+e\cos f)}{h^{3/2}} \cdot \boldsymbol{I}_{3\times 3} & 0_{3\times 3} \\ -\frac{\mu e \sin f}{h^{\frac{3}{2}}} \cdot \boldsymbol{I}_{3\times 3} & \frac{h^{3/2}}{\mu(1+e\cos f)} \cdot \boldsymbol{I}_{3\times 3} \end{bmatrix} \tag{2}$$

where e and f are the eccentricity and true anomaly of the chief, respectively. The term μ is the universal gravitational constant times the mass of Earth, and h is the constant orbital angular momentum of the chief. The Tschauner–Hempel equations for the relative motion of a spacecraft are as follows [7,8]:

$$\tilde{x}'' = 3\tilde{x}/\rho(f) + 2\tilde{y}' \tag{3a}$$

$$\tilde{y}'' = -2\tilde{x}' \tag{3b}$$

$$\tilde{z}'' = -\tilde{z} \tag{3c}$$

where $\rho(f)$ is defined as follows:

$$\rho(f) = 1 + e\cos f \tag{4}$$

B. Carter Solution of Tschauner–Hempel Equations

It is easily seen that the z component is the equation of a harmonic oscillator. It can be analytically integrated as $\tilde{z} = c_5 \cos f + c_6 \sin f$. However, the x and y components are coupled with each other. After some manipulation, the problem is reduced to solving the following second-order differential equation [7,8]:

$$\tilde{x}'' + (4 - 3/\rho(f))\tilde{x} = -2c_3 \tag{5}$$

where c_3 is an arbitrary constant of integration. The homogeneous form of the preceding differential equation is as follows:

$$\tilde{x}'' + (4 - 3/\rho(f))\tilde{x} = 0 \tag{6}$$

One of the solutions of Eq. (6) is

$$\varphi_1 = \rho(f)\sin f \tag{7}$$

In 1954, Lawden [4] proposed a second solution:

$$\varphi_2 = \rho(f)\sin f \cdot I(f) \tag{8}$$

where the integral $I(f)$ is defined as follows:

$$I(f) = \int \frac{1}{\sin^2 f \cdot \rho^2(f)}\,\mathrm{d}f \tag{9}$$

This integral was evaluated by Carter and Humi [5] in terms of elementary functions:

$$I(f) = \begin{cases} -\cot f, \quad e = 0 \\ \dfrac{6e^2}{(1-e^2)^{5/2}}\tan^{-1}\left(\dfrac{(e-1)G(f)}{(1-e^2)^{1/2}}\right) - N(f), \quad 0 < e < 1 \\ \dfrac{-2\cos^3 f + 4\cos^2 f + \cos f - 2}{5\sin f(1+\cos f)^2}, \quad e = 1 \\ \dfrac{3e^2}{(e^2-1)^{3/2}}\ln\left|\dfrac{(e-1)G(f) - (e^2-1)^{1/2}}{(e-1)G(f) + (e^2-1)^{1/2}}\right| - N(f), \quad e > 1 \end{cases} \tag{10}$$

where

$$G(f) = \sin f/(1+\cos f)$$

and

$$N(f) = \frac{1}{2(e-1)^2}\left(\frac{(5e^3-3e^2+3e-1)G^2(f) - (e^2-1)(e-1)}{(e-1)(e+1)^2G^3(f) - (e+1)^3G(f)} + G(f)\right) \tag{11}$$

As Carter and Humi [5] mentioned, φ_2 becomes singular at $\sin f = 0$. To solve this problem, Carter [6] proposed another solution of the form

$$\varphi_2 = 2e\rho(f)\sin f \cdot J(f) - \frac{\cos f}{\rho(f)} \tag{12}$$

where

$$J(f) = \int \frac{\cos f}{\rho^3(f)}\,\mathrm{d}f \tag{13}$$

The term $J(f)$ can be evaluated in the following way:

$$J(f) = \begin{cases} -(1-e^2)^{-5/2}\left[\dfrac{3e}{2}E - (1+e^2)\sin E + \dfrac{e}{2}\sin E\cos E\right], \qquad 0 \le e < 1 \\ \dfrac{1}{4}\tan\left(\dfrac{f}{2}\right) - \dfrac{1}{20}\tan^5\left(\dfrac{f}{2}\right), \qquad e = 1 \\ -(e^2-1)^{-5/2}\left[\dfrac{3e}{2}H - (1+e^2)\sinh H + \dfrac{e}{2}\sinh H\cosh H\right], \qquad e > 1 \end{cases} \tag{14}$$

where E and H denote the eccentric anomaly for the elliptic orbit and hyperbolic anomaly for the hyperbolic orbit, respectively. It is found that the Wronskian of φ_1 and φ_2, defined as $\varphi_1\varphi_2' - \varphi_2\varphi_1'$, is a nonzero constant; hence, the two solutions φ_1 and φ_2 are linearly independent. The third solution φ_3 can be obtained by the method of variation of the parameter in the following way:

$$\varphi_3 = 2[\varphi_1 S(\varphi_2) - \varphi_2 S(\varphi_1)] = -2\rho(f)J(f)\sin f - \frac{\cos^2 f}{\rho(f)} - \cos^2 f \tag{15}$$

where the notation $S(\varphi)$ indicates an antiderivative of a function φ of f, namely,

$$S(\varphi) = \int \varphi(f)\,\mathrm{d}f \tag{16}$$

Finally, Carter [6] obtained the complete solution of the Tschauner–Hempel equations for $e > 0$:

$$\tilde{x}(f) = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 \tag{17}$$

$$\tilde{y}(f) = -2c_1 S(\varphi_1) - 2c_2 S(\varphi_2) - c_3 S(2\varphi_3 + 1) + c_4 \tag{18}$$

$$\tilde{z}(f) = c_5\cos f + c_6\sin f \tag{19}$$

where $c_i (i = 1, \cdots, 6)$ are arbitrary constants, and the three antiderivative functions are the following:

$$S(\varphi_1) = -\cos f - \frac{e}{2}\cos^2 f \tag{20}$$

$$S(\varphi_2) = -\rho^2(f)J(f) \tag{21}$$

$$S(2\varphi_3 + 1) = \frac{2}{e}[\rho^2(f)J(f) - \sin f] - \sin f\cos f \tag{22}$$

It is observed that the antiderivative $S(2\varphi_3 + 1)$ becomes singular because of the troublesome appearance of e in the denominator. To remove this annoyance, Carter [7] modified the integral J as follows:

$$J(f) = \frac{\sin f}{\rho^3(f)} - 3eK(f) \tag{23}$$

where

$$K(f) = \int \frac{\sin^2 f}{\rho^4(f)}\,\mathrm{d}f \tag{24}$$

The evaluation of K leads to the following results:

$$K(f)=\begin{cases}(1-e^2)^{-5/2}\left(\dfrac{1}{2}E-\dfrac{1}{2}\sin E\cos E-\dfrac{e}{3}\sin^3 E\right), & 0\le e<1\\ \dfrac{1}{10}\tan^5\left(\dfrac{f}{2}\right)+\dfrac{1}{6}\tan^3\left(\dfrac{f}{2}\right), & e=1\\ (e^2-1)^{-5/2}\left(\dfrac{1}{2}H-\dfrac{1}{2}\sinh H\cosh H-\dfrac{e}{3}\sinh^3 H\right), & e>1\end{cases} \tag{25}$$

In this manner, Carter [7] successfully solved the Tschauner–Hempel equations for arbitrary Keplerian orbits. It is worth pointing out, however, that the solution of $K(f)$ at $e=1$ in the original paper (i.e., equation 65) of Carter [7] is wrong. The correct one is obtained and shown in Eq. (25) here. The solution obtained by Carter [7] is not preferred by many engineers who either do not know or do not like eccentric or hyperbolic anomalies.

C. Yamanaka and Ankersen Solution of Tschauner–Hempel Equations

Yamanaka and Ankersen [8] also studied the Tschauner–Hempel equations in 2002. They tried to present a simpler solution than the one of Carter [7]. The first solution φ_1 that they chose for the homogeneous second-order differential equation [i.e., Eq. (6)] was the same as the one of Carter [7]. They used an alternative solution, however, for φ_2:

$$\varphi_2=3e^2\rho(f)\sin f\cdot L(f)+\rho(f)\cos f-2e \tag{26}$$

that was written in terms of a new integral:

$$L(f)=\int\frac{1}{\rho^2(f)}\,\mathrm{d}f \tag{27}$$

This integral is easily evaluated, and the result is

$$L(f)=\sqrt{\frac{\mu}{p^3}}t \tag{28}$$

where t is the time.

It is found that φ_1 and φ_2 satisfy the following identity:

$$\varphi_1\varphi_2'-\varphi_2\varphi_1'=e^2-1 \tag{29}$$

Consequently, φ_1 and φ_2 are linearly independent for the case: $e\neq 1$. The third solution φ_3 can be obtained by using the same method as Carter [7]:

$$\varphi_3=\frac{2}{e^2-1}[\varphi_1 S(\varphi_2)-\varphi_2 S(\varphi_1)]=-\frac{\rho(f)\cos f}{e} \tag{30}$$

where the singularity at $e=0$ in φ_3 will be eliminated when the complete solution is assembled. In this manner, Yamanaka and Ankersen [8] finally presented another set of solutions for the Tschauner–Hempel equations:

$$\tilde{x}=k_1\rho(f)\sin f+k_2\rho(f)\cos f+k_3[2-3e\rho(f)L(f)\sin f] \tag{31}$$

$$\tilde{y}=-k_1\cos f(1+\rho(f))+k_2\sin f(1+\rho(f))+3k_3\rho^2(f)L(f)+k_4 \tag{32}$$

$$\tilde{z}=k_5\cos f+k_6\sin f \tag{33}$$

where $k_i(i=1,\cdots,6)$ represent the constant coefficients. It is noticed that the new solution is not written in terms of the eccentric anomaly or hyperbolic anomaly.

III. Modified Solutions of Tschauner–Hempel Equations

An obvious fact to anyone familiar with linear differential equations is that any two linearly independent solutions of the same differential equations lead to equivalent solutions. Hence, the two existing solutions obtained from Carter [7] and Yamanaka and Ankersen [8] are equivalent for $e\neq 1$. In this section, the equivalence of these two solutions will be proved for $e\neq 1$ and a new form of solution will be presented.

A. Evaluation of $J(f)$ in a New Perspective

The solution of Carter [7] is obtained using the integral J defined by Eq. (13). The closed-form evaluation of this integral was not obvious in terms of the true anomaly when $e\neq 1$. This is the reason that Carter evaluated the integral J in terms of the eccentric anomaly or hyperbolic anomaly [i.e., Eq. (14)]. This Note provides, however, a different but similar integral $D(f)$, defined as follows. It is easily evaluated using the true anomaly as the independent variable:

$$D(f)=\int\frac{3e+(2+e^2)\cos f}{(1+e\cos f)^3}\,\mathrm{d}f \tag{34}$$

Rearranging the right-hand side of Eq. (34) yields the following:

$$D(f)=2(1-e^2)\int\frac{\cos f}{(1+e\cos f)^3}\,\mathrm{d}f+3e\int\frac{\mathrm{d}f}{(1+e\cos f)^2} \tag{35}$$

In terms of J and L, this equation may be written as follows:

$$D(f)=2(1-e^2)J(f)+3eL(f) \tag{36}$$

As a consequence, the evaluation of J can be performed as follows:

$$J(f)=\frac{D(f)-3eL(f)}{2(1-e^2)},\qquad e\neq 1 \tag{37}$$

The direct evaluation of D from Eq. (34) results in

$$D(f)=\frac{\sin f(2+e\cos f)}{(1+e\cos f)^2} \tag{38}$$

which can be readily checked by differentiating the right-hand side with respect to the true anomaly. Substituting this expression into Eq. (37) yields the following:

$$J(f)=\frac{1}{2(1-e^2)}\left(\frac{\sin f(2+e\cos f)}{(1+e\cos f)^2}-3e\sqrt{\frac{\mu}{p^3}}t\right),\qquad e\neq 1 \tag{39}$$

As for the parabolic orbit, the solution of Carter [6]

$$J(f)=\frac{1}{4}\tan\left(\frac{f}{2}\right)-\frac{1}{20}\tan^5\left(\frac{f}{2}\right),\qquad e=1 \tag{40}$$

is adopted.

B. Proof of the Equivalence Between Two Existing Solutions

This Note will prove that the solutions obtained from Carter [7] and Yamanaka and Ankersen [8] are actually equivalent for $e\neq 1$. Equivalence here means the two solutions are the same except for a constant scale factor. To show this clearly, we substitute Eq. (37) into Eq. (12) to obtain

$$\varphi_2^C = 2e\rho(f)\sin f \cdot \frac{D(f) - 3eL(f)}{2(1-e^2)} - \frac{\cos f}{\rho(f)} \tag{41}$$

where the superscript C indicates the solution of Carter [7]. Inserting the expression of D from Eq. (38) into Eq. (41) leads to

$$\varphi_2^C = \frac{1}{1-e^2}[3e^2\rho(f)\sin f \cdot L(f) + \rho(f)\cos f - 2e] \tag{42}$$

Equation (26), introduced by Yamanaka and Ankersen [8], is related to the followнig expression:

$$\varphi_2^C = \frac{1}{1-e^2}\varphi_2^{\mathrm{YA}} \tag{43}$$

where the superscript YA represents the solution of Yamanaka and Ankersen [8] and shows that the two solutions φ_2^C and φ_2^{YA} are linearly dependent. We therefore conclude that the solutions of Carter [7] and Yamanaka and Ankersen [8] are equivalent.

The solution of Yamanaka and Ankersen [8] is valid for elliptic orbits: $0 \le e < 1$. Some investigators may think that the solution of [8] is not applicable to other types of orbits. According to the proof of Yamanaka and Ankersen [8] for the linear independence of φ_1 and φ_2, however, their solution also applies for hyperbolic orbits: $e > 1$. This is because the Wronskian of φ_1 and φ_2 is $\varphi_1\varphi_2' - \varphi_2\varphi_1' = e^2 - 1$, which is nonzero for the case: $e > 1$. This shows that the solution of Yamanaka and Ankersen [8] is applicable to both elliptic and hyperbolic orbits. Solutions φ_1 and φ_2 are not linearly independent, however, if $e = 1$. This shows that the solution of Yamanaka and Ankersen [8] unfortunately fails for parabolic orbits. This is a limitation of their method. The solution of Carter [7], however, using the new evaluation of the integral J will be valid for any Keplerian orbits: $e \ge 0$. This is the main result of this Note.

C. Modified Solution of Tschauner–Hempel Equations in Terms of True Anomaly

Summarizing, the general solution of Tschauner–Hempel equations in terms of the true anomaly is

$$\begin{cases} \tilde{x}(f) = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 \\ \tilde{y}(f) = -2c_1S(\varphi_1) - 2c_2S(\varphi_2) - c_3S(2\varphi_3+1) + c_4 \\ \tilde{z}(f) = c_5\cos f + c_6\sin f \\ \tilde{x}'(f) = c_1\varphi_1' + c_2\varphi_2' + c_3\varphi_3' \\ \tilde{y}'(f) = -2c_1\varphi_1 - 2c_2\varphi_2 - c_3(2\varphi_3+1) \\ \tilde{z}'(f) = -c_5\sin f + c_6\cos f \end{cases} \tag{44}$$

where $c_i(i = 1, \cdots, 6)$ are arbitrary constants; and the terms φ_1, φ_1', and $S(\varphi_1)$ are

$$\begin{cases} \varphi_1 = \rho(f)\sin f \\ \varphi_1' = \rho(f)\cos f - e\sin^2 f \\ S(\varphi_1) = -\cos f - \dfrac{1}{2}e\cos^2 f \end{cases} \tag{45}$$

The other six terms φ_2, φ_3, φ_2', φ_3', $S(\varphi_2)$, and $S(2\varphi_3 + 1)$ are written in different forms that depend on the value of eccentricity. For elliptic and hyperbolic orbits ($e \ne 1$),

$$\begin{cases} \varphi_2 = \dfrac{e\varphi_1}{1-e^2}[D(f) - 3eL(f)] - \dfrac{\cos f}{\rho(f)} \\ \varphi_3 = -\dfrac{\varphi_1}{1-e^2}[D(f) - 3eL(f)] - \dfrac{\cos^2 f}{\rho(f)} - \cos^2 f \\ \varphi_2' = \dfrac{e\varphi_1'}{1-e^2}[D(f) - 3eL(f)] + \dfrac{e\sin f\cos f}{\rho^2(f)} + \dfrac{\sin f}{\rho(f)} \\ \varphi_3' = 2[\varphi_1'S(\varphi_2) - \varphi_2'S(\varphi_1)] \\ S(\varphi_2) = -\dfrac{\rho^2(f)[D(f) - 3eL(f)]}{2(1-e^2)} \\ S(2\varphi_3+1) = \dfrac{e\sin f(2+e\cos f)}{1-e^2} - \dfrac{3\rho^2(f)L(f)}{1-e^2} \end{cases} \tag{46}$$

For parabolic orbits ($e = 1$),

$$\begin{cases} \varphi_2 = 2\varphi_1J_1(f) - \dfrac{\cos f}{\rho(f)} \\ \varphi_3 = -2\varphi_1J_1(f) - \dfrac{\cos^2 f}{\rho(f)} - \cos^2 f \\ \varphi_2' = 2\varphi_1'J_1(f) + \dfrac{\sin f\cos f}{\rho^2(f)} + \dfrac{\sin f}{\rho(f)} \\ \varphi_3' = 2[\varphi_1'S(\varphi_2) - \varphi_2'S(\varphi_1)] \\ S(\varphi_2) = -\rho^2(f)J_1(f) \\ S(2\varphi_3+1) = 2[\rho^2(f)J_1(f) - \sin f] - \sin f\cos f \end{cases} \tag{47}$$

where D and L are found from Eqs. (38) and (28), respectively; and J_1 is as follows:

$$J_1(f) = \frac{1}{4}\tan\left(\frac{f}{2}\right) - \frac{1}{20}\tan^5\left(\frac{f}{2}\right) \tag{48}$$

This new form of the solution is expressed in terms of the true anomaly and transition time. There is also no singularity in this new solution. The state transition matrix can be easily established for Tschauner–Hempel equations by using the new solution given by Eqs. (45–48):

$$\phi_{\mathrm{LVLH}}(f, f_0) = \phi(f) \cdot \phi^{-1}(f_0) \tag{49}$$

where

$$\phi(f) = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 & 0 & 0 & 0 \\ -2S(\varphi_1) & -2S(\varphi_2) & -S(2\varphi_3+1) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos f & \sin f \\ \varphi_1' & \varphi_2' & \varphi_3' & 0 & 0 & 0 \\ -2\varphi_1 & -2\varphi_2 & -2\varphi_3 - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin f & \cos f \end{bmatrix} \tag{50}$$

$$\phi^{-1}(f) = \begin{bmatrix} 4S(\varphi_2) + \varphi_2' & 0 & 0 & -\varphi_2 & 2S(\varphi_2) & 0 \\ -4S(\varphi_1) - \varphi_1' & 0 & 0 & \varphi_1 & -2S(\varphi_1) & 0 \\ -2 & 0 & 0 & 0 & -1 & 0 \\ -2S(2\varphi_3+1) - \varphi_3' & 1 & 0 & \varphi_3 & -S(2\varphi_3+1) & 0 \\ 0 & 0 & \cos f & 0 & 0 & -\sin f \\ 0 & 0 & \sin f & 0 & 0 & \cos f \end{bmatrix} \tag{51}$$

Note that the ordering here is different from that used by Carter [7]. The terms φ_1, φ_2, φ_3, φ_1', φ_2', φ_3', $S(\varphi_1)$, $S(\varphi_2)$, and $S(2\varphi_3 + 1)$ are found from Eqs. (45–47).

From Eq. (1), the state transition matrix is obtained as follows:

$$\phi_{\mathrm{LVLH}}(t, t_0) = A^{-1}(f) \cdot \phi(f) \cdot \phi^{-1}(f_0) \cdot A(f_0) \tag{52}$$

where $A(f)$ is found in Eq. (2), and $A^{-1}(f)$ is

$$A^{-1}(f) = \begin{bmatrix} \frac{h^{3/2}}{\mu(1+e\cos f)} \cdot \boldsymbol{I}_{3\times3} & 0_{3\times3} \\ \frac{\mu e \sin f}{h^{3/2}} \cdot \boldsymbol{I}_{3\times3} & \frac{\mu(1+e\cos f)}{h^{3/2}} \cdot \boldsymbol{I}_{3\times3} \end{bmatrix} \tag{53}$$

D. Numerical Examples

To verify the accuracy of the new solution, we compare it with numerical integration of the Tschauner–Hempel equations. We use a fourth-order Runge–Kutta method with a fixed step size. The chaser state is determined using the state transition matrix defined in Eqs. (50–53). We present three cases: $e = 0.1$, 1.0, and 2.0. The remaining five orbital elements of the chief are a semiparameter of 20,000 km, an inclination of 45 deg, a right ascension of 60 deg, an argument of perigee of 30 deg, and an initial true anomaly of 0 deg. The initial relative position and velocity of the deputy are $[x, y, z] = [1, 2, 1]$ km and $[\dot{x}, \dot{y}, \dot{z}] = [0.01, -0.02, 0.01]$ m/s. The relative position error Err_p and relative velocity error Err_v are calculated from the following expressions:

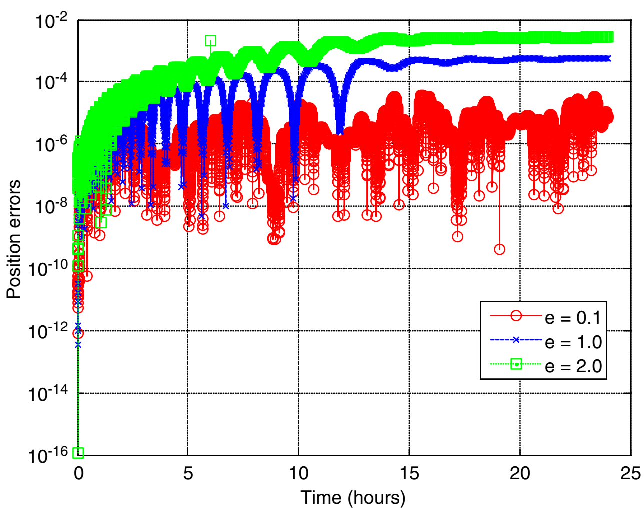

Fig. 1 Propagation errors of position.

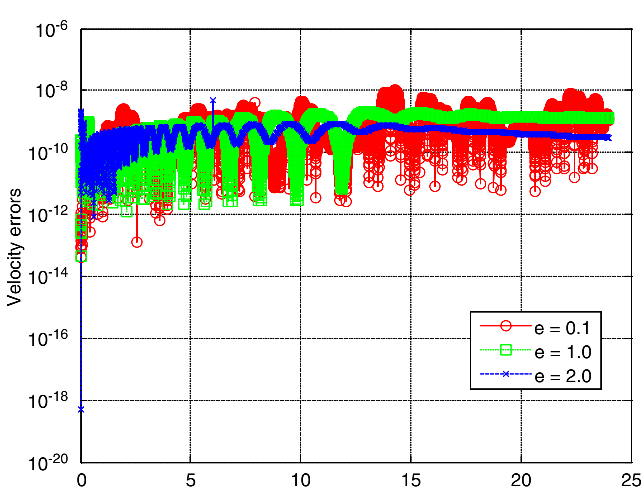

Fig. 2 Propagation errors of velocity.

$$\mathrm{Err}_p = \frac{\sqrt{(x_{\mathrm{STM}} - x_{\mathrm{Num}})^2 + (y_{\mathrm{STM}} - y_{\mathrm{Num}})^2 + (z_{\mathrm{STM}} - z_{\mathrm{Num}})^2}}{\sqrt{x_{\mathrm{Num}}^2 + y_{\mathrm{Num}}^2 + z_{\mathrm{Num}}^2}} \tag{54}$$

$$\mathrm{Err}_v = \frac{\sqrt{(\dot{x}_{\mathrm{STM}} - \dot{x}_{\mathrm{Num}})^2 + (\dot{y}_{\mathrm{STM}} - \dot{y}_{\mathrm{Num}})^2 + (\dot{z}_{\mathrm{STM}} - \dot{z}_{\mathrm{Num}})^2}}{\sqrt{\dot{x}_{\mathrm{Num}}^2 + \dot{y}_{\mathrm{Num}}^2 + \dot{z}_{\mathrm{Num}}^2}} \tag{55}$$

where the subscripts STM and Num represent the states calculated from the state transition matrix and numerical integration, respectively. The plots of the relative errors is shown in Figs. 1 and 2. It can be observed that the relative position errors and velocity errors in all three cases are below 1 and 10^{-6}%, respectively. This demonstrates that the new solution is consistent with numerical computations.

IV. Conclusions

This work investigates the existing solutions of the Tschauner–Hempel equations that are used to describe the motion of a spacecraft relative to another in an arbitrary Keplerian orbit. Two typical solutions obtained by Carter [6,7] and Yamanaka and Ankersen [8] are carefully analyzed and compared. Carter's solution [6,7] applies for all types of Keplerian orbits, but it is expressed in terms of an eccentric anomaly or hyperbolic anomaly when the eccentricity is not unity. The solution proposed by Yamanaka and Ankersen [8], presented in terms of the true anomaly and transition time, however, fails when the eccentricity is unity. By designing a new integral, it is proven that the two existing solutions are, in fact, equivalent for $e \neq 1$. According to this finding, the methods used in these two solutions are incorporated to produce a modified form of solution for the Tschauner–Hempel equations. This new form of solution is valid for all kinds of Keplerian orbits and is relatively simple for engineering use.

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