THE EULER CIRCLE COMPLEX ANALYSIS SAMPLE PROBLEM

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This document contains my solutions to select The Euler Circle Complex Analysis Problems. The first one is same one as in the Poster.

1. Sample Problem 1: Jensen's Formula

Question 1.1. *Prove the following version of Jensen's Formula: Let f be an entire function that is not identically zero, and let $\nu(t)$ be the number of zeros of f with modulus $< t$, counted with multiplicity. Then* $\log|f(0)| = -\int_0^r \frac{\nu(t)}{t}\,dt + \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta$.[1]

Proof. Let D denote the set of zeros of f. Since f is entire and not identically zero, f has only finitely many zeros in $\overline{B_r(0)}$. Indeed, if $\#\left(D \cap \overline{B_r(0)}\right) = \infty$, then the Bolzano-Weierstrass theorem and the fact that $\overline{B_r(0)}$ is closed would imply that D would have a limit point in $\overline{B_r(0)}$. By the identity theorem, f would then be zero on $\overline{B_r(0)}$, a contradiction.

Let the zeros be $a_1, \cdots, a_n$ (counted with multiplicity). Arrange the elements of the set $\{|a_j| : 1 \le j \le n\}$ as $r_1 < \cdots r_k$ where $k \le n$. For notational convenience, we define $r_{k+1} = r$. Further, let m_j denote the number of zeros of f with modulus r_j. By Jensen's formula (as applied to f on $\overline{B_r(0)}$), it suffices to show that:

$$\sum_{j=1}^{n} \log\frac{\rho}{|a_j|} = \int_0^r \frac{\nu(t)}{t}\,dt.$$

First, note that

$$\int_0^r \frac{\nu(t)}{t}\,dt = \sum_{j=1}^{k}\int_{r_j}^{r_{j+1}} \frac{\nu(t)}{t}\,dt = \sum_{j=1}^{k} M_j \int_{r_j}^{r_{j+1}} \frac{1}{t}\,dt = \sum_{j=1}^{k} M_j \log\frac{r_{j+1}}{r_j} = \log\left(\prod_{j=1}^{k}\left(\frac{r_{j+1}}{r_j}\right)^{M_j}\right),$$

where $M_j = \sum_{a=1}^{j} m_j$. This is because for $r_j < t < r_{j+1}$, $\nu(t) = M_j$ by the definition of ν. Further,

$$\prod_{j=1}^{k}\left(\frac{r_{j+1}}{r_j}\right)^{M_j} = \left(\frac{r_2}{r_1}\right)^{m_1}\left(\frac{r_3}{r_2}\right)^{m_1+m_2}\left(\frac{r_4}{r_3}\right)^{m_1+m_2+m_3}\cdots\left(\frac{r}{r_k}\right)^{m_1+\cdots+m_k} = \prod_{j=1}^{k}\left(\frac{r}{r_j}\right)^{m_j}.$$

Therefore,

$$\int_0^r \frac{\nu(t)}{t}\,dt = \sum_{j=1}^{k} m_j \log\frac{r}{r_j} = \sum_{j=1}^{n}\log\frac{r}{|a_j|},$$

as desired. ∎

[1] Course Description: `https://eulercircle.com/classes/complex-analysis/`. Complex Analysis was spread pover two quarters: Winter and Spring 2025.

2. Sample Problem 2: The Cauchy Integral Formula

Question 2.1. *Prove the differential version of the Cauchy integral formula.*

Proof. By the induction hypothesis, we have:

$$\frac{f^{(n-1)}(z_0+h)-f^{(n-1)}(z_0)}{h} = \frac{(n-1)!}{2\pi i h}\left(\int_\Gamma \frac{f(z)}{(z-z_0-h)^n}\,dz - \int_\Gamma \frac{f(z)}{(z-z_0)^n}\,dz\right).$$

Simplifying,

$$\begin{aligned}\int_\Gamma \frac{f(z)}{h}\left(\frac{1}{(z-z_0-h)^n}-\frac{1}{(z-z_0)^n}\right)dz &= \int_\Gamma \frac{f(z)}{h}\left(\frac{(z-z_0)^n-(z-z_0-h)^n}{(z-z_0-h)^n(z-z_0)^n}\right)dz \\ &= \int_\Gamma \frac{(z-z_0)^n-\sum_{i=0}^n \binom{n}{i}(z-z_0)^i(-h)^{n-i}}{(z-z_0-h)^n(z-z_0)^n}\frac{f(z)}{h}\,dz \\ &= \int_\Gamma \frac{\sum_{i=0}^{n-1}\binom{n}{i}(z-z_0)^i h^{n-i-1}}{(z-z_0-h)^n(z-z_0)^n} f(z)\,dz.\end{aligned}$$

Therefore, we are left with

$$f^{(n)}(z_0) = \int_\Gamma \frac{n}{(z-z_0-h)^n(z-z_0)} f(z)\,dz + \int_\Gamma \frac{\sum_{i=0}^{n-2}\binom{n}{i}(z-z_0)^i h^{n-i-1}}{(z-z_0-h)^n(z-z_0)^n} f(z)\,dz.$$

Let M be the maximum value of $|f(z)|$ over Γ. Then,

$$\left|\frac{\sum_{i=0}^{n-2}\binom{n}{i}(z-z_0)^i h^{n-i-1}}{(z-z_0-h)^n(z-z_0)^n} f(z)\right| \le \frac{Mf(h)}{\rho^n(\rho-h)^n},$$

where

$$f(h) := \sup\left\{\left|\sum_{i=0}^{n-2}\binom{n}{i}(z-z_0)^i h^{n-i-1}\right| : z\in\Gamma\right\}.$$

Clearly $f(h)$ goes to zero h tends to 0, so the ML inequality tells us that

$$\lim_{h\to 0}\left|\int_\Gamma \frac{\sum_{i=0}^{n-2}\binom{n}{i}(z-z_0)^i h^{n-i-1}}{(z-z_0-h)^n(z-z_0)^n} f(z)\,dz\right| = 0 \text{ as } \lim_{h\to 0}\frac{2\pi Mf(h)}{\rho^{n-1}(\rho-h)^n} = 0.$$

Next, we deal with the first integral. First, note the decomposition

$$\frac{1}{(z-z_0-h)^n(z-z_0)} = \frac{1}{(z-z_0)^{n+1}} + \frac{(z-z_0)^n-(z-z_0-h)^n}{(z-z_0)^{n+1}(z-z_0-h)^n} = \frac{1}{(z-z_0)^{n+1}} + \frac{\sum_{i=0}^{n-1}\binom{n}{i}(z-z_0)^i h^{n-i}}{(z-z_0)^n(z-z_0-h)^n}.$$

Hence,

$$\int_\Gamma \frac{n}{(z-z_0-h)^n(z-z_0)} f(z)\,dz = \int_\Gamma \frac{n}{(z-z_0)^{n+1}} f(z)\,dz + \int_\Gamma \frac{\sum_{i=0}^{n-1}\binom{n}{i}(z-z_0)^i h^{n-i}}{(z-z_0)^n(z-z_0-h)^n} nf(z)\,dz.$$

Invoking the ML inequality again for the second integral

$$\left|\int_\Gamma \frac{\sum_{i=0}^{n-1}\binom{n}{i}(z-z_0)^i h^{n-i}}{(z-z_0)^n(z-z_0-h)^n} nf(z)\,dz\right| \le \frac{Mng(h)}{\rho^n(\rho-h)^n}\times 2\pi\rho = \frac{2\pi Mng(h)}{\rho^{n-1}(\rho-h)^n},$$

where

$$g(h) := \max\left|\sum_{i=0}^{n-1}\binom{n}{i}(z-z_0)^i h^{n-i}\right|.$$

Since $\lim_{h\to 0} g(h) = 0 \implies \lim_{h\to 0} \frac{2\pi M n g(h)}{\rho^{n-1}(\rho-h)^n} = 0$. Therefore, the second integral vanishes in the limit as $h \to 0$, and so

$$\lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} = \lim_{h\to 0} \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

as desired. ■

3. Sample Problem 3: Convergence of Dirichlet Series

Question 3.1. *Given a Dirichlet series* $\sum_{n=1}^\infty \frac{a_n}{n^s}$, *let*

$$\sigma_c = \left\{ \sigma : \sum_{n=1}^\infty \frac{a_n}{n^s} \text{ converges for some } s \text{ with } \Re s = \sigma \right\}.$$

Prove that if $\Re s > \sigma_c$, *then* $\sum_{n=1}^\infty \frac{a_n}{n^s}$ *converges. We call* σ_c *the abscissa of convergence.*

Proof. Define $S_c = \{\sigma : \exists s \in \mathbb{C} : \sum_{n=1}^\infty a_n n^{-s} \text{ converges and } \Re(s) = \sigma\}$; note that $\sigma_c = \inf(S)$. Let $s \in \mathbb{C}$ be such that $\Re(s) > \sigma_c$. Then there exists a $\delta \in \mathbb{C}$ such that $\Re(\delta) \in S$ and $\Re(s) > \Re(\delta)$. We must show that for all $\varepsilon > 0$, there exists a $N \in \mathbb{Z}^+$ such that for all $M_1 > M_2 \geq N$, we have

$$\left| \sum_{N=M_1}^{M_2} a_N N^{-s} \right| = \left| \sum_{N=M_1}^{M_2} a_N N^{-\delta} \cdot N^{-(s-\delta)} \right| < \varepsilon.$$

Taking cue from this, define $x_n = a_n n^{-\delta}$ and $y_n = n^{-(s-\delta)}$. Abel's summation lemma states that:

$$\sum_{j=M_1}^{M_2} x_j y_j = (S_{M_2} y_{M_2} - S_{M_1-1} y_{M_1}) + \sum_{j=M_1}^{M_2-1} S_j (y_j - y_{j+1}),$$

where $S_j = \sum_{\mu=1}^j x_\mu = \sum_{\mu=1}^j a_\mu \mu^{-\delta}$. We take the absolute value of both sides and apply the triangle inequality to the RHS to get:

$$\left| \sum_{j=M_1}^{M_2-1} x_j y_j \right| \leq |S_{M_2} y_{M_2}| + |S_{M_1-1} y_{M_1}| + \left| \sum_{j=M_1}^{M_2-1} S_j (y_j - y_{j+1}) \right|.$$

Note that $(S_n)_{n=1}^\infty$ is a Cauchy sequence, so it is bounded. Thus, there exists a $C \in \mathbb{R}^+$ such that $|S_n| < C$ for all n. Hence,

$$L = \left| \sum_{j=M_1}^{M_2-1} S_j (y_j - y_{j+1}) \right| \leq \sum_{j=M_1}^{M_2-1} |S_j| |y_j - y_{j+1}| \leq C \sum_{j=M_1}^{M_2-1} |y_{j+1} - y_j|.$$

The last term hints at the mean value theorem. Indeed, define $f_j : [j, j+1]$ by $f_j(x) = x^{-(s-\delta)}$ for all $x \in [j, j+1]$. Then, by the mean value theorem, we have

$$|f(j+1) - f(j)| = |y_{j+1} - y_j| \leq |f'(\xi_j)| \leq |s-\delta| j^{-(\mathfrak{Re}(s-\delta)+1)},$$

for some $j \leq \xi_j \leq j+1$. Thus, $L \leq C' \sum_{j=M_1}^{M_2-1} j^{-(\mathfrak{Re}(s-\delta)+1)}$ where C' is a constant that depends only on s and δ. By the $p-$series test, we know that the series $\sum_{n=1}^\infty n^{-(\mathfrak{Re}(s-\delta)+1)}$ converges as $\mathfrak{Re}(s-\delta) > 0$. Therefore, for every $\varepsilon > 0$, there exists a N such that for all $M_2 > M_1 \geq N$, $\left| \sum_{j=M_1}^{M_2-1} j^{-(\mathfrak{Re}(s-\delta)+1)} \right| < \varepsilon$ and so $L \leq C'\varepsilon$.

Next, note that

$$|S_{M_2} y_{M_2}| + |S_{M_1-1} y_{M_1}| \leq C(|y_{M_2}| + |y_{M_1}|) = C\left(M_1^{-\mathfrak{Re}(s-\delta)} + M_2^{-\mathfrak{Re}(s-\delta)} \right) \leq C,$$

where the last inequality follows as the function $h(x) := x^{-\mathfrak{Re}(s-\delta)}$ is monotonically decreasing. In conclusion, we have $\left|\sum_{N=M_1}^{M_2} x_j y_j\right| < K\varepsilon$ for all $M_1 > M_2 \geq N$ where K is a constant that is dependent only on s and δ, which completes the proof.

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