Vector Spaces

(Q1)

- $v + 0 \cdot v = v + v \cdot 0 = v(1+0)$
  $= v \times 1 = v.$
- $v + (-1) \cdot v = v + v \cdot (-1) = v \cdot (1 + (-1))$
  $= v \cdot 0$
  $= v$ (by above)
- $(a \cdot 0) + a = a \cdot (0+1) = a \cdot 1 = a$
- Assume that $a \neq 0$, the a^{-1} exists in F:
  $a \cdot v = 0$
  $a^{-1} \cdot (a \cdot v) = a^{-1} \cdot 0$
  $(a^{-1} \cdot a) \cdot v = 0$ (by property 3)
  $1 \cdot v = 0$
  $v = 0$
  If $a = 0$, then we are done.

(Q2)

Addition axioms of a vector space:

- For all $a \in F$ and $b \in F$, $(a+b) \in F$ (A0)
- $0 \in F$ (by A1) such that for all $a \in F$, $a + 0 = a = 0 + a$.
- For all $a \in F$, there exists $a^{-1} \in F$ such that $a + a^{-1} = 0 = a^{-1} + a$, given by: $a^{-1} = -a$ (due to A2)
- Commutativity and Associativity are given by A3 and A4.

Scalar multiplication is given by the usual $\times : F \times F \to F$ defined in F:

- (1) is given by M1
- (2) is given by M3
- (3) and (4) are given by D1

(Q3) First, note that $R^n = [(x_1, \cdots, x_n) : x_i \in R]$
We prove that R^n is a vector space over Q or C:

I Addition axioms

1. (Closure): $(x_1, \cdots, x_n) \in R^n$ and $(y_1, \cdots, y_n) \in R^n$
$((x_1, \cdots, x_n) + (y_1, \cdots, y_n)) = (x_1 + y_1, \cdots, x_n + y_n)$
Note that $x_i + y_i = z_i$ for some $z_i \in R$
(R is a field and is closed under $+$). Thus,
$(z_1, \cdots, z_n) \in R^n$

2. (Identity): $(0, \cdots, 0) \in R^n$ $[0 \in R]$ and,
since $0 + x = x + 0 = x$ $\forall x \in R$, we have
$(0, \cdots, 0) + (x_1, \cdots x_n) = (0 + x_1, \cdots, 0 + x_n)$
$= (x_1, \cdots, x_n)$.

3. Associativity follows from R, as does commutativity

4. (Inverses): For any $(x_1, \cdots, x_n) \in R^n$ consider
$(-x_1, \cdots, -x_n) \in R^n$. Now, $(x_1, \cdots, x_n) + (-x_1, \cdots, -x_n)$
$= (x_1 + (-x_1), \cdots, x_n + (-x_n)) = (0, \cdots, 0)$. Thus,
inverses exist.

II Multiplication Axioms (when $F = Q$ or R).

- Note that, $a \times b \in R$ where $a \in Q$ and $b \in R$
and $a \times b \in R$ where $a, b \in R$ (Closure of R)

1. (Closure): It essentially follows from the above property.

2. $a \cdot (b \cdot V) = a \cdot (b \cdot (x_1, \cdots, x_n))$
$= a \cdot (bx_1, \cdots, bx_n)$
$= (abx_1, \cdots, abx_n)$
$= ((ab)x_1, \cdots, (ab)x_n)$
$= (ab)(x_1, \cdots, x_n)$
$= (ab)(V)$

3. (Identity): $1 \in Q$ and $1 \in R$ and thus,
$(1, \cdots, 1) \in R^n \ni (1, \cdots, 1) \cdot (x_1, \cdots, x_n)$
$= (1 \cdot x_1, \cdots, 1 \cdot x_n)$
$= (x_1, \cdots, x_n)$

Question 3 contd.

4. For all $a \in F$ and $u = (x_1, \cdots, x_n)$ and $v = (y_1, \cdots, y_n) \in R^n$, we have

$$a \cdot (u+v) = a \cdot (x_1 + y_1, \cdots, x_n + y_n)$$
$$= (a(x_1+y_1), \cdots, a(x_n+y_n))$$
$$= (ax_1 + ay_1, \cdots, ax_n + ay_n)$$
$$= (ax_1, \cdots, ax_n) + (ay_1, \cdots, ay_n)$$
$$= a(x_1, \cdots, x_n) + a(y_1, \cdots, y_n)$$
$$= au + av.$$

5. $(a+b) \cdot v = a \cdot v + b \cdot v$ is similar to (4).

II R^n over C fails to form a vector space as it is not closed under scalar multiplication. consider, $(1, 1, \cdots, 1) \in R^n$ and $i \in C \Rightarrow i \times (1, \cdots, 1) = (i, \cdots, i) \notin R^n$ since $i \notin R$.

(Q4) The natural scalar multiplication is:

- $c \cdot (a_0 + a_1 x + \cdots + a_n x^n)$
$$= c \cdot a_0 + c a_1 x + \cdots + c \cdot a_n x^n$$

The identity scalar is 1 since $(1 \in F)$

$$1 \cdot (a_0 + \cdots + a_n x^n) = 1 \cdot a_0 + \cdots + 1 \cdot a_n x^n$$
$$= (1 \cdot a_0) + \cdots + (1 \cdot a_n) \cdot x^n$$
$$= a_0 + \cdots + a_n x^n$$

Here, the 1 is the identity from the field F (due to M1).

(II) Proof that $F[x]_{=n}$ is not a vector space.

Consider elements of $F[x]_{=n}$.

$a_0 + a_1 x + \cdots + a_n x^n$ $(a_n \neq 0)$ and
$a_0 + a_1 x + \cdots + -a_n x^n$ (Note that $(-a_n) \in F$ since F is a field and additive inverses exist).

$$(a_0 + a_1 x + \cdots + a_n x^n) + (a_0 + \cdots - a_n x^n)$$
$$= \cancel{(a_0 + a_1 x + \cdots + a}$$
$$= 2a_0 + 2a_1 x + \cdots + 2(a_{n-1}) x^{n-1}.$$

Note that this not an element of $F[x]_{=n}$, since, at maximum, it can be of degree $n-1$. Thus, $F[x]_{=n}$ does not satisfy closure under vector addition, and is thus not a vector space.

(Q7)

- Forward Direction $\Rightarrow$ Straight forward using the properties of the vector space.
- Backward Direction

If $U \subseteq V$ is such that it is

a) Closed under scalar multiplication

b) Closed under vector addition.

We show that U satifies all the vector space properties:

Addition Axioms

1. $0 \in F$ (F is a field) and thus for all $u \in U$, $(0 \cdot u) \in U$ (Since U is closed under scalar multiplication). Now, by problem 1, $0 \cdot u = 0$ (The zero vector).
2. By assumption U is closed under vector addition.
3. $u \in U \Rightarrow (-1) \cdot u \in U$ (U is closed under scalar multiplication, and F is a field implies that $(-1) \in F$). By problem 1, $(-1) \cdot u = -u \in U$. Thus inverses exist.
4. Since $u_1, u_2, u_3 \in U \Rightarrow u_1, u_2, u_3 \in V$, thus since V is a vector space, we have $$u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3.$$
5. Using the same argument as (4), we can show $u_1 + u_2 = u_2 + u_1$ for all $u_1, u_2 \in U$.

Multiplication Axioms

1. $u \in U \Rightarrow u \in V \Rightarrow 1 \cdot u = u$ (Multiplicative Identity)
2. For all $a, b \in F$ and $u \in U$, $u \in U \Rightarrow u \in V$, and since U is a vector space, $(a \cdot (b \cdot v)) = (ab) \cdot v$.

Axioms (3) and (4) can be shown using a similar argument as in (2).

- We can reduce the condition for subspaces

→ A subset $\mathcal{U}$ of a vector space V is a subspace of V if and only if for all $c_1, c_2 \in F$ and $u_1, u_2 \in \mathcal{U}$, $(c_1u_1 + c_2u_2) \in \mathcal{U}$.

→ Proof.

- The forward direction is straightforward =

1. For all $c_1, c_2 \in F$ and $u_1, u_2 \in \mathcal{U}$, we have, $c_1u_1 \in \mathcal{U}$ and $c_2u_2 \in \mathcal{U}$ ($\mathcal{U}$ is closed under scalar multiplication).

Thus, $(c_1u_1) + (c_2u_2) \in \mathcal{U}$ ($\mathcal{U}$ is closed under vector addition).

- The backward condition, we show that it implies that $\mathcal{U}$ is closed under addition and multiplication, and then use prop. 5.2

→ $c_1u_1 + c_2u_2 \in \mathcal{U}$, setting $c_2 = 0$ (the zero in F), we get, $c_1u_1 + 0 \cdot u_2 = c_1u_1 + 0$ (Problem 1) $= c_1u_1$

Thus, $c_1 \in F$ and $u_1 \in \mathcal{U} \Rightarrow (c_1u_1) \in F$

→ $c_1u_1 + c_2u_2 \in \mathcal{U}$, setting $c_1 = 1 = c_2$ (the identity in F) we get, $1 \cdot u_1 + 1 \cdot u_2 = u_1 + u_2$ (Problem 1)

Thus, $u_1, u_2 \in \mathcal{U} \Rightarrow (u_1 + u_2) \in \mathcal{U}$.

(Q8)

- This ~~statement~~ is false =

Take $n = 2$, and $v_1 = (1,2)$ and $v_2 = (2,4)$

$(1,2) \in R^2$ and $(2,4) \in R^2$ but $(1,2)$ and $(2,4)$ are not linearly independent as

$2(1,2) + -1 \times (2,4) = 0$.

- This statement is false. Take the same example as last time: $n=2$, $v_1 = (1,2)$ and $(2,4) = v_2$. Note that $\text{span}[(1,2), (2,4)]$

$$= [\alpha(1,2) + \beta(2,4) : \alpha, \beta \in R]$$
$$= [\alpha(1,2) + 2\beta(1,2) : \alpha, \beta \in R]$$
$$= [(\alpha + 2\beta)\cdot(1,2) : \alpha, \beta \in R]$$
$$= [\gamma \cdot (1,2) : \gamma \in R]$$
$$= \text{span}[(1,2)].$$
$$\neq R^2$$ (as $\dim(\text{span}(1,2)) = 1$ but $\dim(R^2) = 2$).