FLOOD RISK ASSESSMENT OF PUNE

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Context and Background

As part of an IGCSE Geography project completed in 10th grade, I undertook a rudimentary flood risk assessment of Pune. I began by using historical satellite imagery (acquired via Google Earth Pro) for the Baner–Balewadi area to analyze changes in impervious and non-impervious surface areas from 2013 to 2023. Then, using tabulated values to estimate runoff coefficients for various surface types, I applied the Rational Runoff Model to predict runoff under rainfall events of different intensities.

Later, I expanded this framework to develop an integrated bank stability index designed to identify sections of riverbanks vulnerable to significant collapse or rapid erosion during extreme precipitation events. To achieve this, I integrate soil hydraulic criteria introduced in various soil models in the latter half of the 20th century with spatial data (example data acquired using a DEM model of the Mutha river provided by Geo Resource Mapping Pvt. Ltd.) and stress theories, representing these factors as a tuple at each point. Manifold learning techniques are then used to reduce these high-dimensional data points into a lower-dimensional representation. In this process, we tweaked concepts such as the REV to smooth the transition from local to global parameters, redefined the Riemannian metric on the data manifold to better suit riverbanks, and expanded on the spatio-temporal aspects of Isomap.

Properties of Porous Media

Hydraulic conductivity K is a fundamental property of a porous medium that governs its ability to transmit fluid. In this study, a porous material is a material consisting of interconnected void spaces: examples include sand, rocks and clay. We specifically model river banks as being composed entirely of such porous materials. To define the hydraulic conductivity we must consider Darcy's law, first penned down by Henry Darcy [7]. While there are many forms of Darcy's law, we start with the simplest, one-dimensional version: Picture a tube XY of the material having length L and constant cross-sectional area A. Furthermore, let μ be the dynamic viscosity of the fluid, and k the permeability of the material. Then

$$Q = \frac{kA}{\mu L}\Delta p, \tag{1}$$

where Q is the volumetric flow rate and Δp is the change in pressure in the tube between points X and Y. See the Apendix for an elementary derivation of this equation. Equation 1 shows that Δp and Q are directly proportional with proportionality constant $\frac{kA}{\mu L}$, which is the hydraulic conductivity.

Following Terzaghi [21], define the effective stress $\boldsymbol{\sigma}_{\text{eff}}$, as the stress that, for a given pore pressure P, produces the same strain response in a porous material as would be observed in a dry sample where $P = 0$. In symbols, Terzaghi's principle states

$$\boldsymbol{\sigma}_{\text{eff}} = \boldsymbol{\sigma} - P\mathbb{I}, \tag{2}$$

where $\boldsymbol{\sigma}$ the total stress, P the pore pressure and $\mathbb{I}$ denotes the identity matrix. While stress is a directional tensor [6], P is isotropic, and thus multiplication by $\mathbb{I}$ is justified. Crucially, low K values for a bank means water cannot drain effectively from surface-level and lower pores, leading to an increase pressure values as the drainage rate cannot keep up with the infiltration rate during extreme precipitation. Though $\boldsymbol{\sigma}$ may increase during such events, typically P increases much faster [11]; by Equation 2, $\boldsymbol{\sigma}_{\text{eff}}$, the frictional resistance between constituent particles decreases. As the maximum shear stress the medium can support τ_f is a function of $\boldsymbol{\sigma}_{\text{eff}}$, this increases the chances of bank collapse.[1]

There are many approaches to understanding the relationship between different parameters of a porous medium and the mechanism by which fluids and solutes are transported within the medium, which we briefly review here. Mualem's pore-size model [13] and Gerke's dual porosity models are a few examples [10]. First, we introduce a few key terms.

Note that K is a function of the volumetric water content θ of the medium, the ratio of the volume of water to the total volume [12]. The minimum value of θ is the residual water content θ_r, and its maximum value is the saturated water content θ_s. We define the saturated hydraulic conductivity $K_{\text{sat}} = K(\theta_s)$, the maximum value of K, and subsequently the relative hydraulic conductivity by $K_r = K/K_{\text{sat}}$. In a similar vein, the effective saturation S_e of a sample is defined by

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r}.$$

The Representative Elementary Volume. The volumetric water content θ as defined is a global property rather than a local property of the sample; later, when assigning quantities to each point in the river bank, it will be useful to allow θ to vary with position $\mathbf{x}$. Taking inspiration from elementary calculus, we may define

$$\theta(\mathbf{x}) = \lim_{V \to 0} \frac{W}{V}, \tag{3}$$

where V is a volume around $\mathbf{x}$ and W is the total volume of the pores in V. We can make Equation 3 more precise, which will be beneficial later. Let $\mathcal{V}(\mathbf{x}) \subseteq \mathbf{S}$ be a connected open neighborhood of $\mathbf{x}$, and define $\chi_{\text{water}} : \mathbf{S} \to \mathbb{R}$ by $\chi_{\text{water}}(\mathbf{x}) = 1$ if $\mathbf{x}$ is in a water-containing pore and $\chi_{\text{water}}(\mathbf{x}) = 0$ if $\mathbf{x}$ is in a dry pore or in a non-void region.[2] Any of $\mathcal{V}(\mathbf{x})$, $\mathcal{V}_{\mathbf{x}}$, or $\mathcal{V}_i(\mathbf{x})$ denotes an open neighborhood of $\mathbf{x}$; $\mathbf{x}$ may be omitted when clear from context. Finally, letting

$$\theta(\mathbf{x}, \mathcal{V}(\mathbf{x})) = \frac{1}{|\mathcal{V}(\mathbf{x})|} \int_{\mathcal{V}(\mathbf{x})} \chi_{\text{water}}(\mathbf{x}')\, dV', \tag{4}$$

we now define

$$\theta(\mathbf{x}) = \lim_{|\mathcal{V}(\mathbf{x})| \to 0} \theta(\mathbf{x}, \mathcal{V}(\mathbf{x})). \tag{5}$$

[1] The exact relationship between τ_f and $\boldsymbol{\sigma}_{\text{eff}}$ depends on the model used. For instance, Mohr–Coulomb theory states that the material is at failure if $\tau = c + \boldsymbol{\sigma}_{\text{eff}} \tan\phi$ where c is the cohesion and ϕ the angle of internal friction.

[2] For instance, when dealing with abstract theory one may set $\mathcal{V}(\mathbf{x}) = B_\varepsilon(\mathbf{x})$, where $B_\varepsilon(\mathbf{x})$ is the open ball of radius ε centered at $\mathbf{x}$; $B_\varepsilon(\mathbf{x}) = \{\mathbf{y} : \|\mathbf{x} - \mathbf{y}\| < \varepsilon\}$. In other computer-based applications, setting $\mathcal{V}(\mathbf{x})$ as an open cube $[-a, a]^3 + \mathbf{x}$ might be more appropriate.

Hypothetically, if χ_{water} were continuous, Equation 5 would reduce to

$$\theta(\mathbf{x}) = \chi_{\text{water}}(\mathbf{x}). \tag{6}$$

However, in reality, χ_{water} is discontinuous; as a result, Equation 6 does not hold. Indeed, let $\mathbf{x}_0$ be a point on a pore-water interface. Approaching $\mathbf{x}_0$ from the water side yields

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \chi_{\text{water}}(\mathbf{x}) = 1.$$

On the other hand, approaching $\mathbf{x}_0$ from the solid side yields

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \chi_{\text{water}}(\mathbf{x}) = 0,$$

meaning that the limit does not exist at $\mathbf{x}_0$. Motivated by [1], we define a localized version of the standard *representative elementary volume* (REV) for volumetric water content, $\theta_{\text{REV}}(\mathbf{x})$, as

$$\theta_{\text{REV}}(\mathbf{x}) = \inf\left\{|\mathcal{V}_1| : |\theta(\mathbf{x},\mathcal{V}_2) - \theta(\mathbf{x},\mathcal{V}_1)| < \epsilon \ \forall |\mathcal{V}_2| \geq |\mathcal{V}_1|\right\}, \tag{7}$$

where ϵ is a fixed tolerance. Then, we may set

$$\tilde{\theta}(\mathbf{x}) = \theta(\mathbf{x}, \mathcal{V}_{\text{REV}}(\mathbf{x})), \tag{8}$$

where $|\mathcal{V}_{\text{REV}}(\mathbf{x})| = \theta_{\text{REV}}(\mathbf{x})$. Note that this notion can be generalized to other porous material properties apart from volumetric water content. For example, we define the (localized) REV for porosity ϕ by replacing χ_{water} with χ_{void}. Here, $\chi_{\text{void}}(\mathbf{x}) = 1$ if $\mathbf{x}$ is in a pore, regardless of whether it contains water, and $\chi_{\text{void}}(\mathbf{x}) = 0$ otherwise. In general, we denote the REV corresponding to a property q of the sample at the point $\mathbf{x}$ by $q_{\text{REV}}(\mathbf{x})$ and define $\tilde{q}(\mathbf{x}) = q(\mathbf{x}, \mathcal{V}_{\text{REV}}(\mathbf{x}))$.

Physically, the REV is the minimum scale at which the microscopic pore-water fluctuations are effectively 'averaged out'. Specifically, one expects the function $\mathcal{V} \mapsto \theta(\mathbf{x},\mathcal{V})$ to exhibit rapid fluctuations for $|\mathcal{V}| < \theta_{\text{REV}}(\mathbf{x})$. This requires that the REV strikes a balance:

$$\text{pore-scale} \ll \text{REV} \ll \text{sample-grid scale}.$$

Moreover, it is important to note that, while the REV allows us to define a meaningful local volumetric water content $\theta(\mathbf{x})$, this field is not guaranteed to be continuous. This caveat is particularly important when assigning to each point in the bank a tuple of positional, geometric, hydraulic, and mechanical properties, since it is desirable that the collection of such tuples form a smooth manifold in the limit as the number of sampling points tends to infinity.

Redefining the Representative Elementary Volume. Al-Raoush and Papadopoulos [1] note that upon a further increase of the spatial scale, soil parameters may become non-stationary. Specifically, they state that the size of the REV ranges from a minimum bound, the transition from the microscale to the macroscale, to a maximum bound, which represents the transition from a homogeneous to a heterogeneous state. Additionally, in their analysis of porosity, coordination number, and contact orientation anisotropy in a sample of Hostun sand, Schmidt et al. [18, 19] observe the REV to be sufficiently large to catch the characteristics of each region while being as small as possible, such that different regions can be distinguished. Moreover, they emphasize that the REV is not representative of the specimen as a whole. Equation 7 addresses this issue.

However, the definition of the REV used by Al-Raoush and Papadopoulos [1] and Schmidt et al. [18,19], *the minimum volume of a soil sample from which a given parameter becomes independent of the size of the sample*, is one-sided: this definition, and (hence) Equation 7, does not incorporate a maximum bound for the allowed volumes of $\mathcal{V}(\mathbf{x})$. We assume that a sample may contain macroscale heterogeneities and need not be homogeneous; in that case, the conditions of Definition 7 may never hold.

That is, the definition mandates the first inequality

$$\text{pore-scale} \ll \text{REV}$$

but not the second

$$\text{REV} \ll \text{sample-grid scale}.$$

Indeed, if q is an arbitrary property of the sample, this means $q(\mathbf{x},\mathcal{V}(\mathbf{x})) = \tilde{q}(\mathbf{x})$ for an arbitrarily large $|\mathcal{V}(\mathbf{x})|$, which would imply a constant q throughout the sample, inadvertently turning a local property into a global property. For example, Schmidt et al. [18, 19] note that for silica sand specimens, the mean value of the four analyzed elements does not approach the global value. They attribute it to the heterogeneity of the specimen and the fact that the four locations were chosen randomly.

Echoing this sentiment, in the 2000 manuscript, *Is soil variation random?*, Webster [25], in addition to questioning the common geostatistical analysis assumption that the properties of interest are random processes, also questions stationarity. Stationarity is the assumption that data from different regions are realizations of the same generating process. An example is Mualem's [13] pore water distribution function f, which will be explored later, after we formally define stationary processes.

Taking this evidence into account, we redefine the REV (for porosity) by introducing a lower bound (REVL) and an upper bound (REVU). We fix a nested, parametrized family of neighborhoods $\{\mathcal{V}^r(\mathbf{x})\}_{r>0} \subset \{\mathcal{V}(\mathbf{x})\}$, with r a scale parameter, so that $r_2 > r_1 \implies \mathcal{V}^{r_2}(\mathbf{x}) \supseteq \mathcal{V}^{r_1}(\mathbf{x})$. As already noted, $\mathcal{V}^r(\mathbf{x}) = B_r(\mathbf{x})$ or $\mathcal{V}^r(\mathbf{x}) = [-r/2, r/2]^3 + \mathbf{x}$; the exact shape is irrelevant.

Definition 1 (Representative elementary volume)**.** Call an interval $[a, b]$ stable if

$$|\theta(\mathbf{x},\mathcal{V}^{r_2}(\mathbf{x})) - \theta(\mathbf{x},\mathcal{V}^{r_1}(\mathbf{x}))| < \epsilon$$

for all $r_1, r_2 \in [a, b]$. Let I be the collection of all stable intervals. Then, define $L_{\text{REVL}}(\mathbf{x}) = \inf\{a \in \mathbb{R} : \exists b \text{ such that } [a, b] \in I\}$, $\mathcal{V}_{\text{REVL}}(\mathbf{x}) = \mathcal{V}^{L_{\text{REVL}}(\mathbf{x})}(\mathbf{x})$, and $V_{\text{REVL}}(\mathbf{x}) = |\mathcal{V}_{\text{REVL}}(\mathbf{x})|$. Similarly, define $L_{\text{REVU}}(\mathbf{x}) = \sup\{b \in \mathbb{R} : \exists a \text{ such that } [a, b] \in I\}$, $\mathcal{V}_{\text{REVU}}(\mathbf{x}) = \mathcal{V}^{L_{\text{REVU}}(\mathbf{x})}(\mathbf{x})$, and $V_{\text{REVU}}(\mathbf{x}) = |\mathcal{V}_{\text{REVU}}(\mathbf{x})|$.

As before, define $\tilde{\theta}(\mathbf{x}) = \theta(\mathbf{x}, \mathcal{V}_{\text{REVL}}(\mathbf{x}))$.

Remark. We assume that for all $\mathcal{V}^r(\mathbf{x}') \subseteq \mathcal{V}_{\text{REVU}}(\mathbf{x})$ such that $|\mathcal{V}^r(\mathbf{x}')| \geq V_{\text{REVL}}(\mathbf{x})$, we have

$$\left|\theta(\mathbf{x}', \mathcal{V}^r(\mathbf{x}')) - \tilde{\theta}(\mathbf{x})\right| < \epsilon. \tag{9}$$

We use Definition 1 to split $\mathbf{S}$ into homogeneous domains $\mathcal{R}_i$.

Definition 2 (Homogeneous domains)**.** For $\mathbf{x} \in \mathbf{S}$, define

- $I^{(1)}(\mathbf{x}) = \{\mathbf{x}' \in \mathcal{V}_{\text{REVU}}(\mathbf{x}) : \mathcal{V}_{\text{REVL}}(\mathbf{x}') \subseteq \mathcal{V}_{\text{REVU}}(\mathbf{x})\}$,
- $S^{(1)}(\mathbf{x}) = \bigcup_{\mathbf{x}' \in I^{(1)}(\mathbf{x})} \mathcal{V}_{\text{REVU}}(\mathbf{x}')$.

Moreover, for $n \geq 2$, let

- $I^{(n)}(\mathbf{x}) = \left\{\mathbf{x}' \in S^{(n-1)}(\mathbf{x}) : \mathcal{V}_{\text{REVL}}(\mathbf{x}') \subseteq S^{(n-1)}(\mathbf{x})\right\}$,
- $S^{(n)}(\mathbf{x}) = \bigcup_{\mathbf{x}' \in I^{(n-1)}(\mathbf{x})} \mathcal{V}_{\text{REVU}}(\mathbf{x}')$ for all $n \geq 2$.

Furthermore, let $\mathcal{R}(\mathbf{x}) = S^{(k)}(\mathbf{x})$, where k is the smallest integer such that $S^{(k)}(\mathbf{x}) = S^{(k-1)}(\mathbf{x})$ and define the equivalence relation $\sim$ on $\mathbf{S}$ by $\mathbf{x} \sim \mathbf{x}' \iff \mathcal{R}(\mathbf{x}) = \mathcal{R}(\mathbf{x}')$. Lastly, let $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be a fundamental domain for $\sim$, and define $\mathcal{R}_i := \mathcal{R}(\mathbf{x}_i)$.

Remark. Could we have swapped the above definition with a region $R \subseteq \mathbf{S}$ such that $|\tilde{\theta}(\mathbf{x_1}) - \tilde{\theta}(\mathbf{x_2})| < \epsilon$ for all $\mathbf{x}_i \in R$?

Stationary and Non-Stationary Processes. For completeness, we define strict and weak spatially stationary processes. First, we define, in the measure theoretic sense, a random field (and thus a spatial process).

Definition 3. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a $X-$*valued random field* is a collection of $X-$valued random variables indexed by elements in a topological space T. In symbols, a $X-$valued random field F is a collection $\{F_t : t \in T\}$, where each F_t is a $X-$valued random variable.

The definitions of the sample space Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$ will be introduced later. For now, in the context of the internal architecture of the river bank, Ω can be thought of as the ensemble of all physically admissible configurations of the porous medium that constitute the river bank. In particular, an element $\omega \in \Omega$ is a unique spatial arrangement of solid grains and voids. Furthermore, we may require the ω share a macroscopic property, such a bulk hydraulic conductivity or total porosity.

Note that T will always be a (topological) subspace of $\mathbb{R}^d$ for $d = 2$ or $d = 3$; such a random field is called a *spatial process*, whole X depends on the property being considered. For instance, $X = [0, 1]$ when discussing porosity.

Definition 4. A random field F is *strictly spatially stationary* if for any finite set $\{t_1, \cdots, t_n\} \subset T$ and any translational vector h such that $\{t_1 + h, \cdots, t_n + h\} \subset T$, the joint distribution of the random variables $(F_{t_1}, \cdots, F_{t_n})$ is equal to that of $(F_{t_1+h}, \cdots, F_{t_n+h})$. Equivalently,

$$\mathbb{P}(F_{t_1} \in B_1, \cdots, F_{t_n} \in B_n) = \mathbb{P}(F_{t_1+h} \in B_1, \cdots, F_{t_n+h} \in B_n),$$

for all Borel subsets $B_i \subseteq X$.

To take an example, let $F_t(\omega) = f_t$, where f_t is the pore water distribution function valid in $\mathcal{V}_{\text{REV}}(t)$. Then Mualem [13], in deriving his model, assumes that F_t is strictly spatially stationary. Next, we give a weaker version of Definition 4, often easier to deal with. Let $\mathbb{E}[F]$ denote the expected value of the random variable F.

Definition 5. A random field F is *weakly spatially stationary* or *second-order spatially stationary* if

$$\mathbb{E}[F_{t_1}] = \mathbb{E}[F_{t_2}]$$

for all $t_1, t_2 \in T$,

$$\mathbb{E}[F_t^2] < \infty$$

for all $t \in T$ and

$$\text{Cov}(F_{t_i}, F_{t_j}) = C(t_i - t_j),$$

for a function C.

We use our definition of homogeneous regions $\mathcal{R}_i$ to introduce a notion of local spatial symmetry, as opposed to global spatial symmetry.

Pore Generation through Point Processes. For the sake of simplicity, and following Mualem [13], we start by assuming that the pores of $\mathbf{S}$ are spheres, though we will later relax this assumption by assigning 3×3 deformation matrices to deform a perfect spherical pore into a general shape. Next, we assume that the process generating pores in a homogeneous region $\mathcal{R}_i$ differs from the process in a distinct region $\mathcal{R}_j$ $(i \neq j)$. Furthermore, the pore radii are modeled by a narrow Gaussian distribution, consistent with the possibly non-zero tolerance ϵ in Definition 1.

The main source of hydraulic resistance to flow between adjacent $\mathcal{R}_i$ arises primarily from geometric mismatch between pores in adjacent $\mathcal{R}_i$ regions: Water moves from a $\mathcal{R}_i$ to a neighboring region $\mathcal{R}_j$ only by first entering a pore p_i of $\mathcal{R}_i$ which overlaps with a pore p_j of $\mathcal{R}_j$. Consequently, hydraulic resistance has two contributions. First, an overlapping pore pair must exist; if no such overlaps occur, then flow between the two regions is impossible. Secondly, the resistance also depends on the intersection geometry. If the intersection area is small, water must be forced through a narrow opening, increasing resistance. This effect is further influenced by the pore radii r_i of p_i and r_j of p_j: if the radii differ significantly (likely, as they are sampled from two different distributions), then the intersection is small relative to the larger pore, increasing local resistance.

Consistent with labeling a region $\mathcal{R}_i$ as homogeneous, let $\mathcal{R}_i$ have bulk hydraulic conductivity K_i. Also, in the long term, we assume steady state flow in each $\mathcal{R}_i$. That is, the volumetric flux $\mathbf{q}_i : \mathcal{R}_i \to \mathbb{R}$ is constant. Darcy's law as applied to $\mathcal{I}_i$ states that $\mathbf{q}_i = -K_i \nabla P_i$, where $P_i(\mathbf{x})$ is the hydraulic gradient at a $\mathbf{x} \in \mathcal{I}_i$. Taking the gradient of both sides yields $\nabla \mathbf{q}_i = -K_i \nabla^2 P_i$, and the assumption that $\mathbf{q}_i$ is constant yields $\nabla^2 P_i = 0$; P_i is a harmonic function.

We model the generation of pores in each region $\mathcal{R}_i$ using a point process ξ_i and summarize the relevant definitions and theorems that will be employed in the analysis below. Roughly, our goal is to compute the expected value of the conductivity K of $\mathbf{S}$ given the K_i, and the distributions ξ.

Definition 6 (Point Process)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X, \mathcal{A})$ a measurable space. A *point process* is a function $\xi : \Omega \times \mathcal{A} \to \mathbb{Z}^+$ such that

- for every $\omega \in \Omega$, $\xi_w := \xi(\omega, \cdot)$ is a locally finite measure on $(X, \mathcal{A})$ and
- for every $A \in \mathcal{A}$, $\xi^A := \xi(\cdot, A)$ is a (positive integer valued) random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

As usual, let $\mathbf{S} \subset \mathbb{R}^3$ be a particular configuration of the porous material, and let $\mathcal{U}$ be the set of all such configurations in consideration. We make the assumption that $\partial\mathbf{S}$ is the same for all $\mathbf{S} \in \mathcal{U}$; call it X. Additionally, define the indicator function $\mathbf{1_S} : X \to \{0, 1\}$ associated to $\mathbf{S}$ by $\mathbf{1_S}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathbf{S}$ and $\mathbf{1_S}(\mathbf{x}) = 0$ otherwise. We collect all the $\mathbf{1_S}$ in the sample space $\Omega = \{\mathbf{1_S} : \mathbf{S} \in \mathcal{U}\}$; let $X = \mathbb{R}^3$ and $\mathcal{A} = \mathcal{B}(\mathbb{R}^3)$, the Borel $\sigma-$algebra on $\mathbb{R}^3$.

To define ξ, let $C_\mathbf{S}$ be the set of the centers of all the pores of $\mathbf{S}$. Then, we set $\xi(\mathbf{1_S}, B) = \#(C_\mathbf{S} \cap B)$, that is, $\xi(C_\mathbf{S}, B)$ is the number of center points of pores of $\mathbf{S}$ in B. In particular, $\mathbb{E}[\xi^B]$ is the average number of center points in a fixed Borel set B of $\mathbb{R}^3$.

Remark. In general, one may write ξ_ω as $\sum_{i=1}^n \delta_{X_i}$ where the X_i are (random) elements of X and δ_{X_i} is the Dirac measure of X_i; in this case, $\xi_\omega = \sum_{X_i \in C_\omega} \delta_{X_i}$.

Definition 7 (Intensity Measure)**.** Let $\xi : \Omega \times \mathcal{A} \to \mathbb{Z}^+$ be a point process. The *intensity measure* $\Lambda : \mathcal{A} \to [0, \infty]$ is a measure on $(X, \mathcal{A})$ defined by

$$\Lambda(A) := \mathbb{E}[\xi^A],$$

for all $A \in \mathcal{A}$.

We denote the Radon–Nikodym derivative of Λ with respect of the Lebesgue measure on $\mathbb{R}^3$ by λ; λ is called the density of Λ. Campbell's theorem, which we introduce next, allows us to calculate the expectation of the sum of values of a function defined on $C_\mathbf{S}$.

Theorem (Campbell's theorem)**.** Let $\xi : \Omega \times \mathcal{A} \to \mathbb{Z}^+$ be a point process and $f : X \to \mathbb{R}$ a measurable function. Then,

$$\mathbb{E}\left[\sum_{x \in C_\omega} f(x)\right] = \int_X f(x)\,\Lambda(dx) = \int_X f(x)\lambda(x)\,dx.$$

Campbell's theorem can be generalized to higher dimensional functions f, that is, f defined on X^n for $n > 1$. To that end, we introduce the n^{th} factorial power and n^{th} factorial moment of a point process, which will be useful when dealing with pore-pore intersections.

Definition 8 (n^{th} factorial moment measure)**.** Let $\xi : \Omega \times \mathcal{A} \to \mathbb{Z}^+$ be a point process. For $n \geq 1$, define $\xi^{(n)} : \Omega \times \mathcal{A}^n \to \mathbb{Z}^+$, the nth factorial power of ξ, by

$$\xi^{(n)}(\omega, A_1 \times \cdots \times A_n) = \sum_{\mathbf{x} \in C_\omega^{(n)}} \prod_{i=1}^{n} \mathbf{1}_{A_i}(x_i),$$

where $C_\omega^{(n)} = \{(x_1, \cdots, x_n) : x_i \neq x_j\}$ for all $A_i \in \mathcal{A}$. Note that $\xi^{(n),\, A_1 \times \cdots \times A_n} := \xi(\cdot, A_1 \times \cdots \times A_n)$ is an positive integer valued random variable on Ω. Thus, we may define $\Lambda^{(n)}$, the *n^{th} factorial moment measure* by

$$\Lambda^{(n)}(A_1 \times \cdots \times A_n) = \mathbb{E}\left[\xi^{(n),\, A_1 \times \cdots \times A_n}\right].$$

Proposition (Multivariable Campbell's theorem)**.** Let $\xi : \Omega \times \mathcal{A} \to \mathbb{Z}^+$ be a point process and $f : X^n \to \mathbb{R}$ a measurable function. Then

$$\mathbb{E}\left[\sum_{x \in C_\omega} f(x)\right] = \int_{X^n} f(x)\,\Lambda^{(n)}(dx_1, \cdots, dx_n).$$

If we were to assume that $\Lambda^{(n)}$ is absolutely continuous with respect to the Lebesgue measure on $X^n = \mathbb{R}^{3n}$, then, by the Radon–Nikodym theorem, there exists $\lambda^{(n)} : X^n \to \mathbb{R}$ such that

$$\Lambda^{(n)}(A_1 \times \cdots \times A_n) = \int_{A_1 \times \cdots \times A_n} \lambda^{(n)}(x_1, \cdots, x_n) dx_1 \cdots dx_n.$$

Definition 9. We call $\lambda^{(n)}$ as defined above the n^{th} *factorial moment density.*

We can use the n^{th} factorial moment density to define a correlation function.

Definition 10. For a point process ξ, define the *pair correlation function* $\rho^{(2)}$ by

$$\rho^{(2)}(x_1, x_2) = \frac{\lambda^{(2)}(x_1, x_2)}{\lambda^{(1)}(x_1)\lambda^{(1)}(x_2)},$$

for all $x_1, x_2 \in X$.

Next, we give a lemma that helps us better understand the intersection of two pores $p_i = B_{r_i}(\mathbf{x}_i)$ and $p_j = B_{r_j}(\mathbf{x}_j)$.

Lemma. When the intersection of the surfaces $\partial B_{r_i}(\mathbf{x}_i)$ and $\partial B_{r_j}(\mathbf{x}_j)$ is non-empty, it is a circle.

Proof. The result follows from a direct computation. Here, we demonstrate an approach that can be generalized easily to abstract inner product spaces.

For an $a \in \mathbb{R}$, define the function $\mathbf{c} : \mathbb{R} \to \mathbb{R}^3$ by $\mathbf{c}(a) := a\mathbf{x}_j + (1-a)\mathbf{x}_i$. Geometrically, the image of $\mathbf{c}$ is the line passing through $\mathbf{x}_i$ and $\mathbf{x}_j$. Furthermore, define $D_a(\mathbf{x}) = \|\mathbf{x} - \mathbf{c}(a)\|^2$ and set $\alpha = \langle \mathbf{x} - \mathbf{x}_i, \mathbf{x} - \mathbf{x}_j \rangle$ for convenience. Expanding the squared norm for $\mathbf{x} \in \partial B_{r_i}(\mathbf{x}_i) \cap \partial B_{r_j}(\mathbf{x}_j)$ we obtain

$$\begin{aligned} D_a(\mathbf{x}) &= \|a(\mathbf{x} - \mathbf{x}_j) + (1-a)(\mathbf{x} - \mathbf{x}_i)\|^2 \\ &= a^2 r_j^2 + (1-a)^2 r_i^2 + 2a(1-a)\alpha. \end{aligned}$$

We have

$$\begin{aligned} d^2 &= \|\mathbf{x}_i - \mathbf{x}_j\|^2 = \|(\mathbf{x}_i - \mathbf{x}) + (\mathbf{x} - \mathbf{x}_j)\|^2 \\ &= \|\mathbf{x}_i - \mathbf{x}\|^2 + \|\mathbf{x} - \mathbf{x}_j\|^2 - 2\alpha \\ &= r_i^2 + r_j^2 - 2\alpha. \end{aligned}$$

Thus, $\alpha = \frac{1}{2}(r_i^2 + r_j^2 - d^2)$, which we substitute back into the expression for $D_a(\mathbf{x})$ to get

$$\begin{aligned} D_a(\mathbf{x}) &= a^2 r_j^2 + (1-a)^2 r_i^2 + a(1-a)(r_i^2 + r_j^2 - d^2) \\ &= r_i^2 + a(r_j^2 - r_i^2 - d^2) + a^2 d^2. \end{aligned}$$

Thus, for $\mathbf{x} \in \partial B_{r_i}(\mathbf{x}_i) \cap \partial B_{r_j}(\mathbf{x}_j)$, the quantity $D_a(\mathbf{x})$ is independent of $\mathbf{x}$; denote this function by $D(a)$. Therefore, if the intersection is non-empty, it is either a point or a circle with center $\mathbf{c}(a_0)$ where $a_0 \in \mathbb{R}$ is such that $D'(a_0) = 0$. Moreover, its radius r is $\sqrt{D(a_0)}$. Taking the derivative, we have

$$a_0 = \frac{d^2 + r_i^2 - r_j^2}{2d^2},$$

and so

$$r = \sqrt{r_i^2 - \frac{(d^2 + r_i^2 - r_j^2)^2}{4d^2}}. \tag{10}$$

Clearly, if $D(a_0) < 0$, then the intersection is empty, as r does not exist. A straightforward algebraic manipulation shows that $D(a_0) \geq 0$ if and only if $r_i + r_j \geq d$, $r_j + d \geq r_i$ and $d + r_i \geq r_j$. Recognizing this as the triangle inequality, the intersection is non-zero if and only if r_i, r_j and d are the side lengths of a triangle. In that case, it is a circle with radius r, where r is as given in Equation 10. This completes the proof. □

Corollary. We have

$$\text{Area}\left(\partial B_{r_i}(\mathbf{x}) \cap \partial B_{r_j}(\mathbf{x})\right) = \pi\left(r_i^2 - \frac{(d^2 + r_i^2 - r_j^2)^2}{4d^2}\right).$$

How do we assign a conductivity or a resistance value to each pore-pore interface p_i, p_j. Sampson's flow, defined below, first obtained by Sampson [17] as a particular solution of the Stokes' equation, provides a useful starting point.

Definition 11. Sampson flow describes the volumetric flow of a Newtonian fluid through a flat, circular orifice of radius r under a pressure difference ΔP assuming

- the orifice is flat, with infinitesimal thickness,
- the flow is steady, incompressible, laminar and,
- the fluid has constant viscosity μ.

Then,

$$Q = \frac{r^3}{24\mu}\Delta P.$$

We treat the intersection (of the pores p_i and p_j) as a locally flat circular orifice and use Sampson flow to assign a hydraulic conductance K_{ij} to each pore-pore interface, providing a first-order approximation of the microscopic contribution of this interface.

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