THE EULER CIRCLE COMBINATORIAL GAME THEORY SAMPLE PROBLEM

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This document contains my solutions to select The Euler Circle Combinatorial Game Theory Problems. The first one is same one as in the Poster.

1. Sample Problem 1: Tiny Analogue of the Number Avoidance Theorem

Question 1.1. *Suppose $a > b \geq 0$ are numbers. Find the canonical form for $\maltese_a \maltese_b$, i.e. the sum of a and b. Explain why this is a tiny analogue of the number avoidance theorem.*[1]

We start by adding together $\maltese_a$ and $\maltese_b$, and simplifying the dominated options first. We have

$$\begin{aligned}
\maltese_a + \maltese_b &= \{0 \mid \{0 \mid -a\}\} + \{0 \mid \{0 \mid -b\}\} \\
&= \{\maltese_a, \maltese_b \mid \{0 \mid -a\} + \maltese_b, \maltese_a + \{0 \mid -b\}\} \\
&= \{\maltese_b \mid \{0 \mid -a\} + \maltese_b, \maltese_a + \{0 \mid -b\}\},
\end{aligned}$$

where the last equality follows since $b < a \implies \maltese_a < \maltese_b$, and so the left option $\maltese_a$ is dominated, and so we can ignore it. Next, we must compare $\{0 \mid -a\} + \maltese_b$ to $\maltese_a + \{0 \mid -b\}$. We do this by putting the two in canonical form.

Lemma 1.2. *Let $a > b > 0$. Then $\{0 \mid -a\} + \maltese_b = \{\maltese_b \mid \{-a \mid \{-a \mid -a - b\}\}\}$. Notice that we have $\{0 \mid -a\} + \maltese_b = \{\maltese_b \mid -a + \maltese_b\}$.*

Proof. Adding the two together, we get

$$\{0 \mid -a\} + \{0 \mid \{0 \mid -b\}\} = \{\maltese_b, \{0 \mid -a\} \mid \{0 \mid -a\} + \{0 \mid -b\}, -a + \maltese_b\}.$$

First, it can be shown that $\{0 \mid -a\} < \maltese_b$ (this is because $\{0 \mid -a\} \in \mathcal{N}$ and $\maltese_b \in \mathcal{L}$), so we can ignore the $\{0 \mid -a\}$. Then, we must compare $\{0 \mid -a\} + \{0 \mid -b\}$ to $-a + \maltese_b$. Let us put the two in a slightly simpler form.

- We have that $\{0 \mid -a\} + \{0 \mid -b\} = \{\{0 \mid -a\}, \{0 \mid -b\} \mid \{-a \mid -a - b\}, \{-b \mid -a - b\}\}$. Clearly, $\{0 \mid -a\} < \{0 \mid -b\}$, and $\{-a \mid -a - b\} < \{-b \mid -a - b\}$, so we can write
$$\{0 \mid -a\} + \{0 \mid -b\} = \{\{0 \mid -b\} \mid \{-a \mid -a - b\}\}.$$
- We have that $-a + \maltese_b = \{-a \mid -a + \{0 \mid -b\}\} = \{-a \mid \{-a \mid -a - b\}\}$, by the translation theorem.

Thus, both have the same right option, but the left option of $\{0 \mid -a\} + \{0 \mid -b\}$ is $\{0 \mid -b\}$, whereas the left option of $-a + \maltese_b$ is $-a$. Thus, when left starts, right will win a larger margin in the latter game, as the former will end at $-b > -a$. Hence, we claim that $-a + \maltese_b < \{0 \mid -a\} + \{0 \mid -b\}$ (it can be shown that this is indeed the case), and so we can ignore the $\{0 \mid -a\} + \{0 \mid -b\}$ altogether, which completes the proof. ∎

Next, we turn our attention to $\{0 \mid -b\} + \maltese_a = \{\maltese_a, \{0 \mid -b\} \mid \{0 \mid -a\} + \{0 \mid -b\}, -b + \maltese_a\}$. Again, since $\maltese_a \in \mathcal{L}$ and $\{0 \mid -b\} \in \mathcal{N}$, we have that $\{0 \mid -b\} < \maltese_a$, and so we can ignore the $\{0 \mid -b\}$. As for the right options, we must compare $\{0 \mid -a\} + \{0 \mid -b\}$ and $-b + \maltese_a$. We

[1] Course Description: `https://eulercircle.com/classes/combinatorial-game-theory/`.

already have that $\{0 \mid -a\} + \{0 \mid -b\} = \{\{0 \mid -b\} \mid \{-a \mid -a-b\}\}$. Similarly, $-b + \maltese_a = -b + \{0 \mid \{0 \mid -a\}\} = \{-b \mid \{-b \mid -a-b\}\} = -\{\{a+b \mid b\} \mid b\}$. It turns out that the game $\{\{a+b \mid b\} \mid b\} + \{\{0 \mid -b\} \mid \{-a \mid -a-b\}\} \in \mathcal{R}$, so $\{0 \mid -a\} + \{0 \mid -b\} < -b + \maltese_a$, and we can ignore the $-b + \maltese_a$. Hence, $\{0 \mid -b\} + \maltese_a = \{\maltese_a \mid \{0 \mid -a\} + \{0 \mid -b\}\}$.

Lastly, we claim that $\{0 \mid -a\} + \maltese_b < \maltese_a + \{0 \mid -b\}$. That is, the game

$$\{a - \maltese_b \mid -\maltese_b\} + \{\maltese_a \mid \{0 \mid -a\} + \{0 \mid -b\}\} \in \mathcal{L}.$$

Proof. Right to start has the following options.

- He can move to $-\maltese_b$, to which left will respond by moving to $\{b \mid 0\}$ in $-\maltese_b$, leaving us with $\{b \mid 0\} + \{\maltese_a \mid \{0 \mid -a\} + \{0 \mid -b\}\}$. It is easy to show that right has no winning moves from here.
- Right can move to $\{0 \mid -a\} + \{0 \mid -b\}$, to which left will respond by moving to $a - \maltese_b$, which will result in $a - \maltese_b + \{0 \mid -a\} + \{0 \mid -b\}$, which we've already shown is a $\mathcal{L}$ position.

Left to start will move to $a - \maltese_b$, which will result in $\{\{a+b \mid a\} \mid a\} + \{\maltese_a \mid \{0 \mid -a\} + \{0 \mid -b\}\}$. One can show that right has no winning moves from here. ∎

Here are some the results that I used in my proof above.

Lemma 1.3. *Let $a > b$. Then $-a + \maltese_b < \{0 \mid -a\} + \{0 \mid -b\}$.*

Proof. We must show that $\{\{0 \mid -b\} \mid \{-a \mid -a-b\}\} + \{\{a+b \mid a\} \mid a\} \in \mathcal{L}$. Right to start has the following options.

- He may move to $\{-a \mid -a-b\}$, in which case left will respond by moving to $\{a+b \mid a\}$, which will result in the 0 game.
- He may move to a, to which left will respond by moving to $\{0 \mid -b\}$, which will result in $\{0 \mid -b\} + a = \{a \mid a-b\} \in \mathcal{L}$.

Left's to start will move to $\{0 \mid -b\}$, which will result in $\{0 \mid -b\} + \{\{a+b \mid a\} \mid a\}$, which is losing for right. ∎

Lemma 1.4. *Let $a > b$. Then $\{0 \mid -a\} + \{0 \mid -b\} < -b + \maltese_a$.*

Proof. We must show that $\{\{0 \mid -b\} \mid \{-a \mid -a-b\}\} + \{\{a+b \mid b\} \mid b\} \in \mathcal{R}$. Left to start has the following options.

- She can move to $\{0 \mid -b\}$, to which right will respond by moving to $-b$ in $\{0 \mid -b\}$, leaving us with $\{\{a \mid 0\} \mid 0\} \in \mathcal{R}$.
- She can move to $\{a+b \mid b\}$, to which right will respond by moving to b in $\{a+b \mid b\}$, leaving us with $\{\{b \mid 0\} \mid \{b-a \mid -a\}\}$. Left has no other option but to move to the $\mathcal{N}$ position $\{b \mid 0\}$.

Right to start will move to b. ∎

In conclusion, we have the following result.

Proposition 1.5. *Let $a > b$ be numbers. Then $\maltese_a + \maltese_b = \{\maltese_b \mid \{0 \mid -a\} + \maltese_b\}$.*

2. Sample Problem 2: Subtraction Game and Periodic Grundy Values

Question 2.1. *Let S be a finite set of positive integers. Consider the S-subtraction game, in which a move consists of removing stones from the pile, where s is some element in S. Show that the Grundy values of this game are eventually periodic, i.e. there is some integer p and some nonnegative integer n_0 so that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all integers $n \geq n_0$.*

Lemma 2.2. *Let S be a set of non-negative integers. Then $\operatorname{mex}(S) \leq |S|$. In other words, the minimal excludant of a set cannot exceed the size of the set itself.*

Proposition 2.3. *There are only finitely many Grundy values of the $S-$subtraction game. In other words, the set $\{\mathcal{G}(n) : n \in \mathbb{Z}^+ \cup \{0\}\}$ a finite set of non-negative integers.*

Proof. From the definition of Grundy values, for all $n \in \mathbb{Z}^+ \cup \{0\}$, we have $\mathcal{G}(n) = \text{mex}(\{\mathcal{G}(n-s) : s \in S \text{ and } n - s \geq 0\})$. Clearly, $|\{\mathcal{G}(n-s) : s \in S \text{ and } n - s \geq 0\}| \leq S \implies \mathcal{G}(n) \leq |S|$ (this follows from the lemma stated above). Thus, $\mathcal{G}(n)$ can only take on finitely values. ∎

Remark 2.4. In fact, we have that the Grundy values are less than or equal to $|S|$, the maximum number of legal moves available to a player. Thus, $\mathcal{G}(n)$ can take on a maximum of $|S|+1$ distinct values ($\mathcal{G}(n) \in \{0, 1, \cdots, |S|\}$).

Proof. For any positive integer m, there exist non-negative integers $n_0(m)$ and $n_1(m)$ such that $n_0(m) < n_1(m)$ and $\mathcal{G}(n_0(m) - s + i) = \mathcal{G}(n_1(m) - s + i)$ for all $s \in S$ and for $0 \leq i \leq m$. This follows from the pigeonhole principle. To see how, for each non-negative integer n we associate the $(m+1)$ by k matrix given by $A(n)_{ij} = \mathcal{G}(n + i - s_j)$ for $0 \leq i \leq m$ and $1 \leq j \leq k$. Now, each Grundy value of the $S-$subtraction game is between 0 and k (both inclusive), so we have $(k+1)^{(m+1)k}$ choices for $A(n)_{ij}$ for each n, which is clearly finite. Thus, somewhere down the line of integers, two integers must share the same matrix.

Now, let us take m to be $s_k - 1$. We denote the corresponding $n_0(m)$ by n_0 and $n_1(m)$ by n_1. Note that we have $\mathcal{G}(n_0+k) = \mathcal{G}(n_1+k)$ for all $0 \leq k \leq m$. We will show that $\mathcal{G}(n_0+m+j) = \mathcal{G}(n_1+m+j)$ for all $j > 0$. We will use induction on j.

For the base case, that is, when $j = 1$, we must show that $\mathcal{G}(n_0 + m + 1) = \mathcal{G}(n_1 + m + 1)$. Note that $\mathcal{G}(n_0 + m + 1) = \text{mex}\{\mathcal{G}(n_0 + m + 1 - s) : s \in S\} = \text{mex}\{\mathcal{G}(n_0 + s_k - s) : s \in S\}$. At this point, observe that $0 \leq s_k - s$ and $1 \leq s \implies -1 \geq -s \implies s_k - 1 \geq s_k - s \implies m \geq s_k - s$ for all $s \in S$. Thus, $0 \leq s_k - s \leq m$, and so by the induction hypothesis, we have that $\mathcal{G}(n_0 + s_k - s) = \mathcal{G}(n_1 + s_k - s)$ for all $s \in S$, meaning that $\mathcal{G}(n_0 + m + 1) = \mathcal{G}(n_1 + m + 1)$.

Next, assume that $\mathcal{G}(n_0+m+x) = \mathcal{G}(n_1+m+x)$ for all $0 \leq x \leq j$, which implies $\mathcal{G}(n_0+m+x) = \mathcal{G}(n_1 + m + x)$ for all $-m \leq x \leq j$. We must show that $\mathcal{G}(n_0 + m + j + 1) = \mathcal{G}(n_1 + m + j + 1)$. Observe that $\mathcal{G}(n_0 + m + j + 1) = \text{mex}\{\mathcal{G}(n_0 + m + j + 1 - s) : s \in S\}$. Note that $1 \leq s \implies 1 - s \leq 0 \implies j + 1 - s \leq j$ and $s \leq s_k \implies s - j \leq s_k \implies j - s \geq -s_k \implies 1 + j - s \geq 1 - s_k \implies j + 1 - s \geq -m$. Thus, $-m \leq j + 1 - s \leq j$ for all $s \in S$. Hence, by the induction hypothesis, $\mathcal{G}(n_0 + m + j + 1 - s) = \mathcal{G}(n_1 + m + j + 1 - s)$ for all $s \in S$ which means that $\mathcal{G}(n_0 + m + j + 1) = \mathcal{G}(n_1 + m + j + 1)$.

In conclusion, we have that $\mathcal{G}(n_0 + x) = \mathcal{G}(n_1 + x)$ for all $x \geq 0$. Since $n_1 > n_0$, we may write $n_1 = n_0 + p$ where $p > 0$. Thus, $\mathcal{G}(n_1 + x) = \mathcal{G}(n_0 + x + p) = \mathcal{G}(n_0 + x)$. Setting $n = n_0 + x$, we get that $\mathcal{G}(n) = \mathcal{G}(n + p)$ for all $n \geq n_0$. ∎

Remark 2.5. In fact, this proof gives us a bound on p. Namely, $p \leq (k+1)^{k(m+1)} = (k+1)^{ks_k} = (|S|+1)^{|S| \times \sup(S)}$.

3. Sample Problem 3: A Proof

Question 3.1. *Show that if $G = \{G^{L_1}, \cdots \mid G^{R_1}, \cdots\}$ is a game born on day n and $s, t \geq n$, then we may replace G^{L_1} by $\{s \mid \{G^{L_1} \mid -t\}\}$ without changing the value.*

Lemma 3.2. *Let G be a game born on day n. Then $G \leq n$.*

Proof. We will use induction on the birthday of G. That is, assuming that $G' \leq g(G')$ for all games G' such that $g(G') < g(G)$, we will show that $G \leq n$. In other words, we must show that $n - G = \{n - 1 \mid \varnothing\} + \{-G^R \mid -G^L\} \in \mathcal{L}$.

If left starts, her strategy will be to move to a $-g^R \in -G^R$, which will reduce the game to $n - g^R$. By the induction hypothesis, since $g(g^R) < g(G) = n$ we must have that $g^R \leq g(g^R) < n \implies g^R < n \implies n - g^R \in \mathcal{L}$, so left has a winning strategy in $n - g^R$. If right starts, he has no other option but to move to a $-g^L \in -G^L$, which will reduce the game to $n - g^L$. Just like last time one can show that $n - g^L \in \mathcal{L}$, which completes the proof. ■

Lemma 3.3. *Let G be a game born on day n, and let $x \geq n$ be an integer. Then $-x - g^R \in \mathcal{R}$ for all right options g^R of G.*

Proof. Recall that $-g^R$ is a option of $-G$, which is a game born on day n, so $-g^R$ is born before day n. Hence, we know that $-g^R \leq n$ by the previous lemma. Since we are given that $n \leq x$, we must have that $-g^R < x \implies -x - g^R \in \mathcal{R}$ by transitity. ■

Proof. We must show that the game $\{\{s \mid \{G^L \mid -t\}\}, G^{L_i} \mid G^{R_i}\} + \{-G^{R_i} \mid -G^{L_i}\}$ is a $\mathcal{P}$ position. If left starts, she has the following options.

- She may choose to move to $\{s \mid \{G^L \mid -t\}\}$ in $\{\{s \mid \{G^L \mid -t\}\}, G^{L_i} \mid G^{R_i}\}$ which leaves us with $\{s \mid \{G^L \mid -t\}\} + \{-G^{R_i} \mid -G^{L_i}\}$. Right will respond by moving to $\{G^L \mid -t\}$ in $\{s \mid \{G^L \mid -t\}\}$, which will reduce the game to $\{G^L \mid -t\} + \{-G^{R_i} \mid -G^{L_i}\}$. Now, if left responds by moving to G^L in $\{G^L \mid -t\}$, right will move to $-G^L$ in $\{-G^{R_i} \mid -G^{L_i}\}$, which will reduce the game to 0, which is losing for left. On the other hand, if left moves to $-G^{R_i}$ in $\{-G^{R_i} \mid -G^{L_i}\}$, right will respond by moving to $-t$ in $\{G^L \mid -t\}$ which will reduce the game to $-t - G^{R_i}$. By the lemma mentioned above, we know that this is a $\mathcal{R}$ position.
- She may choose to move to G^{L_i} for $i \geq 2$ in $\{\{s \mid \{G^L \mid -t\}\}, G^{L_i} \mid G^{R_i}\}$. Right will respond by moving to $-G^{L_i}$ in $\{-G^{R_i} \mid -G^{L_i}\}$, which will make the game 0.
- She may choose to move to $-G^{R_i}$ in $\{-G^{R_i} \mid -G^{L_i}\}$. Right will respond by moving to G^{R_i} in $\{\{s \mid \{G^L \mid -t\}\}, G^{L_i} \mid G^{R_i}\}$, which will make the game 0 again.

In all the cases, left loses. If right starts, he has the following options.

- He may choose to move to $-G^L$ in $\{-G^{R_i} \mid -G^{L_i}\}$ which leaves us with $\{\{s \mid \{G^L \mid -t\}\}, G^{L_i} \mid G^{R_i}\} - G^L$. Left will respond by moving to $\{s \mid \{G^L \mid -t\}\}$ in the first game, after which we have $\{s \mid \{G^L \mid -t\}\} - G^L$. Right has two options. If he moves to $\{G^L \mid -t\}$ he will loose, as left will respond by moving to G^L in $\{G^L \mid -t\}$, which will reduce the game to 0. On the other hand, if he moves to some right option of $-G^L$, call it $-G^{LR}$, left will respond by moving to s in $\{s \mid \{G^L \mid -t\}\}$, which will reduce the game to $s - G^{LR}$. Note that this game is of the form $s - h$, where h is a left option of a left option of G. Thus, we know that h is born before G, and so by the above lemma we have that $h \leq n$ and since $n < s$ we have that $h < s \implies s - h \in \mathcal{L}$, which means that left has a winning strategy in $s - G^{LR}$ so she wins.

■