Modular Forms: E_2 and the Modular Discriminant

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Euler Circle

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Modular Forms: The Definition.

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- $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,

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What is the purpose of the so-called **modularity condition**? Why does $\mathrm{SL}_2(\mathbb{Z})$ show up? We'll take a step back and consider functions defined on lattices.

Homogeneous Functions on Lattices: Definition.

Recall the definition of a lattice, which generalizes the relationship of $\mathbb{Z}$ as a subset of $\mathbb{R}$.

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Let ω_1 and ω_2 be two complex numbers that are linearly independent over $\mathbb{R}$. The **lattice** generated by ω_1 and ω_2, denoted by $\Lambda(\omega_1, \omega_2)$, is defined by the set $\{n_1\omega_2 + n_2\omega_2 : n_i \in \mathbb{Z}\}$.

Remark

Notice that when we say ω_1 and ω_2 are *linearly independent* over $\mathbb{R}$, we mean $\omega_1/\omega_2 \notin \mathbb{R}$.

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Now, we consider a special class of functions $F : \mathcal{L} \to \mathbb{C}$, where $\mathcal{L}$ is the set of all lattices. Recall that we may scale lattices by a constant: $a\Lambda := \{a\lambda : \lambda \in \Lambda\}$.

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Definition

Let $k \in \mathbb{Z}$. A function $F : \mathcal{L} \to \mathbb{C}$ is said to be homogeneous of weight k if

$$F(\alpha\Lambda) = \alpha^{-k} F(\Lambda),$$

for all $\Lambda \in \mathcal{L}$ and $\alpha \in \mathbb{C}$.

Homogeneous Functions on Lattices: Unpacking Homogeneity.

Working with functions defined on sets is a bit clunky, so instead given such a F, we define G_F to be $G_F(\omega_1, \omega_2) := F(\Lambda(\omega_1, \omega_2))$ for all pairs of linearly independent complex numbers (ω_1, ω_2).

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The homogeneity of F implies that

$$G(\alpha\omega_1, \alpha\omega_2) = F(\Lambda(\alpha\omega_1, \alpha\omega_2)) = F(\alpha\Lambda(\omega_1, \omega_2)) = \alpha^k G(\omega_1, \omega_2).$$

Taking α to be ω_1^{-1}, we have $G(1, \omega_2/\omega_2) = \omega_1^k G(\omega_1, \omega_2) = F(\Lambda(1, \omega_2/\omega_1))$, so G is completely determined by where F sends lattices generated by $\{1, \tau\}$ for some $\tau \in \mathbb{C}$.

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This allows us to consider a uni-variate function $f(\tau) := G(1, \tau) = F(\Lambda(1, \tau))$. However, there are constraints on f: given two complex numbers $\tau_1 \neq \tau_2$, we may have $\Lambda(1, \tau_1) = \Lambda(1, \tau_2)$, meaning we have $f(\tau_1) = f(\tau_2)$. And when does this happen?

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Proposition

Two pairs $(1, \tau_1)$ and $(1, \tau_2)$ are such that $\Lambda(1, \tau_1) = \Lambda(1, \tau_2)$ if and only if there exists integers a, b, c and d such that $ad - bc = 1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \tau_2 \\ 1 \end{pmatrix}.$$

With this in mind, we have

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With this in mind, we have

$$\begin{aligned} f(\tau_1) = F(\Lambda(1,\tau_1)) &= F(\Lambda(c\tau_1 + d, a\tau_1 + b)) \\ &= F\left((c\tau_1 + d)\Lambda\left(1, \frac{a\tau_1 + b}{c\tau_1 + d}\right)\right) \\ &= (c\tau_1 + c)^k F\left(\Lambda\left(1, \frac{a\tau_1 + b}{c\tau_1 + d}\right)\right) \\ &= (c\tau_1 + d)^k f\left(\frac{a\tau_1 + b}{c\tau_1 + d}\right), \end{aligned}$$

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meaning that f satisfies the modularity condition!

Creating Modular Forms Using the Weierstrass $\wp$ Function.

Definition

Let ω_1 and ω_2 be two complex numbers linearly independent over $\mathbb{R}$. Then, the Weierstrass elliptic function corresponding to the lattice $\Lambda(\omega_1, \omega_2)$ generated by ω_1 and ω_2, denoted by $\wp_{\Lambda(\omega_1,\omega_2)}$, is defined by

$$\wp_{\Lambda(\omega_1,\omega_2)}(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda(\omega_1,\omega_2)\setminus\{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right),$$

for all $z \in \mathbb{C} - \Lambda(\omega_1, \omega_2)$.

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Out of the many relationships satisfied by the Weierstrass elliptic function, one of them is that $\wp_{\Lambda(\alpha\omega_1,\alpha\omega_2)}(\alpha z) = \alpha^{-2}\wp_{\Lambda(\omega_1,\omega_2)}(z)$ for all $\alpha \in \mathbb{C}^*$.

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Recall that the Laurent series expansion for $\wp_{\Lambda(\omega_1,\omega_2)}$ about $z = 0$ (where it has a double pole, as at every other lattice point), has the form

$$\frac{1}{z^2} + \sum_{k=1}^{\infty} c_{2k}(\omega_1, \omega_2) z^{2k},$$

where c_{2k} is a complex-valued function that takes as input two linearly independent complex numbers for each $k \in \mathbb{N}$.

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$$\begin{aligned}
\wp_{\Lambda(\alpha\omega_1,\alpha\omega_2)}(\alpha z) &= \frac{1}{(\alpha z)^2} + \sum_{k=1}^{\infty} c_{2k}(\alpha\omega_1,\alpha\omega_2)(\alpha z)^{2k} \\
&= \alpha^{-2}\left(\frac{1}{z^2} + \sum_{k=1}^{\infty} \alpha^{2k+2} c_{2k}(\alpha\omega_1,\alpha\omega_2) z^{2k}\right) \\
&= \alpha^{-2}\left(\frac{1}{z^2} + \sum_{k=1}^{\infty} c_{2k}(\omega_1,\omega_2) z^{2k}\right),
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meaning that $c_{2k+2}(\alpha\omega_1,\alpha\omega_2) = \alpha^{-(2k+2)} c_{2k}(\omega_1,\omega_2)$ for all $\alpha \in \mathbb{C}^*$.

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Remark

We have that c_{2k} is a homogeneous function on lattices of weight $2k+2$. From our previous work, we know that the function defined by $f_{2k}(\tau) = c_{2k}(1,\tau)$ for all $\tau \in \mathbb{H}$ is a modular form of weight $2k+2$!

Computing these coefficients, we get the following.

Proposition

Using the notation above, we have that $c_{2k}(\omega_1, \omega_2) = (2k+1)E_{2k+2}(\omega_1, \omega_2)$, *where*

$$E_n(\omega_1, \omega_2) := \sum_{\lambda \in \Lambda(\omega_1, \omega_2) - \{0\}} \frac{1}{\lambda^k},$$

for all $n \geq 3$.

Due to time constraints, we will not prove this here.

Definition

Let $k \geq 4$ be even. Define E_k, called the Eisenstein series of weight k, as a complex valued function define on the upper half plane

$$E_k(\tau) := \sum_{\omega \in \Lambda(1, \tau) - \{0\}} \frac{1}{\omega^k},$$

for all $\tau \in \mathbb{H}$.

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To prove this, we need a lemma.

Lemma

For each $\tau \in \mathbb{H}$, there is a $\delta_\tau \in (0,1)$ such that

$$|m\tau + n| \geq \delta_\tau |mi + n|,$$

for all integers m and n.

Proof.

We may assume that $m \neq 0$, for if $m = 0$, then one can take δ_τ to be any element of $(0, 1)$ for all τ. Dividing both sides by $|m|$ and then $|i + n/m|$, our problem is now equivalent to finding a δ_τ such that

$$\left|\frac{\tau + n/m}{i + n/m}\right| \geq \delta_\tau,$$

for all integers n and $m \neq 0$.

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$$\left|\frac{\tau + n/m}{i + n/m}\right| \geq \delta_\tau,$$

for all integers n and $m \neq 0$. Taking cue from the above expression, define the real valued function $f_\tau(x) = \left|\frac{\tau - x}{i - x}\right|$ for all real numbers x. Note that f is continuous at all $x \in \mathbb{R}$, and as $x \to \pm\infty$, we have $f_\tau(x) \to 1$. Moreover, f_τ maps to the positive real numbers. This implies that there exists a $R_\tau > 0$ such that for $|x| > R_\tau$, we have $f_\tau(x) \geq 1/2$. On the other hand, as the image of a compact set under a continuous function is compact, there exists a $c(\tau) > 0$ such that $f_\tau(x) \geq c(\tau)$ for all $x \in [-R_\tau, R_\tau]$. Therefore, for all $x \in \mathbb{R}$, we have $f_\tau(x) \geq \delta$ where $\delta = \min\{c(\tau), 1/2\}$, as desired. ■

Eisenstein Series as a Modular Form: Convergence.

To finish off the proof the absolute convergence of the Eisenstein series, we use the following general result regarding the convergence of general lattice sums.

Proposition

Let d be a positive integer. For $s > 0$, the infinite series $\sum_{\mathbf{a} \in \mathbb{Z}^d - \{\mathbf{0}\}} \frac{1}{\|\mathbf{a}\|^s}$ converges for $s > d$ and diverges for $0 < s \leq d$.

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Now, we prove the absolute convergence of G_k.

Proof.

We must show that $\sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{|m\tau + n|^k}$ converges for $k \geq 4$. By the previous lemma, we may choose a $\delta_\tau \in (0, 1)$ such that $|m\tau + n|^k \geq \delta_\tau^k |mi + n|^k$ and so $\frac{1}{|m\tau+n|^k} \leq \frac{1}{\delta_\tau^k |mi+n|^k}$, so $\sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{|m\tau+n|^k} \leq \delta_\tau^{-k} \sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{|mi+n|^k} = \delta_\tau^{-k} \sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(m^2+n^2)^{k/2}}$. From the previous theorem, we know that as $k > 2$, $\sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(m^2+n^2)^{k/2}}$ converges, which completes the proof. ■

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Remark

The modularity condition is equivalent to

- $f(\tau + 1) = f(\tau)$,
- $f(-1/\tau) = \tau^k f(\tau)$.

This is because the two maps $\tau \mapsto \tau + 1$ and $\tau \mapsto -\tau^{-1}$ generate all of the transformation that correspond to $\mathrm{SL}_2(\mathbb{Z})$.

The Weierstrass $\wp$ Function Again: Definition of Δ.

Recall from class that $\wp(z)$ satisfies the following differential equation

$$(\wp'_{\Lambda(1,\tau)}(z))^2 = 4(\wp_{\Lambda(1,\tau)}(z))^3 - 60E_4(\tau)\wp_{\Lambda(1,\tau)} - 140E_6(\tau).$$

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- Notice this differential equation is of the form of a elliptic curve over $\mathbb{C}$: $y^2 = x^3 + ax + b$, where $(x, y) = (\wp_{\Lambda(1,\tau)}(z), \wp'_{\Lambda(1,\tau)}(z))$, $a = -15E_4(\tau)$ and $b = -35E_6(\tau)$.
- Indeed, this effectively sets up a bijection between elliptic curves and elliptic functions.
- The discriminant of a general elliptic curve is $\Delta_{\text{disc}} = 4a^3 + 27b^2$, and $\Delta_{\text{disc}} = 0$ if and only if the cubic equation $x^3 + ax + b = 0$ has repeated roots.

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Definition

Define the modular discriminant, denoted by Δ, as a complex valued function defined on the upper half plane by

$$\Delta(\tau) = (-60G_4(\tau))^3 - 27(140G_6(\tau))^2 = -216000G_4^3(\tau) + 529200G_6^2(\tau) = 16\Delta_{\text{disc}}(\tau).$$

In terms of the normalized Eistenstin series, we have

$$\Delta(\tau) = \frac{1}{1728}(E_4^3 - E_6^2).$$

The Modular Discriminant.

Remark

From the properties of $\wp'_{\Lambda(1,\tau)}(z)$, we must have that $\Delta(\tau)$ is non-zero on its domain.

A key identity satisfied by Δ has to do with infinite products.

Theorem

Δ is a modular form of weight 12.

Theorem

We have that

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

where $q = e^{2\pi i \tau}$.

We will not prove this here, but see my paper.

What can we do assuming the product representation for Δ?
Taking the logarithmic derivative, we get

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and so

$$\frac{d}{d\tau}(\log(\Delta(\tau))) = 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq}{1 - q^n}\right).$$

By writing $\sum_{n=1}^{\infty} \frac{nq}{1-q^n}$ as $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nq^{mn}$ and switching the order of summation, we get that $\sum_{n=1}^{\infty} \frac{nq}{1-q^n} = \sum_{n=1}^{\infty} \sigma_1(n) q^n$,

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$$\frac{\Delta'(\tau)}{\Delta(\tau)} = 2\pi i\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right) = 2\pi i E_2(\tau).$$

The Connection Between G_2 and Δ.

As Δ is a modular form of weight 12, we have

$$\Delta(-1/\tau) = \tau^{12}\Delta(\tau).$$

Differentiating both sides with respect to τ, we get

$$\Delta'(-1/\tau) = 12t^{13}\Delta(\tau) + t^{14}\Delta'(\tau).$$

Then,

$$E_2(-1/\tau) = \frac{\Delta'(-1/\tau)}{2\pi i\Delta(-1/\tau)} = \frac{12t^{13}\Delta(\tau) + t^{14}\Delta'(\tau)}{2\pi i(\tau^{12}\Delta(\tau))} = \frac{6\tau}{\pi i} + \tau^2 E_2(\tau).$$

All in all,

$$E_2(-1/\tau) = \frac{6\tau}{\pi i} + \tau^2 E_2(\tau),$$

meaning E_2 is almost a modular form!

Remark

This property of E_2 is used to create a differential operator, which takes a modular form of weight k to one with weight $k+2$, called the Serre operator.