Self Directed Additional Coursework in High School

Siddharth Kothari

Website: `https://spacersid1331-design.github.io/index.html`

This document outlines additional reading and coursework I have undertaken since 8th grade as part of a sustained effort to deepen my understanding of mathematics. Engaging in these studies enabled me to contribute meaningfully to discussions at The Euler Circle and Math Circle India, while also sparking curiosity in broader areas such as Representation Theory, Algebraic Number Theory, and Algebraic Geometry.[1] This progression led me to pursue independent projects in these fields and to take graduate-level course at IISER.[2]

The material that follows is arranged by subject area. Each section contains concise lists of topics studied and the corresponding resources consulted, such as textbooks, lecture notes, problem sets, and related projects done by me including blogs, videos, and expository articles. These lists are meant to serve as a structured record of the material covered, rather than as complete expositions.

Contents

1 Generating an Interest Mathematics Middle School **2**

2 Linear Algebra Freshman Summer and Sophomore Fall **2**
- 2.1 Concepts and Vocabulary . . . 2
- 2.2 Resources . . . 3

3 Abstract Algebra (Groups) Sophomore Fall **4**
- 3.1 Concepts and Vocabulary . . . 4
- 3.2 Resources . . . 4

4 Real Analysis Sophomore Fall and Sophomore Winter **5**
- 4.1 Concepts and Vocabulary: . . . 5
- 4.2 Resources . . . 5

5 Algebraic Geometry and Adjacent Fields Sophomore Summer **6**
- 5.1 Concepts and Vocabulary . . . 6
- 5.2 Resources . . . 6

6 Algebraic Number Theory Junior Spring to Present **6**
- 6.1 Concepts and Vocabulary . . . 6
- 6.2 Resources . . . 7

7 Differential Geometry Junior Fall **8**
- 7.1 Concepts and Vocabulary . . . 8
- 7.2 Resources . . . 8

8 Representation Theory Project **8**

[1] Math Circle India: `https://www.icts.res.in/outreach/maths-circle-india`. An initiative of ICTS-TIFR (Tata Institute of Fundamental Research).

[2] IISER: Indian Institute of Science Education and Research, Pune is a premier institute dedicated to research and teaching in the basic sciences, established in 2006 by the Ministry of Human Resource Development, India: `https://www.iiserpune.ac.in/`.

9 Euler Circle **9**

10 Exploration: Are Mirrors Perfect? **9**

1 Generating an Interest Mathematics <u>Middle School</u>

The mathematics courses below were offered by the GENWISE (`https://genwise.in/`) foundation, a program for gifted/motivated students. Courses were facilitated by Jerry Burkhart of 5280 Math Education LLC (`https://www.5280math.com/`), and involved weekly submission of assignments based on open-ended problems.

- **Aug–Oct 2021**
  - **GW Fractals and Dynamics**
  - **GW Math Study Circle (Edition 1)**
  - **GW Math Adventures in Algebra**
  - **GW Math Study Circle (Edition 2)**
- **Oct–Feb 2021–2022**
  - **GW Strange and Beautiful Numbers**
  - The course covered topics in numbers systems not taught in traditional setups.
    * **Elementary Number Theory**: Prime factorization via Euclid's game, Euclidean algorithm, using modular arithmetic to solve certain Diophantine equations, patterns in addition and multiplication tables modulo n.
    * **Rational Numbers**: Continued fractions (including their connection with the Euclidean Algorithm), continued fraction representations of irrational numbers such as $\sqrt{2}$ and ϕ (the golden ratio), the mediant of two fractions (including proofs that the mediant lies between the two fractions), and using mediants to gain intuition for equivalence classes of rational numbers.
    * **The Stern–Brocot Tree**: paths to a general rational q from 1 using Diophantine equations, approximating irrational numbers using sequences of rational numbers, constructing Ford circles and calculating their total area with an error of only 9.5×10^{-7} using elementary techniques (without the Riemann zeta function). Full calculation (with instructor comments) using iterated gcd functions: `https://bit.ly/4pV1yeq`.

2 Linear Algebra <u>Freshman Summer and Sophomore Fall</u>

2.1 Concepts and Vocabulary

- **Vector spaces:** Motivation and definition of a vector space, subspace, span (of a set of vectors), spanning set of a vector space, linear independence and linear dependence, basis and dimension of a vector space; prototypical examples such as the real Euclidean space ($\mathbb{R}^n$), and more involved examples such as the space of all continuous functions on $[a, b]$.

- **Linear transformations:** Definition of a linear transformation (additivity and homogeneity), appreciation of the relationship between a basis and linear transformations, the space of all linear transformations ($L(V, W)$), kernel and range of a linear transformation, and injective linear maps, application of the first isomorphism theorem for groups to prove the rank-nullity theorem, isomorphism of vector spaces.

- **Matrices and transformations:** The idea that every linear transformation is associated with a matrix and vice versa, multiplication of matrices and function composition, arithmetic of matrices (including transpose and addition), basic types including square, upper triangular, lower triangular, diagonal, symmetric, and skew-symmetric; the idea that a single transformation corresponds to more than one matrix, change of basis matrix, determinant of an n by n matrix and its connection to how areas change under a transformation, and (intuition for) properties of a determinant. Basic applications of the latter include the Vandermonde matrix and Lagrange interpolation.

- **Matrices continued:** Rank of a matrix and its relationship to independence, trace of a matrix, systems of equations and matrices, Gauss–Jordan elimination, inverse of a matrix, a formula for the latter using determinants and adjoints, and Cramer's rule.

- **Transformations continued:** Eigenvalues and eigenvectors, proof of the existence of eigenvalues, computing eigenvalues via the characteristic polynomial, an eigen-basis, properties of eigenvalues (e.g., the product of eigenvalues is the determinant), diagonalizability and defective matrices, computing the eigendecomposition, the matrix exponential, and using it to solve linear systems of ordinary differential equations.

- **Vector spaces continued:** Exploring the geometry of $\mathbb{R}^n$ via the dot product, generalizing the latter to an inner product and the norm induced by an inner product, the Cauchy–Schwarz inequality (including its applications in optimization problems), orthogonality in inner product spaces, appreciation of an orthogonal basis and the Gram–Schmidt algorithm for computing it; Parseval's identity and its application in computing simple values of the Riemann zeta function.

- **An application:** Solving over-determined systems of equations and their connection with projection onto subspaces of $\mathbb{R}^n$, using this idea to calculate a line of best fit for a given data set (least squares). Generalizing this to formulate *the best approximation theorem*[3] in an arbitrary finite-dimensional vector space, and using it to
  - calculate polynomial approximations of functions such as $\sin x$ and e^x (using the *Legendre polynomials* basis), and
  - do the opposite by computing the *Fourier series* of functions (using the trigonometric polynomial basis).

2.2 Resources

2.2.1 Books and Courses

[1] *Introduction to Linear Algebra, 5th edition* by Gilbert Strang.

[2] *Fundamentals of Mathematics* by Maxim Gilula.

[3] *Interactive Linear Algebra* by Dan Margalit and Joseph Rabinoff.

[4] *This is Linear Algebra* textbook by Ohio State University press.

[5] Completed *Fundamentals of Mathematics Fall Quarter 2023* taught by Maxim Gilula from The Euler Circle. Included *9 lecture hours* and *12 hours of discussions.*

2.2.2 Articles and Papers

[6] On Linear algebra and Approximation Theory.

[7] On Using Legendre polynomials in Approximation Theory.

[8] Problems from: `https://web.pdx.edu/~erdman/LINALG/Linalg_pdf.pdf`.

[3]Personally, this is one of my favorite theorems, as it beautifully demonstrates the power of abstraction. It's surprising that the idea behind Fourier series is *precisely* the same as: 'Given a line and a point, what's the line that connects the two such that it is of minimum length? It's the one that's *perpendicular* to the given line.'

2.2.3 Online Resources

[9] *Essence of Linear Algebra* by 3blue1brown.

[10] Completed 17 lessons from *Introduction to Linear Algebra* Brilliant quiz *Advanced Math Level 4*: `https://brilliant.org/courses/linear-algebra/`.

[11] Lectures 14 to 23 from *MIT OCW Linear Algebra 18.06 Spring 2010* taught by Gilbert Strang: `https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/`.

2.2.4 Outreach

[12] Co-hosted a 8-session linear algebra course at Schoolhouse.world with 10 participants. Series page: `https://schoolhouse.world/series/56477`.

3 Abstract Algebra (Groups) Sophomore Fall

3.1 Concepts and Vocabulary

- **General group theory:** Motivation and definition of a group; order of a group and of an element; subgroup, familiarity with the following types of groups: abelian, cosets and normal subgroup, quotient group (e.g, such as $\mathbb{Z}/n\mathbb{Z}$), simple group, p-group, cyclic group.
- **Relations between groups:** Homomorphims (between two groups), isomorphism, automorphisms, endomorphisms; kernel and range of a homomorphism; the three isomorphism theorems.
- **Results:** Elementary theorems such as Lagrange's theorem; Cauchy's theorem as a partial converse of Lagrange's theorem, the Sylow theorems and their application to show results such as the structure theorem of finite abelian groups. Applications of these theorems in number theory, such as proving the Euler-Fermat theorem and the Chinese Remainder Theorem.
- **Group actions:** Definition and motivation of a group action; orbit, stabilizers and the orbit-stabalizer theorem; Burnside's lemma; applications of group actions to combinatorics.
- **Examples of groups:** Group of integers modulo n ($\mathbb{Z}/n\mathbb{Z}$), multiplicative group of integers modulo p ($(\mathbb{Z}/p\mathbb{Z})^\times$), permutation and alternating groups (S_n and A_n), general and special linear group of n by n matrices with entries in $\mathbb{R}$.

3.2 Resources

3.2.1 Books and Courses

[1] *Abstract Algebra* by Simon Rubinstein-Salzedo, Chapters 1 to 4.

[2] *Fundamentals of Mathematics* by Maxim Gilula, Chapters 1 to 4.

[3] Completed *Fundamentals of Mathematics Fall Quarter 2023* taught by Maxim Gilula from The Euler Circle. Included *6 lecture hours* and *8 hours of discussions*. All problem sets: `https://github.com/spacersid1331-design/math-docs/tree/main/abstract-algebra`.

3.2.2 Articles and Papers

[4] Conjugation in a Group by Keith Conrad.

[5] Proof of Cauchy's Theorem by Keith Conrad.

[6] On the Sylow Theorems by Keith Conrad.

3.2.3 Online resources

[7] Group actions by Andrew Misseldine: `https://www.youtube.com/watch?v=7fdJiMH8PUI`.

4 Real Analysis Sophomore Fall and Sophomore Winter

4.1 Concepts and Vocabulary:

- **Basic real analysis:** Properties of real numbers such as the existence of the infimum and supremum of a set; the notion of a function (injective, surjective, and bijective; maximizer and minimizer of a function) and of a sequence (monotonically increasing and decreasing, and subsequences) of real numbers; epsilon–delta definition of a limit of a function, limit of a sequence, and a Cauchy sequence; construction of the real numbers as equivalence classes of rational Cauchy sequences; definition of continuity and the derivative of a function (at a point), and using it to prove the sum, power, product, reciprocal, and quotient rules; definition of the integral of a function via Riemann sums (upper, lower, trapezoid, as well as terms such as the norm of a partition) and proof of the fundamental theorem of calculus via step-functions; theorems concerning functions including Rolle's theorem, Lagrange's mean value theorem, the extreme value theorem, and the intermediate value theorem (including its application in the root bisection algorithm).

- **Metric spaces:** Definition and motivation of a metric space; examples of metric spaces such as $\mathbb{R}$; definition of a limit of a sequence of points and of a Cauchy sequence in a general metric space; the generalized Bolzano–Weierstrass theorem in $\mathbb{R}^n$ (proof using epsilon nets); open and closed sets (including open and closed balls) and properties of open and closed sets (unions, intersections), along with the basic definition of a topological space; idea of a dense subset of a metric space, closure (of a set), and the notion of a complete metric space.

- **Relationship between metric spaces:** Functions between metric spaces; definition of continuity of a function at a point and uniform continuity (over a set); compactness, both in the open cover and sequential sense; the Heine–Borel theorem and its proof; the generalized extreme value theorem for a continuous function defined over a compact set; the contraction mapping theorem and its applications in solving equations such as $\cos(x) = x$ and other fixed-point equations (NOT for differential equations).

4.2 Resources

4.2.1 Books and Courses

[1] *Introductory Functional Analysis with Applications* by Erwin Kreyzig, parts of chapter one

[2] *A Slice of Analysis* from *Fundamentals of Mathematics* by Maxim Gilula.

[3] Chapter 1 and Chapter from *Applied Analysis* by John K Hunter and Bruno Nachtergaele.

4.2.2 Articles and Papers

[4] *Metric Spaces* by Keith Conrad.

[5] *Contraction Mapping Theorem* by Keith Conrad.

[6] *Adherent, Accumulation, and Isolated Points in Metric Spaces*.

4.2.3 Online Resources

[7] Stack exchange question that I posted on compactness and uniform continuity. Link: `https://bit.ly/4pCYKBX`.

4.2.4 Outreach

[8] 2023 Breakthrough Junior Challenge submission explaining integration using ∞. Ranked top 20% in the peer-to-peer review. Link: `https://www.youtube.com/watch?v=kkljcKRgh9k`.

5 Algebraic Geometry and Adjacent Fields Sophomore Summer

5.1 Concepts and Vocabulary

- **Ring Theory**: Definition of a ring; examples of rings including $\mathbb{Z}$ and $\mathbb{F}[x_1, \cdots, x_n]$; basic notions including zero-divisors, unit elements, subrings, ideals, prime elements, prime ideals, primary ideals, and maximal ideals; the notion of a ring homomorphism as an extension of a group homomorphism, and associated structures including the kernel, image, injectivity, surjectivity, and isomorphism, as well as related results such as the fact that the kernel of a map forms an ideal; types of rings including integral domains, principal ideal domains, Euclidean domains, and Noetherian rings; related results such as the Hilbert basis theorem; localization and local rings.

  Example problems: *Prove that $\mathbb{Z}[x]$ is not a principal domain*, and *Prove that if I and J are ideals of R with $I \subseteq J$, then J/I is an ideal of R/I, and $(R/I)/(J/I) \cong R/J$.*

- **Category Theory and Sheaves**: Definition of a presheaf (identity and glueability), pushforward of a sheaf, morphisms of presheaves and sheaves (kernel of a morphism of presheaves is a presheaf), properties determined at the level of stalks, and sheafification (isomorphisms are determined by stalks, compatible germs).

  Example problem: *Prove that a section of a sheaf of sets is determined by its germs.*

  Personal Reflection: *Imagine two villages, Blobwille and Stickwille. How do we decide which one deserves a railway station for transport? Both have the same population. And a similar number of car owners. It's a fierce competition!*

  Time to consult the map. Blobwille has a homogenous population density—no bulges. But Stickwille is different. Look at that rectangular piece poking out of Stickwille's stomach, lined with houses—we won't have to spend on an expansive bus infrastructure to motivate those people to ride the train!

  What just happened? That rectangular strip, packed with houses, was a specific neighborhood of Stickwille—a local property—but the outcome is a global one: The entire village of Stickwille gets a railway station.

  Isn't that a bit like Algebraic Geometry?

5.2 Resources

5.2.1 Books

[1] Part 3: Ring Theory and Algebraic Geometry from *Abstract Algebra* by Simon Rubinstein-Salzedo.

[2] Chapter 1 and Chapter 2 from *Foundations of Algebraic Geometry* by Ravi Vakil.

[3] *A Term of Commutative Algebra* by Allen Altman and Steven Kleiman.

6 Algebraic Number Theory Junior Spring to Present

6.1 Concepts and Vocabulary

- **Galois Theory**: Definition of a field; finite fields; a field extension (including a simple radical and an algebraic extension); motivation for normal and separable extensions; the tower law; algebraic elements and transcendental elements; minimal polynomials; definition and properties of the splitting field of a polynomial; algebraic closure of a field (including its construction); field embeddings and motivation for the automorphism group of a field extension; Galois conjugates; definition of a Galois extension; and the Galois correspondence.

- **Algebraic Number Theory**: Ring extensions and the definition of an integral element of a ring extension; computing the ring of integers for algebraic number fields such as $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer, and $\mathbb{Q}(\sqrt[3]{5})$; norm, trace, and the discriminant, as well as an integral basis for a number field; motivation for a principal ideal domain as a direct generalization of the integers (starting by translating divisibility of elements of a PID into divisibility of ideals); the fundamental question of elementary algebraic number theory, namely the failure of unique factorization in certain contexts; Kummer's idea of ideal numbers to recover unique factorization using ideals; definition of the class group as a device to measure the extent to which a domain fails to be a UFD; a more rigorous treatment including Dedekind domains as the setting of algebraic number theory, invertible ideals in Dedekind domains, and the various characterizations of a Dedekind domain (such as every integral domain with unique ideal factorization); valuations and their role in ideal factorization, a discrete valuation ring; localization and semilocal domains; ramification and splitting of primes in Dedekind extensions, including totally ramified, unramified, unramified at $\mathfrak{q}$, and unramified above $\mathfrak{p}$; intuition for Dirichlet's unit theorem.

- **Sum of Squares Project**: The Pythagoras number of a number ring R is the smallest number n such that every $x \in R$ can be expressed as a sum of n perfect squares in R. In the process of investigating the Pythagoras number for families of number rings in higher-degree number fields like Lehmer's cyclic quintic fields.

6.2 Resources

6.2.1 Books and Courses

[1] *Algebraic Number Theory* by Frazer Jarvis.

[2] *Galois Theory Through Exercises* by Juliusz Brzezinski.

[3] Lectures and Problem Sets $1-8$ from *OCW 18.785: Number Theory 1* by Andrew Sutherland.

6.2.2 Articles and Papers

[4] *Galois Correspondence* by Keith Conrad.

[5] *Simple Radical Extensions* by Keith Conrad.

[6] *Trace and Norm 1* by Keith Conrad.

[7] *Trace and Norm 2* by Keith Conrad.

[8] *Splitting Fields* by Keith Conrad.

[9] *Finite Fields* by Keith Conrad.

[10] *Factoring in Quadratic Fields* by Keith Conrad.

6.2.3 Online Resources

[11] *Galois Theory Playlist* (Part of MATH 250A Fall 2021) by Richard E Borcherds.

6.2.4 References for Sum of Squares Project

[12] *Indecomposable Totally Positive Numbers in Real Quadratic Orders* by Andreas Dress and Rudolf Scharlau.

[13] *Colmez Cones for Fundamental Units of Totally Real Cubic Fields* by Francisco Diaz y Diaz and Eduardo Friedman.

[14] *Universal Quadratic Forms, Small Norms and Traces in Families of Number Fields* by Vítězslav Kala and Magdaléna Tinková.

[15] *On the Pythagoras Number of the Simplest Cubic Fields* by Magdaléna Tinková.

[16] *Pythagoras Numbers of Orders in Biquadratic Fields* by Jakub Krasenský, Martin Raška, and Ester Sgallová.

[17] Bounds on the Pythagoras number and indecomposables in biquadratic fields by Magdaléna Tinková.

[18] *On biquadratic Fields: When 5 Squares are Not Enough* by Daniel Dombek.

[19] *Universal quadratic forms and indecomposables in number fields: A survey* by Vítězslav Kala.

6.2.5 Outreach

[20] Blog: From Integers to Rings and Ideals: An Introduction to Algebraic Number Theory. Link: `https://bit.ly/4qyrJYk`.

7 Differential Geometry <u>**Junior Fall**</u>

7.1 Concepts and Vocabulary

- **Curves:** Unit-speed, arc-length, reparameterization, curvature, torsion, normal and binormal, Frenet-Serret frame.
- **Surfaces:** Definition of surfaces, surface patches, and surfaces of revolution; the first fundamental form and how it relates to areas and lengths of curves; the Gauss map and Weingarten map, and their connection to the second fundamental form, normal and geodesic curvatures; the Gaussian curvature and mean curvature of a surface; geodesic equations.

7.2 Resources

7.2.1 Books

[1] *Elementary Differential Geometry* by Andrew Pressley.

8 Representation Theory Project

Inspired by the representation theory course[4] that I took at IISER, I decided to derive the representation theory of matrix groups independently, starting with $\mathrm{GL}_n(\mathbb{F}_p)$, the group of $n \times n$ matrices with non-zero determinant with entries in the finite field $\mathbb{F}_p$. In the process, I discovered subgroups such as $\mathcal{B}_n(\mathbb{F}_p)$, $\mathcal{U}_n(\mathbb{F}_p)$ and $\mathrm{PGL}_n(\mathbb{F}_p)$, using normalizers to construct more subgroups, which are used to induce representations to the main group. The highlight was discovering non-monomial representations. I discuss my progress with Dr. Chandrasheel Bhagwat[5] of the IISER Mathematics Department.

See my progress here: `https://www.overleaf.com/read/jbwstkwsjyvv#e5a12f`.

[4] Course description: `https://bit.ly/4b65SCT`.

[5] Dr. Bhagwat's Page: `https://sites.google.com/site/chandrasheelbhagwat/home?authuser=0`.

9 Euler Circle

- **Linear and Abstract Algebra** Fall 2023.
- **Independent Research and Paper Writing** Summer 2024.

  Paper: `https://simonrs.com/eulercircle/irpw2024/siddharth-fourier-paper.pdf`.
- **Combinatorial Game Theory** Fall 2024.

  Paper: `https://simonrs.com/eulercircle/cgt2024/siddharth-surreal.pdf`.
- **Complex Analysis** Winter and Spring 2025.

  Paper: `https://simonrs.com/eulercircle/complexanalysis2025/siddharth-modforms.pdf`.
- **Infinite Series** Winter 2026. To commence.

10 Exploration: Are Mirrors Perfect?

In my freshman year, I thought: *Can we prove that incident rays parallel to the principal axis of a spherical concave mirror converge to a single point?* This is something I had been taught since middle school, and I wanted to see it for myself. Using coordinate geometry, I got the expression

$$F = \frac{r^2}{2\sqrt{r^2 - b^2}},$$

where r is the mirror's radius of curvature and b is the distance between principal axis of the mirror and the incident ray. It was dependent on the incident ray—there was no single focus! That got me curious: *what was the perfect mirror?* See the full exploration into mirrors and differential equations here: `https://www.overleaf.com/read/trpvtpmpbbks#986ed1`.