7. More linear Transformations

(Q1)

(i) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (2 × 2 matrix)

(ii) $\begin{bmatrix} 8 & 7 \\ 5 & 4 \end{bmatrix}$ (2 × 2 matrix)

(iii) $\begin{bmatrix} 5 & 7 & 4 \\ 7 & 14 & 9 \\ 4 & 9 & 6 \end{bmatrix}$ (3 × 3 matrix)

(iv) T_B is defined by:

- $T(1,0,0) = (1,2,0)$
- $T(0,1,0) = (3,2,1)$
- $T(0,0,1) = (2,1,1)$

T_A is defined by:

- $T(1,0,0) = (1,3,2)$
- $T(0,1,0) = (2,2,1)$
- $T(0,0,1) = (0,1,1)$

Thus,

- $T_B(a,b,c) = a(1,2,0) + b(3,2,1) + c(2,1,1)$
- $T_A(a,b,c) = a(1,3,2) + b(2,2,1) + c(0,1,1)$

- $T_A(T_B(a,b,c)) = T_A((a,2a,0) + (3b,2b,b) + (2c,c,c))$

$$= T_A((a+3b+2c,\ 2a+2b+c,\ 0+b+c))$$

$$= T_A((a+3b+2c,\ 2a+2b+c,\ b+c))$$

$$= a+3b+2c\,(1,3,2) + (2a+2b+c)(2,2,1) + (b+c)(0,1,1)$$

$= (a+3b+2c,\ 3a+9b+6c,\ 2a+6b+4c)$
$+ (4a+4b+2c,\ 4a+4b+2c,\ 2a+2b+c)$
$+ (0,\ b+c,\ b+c)$

$= (5a+7b+4c,\ 7a+14b+9c,\ 4a+9b+6c)$
$= a(5,7,4) + b(7,14,9) + c(4,9,6)$
$= a(5,7,4) + b(7,14,9) + c(4,9,6)$
(this is exactly what the matrix multiplication results in.

(Q2) Note that the standard matrix for a θ counterclockwise rotation is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Substituting $-\theta$ in for θ, we get

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Thus, this matrix rotates the v θ clockwise about the origin, (Av is the rotated version of v).

(Q3)

(i) The coordinates of a vector $v \in V$ wrt to a basis $[v_i]_{i=n}^{n}$ is defined to be $(\alpha_1, \cdots, \alpha_n)$ such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Note that the coordinates of the vector v depends on the choice of the basis, different bases gives different coordinates.

Thus, 'we can think of' depends on the choice of the basis, since we're thinking about the coordinates of a vector.

(ii) This statement can be recasted in the language of coordinates, if B and W are bases for the vector space V and $v \in V$, then, for all $v \in V$

$$[v]_B = [v]_W \text{ iff } b_i = w_i \text{ for all } 1 \leq i \leq n$$

(Note: $B = [b_1, \cdots, b_n]$ and $W = [w_1, \cdots, w_n]$).

- Assume, that for all $v \in V$, we have: $[v]_B = [v]_W$. This implies;
  ~~$a_1v_1 + a_2v_2 + \cdots + a_nv_n =$~~
- $a_1b_1 + a_2b_2 + \cdots + a_nb_n = a_1w_1 + a_2w_2 + \cdots + a_nw_n$ for all collections of a_i's. (This follows from the fact that $\text{span}(W) = V = \text{span}(B)$ and that each $v \in V$ can be written uniquely as a linear combination of the basis elements).
- Now, set $a_j = 0$ if $j \neq i$ and $a_j = 1$ if $j = i$ for each i st $1 \leq i \leq n$. This shows that, $w_i = b_i$ from the above expression. Thus, the forward direction is complete.

- For the backward direction it is enough to note that each vector in V can be written uniquely as a linear combination of a set of basis elements.

(Q8) $A \sim B$ if and only if $A = PBP^{-1}$

(1) • $A \sim A$ as $A = IAI^{-1}$ (I is the $n \times n$ identity matrix).

(2) • If $A \sim B$, $\exists P : A = PBP^{-1}$

$A = PBP^{-1}$

$A(P) = (PBP^{-1})(P)$

$AP = PB(P^{-1}P)$

$AP = PBI \Rightarrow AP = PB$

$(P^{-1})(AP) = (P^{-1})(PB)$

$P^{-1}AP = (P^{-1}P)(B)$

$P^{-1}AP = B$

$(P^{-1})A(P^{-1})^{-1} = B$

Let $L = P^{-1}$, then

$B = LAL^{-1}$. Thus, $B \sim A$.

(3) • If $A \sim B$ and $B \sim C$, then:

$\exists P : A = PBP^{-1}$

$\exists L : B = LCL^{-1}$

$$A = PBP^{-1} = P(LCL^{-1})P^{-1}$$
$$= (PL)(C)(L^{-1}P^{-1})$$
$$= (PL)(C)(PL)^{-1}$$

Let $D = PL$, $A = DCD^{-1}$.

Thus, matrix similarity is a equivalence relation.

(Q9) Let $A \sim I$ for some $n \times n$ A.

$$A \sim I \Rightarrow \exists B : A = BIB^{-1}$$
$$A = BB^{-1}$$
$$A = I.$$

Thus, $A \sim I \Rightarrow A = I$.

(Q10) (Solution using eigenvalues).

$$A = \begin{bmatrix} 0 & 0.5 \\ 2 & 0 \end{bmatrix}$$

$$\text{char}(A) = \begin{vmatrix} -\lambda & 0.5 \\ 2 & -\lambda \end{vmatrix}$$

$$= (-\lambda)(-\lambda) - 2 \times 0.5$$

$$\text{char}(A) = \lambda^2 - 1$$

Eigenvalues are $\lambda = \pm 1$

→ $\lambda = 1$,

$$\begin{bmatrix} 0 & 0.5 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$0.5y = x \Rightarrow y = 2x$

$2x = y$

$2(0.5y) = y$

$y = y$ ✓

Thus, $\begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

→ $\lambda = -1$,

$$\begin{bmatrix} 0 & 0.5 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix}$$

$0.5y = -x$

$2x = -y, y = -2x$

Thus, $\begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

The eigenvectors corresponding to $\lambda = 1$ is $(1, 2)$ and the eigenvector corresponding to $\lambda = -1$ is $(1, -2)$

Thus, this linear transformation is described by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ wrt to the eigenbasis of $[(1,2),(1,-2)]$.

Thus, using the change of basis matrix,

$$\begin{bmatrix} 0 & 0.5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{det} \times \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{-4}\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0.5 \\ 2 & 0 \end{bmatrix}^{10} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \times \frac{1}{4}$$

$$= I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, $A^{10} = \underline{I}$.

(Q5)

- An $(m \times n)$ matrix multiplied by a $(n \times k)$ matrix is (by def) a $m \times k$ matrix. Thus, a $(n \times n)$ matrix multiplied by another $(n \times n)$ matrix is defined and is also an $(n \times n)$ matrix.

- (Multiplicative Identity). Note that $i_{(i,j)} = \delta_{ij}$ (by def of I_n).

Now, $(AI)_{i,j}$ = dot product of i^{th} row of A and j^{th} ~~row~~ column of I. Note that the j^{th} ~~row~~ column of I has nonzero (1) entry only at the j^{th} position. Thus, $(AI)_{i,j} = a_{ij}$

$$[a_{i1}, a_{i2}, \cdots, a_{in}] \cdot [0, 0, \cdots, 1, \cdots 0] = a_{ij}$$

$$\Rightarrow AI = A \quad (\text{since } (AI)_{i,j} = a_{i,j} \ \forall i,j : 1 \le i,j \le n)$$

- Inverses: A^{-1} exists iff $\det(A) \neq 0$.
- Commutativity: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+fc & eb+fd \\ ag+ch & bg+hd \end{bmatrix}$$

Thus, we have

- $ae + bg = ae + fc \Rightarrow bg = fc$
- $af + bh = eb + fd \Rightarrow 2.$
- $ag + ch = ce + dg \Rightarrow 2.$
- $cf + dh = bg + hd \Rightarrow cf = bg$