

General formulas:

$$\Sigma = \frac{1}{2} v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Circular

$$v_I = \sqrt{\frac{\mu}{r}}$$

Escape

$$\underline{h} = \underline{r} \times \underline{v} \quad h = \sqrt{\mu P} = r v_0 = r^2 \dot{\theta}$$

$$P = \frac{h^2}{\mu} = a(1-e^2) \quad T = \sqrt{\frac{4\pi^2 a^3}{\mu}}$$

Hyperbolic excess speed

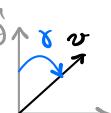
$$v_{II} = \sqrt{2} v_I \quad v_{III} = \sqrt{-\frac{\mu}{a}} = v_\infty = \sqrt{\mu}$$

Rocket parameter

$$v_r = \frac{\mu}{r} \cdot e \cdot \sin \theta = \dot{r} = v \sin \gamma$$

$$v_\theta = \frac{\mu}{r^2} (1+e \cos \theta) = \sqrt{\frac{\mu}{r}} (1+e \cos \theta) = r \dot{\theta} = v \cos \gamma$$

$$e = \frac{\dot{r} \times \underline{h}}{\mu} - \frac{r}{||\underline{r}||} = \frac{1}{\mu} \left[(v^2 - \frac{\mu}{r}) \underline{r} - r v_r \underline{v} \right]$$



Elliptical: $0 < e < 1 \quad a > 0 \quad \epsilon > 0$

$$a = \frac{r_p + r_a}{2}$$

$$e = \frac{r_a - r_p}{r_a + r_p} \quad b = a \sqrt{1 - e^2}$$

Parabolic trajectories: $e = 1 \quad a \rightarrow \infty \quad \epsilon = 0$

$$\Sigma = \frac{1}{2} v^2 - \frac{\mu}{r} = 0 \quad r_p > 0 \quad r_a < 0$$

$$a = \frac{P}{1-e^2} \rightarrow \infty$$

Hyperbolic trajectories: $e > 1 \quad a < 0 \quad \epsilon > 0 \quad P > 0$

$$a_\infty = a \cos(-\frac{1}{e}) \quad \delta = 2a \sin(\frac{1}{e}) \quad v_\infty = \sqrt{\mu/a}$$

$$\bar{a} = |a| \quad \bar{b} = \bar{a} \sqrt{e^2 - 1} \quad b = -e \bar{b} \quad \Delta = \bar{b} = \bar{a} \sqrt{e^2 - 1}$$

$$\text{Time laws: } \text{Ellipse: } \tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{\epsilon}{2} \quad E - e \sin E = \sqrt{\frac{\mu}{a^3}} (t - t_p) \quad M = \sqrt{\frac{\mu}{a^3}} (t - t_p) \quad \dot{M} = \frac{\dot{E}}{ab} \quad \dot{E} = \frac{\dot{E}}{br}$$

$$\text{if } \theta > \pi \quad \text{if } \theta < \pi \quad \rightarrow \theta^* = 2\pi - \theta \rightarrow \Delta t_{\theta \rightarrow 0} = T - \Delta t_{\theta \rightarrow \theta^*}$$

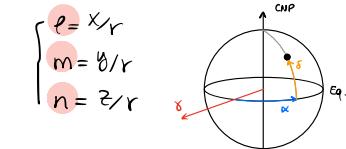
$$\text{Hyperbola: } \tan \frac{\theta}{2} = \sqrt{\frac{1+e}{e-1}} \tan \epsilon \left(\frac{F}{2} \right) \quad e \sin \theta F - F = \sqrt{\frac{\mu}{a^3}} (t - t_p) \quad \dot{E} = \frac{\dot{E}}{br} \quad \dot{M} = -i \cdot \frac{\dot{E}}{\bar{a} \bar{b}}$$

$$\text{Parabola: } D = \tan \left(\frac{\theta}{2} \right) = \tan(\epsilon) \quad \Delta t = t - t_p = \sqrt{\frac{P^3}{\mu}} \frac{1}{2} (D + \frac{D^3}{3}) \quad \dot{E} = i \cdot \dot{E} \quad \dot{M} = -i \cdot \dot{E}$$

Right ascension and declination:

$$\hat{r} = \cos \alpha \cos \delta \hat{i} + \sin \alpha \cos \delta \hat{j} + \sin \delta \hat{k} = \hat{e}_\alpha + m \hat{j} + n \hat{k}$$

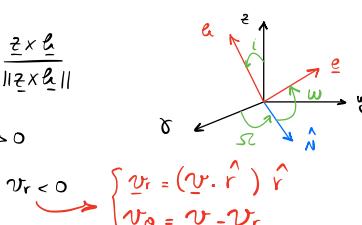
$$\delta = \arcsin \left(\frac{z}{r} \right) \quad \alpha = \begin{cases} \arccos \left(\frac{x}{r \cos \delta} \right) & \text{if } \frac{y}{r} \geq 0 \\ 2\pi - \arccos \left(\frac{x}{r \cos \delta} \right) & \text{if } \frac{y}{r} < 0 \end{cases}$$



Orbital Elements: $r, v \rightarrow a, e, \iota, \Omega, \omega, \theta$

$$a = \frac{1}{(\frac{2}{r} - \frac{v^2}{\mu})} \quad e = \frac{1}{\mu} \left[(v^2 - \frac{\mu}{r}) \underline{r} - r v_r \underline{v} \right] \quad \underline{h} = \underline{r} \times \underline{v} \quad \iota = a \cos \left(\frac{z \cdot \underline{h}}{a} \right) \quad \hat{N} = \frac{\underline{z} \times \underline{h}}{||\underline{z} \times \underline{h}||}$$

$$\Omega = \begin{cases} \arccos(\hat{N} \cdot \hat{x}) & \text{if } \hat{N} \cdot \hat{y} > 0 \\ 2\pi - \arccos(\hat{N} \cdot \hat{x}) & \hat{N} \cdot \hat{y} < 0 \end{cases} \quad \omega = \begin{cases} \arccos \left(\frac{e \cdot \hat{N}}{e} \right) & e \cdot \hat{z} > 0 \\ 2\pi - \arccos \left(\frac{e \cdot \hat{N}}{e} \right) & e \cdot \hat{z} < 0 \end{cases} \quad \theta = \begin{cases} \arccos \left(\frac{r \cdot \hat{e}}{r \cdot e} \right) & v_r > 0 \\ 2\pi - \arccos \left(\frac{r \cdot \hat{e}}{r \cdot e} \right) & v_r < 0 \end{cases}$$



Transformations:

Programma calcolatrice

$$R_{ECEI/CNP} = \begin{bmatrix} \cos \omega \cos \Omega - \sin \omega \cos \iota \sin \delta & \cos \omega \sin \Omega + \sin \omega \cos \iota \cos \delta & \sin \omega \sin \iota \\ -\sin \omega \cos \Omega - \cos \omega \cos \iota \sin \delta & -\sin \omega \sin \Omega + \cos \omega \cos \iota \cos \delta & \cos \omega \sin \iota \\ \sin \omega \sin \Omega & -\sin \omega \sin \Omega & \cos \iota \end{bmatrix} \quad R_{CNP/EU} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v_{ECEI} = \frac{\mu}{r} \begin{bmatrix} e \sin \Omega \\ \pm e \cos \Omega \\ 0 \end{bmatrix} \quad v_{CNP} = \frac{\mu}{r} \begin{bmatrix} -\sin \phi \\ e + \cos \phi \\ 0 \end{bmatrix}$$

Coordinate systems:

Earth-centered ecliptic:

ECEI (CNP):

Topocentric horizon:

β (celestial longitude)

α (right ascension) $0 \leq \alpha \leq 2\pi$ δ (declination) $-\frac{\pi}{2} \leq \delta \leq \frac{\pi}{2}$

A (azimuth)

λ (longitude)

ψ (geographic lat.) or ϕ (geodesic lat.)

γ : intersection between ecliptic and equatorial plane.

Σ : intersection between observer meridian and equatorial plane.

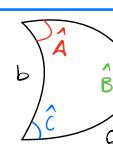
Spherical geometry:

$$\cos a = \cos b \cos c + \sin b \sin c \cos \hat{A}$$

$$\cos \hat{A} = -\cos b \cos \hat{C} + \sin b \sin \hat{C} \cos a$$

$$\frac{\sin a}{\sin \hat{A}} = \frac{\sin b}{\sin \hat{B}} = \frac{\sin c}{\sin \hat{C}}$$

$$\cotan(a) \cdot \sin(b) = \cos(b) \cos(\hat{c}) + \sin(\hat{c}) \cotan(\hat{A})$$

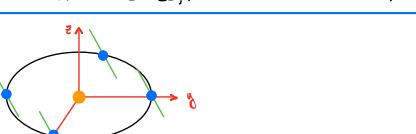


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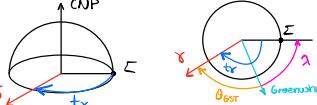
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Measurements of time:

Local sidereal time: $LST = t_\gamma = \Sigma \gamma = \Omega_{LST} = \Omega_{GST} + \lambda$ Greenwich sidereal time



1 year = 365,25 solar days = 366,24 sidereal days

Apparent local time: t : hour angle of real sun

$$LAT = t + 12\epsilon$$

$$LAT = GAT + \lambda$$

Universal time: LMT wrt Greenwich

Local civil time: $LCT = GMT + \lambda$ $n = \text{ceil} \left(\text{sign}(\lambda) \frac{12\lambda - 7,5^\circ}{15^\circ} \right)$

$LAT = LMT + E$

$$\frac{E - E_1}{GMT - GMT_1} = \frac{E_2 - E_1}{GMT_2 - GMT_1}$$

$\downarrow Q \quad \downarrow 24 \quad \downarrow Q$

E is green

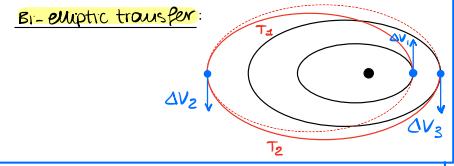
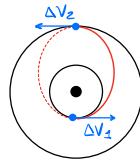
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Impulsive maneuvers: $\Delta T = I_{sp} \cdot g_a \ln\left(\frac{m_f}{m_i}\right)$ $I_{sp} = \frac{T}{m_i g_a}$ sea level $\Delta w = w_{\text{propellant}} = m_f - m_i$ $\frac{m_f}{m_i} = 1 - \exp\left(-\frac{\Delta T}{I_{sp} g_a}\right)$

Hohmann: 2 circ, coplanar, copocal orbits.

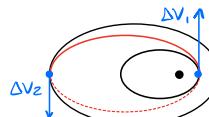
$$\Delta t = \text{TOF} = \frac{T}{2} \quad \ell = \sqrt{2\mu \frac{r_a r_p}{r_a + r_p}} = r \cdot v_\theta = r \cdot v \cdot \cos\delta \quad \delta = 0 \quad \text{perihelion or apophelia}$$

$$v = \frac{\ell}{r} \quad \text{perihelion or apophelia} \quad \Delta V = |V_A| + |V_B|$$



Generalized Hohmann: 2 elliptic, coplanar, copocal orbits
From pericenter or apocenter

$\Delta t \rightarrow \frac{T}{2}$ of transfer orbit. Same as Hohmann



Phasing maneuver: For circular orbits: $T_{ph} = T_1 \left(1 - \text{sign}(\Delta\theta) \frac{|\Delta\theta|}{360^\circ \cdot n_{rev}}\right)$

$$\Delta\theta = \Omega_2 - \Omega_1$$

For elliptic orbits: $T_{ph} = \Delta t_{\text{required}}$

$$\Omega_T = \sqrt{\frac{T_{ph}^2 \mu}{4\pi^2}}$$

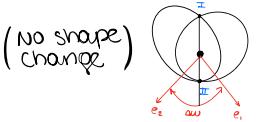
$$\begin{cases} \text{if } \Delta\theta > 0 \\ \text{if } \Delta\theta < 0 \end{cases} \begin{cases} r_{a_T} = r_{P_2} \\ r_{P_T} = 2a_T - r_{a_T} \end{cases} \quad \Delta V_{tot} = 2|\Delta V_1|$$

(secant) Non-Hohmann transfer with common apse line:

- transfer orbit is coaxial

$$\text{transfer orbit: } \begin{cases} r_A = \frac{P_T}{1 + e_T \cos\Omega_1} \\ r_B = \frac{P_T}{1 + e_T \cos\Omega_2} \end{cases} \rightarrow r_A, e_T \rightarrow \ell_A = \sqrt{\mu P_T} \quad \begin{cases} v_r = \frac{\mu}{r} e \sin\delta \\ v_\theta = \frac{\mu}{r} (e + e \cos\delta) \end{cases}$$

$$\begin{aligned} \Delta V_1 &= v_{r_A} \hat{r} + v_{\theta_A} \hat{\theta} \quad (i=1, \dots, 4) \\ \Delta V_2 &= v_{r_B} - v_{r_A} \quad \Rightarrow ||\Delta V_1|| = ||\Delta V_2|| \quad ||\Delta V_1|| + ||\Delta V_2|| = ||v_{r_A} - v_{r_B}|| + ||v_{\theta_A} - v_{\theta_B}|| = \sqrt{v_{r_A}^2 v_{r_B}^2 - 2v_{r_A} v_{r_B} \cos\Delta\theta} + \sqrt{v_{\theta_A}^2 v_{\theta_B}^2} \\ \text{Flight path angle of } \Delta V_i &\rightarrow \tan\Delta\theta_i = \frac{\Delta V_{\perp,i}}{\Delta V_{\parallel,i}} \quad (i=1, 2) \end{aligned}$$



Apsis line rotation: Δw counter-clockwise > 0
 \rightarrow orbits have 2 intersections at most.

$$\Delta w = w_2 - w_1 = \Omega_2 - \Omega_1$$

$$\begin{cases} a \cos\Omega_1 + b \sin\Omega_1 = C \\ b = -e_1^2 e_2 \sin\eta \\ C = e_1^2 - e_2^2 \end{cases}$$

Case 1: Given $\eta = \Delta w$, $e_1, e_2, \Omega_1, \Omega_2$ or calculate this from data.

$$\text{For } \Delta V_1: \begin{cases} v_{r_{\Omega_1}} = \frac{\ell_1}{r_{\Omega_1}} \\ v_{\theta_{\Omega_1}} = \frac{\ell_1}{r_{\Omega_1}} e_1 \sin\Omega_1 \\ \tau_{\Omega_1} = \text{atan}\left(\frac{v_{\theta_{\Omega_1}}}{v_{r_{\Omega_1}}}\right) \end{cases} \rightarrow \begin{cases} v_{r_1} = v_{r_{\Omega_1}} \hat{r} + v_{\theta_{\Omega_1}} \hat{\theta} \\ \text{Repeat for } v_{\theta_2} \\ \Delta V_1 = v_{\theta_2} - v_{\theta_1} \end{cases}$$

$$\begin{cases} \text{Flight path angle of } \Delta V_1 \\ \Delta\theta = \text{atan}\left(\frac{\Delta V_{\perp,1}}{\Delta V_{\parallel,1}}\right) \quad \text{if } v_{\theta_1} > 0 \\ \Delta\theta = \pi - \text{atan}\left(\frac{\Delta V_{\perp,1}}{\Delta V_{\parallel,1}}\right) \quad \text{if } v_{\theta_1} < 0 \end{cases}$$

Case 2: Given $\Omega_{\Omega_1}, \Delta V$ → find $\eta = \Delta w = \Omega_1 - \Omega_2$ and e_1, e_2

$$\Delta V = \frac{\Delta V_r \hat{r} + \Delta V_\theta \hat{\theta}}{\Delta V \sin\gamma \Delta V \cos\gamma} \rightarrow \tan\Omega_{\Omega_1} = \frac{v_{\theta_{\Omega_1}}^2 / r_{\Omega_1}}{(v_{\theta_1} + \Delta V_\theta)(v_{r_1} + \Delta V_r)} = \frac{(v_{\theta_1} + \Delta V_\theta)(v_{r_1} + \Delta V_r)}{(v_{\theta_1} + \Delta V_\theta)^2 e_1 \cos\Omega_1 + \Delta V_\theta(4v_{\theta_1} + 2v_{r_1})}$$

$$\begin{cases} e_2 = e_1 + r_{\Omega_1} \Delta V_\theta \\ e_2 = \frac{1}{\sin\Omega_2} \frac{\ell_2}{\mu} (v_{r_2} + \Delta V_r) \\ \eta = \Omega_1 - \Omega_2 \end{cases} \quad \Delta V \text{ has to invert } v_r \rightarrow v_\theta \text{ is unchanged.}$$

Plane change: copocal orbits, on 2 planes (if v_r changes, shape changes)

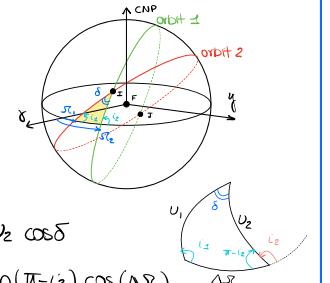
$$\begin{cases} \text{1) } v_{r_1} = v_{r_1} \hat{r} + v_{\theta_1} \hat{\theta} \quad \Delta V = v_{\theta_2} - v_{\theta_1} \\ \text{2) } v_{r_2} = v_{r_2} \hat{r} + v_{\theta_2} \hat{\theta} \end{cases}$$

$$\Delta V = \sqrt{(v_{r_2} - v_{r_1})^2 + v_{\theta_2}^2 + v_{\theta_1}^2 - 2v_{\theta_2} v_{\theta_1} \cos\delta}$$

$$\text{or } \Delta V = \sqrt{v_{r_1}^2 + v_{r_2}^2 - 2v_r v_{r_2} (\cos\Delta\gamma - \cos\gamma_1 \cos\gamma_2 (1 - \cos\delta))}$$

Particular cases:

- 1) No plane change $\rightarrow \delta = 0$
- 2) No shape change $\rightarrow v_{r_1} = v_{r_2}$
- 3) $\Delta r = r_{\Omega_2} - r_{\Omega_1}$
 $v_{\theta_1} = w_1 + \Omega_1^*$ $v_{\theta_2} = w_2 + \Omega_2^*$ → Ω doesn't change
Find $\delta \rightarrow$ find ΔV



Planet phasing: $\phi = \Omega_2 - \Omega_1 + (u_2 - u_1) t$ $T_{syn} = \frac{2\pi}{|u_2 - u_1|}$ $\Delta t_{Hohmann} = \frac{1}{2} \sqrt{\frac{4\pi^2 (r_1 + r_2)^2}{\mu \sin(\frac{\Delta\phi}{2})}}$

$$\begin{cases} \phi_{\text{initial}} = \pi - u_1 \Delta t_{1,2} \\ \phi_{\text{final}} = \pi - u_1 \Delta t_{1,2} \end{cases}$$

$$\begin{cases} \text{1) elliptical} \\ \begin{cases} u_1 = w_1 + \phi_1 \\ u_2 = w_2 + \phi_2 \end{cases} \quad \text{if } l_2 \neq 0 \\ \begin{cases} \lambda_1 = J_2 L_1 + w_1 + \phi_1 \\ \lambda_2 = J_2 L_2 + w_2 + \phi_2 \end{cases} \quad \text{if } l_2 = 0 \end{cases}$$

Patched conics method:

1) Interplanetary transfer (Hohmann) $\rightarrow \mu_{sun}$

$$\Delta V_1 \rightarrow \text{find } v_{\infty}^+ \text{ at planet 1.} \quad v_{sc}^+ = v_{\infty}^+ + v_{\text{planet 1}}$$

$$\Delta V_2 \rightarrow \text{find } v_{\infty}^- \text{ at planet 2.} \quad v_{sc}^- = v_{\infty}^- + v_{\text{planet 2}}$$

In case of Hohmann: $v_{\infty}^+ // v_{\infty}^-$

ΔV opposite to e

Don't align pericenter with frames.

2) Planet SOI: $r_{SOI} = R_{\text{sun-planet}} \left(\frac{m_p}{m_s}\right)^{2/5}$

$$v_{sc} = v_{rel} + v_{\text{planet}}$$

$\hookrightarrow v_{\infty}$: hyperbolic excess speed

$$\Sigma_{\infty} = \Sigma_p = \frac{1}{2} v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \rightarrow a_{hyp}$$

$$v_{\infty} = v_{r\infty} = \frac{\mu}{r} \sqrt{e^2 - 1} \rightarrow \text{find } e \text{ without } r_p$$

$$e_{hyp} = \frac{r_p v_{\infty}}{1 - e^2_{hyp}}$$

$$h_{hyp} = r_p \cdot \sqrt{v_{\infty}^2 + \frac{2\mu_p}{r_p}} = \Delta r_{hyp}$$

$$v_{p,hyp} = \frac{e_{hyp}}{r_p}$$

$$\delta = 2 \sin\left(\frac{1}{1 + \frac{r_p v_{\infty}}{\mu}}\right) = 2 \text{atan}\left(-\frac{a_{hyp}}{\Delta}\right)$$

$$\beta = 90^\circ - \frac{\delta}{2} = \text{acos}\left(\frac{1}{e_{hyp}}\right)$$

$$\Delta = \bar{b} = \bar{a} \sqrt{e^2 - 1} = r_p \cdot \sqrt{1 + \frac{2\mu_p}{r_p v_{\infty}}}$$

$$r_{\text{maneuver}} = r_p \begin{bmatrix} \cos\delta \\ \sin\delta \\ 0 \end{bmatrix}$$

counter-clockwise rotation:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\delta & -\sin\delta & 0 \\ \sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} \Delta V &= v_{sc}^+ - v_{sc}^- = v_{\infty}^+ - v_{\infty}^- \\ \Delta V &= 2v_{\infty} \sin\left(\frac{\delta}{2}\right) \end{aligned}$$

$$\begin{cases} \text{Flight path} \\ \begin{cases} \Phi_1 = \text{atan}\left(\frac{v_{\infty}^+}{v_{\infty}^-}\right)_s = \text{atan}\left(\frac{v_{sc}^+ \sin\delta}{v_{sc}^- \cos\delta - v_p}\right) \\ \Phi_2 = \Phi_1 + \delta \end{cases} \end{cases}$$

