Introduction

The following is extracted from an assignment completed as part of the *Strange and Beautiful Numbers* course, taken towards the end of 8th grade. I attempt to compute the area of Ford circles (a fractal) without formal knowledge of the Euler totient function and do so correctly, discovering limits in the process. The course instructor's comment was: "Your creativity in implementing a way to calculate the sum of the gcds of 1 while using extremely small values for the divisors was inspiring."

Calculating the area of the Ford circles in the unit square

The area of a circle with radius r is πr^2. The only thing we need to input is the radius of the circle. Recall that the radius of Ford circle for a fraction a/b is $1/2b^2$, meaning that the area of a Ford circle for a fraction a/b will be :

$$\pi \times \left(\frac{1}{2b^2}\right)^2 = \frac{\pi}{4b^4}.$$

This shows us that rational numbers that have the denominators will have Ford circles of the same area. Now if there were no two rational numbers that have the same denominator in the left half of the tree, the area of the ford circles would be: $\sum_{b=0}^{x} \frac{\pi}{4b^4}$. The greater the value of x, the closer $\sum_{b=0}^{x} \frac{\pi}{4b^4}$ will be to the total area of all the ford circles within the unit square.

But we know that there are multiple rational numbers in the tree which have the same denominator. We need to multiply $\frac{\pi}{4b^4}$ by the number of fractions with the denominator b in the left half of the tree.

A formula that gives us the number of fractions with the denominator *b*

I tried to understand how may fractions with the denominator b could be present in the left half of the tree by using the facts that I learnt earlier:

- The tree contains all rational numbers, and exactly the fractions that are in the simplest form.
- The tree contains each rational number exactly once.

If the tree contained all the *fractions* (as opposed to rational numbers) then there would be b *fractions* with denominator b. For example, if I tried to figure out the number of proper *fractions* with the denominator 8, we get:

$$\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8}$$

But the only *fractions* that are in simplest form are $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ and these are the only fractions that occur in the tree. Therefore, we need a formula that gives the number of proper fractions that are in the simplest form with denominator b.

Upon a closer look, I realized this is just the number of integers less than b that are coprime to it. Let us call this number $N(b)$.

This is because the numerator and denominator of a fraction that isn't in simplest form have a gcd that is greater than one, whereas the numerators and denominators of a fraction that is in simplest form have a gcd of one i.e., they are coprime to each other.

In order to graph the area of the ford circles, we have to find a formula or a function that outputs the number of co-primes for a given integer. I started out with writing down the summation:

$$S = \sum_{n=1}^{b} \gcd(n, b)$$

This expression has to be tweaked: In other words, we must perform an operation on $\gcd(n, b)$ so that if $\gcd(n, b)$ is greater than one it outputs 0 and returns $\gcd(n, b)$ it as it is if the answer if $\gcd(n, b) = 1$.

I couldn't find any operation that does that, but I did manage to find a method that reduces all the integers that are greater than one to infinitesimal amounts and leaves 1 alone:

$$N(b) \approx \operatorname{round}\left(\sum_{n=1}^{b} gcd(n,b)^{-\,gcd(n,b)}\right).$$

Here, I've used the property that 1 raised to a number greater than 0 is 1, while any other number is raised to a power that is positive results in a number than is greater than the number itself. The greater the power, the larger the number.

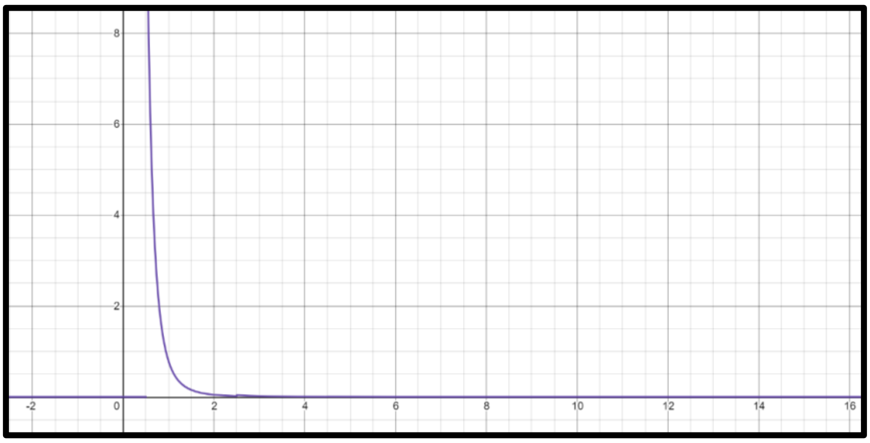
The more "to the power of $gcd(n, b)$" you use, the more accurate the result becomes.

$$N(b) \approx \operatorname{round}\left(\sum_{n=1}^{b} \left(\frac{1}{gcd(n,b)}\right)^{gcd(n,b)^{gcd(n,b)}}\right)$$

What I inputted in Desmos:

$$\left(\frac{\operatorname{round}\left(\sum_{n=1}^{r}\left(\frac{1}{\gcd(n,r)}\right)^{\gcd(n,r)^{\gcd(n,r)^{\gcd(n,r)^{\gcd(n,r)}}}}\right)}{r^{4}}\right)\cdot\left(\frac{\pi}{4}\right)$$

The graph of the function $f(o)$:



You can see that the blue line gets closer and closer to 0 as the value of o increases. Over an infinite amount of time, distance between the blue line and the x −axis will be zero.

This means that the larger and larger the denominator of a fraction gets, the area of the Ford circles related with that denominators gets lesser and lesser.

If we define a function $f(r)$ as:

$$\boxed{\left(\frac{\left(\mathrm{round}\left(\sum_{n=1}^{r}\left(\frac{1}{\gcd(n,r)}\right)^{\gcd(n,r)^{\gcd(n,r)^{\gcd(n,r)^{\gcd(n,r)}}}}\right)\right)}{r^4}\right)\cdot\left(\frac{\pi}{4}\right)}$$

And define the function $t(v)$ as:

$$t(v) \coloneqq \sum_{p=1}^{v} f(p).$$

Then as the value of k increases in $t(k)$, the more accurate the total area the Ford circles will be. I have tabulated some of my results:

i	$t(i)$
1	0.78539816
2	0.83448555
3	0.8538781
4	0.86001402
5	0.86504057
6	0.8662526
7	0.86821528
8	0.86898227
9	0.86970051
10	0.87001467
100	0.8722602
500	0.87228309
1000	0.8722838
1500	0.87228394
2000	0.87228398
5000	0.87228309

This sequence seems to be getting closer to some number i.e., it has a limit. If it is a convergent sequence, then it is safe to say that the total area of all the ford circles in the unit square in roughly $\mathbf{0.8722}$ (These digits have remained unchanged for a long period of time).

This made sense to me—the circles that you add in each stage of the ford circles keep on becoming smaller and smaller—just like how in the sequence $4.9, 4.99, 4.999, 4.9999$ the amount added in each term keeps on becoming smaller and smaller. It makes sense that the total area of the ford circles is a limit of a certain sequence.

Conclusion: Extending this to Other Fractals (Picking off from the Study Circle)

Let me take the example of another fractal—the cantor set. What is the total area of the cantor set ? The sum of the lengths of each stage which is given by the expression:

$$\frac{2^n}{3^{n+1}}.$$

We can put this into a summation notation to express the sum of the lengths of the first x stages of the cantor set (x will hence be the upper bound of the summation):

$$\sum_{n=0}^{x} \frac{2^n}{3^{n+1}}.$$

The larger the value of x, the more accurate the total area removed will be.

x	**Are of Cantor Set Removed**
0	0.3333333333333333
1	0.5555555555555556
2	0.7037037037037037
3	0.8024691358024691
4	0.8683127572016461
5	0.9122085048010974
6	0.9414723365340649
7	0.9609815576893767
8	0.9739877051262511
9	0.9826584700841674
10	0.9884389800561116

Speeding up the process…

20	0.9997995142267857
30	0.9999965232699664
40	0.9999999397081822
50	0.9999999989544479
60	0.9999999999818686
70	0.9999999999996857
80	0.9999999999999947
90	0.9999999999999999
100	0.9999999999999999

The output gets closer and closer to one but will never get there. Hence we can conclude that limit of this output sequence is one, that means is we iterate the cantor set for an infinite number times, then the total length of the cantor set will be 0 (If we started with a line that had a length of one).