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State Transition Matrix of Relative Motion for the Perturbed Noncircular Reference Orbit

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A precise analytic solution that includes the effects of the reference orbit eccentricity and differential perturbations is needed for the relative motion of formation-flying satellites. As a result of the spherical Earth and circular reference orbit assumptions, Hill's equations, which have often been used for describing relative motion, are insufficient for the long-term prediction of the relative motion. A new approach, called the geometric method, is developed to obtain the state transition matrix for the relative motion that includes the effects caused by the reference orbit eccentricity and the differential gravitational perturbations. The geometric method uses the relationship between the relative states and differential orbital elements to obtain the state transition matrix instead of directly solving the complex relative motion differential equations. The state transition matrices are derived for both mean and osculating elements with the primary gravitational perturbation that results from the equatorial bulge term J_2. Although the results are based on the J_2 effects, the approach can be extended easily to include other perturbing forces.

I. Introduction

THE equations describing the relative motion of satellites are needed for rendezvous and formation-flying satellites. The rendezvous problem is of short duration, and there are frequent thruster firings. Therefore, the long-term accuracy of the equations of motion is not as important in the rendezvous problem as in the formation-flying problem, and the Clohessy–Wiltshire equations[1] (Hill's equations) or a modified version incorporating small eccentric effects are usually sufficient. However, Hill's equations are based on the assumptions that the reference orbit is circular, the Earth is spherically symmetric, and the target orbit is very close to the reference orbit such that there is no external perturbing force and the nonlinear terms in the relative motion can be neglected. These assumptions result in unacceptable errors in the long-term prediction of the relative motion for formation-flying satellites. To minimize fuel consumption and maximize lifetime, a more accurate solution for the relative motion is needed.

Closed-form solutions of the relative motion for an elliptic reference orbit without perturbations were derived independently in the 1960s by Lawden[2] and Tschauner and Hempel.[3] The Tschauner–Hempel solution has numerical singularities in that there are terms with the eccentricity in the denominator. The Lawden solution is in mixed variables. In 1996 Garrison et al.[4] used the same basic approach that is used in this paper to derive another closed-form solution, but they used different variables with no perturbations. In addition, there have been several expanded solutions in powers of the eccentricity for unperturbed noncircular reference orbit. Melton[5] used a novel approach to obtain a solution as a function of time or mean anomaly. Vaddi et al.[6] derived corrections to the initial conditions for the first-order effects of the eccentricity and J_2 effects for zero eccentricity. However, there has been no approach that incorporates the primary gravitational perturbation J_2 when the reference orbit is eccentric.

The purpose of this paper is to derive the state transition matrices with both osculating and mean elements for the relative motion of two neighboring satellites when the reference satellite is in an elliptic orbit and both satellites are subjected to the J_2 perturbation. Using the geometric transformation, the state transition matrices are obtained in closed form for the mean elements and for the osculating elements without directly solving the differential equations under the existence of gravitational perturbation J_2.

In this paper the orbital elements of the reference satellite, named as the Chief, and the relative position and velocity vectors of the target satellite, named as the Deputy, are defined as $\boldsymbol{e} = (a, \theta, i, q_1, q_2, \Omega)^T$ and $\boldsymbol{X} = (x, \dot{x}, y, \dot{y}, z, \dot{z})^T$, where θ is the argument of latitude, $q_1 = e\cos\omega$, and $q_2 = e\sin\omega$. This set is used because the true anomaly and the argument of perigee are undefined for a circular orbit and there are numerical problems in the eccentricity mean to osculating transformation for small eccentricity. After obtaining the orbital elements of the Deputy by a Taylor-series expansion about the orbital elements of the Chief, the relative orbital elements between them are obtained by $\delta\boldsymbol{e} = \boldsymbol{e}_d - \boldsymbol{e}_c$. All of the orbital elements without subscript are for the Chief, and $\boldsymbol{X}$ and $\delta\boldsymbol{e}$ are for the Deputy. To obtain more accurate results, the curvilinear coordinate system represented by unit vectors $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ with the origin at the Chief is used instead of the local vertical local horizontal (LVLH) Cartesian frame. That means that x is the difference in the radii and y and z are the curvilinear distances along the imaginary circular orbit on the reference orbital plane and perpendicular to the reference orbit, respectively, at the instantaneous time as shown in Fig. 1.

II. Geometric Method

Using the osculating elements for the Chief and the Deputy under the influence of J_2 and the total angular velocity $\boldsymbol{\varpi} = \dot{\theta}\hat{e}_z + \dot{\Omega}\hat{k} + \dot{i}(\cos\theta\hat{e}_x - \sin\theta\hat{e}_y)$, where $\{\hat{i}, \hat{j}, \hat{k}\}$ are unit vectors in the Earth-centered-inertial (ECI) reference frame, the geometric transformation between $\boldsymbol{X}(t)$ and $\delta\boldsymbol{e}(t)$ is represented by

$$\boldsymbol{X}(t) = \{A(t) + \alpha B(t)\}\delta\boldsymbol{e}(t) \tag{1}$$

where $\alpha = 3J_2R_e^2$, R_e is the equatorial radius of the Earth, and the matrix $B(t)$ contains only the terms perturbed by J_2. Let ϕ_e be the state transition matrix for the relative osculating elements, that is, $\delta\boldsymbol{e}(t) = \phi_e\delta\boldsymbol{e}(t_0)$. Therefore, from the solution $\boldsymbol{X}(t) = \Phi_{J_2}(t, t_0)\boldsymbol{X}(t_0)$, the state transition matrix for the relative motion $\Phi_{J_2}(t, t_0)$ is

$$\Phi_{J_2}(t, t_0) = \{A(t) + \alpha B(t)\}\phi_e(t, t_0)\{A(t_0) + \alpha B(t_0)\}^{-1} \tag{2}$$

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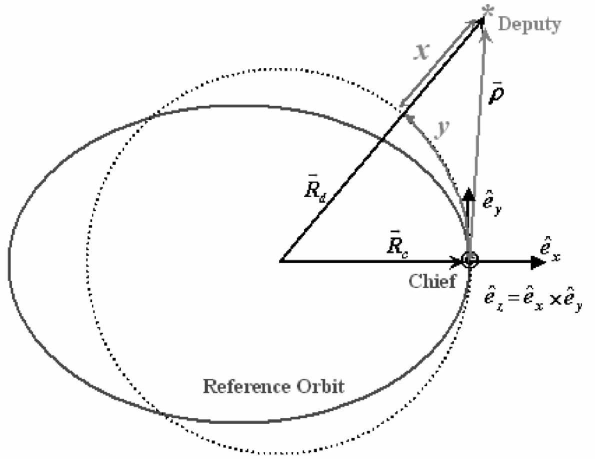


Fig. 1 Curvilinear coordinate system.

Using the transformation matrix $D(t)$ from the relative mean elements to the relative osculating elements and the state transition matrix $\bar{\phi}_{\bar{e}}(t, t_0)$ for the relative mean elements,

$$\delta \boldsymbol{e}_{\text{osc}}(t) = D(t)\delta \boldsymbol{e}_{\text{mean}} = D(t)\bar{\phi}_{\bar{e}}(t, t_0)\delta \bar{\boldsymbol{e}}(t_0) = D(t)\bar{\phi}_{\bar{e}}(t, t_0)D^{-1}(t_0)\delta \boldsymbol{e}_{\text{osc}}(t_0)$$

$$D(t) = \frac{\partial \boldsymbol{e}_{\text{osc}}}{\partial \boldsymbol{e}_{\text{mean}}} \tag{3}$$

$\Phi_{J_2}(t, t_0)$ becomes

$$\Phi_{J_2}(t, t_0) = \{A(t) + \alpha B(t)\}D(t)\bar{\phi}_{\bar{e}}(t, t_0)D^{-1}(t_0) \times \{A(t_0) + \alpha B(t_0)\}^{-1} \tag{4}$$

The state transition matrix for mean elements is

$$\bar{\Phi}_{J_2}(t, t_0) = \{\bar{A}(t) + \alpha \bar{B}(t)\}\bar{\phi}_{\bar{e}}(t, t_0)\{\bar{A}(t_0) + \alpha \bar{B}(t_0)\}^{-1} \tag{5}$$

Because the mean and osculating elements are equal when $J_2 = 0$, $\bar{A}(t) = A(t)$. The angular velocities for mean and for osculating elements are different; consequently, $\bar{B}(t) \neq B(t)$.

Transformation Matrix $\Sigma(t) \equiv \{A(t) + \alpha B(t)\}$

Using the approach given in Alfriend et al.,[7] the position and the velocity of the Chief and the Deputy are

$$\boldsymbol{R}_c = R\hat{e}_x$$

$$\boldsymbol{V} = (\dot{R})\hat{e}_x + (R\varpi_n)\hat{e}_y + (-R\varpi_t)\hat{e}_z \equiv \left(V_{rJ_2}\right)\hat{e}_x + \left(V_{tJ_2}\right)\hat{e}_y + \left(V_{nJ_2}\right)\hat{e}_z$$

$$\boldsymbol{R}_d = \boldsymbol{R}_c + \boldsymbol{\rho} = (R + x)\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\boldsymbol{V}_d = \left(V_{rJ_2} + \dot{x} - y\varpi_n + z\varpi_t\right)\hat{e}_x + \left(V_{tJ_2} + \dot{y} + x\varpi_n - z\varpi_r\right)\hat{e}_y + \left(V_{nJ_2} + \dot{z} - x\varpi_t + y\varpi_r\right)\hat{e}_z \tag{6}$$

where

$$\boldsymbol{\varpi} \equiv (\varpi_r)\hat{e}_x + (\varpi_t)\hat{e}_y + (\varpi_n)\hat{e}_z \tag{7}$$

This same approach was used by Garrison et al.[4] with a different set of variables to develop the state transition matrix without J_2. Also, the position and velocity of the Deputy can be obtained in curvilinear coordinates by using the Taylor-series expansion about the Chief and the geometric transformation:

$$\boldsymbol{R}_d = \begin{pmatrix} R + \delta R \\ 0 \\ 0 \end{pmatrix} + R\begin{pmatrix} 0 \\ \delta\theta + c_i\delta\Omega \\ s_\theta\delta i - c_\theta s_i\delta\Omega \end{pmatrix}$$

$$\boldsymbol{V}_d = \begin{pmatrix} V_{rJ_2} + \delta V_{rJ_2} \\ V_{tJ_2} + \delta V_{tJ_2} \\ V_{nJ_2} + \delta V_{nJ_2} \end{pmatrix} + V_{rJ_2}\begin{pmatrix} 0 \\ \delta\theta + c_i\delta\Omega \\ s_\theta\delta i - c_\theta s_i\delta\Omega \end{pmatrix} + V_{tJ_2}\begin{pmatrix} -\delta\theta - c_i\delta\Omega \\ 0 \\ c_\theta\delta i + s_\theta s_i\delta\Omega \end{pmatrix} + V_{nJ_2}\begin{pmatrix} -s_\theta\delta i + c_\theta s_i\delta\Omega \\ -c_\theta\delta i - s_\theta s_i\delta\Omega \\ 0 \end{pmatrix} \tag{8}$$

where $s_\gamma = \sin\gamma$ and $c_\gamma = \cos\gamma$ for the angle γ. The variables with δ are obtained by a Taylor-series expansion about the Chief. The velocity is divided into two parts. The first part is expressed in terms of the orbital elements and has the same form as that for unperturbed motion. The other, denoted by Δ, is the variation caused by only J_2.

$$V_{jJ_2} = V_j + \Delta V_j$$

$$\delta V_{jJ_2} = \delta V_j + \delta\Delta V_j \qquad (j = r, t, n) \tag{9}$$

From the orbit equation

$$\delta R = (R/a)\delta a + \left[(R^2/p)(q_1\sin\theta - q_2\cos\theta)\right]\delta\theta - \left(2Raq_1/p + R^2\cos\theta/p\right)\delta q_1 - \left(2Raq_2/p + R^2\sin\theta/p\right)\delta q_2 \tag{10}$$

Therefore, from Eqs. (6–10) the general relationship between $\boldsymbol{X}(t)$ and $\delta\boldsymbol{e}(t)$ becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \left(\frac{R}{a}\right)\delta a + \left[\frac{R^2}{p}(q_1 s_\theta - q_2 c_\theta)\right]\delta\theta - \left(\frac{2Raq_1}{p} + \frac{R^2 c_\theta}{p}\right)\delta q_1 - \left(\frac{2Raq_2}{p} + \frac{R^2 s_\theta}{p}\right)\delta q_2 \\ R\delta\theta + Rc_i\delta\Omega \\ R(s_\theta\delta i - c_\theta s_i\delta\Omega) \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \delta V_r \\ \delta V_t \\ \delta V_n \end{pmatrix} + V_r\begin{pmatrix} 0 \\ \delta\theta + c_i\delta\Omega \\ s_\theta\delta i - c_\theta s_i\delta\Omega \end{pmatrix} + V_t\begin{pmatrix} -\delta\theta - c_i\delta\Omega \\ 0 \\ c_\theta\delta i + s_\theta s_i\delta\Omega \end{pmatrix} + V_n\begin{pmatrix} -s_\theta\delta i + c_\theta s_i\delta\Omega \\ -c_\theta\delta i - s_\theta s_i\delta\Omega \\ 0 \end{pmatrix} + \varpi_n\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} + \begin{pmatrix} \delta\Delta V_r \\ \delta\Delta V_t \\ \delta\Delta V_n \end{pmatrix}$$
$$+ \Delta V_r\begin{pmatrix} 0 \\ \delta\theta + c_i\delta\Omega \\ s_\theta\delta i - c_\theta s_i\delta\Omega \end{pmatrix} + \Delta V_t\begin{pmatrix} -\delta\theta - c_i\delta\Omega \\ 0 \\ c_\theta\delta i + s_\theta s_i\delta\Omega \end{pmatrix} + \Delta V_n\begin{pmatrix} -s_\theta\delta i + c_\theta s_i\delta\Omega \\ -c_\theta\delta i - s_\theta s_i\delta\Omega \\ 0 \end{pmatrix} + \varpi_r\begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} + \varpi_t\begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \tag{11}$$

To obtain the matrix $\Sigma(t)$ for the osculating elements, use the disturbing function through J_2

$$\Re = -\frac{\mu J_2 R_e^2}{2a^3(1-e^2)^3}(1+e\cos f)^3(3\sin^2 i\sin^2\theta - 1) \tag{12}$$

$$\frac{\partial}{\partial e}\left(\frac{a}{R}\right) = \left(\frac{a}{R}\right)^2\cos f$$

$$\frac{\partial f}{\partial e} = \left(\frac{a}{R} + \frac{1}{1-e^2}\right)\sin f = \frac{\sin f(2+e\cos f)}{1-e^2}$$

$$\frac{\partial R}{\partial e} = a\frac{\partial}{\partial e}\left(\frac{R}{a}\right) = -a\left(\frac{a}{R}\right)^{-2}\left(\frac{a}{R}\right)^2\cos f = -a\cos f$$

$$\frac{\partial R}{\partial f} = \frac{a(1-e^2)e\sin f}{(1+e\cos f)^2} = \frac{R^2 e\sin f}{a(1-e^2)}$$

$$\frac{\partial f}{\partial M} = \frac{\partial f}{\partial E}\frac{\partial E}{\partial M} = \left(\frac{a}{R}\right)^2\sqrt{1-e^2} = \frac{(1+e\cos f)^2}{(1-e^2)^{\frac{3}{2}}} \tag{13}$$

The time variations of the orbital elements caused by J_2 are obtained from Lagrange's planetary equations.[8]

$$\dot{i} = -\frac{\alpha}{4}\frac{\sqrt{\mu}(1+e\cos f)^3}{a^{\frac{7}{2}}\eta^7}(\sin 2i\sin 2\theta)$$

$$\dot{\Omega} = -\frac{\alpha\sqrt{\mu}}{a^{\frac{7}{2}}\eta^7}(1+e\cos f)^3\cos i\sin^2\theta \tag{14}$$

where $n = \sqrt{(\mu/a^3)}$ and $\eta^2 = 1 - e^2$. For convenience, all of the variables of the Chief and the Deputy without subscripts are osculating in this section. The total angular velocity for osculating elements including the effect of J_2 is

$$\varpi = (\dot{\Omega}\sin\theta\sin i + \dot{i}\cos\theta)\hat{e}_x + (\dot{\Omega}\cos\theta\sin i - \dot{i}\sin\theta)\hat{e}_y + (\dot{\theta} + \dot{\Omega}\cos i)\hat{e}_z \tag{15}$$

Because the orbit plane is defined by the position and velocity, the velocity of the Chief must be in the reference orbital plane. Applying this constraint gives

$$\dot{\Omega}\cos\theta\sin i = \dot{i}\sin\theta \tag{16}$$

and the angular velocity becomes

$$\varpi_r = \frac{\dot{\Omega}\sin i}{\sin\theta}, \qquad \varpi_t = 0, \qquad \varpi_n = \dot{\theta} + \dot{\Omega}\cos i \tag{17}$$

The total angular momentum that is always perpendicular to the instantaneous orbital plane is

$$h = R^2\varpi_n = R^2(\dot{\theta} + \dot{\Omega}\cos i) = \sqrt{\mu p} \tag{18}$$

The time rate of the true anomaly is[9]

$$\dot{f} = \frac{\sqrt{\mu}(1+ec_f)^2}{a^{\frac{3}{2}}\eta^3} + \frac{\alpha}{2}\frac{\sqrt{\mu}(1+ec_f)^3}{ea^{\frac{7}{2}}\eta^7}\Big[\big(3s_i^2 s_\theta^2 - 1\big)(1+ec_f)c_f + (2+ec_f)s_f s_i^2 s_{2\theta}\Big] \tag{19}$$

Using the vis-viva integral and other relationships,[8] the velocity in terms of the osculating elements is

$$V_{rJ_2} = \dot{R} = \sqrt{\mu/p}(q_1\sin\theta - q_2\cos\theta)$$

$$V_{tJ_2} = R\varpi_n = \sqrt{\mu/p}(1 + q_1\cos\theta + q_2\sin\theta)$$

$$V_{nJ_2} = 0, \qquad \Delta V_j = 0 \qquad (j = r, t, n) \tag{20}$$

Also, from Eqs. (9), (10), and (20)

$$\delta R = (R/a)\delta a + (RV_r/V_t)\delta\theta - \big(2Raq_1/p + R^2\cos\theta/p\big)\delta q_1 - \big(2Raq_2/p + R^2\sin\theta/p\big)\delta q_2 \tag{21}$$

Substituting Eqs. (17), (20), and (21) into Eq. (11) for the general relationship yields the transformation matrix $\Sigma(t) \equiv \{A(t) + \alpha B(t)\}$ for osculating elements. The simplified form of this matrix is in Appendix A. The matrix $A(t)$ has the same form as the matrix $A(t)$ for the unperturbed noncircular case.[7] However, they are not numerically the same because the J_2 effect is implied in all of the orbital elements and velocity. The inverse matrix $\Sigma^{-1}(t)$ is given in Appendix B. $A^{-1}(t)$ can be obtained by setting $J_2 = 0$ in $\Sigma^{-1}(t)$.

Matrix $\bar{B}(t)$

Because the mean elements describe the long-term secular effects, it might be advantageous to use them for describing the relative motion. For the mean elements the angular velocity will be different because the constraint that the velocity must be in the orbit plane no longer applies. Using the same process just described, but with only secular variations[10] caused by J_2 after neglecting J_2^2 terms,

$$\dot{a}^{(s)} = \dot{e}^{(s)} = \dot{i}^{(s)} = 0, \qquad \dot{\omega}^{(s)} = \frac{\alpha}{4}\frac{\sqrt{\mu}}{a_0^{\frac{7}{2}}\eta_0^4}\big(5\cos^2 i_0 - 1\big)$$

$$\dot{\Omega}^{(s)} = -\frac{\alpha}{2}\frac{\sqrt{\mu}}{a_0^{\frac{7}{2}}\eta_0^4}\cos i_0, \qquad \dot{M}^{(s)} = \frac{\sqrt{\mu}}{a_0^{\frac{3}{2}}} + \frac{\alpha}{4}\frac{\sqrt{\mu}}{a_0^{\frac{7}{2}}\eta_0^3}\big(3\cos^2 i_0 - 1\big) \tag{22}$$

where $\eta_0^2 = 1 - q_{10}^2 - q_{20}^2$. The subscript 0 means the values at t_0 and all of the elements are mean in this section. Because $\dot{i}^{(s)} = 0$, the total angular velocity for the mean elements becomes

$$\varpi = \big(\dot{\Omega}^{(s)}\sin\theta\sin i\big)\hat{e}_x + \big(\dot{\Omega}^{(s)}\cos\theta\sin i\big)\hat{e}_y + \big(\dot{\theta}^{(s)} + \dot{\Omega}^{(s)}\cos i\big)\hat{e}_z \tag{23}$$

Also, the variations of the velocity become

$$\Delta V_r = -(R^2/p)(q_1\sin\theta - q_2\cos\theta)\dot{\omega}^{(s)}$$

$$\Delta V_t = R\cos i\,\dot{\Omega}^{(s)}, \qquad \Delta V_n = -R\cos\theta\sin i\,\dot{\Omega}^{(s)} \tag{24}$$

Therefore, from Eqs. (23) and (24) the relationship $\bar{\Sigma}(t) \equiv \{\bar{A}(t) + \alpha\bar{B}(t)\}$ between $\boldsymbol{X}_{\text{mean}}$ and $\delta\boldsymbol{e}_{\text{mean}}$ can be obtained from Eq. (11). Although $\bar{\Sigma}(t)$ is a function of the mean elements, $\bar{A}(t)$ has the same form as $A(t)$. Thus, only the matrix $\bar{B}(t)$ is provided in Appendix C.

State Transition Matrix for Relative Mean Orbital Elements $\bar{\phi}_{\bar{e}}\,(t, t_0)$

With only the secular variations by J_2 in Eq. (12), the mean elements are

$$a = a_0, \qquad \lambda = \lambda_0 + \big(\dot{\omega}^{(s)} + \dot{M}^{(s)}\big)(t - t_0), \qquad i = i_0$$

$$q_1 = e\cos\omega = q_{10}\cos\big[\dot{\omega}^{(s)}(t-t_0)\big] - q_{20}\sin\big[\dot{\omega}^{(s)}(t-t_0)\big]$$

$$q_2 = e\sin\omega = q_{10}\sin\big[\dot{\omega}^{(s)}(t-t_0)\big] + q_{20}\cos\big[\dot{\omega}^{(s)}(t-t_0)\big]$$

$$\Omega = \Omega_0 + \dot{\Omega}^{(s)}(t - t_0) \tag{25}$$

where all of the elements without subscript are the mean elements. Now, to obtain the equation for θ define

$$\theta = f + \omega = \text{true argument of latitude}$$

$$\lambda = M + \omega = \text{mean argument of latitude}$$

$$F = E + \omega = \text{eccentric argument of latitude} \tag{26}$$

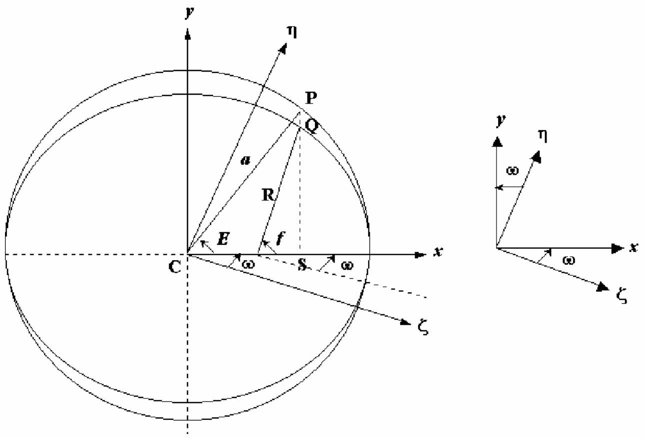


Fig. 2 Relationship between F and θ.

From Kepler's equation $M = E - e\sin E$, the modified Kepler's equation between λ and F becomes

$$\lambda = (E+\omega) - e\sin E = F - \frac{\sqrt{1-e^2}\,e\sin(\theta-\omega)}{1+e\cos(\theta-\omega)}$$

$$= F - q_1\sin F + q_2\cos F \tag{27}$$

Figure 2 is used to obtain the relationship between the eccentric argument of latitude F and the true argument of latitude θ. The point P in $\{\varsigma, \eta\}$ with the angle F is

$$\varsigma = a\cos F, \qquad \eta = a\sin F \tag{28}$$

Also, P in $\{x, y\}$ with the angle f is

$$x = \mathrm{CS} = ae + R\cos f, \qquad y = \frac{SQ}{\sqrt{1-e^2}} = \frac{R\sin f}{\sqrt{1-e^2}} \tag{29}$$

Then, from the relationship between two coordinate systems

$$\begin{pmatrix} a\cos F \\ a\sin F \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} ae + R\cos f \\ \dfrac{R\sin f}{\sqrt{1-e^2}} \end{pmatrix}$$

$$= \begin{pmatrix} ae\cos\omega + R\cos\omega\cos f - \dfrac{R\sin\omega\sin f}{\sqrt{1-e^2}} \\ ae\sin\omega + R\sin\omega\cos f + \dfrac{R\cos\omega\sin f}{\sqrt{1-e^2}} \end{pmatrix} \tag{30}$$

By defining

$$\beta \equiv 1\Big/\left[(1-e^2) + \sqrt{1-e^2}\right] \tag{31}$$

the eccentric argument of latitude in terms of the mean elements becomes

$$\tan F = \frac{R\left(1+\beta q_1^2\right)\sin\theta - \beta R q_1 q_2\cos\theta + a q_2}{R\left(1+\beta q_2^2\right)\cos\theta - \beta R q_1 q_2\sin\theta + a q_1} \tag{32}$$

where

$$\beta = \frac{1}{\left(1-q_1^2-q_2^2\right) + \sqrt{1-q_1^2-q_2^2}} = \frac{1}{\eta^2+\eta}$$

$$R = \frac{a\left(1-q_1^2-q_2^2\right)}{1+q_1\cos\theta+q_2\sin\theta} = \frac{a\eta^2}{1+q_1\cos\theta+q_2\sin\theta} \tag{33}$$

Therefore, the relationship between λ and θ is

$$\lambda = F - \frac{\sqrt{1-q_1^2-q_2^2}\,(q_1\sin\theta - q_2\cos\theta)}{1+q_1\cos\theta+q_2\sin\theta} \tag{34}$$

After defining the function G and using the variations in Eq. (22),

$$G \equiv \lambda - \lambda_0 = (M - M_0) + (\omega - \omega_0) = \left(\dot{M}^{(s)} + \dot{\omega}^{(s)}\right)(t - t_0) \tag{35}$$

the state transition matrix for the relative mean elements is obtained by taking the Taylor-series expansions of Eqs. (25) and (35) about the Chief,

$$\delta\boldsymbol{e}_{\text{mean}}(t) = \bar{\phi}_{\bar{e}}(t, t_0)\,\delta\boldsymbol{e}_{\text{mean}}(t_0) \tag{36}$$

This matrix $\bar{\phi}_{\bar{e}}(t, t_0)$ is given in Appendix D as a function of time and the initial mean elements.

Transformation Matrix Between the Relative Mean and Osculating Elements $D(t)$

The osculating elements can be obtained as functions of the mean elements

$$\boldsymbol{e}_{\text{osc}} = \boldsymbol{e}_{\text{mean}} + \Delta\boldsymbol{e}_p \tag{37}$$

where

$$\Delta\boldsymbol{e}_p = -J_2(\boldsymbol{e}, W_1) \tag{38}$$

$\Delta\boldsymbol{e}_p$ denotes the long- and the short-period variations caused by the primary gravitational perturbation J_2 and are obtained from the generating function[11] via

$$W_1 = W_1^{lp} + W_1^{sp1} + W_1^{sp2} \tag{39}$$

where, in the normalized Delaunay variables $(l, g, h, L, G, H)^T$ and the true anomaly,

$$W_1^{lp} = -\left(\frac{1}{32G^3}\right)\left(1-\frac{G^2}{L^2}\right)\left(1-5\frac{H^2}{G^2}\right)^{-1}$$

$$\times\left(1 - 16\frac{H^2}{G^2} + 15\frac{H^4}{G^4}\right)\sin 2g$$

$$W_1^{sp^1} = \left(\frac{1}{4G^3}\right)\left(-1+\frac{3H^2}{G^2}\right)(f - 1 + e\sin f)$$

$$W_1^{sp^2} = \frac{3}{8G^3}\left(1-\frac{H^2}{G^2}\right)\Bigg[\sin(2f+2g) + e\sin(f+2g)$$

$$+ \frac{e}{3}\sin(3f+2g)\Bigg] \tag{40}$$

The Delaunay variables are defined as

$$L = \sqrt{\mu a}, \qquad G = L(1-e^2)^{\frac{1}{2}}, \qquad H = G\cos i$$

$$l = \text{mean anomaly}, \qquad g = \text{argumant of perigee}$$

$$h = \text{right ascension} \tag{41}$$

Note that the short-period portion has been separated into two parts and the mean to osculating transformation is a single transformation rather than two transformations of mean to long period and long period to short period as is usually done.[11] Using this generating function, $\Delta\boldsymbol{e}_p$ is obtained by the transformation, and finally $\boldsymbol{e}_{\text{osc}}$ is obtained by

$$\boldsymbol{e}_{\text{osc}} = \boldsymbol{e}_{\text{mean}} - \left(J_2 R_e^2\right)\left\{\boldsymbol{e}^{(lp)} + \boldsymbol{e}^{(sp1)} + \boldsymbol{e}^{(sp2)}\right\} \tag{42}$$

Then, the transformation matrix $D(t)$ for the relative elements is obtained by the partial derivatives of the osculating elements with respect to the mean elements:

$$D(t) = \frac{\partial\boldsymbol{e}_{\text{osc}}}{\partial\boldsymbol{e}_{\text{mean}}} = I - \left(J_2 R_e^2\right)\left[D^{(lp)}(t) + D^{(sp1)}(t) + D^{(sp2)}(t)\right] \tag{43}$$

The quantities $\boldsymbol{e}^{(lp)}$, $\boldsymbol{e}^{(sp1)}$, $\boldsymbol{e}^{(sp2)}$, $D^{(lp)}(t)$, $D^{(sp1)}(t)$, and $D^{(sp2)}(t)$ are in Appendix E in terms of the mean elements $\boldsymbol{e}_{\text{mean}}$.

III. Numerical Evaluation

To evaluate the proposed method, the predicted relative motion by the geometric method is compared with the numerical results obtained by numerically integrating the equations of motion of both satellites in the Earth-centered-inertial (ECI) reference frame with a $J_2 \sim J_5$ gravity field, transforming the position and velocity vectors from ECI frame to the Chief LVLH frame, transforming them to the curvilinear frame, and differencing them to obtain the relative position and velocity vectors. The $J_2 \sim J_5$ gravity field has been selected rather than just J_2 because it is more representative of the real world. For the comparison of the theories, the initial conditions are chosen such that the projection of the relative orbit in the horizontal plane is a circle when the Chief orbit is circular with the same semimajor axis. The out-of-plane motion is created by a differential inclination, not differential right ascension. The 0.005 eccentricity will result in a slight drift away from this desired orbit even if there is no perturbation[12] because the Deputy initial conditions based on a circular reference orbit generate a small differential semimajor axis.

The initial conditions are given in Table 1 along with the differences in orbital elements. First, the errors in the geometric method are evaluated for the case of a spherically symmetric Earth. For validation, the geometric method was compared to the Garrison's theory[4] with $J_2 = 0$, and they were numerically equal, demonstrating that it gives another representation for the relative motion for an eccentric reference orbit when $J_2 = 0$. The errors in the geometric method are shown in Fig. 3, and they are centimeter level. Because the geometric method incorporates the Chief orbit eccentricity, these small errors are due to the linearization.

For the perturbed relative motion the initial conditions in Table 1 are considered as perturbed initial osculating elements for the Chief and the osculating values of the relative position and velocity for the Deputy. The initial mean conditions for the Chief and the Deputy are obtained by the preceding closed-form transformation equations and shown in Table 2. There will be a small error in the initial mean

Table 1 Initial conditions for an unperturbed near-circular orbit

Satellite	Condition value
Chief	
a, km	7100
θ, deg	180
i, deg	70
q_1	4.698×10^{-3}
q_2	1.710×10^{-3}
Ω, deg	45
Deputy	
x, m	0
$\dot{x}$, m/s	0.264
y, m	500
$\dot{y}$, m/s	0
z, m	0
$\dot{z}$, m/s	0.528
δa, m	−0.839
$\delta\theta$, deg	4.016×10^{-3}
δi, deg	-4.054×10^{-3}
δq_1	1.199×10^{-7}
δq_2	3.554×10^{-5}
$\delta\Omega$, deg	0

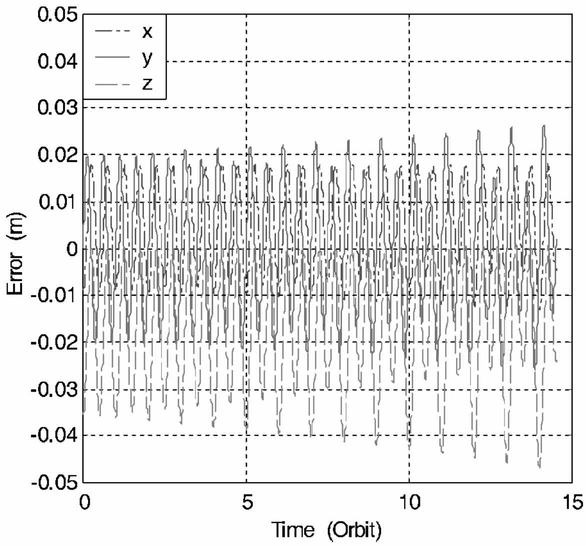
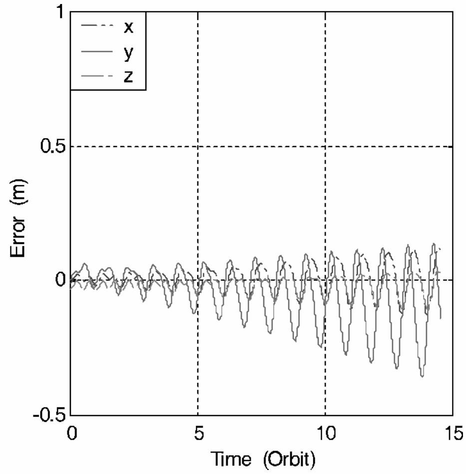
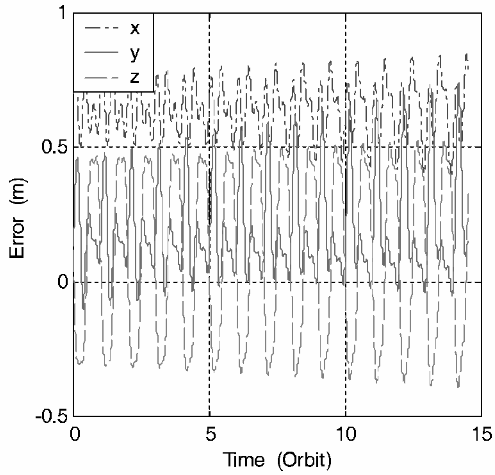


Fig. 3 Position errors in the geometric method for a near-circular orbit and a spherically symmetric Earth.



a) For osculating elements



b) For mean elements

Fig. 4 Position errors in the geometric method for a perturbed near-circular orbit.

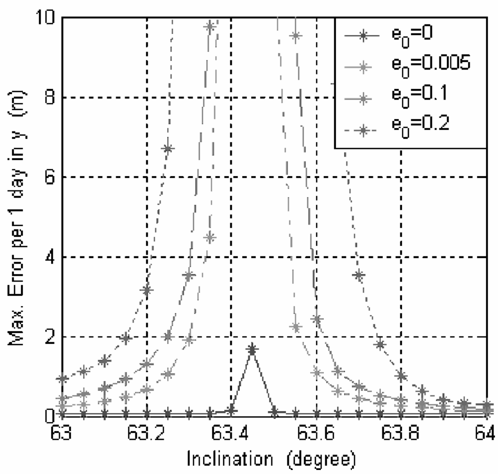


Fig. 5 Maximum in-track errors per 1 day near the critical inclination.

conditions because the osculating to mean transformation is just first order. The error is $\mathcal{O}(J_2^2)$.

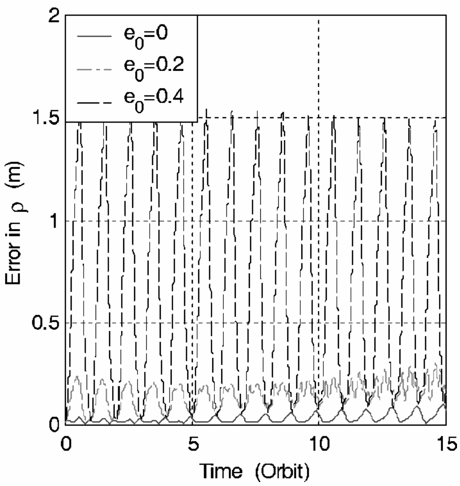
Compared with the numerical solution that includes the $J_2 \sim J_5$ gravitational perturbations, the errors in the geometric method with only J_2 are shown in Fig. 4 for a perturbed near-circular orbit. Figure 4a shows the errors for the osculating elements, and Fig. 4b shows the errors for the mean elements. The small and essentially negligible in-track secular growth shown in both figures results from the neglected terms of $\mathcal{O}(J_2^2)$. The larger errors in the mean element solution shown in Fig. 4b occur because a mean element solution is compared to the exact osculating solution. Thus, the differences between the errors in two figures are the effects of the long- and the short-period terms on the relative motion.

Table 2 Initial mean conditions for a perturbed near-circular orbit

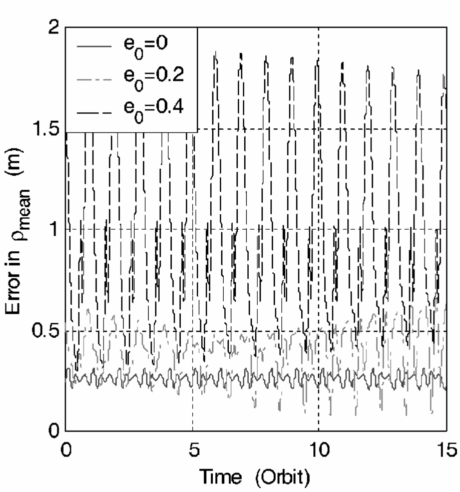
Satellite	Mean condition value
Chief	
a, km	7091.870
θ, deg	180.0002
i, deg	69.9880
q_1	5.230×10^{-3}
q_2	1.709×10^{-3}
Ω, deg	45.0001
Deputy	
x, m	0.710
$\dot{x}$, m/s	0.264
y, m	500.135
$\dot{y}$, m/s	-1.491×10^{-3}
z, m	0.151
$\dot{z}$, m/s	0.527
δa, m	-0.415
$\delta\theta$, deg	4.019×10^{-3}
δi, deg	-4.056×10^{-3}
δq_1	1.601×10^{-7}
δq_2	3.561×10^{-5}
$\delta\Omega$, deg	1.279×10^{-6}

This theory is valid for all inclinations except near the critical inclinations and near-equatorial orbits. A new theory[13] using equinoctial elements has been developed for the near-equatorial orbits. The theory is invalid at the critical inclination because the Brouwer theory is not valid at the critical inclination because of the $(1 - 5\cos^2 i)$ terms in the denominators in the mean to osculating transformation. Figure 5 shows the maximum in-track error after one day as a function of inclination near the critical inclination for various eccentricities with the Chief semimajor axis of 12,000 km. From these results it is concluded that the geometric method is valid except within 0.25 deg of the critical inclination. In the analytic orbit theories used by U.S. Space Command and Naval Space Command for maintaining the space object catalog, the critical inclination problem is handled by letting $(1 - 5\cos^2 i) = \varepsilon\,\mathrm{sign}(1 - 5\cos^2 i)$ whenever $|1 - 5\cos^2 i| < \varepsilon$. This avoids the small denominator problem. This option was not explored.

Figure 6 shows the errors for various Chief eccentricities using the relative orbit initial conditions in Table 2 with the semimajor axis of 12,000 km. The error ρ in denotes the error in relative position of the Deputy, $\varepsilon_\rho = \sqrt{(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2)}$. Although the errors increase some as the Chief eccentricity increases, they are small. Because this solution is valid for all eccentricities and to $\mathcal{O}(J_2)$, it is likely that these errors result from the neglected higher-order gravitational terms, which are $\mathcal{O}(J_2^2)$, and whose magnitude increases with increasing eccentricity.

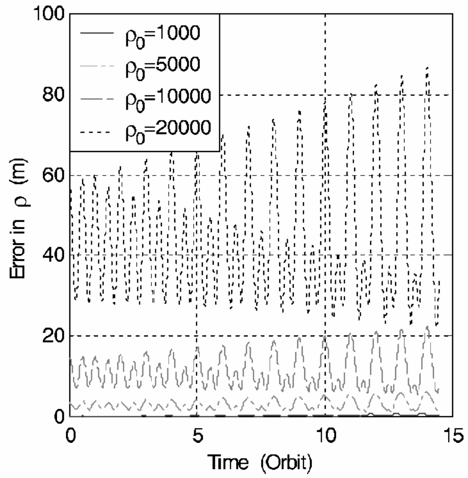


a) For osculating elements

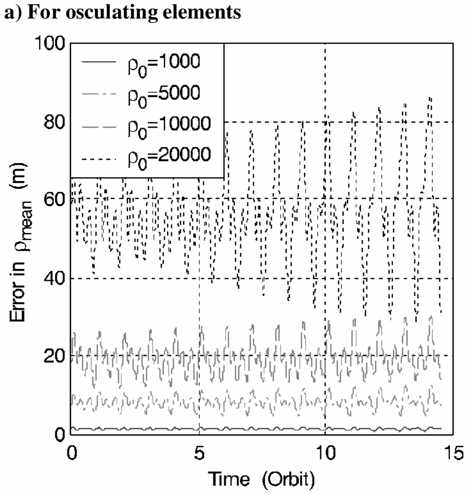


b) For mean elements

Fig. 6 Geometric method errors for various eccentricities.



a) For osculating elements



b) For mean elements

Fig. 7 Geometric method errors for various radii of the projected circular relative orbits.

To evaluate the effect of the linearization, the Chief initial conditions are selected as given in Table 2, and the Deputy initial conditions are selected to result in a projected circular orbit for a circular Chief orbit. As shown in Fig. 7, the error increases as the size of the relative orbit increases and has a small growth in the magnitude as time passes, where ρ_0 is the initial radius of the projected circular relative orbit. They show the small initial biased errors as a result of the linearization in the geometric method.

IV. Conclusions

Using the geometric method, a state transition matrix (STM) of the relative motion of neighboring satellites has been derived for the case of an elliptic Chief orbit and the influence of the gravitational perturbation J_2. The closed-form STMs for both mean elements and osculating elements have been derived. The comparison with the numerical solution that includes the effects of the gravitational perturbations $J_2 \sim J_5$ shows that the geometric method, which includes the Chief orbit eccentricity and the first-order J_2 effects, provides a very precise solution to the relative motion without having to directly solve the complex relative motion differential equations. This method can easily be extended to include the effects of other perturbing forces such as higher-order gravitational terms or differential atmospheric drag. These STMs are valid for any eccentricity, and simpler approximated forms can be obtained if the eccentricity is very small. For example, if the mean eccentricity is 0.0001 then a much simplified STM might be reasonable only with first-order non-J_2 terms and zeroth-order J_2 terms in eccentricity. This STM could be developed by dropping the appropriate terms.

Appendix A: Transformation Matrix $\boldsymbol{\Sigma(t) \equiv \{A(t) + \alpha B(t)\}}$

$$\Sigma_{11} = \frac{R}{a}, \qquad \Sigma_{12} = \frac{RV_r}{V_t}$$

$$\Sigma_{13} = 0, \qquad \Sigma_{14} = -\frac{2Raq_1}{p} - \frac{R^2\cos\theta}{p}$$

$$\Sigma_{15} = -\frac{2Raq_2}{p} - \frac{R^2\sin\theta}{p}, \qquad \Sigma_{16} = 0$$

$$\Sigma_{21} = -\frac{1}{2}\frac{V_r}{a}, \qquad \Sigma_{22} = \left(\frac{V_t}{p}\right)(R-p)$$

$$\Sigma_{23} = 0, \qquad \Sigma_{24} = \frac{V_r a q_1}{p} + \frac{RV_t}{p}\sin\theta$$

$$\Sigma_{25} = \frac{V_r a q_2}{p} - \frac{RV_t}{p}\cos\theta, \qquad \Sigma_{26} = 0$$

$$\Sigma_{31} = 0, \qquad \Sigma_{32} = R, \qquad \Sigma_{33} = 0, \qquad \Sigma_{34} = 0$$

$$\Sigma_{35} = 0, \qquad \Sigma_{36} = R\cos i$$

$$\Sigma_{41} = -\frac{3}{2}\frac{V_t}{a}, \qquad \Sigma_{42} = -V_r$$

$$\Sigma_{43} = -\alpha\left(\frac{V_t\sin i\cos i\sin^2\theta}{pR}\right)$$

$$\Sigma_{44} = \frac{V_t}{p}(3aq_1 + 2R\cos\theta), \qquad \Sigma_{45} = \frac{V_t}{p}(3aq_2 + 2R\sin\theta)$$

$$\Sigma_{46} = V_r\cos i + \alpha\left(\frac{V_t\sin^2 i\cos i\sin\theta\cos\theta}{pR}\right)$$

$$\Sigma_{51} = 0, \qquad \Sigma_{52} = 0, \qquad \Sigma_{53} = R\sin\theta, \qquad \Sigma_{54} = 0$$

$$\Sigma_{55} = 0, \qquad \Sigma_{56} = -R\sin i\cos\theta$$

$$\Sigma_{61} = 0, \qquad \Sigma_{62} = \alpha\left(\frac{V_t\sin i\cos i\sin\theta}{pR}\right)$$

$$\Sigma_{63} = V_r\sin\theta + V_t\cos\theta, \qquad \Sigma_{64} = 0, \qquad \Sigma_{65} = 0$$

$$\Sigma_{66} = -(V_r\cos\theta - V_t\sin\theta)\sin i + \alpha\left(\frac{V_t\sin i\cos^2 i\sin\theta}{pR}\right)$$

Appendix B: Inverse Matrix $\boldsymbol{\Sigma^{-1}(t) \equiv \{A(t) + \alpha B(t)\}^{-1}}$

$$\Sigma^{-1}_{11} = -\left(\frac{2a}{R^3V_t^2}\right)\left[3aV_t^2(R-p) - 2\left(apV_r^2 + R^2V_t^2\right)\right]$$

$$\Sigma^{-1}_{12} = \frac{2a^2pV_r}{R^2V_t^2}$$

$$\Sigma^{-1}_{13} = \left(\frac{2aV_r}{R^3V_t^3}\right)\left[2aV_t^2(R-p) - \left(apV_r^2 + R^2V_t^2\right)\right]$$

$$\Sigma^{-1}_{14} = -\left(\frac{2a}{R^2V_t^3}\right)\left[2aV_t^2(R-p) - \left(apV_r^2 + R^2V_t^2\right)\right]$$

$$\Sigma^{-1}_{15} = -\alpha\left(\frac{2a\sin i\cos i\sin\theta}{pR^4V_t^2}\right)\left[2aV_t^2(R-p) - \left(apV_r^2 + R^2V_t^2\right)\right]$$

$$\Sigma^{-1}_{16} = 0$$

$$\Sigma^{-1}_{21} = 0, \qquad \Sigma^{-1}_{22} = 0, \qquad \Sigma^{-1}_{23} = \frac{1}{R} + \alpha\left(\frac{\cos^2 i\sin^2\theta}{pR^2}\right)$$

$$\Sigma^{-1}_{24} = 0, \qquad \Sigma^{-1}_{25} = \left(\frac{V_r\sin\theta + V_t\cos\theta}{RV_t}\right)\left(\frac{\cos i}{\sin i}\right)$$

$$\Sigma^{-1}_{26} = -\left(\frac{\sin\theta}{V_t}\right)\left(\frac{\cos i}{\sin i}\right)$$

$$\Sigma^{-1}_{31} = 0, \qquad \Sigma^{-1}_{32} = 0, \qquad \Sigma^{-1}_{33} = -\alpha\left(\frac{\sin i\cos i\sin\theta\cos\theta}{pR^2}\right)$$

$$\Sigma^{-1}_{34} = 0, \qquad \Sigma^{-1}_{35} = -\frac{(V_r\cos\theta - V_t\sin\theta)}{RV_t}, \qquad \Sigma^{-1}_{36} = \frac{\cos\theta}{V_t}$$

$$\Sigma^{-1}_{41} = \frac{p(2V_r\sin\theta + 3V_t\cos\theta)}{R^2V_t}, \qquad \Sigma^{-1}_{42} = \left(\frac{p}{RV_t}\right)\sin\theta$$

$$\Sigma^{-1}_{43} = -\left(\frac{1}{R^2V_t^2}\right)\left[pV_r(V_r\sin\theta + V_t\cos\theta) - (R-p)V_t^2\sin\theta\right] + \alpha\left(\frac{\cos^2 i\sin^2\theta}{pR^3V_t}\right)[pV_r\cos\theta + (R-p)V_t\sin\theta]$$

$$\Sigma^{-1}_{44} = \left(\frac{p}{RV_t^2}\right)(V_r\sin\theta + 2V_t\cos\theta)$$

$$\Sigma^{-1}_{45} = \left(\frac{1}{R^2V_t^2}\right)(V_r\sin\theta + V_t\cos\theta)[pV_r\cos\theta + (R-p)V_t\sin\theta] \times\left(\frac{\cos i}{\sin i}\right) + \alpha\left(\frac{\sin i\cos i\sin\theta}{V_tR^3}\right)(V_r\sin\theta + 2V_t\cos\theta)$$

$$\Sigma^{-1}_{46} = -\left(\frac{\sin\theta}{RV_t^2}\right)[pV_r\cos\theta + (R-p)V_t\sin\theta]\left(\frac{\cos i}{\sin i}\right)$$

$$\Sigma^{-1}_{51} = -\frac{p(2V_r\cos\theta - 3V_t\sin\theta)}{R^2V_t}, \qquad \Sigma^{-1}_{52} = -\left(\frac{p}{RV_t}\right)\cos\theta$$

$$\Sigma^{-1}_{53} = \left(\frac{1}{R^2V_t^2}\right)\left[pV_r(V_r\cos\theta - V_t\sin\theta) - (R-p)V_t^2\cos\theta\right] + \alpha\left(\frac{\cos^2 i\sin^2\theta}{pR^3V_t}\right)[pV_r\sin\theta - (R-p)V_t\cos\theta]$$

$$\Sigma^{-1}_{54} = -\left(\frac{p}{RV_t^2}\right)(V_r\cos\theta - 2V_t\sin\theta)$$

$$\Sigma^{-1}_{55} = \left(\frac{1}{R^2V_t^2}\right)(V_r\sin\theta + V_t\cos\theta)[pV_r\sin\theta - (R-p)V_t\cos\theta] \times\left(\frac{\cos i}{\sin i}\right) - \alpha\left(\frac{\sin i\cos i\sin\theta}{V_tR^3}\right)(V_r\cos\theta - 2V_t\sin\theta)$$

$$\Sigma^{-1}_{56} = -\left(\frac{\sin\theta}{RV_t^2}\right)[pV_r\sin\theta - (R-p)V_t\cos\theta]\left(\frac{\cos i}{\sin i}\right)$$

$$\Sigma^{-1}_{61} = 0, \qquad \Sigma^{-1}_{62} = 0, \qquad \Sigma^{-1}_{63} = -\alpha\left(\frac{\cos i\sin^2\theta}{pR^2}\right)$$

$$\Sigma^{-1}_{64} = 0, \qquad \Sigma^{-1}_{65} = -\left(\frac{V_r\sin\theta + V_t\cos\theta}{RV_t\sin i}\right), \qquad \Sigma^{-1}_{66} = \frac{\sin\theta}{V_t\sin i}$$

Appendix C: Matrix $\bar{B}(t)$

$$\bar{B}_{11} = 0, \qquad \bar{B}_{12} = 0, \qquad \bar{B}_{13} = 0, \qquad \bar{B}_{14} = 0$$

$$\bar{B}_{15} = 0, \qquad \bar{B}_{16} = 0$$

$$\bar{B}_{21} = \frac{5}{8}\frac{nRV_r}{ap^2V_t}(5\cos^2 i - 1)$$

$$\bar{B}_{22} = -\frac{1}{4}\frac{nR}{p^3V_t^2}(5\cos^2 i - 1)\left[2pV_r^2 - V_t^2(R-p)\right]$$

$$\bar{B}_{23} = \frac{5}{2}\frac{nRV_r}{p^2V_t}(\sin i\cos i)$$

$$\bar{B}_{24} = \frac{1}{4}\frac{nR}{p^3V_t}(5\cos^2 i - 1)\{2V_r(R\cos\theta - aq_1) - RV_t\sin\theta\}$$

$$\bar{B}_{25} = \frac{1}{4}\frac{nR}{p^3V_t}(5\cos^2 i - 1)\{2V_r(R\sin\theta - aq_2) + RV_t\cos\theta\}$$

$$\bar{B}_{26} = 0$$

$$\bar{B}_{31} = 0, \qquad \bar{B}_{32} = 0, \qquad \bar{B}_{33} = 0$$

$$\bar{B}_{34} = 0, \qquad \bar{B}_{35} = 0, \qquad \bar{B}_{36} = 0$$

$$\bar{B}_{41} = \frac{7}{4}\frac{nR}{ap^2}\cos^2 i, \qquad \bar{B}_{42} = -\frac{1}{4}\frac{nRV_r}{p^2V_t}(5\cos^2 i - 1)$$

$$\bar{B}_{43} = \frac{1}{2}\frac{nR}{p^2}(\sin i\cos i), \qquad \bar{B}_{44} = -2\frac{nRaq_1}{p^3}\cos^2 i$$

$$\bar{B}_{45} = -2\frac{nRaq_2}{p^3}\cos^2 i, \qquad \bar{B}_{46} = -\frac{1}{4}\frac{nRV_r\cos i}{p^2V_t}(5\cos^2 i - 1)$$

$$\bar{B}_{51} = 0, \qquad \bar{B}_{52} = 0, \qquad \bar{B}_{53} = 0, \qquad \bar{B}_{54} = 0$$

$$\bar{B}_{55} = 0, \qquad \bar{B}_{56} = 0$$

$$\bar{B}_{61} = -\frac{7}{4}\frac{nR}{ap^2}(\cos\theta\sin i\cos i), \qquad \bar{B}_{62} = 0$$

$$\bar{B}_{63} = -\frac{1}{4}\frac{nR}{p^2V_t}\left\{V_r\sin\theta(5\cos^2 i - 1) + 2V_t\cos\theta\sin^2 i\right\}$$

$$\bar{B}_{64} = 2\frac{nRaq_1}{p^3}(\cos\theta\sin i\cos i)$$

$$\bar{B}_{65} = 2\frac{nRaq_2}{p^3}(\cos\theta\sin i\cos i)$$

$$\bar{B}_{66} = \frac{1}{4}\frac{nRV_r}{p^2V_t}\cos\theta\sin i(5\cos^2 i - 1)$$

where

$$\alpha = 3J_2R_e^2, \qquad n = \sqrt{\mu/a^3}, \qquad p = a\left(1 - q_1^2 - q_2^2\right)$$

$$R = p/(1 + q_1\cos\theta + q_2\sin\theta)$$

$$V_r = \sqrt{\mu/p}(q_1\sin\theta - q_2\cos\theta)$$

$$V_t = \sqrt{\mu/p}(1 + q_1\cos\theta + q_2\sin\theta)$$

Appendix D: State Transition Matrix for Relative Mean Elements $\bar{\phi}_{\bar{e}}(t, t_0)$

$$\bar{\phi}_{\bar{e}\,11} = 1, \qquad \bar{\phi}_{\bar{e}\,12} = 0, \qquad \bar{\phi}_{\bar{e}\,13} = 0$$

$$\bar{\phi}_{\bar{e}\,14} = 0, \qquad \bar{\phi}_{\bar{e}\,15} = 0, \qquad \bar{\phi}_{\bar{e}\,16} = 0$$

$$\bar{\phi}_{\bar{e}\,21} = -\frac{(t-t_0)}{G_\theta}\left\{\left(\frac{3}{2}\frac{n_0}{a_0}\right) + \frac{7\alpha}{8}\left(\frac{n_0}{a_0p_0^2}\right)\left[\eta_0\left(3\cos^2 i_0 - 1\right) + K\left(5\cos^2 i_0 - 1\right)\right]\right\}$$

$$\bar{\phi}_{\bar{e}\,22} = -\frac{G_{\theta_0}}{G_\theta}$$

$$\bar{\phi}_{\bar{e}\,23} = -\frac{(t-t_0)}{G_\theta}\left\{\frac{\alpha}{2}\left(\frac{n_0}{p_0^2}\right)(\sin i_0\cos i_0)(3\eta_0 + 5K)\right\}$$

$$\bar{\phi}_{\bar{e}\,24} = -\frac{1}{G_\theta}\left(G_{q10} + c_{\{\dot{\omega}^{(s)}(t-t_0)\}}G_{q1} + s_{\{\dot{\omega}^{(s)}(t-t_0)\}}G_{q2}\right) + \frac{(t-t_0)}{G_\theta}\left(\frac{\alpha}{4}\right)\left(\frac{a_0n_0q_{10}}{p_0^3}\right)\left[3\eta_0\left(3\cos^2 i_0 - 1\right) + 4K\left(5\cos^2 i_0 - 1\right)\right]$$

$$\bar{\phi}_{\bar{e}\,25} = -\frac{1}{G_\theta}\left(G_{q20} - s_{\{\dot{\omega}^{(s)}(t-t_0)\}}G_{q1} + c_{\{\dot{\omega}^{(s)}(t-t_0)\}}G_{q2}\right) + \frac{(t-t_0)}{G_\theta}\left(\frac{\alpha}{4}\right)\left(\frac{a_0n_0q_{20}}{p_0^3}\right)\left[3\eta_0\left(3\cos^2 i_0 - 1\right) + 4K\left(5\cos^2 i_0 - 1\right)\right]$$

$$\bar{\phi}_{\bar{e}\,26} = 0$$

$$\bar{\phi}_{\bar{e}\,31} = 0, \qquad \bar{\phi}_{\bar{e}\,32} = 0, \qquad \bar{\phi}_{\bar{e}\,33} = 1$$

$$\bar{\phi}_{\bar{e}\,34} = 0, \qquad \bar{\phi}_{\bar{e}\,35} = 0, \qquad \bar{\phi}_{\bar{e}\,36} = 0$$

$$\bar{\phi}_{\bar{e}\,41} = \frac{7\alpha}{8}\left(\frac{n_0}{a_0p_0^2}\right)\left(q_{10}s_{\{\dot{\omega}^{(s)}(t-t_0)\}} + q_{20}c_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) \times\left(5\cos^2 i_0 - 1\right)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,42} = 0$$

$$\bar{\phi}_{\bar{e}\,43} = \frac{5\alpha}{2}\left(\frac{n_0}{p_0^2}\right)\left(q_{10}s_{\{\dot{\omega}^{(s)}(t-t_0)\}} + q_{20}c_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) \times(\sin i_0\cos i_0)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,44} = c_{\{\dot{\omega}^{(s)}(t-t_0)\}} - \alpha\left(\frac{a_0n_0q_{10}}{p_0^3}\right)\left(q_{10}s_{\{\dot{\omega}^{(s)}(t-t_0)\}} + q_{20}c_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) \times\left(5\cos^2 i_0 - 1\right)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,45} = -s_{\{\dot{\omega}^{(s)}(t-t_0)\}} - \alpha\left(\frac{a_0n_0q_{20}}{p_0^3}\right)\left(q_{10}s_{\{\dot{\omega}^{(s)}(t-t_0)\}} + q_{20}c_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) \times\left(5\cos^2 i_0 - 1\right)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,46} = 0$$

$$\bar{\phi}_{\bar{e}\,51} = -\frac{7\alpha}{8}\left(\frac{n_0}{a_0 p_0^2}\right)\left(q_{10}c_{\{\dot{\omega}^{(s)}(t-t_0)\}} - q_{20}s_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) \times\left(5\cos^2 i_0 - 1\right)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,52} = 0$$

$$\bar{\phi}_{\bar{e}\,53} = -\frac{5\alpha}{2}\left(\frac{n_0}{p_0^2}\right)\left(q_{10}c_{\{\dot{\omega}^{(s)}(t-t_0)\}} - q_{20}s_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) \times(\sin i_0 \cos i_0)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,54} = s_{\{\dot{\omega}^{(s)}(t-t_0)\}} + \alpha\left(\frac{a_0 n_0 q_{10}}{p_0^3}\right)\left(q_{10}c_{\{\dot{\omega}^{(s)}(t-t_0)\}} - q_{20}s_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) \times\left(5\cos^2 i_0 - 1\right)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,55} = c_{\{\dot{\omega}^{(s)}(t-t_0)\}} + \alpha\left(\frac{a_0 n_0 q_{20}}{p_0^3}\right)\left(q_{10}c_{\{\dot{\omega}^{(s)}(t-t_0)\}} - q_{20}s_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) \times\left(5\cos^2 i_0 - 1\right)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,56} = 0$$

$$\bar{\phi}_{\bar{e}\,61} = \frac{7\alpha}{4}\left(\frac{n_0 \cos i_0}{a_0 p_0^2}\right)(t-t_0), \qquad \bar{\phi}_{\bar{e}\,62} = 0$$

$$\bar{\phi}_{\bar{e}\,63} = \frac{\alpha}{2}\left(\frac{n_0 \sin i_0}{p_0^2}\right)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,64} = -2\alpha\left(\frac{n_0 a_0 q_{10}\cos i_0}{p_0^3}\right)(t-t_0)$$

$$\bar{\phi}_{\bar{e}\,65} = -2\alpha\left(\frac{n_0 a_0 q_{20}\cos i_0}{p_0^3}\right)(t-t_0), \qquad \bar{\phi}_{\bar{e}\,66} = 1$$

$$G_a = \frac{\partial G}{\partial a} = 0, \qquad G_{a_0} = \frac{\partial G}{\partial a_0} = 0$$

$$G_\theta = \frac{\partial G}{\partial \theta} = \frac{nR}{V_t}, \qquad G_{\theta_0} = \frac{\partial G}{\partial \theta_0} = -\frac{n_0 R_0}{V_{t0}}$$

$$G_i = \frac{\partial G}{\partial i} = 0, \qquad G_{i_0} = \frac{\partial G}{\partial i_0} = 0$$

$$G_{q_1} = \frac{\partial G}{\partial q_1} = \frac{q_2}{\eta(1+\eta)} + \frac{q_1 V_r}{\eta V_t} - \frac{\eta R(a+R)}{p^2}(q_2 + \sin\theta)$$

$$G_{q_{10}} = \frac{\partial G}{\partial q_{10}} = -\frac{q_{20}}{\eta_0(1+\eta_0)} - \frac{q_{10}V_{r0}}{\eta_0 V_{t0}} + \frac{\eta_0 R_0(a_0+R_0)}{p_0^2}(q_{20} + \sin\theta_0)$$

$$G_{q_2} = \frac{\partial G}{\partial q_2} = -\frac{q_1}{\eta(1+\eta)} + \frac{q_2 V_r}{\eta V_t} + \frac{\eta R(a+R)}{p^2}(q_1 + \cos\theta)$$

$$G_{q_{20}} = \frac{\partial G}{\partial q_{20}} = \frac{q_{10}}{\eta_0(1+\eta_0)} - \frac{q_{20}V_{r0}}{\eta_0 V_{t0}} - \frac{\eta_0 R_0(a_0+R_0)}{p_0^2}(q_{10} + \cos\theta_0)$$

$$G_\Omega = \frac{\partial G}{\partial \Omega} = 0, \qquad G_{\Omega_0} = \frac{\partial G}{\partial \Omega_0} = 0$$

where

$$s_{\{\dot{\omega}^{(s)}(t-t_0)\}} = \sin\left[\dot{\omega}^{(s)}(t-t_0)\right], \qquad c_{\{\dot{\omega}^{(s)}(t-t_0)\}} = \cos\left[\dot{\omega}^{(s)}(t-t_0)\right]$$

$$\eta_0^2 = 1 - q_{10}^2 - q_{20}^2, \qquad \eta^2 = 1 - q_1^2 - q_2^2$$

$$V_r = \sqrt{\mu/p}(q_1\sin\theta - q_2\cos\theta)$$

$$V_t = \sqrt{\mu/p}(1 + q_1\cos\theta + q_2\sin\theta)$$

$$\dot{\omega}^{(s)} = \tfrac{3}{4}J_2(R_e/p_0)^2 n_0\left(5\cos^2 i_0 - 1\right)$$

$$K = 1 + G_{q_1}\left(q_{10}s_{\{\dot{\omega}^{(s)}(t-t_0)\}} + q_{20}c_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right) - G_{q_2}\left(q_{10}c_{\{\dot{\omega}^{(s)}(t-t_0)\}} - q_{20}s_{\{\dot{\omega}^{(s)}(t-t_0)\}}\right)$$

Appendix E: Transformation Matrix from Mean to Osculating Elements $D(t)$

$$D_{11}^{(lp)} = -\left(\frac{1}{a}\right)a^{(lp)}, \qquad D_{12}^{(lp)} = D_{13}^{(lp)} = D_{14}^{(lp)} = D_{15}^{(lp)} = D_{16}^{(lp)} = 0$$

$$D_{21}^{(lp)} = -\left(\frac{2}{a}\right)\theta^{(lp)}, \qquad D_{22}^{(lp)} = -\left(\frac{\sin^2 i}{16a^2\eta^4}\right)(1 - 10\Theta\cos^2 i)[2(q_1\cos\theta - q_2\sin\theta) + \varepsilon_1\cos 2\theta]$$

$$D_{23}^{(lp)} = \left(\frac{\sin 2i}{16a^2\eta^4}\right)\{5q_1q_2(11 + 112\Theta\cos^2 i + 520\Theta^2\cos^4 i + 800\Theta^3\cos^6 i) - [2q_1q_2 + (2+\varepsilon_2)(q_1\sin\theta + q_2\cos\theta)] \times[(1 - 10\Theta\cos^2 i) + 10\Theta\sin^2 i(1 + 5\Theta\cos^2 i)]\}$$

$$D_{24}^{(lp)} = \left(\frac{1}{16a^2\eta^6}\right)\left\{\left(\eta^2 + 4q_1^2\right)\left[q_2(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i) - \sin^2 i(1 - 10\Theta\cos^2 i)(3q_2 + 2\sin\theta)\right] - 2\sin^2 i(1 - 10\Theta\cos^2 i)[4q_2 + \sin\theta(1+\varepsilon_1)]q_1\cos\theta\right\}$$

$$D_{25}^{(lp)} = \left(\frac{1}{16a^2\eta^6}\right)\left\{\left(\eta^2 + 4q_2^2\right)\left[q_1(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i) - \sin^2 i(1 - 10\Theta\cos^2 i)(3q_1 + 2\cos\theta)\right] - 2\sin^2 i(1 - 10\Theta\cos^2 i)[4q_1 + \cos\theta(1+\varepsilon_1)]q_2\sin\theta\right\}$$

$$D_{26}^{(lp)} = 0$$

$$D_{31}^{(lp)} = -\left(\frac{2}{a}\right)i^{(lp)}, \qquad D_{32}^{(lp)} = 0, \qquad D_{33}^{(lp)} = \left(\frac{q_1^2 - q_2^2}{16a^2\eta^4}\right)[\cos 2i(1 - 10\Theta\cos^2 i) + 5\Theta\sin^2 2i(1 + 5\Theta\cos^2 i)]$$

$$D_{34}^{(lp)} = \left(\frac{q_1\sin 2i}{16a^2\eta^6}\right)(1 - 10\Theta\cos^2 i)\left[\eta^2 + 2\left(q_1^2 - q_2^2\right)\right], \qquad D_{35}^{(lp)} = -\left(\frac{q_2\sin 2i}{16a^2\eta^6}\right)(1 - 10\Theta\cos^2 i)\left[\eta^2 - 2\left(q_1^2 - q_2^2\right)\right], \qquad D_{36}^{(lp)} = 0$$

$$D_{41}^{(lp)} = -\left(\frac{2}{a}\right)q_1{}^{(lp)}, \qquad D_{42}^{(lp)} = 0, \qquad D_{43}^{(lp)} = -\left(\frac{q_1\sin 2i}{16a^2\eta^4}\right)\begin{bmatrix}\eta^2[(1 - 10\Theta\cos^2 i) + 10\Theta\sin^2 i(1 + 5\Theta\cos^2 i)]\\ + 5q_2^2(11 + 112\Theta\cos^2 i + 520\Theta^2\cos^4 i + 800\Theta^3\cos^6 i)\end{bmatrix}$$

$$D_{44}^{(lp)} = -\left(\frac{1}{16a^2\eta^6}\right)\begin{bmatrix}\eta^2\sin^2 i(1 - 10\Theta\cos^2 i)\left(\eta^2 + 2q_1^2\right)\\ + q_2^2(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i)\left(\eta^2 + 4q_1^2\right)\end{bmatrix}$$

$$D_{45}^{(lp)} = -\left(\frac{q_1q_2}{8a^2\eta^6}\right)\left[\eta^2\sin^2 i(1 - 10\Theta\cos^2 i) + (3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i)\left(\eta^2 + 2q_2^2\right)\right], \qquad D_{46}^{(lp)} = 0$$

$$D_{51}^{(lp)} = -\left(\frac{2}{a}\right)q_2{}^{(lp)}, \qquad D_{52}^{(lp)} = 0, \qquad D_{53}^{(lp)} = \left(\frac{q_2\sin 2i}{16a^2\eta^4}\right)\begin{bmatrix}\eta^2(1 - 10\Theta\cos^2 i) + 10\Theta\eta^2\sin^2 i(1 + 5\Theta\cos^2 i)\\ + 5q_1^2(11 + 112\Theta\cos^2 i + 520\Theta^2\cos^4 i + 800\Theta^3\cos^6 i)\end{bmatrix}$$

$$D_{54}^{(lp)} = \left(\frac{q_1q_2}{8a^2\eta^6}\right)\left[\eta^2\sin^2 i(1 - 10\Theta\cos^2 i) + (3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i)\left(\eta^2 + 2q_1^2\right)\right]$$

$$D_{55}^{(lp)} = \left(\frac{1}{16a^2\eta^6}\right)\begin{bmatrix}\eta^2\sin^2 i(1 - 10\Theta\cos^2 i)\left(\eta^2 + 2q_2^2\right)\\ + q_1^2(3 - 55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i)\left(\eta^2 + 4q_2^2\right)\end{bmatrix}, \qquad D_{56}^{(lp)} = 0$$

$$D_{61}^{(lp)} = -\left(\frac{2}{a}\right)\Omega^{(lp)}, \qquad D_{62}^{(lp)} = 0, \qquad D_{63}^{(lp)} = -\left(\frac{q_1q_2\sin i}{8a^2\eta^4}\right)\left[(11 + 80\Theta\cos^2 i + 200\Theta^2\cos^4 i) + 160\Theta\cos^2 i(1 + 5\Theta\cos^2 i)^2\right]$$

$$D_{64}^{(lp)} = \left(\frac{q_2\cos i}{8a^2\eta^6}\right)(11 + 80\Theta\cos^2 i + 200\Theta^2\cos^4 i)\left(\eta^2 + 4q_1^2\right), \qquad D_{65}^{(lp)} = \left(\frac{q_1\cos i}{8a^2\eta^6}\right)(11 + 80\Theta\cos^2 i + 200\Theta^2\cos^4 i)\left(\eta^2 + 4q_2^2\right)$$

$$D_{66}^{(lp)} = 0$$

$$D_{11}^{(sp1)} = -\left(\frac{1}{a}\right)a^{(sp1)}, \qquad D_{12}^{(sp1)} = -\left(\frac{3\varepsilon_3}{2a\eta^6}\right)(1 - 3\cos^2 i)(1 + \varepsilon_2)^2, \qquad D_{13}^{(sp1)} = \left(\frac{3\sin 2i}{2a\eta^6}\right)\left[(1 + \varepsilon_2)^3 - \eta^3\right]$$

$$D_{14}^{(sp1)} = \left(\frac{3(1 - 3\cos^2 i)}{2a\eta^8}\right)\left[2q_1(1 + \varepsilon_2)^3 + \eta^2(1 + \varepsilon_2)^2\cos\theta - \eta^3 q_1\right]$$

$$D_{15}^{(sp1)} = \left(\frac{3(1 - 3\cos^2 i)}{2a\eta^8}\right)\left[2q_2(1 + \varepsilon_2)^3 + \eta^2(1 + \varepsilon_2)^2\sin\theta - \eta^3 q_2\right], \qquad D_{16}^{(sp1)} = 0$$

$$D_{21}^{(sp1)} = -\left(\frac{2}{a}\right)\theta^{(sp1)}, \qquad D_{22}^{(sp1)} = \left[\frac{(1 - 3\cos^2 i)}{4a^2\eta^4(1 + \eta)}\right]\left[\varepsilon_2(1 + \varepsilon_2 - \eta) - \varepsilon_3^2\right] + \left[\frac{3(1 - 5\cos^2 i)}{4a^2\eta^4(1 + \varepsilon_2)^2}\right]\left[(1 + \varepsilon_2)^3 - \eta^3\right]$$

$$D_{23}^{(sp1)} = \left[\frac{3\varepsilon_3\sin 2i}{4a^2\eta^4(1 + \eta)}\right][(1 + \varepsilon_2) + (5 + 4\eta)] + \left(\frac{15\sin 2i}{4a^2\eta^4}\right)(\theta - \lambda)$$

$$D_{24}^{(sp1)} = \left[\frac{(1 - 3\cos^2 i)}{4a^2\eta^6(1 + \eta)^2}\right]\left\{\eta^2[\varepsilon_1\sin\theta + (1 + \eta)(\varepsilon_2\sin\theta + \varepsilon_3\cos\theta)] + q_1\varepsilon_3[4(\varepsilon_1 + \varepsilon_2) + \eta(2 + 5\varepsilon_2)]\right\}$$
$$+ \left[\frac{3(1 - 5\cos^2 i)}{4a^2\eta^6}\right]\left\{4q_1[(\theta - \lambda) + \varepsilon_3] + \eta^2\sin\theta\right\} - \left[\frac{3(1 - 5\cos^2 i)}{4a^2\eta^4}\right](\lambda_{q_1})$$

$$D_{25}^{(sp1)} = -\left[\frac{(1 - 3\cos^2 i)}{4a^2\eta^6(1 + \eta)^2}\right]\left\{\eta^2[\varepsilon_1\cos\theta + (1 + \eta)(\varepsilon_2\cos\theta - \varepsilon_3\sin\theta)] - q_2\varepsilon_3[4(\varepsilon_1 + \varepsilon_2) + \eta(2 + 5\varepsilon_2)]\right\}$$
$$+ \left[\frac{3(1 - 5\cos^2 i)}{4a^2\eta^6}\right]\left\{4q_2[(\theta - \lambda) + \varepsilon_3] - \eta^2\cos\theta\right\} - \left[\frac{3(1 - 5\cos^2 i)}{4a^2\eta^4}\right](\lambda_{q_2})$$

$$D_{26}^{(sp1)} = 0$$

$$D_{31}^{(sp1)} = -\left(\frac{2}{a}\right) i^{(sp1)}, \qquad D_{32}^{(sp1)} = D_{33}^{(sp1)} = D_{34}^{(sp1)} = D_{35}^{(sp1)} = D_{36}^{(sp1)} = 0$$

$$D_{41}^{(sp1)} = -\left(\frac{2}{a}\right) q_1{}^{(sp1)}$$

$$D_{42}^{(sp1)} = -\left[\frac{(1-3\cos^2 i)}{4a^2\eta^4}\right]\left[(1+\varepsilon_2)(2\sin\theta + \varepsilon_2\sin\theta + 2\varepsilon_3\cos\theta) + \varepsilon_3(q_1+\cos\theta) + \eta^2\sin\theta\right] - \left[\frac{3q_2(1-5\cos^2 i)}{4a^2\eta^4(1+\varepsilon_2)^2}\right]\left[(1+\varepsilon_2)^3 - \eta^3\right]$$

$$D_{43}^{(sp1)} = \left[\frac{3q_1\sin 2i}{4a^2\eta^2(1+\eta)}\right] + \left(\frac{3\sin 2i}{4a^2\eta^4}\right)\left\{(1+\varepsilon_2)[q_1 + (2+\varepsilon_2)\cos\theta] - 5q_2\varepsilon_3 + \eta^2\cos\theta\right\} - \left(\frac{15q_2\sin 2i}{4a^2\eta^4}\right)(\theta-\lambda)$$

$$D_{44}^{(sp1)} = \left[\frac{(1-3\cos^2 i)}{4a^2\eta^2(1+\eta)}\right] + \left[\frac{(1-3\cos^2 i)}{8a^2\eta^6}\right]\left\{\begin{matrix}\eta^2[5+2(5q_1\cos\theta + 2q_2\sin\theta) + (3+2\varepsilon_2)\cos 2\theta] \\ +2q_1[4(1+\varepsilon_2)(2+\varepsilon_2)\cos\theta + (3\eta+4\varepsilon_2)q_1]\end{matrix}\right\}$$

$$+\left[\frac{(1-3\cos^2 i)q_1^2(4+5\eta)}{4a^2\eta^6(1+\eta)^2}\right] - \left[\frac{3q_2(1-5\cos^2 i)}{4a^2\eta^6}\right]\left[4q_1\varepsilon_3 + \eta^2\sin\theta\right] - \left[\frac{3q_1q_2(1-5\cos^2 i)}{a^2\eta^6}\right](\theta-\lambda) + \left[\frac{3q_2(1-5\cos^2 i)}{4a^2\eta^4}\right](\lambda_{q_1})$$

$$D_{45}^{(sp1)} = \left[\frac{(1-3\cos^2 i)}{8a^2\eta^6}\right]\left\{\begin{matrix}\eta^2[2(q_1\sin\theta + 2q_2\cos\theta) + (3+2\varepsilon_2)\sin 2\theta] \\ +2q_2[4(1+\varepsilon_2)(2+\varepsilon_2)\cos\theta + (3\eta+4\varepsilon_2)q_1]\end{matrix}\right\} + \left[\frac{(1-3\cos^2 i)q_1q_2(4+5\eta)}{4a^2\eta^6(1+\eta)^2}\right]$$

$$-\left[\frac{3(1-5\cos^2 i)}{4a^2\eta^6}\right]\left[\varepsilon_3(\eta^2+4q_2^2) - \eta^2 q_2\cos\theta\right] - \left[\frac{3(1-5\cos^2 i)}{4a^2\eta^6}\right]\left[\left(\eta^2+4q_2^2\right)(\theta-\lambda)\right] + \left[\frac{3q_2(1-5\cos^2 i)}{4a^2\eta^4}\right](\lambda_{q_2})$$

$$D_{46}^{(sp1)} = 0$$

$$D_{51}^{(sp1)} = -\left(\frac{2}{a}\right) q_2{}^{(sp1)}$$

$$D_{52}^{(sp1)} = \left[\frac{(1-3\cos^2 i)}{4a^2\eta^4}\right]\left[(1+\varepsilon_2)(2\cos\theta + \varepsilon_2\cos\theta - 2\varepsilon_3\sin\theta) - \varepsilon_3(q_2+\sin\theta) + \eta^2\cos\theta\right] + \left[\frac{3q_1(1-5\cos^2 i)}{4a^2\eta^4(1+\varepsilon_2)^2}\right]\left[(1+\varepsilon_2)^3 - \eta^3\right]$$

$$D_{53}^{(sp1)} = \left[\frac{3q_2\sin 2i}{4a^2\eta^2(1+\eta)}\right] + \left(\frac{3\sin 2i}{4a^2\eta^4}\right)\left\{(1+\varepsilon_2)[q_2 + (2+\varepsilon_2)\sin\theta] + 5q_1\varepsilon_3 + \eta^2\sin\theta\right\} - \left(\frac{15q_1\sin 2i}{4a^2\eta^4}\right)(\theta-\lambda)$$

$$D_{54}^{(sp1)} = \left[\frac{(1-3\cos^2 i)}{8a^2\eta^6}\right]\left\{\begin{matrix}\eta^2[2(2q_1\sin\theta + q_2\cos\theta) + (3+2\varepsilon_2)\sin 2\theta] \\ +2q_1[4(1+\varepsilon_2)(2+\varepsilon_2)\sin\theta + (3\eta+4\varepsilon_2)q_2]\end{matrix}\right\} + \left[\frac{(1-3\cos^2 i)q_1q_2(4+5\eta)}{4a^2\eta^6(1+\eta)^2}\right] + \left[\frac{3(1-5\cos^2 i)}{4a^2\eta^6}\right]$$

$$\times\left[\varepsilon_3\left(\eta^2+4q_1^2\right) + \eta^2 q_1\sin\theta\right] + \left[\frac{3(1-5\cos^2 i)}{4a^2\eta^6}\right]\left[\left(\eta^2+4q_1^2\right)(\theta-\lambda)\right] - \left[\frac{3q_1(1-5\cos^2 i)}{4a^2\eta^4}\right](\lambda_{q_1})$$

$$D_{55}^{(sp1)} = \left[\frac{(1-3\cos^2 i)}{4a^2\eta^2(1+\eta)}\right] + \left[\frac{(1-3\cos^2 i)}{8a^2\eta^6}\right]\left\{\begin{matrix}\eta^2[5+2(2q_1\cos\theta + 5q_2\sin\theta) - (3+2\varepsilon_2)\cos 2\theta] \\ +2q_2[4(1+\varepsilon_2)(2+\varepsilon_2)\sin\theta + (3\eta+4\varepsilon_2)q_2]\end{matrix}\right\} + \left[\frac{(1-3\cos^2 i)q_2^2(4+5\eta)}{4a^2\eta^6(1+\eta)^2}\right]$$

$$+\left[\frac{3q_1(1-5\cos^2 i)}{4a^2\eta^6}\right]\left[4q_2\varepsilon_3 - \eta^2\cos\theta\right] + \left[\frac{3q_1q_2(1-5\cos^2 i)}{a^2\eta^6}\right](\theta-\lambda) - \left[\frac{3q_1(1-5\cos^2 i)}{4a^2\eta^4}\right](\lambda_{q_2})$$

$$D_{56}^{(sp1)} = 0$$

$$D_{61}^{(sp1)} = -\left(\frac{2}{a}\right)\Omega^{(sp1)}, \qquad D_{62}^{(sp1)} = \left[\frac{3\cos i}{2a^2\eta^4(1+\varepsilon_2)^2}\right]\left[(1+\varepsilon_2)^3 - \eta^3\right], \qquad D_{63}^{(sp1)} = -\left(\frac{3\varepsilon_3\sin i}{2a^2\eta^4}\right) - \left(\frac{3\sin i}{2a^2\eta^4}\right)(\theta-\lambda)$$

$$D_{64}^{(sp1)} = \left(\frac{3\cos i}{2a^2\eta^6}\right)\left(4q_1\varepsilon_3 + \eta^2\sin\theta\right) + \left(\frac{6q_1\cos i}{a^2\eta^6}\right)(\theta-\lambda) - \left(\frac{3\cos i}{2a^2\eta^4}\right)(\lambda_{q_1})$$

$$D_{65}^{(sp1)} = \left(\frac{3\cos i}{2a^2\eta^6}\right)\left(4q_2\varepsilon_3 - \eta^2\cos\theta\right) + \left(\frac{6q_2\cos i}{a^2\eta^6}\right)(\theta-\lambda) - \left(\frac{3\cos i}{2a^2\eta^4}\right)(\lambda_{q_2}), \qquad D_{66}^{(sp1)} = 0$$

$$D_{11}^{(sp2)} = -\left(\frac{1}{a}\right)a^{(sp2)}, \qquad D_{12}^{(sp2)} = \left(\frac{3\sin^2 i}{2a\eta^6}\right)(1+\varepsilon_2)^2[2(1+\varepsilon_2)\sin 2\theta + 3\varepsilon_3\cos 2\theta], \qquad D_{13}^{(sp2)} = -\left(\frac{3\sin 2i\cos 2\theta}{2a\eta^6}\right)(1+\varepsilon_2)^3$$

$$D_{14}^{(sp2)} = -\left(\frac{9\sin^2 i\cos 2\theta}{2a\eta^8}\right)(1+\varepsilon_2)^2\left[2q_1(1+\varepsilon_2)+\eta^2\cos\theta\right]$$

$$D_{15}^{(sp2)} = -\left(\frac{9\sin^2 i\cos 2\theta}{2a\eta^8}\right)(1+\varepsilon_2)^2\left[2q_2(1+\varepsilon_2)+\eta^2\sin\theta\right], \qquad D_{16}^{(sp2)} = 0$$

$$D_{21}^{(sp2)} = -\left(\frac{2}{a}\right)\theta^{(sp2)}, \qquad D_{22}^{(sp2)} = -\left(\frac{1}{8a^2\eta^4}\right)\left\{\begin{array}{l}3(3-5\cos^2 i)[(q_1\cos\theta - q_2\sin\theta) + 2\cos 2\theta + (q_1\cos 3\theta + q_2\sin 3\theta)]\\ -\sin^2 i[5(q_1\cos\theta - q_2\sin\theta) + 16\cos 2\theta + 9(q_1\cos 3\theta + q_2\sin 3\theta)]\end{array}\right\}$$

$$D_{23}^{(sp2)} = -\left(\frac{\sin 2i}{8a^2\eta^4}\right)[10(q_1\sin\theta + q_2\cos\theta) + 7\sin 2\theta + 2(q_1\sin 3\theta - q_2\cos 3\theta)]$$

$$\begin{aligned}
D_{24}^{(sp2)} = &-\left[\frac{(3-5\cos^2 i)}{8a^2\eta^6}\right]\left\{4q_1[3\sin 2\theta + q_2(3\cos\theta - \cos 3\theta)] + \left(\eta^2+4q_1^2\right)(3\sin\theta + \sin 3\theta)\right\} - \left[\frac{\sin^2 i}{8a^2\eta^2(1+\eta)}\right](3\sin\theta + \sin 3\theta)\\
&-\left[\frac{\sin^2 i}{32a^2\eta^4(1+\eta)}\right]\left\{\begin{array}{l}36q_2 - 4(2+3\eta)\sin\theta - (39+12\eta+\eta^2)\sin 3\theta + 9\varepsilon_1\sin 5\theta + 12q_2(2q_1\cos\theta + q_2\sin\theta)\\ +9q_1(q_1\sin 3\theta - q_2\cos 3\theta) + 18(3q_1\sin 4\theta + 2q_2\cos 4\theta) - 3q_1(q_1\sin 5\theta - 11q_2\cos 5\theta)\\ +24[(1+\varepsilon_2)(2+\varepsilon_2)\sin\theta + \varepsilon_3(3+2\varepsilon_2)\cos\theta]\cos 2\theta\end{array}\right\}\\
&-\left[\frac{3\sin^2 i}{32a^2\eta^4(1+\eta)^2}\right]\left[4\sin\theta - 6q_1\sin 4\theta - q_1(q_1\sin 5\theta + q_2\cos 5\theta)\right]\\
&+\left[\frac{q_1\sin^2 i}{8a^2\eta^6(1+\eta)}\right]\left[20(1+\eta)(q_1\sin\theta + q_2\cos\theta) + 32(1+\eta)\sin 2\theta + 3(4+3\eta)(q_1\sin 3\theta - q_2\cos 3\theta)\right]\\
&-\left[\frac{q_1\sin^2 i(4+5\eta)}{32a^2\eta^6(1+\eta)^2}\right]\left\{\begin{array}{l}24(q_1\sin\theta + q_2\cos\theta) + 24\varepsilon_3(1+\varepsilon_2)(2+\varepsilon_2)\cos 2\theta - (27+3\eta)(q_1\sin 3\theta - q_2\cos 3\theta)\\ -18\sin 4\theta - 3(q_1\sin 5\theta + q_2\cos 5\theta)\\ +12q_2[(3+\varepsilon_2)q_1 + 3(q_1\cos 4\theta + q_2\sin 4\theta) + q_1(q_1\cos 5\theta + q_2\sin 5\theta)]\end{array}\right\}
\end{aligned}$$

$$\begin{aligned}
D_{25}^{(sp2)} = &-\left[\frac{(3-5\cos^2 i)}{8a^2\eta^6}\right]\left\{4q_2[3\sin 2\theta + q_1(3\sin\theta + \sin 3\theta)] + \left(\eta^2+4q_2^2\right)(3\cos\theta - \cos 3\theta)\right\} - \left[\frac{\sin^2 i}{8a^2\eta^2(1+\eta)}\right](3\cos\theta - \cos 3\theta)\\
&-\left[\frac{\sin^2 i}{32a^2\eta^4(1+\eta)}\right]\left\{\begin{array}{l}36q_1 - 4(2+3\eta)\cos\theta + (39+12\eta+\eta^2)\cos 3\theta + 9\varepsilon_1\cos 5\theta + 12q_1(q_1\cos\theta + 2q_2\sin\theta)\\ +9q_2(q_1\sin 3\theta - q_2\cos 3\theta) + 18(2q_1\cos 4\theta + 7q_2\sin 4\theta) + 3q_2(11q_1\sin 5\theta - q_2\cos 5\theta)\\ +24[\varepsilon_3(3+2\varepsilon_2)\sin\theta - (1+\varepsilon_2)(2+\varepsilon_2)\cos\theta]\cos 2\theta\end{array}\right\}\\
&-\left[\frac{3\sin^2 i}{32a^2\eta^4(1+\eta)^2}\right][4\cos\theta - 6q_2\sin 4\theta - q_2(q_1\sin 5\theta + q_2\cos 5\theta)]\\
&+\left[\frac{q_2\sin^2 i}{8a^2\eta^6(1+\eta)}\right][20(1+\eta)(q_1\sin\theta + q_2\cos\theta) + 32(1+\eta)\sin 2\theta + 3(4+3\eta)(q_1\sin 3\theta - q_2\cos 3\theta)]\\
&-\left[\frac{q_2\sin^2 i(4+5\eta)}{32a^2\eta^6(1+\eta)^2}\right]\left\{\begin{array}{l}24(q_1\sin\theta + q_2\cos\theta) + 24\varepsilon_3(1+\varepsilon_2)(2+\varepsilon_2)\cos 2\theta\\ -(27+3\eta)(q_1\sin 3\theta - q_2\cos 3\theta) - 18\sin 4\theta - 3(q_1\sin 5\theta + q_2\cos 5\theta)\\ +12q_2[(3+\varepsilon_2)q_1 + 3(q_1\cos 4\theta + q_2\sin 4\theta) + q_1(q_1\cos 5\theta + q_2\sin 5\theta)]\end{array}\right\}
\end{aligned}$$

$$D_{26}^{(sp2)} = 0$$

$$D_{31}^{(sp2)} = -\left(\frac{2}{a}\right)i^{(sp2)}, \qquad D_{32}^{(sp2)} = \left(\frac{3\sin 2i}{8a^2\eta^4}\right)[(q_1\sin\theta + q_2\cos\theta) + 2\sin 2\theta + (q_1\sin 3\theta - q_2\cos 3\theta)]$$

$$D_{33}^{(sp2)} = -\left(\frac{\cos 2i}{4a^2\eta^4}\right)[3(q_1\cos\theta - q_2\sin\theta) + 3\cos 2\theta + (q_1\cos 3\theta + q_2\sin 3\theta)]$$

$$D_{34}^{(sp2)} = -\left(\frac{\sin 2i}{8a^2\eta^6}\right)\left\{4q_1[3\cos 2\theta - q_2(3\sin\theta - \sin 3\theta)] + \left(\eta^2 + 4q_1^2\right)(3\cos\theta + \cos 3\theta)\right\}$$

$$D_{35}^{(sp2)} = -\left(\frac{\sin 2i}{8a^2\eta^6}\right)\left\{4q_2[3\cos 2\theta + q_1(3\cos\theta + \cos 3\theta)] - \left(\eta^2 + 4q_2^2\right)(3\sin\theta - \sin 3\theta)\right\}$$

$$D_{36}^{(sp2)} = 0$$

$$D_{41}^{(sp2)} = -\left(\frac{2}{a}\right)q_1{}^{(sp2)}$$

$$D_{42}^{(sp2)} = \left[\frac{3q_2(3-5\cos^2 i)}{8a^2\eta^4}\right][(q_1\cos\theta - q_2\sin\theta) + 2\cos 2\theta + (q_1\cos 3\theta + q_2\sin 3\theta)]$$

$$+\left(\frac{3\sin^2 i}{32a^2\eta^4}\right)\left\{\begin{array}{l} 2\begin{bmatrix} 2q_2\varepsilon_2 - 9q_2(q_1\cos 3\theta + q_2\sin 3\theta) + 12(q_1\sin 4\theta - q_2\cos 4\theta) \\ -5q_2(q_1\cos 5\theta + q_2\sin 5\theta) \end{bmatrix} \\ + \left[4\left(1+3q_1^2\right)\sin\theta + 40q_1\sin 2\theta + (28+17\varepsilon_1)\sin 3\theta + 5\varepsilon_1\sin 5\theta\right] \end{array}\right\}$$

$$D_{43}^{(sp2)} = -\left(\frac{\sin 2i}{32a^2\eta^4}\right)\left\{\begin{array}{l} 2\begin{bmatrix} 36q_1(q_1\cos\theta - q_2\sin\theta) + 30(q_1\cos 2\theta - q_2\sin 2\theta) - q_2(q_1\sin 3\theta - q_2\cos 3\theta) \\ +9(q_1\cos 4\theta + q_2\sin 4\theta) + 3q_2(q_1\sin 5\theta - q_2\cos 5\theta) \end{bmatrix} \\ +[6q_1(3+2q_1\cos\theta) + 12(1-4\varepsilon_1)\cos\theta + (28+17\varepsilon_1)\cos 3\theta + 3\varepsilon_1\cos 5\theta] \end{array}\right\}$$

$$D_{44}^{(sp2)} = \left[\frac{q_2(3-5\cos^2 i)}{8a^2\eta^6}\right]\left\{4q_1[3\sin 2\theta + q_2(3\cos\theta - \cos 3\theta)] + \left(\eta^2 + 4q_1^2\right)(3\sin\theta + \sin 3\theta)\right\}$$

$$-\left(\frac{\sin^2 i}{8a^2\eta^4}\right)\left\{\begin{array}{l} [8q_1\cos 3\theta - 3q_2(\sin\theta - \sin 3\theta)] \\ +3[5+\varepsilon_2 + 3\cos 2\theta + 3(q_1\cos 3\theta + q_2\sin 3\theta)]\cos 2\theta \end{array}\right\}$$

$$-\left(\frac{3q_1\sin^2 i}{4a^2\eta^6}\right)\left\{\begin{array}{l} 2q_1[(q_1\cos\theta - q_2\sin\theta) + (q_1\cos 3\theta + q_2\sin 3\theta)] \\ + \begin{bmatrix} 9\cos\theta - \cos 3\theta + 2q_1(5+\varepsilon_2) + 6(q_1\cos 2\theta + q_2\sin 2\theta) \\ +2q_1(q_1\cos 3\theta + q_2\sin 3\theta) \end{bmatrix}\cos 2\theta \end{array}\right\}$$

$$D_{45}^{(sp2)} = \left[\frac{(3-5\cos^2 i)}{8a^2\eta^6}\right]\left\{\left(\eta^2 + 4q_2^2\right)[3\sin 2\theta + q_1(3\sin\theta + \sin 3\theta)] + 2q_2\left(\eta^2 + 2q_2^2\right)(3\cos\theta - \cos 3\theta)\right\}$$

$$+\left(\frac{\sin^2 i}{16a^2\eta^4}\right)[6(q_1\sin\theta + 2q_2\cos\theta) - (9q_1\sin 3\theta + q_2\cos 3\theta) - 9\sin 4\theta - 3(q_1\sin 5\theta + q_2\cos 5\theta)]$$

$$-\left(\frac{3q_2\sin^2 i}{8a^2\eta^6}\right)\left\{\begin{array}{l} 2q_1\begin{bmatrix} 3 + 2(2q_1\cos\theta - q_2\sin\theta) + 10\cos 2\theta + 3(q_1\cos 3\theta + q_2\sin 3\theta) \\ +(q_1\cos 5\theta + q_2\sin 5\theta) \end{bmatrix} \\ +[8\cos\theta + 9\cos 3\theta + 6(q_1\cos 4\theta + q_2\sin 4\theta) - \cos 5\theta] \end{array}\right\}$$

$$D_{46}^{(sp2)} = 0$$

$$D_{51}^{(sp2)} = -\left(\frac{2}{a}\right)q_2{}^{(sp2)}$$

$$D_{52}^{(sp2)} = -\left[\frac{3q_1(3-5\cos^2 i)}{8a^2\eta^4}\right][(q_1\cos\theta - q_2\sin\theta) + 2\cos 2\theta + (q_1\cos 3\theta + q_2\sin 3\theta)]$$

$$+\left(\frac{3\sin^2 i}{32a^2\eta^4}\right)\left\{\begin{array}{l} 2\begin{bmatrix} 2q_1\varepsilon_2 + 9q_1(q_1\cos 3\theta + q_2\sin 3\theta) - 12(q_1\cos 4\theta + q_2\sin 4\theta) \\ -5q_1(q_1\cos 5\theta + q_2\sin 5\theta) \end{bmatrix} \\ + \left[4\left(1+3q_2^2\right)\cos\theta + 40q_2\sin 2\theta - (28+17\varepsilon_1)\cos 3\theta + 5\varepsilon_1\cos 5\theta\right] \end{array}\right\}$$

$$D_{53}^{(sp2)} = -\left(\frac{\sin 2i}{32a^2\eta^4}\right)\left\{\begin{array}{l} 2\begin{bmatrix} 36q_1(q_1\sin\theta + q_2\cos\theta) + 30(q_1\sin 2\theta + q_2\cos 2\theta) + q_1(q_1\sin 3\theta - q_2\cos 3\theta) \\ +9(q_1\sin 4\theta - q_2\cos 4\theta) + 3q_1(q_1\sin 5\theta - q_2\cos 5\theta) \end{bmatrix} \\ -[6q_2(3+2q_2\sin\theta) + 12(1+2\varepsilon_1)\sin\theta - (28+17\varepsilon_1)\sin 3\theta + 3\varepsilon_1\sin 5\theta] \end{array}\right\}$$

$$D_{54}^{(sp2)} = -\left[\frac{(3-5\cos^2 i)}{8a^2\eta^6}\right]\left\{\left(\eta^2+4q_1^2\right)[3\sin 2\theta + q_2(3\cos\theta - \cos 3\theta)] + 2q_1\left(\eta^2+2q_1^2\right)(3\sin\theta + \sin 3\theta)\right\}$$

$$-\left(\frac{\sin^2 i}{16a^2\eta^4}\right)[6(2q_1\sin\theta + q_2\cos\theta) + (q_1\sin 3\theta + 9q_2\cos 3\theta) + 9\sin 4\theta - 3(q_1\sin 5\theta + q_2\cos 5\theta)]$$

$$+\left(\frac{3q_1\sin^2 i}{8a^2\eta^6}\right)\left\{\begin{array}{l}2q_2\left[\begin{array}{l}3-2(q_1\cos\theta - 2q_2\sin\theta) - 10\cos 2\theta - 3(q_1\cos 3\theta + q_2\sin 3\theta)\\ +(q_1\cos 5\theta + q_2\sin 5\theta)\end{array}\right]\\ +[8\sin\theta - 9\sin 3\theta - 6(q_1\sin 4\theta - q_2\cos 4\theta) - \sin 5\theta]\end{array}\right\}$$

$$D_{55}^{(sp2)} = -\left[\frac{q_1(3-5\cos^2 i)}{8a^2\eta^6}\right]\left\{4q_2[3\sin 2\theta + q_1(3\sin\theta + \sin 3\theta)] + \left(\eta^2+4q_2^2\right)(3\cos\theta - \cos 3\theta)\right\}$$

$$-\left(\frac{\sin^2 i}{8a^2\eta^4}\right)\left\{\begin{array}{l}[8q_2\sin 3\theta + 3q_1(\cos\theta + \cos 3\theta)]\\ +3[5+\varepsilon_2 - 3\cos 2\theta - (q_1\cos 3\theta - q_2\sin 3\theta)]\cos 2\theta\end{array}\right\}$$

$$-\left(\frac{3\sin^2 i}{4a^2\eta^6}\right)\left[\begin{array}{l}9\sin\theta - \sin 3\theta + 2q_2(5+\varepsilon_2) + 6(q_1\sin 2\theta - q_2\cos 2\theta)\\ +2q_1(q_1\sin 3\theta - q_2\cos 3\theta)\end{array}\right](q_2\cos 2\theta)$$

$$D_{56}^{(sp2)} = 0$$

$$D_{61}^{(sp2)} = -\left(\frac{2}{a}\right)\Omega^{(sp2)}, \qquad D_{62}^{(sp2)} = -\left(\frac{3\cos i}{4a^2\eta^4}\right)[(q_1\cos\theta - q_2\sin\theta) + 2\cos 2\theta + (q_1\cos 3\theta + q_2\sin 3\theta)]$$

$$D_{63}^{(sp2)} = \left(\frac{\sin i}{4a^2\eta^4}\right)[3(q_1\sin\theta + q_2\cos\theta) + 3\sin 2\theta + (q_1\sin 3\theta - q_2\cos 3\theta)]$$

$$D_{64}^{(sp2)} = -\left(\frac{\cos i}{4a^2\eta^6}\right)\left\{4q_1[3\sin 2\theta + q_2(3\cos\theta - \cos 3\theta)] + \left(\eta^2+4q_1^2\right)(3\sin\theta + \sin 3\theta)\right\}$$

$$D_{65}^{(sp2)} = -\left(\frac{\cos i}{4a^2\eta^6}\right)\left\{4q_2[3\sin 2\theta + q_1(3\sin\theta + \sin 3\theta)] + \left(\eta^2+4q_2^2\right)(3\cos\theta - \cos 3\theta)\right\}, \qquad D_{66}^{(sp2)} = 0$$

where,

$$a^{(lp)} = 0, \qquad \lambda^{(lp)} = \left[\frac{q_1q_2\sin^2 i}{8a^2\eta^2(1+\eta)}\right](1-10\Theta\cos^2 i) + \left(\frac{q_1q_2}{16a^2\eta^4}\right)(3-55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i)$$

$$\theta^{(lp)} = \lambda^{(lp)} - \left(\frac{\sin^2 i}{16a^2\eta^4}\right)(1-10\Theta\cos^2 i)\left\{q_1q_2\left[3+\frac{2\eta^2}{(1+\eta)}\right] + 2(q_1\sin\theta + q_2\cos\theta) + \frac{\varepsilon_1\sin 2\theta}{2}\right\}$$

$$i^{(lp)} = \left(\frac{\sin 2i}{32a^2\eta^4}\right)(1-10\Theta\cos^2 i)\left(q_1^2 - q_2^2\right)$$

$$q_1^{(lp)} = -\left(\frac{q_1\sin^2 i}{16a^2\eta^2}\right)(1-10\Theta\cos^2 i) - \left(\frac{q_1q_2^2}{16a^2\eta^4}\right)(3-55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i)$$

$$q_2^{(lp)} = \left(\frac{q_2\sin^2 i}{16a^2\eta^2}\right)(1-10\Theta\cos^2 i) + \left(\frac{q_1^2q_2}{16a^2\eta^4}\right)(3-55\cos^2 i - 280\Theta\cos^4 i - 400\Theta^2\cos^6 i)$$

$$\Omega^{(lp)} = \left(\frac{q_1q_2\cos i}{8a^2\eta^4}\right)(11 + 80\Theta\cos^2 i + 200\Theta^2\cos^4 i)$$

$$a^{(sp1)} = \left[\frac{(1-3\cos^2 i)}{2a\eta^6}\right]\left[(1+\varepsilon_2)^3 - \eta^3\right], \qquad \lambda^{(sp1)} = \left[\frac{\varepsilon_3(1-3\cos^2 i)}{4a^2\eta^4(1+\eta)}\right]\left[(1+\varepsilon_2)^2 + (1+\varepsilon_2) + \eta^2\right] + \left[\frac{3(1-5\cos^2 i)}{4a^2\eta^4}\right](\theta - \lambda + \varepsilon_3)$$

$$\theta^{(sp1)} = \lambda^{(sp1)} - \left[\frac{\varepsilon_3(1-3\cos^2 i)}{4a^2\eta^4(1+\eta)}\right]\left[(1+\varepsilon_2)^2 + \eta(1+\eta)\right], \qquad i^{(sp1)} = 0$$

$$q_1^{(sp1)} = \left[\frac{(1-3\cos^2 i)}{4a^2\eta^4(1+\eta)}\right]\left\{\left[(1+\varepsilon_2)^2+\eta^2\right][q_1+(1+\eta)\cos\theta]+(1+\varepsilon_2)[(1+\eta)\cos\theta+q_1(\eta-\varepsilon_2)]\right\} - \left[\frac{3q_2(1-5\cos^2 i)}{4a^2\eta^4}\right](\theta-\lambda+\varepsilon_3)$$

$$q_2^{(sp1)} = \left[\frac{(1-3\cos^2 i)}{4a^2\eta^4(1+\eta)}\right]\left\{\left[(1+\varepsilon_2)^2+\eta^2\right][q_2+(1+\eta)\sin\theta]+(1+\varepsilon_2)[(1+\eta)\sin\theta+q_2(\eta-\varepsilon_2)]\right\} + \left[\frac{3q_1(1-5\cos^2 i)}{4a^2\eta^4}\right](\theta-\lambda+\varepsilon_3)$$

$$\Omega^{(sp1)} = \left(\frac{3\cos i}{2a^2\eta^4}\right)[(\theta-\lambda)+\varepsilon_3], \qquad a^{(sp2)} = -\left(\frac{3\sin^2 i}{2a\eta^6}\right)(1+\varepsilon_2)^3\cos 2\theta$$

$$\lambda^{(sp2)} = -\left[\frac{3\varepsilon_3\sin^2 i\cos 2\theta}{4a^2\eta^4(1+\eta)}\right](1+\varepsilon_2)(2+\varepsilon_2) - \left[\frac{\sin^2 i}{8a^2\eta^2(1+\eta)}\right][3(q_1\sin\theta+q_2\cos\theta)+(q_1\sin 3\theta-q_2\cos 3\theta)]$$
$$-\left[\frac{(3-5\cos^2 i)}{8a^2\eta^4}\right][3(q_1\sin\theta+q_2\cos\theta)+3\sin 2\theta+(q_1\sin 3\theta-q_2\cos 3\theta)]$$

$$\theta^{(sp2)} = \lambda^{(sp2)} - \left[\frac{\sin^2 i}{32a^2\eta^4(1+\eta)}\right]\left\{\begin{array}{l}36q_1q_2-4(3\eta^2+5\eta-1)(q_1\sin\theta+q_2\cos\theta)+12\varepsilon_2q_1q_2\\ -32(1+\eta)\sin 2\theta-(\eta^2+12\eta+39)(q_1\sin 3\theta-q_2\cos 3\theta)\\ +36q_1q_2\cos 4\theta-18\left(q_1^2-q_2^2\right)\sin 4\theta-3\left(q_1^2-3q_2^2\right)q_1\sin 5\theta\\ +3\left(3q_1^2-q_2^2\right)q_2\cos 5\theta\end{array}\right\}$$

$$i^{(sp2)} = -\left(\frac{\sin 2i}{8a^2\eta^4}\right)[3(q_1\cos\theta-q_2\sin\theta)+3\cos 2\theta+(q_1\cos 3\theta+q_2\sin 3\theta)]$$

$$q_1^{(sp2)} = \left[\frac{q_2(3-5\cos^2 i)}{8a^2\eta^4}\right][3(q_1\sin\theta+q_2\cos\theta)+3\sin 2\theta+(q_1\sin 3\theta-q_2\cos 3\theta)]$$
$$+\left(\frac{\sin^2 i}{8a^2\eta^4}\right)\left[3\left(\eta^2-q_1^2\right)\cos\theta+3q_1q_2\sin\theta-\left(\eta^2+3q_1^2\right)\cos 3\theta-3q_1q_2\sin 3\theta\right]$$
$$-\left(\frac{3\sin^2 i\cos 2\theta}{16a^2\eta^4}\right)\left[\begin{array}{l}10q_1+\left(8+3q_1^2+q_2^2\right)\cos\theta+2q_1q_2\sin\theta+6(q_1\cos 2\theta+q_2\sin 2\theta)\\ +\left(q_1^2-q_2^2\right)\cos 3\theta+2q_1q_2\sin 3\theta\end{array}\right]$$

$$q_2^{(sp2)} = -\left[\frac{q_1(3-5\cos^2 i)}{8a^2\eta^4}\right][3(q_1\sin\theta+q_2\cos\theta)+3\sin 2\theta+(q_1\sin 3\theta-q_2\cos 3\theta)]$$
$$-\left(\frac{\sin^2 i}{8a^2\eta^4}\right)\left[3\left(\eta^2-q_2^2\right)\sin\theta+3q_1q_2\cos\theta+\left(\eta^2+3q_2^2\right)\sin 3\theta+3q_1q_2\cos 3\theta\right]$$
$$-\left(\frac{3\sin^2 i\cos 2\theta}{16a^2\eta^4}\right)\left[\begin{array}{l}10q_2+\left(8+q_1^2+3q_2^2\right)\sin\theta+2q_1q_2\cos\theta+6(q_1\sin 2\theta-q_2\cos 2\theta)\\ +\left(q_1^2-q_2^2\right)\sin 3\theta-2q_1q_2\cos 3\theta\end{array}\right]$$

$$\Omega^{(sp2)} = -\left(\frac{\cos i}{4a^2\eta^4}\right)[3(q_1\sin\theta+q_2\cos\theta)+3\sin 2\theta+(q_1\sin 3\theta-q_2\cos 3\theta)]$$

$$\lambda_{q_1} = \left(\frac{\partial\lambda}{\partial q_1}\right) = \frac{q_2}{\eta(1+\eta)} + \frac{q_1V_r}{\eta V_t} - \frac{\eta R(a+R)}{p^2}(q_2+\sin\theta), \qquad \lambda_{q_2} = \left(\frac{\partial\lambda}{\partial q_2}\right) = -\frac{q_1}{\eta(1+\eta)} + \frac{q_2V_r}{\eta V_t} + \frac{\eta R(a+R)}{p^2}(q_1+\cos\theta)$$

$$\Theta = \frac{1}{(1-5\cos^2 i)}, \qquad \eta^2 = 1-q_1^2-q_2^2, \qquad R = \frac{p}{1+q_1\cos\theta+q_2\sin\theta}, \qquad V_r = \sqrt{\frac{\mu}{p}}(q_1\sin\theta-q_2\cos\theta)$$

$$V_t = \sqrt{\frac{\mu}{p}}(1+q_1\cos\theta+q_2\sin\theta), \qquad \varepsilon_1 = \sqrt{q_1^2+q_2^2}, \qquad \varepsilon_2 = q_1\cos\theta+q_2\sin\theta, \qquad \varepsilon_3 = q_1\sin\theta-q_2\cos\theta$$

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