Higher dimensional semantics of axiomatic dependent type theory

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Abstract

Axiomatic type theory is a dependent type theory without computation rules. The term equality judgements that usually characterise these rules are replaced by computation *axioms*, i.e., additional term judgements that are typed by identity types. This paper is devoted to providing an effective description of its semantics, from a higher categorical perspective: given the challenge of encoding intensional type formers into 1-dimensional categorical terms and properties, a challenge that persists even for axiomatic type formers, we adopt Richard Garner's approach in the 2-dimensional study of dependent types. We prove that the type formers of axiomatic theories can be encoded into natural 2-dimensional category theoretic data, obtaining a presentation of the semantics of axiomatic type theory via 2-categorical models called *display map 2-categories*. In the axiomatic case, the 2-categorical requirements identified by Garner for interpreting intensional type formers are relaxed. Therefore, we obtain a presentation of the semantics of the axiomatic theory that generalises Garner's one for the intensional case. Our main result states that the interpretation of axiomatic theories within display map 2-categories is well-defined and enjoys the soundness property. We use this fact to provide a semantic proof that the computation rule of intensional identity types is not admissible in axiomatic type theory. This is achieved via a revisitation of Hofmann and Streicher's *groupoid model* that believes axiomatic identity types but does not believe intensional ones.

Contents

1 Introduction and motivation

This paper presents an analysis of the semantics of dependent types from a categorical perspective, with the aim of applying it to the study of axiomatic type theories.

1.1 The problem of the semantics of intensional type formers

In formulating the semantics of theories of dependent types, we typically encounter two distinct, but related, approaches: *syntactic* and *categorical*. The syntactic approach directly mirrors the structure of the syntax of the theory, defining how judgements in the conclusions of inference rules are interpreted based on the interpretations assigned to the premises—in other words, we equip a model with a choice function for every inference rule: a function that chooses, for the interpretation of every instance of the premises, an interpretation of the corresponding instance of the conclusion. On the other hand, the categorical approach adds structure to the model, structure that allows to automatically recover choice functions analogous to those provided in the syntactic approach.

For example, given a display map category $(\mathbf{C}, \mathcal{D})$—we will recall this notion later—consider the case of extensional $=$-types. As we mentioned, to model a theory featuring this type former we may just replicate its syntax within the "language" of $(\mathbf{C}, \mathcal{D})$, according to which a *display map* is the interpretation of a type judgement $A : \textsc{Type}$ and its *sections* interpret the corresponding term judgements $t : A$. Hence, a display map interpreting a given type $A : \textsc{Type}$ is required to be endowed with the choice of another display map interpreting the identity type $\lfloor x, x' : A \rfloor \; x = x' : \textsc{Type}$; in other words, the display map category $(\mathbf{C}, \mathcal{D})$ is equipped with a choice function validating the *formation rule* of extensional $=$-types. Analogously, additional choice functions will validate the other rules of this type former. But alternatively, as explained in [27], we can simply require that the diagonal arrow $\Gamma.A \to \Gamma.A.A[P_A]$ is a display map whenever P_A is a display map $\Gamma.A \to \Gamma$ in $\mathcal{D}$, because this categorical requirement on $(\mathbf{C}, \mathcal{D})$ is equivalent to equipping $(\mathbf{C}, \mathcal{D})$ with the necessary choice functions to validate the extensional $=$-type former. To provide the reader with another example, if we add Σ-types to the theory, then the categorical condition characterising its semantics is the following: it stipulates that display maps must be closed under composition up to isomorphism. This categorical characterisation of Σ-types is the basis of the analysis contained in [36] and [21].

These are examples of how extensional type formers admit a concise categorical characterisation of their models: it consists of *closure properties on the class of their display maps*. However, in case we drop extensional identity types and we ask ourselves how to handle $=$- and Σ-types, things become more complicated, and such a categorical characterisation of their semantics is harder to find. To simplify the problem and recover a categorical characterisation of the semantics of intensional type formers, Garner proposes a 2-categorical formulation in [17]. In this scenario, a synthetic and conceptually simple characterisation of identities, dependent sums, and other type constructors *exists*. This stems from the wider range of potential closure properties—whether strong or weak—to require on the class $\mathcal{D}$, that we can access: properties that enable us to distinguish between varying strengths of type formers. Notably, if dimension 2 were omitted, these properties would collapse into the ones we have just discussed, recovering the extensional version of our type constructors.

In this paper, we adopt Garner's perspective to study the semantics of *axiomatic type theory* from a categorical point of view, specifically through a 2-categorical one.

1.2 Axiomatic dependent type theory

In recent years, there has been a growing interest in various weakenings for theories of dependent types, particularly those weakenings with respect to the strength of the computation rules of the type formers. A type former is meant to encode a piece of logic—e.g. the rules of existential quantification for a notion of Σ-type former—into the *propositions-as-types* formalism of dependent type theories. It is the ensemble of a *formation* rule, an *introduction* rule, and an *elimination* rule, together with one or more *computation* rules that explain the interaction between terms obtained by introducing and eliminating, and that allow one to simplify or rephrase proofs. In both extensional and intensional type theory (ETT and ITT respectively), computation rules consist of term equality judgements, of the form $t \equiv t'$, where $t : A$ and $t' : A$ for some type judgement $A : \textsc{Type}$. However, term equality judgements are not the only available way to state that "two terms of a given type are equal". If a theory has some flavour of the *identity type* former, one may judge a specific term judgement $p : t = t'$, typed by the *identity type* $t = t' : \textsc{Type}$, to state that "there is a proof that the two terms $t : A$ and $t' : A$ are equal". This leads us to a *weaker* equality judgement, expressed as $p : t = t'$. We say that $t \equiv t'$ is a *judgemental* equality, because it consists of a term equality judgement, and that $p : t = t'$ is a *propositional* equality, because the judgement $t = t' : \textsc{Type}$—unlike the judgement $t \equiv t'$—is a type, hence an actual statement or proposition, and p is a proof, or a witness, that "the statement $t = t'$ holds".

Judgemental equalities are stronger than propositional equalities because of $=$-introduction rule. Propositional equalities are as strong as judgemental equalities if $=$-types are extensional. On the other hand, propositional equalities are actually weaker than judgemental equalities if $=$-types are intensional: the groupoid model [24, 25, 40] is a model of intensional identity types that does not believe in the uniqueness of identity proof, whereas every extensional model will do so. Hence the groupoid model does not have extensional identity types and therefore if will not necessarily believe that $t \equiv t'$ whenever the type $t = t'$ is inhabited. As we are going to see, propositional equalities become even weaker if $=$-types are just *axiomatic*.

Axiomatic $=$-types are "very intensional" $=$-types. They consist of the usual rules of intensional $=$-types, except for the judgemental equality of its computation rule, *which is replaced by a propositional equality*: whenever we are given judgements:

$$\lfloor x, x' : A;\ p : x = x' \rfloor\ C(x, x', p) : \textsc{Type} \qquad \text{and} \qquad \lfloor x : A \rfloor\ c(x) : C(x, x, r(x))$$

and hence, by elimination, a term judgement $\lfloor x, x' : A;\ p : x = x' \rfloor\ \mathsf{J}(c, x, x', p) : C(x, x', p)$, then, in place of asking that the term equality judgement $\lfloor x : A \rfloor\ \mathsf{J}(c, x, x, r(x)) \equiv c(x)$ holds, we only ask that an additional term judgement of the form:

$$\lfloor x : A \rfloor\ \mathsf{H}(c, x) : \mathsf{J}(c, x, x, r(x)) = c(x)$$

holds. This term judgement replacing the computation rule is called *computation axiom*: it requires that the statement $\mathsf{J}(c, x, x, r(x)) = c(x)$ holds. This is depicted in detail in Fig. 1.

In general, when a dependent type theory has a type former whose computation rules are replaced by computation axioms, one says that the type former is in *axiomatic* form. This paper is about semantics of *axiomatic type theory* (ATT), a weakening of ITT with computation

$$\text{Form Rule}\ \frac{A : \textsc{Type}}{\lfloor x, x' : A \rfloor\ x = x' : \textsc{Type}} \qquad \text{Elim Rule}\ \frac{\begin{array}{c} A : \textsc{Type} \\ \lfloor x, x';\ p : x = x' \rfloor\ C(x, x', p) : \textsc{Type} \\ \lfloor x \rfloor\ c(x) : C(x, x, \mathsf{r}(x)) \end{array}}{\lfloor x, x';\ p \rfloor\ \mathsf{J}(c, x, x', p) : C(x, x', p)}$$

$$\text{Intro Rule}\ \frac{A : \textsc{Type}}{\lfloor x : A \rfloor\ \mathsf{r}(x) : x = x} \qquad \text{Comp Axiom}\ \frac{\begin{array}{c} A : \textsc{Type} \\ \lfloor x, x';\ p : x = x' \rfloor\ C(x, x', p) : \textsc{Type} \\ \lfloor x \rfloor\ c(x) : C(x, x, \mathsf{r}(x)) \end{array}}{\lfloor x \rfloor\ \mathsf{H}(c, x) : \mathsf{J}(c, x, x, \mathsf{r}(x)) = c(x)}$$

Figure 1: Axiomatic $=$-types

axioms. We will be focusing on two of its type formers, i.e. $=$-types and Σ-types, leaving the analysis of the others for an extended version.

There are several motivations and advantages to work on ATT:

$\succ$ Axiomatic type theory is *objective* [35, 43] in the way operations, such as the sum of natural numbers or the composition of identity proofs, can be defined. In ITT, depending on how the elimination rule of these type formers is applied, will satisfy either $n + 0 \equiv n$ and $0 + n = n$ or alternatively $n + 0 = n$ and $0 + n \equiv n$, if n is a natural number—and similarly for composition of identity proofs. In ATT, such an ambiguity is removed from the outset: every notion of natural sum and every notion of identity proof composition will satisfy all these equalities just in propositional form.

At the same time, while it only has one notion of equality like ETT, axiomatic type theory does not deduce the uniqueness of identity proofs, and it is therefore compatible with homotopy type theory.

$\succ$ Axiomatic type formers are *homotopy invariant*, as explained in [5, 6]. Whenever a type is homotopy equivalent to one derived from the formation rule of a given type former, it adheres to all the rules associated with that type former.

$\succ$ As shown in [43], by replacing computation rules with computation axioms, the *type checking*, i.e. the decidability of the derivability of a term judgement $t : A$, holds and can be done in quadratic time. This improves the non-elementary one of ordinary ITT.

$\succ$ Compared to ITT, axiomatic type theory enjoy a *broader concept of semantics*, in the sense that it has more models—every model of ITT is also a model of ATT but, as we show in the last section, there are models of the latter that do not validate computation rules—making it simpler—in practice there are less constraints regarding judgemental equalities to be satisfied and checked—to construct concrete models, for example, to obtain independence results.

Nevertheless, ATT is as expressive as ITT and encodes the whole intuitionistic predicate logic.

$\succ$ Accordingly, several *conservativity results* of intensional and extensional type theories over the axiomatic one hold: for example, ETT extended with function extensionality and uniqueness of identity proof rules is conservative over ATT [46, 37]. Building on Hofmann's work [22], this topic has been studied extensively by Bocquet [9], by Boulier and Winterhalter [11, 46], and by Spadetto [37], showing how the axiomatic theory does not lose much deductive strength with respect to the intensional and extensional ones. Therefore, one may to some extent interchangeably use axiomatic type formers in order to study intensional and extensional ones, and vice versa.

1.3 Related work

Hints of the emergence of an interest in propositional computation rules, or computation axioms, are scattered everywhere in the homotopy type theoretic literature, starting from the work of Awodey, Gambino, Sojakova [5, 6], that presented an initial study of an axiomatic type former, namely axiomatic well-founded tree types, or axiomatic W-types. Coquand and others [15, 14] conducted initial analyses related to axiomatic $=$-types. Another recent paper introduced a univalent model of ITT where the computation axiom for identity types is validated, although its judgemental version is not [8]. Following this, the type constructor has been thoroughly examined by van den Berg and Moerdijk [42, 44], who introduced and explored a semantic concept for dependent type theories with propositional identity types, using the notion of a *path category* and showing its link with axiomatic $=$-types. Otten and Spadetto [34] explain how this notion of semantics is actually an instance of the usual one via display map categories (formulated as full comprehension categories). Among other works on axiomatic type formers, we note the previously mentioned studies on conservativity over such type theories by Bocquet [9], by Boulier and Winterhalter [11], and by Spadetto [37], Bocquet's work on the coherence property of the class of models of axiomatic type formers [10], and Vidmar's work on extending natural models [4] to type theories with propositional expansion rules, or expansion axioms [45].

1.4 Contributions

In this paper, we adopt Garner's perspective [17] to study the semantics of ATT from a categorical point of view, specifically from a 2-categorical one. In detail, we prove that a *display map 2-category* with some categorical structure—obtained reformulating the syntax of the axiomatic type formers in 2-categorical terms—is sufficient to recover the semantic counterparts of these type constructors as in the *syntactic approach*, particularly inducing an ordinary (split) display map category—i.e. an actual model of the structural rules of dependent type theory—that models the theory of dependent types in axiomatic form—see Section 4 and Theorem 4.9. This provides an answer to the problem of formulating models of ATT in categorical terms, i.e. within the *categorical approach*, in a way that includes and generalises the class of intensional models identified by Garner: display maps are not necessarily normal isofibrations, as in the intensional case, but merely cloven isofibrations and the axiomatic form of Σ-types does not require that display maps be closed under composition up to injective equivalence, as for intensional models, but merely up to homotopy equivalence.

In broad terms, this work falls within the field of categorical semantics of dependent type theories, and aims to contribute to the central objective of developing a general and feasible notion of semantics for variants and generalisations of dependent type theory.

1.5 Outline

In Section 2 we recall the notion of a (split) display map category and briefly explain how such a structure provides a model of the structural rules of dependent type theory. In Section 3 we specialise this notion to model ATT, using the syntactic approach. In Section 4, we formulate the notion of a display map 2-category as a structure that encodes the structural and logical aspects of ATT as 2-categorical data on display maps. Accordingly, we show that fulfilling such data allows a display map 2-category to induce a model of ATT as described in Section 3. In Section 5, we use this result to identify a model of ATT—based on the groupoid model—that is not a model of ITT. In Section 7, we discuss the position of this work within contemporary research on the categorical semantics of generalised dependent type theory.

2 Recap on display map categories

In this section, we recall the notion of display map categories [41, 27, 32] one of several equivalent categorical structures that provide a sound and complete semantics for dependent type theories. We specialise it to provide such a notion of semantics for ATT with $=$-types and Σ-types. We leave the semantics of the other axiomatic type formers for an extended version.

Definition 2.1. A *(split) display map category* $(\mathbf{C}, \mathcal{D})$ is a category $\mathbf{C}$ with a chosen terminal object 1, together with a class $\mathcal{D}$ of arrows of $\mathbf{C}$, called *display maps* and denoted as $\Gamma.A \to \Gamma$ and labeled as P_A, such that:

- for every display map $\Gamma.A \to \Gamma$ and every arrow $f : \Delta \to \Gamma$, there is a choice of a display map $\Delta.A[f] \to \Delta$ such that the square:

$$\begin{tikzcd} \Delta.A[f] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma.A \arrow[d] \\ \Delta \arrow[r, "f"] & \Gamma \end{tikzcd}$$

  is a pullback; as the former display map is labeled as P_A, the latter will be labeled as $P_{A[f]}$; the arrow $\Delta.A[f] \to \Gamma.A$ will be denoted as $f.A$, or as $f^\bullet$ in absence of ambiguity; we will adopt the notation $A^\blacktriangledown$ in place of $A[P_A]$;

- the equalities:

$$\begin{aligned} P_{A[1_A]} &= P_A & (1_\Gamma).A &= 1_{\Gamma.A} \\ P_{A[fg]} &= P_{A[f][g]} & (f.A)(g.A[f]) &= (fg).A \end{aligned}$$

  hold for every choice of composable arrows f and g and every display map $\Gamma.A \to \Gamma$, where Γ is the target of f.

In this paper, the term "display map category" specifically refers to a "*split* display map category". That is, we always assume that the display map categories we discuss satisfy the second of the two conditions in Definition 2.1. Under this assumption, the notion of a display map category is entirely equivalent to other notions commonly used as models of dependent type theories, such as categories with attributes [12, 31, 28], categories with families [13, 16, 20, 23], and full split comprehension categories [26, 30].

Such a display map category $(\mathbf{C}, \mathcal{D})$ constitutes a sound model of the structural part of dependent type theory: as we mentioned in Section 1, a context is encoded as an object Γ of $\mathbf{C}$, a type in that context is interpreted as a display map P_A of codomain Γ, and a term of that type as a section of P_A. Then we can observe e.g. that the fundamental structural rules:

$$\frac{}{\lfloor \_ \rfloor} \qquad \frac{\lfloor \gamma : \Gamma \rfloor \ A(\gamma) : \textsc{Type}}{\lfloor \gamma : \Gamma, x : A(\gamma) \rfloor} \qquad \frac{\lfloor \gamma : \Gamma \rfloor \ A(\gamma) : \textsc{Type}}{\lfloor \gamma : \Gamma, x : A(\gamma) \rfloor \ x : A(\gamma)}$$

are validated: the terminal object 1 interprets the *empty context* _ and, if the object Γ is the interpretation of the context γ and the display map $P_A : \Gamma.A \to \Gamma$ is the interpretation of the type-in-context A, then the object $\Gamma.A$ interprets the *extended context* γ, x and the unique section $\delta_A : \Gamma.A \to \Gamma.A.A^\blacktriangledown$ of $P_{A^\blacktriangledown}$ such that the diagram:

$$\begin{tikzcd} \Gamma.A \arrow[d] \arrow[dr, equal] & \\ \Gamma.A.A^\blacktriangledown \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma.A \arrow[d] \\ \Gamma.A \arrow[r, "P_A"] & \Gamma \end{tikzcd}$$

commutes interprets the *variable term-in-context* x. Moreover, substitution—and hence weakening—is interpreted by using the cleavage of $(\mathbf{C}, \mathcal{D})$: if the section $a : \Gamma \to \Gamma.A$ of P_A is the interpretation of a term judgement $\lfloor \gamma : \Gamma \rfloor\ a(\gamma) : A(\gamma)$ and the arrow $f : \Delta \to \Gamma$ is the interpretation of a substitution $\lfloor \delta : \Delta \rfloor\ f(\delta) : \Gamma$, then the judgements:

$$\lfloor \delta : \Delta \rfloor\ A(f(\delta)) : \textsc{Type} \quad \text{and} \quad \lfloor \delta : \Delta \rfloor\ a(f(\delta)) : A(f(\delta))$$

will be interpreted by the display map $P_{A[f]}$ and by its unique section $a[f] : \Delta \to \Delta.A[f]$ such that the square:

$$\begin{tikzcd}
\Delta \arrow[r, "f"] \arrow[d, "{a[f]}"'] & \Gamma \arrow[d, "a"] \\
\Delta.A[f] \arrow[r, "f^{\bullet}"] & \Gamma.A
\end{tikzcd}$$

commutes, respectively.

2.1 Encoding parallel terms into a substitution

If a and b are sections of a display map $P_A : \Gamma.A \to \Gamma$, we denote as $a; b$ the unique arrow $\Gamma \to \Gamma.A.A^{\blacktriangledown}$ such that the diagram:

$$\begin{tikzcd}
\Gamma \arrow[dr] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\
& \Gamma.A.A^{\blacktriangledown} \arrow[r, "P_A^{\bullet}"] \arrow[d, "P_{A^{\blacktriangledown}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d] \\
& \Gamma.A \arrow[r] & \Gamma
\end{tikzcd}$$

commutes. If a and b interpret given terms $a(\gamma) : A(\gamma)$ and $b(\gamma) : A(\gamma)$ in context $\gamma : \Gamma$, then the substitution $\lfloor \gamma : \Gamma \rfloor\ \gamma, a(\gamma), b(\gamma) : \Gamma.A.A^{\blacktriangledown}$ is interpreted by the arrow $a; b$.

We recall that the square:

$$\begin{tikzcd}
\Delta \arrow[r, "f"] \arrow[d, "{a[f];b[f]}"'] & \Gamma \arrow[d, "a;b"] \\
\Delta.A[f].A[f]^{\blacktriangledown} \arrow[r, "f^{\bullet\bullet}"] & \Gamma.A.A^{\blacktriangledown}
\end{tikzcd}$$

commutes for every arrow $f : \Delta \to \Gamma$. This fact is crucial to prove that the stability conditions relative to H_c and σ_c type-check, in Definitions 3.1 and 3.2 respectively. Hence, we recall a proof of this fact in Appendix A.

In the next section we show how to use display map categories to model axiomatic type theory via the syntactic approach.

3 Syntactic approach to the semantics of axiomatic type theory

In this section we specialise the notion of display map categories to make them into models of axiomatic type theory. This consists in endowing a display map category with additional structure constituting the semantic counterpart of the type former. As we mentioned in Section 1, this additional structure may be formulated in alignment with the syntax, by means of choice functions, each associated to a given rule of the theory, assigning to the interpretation of the premises of that rule an interpretation of its consequence. These choice functions constitute an encoding of type formers into the given display map categories. If a display map category $(\mathbf{C}, \mathcal{D})$ is endowed with such a function for every logical rule in the theory, then the interpretation in $(\mathbf{C}, \mathcal{D})$ of every type judgement and every term judgement of the theory is defined and respects all the type equality judgements and all the term equality judgements of the theory. In other words, the interpretation of the theory in $(\mathbf{C}, \mathcal{D})$ is defined and is *sound*.

Referring to the rules of Fig. 1 for axiomatic =-types, we give the following:

Definition 3.1 (Semantics of axiomatic =-types—syntactic formulation)**.** Let $(\mathbf{C}, \mathcal{D})$ be a display map category.

Let us assume that, for every object Γ and every display map P_A of codomain Γ, there is a choice of:

- $\succ$ (*Form Rule*) a display map $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \to \Gamma.A.A^{\blacktriangledown}$;
- $\succ$ (*Intro Rule*) a section:
$$\mathsf{refl}_A : \Gamma.A \to \Gamma.A.\mathsf{Id}_A[\delta_A]$$
of the display map $\Gamma.A.\mathsf{Id}_A[\delta_A] \to \Gamma.A$;

and that, for every object Γ, every display map P_A of codomain Γ, every display map P_C of codomain $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$, and every section $c : \Gamma.A \to \Gamma.A.C[\mathsf{r}_A]$ of $P_{C[\mathsf{r}_A]}$—where r_A is the composition:

$$\Gamma.A \xrightarrow{\ \mathsf{refl}_A\ } \Gamma.A.\mathsf{Id}_A[\delta_A] \xrightarrow{\ \delta_A^{\bullet}\ } \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$$

—there is a choice of:

- $\succ$ (*Elim Rule*) a section:
$$\mathsf{J}_c : \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \to \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A.C$$
of the display map $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A.C \to \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$;
- $\succ$ (*Comp Axiom*) a section:
$$\mathsf{H}_c : \Gamma.A \to \Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c]$$
of the display map $\Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c] \to \Gamma.A$, where the arrow:
$$\mathsf{J}_c[\mathsf{r}_A]; c : \Gamma.A \to \Gamma.A.C[\mathsf{r}_A].C[\mathsf{r}_A]^{\blacktriangledown}$$
is built as explained in Subsection 2.1.

Moreover, let us assume that the following *stability conditions*:

$$\begin{aligned} \mathsf{Id}_A[f^{\bullet\bullet}] &= \mathsf{Id}_{A[f]} & \qquad \mathsf{J}_c[f^{\bullet\bullet\bullet}] &= \mathsf{J}_{c[f^{\bullet}]} \\ \mathsf{refl}_A[f^{\bullet}] &= \mathsf{refl}_{A[f]} & \qquad \mathsf{H}_c[f^{\bullet}] &= \mathsf{H}_{c[f^{\bullet}]} \end{aligned}$$

hold[1] for every arrow $f : \Delta \to \Gamma$.

Then we say that $(\mathbf{C}, \mathcal{D})$ is *endowed with axiomatic =-types*.

Analogously, referring to the rules of Fig. 2 for axiomatic Σ-types, we give the following:

[1] We refer the reader to Appendix A for additional details on the type-checking of the stability conditions.

$$\text{Form Rule}\ \frac{\begin{array}{c} A : \textsc{Type} \\ \lfloor x : A \rfloor\ B(x) : \textsc{Type} \end{array}}{\Sigma_{x:A} B(x) : \textsc{Type}} \qquad \text{Elim Rule}\ \frac{\begin{array}{c} A : \textsc{Type} \qquad \lfloor x : A \rfloor\ B(x) : \textsc{Type} \\ \lfloor u : \Sigma_{x:A} B(x) \rfloor\ C(u) : \textsc{Type} \\ \lfloor x : A;\ y : B(x) \rfloor\ c(x,y) : C(\langle x, y \rangle) \end{array}}{\lfloor u : \Sigma_{x:A} B(x) \rfloor\ \mathsf{split}(c, u) : C(u)}$$

$$\text{Intro Rule}\ \frac{\begin{array}{c} A : \textsc{Type} \\ \lfloor x : A \rfloor\ B(x) : \textsc{Type} \end{array}}{\begin{array}{c} \lfloor x : A;\ y : B(x) \rfloor \\ \langle x, y \rangle : \Sigma_{x:A} B(x) \end{array}} \qquad \text{Comp Axiom}\ \frac{\begin{array}{c} A : \textsc{Type} \qquad \lfloor x : A \rfloor\ B(x) : \textsc{Type} \\ \lfloor u : \Sigma_{x:A} B(x) \rfloor\ C(u) : \textsc{Type} \\ \lfloor x : A;\ y : B(x) \rfloor\ c(x,y) : C(\langle x, y \rangle) \end{array}}{\begin{array}{c} \lfloor x : A;\ y : B(x) \rfloor \\ \sigma(c, x, y) : \mathsf{split}(c, \langle x, y \rangle) = c(x, y) \end{array}}$$

Figure 2: Axiomatic Σ-types

Definition 3.2 (Semantics of axiomatic Σ-types—syntactic formulation)**.** Let $(\mathbf{C}, \mathcal{D})$ be a display map category endowed with axiomatic $=$-types.

Let us assume that, for every object Γ, every display map P_A of codomain Γ, and every display map P_B of codomain $\Gamma.A$, there is a choice of:

- (*Form Rule*) a display map $\Gamma.\mathbf{\Sigma}_A^B \to \Gamma$;
- (*Intro Rule*) a section:
  $$\mathsf{pair}_A^B : \Gamma.A.B \to \Gamma.A.B.\mathbf{\Sigma}_A^B[P_A P_B]$$
  of the display map $\Gamma.A.B.\mathbf{\Sigma}_A^B[P_A P_B] \to \Gamma.A.B$;

and that, for every object Γ, every display map P_A of codomain Γ, every display map P_B of codomain $\Gamma.A$, every display map P_C of codomain $\Gamma.\Sigma_A$, and every section $c : \Gamma.A.B \to \Gamma.A.B.C[\mathsf{p}_A^B]$ of $P_{C[\mathsf{p}_A^B]}$—where p_A^B is the composition:

$$\Gamma.A.B \xrightarrow{\ \mathsf{pair}_A^B\ } \Gamma.A.B.\mathbf{\Sigma}_A^B[P_A P_B] \xrightarrow{(P_A P_B)^{\bullet}} \Gamma.\mathbf{\Sigma}_A^B$$

—there is a choice of:

- (*Elim Rule*) a section:
  $$\mathsf{split}_c : \Gamma.\mathbf{\Sigma}_A^B \to \Gamma.\mathbf{\Sigma}_A^B.C$$
  of the display map $\Gamma.\mathbf{\Sigma}_A^B.C \to \Gamma.\mathbf{\Sigma}_A^B$;
- (*Comp Axiom*) a section:
  $$\sigma_c : \Gamma.A.B \to \Gamma.A.B.\mathsf{Id}_{C[\mathsf{p}_A^B]}[\mathsf{split}_c[\mathsf{p}_A^B]; c]$$
  of the display map $\Gamma.A.B.\mathsf{Id}_{C[\mathsf{p}_A^B]}[\mathsf{split}_c[\mathsf{p}_A^B]; c] \to \Gamma.A.B$, where the arrow:
  $$\mathsf{split}_c[\mathsf{p}_A^B]; c : \Gamma.A.B \to \Gamma.A.B.C[\mathsf{p}_A^B].C[\mathsf{p}_A^B]^{\blacktriangledown}$$
  is built as explained in Subsection 2.1.

$$\text{Form Rule}\ \frac{\begin{array}{c}A : \textsc{Type}\\ \lfloor x : A \rfloor\ B(x) : \textsc{Type}\end{array}}{\Pi_{x:A}B(x) : \textsc{Type}} \qquad \text{Intro Rule}\ \frac{\begin{array}{c}A : \textsc{Type} \qquad \lfloor x : A \rfloor\ B(x) : \textsc{Type}\\ \lfloor x : A \rfloor\ v(x) : B(x)\end{array}}{\lambda x.v(x) : \Pi_{x:A}B(x)}$$

$$\text{Elim Rule}\ \frac{\begin{array}{c}A : \textsc{Type}\\ \lfloor x : A \rfloor\ B(x) : \textsc{Type}\end{array}}{\begin{array}{c}\lfloor z : \Pi_{x:A}B(x);\ x : A \rfloor\\ \mathsf{ev}(z,x) : B(x)\end{array}} \qquad \text{Comp Axiom}\ \frac{\begin{array}{c}A : \textsc{Type} \qquad \lfloor x : A \rfloor\ B(x) : \textsc{Type}\\ \lfloor x : A \rfloor\ v(x) : B(x)\end{array}}{\begin{array}{c}\lfloor x : A \rfloor\\ \beta(v,x) : \mathsf{ev}(\lambda x.v(x),x) = v(x)\end{array}}$$

$$\text{Intro Rule}\ \frac{A : \textsc{Type} \qquad \lfloor x : A \rfloor\ B(x) : \textsc{Type}}{\begin{array}{c}\lfloor z, z';\ q : \Pi_{x:A}\mathsf{ev}(z,x) = \mathsf{ev}(z',x) \rfloor\\ \mathsf{funext}(z,z',q) : z = z'\end{array}}$$

$$\text{Exp Axiom}\ \frac{\begin{array}{c}A : \textsc{Type}\\ \lfloor x : A \rfloor\ B(x) : \textsc{Type}\end{array}}{\begin{array}{c}\lfloor z, z';\ p : z = z' \rfloor\\ \eta^{\Pi}(z,z',p) : p =\\ \mathsf{funext}(z,z',\lambda x.\mathsf{ev}(p,x))\end{array}} \qquad \text{Comp Axiom}\ \frac{A : \textsc{Type} \qquad \lfloor x : A \rfloor\ B(x) : \textsc{Type}}{\begin{array}{c}\lfloor z, z';\ q : \Pi_{x:A}\mathsf{ev}(z,x) = \mathsf{ev}(z',x) \rfloor\\ \beta^{\Pi}(z,z',q) :\\ \lambda x.\mathsf{ev}(\mathsf{funext}(z,z',q),x) = q\end{array}}$$

Figure 3: Axiomatic Π-types & axiomatic function extensionality

Moreover, let us assume that the following *stability conditions*:

$$\begin{array}{cc}\mathsf{\Sigma}_A^B[f] = \mathsf{\Sigma}_{A[f]}^{B[f^\bullet]} & \mathsf{split}_c[f^\bullet] = \mathsf{split}_{c[f^{\bullet\bullet}]}\\ \mathsf{pair}_A^B[f^{\bullet\bullet}] = \mathsf{pair}_{A[f]}^{B[f^\bullet]} & \sigma_c[f^{\bullet\bullet}] = \sigma_{c[f^{\bullet\bullet}]}\end{array}$$

hold[1] for every arrow $f : \Delta \to \Gamma$.

Then we say that $(\mathbf{C}, \mathcal{D})$ is *endowed with axiomatic* Σ*-types.*

Now, referring to the rules of Fig. 2 for axiomatic Σ-types, we give the following:

Definition 3.3 (Semantics of axiomatic Π-types & axiomatic function extensionality—syntactic formulation)**.** Let $(\mathbf{C}, \mathcal{D})$ be a display map category endowed with axiomatic =-types.

Let us assume that, for every object Γ, every display map P_A of codomain Γ, and every display map P_B of codomain $\Gamma.A$, there is a choice of:

- ≻ (*Form Rule*) a display map $\Gamma.\mathsf{\Pi}_A^B \to \Gamma$;
- ≻ (*Elim Rule*) a section:

$$\mathsf{ev}_A^B : \Gamma.\mathsf{\Pi}_A^B.A[P_{\mathsf{\Pi}_A^B}] \to \Gamma.\mathsf{\Pi}_A^B.A[P_{\mathsf{\Pi}_A^B}].B[P_{\mathsf{\Pi}_A^B}^\bullet]$$

of the display map $\Gamma.\mathsf{\Pi}_A^B.A[P_{\mathsf{\Pi}_A^B}].B[P_{\mathsf{\Pi}_A^B}^\bullet] \to \Gamma.\mathsf{\Pi}_A^B.A[P_{\mathsf{\Pi}_A^B}]$;

and that, for every object Γ, every display map P_A of codomain Γ, every display map P_B of codomain $\Gamma.A$, and every section $\upsilon : \Gamma.A \to \Gamma.A.B$ of P_B there is a choice of:

- (*Intro Rule*) a section:
$$\mathsf{abst}_\upsilon : \Gamma \to \Gamma.\mathsf{\Pi}_A^B$$
of the display map $\Gamma.\mathsf{\Pi}_A^B \to \Gamma$;

- (*Comp Axiom*) a section:
$$\beta_\upsilon : \Gamma.A \to \Gamma.A.\mathsf{Id}_B[\mathsf{ev}_A^B[\mathsf{abst}_\upsilon^\bullet]; \upsilon]$$
of the display map $\Gamma.A.\mathsf{Id}_B[\mathsf{ev}_A^B[\mathsf{abst}_\upsilon^\bullet]; \upsilon] \to \Gamma.A$, where the arrow:
$$\mathsf{ev}_A^B[\mathsf{abst}_\upsilon^\bullet]; \upsilon : \Gamma.A \to \Gamma.A.B.B^\blacktriangledown$$
is built as explained in Subsection 2.1.

Moreover, let us assume that the following *stability conditions*:
$$\mathsf{\Pi}_A^B[f] = \mathsf{\Pi}_{A[f]}^{B[f^\bullet]} \qquad \mathsf{abst}_\upsilon[f] = \mathsf{abst}_{\upsilon[f^\bullet]}$$
$$\mathsf{ev}_A^B[f^{\bullet\bullet}] = \mathsf{ev}_{A[f]}^{B[f^\bullet]} \qquad \beta_\upsilon[f^\bullet] = \beta_{\upsilon[f^\bullet]}$$
hold[1] for every arrow $f : \Delta \to \Gamma$.

Then we say that $(\mathbf{C}, \mathcal{D})$ is *endowed with axiomatic* Π*-types*.

Additionally, let us assume that, for every object Γ, every display map P_A of codomain Γ, every display map P_B of codomain $\Gamma.A$, and every pair of sections z, z' of $P_{\mathsf{\Pi}_A^B}$, there is a choice of:

- (*Intro Rule*) a section:
$$\mathsf{funext}_q : \Gamma \to \Gamma.\mathsf{Id}_{\mathsf{\Pi}_A^B}[z; z']$$
of the display map $\Gamma.A.\mathsf{Id}_{\mathsf{\Pi}_A^B}[z; z'] \to \Gamma.A$ for every section:
$$q : \Gamma \to \Gamma.\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}_A^B[z^\bullet]; \mathsf{ev}_A^B[z'^\bullet]]}$$
of the display map $\Gamma.\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}_A^B[z^\bullet]; \mathsf{ev}_A^B[z'^\bullet]]} \to \Gamma$;

- (*Comp Axiom*) a section:
$$\beta_q^\Pi : \Gamma \to \Gamma.\mathsf{Id}_{\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}_A^B[z^\bullet]; \mathsf{ev}_A^B[z'^\bullet]]}}[\ \mathsf{happly}_{z;z'}[\mathsf{funext}_q]\ ;\ q\ ]$$
of the display map $\Gamma.\mathsf{Id}_{\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}_A^B[z^\bullet]; \mathsf{ev}_A^B[z'^\bullet]]}}[\ \mathsf{happly}_{z;z'}[\mathsf{funext}_q]\ ;\ q\ ] \to \Gamma$ for every section:
$$q : \Gamma \to \Gamma.\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}_A^B[z^\bullet]; \mathsf{ev}_A^B[z'^\bullet]]}$$
of the display map $\Gamma.\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}_A^B[z^\bullet]; \mathsf{ev}_A^B[z'^\bullet]]} \to \Gamma$;

- (*Exp Axiom*) a section:
$$\eta_p^\Pi : \Gamma \to \Gamma.\mathsf{Id}_{\mathsf{Id}_{\mathsf{\Pi}_A^B}[z;z']}[\ p\ ;\ \mathsf{funext}_{\mathsf{happly}_{z;z'}[p]}\ ]$$
of the display map $\Gamma \to \Gamma.\mathsf{Id}_{\mathsf{Id}_{\mathsf{\Pi}_A^B}[z;z']}[\ p\ ;\ \mathsf{funext}_{\mathsf{happly}_{z;z'}[p]}\ ]$ for every section:
$$p : \Gamma \to \Gamma.\mathsf{Id}_{\mathsf{\Pi}_A^B}[z; z']$$
of the display map $\Gamma.\mathsf{Id}_{\mathsf{\Pi}_A^B}[z; z'] \to \Gamma$;

where:

- s and t are $P_{(\mathsf{\Pi}^B_A)^\blacktriangledown} P_{\mathsf{Id}_{\mathsf{\Pi}^B_A}}$ and $P^\bullet_{\mathsf{\Pi}^B_A} P_{\mathsf{Id}_{\mathsf{\Pi}^B_A}}$ respectively;
- the section:
$$\mathsf{happly} : \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A} \to \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}.\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}\mathsf{s}]}^{\mathsf{Id}_{B[(P_{\mathsf{\Pi}^B_A}\mathsf{s})^\bullet]}[\mathsf{ev}^B_A[\mathsf{s}^\bullet],\mathsf{ev}^B_A[\mathsf{t}^\bullet]]}$$
of the display map $\Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}.\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}\mathsf{s}]}^{\mathsf{Id}_{B[(P_{\mathsf{\Pi}^B_A}\mathsf{s})^\bullet]}[\mathsf{ev}^B_A[\mathsf{s}^\bullet],\mathsf{ev}^B_A[\mathsf{t}^\bullet]]} \to \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}$ is defined in Remark 3.4;
- and $\mathsf{happly}_{z;z'}$ is $\mathsf{happly}[(z;z')^\bullet]$.

Moreover, let us assume that the following *stability conditions*:

$$\begin{aligned}\mathsf{funext}_q[f] &= \mathsf{funext}_{q[f]}\\ \beta^\Pi_q[f] &= \beta^\Pi_{q[f]}\\ \eta^\Pi_q[f] &= \eta^\Pi_{q[f]}\end{aligned}$$

hold[1] for every arrow $f : \Delta \to \Gamma$.

Then we say that $(\mathbf{C}, \mathcal{D})$ is *endowed with axiomatic* Π*-types & axiomatic function extensionality.*

Remark 3.4. Let s and t be the arrows:

$$P_{(\mathsf{\Pi}^B_A)^\blacktriangledown} P_{\mathsf{Id}_{\mathsf{\Pi}^B_A}} \qquad \text{and} \qquad P^\bullet_{\mathsf{\Pi}^B_A} P_{\mathsf{Id}_{\mathsf{\Pi}^B_A}}$$

respectively. By using the *stability conditions* and the commutativity of Diagram (9), the re-indexing of the display map:

$$\Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}.\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}\mathsf{s}]}^{\mathsf{Id}_{B[(P_{\mathsf{\Pi}^B_A}\mathsf{s})^\bullet]}[\mathsf{ev}^B_A[\mathsf{s}^\bullet],\mathsf{ev}^B_A[\mathsf{t}^\bullet]]} \to \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}$$

via the arrow $\mathsf{r}_{\mathsf{\Pi}^B_A} : \Gamma.\mathsf{\Pi}^B_A \to \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}$ is proven to be the display map:

$$\Gamma.\mathsf{\Pi}^B_A.\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}]}^{\mathsf{Id}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A;\mathsf{ev}^B_A]} \to \Gamma.\mathsf{\Pi}^B_A \ .$$

Now, if we build a section c of the latter display map, then a section happly of the former display map can be defined as J_c. Additionally, if we build a section υ of the display map:

$$\Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}].\mathsf{Id}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A;\mathsf{ev}^B_A] \to \Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}]$$

then c can be defined as abst_υ. Finally, we define υ as the unique section $\mathsf{r}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A]$ of this display map that makes the diagram:

$$\begin{tikzcd}
\Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}] \arrow[r] \arrow[d, "\mathsf{ev}^B_A"] & \Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}].\mathsf{Id}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A;\mathsf{ev}^B_A] \arrow[d, "(\mathsf{ev}^B_A;\mathsf{ev}^B_A)^\blacktriangledown"] \\
\Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}].B[P^\bullet_{\mathsf{\Pi}^B_A}] \arrow[r, "\mathsf{r}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}"] & \Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}].B[P^\bullet_{\mathsf{\Pi}^B_A}].B[P^\bullet_{\mathsf{\Pi}^B_A}]^\blacktriangledown.\mathsf{Id}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}
\end{tikzcd}$$

commute. Summarising, the section happly is defined as:

$$\mathsf{J}_{\mathsf{abst}_{\mathsf{r}_{B[P^{\bullet}_{\mathsf{\Pi}^B_A}]}}[\mathsf{ev}^B_A]}$$

and it is the interpretation of the term judgement of type $\Pi_{x:A}^{\mathsf{ev}(z,x)=\mathsf{ev}(z',x)}$ in context:

$$\lfloor z, z' : \Pi_{x:A} B(x);\ p : z = z' \rfloor$$

obtained by elimination via the term judgement $\lfloor z : \Pi_{x:A} B(x) \rfloor\ \lambda x.\mathsf{r}(\mathsf{ev}(z,x))$.

Now, by using again the *stability conditions* and the commutativity of Diagram (9), the re-indexing of the display map:

$$\Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}^B_A}.\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}\mathsf{s}]}^{\mathsf{Id}_{B[(P_{\mathsf{\Pi}^B_A}\mathsf{s})^{\bullet}]}[\mathsf{ev}^B_A[\mathsf{s}^{\bullet}],\mathsf{ev}^B_A[\mathsf{t}^{\bullet}]]} \to \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}^B_A}$$

via the arrow $(z; z')^{\bullet} p : \Gamma.\mathsf{\Pi}^B_A \to \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}^B_A}$—for some section: $p : \Gamma \to \Gamma.\mathsf{Id}_{\mathsf{\Pi}^B_A}[z; z']$ of the display map $\Gamma.\mathsf{Id}_{\mathsf{\Pi}^B_A}[z; z'] \to \Gamma$—is proven to be the display map:

$$\Gamma.\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^{\bullet}];\mathsf{ev}^B_A[z'^{\bullet}]]} \to \Gamma$$

hence $\mathsf{happly}_{z;z'}[p]$ is a section of this display map.

Let $(\mathbf{C}, \mathcal{D})$ be a display map category endowed with axiomatic =-types, axiomatic Σ-types, and axiomatic Π-types & axiomatic function extensionality—see Definitions 3.1 to 3.3, respectively. Then a notion of interepretation of ATT, with =-types and Σ-types, in $(\mathbf{C}, \mathcal{D})$—in the sense of Section 2—is defined. The proof of this fact is standard and adapts the argument provided by Streicher [39] and Hofmann [23]. We also refer the reader to [12, 38, 20] for further details. The argument consists in defining an a priori partial interpretation function, whose domain consists of the *pre-judgements* of the raw syntax of ATT, and which can be shown by induction on derivations to be well-defined on the actual judgements of ATT. In detail:

- a context $\gamma : \Gamma$ of ATT is interpreted as an object Γ of $\mathbf{C}$;
- if Γ is the interpretation of the context γ, then a type judgment $\lfloor \gamma : \Gamma \rfloor\ A(\gamma) : \textsc{Type}$ of ATT is interpreted as a display map $\Gamma.A \to \Gamma$ of $\mathcal{D}$;
- if $\Gamma.A \to \Gamma$ is the interpretation of the type judgment $\lfloor \gamma : \Gamma \rfloor\ A(\gamma) : \textsc{Type}$, then a term judgment $\lfloor \gamma : \Gamma \rfloor\ a(\gamma) : A(\gamma)$ of ATT is interpreted as a section $\Gamma \xrightarrow{a} \Gamma.A$ of $\Gamma.A \xrightarrow{P_A} \Gamma$.

We only mention that the interpretation function satisfies the following expected clauses. Let us assume that Γ is the interpretation of the context $\gamma : \Gamma$, P_A is the interpretation of the type judgement $\lfloor \gamma : \Gamma \rfloor\ A(\gamma) : \textsc{Type}$, and P_B—when required—is the interpretation of the type judgement $\lfloor \gamma : \Gamma;\ x : A(\gamma) \rfloor\ B(\gamma, x) : \textsc{Type}$. Then:

Clauses for type judgement interpretation.

- the display map $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \to \Gamma.A.A^{\blacktriangledown}$ is the interpretation of $\lfloor \gamma;\ x, x' : A(\gamma) \rfloor\ x = x' : \textsc{Type}$;
- the display map $\Gamma.\mathsf{\Sigma}^B_A \to \Gamma$ is the interpretation of $\lfloor \gamma \rfloor\ \Sigma_{x:A(\gamma)} B(\gamma, x) : \textsc{Type}$;
- the display map $\Gamma.\mathsf{\Pi}^B_A \to \Gamma$ is the interpretation of $\lfloor \gamma \rfloor\ \Pi_{x:A(\gamma)} B(\gamma, x) : \textsc{Type}$;

Clauses for term judgement interpretation.

- the section refl_A of $\Gamma.A.\mathsf{Id}_A[\delta_A] \to \Gamma.A$ is the interpretation of $\lfloor\gamma;\ x : A(\gamma)\rfloor\ \mathsf{r}(x) : x = x$;
- the section pair^B_A of the display map $\Gamma.A.B.\mathsf{\Sigma}^B_A[P_A P_B] \to \Gamma.A.B$ is the interpretation of $\lfloor\gamma;\ x : A(\gamma);\ y : B(\gamma, x)\rfloor\ \langle x, y\rangle : \Sigma_{x:A(\gamma)} B(\gamma, x)$;
- the section ev^B_A of the display map $\Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}].B[P^\bullet_{\mathsf{\Pi}^B_A}] \to \Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}]$ is the interpretation of $\lfloor\gamma;\ z : \Pi_{x:A(\gamma)} B(\gamma, x);\ x : A\rfloor\ \mathsf{ev}(z, x) : B(x)$.

Additionally, let us add to the assumptions, for the following two clauses, that P_C is the interpretation of the type judgement $\lfloor\gamma; x, x';\ p : x = x'\rfloor\ C(\gamma, x, x', p) : \textsc{Type}$ and $\Gamma.A \xrightarrow{c} \Gamma.A.C[\mathsf{r}_A]$ is the interpretation of the term judgement $\lfloor\gamma; x\rfloor\ c(\gamma, x) : C(\gamma, x, x, \mathsf{r}(x))$. Then:

- the section J_c of the display map $\Gamma.A.A^\blacktriangledown.\mathsf{Id}_A.C \to \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A$ is the interpretation of $\lfloor\gamma; x, y; p\rfloor\ \mathsf{J}(c, \gamma, x, x', p) : C(\gamma, x, x', p)$;
- the section H_c of $\Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c] \to \Gamma.A$ is the interpretation of the term judgement $\lfloor\gamma; x\rfloor\ \mathsf{H}(c, \gamma, x) : \mathsf{J}(c, \gamma, x, x, \mathsf{r}(x)) = c(\gamma, x)$.

Additionally, let us add to the assumptions, for the following two clauses, that P_C is the interpretation of the type judgement $\lfloor\gamma;\ u : \Sigma_{x:A(\gamma)} B(\gamma, x)\rfloor\ C(\gamma, u) : \textsc{Type}$ and $\Gamma.A.B \xrightarrow{c} \Gamma.A.B.C[\mathsf{p}_A]$ is the interpretation of the term judgement $\lfloor\gamma;\ x;\ y\rfloor\ c(\gamma, x, y) : C(\gamma, \langle x, y\rangle)$. Then:

- the section split_c of $\Gamma.\mathsf{\Sigma}^B_A.C \to \Gamma.\mathsf{\Sigma}^B_A$ is the interpretation of $\lfloor\gamma; u\rfloor\ \mathsf{split}(c, \gamma, u) : C(\gamma, u)$;
- the section σ_c of the display map $\Gamma.A.B.\mathsf{Id}_{C[\mathsf{p}^B_A]}[\mathsf{split}_c[\mathsf{p}^B_A]; c] \to \Gamma.A.B$ is the interpretation of $\lfloor\gamma; x; y\rfloor\ \sigma(c, \gamma, x, y) : \mathsf{split}(c, \gamma, \langle x, y\rangle) = c(\gamma, x, y)$.

Additionally, let us add to the assumptions, for the following two clauses, that $\upsilon : \Gamma.A \to \Gamma.A.B$ is the interpretation of the term judgement $\lfloor\gamma; x\rfloor\ \upsilon(\gamma, x) : B(\gamma, x)$. Then:

- the section abst_υ of $\Gamma \to \Gamma.\mathsf{\Pi}^B_A$ is the interpretation of $\lfloor\gamma\rfloor\ \lambda x.\upsilon(\gamma, x) : \Pi_{x:A(\gamma)} B(\gamma, x)$;
- the section β_q of the display map $\Gamma.A \to \Gamma.A.\mathsf{Id}_B[\mathsf{ev}^B_A[\mathsf{abst}^\bullet_\upsilon]; \upsilon]$ is the interpretation of $\lfloor\gamma; x\rfloor\ \beta(\upsilon, \gamma, x) : \mathsf{ev}(\lambda x.\upsilon(\gamma, x), x) = \upsilon(\gamma, x)$.

Additionally, let us add to the assumptions, for the following two clauses, that $z, z' : \Gamma \to \Gamma.\Pi_{x:A(\gamma)} B(\gamma, x)$ are the interpretations of $\lfloor\gamma\rfloor\ z(\gamma), z'(\gamma) : \Pi_{x:A(\gamma)} B(\gamma, x)$ and that $q : \Gamma \to \Gamma.\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^\bullet]; \mathsf{ev}^B_A[z'^\bullet]]}$ is the interpretation of $\lfloor\gamma\rfloor\ q(\gamma) : \Pi_{x:A(\gamma)} \mathsf{ev}(z(\gamma), x) = \mathsf{ev}(z'(\gamma), x)$. Then:

- the section funext_q of the display map $\Gamma.\mathsf{Id}_{\mathsf{\Pi}^B_A}[z; z'] \to \Gamma$ is the interpretation of the term judgement $\lfloor\gamma\rfloor\ \mathsf{funext}(z(\gamma), z'(\gamma), q(\gamma)) : z(\gamma) = z'(\gamma)$;
- the section β^Π_q of the display map $\Gamma.\mathsf{Id}_{\mathsf{\Pi}_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^\bullet]; \mathsf{ev}^B_A[z'^\bullet]]}}[\ \mathsf{happly}_{z;z'}[\mathsf{funext}_q]\ ;\ q\ ] \to \Gamma$ is the interpretation of $\lfloor\gamma\rfloor\ \beta^\Pi(z(\gamma), z'(\gamma), q(\gamma)) : \lambda x.\mathsf{ev}(\mathsf{funext}(z(\gamma), z'(\gamma), q(\gamma)), x) = q(\gamma)$.

Finally, let us add to the assumptions, for the last clause, that $z, z' : \Gamma \to \Gamma.\Pi_{x:A(\gamma)} B(\gamma, x)$ are the interpretations of $\lfloor\gamma\rfloor\ z(\gamma), z'(\gamma) : \Pi_{x:A(\gamma)} B(\gamma, x)$ and that $p : \Gamma \to \Gamma.\mathsf{Id}_{\mathsf{\Pi}^B_A}[z; z']$ is the interpretation of $\lfloor\gamma\rfloor\ p(\gamma) : z(\gamma) = z'(\gamma)$. Then:

- the section η^Π_p of the display map $\Gamma \to \Gamma.\mathsf{Id}_{\mathsf{Id}_{\mathsf{\Pi}^B_A}[z; z']}[\ p\ ;\ \mathsf{funext}_{\mathsf{happly}_{z;z'}[p]}\ ]$ is the interpretation of $\lfloor\gamma\rfloor\ \eta^\Pi(z(\gamma), z'(\gamma), p(\gamma)) : p(\gamma) = \mathsf{funext}(z(\gamma), z'(\gamma), \lambda x.\mathsf{ev}(p(\gamma), x))$.

This notion of interpretation of ATT in $(\mathbf{C}, \mathcal{D})$ satisfies the following:

Theorem 3.5 (Soundness property)**.** *The interpretation of* ATT *in* $(\mathbf{C}, \mathcal{D})$ *is sound, that is:*

- *whenever* ATT *infers* $\lfloor \gamma : \Gamma \rfloor\ A(\gamma) : \textsc{Type}$*, then its interpretation—let us indicate it as* P_A*—is defined;*
- *whenever* ATT *infers* $\lfloor \gamma : \Gamma \rfloor\ a(\gamma) : A(\gamma)$*, then its interpretation—let us indicate it as* $a : \Gamma \to \Gamma.A$*—is defined;*
- *whenever* ATT *infers* $\lfloor \gamma : \Gamma \rfloor\ A(\gamma) \equiv A'(\gamma)$ *for some type judgements* $\lfloor \gamma : \Gamma \rfloor\ A(\gamma) : \textsc{Type}$ *and* $\lfloor \gamma : \Gamma \rfloor\ A'(\gamma) : \textsc{Type}$*, then their interpretations* P_A *and* $P_{A'}$ *coincide;*
- *whenever* ATT *infers* $\lfloor \gamma : \Gamma \rfloor\ a(\gamma) \equiv a'(\gamma)$ *for some term judgements* $\lfloor \gamma : \Gamma \rfloor\ a(\gamma) : A(\gamma)$ *and* $\lfloor \gamma : \Gamma \rfloor\ a'(\gamma) : A(\gamma)$ *and some type judgement* $\lfloor \gamma : \Gamma \rfloor\ A(\gamma) : \textsc{Type}$*, then their interpretations* a *and* a'*—as sections of* P_A*—coincide.*

Proof. The proof of such a result is standard and we refer the reader to [39, 38, 20, 23] for details. The argument is by induction on the derivation of type, term, type equality, and term equality judgements.

A preliminary step is proving three properties called *weakening property*, *substitution property*, and *general substitution property*:

- *Weakening property.* Let Γ be the interpretation of $\gamma : \Gamma$, $P_{\Delta_{k+1}}$ be the interpretation of:

  $$\lfloor \gamma : \Gamma, \delta_1 : \Delta_1(\gamma), ..., \delta_k(\gamma, \delta_1, ..., \delta_{k-1}) \rfloor\ \Delta_{k+1}(\gamma, \delta_1, ..., \delta_k) : \textsc{Type}$$

  for all $k \in \{0, ..., n-1\}$ for some $n \geq 0$, and P_D be the interpretation of $\lfloor \gamma : \Gamma \rfloor\ D(\gamma) : \textsc{Type}$.

  Let P_A be the interpretation of $\lfloor \gamma : \Gamma, \delta : \Delta(\gamma) \rfloor\ A(\gamma, \delta) : \textsc{Type}$, and a be the interpretation of $\lfloor \gamma : \Gamma, \delta : \Delta(\gamma) \rfloor\ a(\gamma, \delta) : A(\gamma, \delta)$.

  Then the interpretations of:

  $$\lfloor \gamma : \Gamma, d : D(\gamma), \delta : \Delta(\gamma) \rfloor\ A(\gamma, \delta) : \textsc{Type} \quad \text{and} \quad \lfloor \gamma : \Gamma, d : D(\gamma), \delta : \Delta(\gamma) \rfloor\ a(\gamma, \delta) : A(\gamma, \delta)$$

  are defined and are $P_{A[P_D.\Delta]}$ and $a[P_D.\Delta]$ respectively.

- *Substitution property.* Let Γ be the interpretation of $\gamma : \Gamma$, P_D be the interpretation of $\lfloor \gamma : \Gamma \rfloor\ D(\gamma) : \textsc{Type}$, $P_{\Delta_{k+1}}$ be the interpretation of:

  $$\lfloor \gamma : \Gamma, d : D(\gamma), \delta_1 : \Delta_1(\gamma, d), ..., \delta_k(\gamma, d, \delta_1, ..., \delta_{k-1}) \rfloor\ \Delta_{k+1}(\gamma, d, \delta_1, ..., \delta_k) : \textsc{Type}$$

  for all $k \in \{0, ..., n-1\}$ for some $n \geq 0$, and t be the interpretation of $\lfloor \gamma : \Gamma \rfloor\ t(\gamma) : D(\gamma)$.

  Let P_A be the interpretation of $\lfloor \gamma : \Gamma, d : D(\gamma), \delta : \Delta(\gamma, d) \rfloor\ A(\gamma, d, \delta) : \textsc{Type}$, and a be the interpretation of $\lfloor \gamma : \Gamma, d : D(\gamma), \delta : \Delta(\gamma, d) \rfloor\ a(\gamma, d, \delta) : A(\gamma, d, \delta)$.

  Then the interpretations of:

  $$\lfloor \gamma : \Gamma, \delta : \Delta(\gamma, t) \rfloor\ A(\gamma, t, \delta) : \textsc{Type} \quad \text{and} \quad \lfloor \gamma : \Gamma, \delta : \Delta(\gamma, t) \rfloor\ a(\gamma, t, \delta) : A(\gamma, t, \delta)$$

  are defined and are $P_{A[t.\Delta]}$ and $a[t.\Delta]$ respectively.

- *General substitution property.* Let Γ be the interpretation of $\gamma : \Gamma$, $P_{\Delta_{k+1}}$ be the interpretation of:

$$\lfloor \gamma : \Gamma, \delta_1 : \Delta_1(\gamma), ..., \delta_k(\gamma, \delta_1, ..., \delta_{k-1}) \rfloor\ \Delta_{k+1}(\gamma, \delta_1, ..., \delta_k) : \text{Type}$$

for all $k \in \{0, ..., n-1\}$ for some $n \geq 0$, let $\Omega \xrightarrow{f} \Gamma$ be the interpretation of the generalised term judgement $\lfloor \omega : \Omega \rfloor\ f(\omega) : \Gamma$.

Let P_A be the interpretation of $\lfloor \gamma : \Gamma, \delta : \Delta(\gamma) \rfloor\ A(\gamma, \delta) : \text{Type}$, and a be the interpretation of $\lfloor \gamma : \Gamma, \delta : \Delta(\gamma) \rfloor\ a(\gamma, \delta) : A(\gamma, \delta)$.

Then the interpretations of:

$$\lfloor \omega : \Omega, \delta : \Delta(f(\omega)) \rfloor\ A(f(\omega), \delta) : \text{Type} \quad \text{and} \quad \lfloor \omega : \Omega, \delta : \Delta(f(\omega)) \rfloor\ a(f(\omega), \delta) : A(f(\omega), \delta)$$

are defined and are $P_{A[f.\Delta]}$ and $a[f.\Delta]$ respectively.

The proof of these properties is by induction on the derivation of type and term judgements. We show the weakening property for the second clause for type judgement interpretation and for the last three clauses for term judgement interpretation (i.e. the clauses involving propositional dependent sums). For the other clauses the argument is analogous. Let the interpretations of:

$$\begin{gathered} \gamma : \Gamma \\ \lfloor \gamma \rfloor\ A(\gamma) : \text{Type} \\ \lfloor \gamma, x : A(\gamma) \rfloor\ B(\gamma, x) : \text{Type} \\ \lfloor \gamma, u \rfloor\ C(\gamma, u) : \text{Type} \\ \lfloor \gamma, x, y \rfloor\ c(\gamma, x, y) : C(\gamma, \langle x, y \rangle) \\ \lfloor \gamma \rfloor\ D(\gamma) : \text{Type} \end{gathered}$$

be Γ, P_A, P_B, P_C, c, and P_D respectively. By inductive hypothesis, the interpretations of $\lfloor \gamma, d : D(\gamma) \rfloor\ A(\gamma) : \text{Type}$, $\lfloor \gamma, d : D(\gamma), x : A(\gamma) \rfloor\ B(\gamma, x) : \text{Type}$, $\lfloor \gamma, d : D(\gamma), u \rfloor\ C(\gamma, u) : \text{Type}$, $\lfloor \gamma, d : D(\gamma), x, y \rfloor\ c(\gamma, x, y) : C(\gamma, \langle x, y \rangle)$ exist and are:

$$P_{A[P_D]}, \quad P_{B[P_D^{\bullet}]}, \quad P_{C[P_D^{\bullet}]}, \quad \text{and} \quad c[P_D^{\bullet\bullet}]$$

respectively. Then, by definition, the interpretations of:

$$\begin{gathered} \lfloor \gamma, d : D(\gamma) \rfloor\ \Sigma_{x:A(\gamma)} B(\gamma, x) : \text{Type} \\ \lfloor \gamma, d : D(\gamma), x, y \rfloor\ \langle x, y \rangle : \Sigma_{x:A(\gamma)} B(\gamma, x) \\ \lfloor \gamma : \Gamma, d : D(\gamma), u : \Sigma_{x:A(\gamma)} B(\gamma, x) \rfloor\ \mathsf{split}(c, \gamma, u) : C(\gamma, u) \\ \lfloor \gamma, d : D(\gamma), x, y \rfloor\ \sigma(c, \gamma, x, y) : \mathsf{split}(c, \gamma, \langle x, y \rangle) = c(\gamma, x, y) \end{gathered}$$

exist and are:

$$P_{\mathbf{\Sigma}_{A[P_D]}^{B[P_D^{\bullet}]}}, \quad \mathsf{pair}_{A[P_D]}^{B[P_D^{\bullet}]}, \quad \mathsf{split}_{c[P_D^{\bullet\bullet}]}, \quad \text{and} \quad \sigma_{c[P_D^{\bullet\bullet}]}$$

respectively. By the *stability conditions* of Definition 3.2 these morphisms coincide with:

$$P_{\mathbf{\Sigma}_A^B[P_D]}, \quad \mathsf{pair}_A^B[P_D^{\bullet\bullet}], \quad \mathsf{split}_c[P_D^{\bullet}], \quad \text{and} \quad \sigma_c[P_D^{\bullet\bullet}]$$

respectively, hence we are done. The proof of the substitution property and the general substitution property are analogous.

These three results are used in the proof by induction of the four points of the statement. In the theory ATT, there are fewer rules ending with a type/term equality judgement compared to ITT, hence the proof of the third and the fourth points is contained in the proof of the soundness of the interpretation of ITT, which is contained e.g. in [23]. Analogously, in ATT the rules ending with a type judgement are the same as in ITT, hence we are done also with the first point of the statement. Regarding the second point, there are additional clauses for term judgement interpretation that we need to check (the others are contained in ATT), namely the computation axioms as well as the expansion axiom for function extensionality. Let us check e.g. the computation axiom for Σ-types, as the others are analogous. Let Γ be the interpretation of the context $\gamma : \Gamma$. By inductive hypothesis, the interpretations of:

$$\lfloor \gamma : \Gamma \rfloor \; A(\gamma) : \textsc{Type}$$
$$\lfloor \gamma : \Gamma, x : A(\gamma) \rfloor \; B(\gamma, x) : \textsc{Type}$$
$$\lfloor \gamma, u \rfloor \; C(\gamma, u) : \textsc{Type}$$
$$\lfloor \gamma, x, y \rfloor \; c(\gamma, x, y) : C(\gamma, \langle x, y \rangle)$$

are defined. Let us call them P_A, P_B, P_C, and c respectively. Then the section:

$$\sigma_c : \Gamma.A.B \to \Gamma.A.B.\mathsf{Id}_{C[\mathsf{p}_A^B]}[\ \mathsf{split}_c[\mathsf{p}_A^B]\ ;\ c\ ]$$

of $P_{\mathsf{Id}_{C[\mathsf{p}_A^B]}[\ \mathsf{split}_c[\mathsf{p}_A^B]\ ;\ c\ ]}$ is defined. Hence the interpretation of:

$$\lfloor \gamma, x, y \rfloor \; \sigma(c, \gamma, x, y) : \mathsf{split}(c, \gamma, \langle x, y \rangle) = c(\gamma, x, y)$$

is defined. □

In light of this result, we may call such a display map category $(\mathbf{C}, \mathcal{D})$ equipped with axiomatic =-types, axiomatic Σ-types, and axiomatic Π-types & axiomatic function extensionality a *model of* ATT. As anticipated, in the next section we show how to provide a categorical presentation of axiomatic type constructors.

4 Categorical formulation of the semantics of axiomatic type theory

As mentioned in the introduction, the issue with the formulation presented in the Section 3 regarding the semantics of axiomatic type formers is that, in this form, it may be regarded as impractical, depending on the context—for example, when seeking concrete models of ATT. As we pointed out, a more practical formulation should rely less on equipping the base structure with choice functions and more on universal properties, which allow these choice functions to be derived automatically. Universal properties are generally easier to identify in mathematical practice. However, the challenge with axiomatic type formers is that one-dimensional universal properties fail to uniquely characterise them. These properties collapse intensional and axiomatic type formers into extensional ones, making it impossible to distinguish between them. Consequently, they effectively characterise only the semantic transcription of the rules of ETT.

This is why the notion we introduce below is 2-dimensional: while 1-cells, as usual, represent substitutions, 2-cells represent (equivalence classes of) *context propositional equalities*, or propositional equalities between substitutions: lists, indicated as $\lfloor \gamma : \Gamma \rfloor \; p(\gamma) : f(\gamma) = g(\gamma)$, of term judgements of the form:

$\lfloor \gamma : \Gamma \rfloor \; p_1(\gamma) : f_1(\gamma) = g_1(\gamma)$

$\lfloor \gamma : \Gamma \rfloor \; p_2(\gamma) : f_2(\gamma) = p_1(\gamma)^* g_2(\gamma)$

$\lfloor \gamma : \Gamma \rfloor \; p_3(\gamma) : f_3(\gamma) = (p_1(\gamma), p_2(\gamma))^* g_3(\gamma)$

...

$$\lfloor \gamma : \Gamma \rfloor \; p_n(\gamma) : f_n(\gamma) = (p_1(\gamma), ..., p_{n-1}(\gamma))^* g_n(\gamma)$$

where $\lfloor \gamma : \Gamma \rfloor \; f(\gamma) : \Delta$ and $\lfloor \gamma : \Gamma \rfloor \; g(\gamma) : \Delta$ are parallel substitutions of ATT and trasport operations along multiple identity proofs are defined by the elimination rule for axiomatic identity types.

Following the approach devised by Garner [17], the presence of an additional dimension enables us to impose more fine-grained universal properties on such a structure. These properties allow us to identify axiomatic type formers without conflating them with extensional ones.

Definition 4.1. A *(split) display map 2-category* $(\mathcal{C}, \mathcal{D})$ is a (2,1)-category $\mathcal{C}$ with a chosen 2-terminal object 1, together with a class $\mathcal{D}$ of 1-cells of $\mathcal{C}$, called *display maps* and denoted as $\Gamma.A \to \Gamma$ and labeled as P_A, such that:

- for every display map $\Gamma.A \to \Gamma$ and every arrow $f : \Delta \to \Gamma$, there is a choice of a display map $\Delta.A[f] \to \Delta$ such that the square:

$$\begin{tikzcd} \Delta.A[f] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma.A \arrow[d] \\ \Delta \arrow[r, "f"] & \Gamma \end{tikzcd}$$

  is a pullback and a 2-pullback; as the former display map is labeled as P_A, the latter will be labeled as $P_{A[f]}$; the arrow $\Delta.A[f] \to \Gamma.A$ will be denoted as $f.A$, or as $f^\bullet$ in absence of ambiguity; we will adopt the notation $A^\blacktriangledown$ in place of $A[P_A]$;

- every display map $\Gamma.A \to \Gamma$ is a *cloven isofibration* i.e. for every 2-cell p as in the diagram below, there is a choice of a 1-cell t_g^p and of a 2-cell τ_g^p as in the diagram below, in such a way that the equality:

$$\begin{tikzcd} \Delta \arrow[r, "g"] \arrow[dr, bend right, "f"'] & \Gamma.A \arrow[d] \\ & \Gamma \arrow[ul, phantom, "\overset{p}{\Longrightarrow}"] \end{tikzcd} \quad = \quad \begin{tikzcd} \Delta \arrow[r, "g"] \arrow[r, bend right, "\mathsf{t}_g^p"'] \arrow[r, phantom, "\Uparrow \tau_g^p"] \arrow[dr, bend right, "f"'] & \Gamma.A \arrow[d] \\ & \Gamma \arrow[ul, phantom, "="] \end{tikzcd}$$

  between the 2-cells p and $P_A * \tau_g^p$ holds and the equalities:

$$\mathsf{t}_{gh}^{p*h} = \mathsf{t}_g^p h \qquad \text{and} \qquad \tau_{gh}^{p*h} = \tau_g^p * h$$

  hold for every 1-cell $h : \Omega \to \Delta$;

- the equalities:

$$\begin{array}{ll} P_{A[1_A]} = P_A & (1_\Gamma).A = 1_{\Gamma.A} \\ P_{A[fg]} = P_{A[f][g]} & (f.A)(g.A[f]) = (fg).A \end{array}$$

  hold for every choice of composable arrows f and g and every display map $\Gamma.A \to \Gamma$, where Γ is the target of f;

- for every choice of display maps:

$$\Gamma.A_1 \to \Gamma, \quad \Gamma.A_1.A_2 \to \Gamma.A_1, \quad ..., \quad \Gamma.A_1. \ldots .A_{n-1}.A_n \to \Gamma.A_1. \ldots .A_{n-1}$$

and for every 1-cell $\Delta \xrightarrow{f} \Gamma$, if we are given a display map $\Gamma.A.C \to \Gamma.A$ [2] and commutative squares:

$$\begin{tikzcd} \Omega' \arrow[r, "f'"] \arrow[d, "g'"'] & \Omega \arrow[d, "g"] \\ \Delta.A[f].C[f.A] \arrow[r, "f.A.C"] & \Gamma.A.C \end{tikzcd} \qquad \begin{tikzcd} \Omega' \arrow[r, "f'"] \arrow[d, "h'"'] & \Omega \arrow[d, "h"] \\ \Delta.A[f] \arrow[r, "f.A"] & \Gamma.A \end{tikzcd}$$

such that $P_A P_C g = P_A h$ and $P_{A[f]} P_C g' = P_{A[f]} h'$ and a 2-cell:

$$\begin{tikzcd} \Omega \arrow[r, "g"] \arrow[dr, bend right, "h"'] & \Gamma.A.C \arrow[d] \\ & \Gamma.A \arrow[from=1-1, to=1-2, phantom, shift right=4, "\xRightarrow{p}"] \end{tikzcd}$$

such that $P_A * p = 1_{P_A h}$, then the following diagram of 2-cells:

$$\begin{tikzcd} \Omega' \arrow[r, "f'"] \arrow[d, bend right=50, "\mathsf{t}^{p[f]}_{g'}"'] \arrow[d, bend left=50, "g'"] \arrow[d, phantom, "\xRightarrow{\tau^{p[f]}_{g'}}"] & \Omega \arrow[d, bend right=50, "\mathsf{t}^{p}_{g}"'] \arrow[d, bend left=50, "g"] \arrow[d, phantom, "\xRightarrow{\tau^{p}_{g}}"] \\ \Delta.A[f].C[f.A] \arrow[r, "f.A.C"] & \Gamma.A.C \end{tikzcd}$$

commutes, i.e. the equalities:

$$\mathsf{t}^p_g[f.A] = \mathsf{t}^{p[f]}_{g'} \qquad \text{and} \qquad \tau^p_g[f.A] = \tau^{p[f]}_{g'}$$

hold.[2]

The notion of a display map 2-category captures the structural aspects of dependent type theory, combined with fragments of the specific logical aspects of ATT. Specifically, the cloven isofibration structure with which the display maps are equipped provides a 2-dimensional interpretation of a consequence of the elimination rule and the computation *axiom* for =-types: transport—along one or more identity proofs. In [17], display maps in a model of ITT are endowed with a *normal* isofibration structure, which in our setting corresponds to the additional *normality* requirements that $\mathsf{t}^{1_{P_A g}}_g = g$ and $\tau^{1_{P_A g}}_g = 1_g$. However, dropping these requirements and reducing display maps to mere cloven isofibrations is fundamental to prevent models of ATT from being models of ITT. Let us briefly analyse how the cloven isofibration structure is induced in the context of the syntax of ATT itself.

If we are given a type judgement $\lfloor \gamma : \Gamma \rfloor\ A(\gamma) : \textsc{Type}$ and substitutions:

$$\lfloor \delta : \Delta \rfloor\ f(\delta) : \Gamma \quad \text{and} \quad \lfloor \delta : \Delta \rfloor\ g(\delta) : \Gamma.A$$

—hence $g(\delta)$ is of the form $g_1(\delta) : \Gamma\,,\, g_2(\delta) : A(g_1(\delta))$—with a context identity proof $\lfloor \delta : \Delta \rfloor\ p(\delta) : f(\delta) = g_1(\delta)$, then t^p_g is the substitution $\lfloor \delta : \Delta \rfloor\ f(\delta) : \Gamma\,,\, p(\delta)^* g_2(\delta) : A(f(\delta))$ and τ^p_g is the context identity proof:

$$\lfloor \delta : \Delta \rfloor\ f(\delta)\,,\, p(\delta)^* g_2(\delta) \quad = \quad g_1(\delta)\,,\, g_2(\delta)$$

[2]Here $A_1. \ldots .A_n$ and $A_1[f].A_2[f^\bullet].A_3[f^{\bullet\bullet}]. \ldots .A_n[f^{\bullet\bullet\cdots\bullet}]$ are indicated as A and $A[f]$ respectively, and $P_{A_n} P_{A_{n-1}} \ldots P_{A_1}$ is indicated as P_A. Re-indexings on 1- and 2-cells are defined using the 1- and 2-pullback structure, analogously to display map categories.

provided by the list:

$$
\begin{array}{lrl}
\lfloor \delta : \Delta \rfloor & p(\delta) : & f(\delta) = g_1(\delta) \\
\lfloor \delta : \Delta \rfloor & \mathsf{r}(p(\delta)^* g_2(\delta)) : & p(\delta)^* g_2(\delta) = p(\delta)^* g_2(\delta)
\end{array}
$$

of identity proofs. Now, if $f(\delta) \equiv g_1(\delta)$ and $p(\delta) \equiv \mathsf{r}(g_1(\delta))$, then t^p_g is $\lfloor \delta : \Delta \rfloor\ g_1(\delta) : \Gamma\,,\ \mathsf{r}(g_1(\delta))^* g_2(\delta) : A(g_1(\delta))$ and τ^p_g is the list:

$$
\begin{array}{lrl}
\lfloor \delta : \Delta \rfloor & \mathsf{r}(g_1(\delta)) : & g_1(\delta) = g_1(\delta) \\
\lfloor \delta : \Delta \rfloor & \mathsf{r}(\mathsf{r}(g_1(\delta))^* g_2(\delta)) : & \mathsf{r}(g_1(\delta))^* g_2(\delta) = \mathsf{r}(g_1(\delta))^* g_2(\delta)
\end{array}
$$

hence in general in this case t^p_g is *not* $g(\delta)$ and τ^p_g is *not* its identity 2-cell: in ATT we can infer that $\mathsf{r}(g_1(\delta))^* g_2(\delta) = g_2(\delta)$—it is a fragment of the computation axiom for =-types—but not that $\mathsf{r}(g_1(\delta))^* g_2(\delta) \equiv g_2(\delta)$. In other words, the display map associated to the type A is a cloven isofibration but not necessarily a normal isofibration.

This fact explains why, in a general display map 2-category, display maps are cloven isofibrations but not normal isofibrations: this is what happens when we consider the syntax of ATT. As we will see—Subsection 4.4—the cloven isofibration structure on display maps enables the interpretation of computation axioms within a display map 2-category. However, the fact that they are generally *not* normal isofibrations is crucial to ensuring that these computation axioms are validated without necessarily validating the corresponding computation rules: this weakening prevents the interpretation of terms H_c and σ_c from collapsing into reflexivities, hence the respective propositional equalities from being judgemental.

4.1 Axiomatic =-types

We specialise the notion of a display map 2-category to obtain a version that incorporates a 2-dimensional semantic interpretation of axiomatic =-types.

Definition 4.2 (Semantics of axiomatic =-types—categorical formulation)**.** Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category.

Let us assume that, for every object Γ and every display map P_A of codomain Γ, there is a choice of a display map $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \to \Gamma.A.A^{\blacktriangledown}$ and of an *arrow object*:

$$
\begin{tikzcd}
\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \arrow[r, shift left, "P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}"] \arrow[r, shift right, "P^{\bullet}_A P_{\mathsf{Id}_A}"'] \arrow[r, phantom, "\Downarrow \alpha_A"] \arrow[dr, bend right, "P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}"'] & \Gamma.A \arrow[d, "P_A"] \\
& \Gamma
\end{tikzcd}
$$

for P_A, i.e. a 2-cell α_A of $\mathcal{C}/\Gamma$ into P_A as in the diagram that induces by postcomposition a family of isomorphisms of categories:

$$
(\mathcal{C}/\Gamma)(\ h\ ,\ P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}\ ) \ \ \to\ \ (\mathcal{C}/\Gamma)(\ h\ ,\ P_A\ )^{\to}
$$

2-natural in h, where h is any 1-cell $\Delta \to \Gamma$, in such a way that the *stability conditions*:

$$
\mathsf{Id}_A[f^{\bullet\bullet}] = \mathsf{Id}_{A[f]} \qquad \text{and} \qquad \alpha_A[f] = \alpha_{A[f]}
$$

hold for every 1-cell $f : \Delta \to \Gamma$.

Then we say that $(\mathcal{C}, \mathcal{D})$ is *endowed with axiomatic =-types.*

Notation and properties. Let a, b be sections of a display map $P_A : \Gamma.A \to \Gamma$ and let p be an arrow $a \Rightarrow b$ in the category $(\mathcal{C}/\Gamma)(1_\Gamma, P_A)$ of sections of P_A. Thenm being α_A an arrow object, there is a unique arrow:

$$\tilde{p} : \Gamma \to \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A$$

in $(\mathcal{C}/\Gamma)(1_\Gamma, P_A P_{A^\blacktriangledown} P_{\mathsf{Id}_A})$ such that $\alpha_A * \tilde{p} = p$. In particular:

$$P_{A^\blacktriangledown} P_{\mathsf{Id}_A} \tilde{p} = a \quad \text{and} \quad P^\bullet_A P_{\mathsf{Id}_A} \tilde{p} = b$$

which means that $P_{\mathsf{Id}_A} \tilde{p} = a; b$ i.e. the square:

$$\begin{tikzcd}
\Gamma \arrow[r, "\tilde{p}"] \arrow[d, equal] & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \arrow[d] \\
\Gamma \arrow[r, "a;b"] & \Gamma.A.A^\blacktriangledown
\end{tikzcd}$$

commutes. We denote as $\{p\}$ the unique section of $P_{\mathsf{Id}_A[a;b]}$ such that:

$$\begin{tikzcd}
\Gamma \arrow[dr, "\{p\}"] \arrow[drr, bend left, "\tilde{p}"] \arrow[ddr, bend right, equal] & & \\
& \Gamma.\mathsf{Id}_A[a;b] \arrow[r] \arrow[d] & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \arrow[d] \\
& \Gamma \arrow[r, "a;b"] & \Gamma.A.A^\blacktriangledown
\end{tikzcd}$$

commutes i.e. the unique section of $P_{\mathsf{Id}_A[a;b]}$ such that:

$$\alpha_A (a;b)^\bullet \{p\} = p \quad . \tag{1}$$

Now, let f be any arrow $\Delta \to \Gamma$ and let us observe that the diagram:

$$\begin{tikzcd}
\Delta \arrow[r, "f"] \arrow[d, "\{p\}[f]"'] & \Gamma \arrow[d, "\{p\}"] \\
\Delta.\mathsf{Id}_{A[f]}[a[f];b[f]] \arrow[r, "f^\bullet"] \arrow[d, "(a[f];b[f])^\bullet"'] & \Gamma.\mathsf{Id}_A[a;b] \arrow[d, "(a;b)^\bullet"] \\
\Delta.A[f].A[f]^\blacktriangledown.\mathsf{Id}_{A[f]} \arrow[r, "f^{\bullet\bullet\bullet}"] \arrow[d, shift right=3] \arrow[d, shift left=3] \arrow[d, phantom, "\overset{\alpha_A[f]}{\Longrightarrow}"] & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \arrow[d, shift right=3] \arrow[d, shift left=3] \arrow[d, phantom, "\overset{\alpha_A}{\Rightarrow}"] \\
\Delta.A[f] \arrow[r, "f^\bullet"] & \Gamma.A
\end{tikzcd}$$

commutes: the upper and lower squares commute by definition, while the middle one commutes because Diagram (9) does. By (1) and since $\alpha_A[f]$ is $\alpha_{A[f]}$, this implies that:

$$\{p\}[f] = \{p[f]\} \tag{2}$$

for every arrow p in the category of the section of P_A and every substitution f for the context of P_A.

Let $(\boldsymbol{\mathcal{C}}, \mathcal{D})$ be a display map 2-category. Clearly $(\boldsymbol{\mathcal{C}}, \mathcal{D})$ is in particular a display map category $(\mathbf{C}, \mathcal{D})$, where the category $\mathbf{C}$ is the underlying category of $\boldsymbol{\mathcal{C}}$: the class $\mathcal{D}$ of display maps over $\boldsymbol{\mathcal{C}}$ according to Definition 4.1 is in particular a class of display maps over $\mathbf{C}$ according to Definition 2.1.

Now, let us assume that the display map 2-category $(\boldsymbol{\mathcal{C}}, \mathcal{D})$ is endowed with axiomatic =-types—Definition 4.2—and let P_A be a display map of codomain Γ. The next paragraphs show that the display map category $(\mathbf{C}, \mathcal{D})$ induced by $(\boldsymbol{\mathcal{C}}, \mathcal{D})$ is endowed with appropriate choice functions validating axiomatic =-types—as in Definition 3.1.

Form Rule for =-types. We already have a choice of a display map $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \to \Gamma.A.A^{\blacktriangledown}$.

Intro Rule for =-types. Being α_A an arrow object, the identity 2-cell:

$$\begin{tikzcd}
\Gamma.A \arrow[r, bend left, "1_{\Gamma.A}"] \arrow[r, bend right, "1_{\Gamma.A}"'] \arrow[r, phantom, "\Downarrow 1_{1_{\Gamma.A}}"] \arrow[dr, bend right, "P_A"'] & \Gamma.A \arrow[d, "P_A"] \\
& \Gamma
\end{tikzcd}$$

of $1_{\Gamma.A}$ in $\boldsymbol{\mathcal{C}}/\Gamma$ factors uniquely through α_A as a 1-cell:

$$\begin{tikzcd}
\Gamma.A \arrow[r, "\mathsf{r}_A"] \arrow[dr, bend right, "P_A"'] & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \arrow[d, "P_{\mathsf{Id}_A} P_{A^{\blacktriangledown}} P_A"] \\
& \Gamma
\end{tikzcd}$$

of $\boldsymbol{\mathcal{C}}/\Gamma$ such that $\alpha_A * \mathsf{r}_A = 1_{1_{\Gamma.A}}$. In particular:

$$P_{A^{\blacktriangledown}}(P_{\mathsf{Id}_A}\mathsf{r}_A) = 1_{\Gamma.A} \qquad P_A^{\bullet}(P_{\mathsf{Id}_A}\mathsf{r}_A) = 1_{\Gamma.A}$$

hence $P_{\mathsf{Id}_A}\mathsf{r}_A = \delta_A$—being:

$$\begin{tikzcd}
\Gamma.A.A^{\blacktriangledown} \arrow[r, "P_A^{\bullet}"] \arrow[d, "P_{A^{\blacktriangledown}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "P_A"] \\
\Gamma.A \arrow[r, "P_A"] & \Gamma
\end{tikzcd}$$

a pullback square—and therefore there is unique a section:

$$\mathsf{refl}_A : \Gamma.A \to \Gamma.A.\mathsf{Id}_A[\delta_A]$$

of $P_{\mathsf{Id}_A[\delta_A]}$ such that $\delta_A^{\bullet}\mathsf{refl}_A = \mathsf{r}_A$.

Elim Rule for =-types. Let $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A.C \to \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$ be a display map and let c be a section $\Gamma.A \to \Gamma.A.C[\mathsf{r}_A]$ of the display map $P_{C[\mathsf{r}_A]}$. The pair $(\alpha_A, 1_{P_A^{\bullet}P_{\mathsf{Id}_A}})$ constitutes an arrow from α_A to $\alpha_A * \mathsf{r}_A(P_A^{\bullet}P_{\mathsf{Id}_A})$ in the category $(\boldsymbol{\mathcal{C}}/\Gamma)(P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}, P_A)^{\to}$. Hence, it is induced, by postcomposition via α_A, by a unique arrow:

$$\varphi_A : 1_{\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A} \Longrightarrow \mathsf{r}_A(P_A^{\bullet}P_{\mathsf{Id}_A})$$

of the category $(\boldsymbol{\mathcal{C}}/\Gamma)(P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}, P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A})$. In other words, the equalities:

$$P_{A^{\blacktriangledown}}P_{\mathsf{Id}_A} * \varphi_A = \alpha_A \quad \text{and} \quad P_A^{\bullet}P_{\mathsf{Id}_A} * \varphi_A = 1_{P_A^{\bullet}P_{\mathsf{Id}_A}}$$

hold. Moreover:

$$P_{A^{\blacktriangledown}}P_{\mathsf{Id}_A} * \varphi_A * \mathsf{r}_A = \alpha_A * \mathsf{r}_A = 1_{1_{\Gamma.A}} = P_{A^{\blacktriangledown}}P_{\mathsf{Id}_A} * 1_{\mathsf{r}_A}$$
$$P_A^{\bullet}P_{\mathsf{Id}_A} * \varphi_A * \mathsf{r}_A = 1_{P_A^{\bullet}P_{\mathsf{Id}_A}} * \mathsf{r}_A = 1_{1_{\Gamma.A}} = P_A^{\bullet}P_{\mathsf{Id}_A} * 1_{\mathsf{r}_A}$$

therefore $\varphi_A * \mathsf{r}_A$ and 1_{r_A} induce, by postcomposition via α_A, the same arrow $\alpha_A * \mathsf{r}_A \to \alpha_A * \mathsf{r}_A$, i.e. $1_{1_{\Gamma.A}} \to 1_{1_{\Gamma.A}}$, in $(\mathcal{C}/\Gamma)(P_A, P_A)^{\to}$ — namely the pair $(1_{1_{\Gamma.A}}, 1_{1_{\Gamma.A}})$. By the universality of α_A, we conclude that:

$$\varphi_A * \mathsf{r}_A = 1_{\mathsf{r}_A}.$$

Now, let $\tilde{\mathsf{J}}_c := (\mathsf{r}_A.C)cP^{\bullet}_A P_{\mathsf{Id}_A}$. We observe that:

$$P_C \tilde{\mathsf{J}}_c = \mathsf{r}_A P_{C[\mathsf{r}_A]} c P^{\bullet}_A P_{\mathsf{Id}_A} = \mathsf{r}_A P^{\bullet}_A P_{\mathsf{Id}_A}$$

hence we can rewrite the codomain of the 2-cell φ_A as follows:

$$\begin{tikzcd} \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \arrow[r, "\tilde{\mathsf{J}}_c"] \arrow[dr, equal, bend right] & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A.C \arrow[d, "P_C"] \arrow[dl, phantom, "\overset{\Longrightarrow}{\varphi_A}"] \\ & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \end{tikzcd}$$

and, using the cloven isofibration structure on P_C, we obtain a section $\mathsf{t}^{\varphi_A}_{\tilde{\mathsf{J}}_c} : \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \to \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A.C$ of P_C, as well as a 2-cell:

$$\begin{tikzcd} \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \arrow[r, bend left, "\mathsf{t}^{\varphi_A}_{\tilde{\mathsf{J}}_c}"] \arrow[r, bend right, "\tilde{\mathsf{J}}_c"'] \arrow[r, phantom, "\Downarrow \tau^{\varphi_A}_{\tilde{\mathsf{J}}_c}"] & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A.C \end{tikzcd}$$

such that $P_C * \tau^{\varphi_A}_{\tilde{\mathsf{J}}_c} = \varphi_A$. We define $\mathsf{J}_c := \mathsf{t}^{\varphi_A}_{\tilde{\mathsf{J}}_c}$.

Comp Axiom for =-types. Referring to the diagram:

$$\begin{tikzcd} \Gamma.A \arrow[r, "\mathsf{r}_A"] \arrow[d, shift right, "\mathsf{J}_c[\mathsf{r}_A]"'] \arrow[d, shift left, "c"] & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \arrow[d, shift right, "\mathsf{J}_c"'] \arrow[d, shift left, "\tilde{\mathsf{J}}_c"] \arrow[d, phantom, "\Rightarrow"] \\ \Gamma.A.C[\mathsf{r}_A] \arrow[r, "\mathsf{r}^{\bullet}_A"] \arrow[d, "P_{C[\mathsf{r}_A]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A.C \arrow[d, "P_C"] \\ \Gamma.A \arrow[r, "\mathsf{r}_A"] & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \end{tikzcd} \tag{3}$$

and observing that $c = \tilde{\mathsf{J}}_c[\mathsf{r}_A]$ [3] and that:

$$P_C * \tau^{\varphi_A}_{\tilde{\mathsf{J}}_c} * \mathsf{r}_A = P_C * \tau^{\varphi_A * \mathsf{r}_A}_{\tilde{\mathsf{J}}_c \mathsf{r}_A} = \varphi_A * \mathsf{r}_A = 1_{\mathsf{r}_A} = \mathsf{r}_A * 1_{1_{\Gamma.A}}$$

we conclude, by the 2-universal property of $\Gamma.A.C[\mathsf{r}_A]$, that there is unique a 2-cell:

$$\begin{tikzcd} \Gamma.A \arrow[r, bend left, "\mathsf{J}_c[\mathsf{r}_A]"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow h_c"] & \Gamma.A.C[\mathsf{r}_A] \end{tikzcd}$$

such that $P_{C[\mathsf{r}_A]} * h_c = 1_{1_{\Gamma.A}}$ and $\mathsf{r}^{\bullet}_A * h_c = \tau^{\varphi_A}_{\tilde{\mathsf{J}}_c} * \mathsf{r}_A$. Then we define H_C as the section:

$$\{h_c\} : \Gamma.A \to \Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c]$$

[3] Since $P_C \tilde{\mathsf{J}}_c \mathsf{r}_A = P_C \mathsf{r}^{\bullet}_A c P^{\bullet}_A P_{\mathsf{Id}_A} \mathsf{r}_A = P_C \mathsf{r}^{\bullet}_A c = \mathsf{r}_A P_{C[\mathsf{r}_A]} c = \mathsf{r}_A 1_{\Gamma.A}$ and $P_{C[\mathsf{r}_A]} c = 1_{\Gamma.A}$ and $\mathsf{r}^{\bullet}_A c = \mathsf{r}^{\bullet}_A c P^{\bullet}_A P_{\mathsf{Id}_A} \mathsf{r}_A = \mathsf{J}_c \mathsf{r}_A$.

satisfying (1), which we recall to be defined as follows: by the universal property of $\alpha_{C[\mathsf{r}_A]}$, there is unique an arrow $\tilde{h}_c : \Gamma.A \to \Gamma.A.C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown.\mathsf{Id}_{C[\mathsf{r}_A]}$ over $\Gamma.A$ such that $\alpha_{C[\mathsf{r}_A]} * \tilde{h}_c = h_c$. In particular:

$$P_{C[\mathsf{r}_A]^\blacktriangledown} P_{\mathsf{Id}_{C[\mathsf{r}_A]}} \tilde{h}_c = \mathsf{J}_c[\mathsf{r}_A] \quad \text{ and } \quad P^\bullet_{C[\mathsf{r}_A]} P_{\mathsf{Id}_{C[\mathsf{r}_A]}} \tilde{h}_c = c$$

which means that $P_{\mathsf{Id}_{C[\mathsf{r}_A]}} \tilde{h}_c = \mathsf{J}_c[\mathsf{r}_A]; c$. Hence the diagram:

$$\begin{tikzcd}
\Gamma.A \arrow[d, "\mathsf{H}_c"'] \arrow[dr, bend left, "\tilde{h}_c"] & \\
\Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown.\mathsf{Id}_{C[\mathsf{r}_A]} \arrow[d] \\
\Gamma.A \arrow[r, "\mathsf{J}_c[\mathsf{r}_A];c"'] & \Gamma.A.C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown
\end{tikzcd}$$

commutes for a unique section:

$$\{h_c\} : \Gamma.A \to \Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c]$$

of $P_{\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A];c]}$.

Proposition 4.3. *Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic =-types. Referring to the data defined in paragraphs* **Form Rule, Intro Rule, Elim Rule, and Comp Axiom, for =-types***, the* stability conditions*:*

$$\begin{array}{cc}
\mathsf{Id}_A[f^{\bullet\bullet}] = \mathsf{Id}_{A[f]} & \mathsf{J}_c[f^{\bullet\bullet\bullet}] = \mathsf{J}_{c[f^\bullet]} \\
\mathsf{refl}_A[f^\bullet] = \mathsf{refl}_{A[f]} & \mathsf{H}_c[f^\bullet] = \mathsf{H}_{c[f^\bullet]}
\end{array}$$

of Defintion 3.1 hold for every arrow $\Delta \xrightarrow{f} \Gamma$.

Proof. See Appendix C. □

4.2 Axiomatic Σ-types

We specialise the notion of a display map 2-category with axiomatic =-types to obtain a version that incorporates a 2-dimensional semantic interpretation of axiomatic Σ-types.

Definition 4.4 (Semantics of axiomatic Σ-types—categorical formulation)**.** Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic =-types.

Let us assume that, for every object Γ, every display map P_A of codomain Γ, and every display map P_B of codomain $\Gamma.A$, there is a choice of a display map $\Gamma.\Sigma^B_A \to \Gamma$ and of an *equivalence*:

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, "(\mathsf{p}^B_A, \pi^B_A, \eta^B_A, \beta^B_A)"] \arrow[d, "P_B"'] & \Gamma.\Sigma^B_A \arrow[d, "P_{\Sigma^B_A}"] \\
\Gamma.A \arrow[r, "P_A"] & \Gamma
\end{tikzcd}$$

from $P_A P_B$ to $P_{\Sigma^B_A}$, i.e. a choice of objects:

$$\mathsf{p}^B_A : \Gamma.A.B \to \Gamma.\Sigma^B_A \quad \text{ and } \quad \pi^B_A : \Gamma.\Sigma^B_A \to \Gamma.A.B$$

of $(\mathcal{C}/\Gamma)(P_A P_B, P_{\Sigma_A^B})$ and $(\mathcal{C}/\Gamma)(P_{\Sigma_A^B}, P_A P_B)$ respectively and of arrows:

$$\eta_A^B : 1_{\Gamma.\Sigma_A^B} \Longrightarrow \mathsf{p}_A^B \pi_A^B \quad \text{and} \quad \beta_A^B : \pi_A^B \mathsf{p}_A^B \Longrightarrow 1_{\Gamma.A.B}$$

of $(\mathcal{C}/\Gamma)(P_{\Sigma_A^B}, P_{\Sigma_A^B})$ and $(\mathcal{C}/\Gamma)(P_A P_B, P_A P_B)$ respectively, in such a way that the *stability conditions*:

$$\Sigma_A^B[f] = \Sigma_{A[f]}^{B[f^\bullet]} \qquad \begin{array}{ll} \mathsf{p}_A^B[f] = \mathsf{p}_{A[f]}^{B[f^\bullet]} & \pi_A^B[f] = \pi_{A[f]}^{B[f^\bullet]} \\ \eta_A^B[f] = \eta_{A[f]}^{B[f^\bullet]} & \beta_A^B[f] = \beta_{A[f]}^{B[f^\bullet]} \end{array}$$

hold for every 1-cell $f : \Delta \to \Gamma$.

Then we say that $(\mathcal{C}, \mathcal{D})$ is *endowed with axiomatic Σ-types*.

Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic $=$-types.

Let us assume that $(\mathcal{C}, \mathcal{D})$ is endowed with axiomatic Σ-types—Definition 4.4—let P_A be a display map of codomain Γ and let P_B be a display map of codomain $\Gamma.A$. The next paragraphs show that the display map category $(\mathbf{C}, \mathcal{D})$ induced by $(\mathcal{C}, \mathcal{D})$ is endowed with appropriate choice functions validatin axiomatic Σ-types—as in Definition 3.2.

Form Rule for Σ-types. We already have a choice of a display map $\Gamma.\Sigma_A^B \to \Gamma$.

Intro Rule for Σ-types. Since $P_{\Sigma_A^B} \mathsf{p}_A^B = P_A P_B$ there is unique a section:

$$\mathsf{pair}_A^B : \Gamma.A.B \to \Gamma.A.B.\Sigma_A^B[P_A P_B]$$

of $P_{\Sigma_A^B[P_A P_B]}$ such that $(P_A P_B)^\bullet \mathsf{refl}_A = \mathsf{p}_A^B$.

Elim Rule for Σ-types. Let $\Gamma.\Sigma_A^B.C \to \Gamma.\Sigma_A^B$ be a display map and let c be a section $\Gamma.A.B \to \Gamma.A.B.C[\mathsf{p}_A^B]$ of the display map $P_{C[\mathsf{p}_A^B]}$. Let $\tilde{\mathsf{split}}_c := (\mathsf{p}_A^B)^\bullet c \pi_A^B$. We observe that:

$$P_C \tilde{\mathsf{split}}_c = \mathsf{p}_A^B P_{C[\mathsf{p}_A^B]} c \pi_A^B = \mathsf{p}_A^B \pi_A^B$$

hence we can rewrite the codomain of the 2-cell η_A^B as follows:

$$\begin{tikzcd} \Gamma.\Sigma_A^B \arrow[r, "\tilde{\mathsf{split}}_c"] \arrow[dr, equal, bend right] & \Gamma.\Sigma_A^B.C \arrow[d, "P_C"] \arrow[dl, phantom, "\overset{\Longrightarrow}{\eta_A^B}"] \\ & \Gamma.\Sigma_A^B \end{tikzcd}$$

and, using the cloven isofibration structure on P_C, we obtain a section $\mathsf{t}^{\eta_A^B}_{\tilde{\mathsf{split}}_c} : \Gamma.\Sigma_A^B \to \Gamma.\Sigma_A^B.C$ of P_C, as well as a 2-cell:

$$\begin{tikzcd} \Gamma.\Sigma_A^B \arrow[r, bend left, "\mathsf{t}^{\eta_A^B}_{\tilde{\mathsf{split}}_c}"] \arrow[r, bend right, "\tilde{\mathsf{split}}_c"'] \arrow[r, phantom, "\Downarrow \tau^{\eta_A^B}_{\tilde{\mathsf{split}}_c}"] & \Gamma.\Sigma_A^B.C \end{tikzcd}$$

such that $P_C * \tau^{\eta_A^B}_{\tilde{\mathsf{split}}_c} = \eta_A^B$. We define $\mathsf{split}_c := \mathsf{t}^{\eta_A^B}_{\tilde{\mathsf{split}}_c}$.

Comp Axiom for Σ-types. Let us condider the 2-cell:

$$\mathsf{split}_c \mathsf{p}_A^B \xrightarrow{\tau^{\eta_A^B}_{\tilde{\mathsf{split}}_c} * \mathsf{p}_A^B} \tilde{\mathsf{split}}_c \mathsf{p}_A^B \xrightarrow{(\mathsf{p}_A^B)^\bullet c * \beta_A^B} (\mathsf{p}_A^B)^\bullet c$$

and observe that:

$$P_C * (\mathsf{split}_c \mathsf{p}_A^B \Longrightarrow (\mathsf{p}_A^B)^\bullet c) = (\eta_A^B * \mathsf{p}_A^B)(\mathsf{p}_A^B * \beta_A^B) = 1_{\mathsf{p}_A^B}$$

since, being the given $(\mathsf{p}_A^B, \pi_A^B, \eta_A^B, \beta_A^B)$ an equivalence, we can assume without loss of generality that it is in fact an adjoint equivalence—up to replacing η_A^B with a parallel 2-cell [29], [7], [18, Chapter 6]. Therefore, referring to the diagram:

$$\begin{tikzcd}
\Gamma.A.B \arrow[rr, equal] \arrow[d, shift right=3, "\mathsf{split}_c[\mathsf{p}_A^B]"'] \arrow[d, shift left=3, "c"] & & \Gamma.A.B \arrow[d, shift right=3, "\mathsf{split}_c \mathsf{p}_A^B"'] \arrow[d, shift left=3, "(\mathsf{p}_A^B)^\bullet c"] \arrow[d, phantom, "\Rightarrow"] \\
\Gamma.A.B.C[\mathsf{p}_A^B] \arrow[rr, "(\mathsf{p}_A^B)^\bullet"] \arrow[d, "P_{C[\mathsf{p}_A^B]}"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \Gamma.\Sigma_A^B.C \arrow[d, "P_C"] \\
\Gamma.A.B \arrow[rr, "\mathsf{p}_A^B"] & & \Gamma.\Sigma_A^B
\end{tikzcd} \tag{4}$$

by the 2-universal property of $\Gamma.A.B.C[\mathsf{p}_A^B]$ there is unique a 2-cell:

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, bend left, "\mathsf{split}_c[\mathsf{p}_A^B]"{name=U}] \arrow[r, bend right, "c"'{name=D}] \arrow[r, phantom, "\Downarrow s_c"] & \Gamma.A.B.C[\mathsf{p}_A^B]
\end{tikzcd}$$

such that $P_{C[\mathsf{p}_A^B]} * s_c = 1_{1_{\Gamma.A.B}}$ and $(\mathsf{split}_c \mathsf{p}_A^B \Rightarrow (\mathsf{p}_A^B)^\bullet c) = (\mathsf{p}_A^B)^\bullet * s_c$. As for the paragraph **Comp Axiom for =-types**, we define σ_c as the unique section:

$$\{s_c\} : \Gamma.A.B \to \Gamma.A.B.\mathsf{Id}_{C[\mathsf{p}_A^B]}[\mathsf{split}_c[\mathsf{p}_A^B]; c]$$

of $P_{\mathsf{Id}_{C[\mathsf{p}_A^B]}[\mathsf{split}_c[\mathsf{p}_A^B];c]}$ such that (1) is satisfied—where A is $C[\mathsf{p}_A^B]$, a and b are $\mathsf{split}_c[\mathsf{p}_A^B]$ and c respectively, and p is s_c.

Proposition 4.5. *Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic =-types and axiomatic Σ-types. Referring to the data defined in paragraphs* **Form Rule, Intro Rule, Elim Rule, and Comp Axiom, for Σ-types***, the* stability conditions*:*

$$\begin{array}{cc}
\Sigma_A^B[f] = \Sigma_{A[f]}^{B[f^\bullet]} & \mathsf{split}_c[f^\bullet] = \mathsf{split}_{c[f^{\bullet\bullet}]} \\
\mathsf{pair}_A^B[f^{\bullet\bullet}] = \mathsf{pair}_{A[f]}^{B[f^\bullet]} & \sigma_c[f^{\bullet\bullet}] = \sigma_{c[f^{\bullet\bullet}]}
\end{array}$$

of Defintion 3.2 hold for every arrow $\Delta \xrightarrow{f} \Gamma$.

Proof. See Appendix C. □

4.3 Axiomatic Π-types

We specialise the notion of a display map 2-category with axiomatic =-types to obtain a version that incorporates a 2-dimensional semantic interpretation of axiomatic Π-types & axiomatic function extensionality.

Definition 4.6 (Semantics of axiomatic Π-types & axiomatic function extensionality—categorical formulation)**.** Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic =-types.

Let us assume that, for every object Γ, every display map P_A of codomain Γ, and every display map P_B of codomain $\Gamma.A$, there is a choice of a display map $\Gamma.\mathsf{\Pi}_A^B \to \Gamma$ in such a way that:

$$\Pi_A^B[f] = \Pi_{A[f]}^{B[f^\bullet]}$$

for every arrow $f : \Delta \to \Gamma$, and in such a way that the assignment:

$$\begin{aligned} \mathcal{D}/\Gamma.A &\to \boldsymbol{\mathcal{C}}/\Gamma \\ P_B &\mapsto P_{\mathsf{\Pi}_A^B} \end{aligned}$$

constitutes a right I_A-relative 2-coadjoint to the 2-functor:

$$\begin{aligned} \boldsymbol{\mathcal{C}}/\Gamma &\to \boldsymbol{\mathcal{C}}/\Gamma.A \\ f &\mapsto f^\bullet \end{aligned}$$

where I_A is the full forgetful 2-functor $\mathcal{D}/\Gamma.A \hookrightarrow \boldsymbol{\mathcal{C}}/\Gamma.A$.

Unfolding this requirement, this means that there is a family of equivalences of categories:

$$(\lambda_A^B, \mathsf{app}_A^B, \mathsf{app}_A^B\lambda_A^B \overset{\beta^{A,B}}{\Longrightarrow} 1, 1 \overset{\eta^{A,B}}{\Longrightarrow} \lambda_A^B\mathsf{app}_A^B) : (\boldsymbol{\mathcal{C}}/\Gamma.A)(f^\bullet, P_B) \simeq (\boldsymbol{\mathcal{C}}/\Gamma)(f, P_{\mathsf{\Pi}_A^B})$$

2-natural in $f : \Delta \to \Gamma$ and P_B—which means that the diagrams of 1-functors:

$$\begin{tikzcd} (\boldsymbol{\mathcal{C}}/\Gamma.A)(f^\bullet, P_B) \arrow[r, shift left, "\lambda_A^B"] \arrow[d, "(_)(g^\bullet)"] & (\boldsymbol{\mathcal{C}}/\Gamma)(f, P_{\mathsf{\Pi}_A^B}) \arrow[l, shift left, "\mathsf{app}_A^B"] \arrow[d, "(_)(g)"] \\ (\boldsymbol{\mathcal{C}}/\Gamma.A)((fg)^\bullet, P_B) \arrow[r, shift left, "\lambda_A^B"] & (\boldsymbol{\mathcal{C}}/\Gamma)(fg, P_{\mathsf{\Pi}_A^B}) \arrow[l, shift left, "\mathsf{app}_A^B"] \end{tikzcd}
\qquad
\begin{tikzcd} (\boldsymbol{\mathcal{C}}/\Gamma.A)(f^\bullet, P_B) \arrow[r, shift left, "\lambda_A^B"] \arrow[d, "1_{\Delta.A[f]};_"] & (\boldsymbol{\mathcal{C}}/\Gamma)(f, P_{\mathsf{\Pi}_A^B}) \arrow[l, shift left, "\mathsf{app}_A^B"] \arrow[d, "1_\Delta;_"] \\ (\boldsymbol{\mathcal{C}}/\Delta.A[f])(1_\Delta^\bullet, P_{B[f^\bullet]}) \arrow[r, shift left, "\lambda_{A[f]}^{B[f.A]}"] & (\boldsymbol{\mathcal{C}}/\Delta)(1_\Delta, P_{\mathsf{\Pi}_{A[f]}^{B[f^\bullet]}}) \arrow[l, shift left, "\mathsf{app}_{A[f]}^{B[f.A]}"] \end{tikzcd}$$

commute and the equalities:

$$\begin{aligned} \beta_h^{A,B} g^\bullet &= \beta_{hg^\bullet}^{A,B} & \qquad 1_{\Delta.A[f]}; \beta_h^{A,B} &= \beta_{1_{\Delta.A[f]};h}^{A[f],B[f.A]} \\ \eta_k^{A,B} g &= \eta_{kg}^{A,B} & 1_\Delta; \eta_k^{A,B} &= \eta_{1_\Delta;k}^{A[f],B[f.A]} \end{aligned}$$

hold, for every $g : \Delta' \to \Delta$, every $h : f^\bullet \to P_B$, and every $k : f \to P_{\mathsf{\Pi}_A^B}$.

Then we say that $(\boldsymbol{\mathcal{C}}, \mathcal{D})$ is *endowed with axiomatic* Π*-types & axiomatic function extensionality.*

Let $(\boldsymbol{\mathcal{C}}, \mathcal{D})$ be a display map 2-category endowed with axiomatic $=$-types.

Let us assume that the display map 2-category $(\boldsymbol{\mathcal{C}}, \mathcal{D})$ is endowed with axiomatic Π-types & axiomatic function extensionality—Definition 4.6—let P_A be a display map of codomain Γ and let P_B be a display map of codomain $\Gamma.A$. The next paragraphs show that the display map category $(\mathbf{C}, \mathcal{D})$ induced by $(\boldsymbol{\mathcal{C}}, \mathcal{D})$ is endowed with appropriate choice functions validating axiomatic Π-types & axiomatic function extensionality—as in Definition 3.3.

Form Rule for Π-types. We already have a choice of a display map $\Gamma.\Pi_A^B \to \Gamma$.

Elim Rule for Π-types. The diagram:

$$\begin{tikzcd}
(\mathcal{C}/\Gamma.A)(P^{\bullet}_{\mathsf{\Pi}^B_A}, P_B) & & (\mathcal{C}/\Gamma)(P_{\mathsf{\Pi}^B_A}, P_{\mathsf{\Pi}^B_A}) \arrow[ll, "\mathsf{app}^B_A"'] \arrow[d, "1_{\Gamma.\mathsf{\Pi}^B_A};-"] \\
(\mathcal{C}/\Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}])(1^{\bullet}_{\Gamma.\mathsf{\Pi}^B_A}, P_{B[P^{\bullet}_{\mathsf{\Pi}^B_A}]}) \arrow[from=1-1, "1_{\Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}]};-"] & & (\mathcal{C}/\Gamma.\mathsf{\Pi}^B_A)(1_{\Gamma.\mathsf{\Pi}^B_A}, P_{\mathsf{\Pi}^{B\blacktriangledown}_A}) \arrow[ll, "\mathsf{app}^{B[P^{\bullet}_{\mathsf{\Pi}^B_A}]}_{A[P_{\mathsf{\Pi}^B_A}]}"]
\end{tikzcd}$$

commutes—observe that $\mathsf{\Pi}^{B\blacktriangledown}_A = \mathsf{\Pi}^B_A[P_{\mathsf{\Pi}^B_A}] = \mathsf{\Pi}^{B[P^{\bullet}_{\mathsf{\Pi}^B_A}]}_{A[P_{\mathsf{\Pi}^B_A}]}$. Hence we may define ev^B_A as the section:

$$\mathsf{app}^{B[P^{\bullet}_{\mathsf{\Pi}^B_A}]}_{A[P_{\mathsf{\Pi}^B_A}]}\delta_{\mathsf{\Pi}^B_A} = \mathsf{app}^{B[P^{\bullet}_{\mathsf{\Pi}^B_A}]}_{A[P_{\mathsf{\Pi}^B_A}]}(1_{\Gamma.\mathsf{\Pi}^B_A};1_{\Gamma.\mathsf{\Pi}^B_A}) = 1_{\Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}]};\mathsf{app}^B_A 1_{\Gamma.\mathsf{\Pi}^B_A}$$

of $P_{B[P^{\bullet}_{\mathsf{\Pi}^B_A}]}$.

Intro Rule for Π-types. If υ is a section of P_B then we define the section abst_υ of $P_{\mathsf{\Pi}^B_A}$ as the image of υ via the map $\lambda^B_A : (\mathcal{C}/\Gamma.A)(1_{\Gamma.A}, P_B) \to (\mathcal{C}/\Gamma)(1_\Gamma, P_{\mathsf{\Pi}^B_A})$.

Comp Axiom for Π-types. We observe that:

$$\mathsf{ev}^B_A[\mathsf{abst}^{\bullet}_\upsilon] = P^{\bullet\bullet}_{\mathsf{\Pi}^B_A}\mathsf{ev}^B_A\mathsf{abst}^{\bullet}_\upsilon$$

because the diagram:

$$\begin{tikzcd}
\Gamma.A \arrow[r, "z^{\bullet}"] \arrow[d, "\mathsf{ev}^B_A[z^{\bullet}]"'] & \Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}] \arrow[d, "\mathsf{ev}^B_A"] & \\
\Gamma.A.B \arrow[r, "z^{\bullet\bullet}"] \arrow[rr, equal, bend right] & \Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}].B[P^{\bullet}_{\mathsf{\Pi}^B_A}] \arrow[r, "P^{\bullet\bullet}_{\mathsf{\Pi}^B_A}"] & \Gamma.A.B
\end{tikzcd}$$

commutes for every section z of $P_{\mathsf{\Pi}^B_A}$. Hence:

$$\begin{aligned}
\mathsf{ev}^B_A[z^{\bullet}] &= P^{\bullet\bullet}_{\mathsf{\Pi}^B_A}\mathsf{ev}^B_A z^{\bullet} \\
&= P^{\bullet\bullet}_{\mathsf{\Pi}^B_A}(1_{\Gamma.\mathsf{\Pi}^B_A.A[P_{\mathsf{\Pi}^B_A}]};\mathsf{app}^B_A 1_{\Gamma.\mathsf{\Pi}^B_A})z^{\bullet} \\
&= (\mathsf{app}^B_A 1_{\Gamma.\mathsf{\Pi}^B_A})z^{\bullet} \\
&= \mathsf{app}^B_A(1_{\Gamma.\mathsf{\Pi}^B_A}z) \\
&= \mathsf{app}^B_A z
\end{aligned}$$

that is:

$$\mathsf{ev}^B_A[z^{\bullet}] = \mathsf{app}^B_A z \tag{5}$$

for every section z of $P_{\mathsf{\Pi}^B_A}$. In particular:

$$\mathsf{ev}^B_A[\mathsf{abst}^{\bullet}_\upsilon] = \mathsf{app}^B_A\mathsf{abst}_\upsilon = \mathsf{app}^B_A\lambda^B_A\upsilon$$

and therefore $\beta_{\upsilon}^{A,B}$ is a morphism:

$$\mathsf{ev}_A^B[\mathsf{abst}_{\upsilon}^{\bullet}] \Rightarrow \upsilon$$

in the category $(\boldsymbol{\mathcal{C}}/\Gamma.A)(1_{\Gamma.A}, P_B)$ of the sections of P_B. As for the paragraphs **Comp Axiom for =-types** and **Comp Axiom for Σ-types**, we define β_{υ} as the unique section:

$$\{\beta_{\upsilon}^{A,B}\} : \Gamma.A \to \Gamma.A.\mathsf{Id}_B[\mathsf{ev}_A^B[\mathsf{abst}_{\upsilon}^{\bullet}];\upsilon]$$

of $P_{\mathsf{Id}_B[\mathsf{ev}_A^B[\mathsf{abst}_{\upsilon}^{\bullet}];\upsilon]}$ such that (1) is satisfied—where A is B, a and b are $\mathsf{ev}_A^B[\mathsf{abst}_{\upsilon}^{\bullet}]$ and υ respectively, and p is $\beta_{\upsilon}^{A,B}$.

Stability of app and λ under re-indexing. By naturalities, if z is a section of $P_{\mathsf{\Pi}_A^B}$, υ is a section of P_B, and f is an arrow $\Delta \to \Gamma$, we observe that:

$$\begin{aligned}
(\mathsf{app}_A^B z)[f^{\bullet}] &= 1_{\Delta.A[f]};(\mathsf{app}_A^B z)f^{\bullet} \\
&= 1_{\Delta.A[f]};\mathsf{app}_A^B(zf) \\
&= \mathsf{app}_{A[f]}^{B[f^{\bullet}]}(1_{\Delta};zf) \\
&= \mathsf{app}_{A[f]}^{B[f^{\bullet}]}(z[f])
\end{aligned}
\qquad\qquad
\begin{aligned}
(\lambda_A^B\upsilon)[f] &= 1_{\Delta};(\lambda_A^B\upsilon)f \\
&= 1_{\Delta};\lambda_A^B(\upsilon f^{\bullet}) \\
&= \lambda_{A[f]}^{B[f^{\bullet}]}(1_{\Delta.A[f]};\upsilon f^{\bullet}) \\
&= \lambda_{A[f]}^{B[f^{\bullet}]}(\upsilon[f^{\bullet}])
\end{aligned}$$

that is:

$$(\mathsf{app}_A^B z)[f^{\bullet}] = \mathsf{app}_{A[f]}^{B[f^{\bullet}]}(z[f]) \qquad\qquad (\lambda_A^B\upsilon)[f] = \lambda_{A[f]}^{B[f^{\bullet}]}(\upsilon[f^{\bullet}]) \quad . \tag{6}$$

In particular:

$$\begin{aligned}
\mathsf{ev}_A^B[f^{\bullet\bullet}] &= (\mathsf{app}_{A[P_{\mathsf{\Pi}_A^B}]}^{B[P_{\mathsf{\Pi}_A^B}^{\bullet}]}\delta_{\mathsf{\Pi}_A^B})[f^{\bullet\bullet}] \overset{(6)}{=} \mathsf{app}_{A[P_{\mathsf{\Pi}_A^B}][f^{\bullet}]}^{B[P_{\mathsf{\Pi}_A^B}^{\bullet}][f^{\bullet\bullet}]}\,\delta_{\mathsf{\Pi}_A^B}[f^{\bullet}] \\
&= \mathsf{app}_{A[f][P_{\mathsf{\Pi}_{A[f]}^{B[f^{\bullet}]}}]}^{B[f^{\bullet}][P_{\mathsf{\Pi}_{A[f]}^{B[f^{\bullet}]}}^{\bullet}]}\,\delta_{\mathsf{\Pi}_A^B[f]} = \mathsf{app}_{A[f][P_{\mathsf{\Pi}_{A[f]}^{B[f^{\bullet}]}}]}^{B[f^{\bullet}][P_{\mathsf{\Pi}_{A[f]}^{B[f^{\bullet}]}}^{\bullet}]}\,\delta_{\mathsf{\Pi}_{A[f]}^{B[f^{\bullet}]}} \\
&= \mathsf{ev}_{A[f]}^{B[f^{\bullet}]}
\end{aligned}$$

that is:

$$\mathsf{ev}_A^B[f^{\bullet\bullet}] = \mathsf{ev}_{A[f]}^{B[f^{\bullet}]} \quad . \tag{7}$$

Characterising happly. Referring to the notation of Remark 3.4, let:

$$\begin{tikzcd}
\Gamma.\mathsf{\Pi}_A^B.(\mathsf{\Pi}_A^B)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}_A^B} \arrow[r, shift left=2, "\tilde{\mathsf{s}}"] \arrow[r, shift right=2, "\tilde{\mathsf{t}}"'] \arrow[r, phantom, "\Downarrow\tilde{\alpha}_{\mathsf{\Pi}_A^B}"] & \Gamma.\mathsf{\Pi}_A^B.(\mathsf{\Pi}_A^B)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}_A^B}.\mathsf{Id}_{A[P_{\mathsf{\Pi}_A^B}\mathsf{s}]}^{B[(P_{\mathsf{\Pi}_A^B}\mathsf{s})^{\bullet}]}
\end{tikzcd}$$

be the re-indexing of $\alpha_{\mathsf{\Pi}_A^B}$ via $P_{\mathsf{\Pi}_A^B}\mathsf{s} : \Gamma.\mathsf{\Pi}_A^B.(\mathsf{\Pi}_A^B)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}_A^B} \to \Gamma$ and let:

$$\mathsf{happ} : \Gamma.\mathsf{\Pi}_A^B.(\mathsf{\Pi}_A^B)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}_A^B}.A[P_{\mathsf{\Pi}_A^B}\mathsf{s}] \to \Gamma.\mathsf{\Pi}_A^B.(\mathsf{\Pi}_A^B)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}_A^B}.A[P_{\mathsf{\Pi}_A^B}\mathsf{s}].\mathsf{Id}_{B[(P_{\mathsf{\Pi}_A^B}\mathsf{s})^{\bullet}]}[\mathsf{app}\,\tilde{\mathsf{s}};\mathsf{app}\,\tilde{\mathsf{t}}]$$

be $\{\mathsf{app}\,\tilde{\alpha}_{\mathsf{\Pi}_A^B}\}$ i.e. the re-indexing along $\mathsf{app}\,\tilde{\mathsf{s}};\mathsf{app}\,\tilde{\mathsf{t}}$ of the unique factorisation of $\mathsf{app}\,\tilde{\alpha}_{\mathsf{\Pi}_A^B}$ through the arrow object $\alpha_{B[(P_{\mathsf{\Pi}_A^B})^{\bullet}]}$—see (1). Now, we observe that:

$$\mathsf{app}\,\tilde{\mathsf{s}} = \mathsf{app}_{A[P_{\mathsf{\Pi}_A^B}\mathsf{s}]}^{B[(P_{\mathsf{\Pi}_A^B}\mathsf{s})^{\bullet}]}\tilde{\mathsf{s}} \overset{(5)}{=} \mathsf{ev}_{A[P_{\mathsf{\Pi}_A^B}\mathsf{s}]}^{B[(P_{\mathsf{\Pi}_A^B}\mathsf{s})^{\bullet}]}[\tilde{\mathsf{s}}^{\bullet}] = \mathsf{ev}_A^B[(P_{\mathsf{\Pi}_A^B}\mathsf{s})^{\bullet\bullet}][\tilde{\mathsf{s}}^{\bullet}] = \mathsf{ev}_A^B[((P_{\mathsf{\Pi}_A^B}\mathsf{s})^{\bullet}\tilde{\mathsf{s}})^{\bullet}] = \mathsf{ev}_A^B[\mathsf{s}^{\bullet}]$$

and analogously $\mathsf{app}\,\tilde{\mathsf{t}} = \mathsf{ev}_A^B[\mathsf{t}^{\bullet}]$. Therefore happ is an arrow:

$$\Gamma.\mathsf{\Pi}_A^B.(\mathsf{\Pi}_A^B)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}_A^B}.A[P_{\mathsf{\Pi}_A^B}\mathsf{s}] \to \Gamma.\mathsf{\Pi}_A^B.(\mathsf{\Pi}_A^B)^{\blacktriangledown}.\mathsf{Id}_{\mathsf{\Pi}_A^B}.A[P_{\mathsf{\Pi}_A^B}\mathsf{s}].\mathsf{Id}_{B[(P_{\mathsf{\Pi}_A^B}\mathsf{s})^{\bullet}]}[\mathsf{ev}_A^B[\mathsf{s}^{\bullet}];\mathsf{ev}_A^B[\mathsf{t}^{\bullet}]]$$

and hence $\lambda\,\mathsf{happ}$ is parallel to happly. We can now state and prove the following:

Theorem 4.7 (The happy theorem)**.** *Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic $=$-types and axiomatic Π-types & axiomatic function extensionality. Then there exists an arrow:*

$$\mathsf{happly} \Rightarrow \lambda\,\mathsf{happ}$$

in the category of the sections of the display map:

$$\Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}.\left(\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}\mathsf{s}]}^{\mathsf{Id}_{B[(P_{\mathsf{\Pi}^B_A}\mathsf{s})^\bullet]}[\mathsf{ev}^B_A[\mathsf{s}^\bullet];\mathsf{ev}^B_A[\mathsf{t}^\bullet]]}\right) \to \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}$$

which is stable under re-indexing.

Proof. Using the appropriate arrow object, it is enough to build a section of the display map:

$$\Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}.\mathsf{Id}_{\left(\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}\mathsf{s}]}^{\mathsf{Id}_{B[(P_{\mathsf{\Pi}^B_A}\mathsf{s})^\bullet]}[\mathsf{ev}^B_A[\mathsf{s}^\bullet];\mathsf{ev}^B_A[\mathsf{t}^\bullet]]}\right)}[\mathsf{happly};\lambda\,\mathsf{happ}] \to \Gamma.\mathsf{\Pi}^B_A.(\mathsf{\Pi}^B_A)^\blacktriangledown.\mathsf{Id}_{\mathsf{\Pi}^B_A}$$

and, by elimination, such a section can be defined as J_c if we build a section c of the display map:

$$\Gamma.\mathsf{\Pi}^B_A.\mathsf{Id}_{\left(\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}\mathsf{s}]}^{\mathsf{Id}_{B[(P_{\mathsf{\Pi}^B_A}\mathsf{s})^\bullet]}[\mathsf{ev}^B_A[\mathsf{s}^\bullet];\mathsf{ev}^B_A[\mathsf{t}^\bullet]]}\right)}[\mathsf{happly};\lambda\,\mathsf{happ}][\mathsf{r}_{\mathsf{\Pi}^B_A}] \to \Gamma.\mathsf{\Pi}^B_A$$

that is:

$$\Gamma.\mathsf{\Pi}^B_A.\mathsf{Id}_{\left(\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}]}^{\mathsf{Id}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A;\mathsf{ev}^B_A]}\right)}[\ \mathsf{happly}[\mathsf{r}_{\mathsf{\Pi}^B_A}]\ ;\ (\lambda\,\mathsf{happ})[\mathsf{r}_{\mathsf{\Pi}^B_A}]\ ] \to \Gamma.\mathsf{\Pi}^B_A \quad .$$

Again, using the appropriate arrow object, it is enough to build an arrow:

$$\begin{aligned}
\mathsf{happly}[\mathsf{r}_{\mathsf{\Pi}^B_A}] \Rightarrow (\lambda\,\mathsf{happ})[\mathsf{r}_{\mathsf{\Pi}^B_A}] &\overset{(6)}{=} \lambda(\mathsf{happ}[\mathsf{r}^\bullet_{\mathsf{\Pi}^B_A}]) = \lambda(\{\mathsf{app}\,\tilde{\alpha}_{\mathsf{\Pi}^B_A}\}[\mathsf{r}^\bullet_{\mathsf{\Pi}^B_A}]) \\
&\overset{(2)}{=} \lambda\{(\mathsf{app}\,\tilde{\alpha}_{\mathsf{\Pi}^B_A})[\mathsf{r}^\bullet_{\mathsf{\Pi}^B_A}]\} \\
&\overset{(6)}{=} \lambda\{\mathsf{app}(\tilde{\alpha}_{\mathsf{\Pi}^B_A}[\mathsf{r}_{\mathsf{\Pi}^B_A}])\} = \lambda\{\mathsf{app}\,1_{\delta_{\mathsf{\Pi}^B_A}}\} = \lambda\{1_{\mathsf{app}\,\delta_{\mathsf{\Pi}^B_A}}\} = \lambda\{1_{\mathsf{ev}^B_A}\}
\end{aligned}$$

in the category of the sections of $\Gamma.\mathsf{\Pi}^B_A.\mathsf{\Pi}_{A[P_{\mathsf{\Pi}^B_A}]}^{\mathsf{Id}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A;\mathsf{ev}^B_A]} \to \Gamma.\mathsf{\Pi}^B_A$ and, since in this category there is an arrow:

$$\mathsf{happly}[\mathsf{r}_{\mathsf{\Pi}^B_A}] = \mathsf{J}_{\mathsf{abst}_{\mathsf{r}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A]}}[\mathsf{r}_{\mathsf{\Pi}^B_A}] \Rightarrow \mathsf{abst}_{\mathsf{r}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A]} = \lambda(\mathsf{r}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A])$$

it is enough to build an arrow:

$$\lambda(\mathsf{r}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A]) \Rightarrow \lambda\{1_{\mathsf{ev}^B_A}\}$$

in the same category. Therefore we are left to build an arrow $\mathsf{r}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A] \Rightarrow \{1_{\mathsf{ev}^B_A}\}$ which amounts to building an arrow:

$$\alpha_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}(\mathsf{ev}^B_A;\mathsf{ev}^B_A)^\bullet\mathsf{r}_{B[P^\bullet_{\mathsf{\Pi}^B_A}]}[\mathsf{ev}^B_A] \Rightarrow 1_{\mathsf{ev}^B_A}$$

in the category $(\boldsymbol{\mathcal{C}}/\Gamma.A)(1_{\Gamma.\Pi^B_A.A[P_{\Pi^B_A}]}, P_{B[P^\bullet_{\Pi^B_A}]})^{\to}$. However:

$$\alpha_{B[P^\bullet_{\Pi^B_A}]}(\mathsf{ev}^B_A;\mathsf{ev}^B_A)^\bullet \mathsf{r}_{B[P^\bullet_{\Pi^B_A}]}[\mathsf{ev}^B_A] = \alpha_{B[P^\bullet_{\Pi^B_A}]}\mathsf{r}_{B[P^\bullet_{\Pi^B_A}]}\mathsf{ev}^B_A = 1_{1_{\Gamma.\Pi^B_A.A[P_{\Pi^B_A}].B[P^\bullet_{\Pi^B_A}]}}\mathsf{ev}^B_A = 1_{\mathsf{ev}^B_A}$$

hence the identity is such an arrow. In conclusion, an arrow:

$$\mathsf{happly}[\mathsf{r}_{\Pi^B_A}] \Rightarrow (\lambda\,\mathsf{happ})[\mathsf{r}_{\Pi^B_A}]$$

is just the one associated to the computation axiom:

$$\mathsf{H}_{\mathsf{abst}_{\mathsf{r}_{B[P^\bullet_{\Pi^B_A}]}[\mathsf{ev}^B_A]}}$$

for $\mathsf{happly}[\mathsf{r}_{\Pi^B_A}]$—which is stable under re-indexing by Proposition 4.3. Then:

$$\mathsf{J}_{\mathsf{H}_{\mathsf{abst}_{\mathsf{r}_{B[P^\bullet_{\Pi^B_A}]}[\mathsf{ev}^B_A]}}}$$

is a section of:

$$\Gamma.\Pi^B_A.(\Pi^B_A)^\blacktriangledown.\mathsf{Id}_{\Pi^B_A}.\mathsf{Id}[\mathsf{happly};\lambda\,\mathsf{happ}] \to \Gamma.\Pi^B_A.(\Pi^B_A)^\blacktriangledown.\mathsf{Id}_{\Pi^B_A}$$

hence, as we said, we can define:

$$\mathsf{happly} \Rightarrow \lambda\,\mathsf{happ}$$

as:

$$\alpha_{\left(\Pi_{A[P_{\Pi^B_A}\mathsf{s}]}^{\mathsf{Id}_{B[(P_{\Pi^B_A}\mathsf{s})^\bullet]}[\mathsf{ev}^B_A[\mathsf{s}^\bullet];\mathsf{ev}^B_A[\mathsf{t}^\bullet]]}\right)}(\mathsf{happly};\lambda\,\mathsf{happ})^\bullet \mathsf{J}_{\mathsf{H}_{\mathsf{abst}_{\mathsf{r}_{B[P^\bullet_{\Pi^B_A}]}[\mathsf{ev}^B_A]}}} \quad .$$

This is enough to verify that $\mathsf{happly} \Rightarrow \lambda\,\mathsf{happ}$ is stable under every re-indexing for the domain of P_A. Happy ending. □

Intro Rule for function extensionality. Let z, z' be sections of the display map $\Gamma.\Pi^B_A \to \Gamma$ and let q be a section of the display map:

$$\Gamma.\Pi_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^\bullet];\mathsf{ev}[z'^\bullet]]} \overset{(5)}{=} \Gamma.\Pi_A^{\mathsf{Id}_B[\mathsf{app}^B_A z;\mathsf{app}^B_A z']} \to \Gamma \quad .$$

Then $\mathsf{app}_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^\bullet];\mathsf{ev}^B_A[z'^\bullet]]}q$ is a section of the display map $\Gamma.A.\mathsf{Id}_B[\mathsf{app}^B_A z;\mathsf{app}^B_A z'] \to \Gamma.A$ and therefore:

$$\alpha_B(\mathsf{app}^B_A z;\mathsf{app}^B_A z')^\bullet \mathsf{app}_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^\bullet];\mathsf{ev}^B_A[z'^\bullet]]}q$$

is an arrow $\mathsf{app}^B_A z \Rightarrow \mathsf{app}^B_A z'$ in the category $(\boldsymbol{\mathcal{C}}/\Gamma.A)(1_{\Gamma.A}, P_B)$ of the sections of P_B. Therefore:

$$(\eta^{A,B}_{z'})^{-1}\lambda^B_A(\alpha_B(\mathsf{app}^B_A z;\mathsf{app}^B_A z')^\bullet \mathsf{app}_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^\bullet];\mathsf{ev}^B_A[z'^\bullet]]}q)(\eta^{A,B}_z)$$

is an arrow $z \Rightarrow z'$ in the category of the sections of $P_{\Pi^B_A}$. We define funext_q as:

$$\{(\eta^{A,B}_{z'})^{-1}\lambda^B_A(\alpha_B(\mathsf{app}^B_A z;\mathsf{app}^B_A z')^\bullet \mathsf{app}_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^\bullet];\mathsf{ev}^B_A[z'^\bullet]]}q)(\eta^{A,B}_z)\}$$

i.e. the re-indexing via $z; z'$ of the unique factorisation of:

$$(\eta^{A,B}_{z'})^{-1}\lambda^B_A(\alpha_B(\mathsf{app}^B_A z;\mathsf{app}^B_A z')^\bullet \mathsf{app}_A^{\mathsf{Id}_B[\mathsf{ev}^B_A[z^\bullet];\mathsf{ev}^B_A[z'^\bullet]]}q)(\eta^{A,B}_z)$$

through $\alpha_{\Pi_A^B}$.

Comp Axiom for function extensionality. If we build an arrow $\mathsf{happly}_{z;z'}[\mathsf{funext}_q] \Rightarrow q$ in the category of the sections of $\Gamma.\Pi_A^{\mathsf{Id}_B[\mathsf{app}_A^B z;\mathsf{app}_A^B z']} \to \Gamma$ then β_q^Π can be defined as the re-indexing via $\mathsf{happly}_{z;z'}[\mathsf{funext}_q];q$ of the unique factorisation of $\mathsf{happly}_{z;z'}[\mathsf{funext}_q] \Rightarrow q$ through $\alpha_{\Pi_A^{\mathsf{Id}_B[\mathsf{app}_A^B z;\mathsf{app}_A^B z']}}$. By Theorem 4.7, we know that:

$$\mathsf{happly}_{z;z'}[\mathsf{funext}_q] = \mathsf{happly}[(z;z')^\bullet \mathsf{funext}_q] \Rightarrow (\lambda\,\mathsf{happ})[(z;z')^\bullet \mathsf{funext}_q]$$

hence we are left to build $(\lambda\,\mathsf{happ})[(z;z')^\bullet \mathsf{funext}_q] \Rightarrow q$.

Let p be a section of the display map $\Gamma.\mathsf{Id}_{\Pi_A^B}[z;z']$. Then:

$$(\lambda\,\mathsf{happ})[(z;z')^\bullet p] \overset{(6)}{=} \lambda(\mathsf{happ}[((z;z')^\bullet p)^\bullet]) = \lambda(\{\mathsf{app}\,\tilde{\alpha}_{\Pi_A^B}\}[((z;z')^\bullet p)^\bullet])$$

where we recall that $\{\mathsf{app}\,\tilde{\alpha}_{\Pi_A^B}\}$ denotes the re-indexing along $\mathsf{app}\,\tilde{\mathsf{s}};\mathsf{app}\,\tilde{\mathsf{t}}$ of the unique factorisation of $\mathsf{app}\,\tilde{\alpha}_{\Pi_A^B}$ through the arrow object $\alpha_{B[(P_{\Pi_A^B})^\bullet]}$. Therefore:

$$(\lambda\,\mathsf{happ})[(z;z')^\bullet p] \overset{(2)}{=} \lambda\{(\mathsf{app}\,\tilde{\alpha}_{\Pi_A^B})[((z;z')^\bullet p)^\bullet]\} \overset{(6)}{=} \lambda\{\mathsf{app}(\tilde{\alpha}_{\Pi_A^B}[(z;z')^\bullet p])\}$$

and, since:

$$\tilde{\alpha}_{\Pi_A^B}[(z;z')^\bullet p] = \alpha_{\Pi_A^B}(z;z')^\bullet p$$

by definition of $\tilde{\alpha}_{\Pi_A^B}$, we deduce that:

$$(\lambda\,\mathsf{happ})[(z;z')^\bullet p] = \lambda\{\mathsf{app}(\alpha_{\Pi_A^B}(z;z')^\bullet p)\} \tag{8}$$

and in particular, when p is funext_q, we obtain that:

$$(\lambda\,\mathsf{happ})[(z;z')^\bullet \mathsf{funext}_q] = \lambda\{\mathsf{app}(\ (\eta_{z'}^{A,B})^{-1}\lambda(\ \alpha_B(\mathsf{app}\,z;\mathsf{app}\,z')^\bullet \mathsf{app}\,q\ )(\eta_z^{A,B})\ )\} \quad .$$

Therefore, in order to build $(\lambda\,\mathsf{happ})[(z;z')^\bullet \mathsf{funext}_q] \Rightarrow q$ it is enough to build:

$$\{\mathsf{app}(\ (\eta_{z'}^{A,B})^{-1}\lambda(\ \alpha_B(\mathsf{app}\,z;\mathsf{app}\,z')^\bullet \mathsf{app}\,q\ )(\eta_z^{A,B})\ )\} \Rightarrow \mathsf{app}\,q$$

which amounts to building:

$$\mathsf{app}(\ (\eta_{z'}^{A,B})^{-1}\lambda(\ \alpha_B(\mathsf{app}\,z;\mathsf{app}\,z')^\bullet \mathsf{app}\,q\ )(\eta_z^{A,B})\ ) \Rightarrow \alpha_B(\mathsf{app}\,z;\mathsf{app}\,z')^\bullet \mathsf{app}\,q$$

in the category $(\mathcal{C}/\Gamma.A)(1_{\Gamma.A}, P_B)^{\to}$ i.e. a commutative diagram of the form:

$$\begin{tikzcd}
\mathsf{app}\,z \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\mathsf{app}(\ (\eta_{z'}^{A,B})^{-1}\lambda(\ \alpha_B(\mathsf{app}\,z;\mathsf{app}\,z')^\bullet \mathsf{app}\,q\ )(\eta_z^{A,B})\ )"'] & \mathsf{app}\,z \arrow[d, Rightarrow, "\alpha_B(\mathsf{app}\,z;\mathsf{app}\,z')^\bullet \mathsf{app}\,q"] \\
\mathsf{app}\,z' \arrow[r, Rightarrow] & \mathsf{app}\,z'
\end{tikzcd}$$

and the pair $(1_{\mathsf{app}\,z}, 1_{\mathsf{app}\,z'})$ constitutes such a commutative diagram. We are done.

Exp Axiom for function extensionality. If we build an arrow $p \Rightarrow \mathsf{funext}_{\mathsf{happly}_{z;z'}[p]}$ in the category of the section of $\Gamma.\mathsf{Id}_{\Pi_A^B} \to \Gamma$ then γ_p^Π can be defined as the section:

$$\{p \Rightarrow \mathsf{funext}_{\mathsf{happly}_{z;z'}[p]}\}$$

of $\Gamma.\mathsf{Id}_{\mathsf{Id}_{\Pi_A^B}}[p;\mathsf{funext}_{\mathsf{happly}_{z;z'}[p]}] \to \Gamma$. By definition $\mathsf{funext}_{\mathsf{happly}_{z;z'}[p]}$ is the section:

$$\{(\eta_{z'}^{A,B})^{-1}\lambda(\ \alpha_B(\mathsf{app}\, z;\mathsf{app}\, z')^{\bullet}(\mathsf{app}\,\mathsf{happly}_{z;z'}[p])\ )(\eta_z^{A,B})\}$$

and, by Theorem 4.7, we know that:

$$\lambda\{\mathsf{app}(\alpha_{\Pi_A^B}(z;z')^{\bullet}p)\} \overset{(8)}{=} (\lambda\,\mathsf{happ})[(z;z')^{\bullet}p] \Rightarrow \mathsf{happly}[(z;z')^{\bullet}p] = \mathsf{happly}_{z;z'}[p]$$

hence:

$$\{\mathsf{app}(\alpha_{\Pi_A^B}(z;z')^{\bullet}p)\} \Rightarrow \mathsf{app}\,\lambda\{\mathsf{app}(\alpha_{\Pi_A^B}(z;z')^{\bullet}p)\} \Rightarrow \mathsf{app}\,\mathsf{happly}_{z;z'}[p].$$

Therefore we are left to exhibit an arrow:

$$\begin{aligned} p \Rightarrow \{(\eta_{z'}^{A,B})^{-1}&\lambda(\ \alpha_B(\mathsf{app}\, z;\mathsf{app}\, z')^{\bullet}\{\mathsf{app}(\alpha_{\Pi_A^B}(z;z')^{\bullet}p)\}\ )(\eta_z^{A,B})\} \\ &= \{(\eta_{z'}^{A,B})^{-1}\lambda\mathsf{app}(\alpha_{\Pi_A^B}(z;z')^{\bullet}p)(\eta_z^{A,B})\} \\ &= \{\alpha_{\Pi_A^B}(z;z')^{\bullet}p\} \\ &= p \end{aligned}$$

and 1_p is such an arrow.

Proposition 4.8. *Let $(\mathcal{C},\mathcal{D})$ be a display map 2-category endowed with axiomatic $=$-types and axiomatic Π-types & axiomatic function extensionality. Referring to the data defined in paragraphs* **Form Rule, Intro Rule, Elim Rule, and Comp Axiom, for Π-types** *and* **Intro Rule, Comp Axiom, and Exp Axiom, for function extensionality**, *the* stability conditions*:*

$$\Pi_A^B[f] = \Pi_{A[f]}^{B[f^{\bullet}]} \qquad \mathsf{abst}_{\upsilon}[f] = \mathsf{abst}_{\upsilon[f^{\bullet}]}$$
$$\mathsf{ev}_A^B[f^{\bullet\bullet}] = \mathsf{ev}_{A[f]}^{B[f^{\bullet}]} \qquad \beta_{\upsilon}[f^{\bullet}] = \beta_{\upsilon[f^{\bullet}]}$$

and:

$$\mathsf{funext}_q[f] = \mathsf{funext}_{q[f]}$$
$$\beta_q^{\Pi}[f] = \beta_{q[f]}^{\Pi}$$
$$\eta_q^{\Pi}[f] = \eta_{q[f]}^{\Pi}$$

of Defintion 3.3 hold for every arrow $\Delta \xrightarrow{f} \Gamma$.

Proof. See Appendix C. □

By Proposition 4.3, Proposition 4.5, and Proposition 4.8, we infer the following:

Theorem 4.9. *Every* display map 2-category $(\mathcal{C},\mathcal{D})$ *endowed with axiomatic $=$-types, axiomatic Σ-types, and axiomatic Π-types & axiomatic function extensionality—Definitions 4.2, 4.4, and 4.6—induces a* display map category $(\mathbf{C},\mathcal{D})$ *endowed with axiomatic $=$-types, axiomatic Σ-types, and axiomatic Π-types & axiomatic function extensionality—Definitions 3.1, 3.2, and 3.3—i.e. a model of* ATT.

By combining Theorem 4.9 with Theorem 3.5, we deduce that an interpretation of ATT in any display map 2-category, endowed with axiomatic =-types and axiomatic Σ-types, is well-defined and sound.

4.4 The role of the cloven isofibration structure

Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category equipped with axiomatic $=$-types and axiomatic Σ-types. As proven in the paragraph **Elim Rule for $=$-types**, the structure of axiomatic $=$-types associated with the display maps in the class $\mathcal{D}$ enables the construction of an *elimination pseudo-term* $\tilde{\mathsf{J}}_c$ *for $=$-types*—for every section c of the display map $P_{C[\mathsf{r}_A]}$. This is not a genuine section of P_C, as the composition $P_C\tilde{\mathsf{J}}_c$ equals the identity of $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$ merely up to the 2-cell φ_A, making it the interpretation of a pseudo-term of type C. Similarly, the structure of axiomatic Σ-types on display maps allows for the construction of an *elimination pseudo-term* $\tilde{\mathsf{split}}_c$ *for Σ-types*—for every section c of the display map $P_{C[\mathsf{p}_A^B]}$—as shown in the paragraph **Elim Rule for Σ-types**. Once again, this is the interpretation of a pseudo-term in the sense that the composition $P_C\tilde{\mathsf{split}}_c$ is the identity of $\Gamma.\boldsymbol{\Sigma}_A^B$ only up to the 2-cell η_A^B.

The structure of cloven isofibrations on display maps plays a crucial role at this point, as it enables us to "strictify" the pseudo-terms $\tilde{\mathsf{J}}_c$ and $\tilde{\mathsf{split}}_c$—by transporting them back along the context identity proofs φ_A and η_A^B—into the genuine elimination terms J_c and split_c, respectively. In other words, these become actual sections of the corresponding display maps involved, at the cost of introducing additional 2-cells $\mathsf{J}_c \Rightarrow \tilde{\mathsf{J}}_c$ and $\mathsf{split}_c \Rightarrow \tilde{\mathsf{split}}_c$.

Paragraphs **Comp Axiom for $=$-types** and **Σ-types** show how, essentially, these two 2-cells represent the respective computation axioms: they lead, through re-indexing and further exploiting the arrow object structure—this time the one of the display maps $P_{C[\mathsf{r}_A]}$ and $P_{C[\mathsf{p}_A^B]}$ respectively—to the identification of sections H_c and σ_c of the display maps associated to the types:

$$\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c] \quad \text{and} \quad \mathsf{Id}_{C[\mathsf{p}_A^B]}[\mathsf{split}_c[\mathsf{p}_A^B]; c]$$

respectively. These sections, therefore, provide interpretations for the terms of the computation axioms for $=$-types and Σ-types, respectively.

As mentioned above, since the isofibration structure on the display maps is cloven but not necessarily normal, this ensures that the computation axioms are satisfied, but not necessarily that the computation rules are. In other words, $\mathsf{J}_c[\mathsf{r}_A]$ and $\mathsf{split}_c[\mathsf{p}_A^B]$ do not, in general, coincide with the respective term c. To clarify this point, let us start by briefly analysing what happens if display maps in $(\mathcal{C}, \mathcal{D})$ are normal, as isofibrations. Referring to the diagram (3), if P_C is normal, then:

$$\mathsf{J}_c\mathsf{r}_A = t_{\tilde{\mathsf{J}}_c\mathsf{r}_A}^{\varphi_A * \mathsf{r}_A} = t_{\tilde{\mathsf{J}}_c\mathsf{r}_A}^{1_{\mathsf{r}_A}} = \tilde{\mathsf{J}}_c\mathsf{r}_A$$

$$(\mathsf{J}_c \Rightarrow \tilde{\mathsf{J}}_c) * \mathsf{r}_A = \tau_{\tilde{\mathsf{J}}_c\mathsf{r}_A}^{\varphi_A * \mathsf{r}_A} = \tau_{\tilde{\mathsf{J}}_c\mathsf{r}_A}^{1_{\mathsf{r}_A}} = 1_{\tilde{\mathsf{J}}_c\mathsf{r}_A}$$

implying that $\mathsf{J}_c[\mathsf{r}_A]$ is in fact c and that h_c is the identity 1-cell of c, respectively. The latter condition, in turn, implies that H_c is $\mathsf{refl}_{C[\mathsf{r}_A]}[c]$. Similarly, referring to the diagram (4), if P_C is normal and, additionally, the 2-cell β_A^B is the identity 2-cell—hence $(\mathsf{p}_A^B)_c^{\bullet}$ coincides with $\tilde{\mathsf{split}}_c\mathsf{p}_A^B$—then:

$$\mathsf{split}_c\mathsf{p}_A^B = t_{\tilde{\mathsf{split}}_c\mathsf{p}_A^B}^{\eta_A^B * \mathsf{p}_A^B} = t_{\tilde{\mathsf{split}}_c\mathsf{p}_A^B}^{1_{\mathsf{p}_A^B}} = \tilde{\mathsf{split}}_c\mathsf{p}_A^B$$

$$(\mathsf{split}_c \Rightarrow \tilde{\mathsf{split}}_c) * \mathsf{p}_A^B = \tau_{\tilde{\mathsf{split}}_c\mathsf{p}_A^B}^{\eta_A^B * \mathsf{p}_A^B} = \tau_{\tilde{\mathsf{split}}_c\mathsf{p}_A^B}^{1_{\mathsf{p}_A^B}} = 1_{\tilde{\mathsf{split}}_c\mathsf{p}_A^B}$$

implying, as before, that $\mathsf{split}_c[\mathsf{p}_A^B]$ is c and that σ_c is $\mathsf{refl}_{C[\mathsf{p}_A^B]}[c]$. In this case, we conclude that the display map 2-category $(\mathcal{C}, \mathcal{D})$ is a model of ITT. However, in the general case where display maps are *not* required to be normal as isofibrations, but only cloven, this situation is not necessary: this is what we show in the next section.

5 Revisiting the groupoid model

The groupoid model of ITT, [24, 25, 40], was devised by Hofmann and Streicher as a model of ITT that does not satisfy the uniqueness of identity proofs rule, providing a semantic proof that the latter is not admissible in ITT. It is considered a prelude towards the development of homotopy type theory. Additionally, it can be considered the quintessential example of a model under the notion of semantics introduced by Garner [17], as every other model according to this notion of semantics can, in a certain sense, be viewed as a generalisation of it. Notably, the display maps of the groupoid model formulated as a display map 2-category are in fact normal isofibrations.

The groupoid model, celebrated for its fundamental and illuminating nature, has inspired other authors to propose modifications aimed at developing models for generalised versions of dependent type theories. Notably, North [33] and, from a different but related perspective, Altenkirch and Neumann [3] considered a weakening of this structure based on demoting groupoids to mere categories, thereby obtaining a foundational model for directed type theory. Our approach follows a similar trajectory. We propose a weakening of the groupoid model with the aim of demoting display maps to mere cloven isofibrations. However, we ensure to choose them in such a way that the re-indexing choices remain split, thereby preserving the integrity of an authentic model.

Let **Grpd** be the (2,1)-category of groupoids, functors, and natural transformations—i.e. natural isomorphisms—with a chosen groupoid 1 made of one object and one morphism, as a specified 2-terminal object of **Grpd**. Let $\mathcal{D}$ be the class of 1-cells whose elements are the functors of the form $P_A : \Gamma.A \to \Gamma$ obtained by applying the Grothendieck construction to the *pseudofunctors* $(A, \phi^A, \psi^A) : \Gamma \to \mathbf{Grpd}$ where Γ is a groupoid, and ϕ^A and ψ^A are the *coherent* families of natural isomorphisms $\phi^A_{p,q} : A_q A_p \Rightarrow A_{qp}$ and $\psi^A_\gamma : A_{1_\gamma} \Rightarrow 1_{A_\gamma}$—where γ is any object of Γ and p and q any composable arrows of Γ—making A into a pseudofunctor. Hence $\Gamma.A$ is the groupoid whose objects are the pairs (γ, x) where γ is an object of Γ and x is an object of A_γ, and the arrows (γ, x) to (γ', x') are the pairs (p_1, p_2) where p_1 is an arrow $\gamma \to \gamma'$ and p_2 is an arrow $A_{p_1}x \to x'$. Composition of arrows, identity arrows, and inverses to arrows, are defined using the families ϕ^A and ψ^A: associativity, unitalities, and invertibilities follow from their coherence laws.[4] The functor P_A is the projection on the first component.

Cloven isofibration structure on display maps. If we are given a 2-cell π of the form:

$$\begin{tikzcd}
\Delta \arrow[r, "g = (g_1, g_2)"] \arrow[dr, bend right, "f"'] & \Gamma.A \arrow[d, "P_A"] \arrow[dl, phantom, "\overset{\pi}{\Longrightarrow}"] \\
& \Gamma
\end{tikzcd}$$

then the mappings:

$$\delta \mapsto (f\delta\ ,\ A_{\pi_\delta^{-1}} g_2\delta) \qquad \text{and } (\delta \xrightarrow{p} \delta') \mapsto (fp\ ,\ A_{fp}A_{\pi_\delta^{-1}} g_2\delta \xrightarrow{\phi^A} A_{(fp)\pi_\delta^{-1}} g_2\delta =$$
$$A_{\pi_{\delta'}^{-1}(g_1p)} g_2\delta \xrightarrow{(\phi^A)^{-1}} A_{\pi_{\delta'}^{-1}} A_{(g_1p)} g_2\delta \xrightarrow{A_{\pi_{\delta'}^{-1}} g_2 p} A_{\pi_{\delta'}^{-1}} g_2\delta')$$

define a functor $\mathsf{t}_g^\pi : \Delta \to \Gamma.A$ whose postcomposition via P_A is f. Moreover, the arrow:

$$(\pi_\delta\ ,\ A_{\pi_\delta} A_{\pi_\delta^{-1}} g_2\delta \xrightarrow{\phi^A} A_{1_{g_1\delta}} g_2\delta \xrightarrow{\psi^A} g_2\delta)$$

[4] See Appendix D for additional details.

from $(f\delta, A_{\pi_\delta^{-1}} g_2\delta)$ to $(g_1\delta, g_2\delta)$ is the δ-component of a natural isomorphism $\tau_g^\pi : \mathsf{t}_g^\pi \Rightarrow g$ whose postcomposition via P_A is π.

These choices of t_g^π and τ_g^π define a cloven isofibration structure on P_A, which in general is *not* normal: if π is the identity and hence f coincides with g_1, then the mapping $\delta \mapsto (f\delta\ ,\ A_{\pi_\delta^{-1}} g_2\delta)$ coincides with $\delta \mapsto (g_1\delta\ ,\ A_{1_{g_1\delta}} g_2\delta)$ which in general is different from $(g_1\delta, g_2\delta)$, since the pseudofunctor A can be non-strict, hence in general t_g^π is different form g. Therefore, the takeaway is that expanding the class of semantic types to encompass all pseudofunctors into the 2-category **Grpd**—not just the strict functors used as semantic types in the original groupoid model—allows the class $\mathcal{D}$ of display maps to include cloven isofibrations that are not necessarily normal.

Re-indexing structure and arrow object structure on display maps. Without delving too deeply into the details[4], we state that the 1-cells of $\mathcal{D}$ can be endowed with a 1- and 2-dimensional re-indexing structure, as well as with a structure of arrow objects, such that the compatibilities outlined in the third and fourth points of Definition 4.1 are satisfied. We only mention that the re-indexing of a display map $P_A : \Gamma.A \to \Gamma$ along a 1-cell $f : \Delta \to \Gamma$ of **Grpd** is obtained by applying the Grothendieck construction to the pseudofunctor $\Delta \xrightarrow{f} \Gamma \xrightarrow{A} \mathbf{Grpd}$, where A is the pseudofunctor inducing P_A. We also mention that P_{Id_A} is induced by the (pseudo)functor $\mathsf{Id}_A : \Gamma.A.A^\blacktriangledown \to \mathbf{Grpd}$ mapping:

$$
\begin{aligned}
(\gamma, x, y) &\mapsto A_\gamma(x, y) \\
(p_1, p_2, p_3) &\mapsto (p \mapsto (x' \xrightarrow{p_2^{-1}} A_{p_1}x \xrightarrow{A_{p_1}p} A_{p_1}y \xrightarrow{p_3} y'))
\end{aligned}
$$

—where the homset $A_\gamma(x, y)$ is endowed with the trivial groupoid structure—and that the (γ, x, y, p)-component of the 2-cell $\alpha_A : P_{A^\blacktriangledown} P_{\mathsf{Id}_A} \Rightarrow P_A^\bullet P_{\mathsf{Id}_A}$ is the arrow:

$$(1_\gamma\ ,\ A_{1_\gamma}x \xrightarrow{\psi^A} x \xrightarrow{p} y\ ) : (\gamma, x) \to (\gamma, y).$$

Following the construction at paragraphs **Elim Rule** and **Comp Axiom for =-types**, one can now reconstruct the choice functions r, φ, and J and observe that J_c acts *on objects* as the mapping:

$$(\gamma, x, y, p) \mapsto (\gamma, x, y, p, C_{\varphi_A^{-1}} c(\gamma, y))$$

whenever $(\gamma, x) \mapsto (\gamma, x, c(\gamma, x))$ is the action on object of a section c of $P_{C[\mathsf{r}_A]}$, for some display map P_C over $\Gamma.A.A^\blacktriangledown.\mathsf{Id}_A$. This allows to conclude that $\mathsf{J}_c[\mathsf{r}_A]$ acts on objects as the mapping:

$$(\gamma, x) \mapsto (\gamma, x, C_{1_{(\gamma,x,x,1_x)}} c(\gamma, x))$$

and therefore, unless C was chosen to be a strict functor $\Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \to \mathbf{Grpd}$, in general $\mathsf{J}_c[\mathsf{r}_A]$ and c *will not coincide.* We conclude that:

Theorem 5.1. *The pair* $(\mathbf{Grpd}, \mathcal{D})$ *is a display map 2-category endowed with axiomatic =-type, which, as a display map category—see Theorem 4.9— is a model of axiomatic =-types that does not validate Comp Rule for =-types.*

In particular Comp Rule for =-types is not admissible in ATT*.*

6 Syntactic 2-category and completeness

7 Conclusion and future work

In this paper we provided a 2-categorical procedure to construct models of ATT, focusing on the =-type former and the Σ-type former. We applied this procedure to identify a weakening of the groupoid model that models ATT without believing Comp Rule for =-types. The semantics provided by this class of models, however, is not complete with respect to ATT. Similarly to Garner's notion of semantics for ITT, every model validates the discreteness rule—namely, every model believes that every type is a 1-type. This arises due to the arrow object structure in display maps. This phenomenon occurs in both the original and our weakened version of the groupoid model, as well as in the versions of the category model proposed in [33] and [3]—using directed =-types, or *hom-types*, in place of =-types.

Hence it is natural to ask whether it is possible to interpret ATT in tricategories—according to the notion presented in [19]—with a class of display maps in such a way that the interpretation is complete without adding the discreteness rule to ATT. In a display map 2-category, two parallel 2-cells are either distinct or coincide. However, in a *display map 3-category*, we may talk about homotopies between two 2-cells, or propositional equalities between identity proofs, without necessarily collapsing provably identical identity proofs into the same 2-cell. Moreover, every identity proof between identity proofs is itself in particular an identity proof: this leads to the idea that the third dimension should not be explicitly used to define the interpretation, which, like for display map 2-categories, will only rely on 2-cells—to be later converted into sections using the arrow object. Yet, as mentioned, we will not have the issue of having to collapse every two parallel 2-cells that are not distinct into the same 2-cell. This is why we believe that the interpretation of ATT in appropriate display map 3-categories might be complete.

This work belongs to the field of research in the categorical semantics of dependent type theories, along with their variants and generalisations—such as ATT itself—and aims to advance one of the main objectives of the area: to establish a concept of a general semantics that is both broad enough to encompass highly general notions of dependent type theories and functional enough to be effectively applied in their study. We argue that this work can be contextualised within the bicategorical approach to the semantics of dependent type theory proposed by Ahrens, North, and van der Weide [1, 2]. This approach aims to provide a general framework for describing the semantics of the structural part of dependent type theory, as well as its generalisation towards directed type theory. Specifically, we believe that display map 2-categories constitute a particular instance of the notion of a *display map bicategory* introduced in that work, under the condition that reductions are symmetric. In fact, in a display map 2-category, the 2-cells are independent of the notion of a =-type former and can, a priori, be regarded as *reduction judgements*—although not directed: it is the semantic =-type former that enables us to interpret and convert them as identity proofs.

We are also interested in investigating whether Garner's idea of encapsulating the semantics of intensional =-types in arrow objects—applied in this article to encode the semantics of axiomatic =-types—could be adapted, through a suitable generalisation of our notion of display map 2-category, to encode the semantics of directed identity types—whether intensional or axiomatic—as proposed by North [33] or by Altenkirch and Neumann [3]. This would aim to address a problem posed by Ahrens, North, and van der Weide [2] themselves: extending the notion of comprehension bicategory to accommodate the interpretation of the hom-type former à la North.

More generally, we believe there is still much to explore regarding the semantics of various extensions and variations of dependent type theory. This is especially true in relation to the weakenings of the identity type constructor—both with respect to its elimination rule and its computation rule—that are currently prominent, as well as their interrelationships. We believe the notion of semantics introduced in this article provides a meaningful contribution to this important line of research in type theory.

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A Type-checking of the stability conditions

This section begins with the proof of the fact, mentioned in Subsection 2.1, that in a display map category $(\mathbf{C}, \mathcal{D})$ the square:

$$
\begin{tikzcd}
\Delta \arrow[r, "f"] \arrow[d, "{a[f];b[f]}"'] & \Gamma \arrow[d, "{a;b}"] \\
\Delta.A[f].A[f]^{\blacktriangledown} \arrow[r, "f^{\bullet\bullet}"] & \Gamma.A.A^{\blacktriangledown}
\end{tikzcd}
\tag{9}
$$

commutes if f is a given arrow $\Delta \to \Gamma$ and a and b are sections of a given display map P_A.

By building the term $b[P_A a]$, i.e. b itself—display map categories are assumed to be split—according to its definition:

$$
\begin{tikzcd}
\Gamma \arrow[r, "a"] \arrow[d, "{b = b[P_A a]}"'] & \Gamma.A \arrow[r, "P_A"] \arrow[d, "{b[P_A]}"] & \Gamma \arrow[d, "b"] \\
\Gamma.A = \Gamma.A[P_A a] \arrow[r, "a^{\bullet}"] \arrow[d, "{P_A = P_{A[P_A a]}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.A^{\blacktriangledown} \arrow[r, "P_A^{\bullet}"] \arrow[d, "P_{A^{\blacktriangledown}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "P_A"] \\
\Gamma \arrow[r, "a"] & \Gamma.A \arrow[r, "P_A"] & \Gamma
\end{tikzcd}
$$

we infer that the upper left-hand square commutes. And since:

$$
P_{A^{\blacktriangledown}} b[P_A] a = a \quad \text{ and } \quad P_A^{\bullet} a^{\bullet} b = (P_A a)^{\bullet} b = b
$$

we conclude the following two presentations:

$$
\begin{aligned}
a;b &= \left(\; \Gamma \xrightarrow{\;b\;} \Gamma.A \xrightarrow{\;a^{\bullet}\;} \Gamma.A.A^{\blacktriangledown} \; \right) \\
&= \left(\; \Gamma \xrightarrow{\;a\;} \Gamma.A \xrightarrow{\;b[P_A]\;} \Gamma.A.A^{\blacktriangledown} \; \right)
\end{aligned}
$$

of the arrow $a;b$.

In order to verify that the diagram (9) commutes, we use the first presentation:

$$
\Delta \xrightarrow{b[f]} \Delta.A[f] \xrightarrow{a[f]^{\bullet}} \Delta.A[f].A[f]^{\blacktriangledown}
$$

for $a[f];b[f]$, and the second one:

$$
\Gamma \xrightarrow{a} \Gamma.A \xrightarrow{b[P_A]} \Gamma.A.A^{\blacktriangledown}
$$

for $a;b$. We are left to verify that:

$$
\begin{tikzcd}
\Delta \arrow[rrr, "f"] \arrow[d, "{b[P_A(af)]}"'] & & & \Gamma \arrow[r, "a"] & \Gamma.A \arrow[d, "{b[P_A]}"] \\
\Delta.A[f] \arrow[r, "{a[f]^{\bullet}}"] & \Delta.A[f].A[f]^{\blacktriangledown} \arrow[r, phantom, "="] & \Delta.A[f].A^{\blacktriangledown}[f^{\bullet}] \arrow[rr, "f^{\bullet\bullet}"] & & \Gamma.A.A^{\blacktriangledown}
\end{tikzcd}
$$

commutes, since $b[P_A(af)] = b[f]$. Now, since the diagram:

$$\begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "{b[P_A(af)]}"] & \Gamma \arrow[r, "a"] & \Gamma.A \arrow[d, "{b[P_A]}"] \\ \Delta.A[f] \arrow[r, "f^\bullet"] & \Gamma.A \arrow[r, "a^\bullet"] & \Gamma.A.A^\blacktriangledown \end{tikzcd}$$

commutes—and this is true because the equality $a^\bullet f^\bullet = (af)^\bullet$ holds—we are done if we verify that:

$$\begin{tikzcd} \Delta.A[f] \arrow[r, "f^\bullet"] \arrow[d, "{a[f]^\bullet}"] & \Gamma.A \arrow[d, "a^\bullet"] \\ \Delta.A[f].A[f]^\blacktriangledown \arrow[r, "f^{\bullet\bullet}"] & \Gamma.A.A^\blacktriangledown \end{tikzcd}$$

commutes. This is actually true as:

$$f^{\bullet\bullet}a[f]^\bullet = (f^\bullet a[f])^\bullet = (af)^\bullet = a^\bullet f^\bullet$$

hence we are done.

In what follows, we provide full verification of the type-checking of the stability conditions of Definitions 3.1 and 3.2. Verification is completely analogous for Definition 3.3.

Type-checking of the stability conditions of Definition 3.1

Let us assume that $(\mathbf{C}, \mathcal{D})$ is endowed with the choice functions Id, refl, J, H as in *Form Rule*, *Intro Rule*, *Elim Rule*, *Comp Axiom* of Definition 3.1. Let f be an arrow $\Delta \to \Gamma$ and let P_A be a display map of codomain Γ.

Stability of Id. The codomain of $P_{\mathsf{Id}_{A[f]}}$ is by definition $\Delta.A[f].A[f]^\blacktriangledown$. We observe that the codomain of $P_{\mathsf{Id}_A[f^{\bullet\bullet}]}$ is $\Delta.A[f].A^\blacktriangledown[f^\bullet]$, which coincides with $\Delta.A[f].A[f]^\blacktriangledown$ since:

$$A^\blacktriangledown[f^\bullet] = A[P_A][f.A] = A[f][P_{A[f]}] = A[f]^\blacktriangledown$$

being $(\mathbf{C}, \mathcal{D})$ is split. As $P_{\mathsf{Id}_{A[f]}}$ and $P_{\mathsf{Id}_A[f^{\bullet\bullet}]}$ have the same codomain, the equality:

$$\mathsf{Id}_A[f^{\bullet\bullet}] = \mathsf{Id}_{A[f]}$$

is meaningful. From now on, let us assume that it is satisfied.

Stability of refl. By definition $\mathsf{refl}_{A[f]}$ is a section of the display map $\Delta.A[f].\mathsf{Id}_{A[f]}[\delta_{A[f]}] \to \Delta.A[f]$. We observe that $\mathsf{refl}_A[f^\bullet]$ is a section of the display map $\Delta.A[f].\mathsf{Id}_A[\delta_A][f^\bullet] \to \Delta.A[f]$, and:

$$\mathsf{Id}_A[\delta_A][f^\bullet] = \mathsf{Id}_A[f^{\bullet\bullet}][\delta_{A[f]}] = \mathsf{Id}_{A[f]}[\delta_{A[f]}].$$

Therefore $\mathsf{refl}_A[f^\bullet]$ is itself a section of $\Delta.A[f].\mathsf{Id}_{A[f]}[\delta_{A[f]}] \to \Delta.A[f]$, hence the equality:

$$\mathsf{refl}_A[f^\bullet] = \mathsf{refl}_{A[f]}$$

is meaningful. From now on, let us assume that it is satisfied. Observe that this implies that the square:

$$\begin{tikzcd} \Delta.A[f] \arrow[r, "f^\bullet"] \arrow[d, "\mathsf{r}_{A[f]}"] & \Gamma.A \arrow[d, "\mathsf{r}_A"] \\ \Delta.A[f].A[f]^\blacktriangledown.\mathsf{Id}_{A[f]} \arrow[r, "f^{\bullet\bullet\bullet}"] & \Delta.A.A^\blacktriangledown.\mathsf{Id}_A \end{tikzcd}$$

commutes.

Stability of J. Let P_C be a display map of codomain $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$ and let c be a section of $\Gamma.A.C[\mathsf{r}_A] \to \Gamma.A$. As J_c is by definition a section of P_C, then $\mathsf{J}_c[f^{\bullet\bullet\bullet}]$ is a section of $P_{C[f^{\bullet\bullet\bullet}]}$. Now, we observe that $P_{C[f^{\bullet\bullet\bullet}]}$ is a display map of codomain:

$$\Delta.A[f].A^{\blacktriangledown}[f^{\bullet}].\mathsf{Id}_A[f^{\bullet\bullet}] = \Delta.A[f].A[f]^{\blacktriangledown}.\mathsf{Id}_{A[f]}$$

and $c[f^{\bullet}]$ is a section of $\Delta.A[f].C[\mathsf{r}_A][f^{\bullet}] \to \Delta.A[f]$ i.e. of:

$$\Delta.A[f].C[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}] \to \Delta.A[f].$$

Therefore $\mathsf{J}_{c[f^{\bullet}]}$ is well defined and it is a section of $P_{C[f^{\bullet\bullet\bullet}]}$. As $\mathsf{J}_c[f^{\bullet\bullet\bullet}]$ and $\mathsf{J}_{c[f^{\bullet}]}$ are both sections of $P_{C[f^{\bullet\bullet\bullet}]}$, the equality:

$$\mathsf{J}_c[f^{\bullet\bullet\bullet}] = \mathsf{J}_{c[f^{\bullet}]}$$

is meaningful. From now on, let us assume that it is satisfied.

Stability of H. Finally, we observe that by definition $\mathsf{H}_{c[f^{\bullet}]}$ is a section of the display map:

$$\Delta.A[f].\mathsf{Id}_{C[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}]}[\mathsf{J}_{c[f^{\bullet}]}[\mathsf{r}_{A[f]}]; c[f^{\bullet}]] \to \Delta.A[f].$$

Analogously, by definition H_c is a section of the display map:

$$\Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c] \to \Gamma.A$$

hence $\mathsf{H}_c[f^{\bullet}]$ is a section of the display map:

$$\Delta.A[f].\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c][f^{\bullet}] \to \Delta.A[f].$$

Now, we observe that:

$$\begin{array}{c}
\Delta.A[f].\mathsf{Id}_{C[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}]}[\mathsf{J}_{c[f^{\bullet}]}[\mathsf{r}_{A[f]}]; c[f^{\bullet}]] \\
\| \\
\Delta.A[f].\mathsf{Id}_{C[\mathsf{r}_A][f^{\bullet}]}[\mathsf{J}_c[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}]; c[f^{\bullet}]] \\
\| \\
\Delta.A[f].\mathsf{Id}_{C[\mathsf{r}_A][f^{\bullet}]}[\mathsf{J}_c[\mathsf{r}_A][f^{\bullet}]; c[f^{\bullet}]] \\
\| \\
\Delta.A[f].\mathsf{Id}_{C[\mathsf{r}_A]}[f^{\bullet}.C[\mathsf{r}_A].C[\mathsf{r}_A]^{\blacktriangledown}][\mathsf{J}_c[\mathsf{r}_A][f^{\bullet}]; c[f^{\bullet}]] \\
\|\ \text{Diagram (9)} \\
\Delta.A[f].\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c][f^{\bullet}]
\end{array}$$

where the last equality follows from an instance of the commutative diagram (9). We conclude that $\mathsf{H}_c[f^{\bullet}]$ and $\mathsf{H}_{c[f^{\bullet}]}$ are sections of the same display map, hence the equality:

$$\mathsf{H}_c[f^{\bullet}] = \mathsf{H}_{c[f^{\bullet}]}$$

is meaningful.

Type-checking of the stability conditions of Definition 3.2

Let us assume that $(\mathbf{C}, \mathcal{D})$ is endowed with axiomatic =-types and with the choice functions Σ, pair, split, σ as in *Form Rule*, *Intro Rule*, *Elim Rule*, *Comp Axiom* of Definition 3.2. Let f

be an arrow $\Delta \to \Gamma$, let P_A be a display map of codomain Γ, and let P_B be a display map of codomain $\Gamma.A$.

Stability of $\mathbf{\Sigma}$. The codomain of both $P_{\Sigma_{A[f]}^{B[f^\bullet]}}$ and $P_{\Sigma_A^B[f]}$ is by definition Δ. Hence the equality:

$$\Sigma_A^B[f] = \Sigma_{A[f]}^{B[f^\bullet]}$$

is meaningful. From now on, let us assume that it is satisfied.

Stability of pair. By definition $\mathsf{pair}_{A[f]}^{B[f^\bullet]}$ is a section of the display map:

$$\Delta.A[f].B[f^\bullet].\Sigma_{A[f]}^{B[f^\bullet]}[P_{A[f]}P_{B[f^\bullet]}] \to \Delta.A[f].B[f^\bullet].$$

We observe that $\mathsf{pair}_A^B[f^{\bullet\bullet}]$ is a section of the display map:

$$\Delta.A[f].B[f^\bullet].\Sigma_A^B[P_A P_B][f^{\bullet\bullet}] \to \Delta.A[f].B[f^\bullet]$$

and that additionally $\Sigma_A^B[P_A P_B][f^{\bullet\bullet}] = \Sigma_A^B[f][P_{A[f]}P_{B[f^\bullet]}] = \Sigma_{A[f]}^{B[f^\bullet]}[P_{A[f]}P_{B[f^\bullet]}]$. Therefore $\mathsf{pair}_A^B[f^{\bullet\bullet}]$ is itself a section of:

$$\Delta.A[f].B[f^\bullet].\Sigma_{A[f]}^{B[f^\bullet]}[P_{A[f]}P_{B[f^\bullet]}] \to \Delta.A[f].B[f^\bullet]$$

hence the equality:

$$\mathsf{pair}_A^B[f^{\bullet\bullet}] = \mathsf{pair}_{A[f]}^{B[f^\bullet]}$$

is meaningful. From now on, let us assume that it is satisfied. Observe that this implies that the square:

$$\begin{tikzcd}
\Delta.A[f].B[f.A] \arrow[r, "f^{\bullet\bullet}"] \arrow[d, "\mathsf{p}_{A[f]}^{B[f^\bullet]}"] & \Gamma.A.B \arrow[d, "\mathsf{p}_A^B"] \\
\Delta.\Sigma_{A[f]}^{B[f^\bullet]} \arrow[r, "f^\bullet"] & \Gamma.\Sigma_A^B
\end{tikzcd}$$

commutes.

Stability of split. Let P_C be a display map of codomain $\Gamma.\Sigma_A^B$ and let c be a section of $\Gamma.A.B.C[\mathsf{p}_A^B] \to \Gamma.A.B$. As split_c is by definition a section of P_C, then $\mathsf{split}_c[f^\bullet]$ is a section of $P_{C[f^\bullet]}$. Now, we observe that $P_{C[f^\bullet]}$ is a display map of codomain:

$$\Delta.\Sigma_A^B[f] = \Delta.\Sigma_{A[f]}^{B[f^\bullet]}$$

and that the arrow $c[f^{\bullet\bullet}]$ is a section of the display map $\Delta.A[f].B[f^\bullet].C[\mathsf{p}_A^B][f^{\bullet\bullet}] \to \Delta.A[f].B[f^\bullet]$ i.e. of:

$$\Delta.A[f].B[f^\bullet].C[f^\bullet][\mathsf{p}_{A[f]}^{B[f^\bullet]}] \to \Delta.A[f].B[f^\bullet].$$

Therefore $\mathsf{split}_{c[f^{\bullet\bullet}]}$ is well defined and it is a section of $P_{C[f^\bullet]}$. As $\mathsf{split}_c[f^\bullet]$ and $\mathsf{split}_{c[f^{\bullet\bullet}]}$ are both sections of $P_{C[f^\bullet]}$, the equality:

$$\mathsf{split}_c[f^\bullet] = \mathsf{split}_{c[f^{\bullet\bullet}]}$$

is meaningful. From now on, let us assume that it is satisfied.

Stability of σ. Finally, we observe that by definition $\sigma_{c[f^{\bullet\bullet}]}$ is a section of the display map:

$$\Delta.A[f].B[f^\bullet].\mathsf{Id}_{C[f^\bullet][\mathsf{p}^{B[f^\bullet]}_{A[f]}]}[\mathsf{split}_{c[f^{\bullet\bullet}]}[\mathsf{p}^{B[f^\bullet]}_{A[f]}];c[f^{\bullet\bullet}]] \to \Delta.A[f].B[f^\bullet].$$

Analogously, by definition σ_c is a section of the display map:

$$\Gamma.A.B.\mathsf{Id}_{C[\mathsf{p}^B_A]}[\mathsf{split}_c[\mathsf{p}^B_A];c] \to \Gamma.A.B$$

hence $\sigma_c[f^{\bullet\bullet}]$ is a section of the display map:

$$\Delta.A[f].B[f^\bullet].\mathsf{Id}_{C[\mathsf{p}^B_A]}[\mathsf{split}_c[\mathsf{p}^B_A];c][f^{\bullet\bullet}] \to \Delta.A[f].B[f^\bullet].$$

Now, we observe that:

$$\begin{array}{c}
\Delta.A[f].B[f^\bullet].\mathsf{Id}_{C[f^\bullet][\mathsf{p}^{B[f^\bullet]}_{A[f]}]}[\mathsf{split}_{c[f^{\bullet\bullet}]}[\mathsf{p}^{B[f^\bullet]}_{A[f]}];c[f^{\bullet\bullet}]] \\
\| \\
\Delta.A[f].B[f^\bullet].\mathsf{Id}_{C[\mathsf{p}^B_A][f^{\bullet\bullet}]}[\mathsf{split}_{c[f^{\bullet\bullet}]}[\mathsf{p}^{B[f^\bullet]}_{A[f]}];c[f^{\bullet\bullet}]] \\
\| \\
\Delta.A[f].B[f^\bullet].\mathsf{Id}_{C[\mathsf{p}^B_A][f^{\bullet\bullet}]}[\mathsf{split}_c[f^\bullet][\mathsf{p}^{B[f^\bullet]}_{A[f]}];c[f^{\bullet\bullet}]] \\
\| \\
\Delta.A[f].B[f^\bullet].\mathsf{Id}_{C[\mathsf{p}^B_A][f^{\bullet\bullet}]}[\mathsf{split}_c[\mathsf{p}^B_A][f^{\bullet\bullet}];c[f^{\bullet\bullet}]] \\
\| \\
\Delta.A[f].B[f^\bullet].\mathsf{Id}_{C[\mathsf{p}^B_A]}[f^{\bullet\bullet}.C[\mathsf{p}^B_A].C[\mathsf{p}^B_A]^\blacktriangledown][\mathsf{split}_c[\mathsf{p}^B_A][f^{\bullet\bullet}];c[f^{\bullet\bullet}]] \\
\|\ \text{Diagram (9)} \\
\Delta.A[f].B[f^\bullet].\mathsf{Id}_{C[\mathsf{p}^B_A]}[\mathsf{split}_c[\mathsf{p}^B_A];c][f^{\bullet\bullet}]
\end{array}$$

where the last equality follows from an instance of the commutative diagram (9). We conclude that $\sigma_c[f^{\bullet\bullet}]$ and $\sigma_{c[f^{\bullet\bullet}]}$ are sections of the same display map, hence the equality:

$$\sigma_c[f^{\bullet\bullet}] = \sigma_{c[f^{\bullet\bullet}]}$$

is meaningful.

B Validating axiomatic type formers in a display map 2-category

This section provides the full construction of the choice functions Form Rule, Intro Rule, Elim Rule, and Comp Axiom for =-types and Σ-types in suitable display map 2-categories, including all the details omitted in Section 4.

Let $(\boldsymbol{\mathcal{C}},\mathcal{D})$ be a display map 2-category endowed with axiomatic =-types—Definition 4.2—and let P_A be a display map of codomain Γ. The next paragraphs show that the display map category $(\mathbf{C},\mathcal{D})$ induced by $(\boldsymbol{\mathcal{C}},\mathcal{D})$ is endowed with appropriate choice functions validating axiomatic =-types—as in Definition 3.1.

Form Rule for =-types. We already have a choice of a display map $\Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \to \Gamma.A.A^\blacktriangledown$.

Intro Rule for =-types. Let us consider the 2-cell:

$$\begin{tikzcd} \Gamma.A \arrow[r, shift left=2, "1_{\Gamma.A}"] \arrow[r, shift right=2, "1_{\Gamma.A}"'] \arrow[r, phantom, "\Downarrow 1_{1_{\Gamma.A}}"] \arrow[dr, bend right, "P_A"'] & \Gamma.A \arrow[d, "P_A"] \\ & \Gamma \end{tikzcd}$$

of $\mathcal{C}/\Gamma$. Being α_A an arrow object, there is unique a 1-cell:

$$\begin{tikzcd} \Gamma.A \arrow[r, "\mathsf{r}_A"] \arrow[dr, bend right, "P_A"'] & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \arrow[d, "P_{\mathsf{Id}_A} P_{A^{\blacktriangledown}} P_A"] \\ & \Gamma \end{tikzcd}$$

of $\mathcal{C}/\Gamma$ such that $\alpha_A * \mathsf{r}_A = 1_{1_{\Gamma.A}}$. In particular:

$$P_{A^{\blacktriangledown}}(P_{\mathsf{Id}_A}\mathsf{r}_A) = 1_{\Gamma.A} \qquad P_A^{\bullet}(P_{\mathsf{Id}_A}\mathsf{r}_A) = 1_{\Gamma.A}$$

hence $P_{\mathsf{Id}_A}\mathsf{r}_A = \delta_A$—being:

$$\begin{tikzcd} \Gamma.A.A^{\blacktriangledown} \arrow[r, "P_A^{\bullet}"] \arrow[d, "P_{A^{\blacktriangledown}}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Gamma.A \arrow[d, "P_A"] \\ \Gamma.A \arrow[r, "P_A"] & \Gamma \end{tikzcd}$$

a pullback square—and therefore there is unique a section:

$$\mathsf{refl}_A : \Gamma.A \to \Gamma.A.\mathsf{Id}_A[\delta_A]$$

of $P_{\mathsf{Id}_A[\delta_A]}$ such that $\delta_A^{\bullet}\mathsf{refl}_A = \mathsf{r}_A$.

Elim Rule for =-types. Let $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A.C \to \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$ be a display map and let c be a section $\Gamma.A \to \Gamma.A.C[\mathsf{r}_A]$ of the display map $P_{C[\mathsf{r}_A]}$. Let us consider the commutative diagram:

$$\begin{tikzcd} P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A} \arrow[r, Rightarrow, "\alpha_A"] \arrow[d, Rightarrow, "\alpha_A"'] & (P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A})\mathsf{r}_A(P_A^{\bullet} P_{\mathsf{Id}_A}) \arrow[d, equal, "\alpha_A * \mathsf{r}_A(P_A^{\bullet} P_{\mathsf{Id}_A})"] \\ P_A^{\bullet} P_{\mathsf{Id}_A} \arrow[r, equal] & (P_A^{\bullet} P_{\mathsf{Id}_A})\mathsf{r}_A(P_A^{\bullet} P_{\mathsf{Id}_A}) \end{tikzcd}$$

—observe that $\alpha_A * \mathsf{r}_A(P_A^{\bullet} P_{\mathsf{Id}_A}) = 1_{P_A^{\bullet} P_{\mathsf{Id}_A}}$ because $\alpha_A * \mathsf{r}_A = 1_{1_{\Gamma.A}}$—where α_A on the left and $\alpha_A * \mathsf{r}_A(P_A^{\bullet} P_{\mathsf{Id}_A})$ are 2-cells of $\mathcal{C}/\Gamma$ induced by the 1-cells:

$$1_{\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A} \quad \text{and} \quad \mathsf{r}_A(P_A^{\bullet} P_{\mathsf{Id}_A})$$

of $\mathcal{C}/\Gamma$ respectively, by postcomposition via α_A. Hence the pair $(\alpha_A, 1_{P_A^{\bullet} P_{\mathsf{Id}_A}})$—whose components are the horizontal arrows of the diagram—that constitutes an arrow of the category $(\mathcal{C}/\Gamma)(P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}, P_A)^{\to}$ is induced, by postcomposition via α_A, by a unique arrow:

$$\varphi_A : 1_{\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A} \Longrightarrow \mathsf{r}_A(P_A^{\bullet} P_{\mathsf{Id}_A})$$

of $(\mathcal{C}/\Gamma)(P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}, P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A})$. In other words the equalities:

$$P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A} * \varphi_A = \alpha_A \quad \text{and} \quad P_A^{\bullet} P_{\mathsf{Id}_A} * \varphi_A = 1_{P_A^{\bullet} P_{\mathsf{Id}_A}}$$

hold. Moreover:

$$P_{A^\blacktriangledown} P_{\mathsf{Id}_A} * \varphi_A * \mathsf{r}_A = \alpha_A * \mathsf{r}_A = 1_{1_{\Gamma.A}} = P_{A^\blacktriangledown} P_{\mathsf{Id}_A} * 1_{\mathsf{r}_A}$$
$$P^\bullet_A P_{\mathsf{Id}_A} * \varphi_A * \mathsf{r}_A = 1_{P^\bullet_A P_{\mathsf{Id}_A}} * \mathsf{r}_A = 1_{1_{\Gamma.A}} = P^\bullet_A P_{\mathsf{Id}_A} * 1_{\mathsf{r}_A}$$

therefore $\varphi_A * \mathsf{r}_A$ and 1_{r_A} induce, by postcomposition via α_A, the same arrow $\alpha_A * \mathsf{r}_A \to \alpha_A * \mathsf{r}_A$, i.e. $1_{1_{\Gamma.A}} \to 1_{1_{\Gamma.A}}$, in $(\mathcal{C}/\Gamma)(P_A, P_A)^\to$ — namely the pair $(1_{1_{\Gamma.A}}, 1_{1_{\Gamma.A}})$. Being α_A an arrow object, we conclude that:

$$\varphi_A * \mathsf{r}_A = 1_{\mathsf{r}_A}.$$

Now, we observe that:

$$P_C(\mathsf{r}_A.C)cP^\bullet_A P_{\mathsf{Id}_A} = \mathsf{r}_A P_{C[\mathsf{r}_A]} c P^\bullet_A P_{\mathsf{Id}_A} = \mathsf{r}_A P^\bullet_A P_{\mathsf{Id}_A}$$

hence we can rewrite the codomain of the 2-cell φ_A as follows:

$$\begin{tikzcd} \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \arrow[r, "(\mathsf{r}_A.C)cP^\bullet_A P_{\mathsf{Id}_A}"] \arrow[dr, equal, bend right] & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A.C \arrow[d, "P_C"] \arrow[dl, phantom, "\Rightarrow_{\varphi_A}"] \\ & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \end{tikzcd}$$

and, using the cloven isofibration structure on P_C, we obtain a section $\mathsf{t}^{\varphi_A}_{\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}} : \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \to \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A.C$ of P_C, as well as a 2-cell:

$$\begin{tikzcd} \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \arrow[r, bend left, "\mathsf{t}^{\varphi_A}_{\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}}"] \arrow[r, bend right, "\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}"'] \arrow[r, phantom, "\Downarrow \tau^{\varphi_A}_{\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}}"] & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A.C \end{tikzcd}$$

such that $P_C * \tau^{\varphi_A}_{\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}} = \varphi_A$. We define $\mathsf{J}_c := \mathsf{t}^{\varphi_A}_{\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}}$.

Comp Axiom for =-types. Referring to the diagram:

$$\begin{tikzcd} \Gamma.A \arrow[rr, "\mathsf{r}_A"] \arrow[d, shift right=3, "\mathsf{J}_c[\mathsf{r}_A]"'] \arrow[d, shift left=3, "c"] & & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \arrow[d, shift right=3, "\mathsf{J}_c"'] \arrow[d, shift left=3, "\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}"] \arrow[d, phantom, "\Rightarrow"] \\ \Gamma.A.C[\mathsf{r}_A] \arrow[rr, "\mathsf{r}^\bullet_A"] \arrow[d, "P_{C[\mathsf{r}_A]}"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A.C \arrow[d, "P_C"] \\ \Gamma.A \arrow[rr, "\mathsf{r}_A"] & & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \end{tikzcd}$$

—where the 2-cell is $\tau^{\varphi_A}_{\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}}$—we observe that:

$$P_C \mathsf{r}^\bullet_A c P^\bullet_A P_{\mathsf{Id}_A} \mathsf{r}_A = P_C \mathsf{r}^\bullet_A c = \mathsf{r}_A P_{C[\mathsf{r}_A]} c = \mathsf{r}_A 1_{\Gamma.A}$$

and, since $P_{C[\mathsf{r}_A]} c = 1_{\Gamma.A}$ and $\mathsf{r}^\bullet_A c = \mathsf{r}^\bullet_A c P^\bullet_A P_{\mathsf{Id}_A} \mathsf{r}_A$, we obtain that c is in fact $(\ \mathsf{r}^\bullet_A c P^\bullet_A P_{\mathsf{Id}_A}\ )[\mathsf{r}_A]$. Moreover:

$$\begin{aligned} P_C \ * \ \tau^{\varphi_A}_{\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A}} \ * \ \mathsf{r}_A &= P_C \ * \ \tau^{\varphi_A * \mathsf{r}_A}_{\mathsf{r}^\bullet_A cP^\bullet_A P_{\mathsf{Id}_A} \mathsf{r}_A} \\ &= P_C \ * \ \tau^{1_{\mathsf{r}_A}}_{\mathsf{r}^\bullet_A c} \\ &= 1_{\mathsf{r}_A} \\ &= \mathsf{r}_A \ * \ 1_{1_{\Gamma.A}} \end{aligned}$$

hence, by the 2-universal property of $\Gamma.A.C[\mathsf{r}_A]$, we conclude that there is unique a 2-cell:

$$\begin{tikzcd} \Gamma.A \arrow[r, bend left, "\mathsf{J}_c[\mathsf{r}_A]"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow h_c"] & \Gamma.A.C[\mathsf{r}_A] \end{tikzcd}$$

such that $P_{C[\mathsf{r}_A]} * h_c = 1_{1_{\Gamma.A}}$ and $\mathsf{r}_A^\bullet * h_c = \tau^{\varphi_A}_{\mathsf{r}_A^\bullet c P_A^\bullet P_{\mathsf{Id}_A}} * \mathsf{r}_A$. Therefore there is unique an arrow:

$$\begin{tikzcd} \Gamma.A \arrow[r, "\tilde{h}_c"] \arrow[dr, equal] & \Gamma.A.C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown.\mathsf{Id}_{C[\mathsf{r}_A]} \arrow[d, "P_{C[\mathsf{r}_A]} P_{C[\mathsf{r}_A]^\blacktriangledown} P_{\mathsf{Id}_{C[\mathsf{r}_A]}}"] \\ & \Gamma.A \end{tikzcd}$$

in $(\mathcal{C}/\Gamma.A)(\ 1_{\Gamma.A}\ ,\ P_{C[\mathsf{r}_A]} P_{C[\mathsf{r}_A]^\blacktriangledown} P_{\mathsf{Id}_{C[\mathsf{r}_A]}}\ )$ such that $\alpha_{C[\mathsf{r}_A]} * \tilde{h}_c = h_c$. Since:

$$P_{C[\mathsf{r}_A]^\blacktriangledown} P_{\mathsf{Id}_{C[\mathsf{r}_A]}} \tilde{h}_c = \mathsf{J}_c[\mathsf{r}_A] \quad \text{ and } \quad P^\bullet_{C[\mathsf{r}_A]} P_{\mathsf{Id}_{C[\mathsf{r}_A]}} \tilde{h}_c = c$$

we obtain that $P_{\mathsf{Id}_{C[\mathsf{r}_A]}} \tilde{h}_c = \mathsf{J}_c[\mathsf{r}_A]; c$ hence the diagram:

$$\begin{tikzcd} \Gamma.A \arrow[d, "\mathsf{H}_c"'] \arrow[dr, bend left, "\tilde{h}_c"] & \\ \Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown.\mathsf{Id}_{C[\mathsf{r}_A]} \arrow[d] \\ \Gamma.A \arrow[r, "\mathsf{J}_c[\mathsf{r}_A]; c"'] & \Gamma.A.C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown \end{tikzcd}$$

commutes for a unique section:

$$\mathsf{H}_c : \Gamma.A \to \Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c]$$

of $P_{\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A];c]}$.

Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic =-types and axiomatic Σ-types—Definitions 4.2 and 4.4—let P_A be a display map of codomain Γ, and let P_B be a display map of codomain $\Gamma.A$. The next paragraphs show that the display map category $(\mathbf{C}, \mathcal{D})$ induced by $(\mathcal{C}, \mathcal{D})$ is endowed with appropriate choice functions validatin axiomatic Σ-types—as in Definition 3.2.

Form Rule for $\boldsymbol{\Sigma}$-types. We already have a choice of a display map $\Gamma.\mathsf{\Sigma}^B_A \to \Gamma$.

Intro Rule for $\boldsymbol{\Sigma}$-types. Since $P_{\mathsf{\Sigma}^B_A} \mathsf{p}^B_A = P_A P_B$ there is unique a section:

$$\mathsf{pair}^B_A : \Gamma.A.B \to \Gamma.A.B.\mathsf{\Sigma}^B_A[P_A P_B]$$

of $P_{\mathsf{\Sigma}^B_A[P_A P_B]}$ such that $(P_A P_B)^\bullet \mathsf{refl}_A = \mathsf{p}^B_A$.

Elim Rule for $\boldsymbol{\Sigma}$-types. Let $\Gamma.\mathsf{\Sigma}^B_A.C \to \Gamma.\mathsf{\Sigma}^B_A$ be a display map and let c be a section $\Gamma.A.B \to \Gamma.A.B.C[\mathsf{p}^B_A]$ of the display map $P_{C[\mathsf{p}^B_A]}$. We observe that:

$$P_C (\mathsf{p}^B_A)^\bullet c \pi^B_A = \mathsf{p}^B_A P_{C[\mathsf{p}^B_A]} c \pi^B_A = \mathsf{p}^B_A \pi^B_A$$

hence we can rewrite the codomain of the 2-cell η^B_A as follows:

$$\begin{tikzcd}
\Gamma.\Sigma^B_A \arrow[r, "(\mathsf{p}^B_A)^\bullet c\pi^B_A"] \arrow[dr, equal, bend right] \arrow[r, phantom, shift right=4ex, "\overset{\Longrightarrow}{\eta^B_A}"] & \Gamma.\Sigma^B_A.C \arrow[d, "P_C"] \\
& \Gamma.\Sigma^B_A
\end{tikzcd}$$

and, using the cloven isofibration structure on P_C, we obtain a section $\mathsf{t}^{\eta^B_A}_{(\mathsf{p}^B_A)^\bullet c\pi^B_A} : \Gamma.\Sigma^B_A \to \Gamma.\Sigma^B_A.C$ of P_C, as well as a 2-cell:

$$\begin{tikzcd}
\Gamma.\Sigma^B_A \arrow[r, bend left, "\mathsf{t}^{\eta^B_A}_{(\mathsf{p}^B_A)^\bullet c\pi^B_A}"] \arrow[r, bend right, "(\mathsf{p}^B_A)^\bullet c\pi^B_A"'] \arrow[r, phantom, "\Downarrow \tau^{\eta^B_A}_{(\mathsf{p}^B_A)^\bullet c\pi^B_A}"] & \Gamma.\Sigma^B_A.C
\end{tikzcd}$$

such that $P_C * \tau^{\eta^B_A}_{(\mathsf{p}^B_A)^\bullet c\pi^B_A} = \eta^B_A$. We define $\mathsf{split}_c := \mathsf{t}^{\eta^B_A}_{(\mathsf{p}^B_A)^\bullet c\pi^B_A}$.

Comp Axiom for Σ-types. Let us condider the 2-cell:

$$\mathsf{split}_c\mathsf{p}^B_A \xRightarrow{\tau^{\eta^B_A}_{(\mathsf{p}^B_A)^\bullet c\pi^B_A} * \mathsf{p}^B_A} (\mathsf{p}^B_A)^\bullet c\pi^B_A\mathsf{p}^B_A \xRightarrow{(\mathsf{p}^B_A)^\bullet c * \beta^B_A} (\mathsf{p}^B_A)^\bullet c$$

and observe that:

$$P_C * (\mathsf{split}_c\mathsf{p}^B_A \Longrightarrow (\mathsf{p}^B_A)^\bullet c) = (\eta^B_A * \mathsf{p}^B_A)(\mathsf{p}^B_A * \beta^B_A) = 1_{\mathsf{p}^B_A}$$

since, up to replacing η^B_A with:

$$1_{\Gamma.\Sigma^B_A} \xrightarrow{\quad\eta^B_A\quad} \xrightarrow{\mathsf{p}^B_A(\beta^B_A)^{-1}\pi^B_A} \xrightarrow{\mathsf{p}^B_A\pi^B_A(\eta^B_A)^{-1}} \mathsf{p}^B_A\pi^B_A$$

we can assume without loss of generality that, being the given $(\mathsf{p}^B_A, \pi^B_A, \eta^B_A, \beta^B_A)$ an equivalence, it is in fact an adjoint equivalence — see [29], [7], [18, Chapter 6]. Therefore, the diagram of 2-cells:

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, equal] \arrow[dd, equal] & \Gamma.A.B \arrow[d, shift right=2ex, "\mathsf{split}_c\mathsf{p}^B_A"'] \arrow[d, shift left=2ex, "(\mathsf{p}^B_A)^\bullet c"] \arrow[d, phantom, "\Rightarrow"] \\
& \Gamma.\Sigma^B_A.C \arrow[d, "P_C"] \\
\Gamma.A.B \arrow[r, "\mathsf{p}^B_A"] & \Gamma.\Sigma^B_A
\end{tikzcd}$$

commutes and, by the 2-universal property of $\Gamma.A.B.C[\mathsf{p}^B_A]$, we conclude that there is unique a 2-cell s_c such that the diagram of 2-cells:

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, equal] \arrow[d, shift right=2ex, "\mathsf{split}_c[\mathsf{p}^B_A]"'] \arrow[d, shift left=2ex, "c"] \arrow[d, phantom, "\overset{s_c}{\Rightarrow}"] & \Gamma.A.B \arrow[d, shift right=2ex, "\mathsf{split}_c\mathsf{p}^B_A"'] \arrow[d, shift left=2ex, "(\mathsf{p}^B_A)^\bullet c"] \arrow[d, phantom, "\Rightarrow"] \\
\Gamma.A.B.C[\mathsf{p}^B_A] \arrow[r, dashed, "(\mathsf{p}^B_A)^\bullet"] \arrow[d, "P_{C[\mathsf{p}^B_A]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.\Sigma^B_A.C \arrow[d, "P_C"] \\
\Gamma.A.B \arrow[r, "\mathsf{r}_A"] & \Gamma.\Sigma^B_A
\end{tikzcd}$$

commutes and $P_{C[\mathsf{p}_A^B]} * s_c = 1_{1_{\Gamma.A.B}}$. Therefore there is unique an arrow:

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, "\tilde{s}_c"] \arrow[dr, equal, bend right] & \Gamma.A.B.C[\mathsf{p}_A^B].C[\mathsf{p}_A^B]^{\blacktriangledown}.\mathsf{Id}_{C[\mathsf{p}_A^B]} \arrow[d, "P_{C[\mathsf{p}_A^B]} P_{C[\mathsf{p}_A^B]^{\blacktriangledown}} P_{\mathsf{Id}_{C[\mathsf{p}_A^B]}}"] \\
& \Gamma.A.B
\end{tikzcd}$$

in $(\mathcal{C}/\Gamma.A.B)(\ 1_{\Gamma.A.B}\ ,\ P_{C[\mathsf{p}_A^B]} P_{C[\mathsf{p}_A^B]^{\blacktriangledown}} P_{\mathsf{Id}_{C[\mathsf{p}_A^B]}}\ )$ such that $\alpha_{C[\mathsf{p}_A^B]} * \tilde{s}_c = s_c$. And since:

$$P_{C[\mathsf{p}_A^B]^{\blacktriangledown}} P_{\mathsf{Id}_{C[\mathsf{p}_A^B]}} \tilde{s}_c = \mathsf{split}_c[\mathsf{p}_A^B] \quad \text{and} \quad P^{\bullet}_{C[\mathsf{p}_A^B]} P_{\mathsf{Id}_{C[\mathsf{p}_A^B]}} \tilde{s}_c = c$$

we obtain that $P_{\mathsf{Id}_{C[\mathsf{p}_A^B]}} \tilde{s}_c = \mathsf{split}_c[\mathsf{p}_A^B]; c$ hence the diagram:

$$\begin{tikzcd}
\Gamma.A.B \arrow[d, "\sigma_c"'] \arrow[dr, bend left, "\tilde{s}_c"] & \\
\Gamma.A.B.\mathsf{Id}_{C[\mathsf{p}_A^B]}[\mathsf{split}_c[\mathsf{p}_A^B]; c] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.B.C[\mathsf{p}_A^B].C[\mathsf{p}_A^B]^{\blacktriangledown}.\mathsf{Id}_{C[\mathsf{p}_A^B]} \arrow[d] \\
\Gamma.A.B \arrow[r, "\mathsf{split}_c[\mathsf{p}_A^B]; c"'] & \Gamma.A.B.C[\mathsf{p}_A^B].C[\mathsf{p}_A^B]^{\blacktriangledown}
\end{tikzcd}$$

commutes for a unique section:

$$\sigma_c : \Gamma.A.B \to \Gamma.A.B.\mathsf{Id}_{C[\mathsf{p}_A^B]}[\mathsf{split}_c[\mathsf{p}_A^B]; c]$$

of $P_{\mathsf{Id}_{C[\mathsf{p}_A^B]}[\mathsf{split}_c[\mathsf{p}_A^B]; c]}$.

C Stability conditions in display map 2-categories

This section contains the proof of Proposition 4.3. The proof of Proposition 4.5 is completely analogous, and uses the *stability conditions* in Definition 4.4 in place of the ones in Definition 4.2. We also provide the proof of Proposition 4.8 for $\mathsf{\Pi}$, ev, abst, and β. The proof of Proposition 4.8 for funext, β^{Π}, and η^{Π} follows the same argument together with Theorem 4.7.

Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic $=$-types. Let f be an arrow $\Delta \to \Gamma$ and let P_A be a display map of codomain Γ.

Stability of Id. The first stability condition $\mathsf{Id}_A[f^{\bullet\bullet}] = \mathsf{Id}_{A[f]}$ holds by definition.

Stability of refl. Now, the following diagram:

$$\begin{tikzcd}
\Delta.A[f].A[f]^{\blacktriangledown}.\mathsf{Id}_{A[f]} \arrow[r, "f^{\bullet\bullet\bullet}"] \arrow[d, shift right=3ex] \arrow[d, shift left=3ex] \arrow[d, phantom, "\overset{\alpha_{A[f]}}{\Longrightarrow}"] & \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \arrow[d, shift right=2ex] \arrow[d, shift left=2ex] \arrow[d, phantom, "\overset{\alpha_A}{\Rightarrow}"] \\
\Delta.A[f] \arrow[r, "f^{\bullet}"'] & \Gamma.A
\end{tikzcd}$$

commutes and $\alpha_A[f] = \alpha_{A[f]}$, by definition. Therefore:

$$\alpha_A * (f^{\bullet\bullet\bullet}\mathsf{r}_{A[f]}) = (\alpha_A * f^{\bullet\bullet\bullet}) * \mathsf{r}_{A[f]} = (f^{\bullet} * \alpha_A[f]) * \mathsf{r}_{A[f]} = (f^{\bullet} * \alpha_{A[f]}) * \mathsf{r}_{A[f]} = \\ f^{\bullet} * (\alpha_{A[f]} * \mathsf{r}_{A[f]}) = 1_{f^{\bullet}}$$

and:

$$\alpha_A * (\mathsf{r}_A f^\bullet) = (\alpha_A * \mathsf{r}_A) * f^\bullet = 1_{f^\bullet}$$

hence the diagram:

$$\begin{tikzcd}
\Delta.A[f] \arrow[r, "f^\bullet"] \arrow[d, "\mathsf{r}_{A[f]}"'] & \Gamma.A \arrow[d, "\mathsf{r}_A"] \\
\Delta.A[f].A[f]^\blacktriangledown.\mathsf{Id}_{A[f]} \arrow[r, "f^{\bullet\bullet\bullet}"] & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A
\end{tikzcd}$$

commutes being α_A an arrow object. As usual, the commutativity of this diagram is equivalent to the stability condition $\mathsf{refl}_A[f^\bullet] = \mathsf{refl}_{A[f]}$.

Stability of J. Let P_C be a display map of codomain $\Gamma.A.A^\blacktriangledown.\mathsf{Id}_A$ and let c be a section of $P_{C[\mathsf{p}_A^B]}$. Let us observe that the diagram:

$$\begin{tikzcd}
\Delta.A[f].A[f]^\blacktriangledown.\mathsf{Id}_{A[f]} \arrow[r, "f^{\bullet\bullet\bullet}"] \arrow[d, "P_{\mathsf{Id}_{A[f]}}"'] & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A \arrow[d, "P_{\mathsf{Id}_A}"] \\
\Delta.A[f].A[f]^\blacktriangledown \arrow[r, "f^{\bullet\bullet}"] \arrow[d, "P^\bullet_{A[f]}"'] & \Gamma.A.A^\blacktriangledown \arrow[d, "P^\bullet_A"] \\
\Delta.A[f] \arrow[r, "f^\bullet"] \arrow[d, "c[f^\bullet]"'] & \Gamma.A \arrow[d, "c"] \\
\Delta.A[f].C[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}] \arrow[r, "f^\bullet.C[\mathsf{r}^\bullet_A]"] \arrow[d, "\mathsf{r}_{A[f]}.C[f^{\bullet\bullet\bullet}]"'] & \Gamma.A.C[\mathsf{r}^\bullet_A] \arrow[d, "\mathsf{r}_A.C"] \\
\Delta.A[f].A[f]^\blacktriangledown.\mathsf{Id}_{A[f]}.C[f^{\bullet\bullet\bullet}] \arrow[r, "f^{\bullet\bullet\bullet}.C"'] & \Gamma.A.A^\blacktriangledown.\mathsf{Id}_A.C
\end{tikzcd}$$

commutes and that the two columns yield:

$$P_{\mathsf{Id}_{A[f]}} \quad \text{and} \quad P_{\mathsf{Id}_A}$$

when postcomposed by $P_{\mathsf{Id}_{A[f]}} P_{C[f^{\bullet\bullet\bullet}]}$ and by $P_{\mathsf{Id}_A} P_C$ respectively.

Moreover, let us observe that:

$$\begin{aligned}
f^\bullet * (P_{A[f]^\blacktriangledown} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f]) &= P_{A^\blacktriangledown} P_{\mathsf{Id}_A} * (f^{\bullet\bullet\bullet} * \varphi_A[f]) \\
&= (P_{A^\blacktriangledown} P_{\mathsf{Id}_A} * \varphi_A) * f^{\bullet\bullet\bullet} \\
&= \alpha_A * f^{\bullet\bullet\bullet}
\end{aligned}$$

$$\begin{aligned}
P_{A[f]} * (P_{A[f]^\blacktriangledown} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f]) &= P_{A[f]} P_{A[f]^\blacktriangledown} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f] \\
&= 1_{P_{A[f]} P_{A[f]^\blacktriangledown} P_{\mathsf{Id}_{A[f]}}}
\end{aligned}$$

which implies that:

$$P_{A[f]^\blacktriangledown} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f] = \alpha_A[f] = \alpha_{A[f]}.$$

Analogously:

$$\begin{aligned} f^{\bullet} * (P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f]) &= P^{\bullet}_A P_{\mathsf{Id}_A} * (f^{\bullet\bullet\bullet} * \varphi_A[f]) \\ &= (P^{\bullet}_A P_{\mathsf{Id}_A} * \varphi_A) * f^{\bullet\bullet\bullet} \\ &= 1_{P^{\bullet}_A P_{\mathsf{Id}_A}} * f^{\bullet\bullet\bullet} \end{aligned}$$

$$\begin{aligned} P_{A[f]} * (P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f]) &= P_{A[f]} P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f] \\ &= 1_{P_{A[f]} P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}}} \end{aligned}$$

which implies that:

$$P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f] = 1_{P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}}}.$$

As:

$$\begin{aligned} P_{A[f]^{\blacktriangledown}} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f] &= \alpha_{A[f]} \\ P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}} * \varphi_A[f] &= 1_{P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}}} \end{aligned}$$

it must be that $\varphi_A[f] = \varphi_{A[f]}$ by the universal property of $\alpha_{A[f]}$.

Therefore:

$$\begin{aligned} \mathsf{J}_c[f^{\bullet\bullet\bullet}] &= \mathsf{t}^{\varphi_A}_{(\mathsf{r}_A.C)cP^{\bullet}_A P_{\mathsf{Id}_A}}[f^{\bullet\bullet\bullet}] \\ &= \mathsf{t}^{\varphi_A[f]}_{(\mathsf{r}_{A[f]}.C[f^{\bullet\bullet\bullet}])c[f^{\bullet}]P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}}} \\ &= \mathsf{t}^{\varphi_{A[f]}}_{(\mathsf{r}_{A[f]}.C[f^{\bullet\bullet\bullet}])c[f^{\bullet}]P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}}} \\ &= \mathsf{J}_{c[f^{\bullet}]}. \end{aligned}$$

Stability of H. Analogously:

$$\tau^{\varphi_A}_{(\mathsf{r}_A.C)cP^{\bullet}_A P_{\mathsf{Id}_A}}[f^{\bullet\bullet\bullet}] = \tau^{\varphi_{A[f]}}_{(\mathsf{r}_{A[f]}.C[f^{\bullet\bullet\bullet}])c[f^{\bullet}]P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}}}$$

hence:

$$\begin{aligned} h_c[f^{\bullet}] &= \tau^{\varphi_A}_{(\mathsf{r}_A.C)cP^{\bullet}_A P_{\mathsf{Id}_A}}[\mathsf{r}_A][f^{\bullet}] \\ &= \tau^{\varphi_A}_{(\mathsf{r}_A.C)cP^{\bullet}_A P_{\mathsf{Id}_A}}[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}] \\ &= \tau^{\varphi_{A[f]}}_{(\mathsf{r}_{A[f]}.C[f^{\bullet\bullet\bullet}])c[f^{\bullet}]P^{\bullet}_{A[f]} P_{\mathsf{Id}_{A[f]}}}[\mathsf{r}_{A[f]}] \\ &= h_{c[f^{\bullet}]}. \end{aligned}$$

In particular, since $\alpha_{C[\mathsf{r}_A]}[f^{\bullet}] * \tilde{h}_c[f^{\bullet}]$ is a section of $P_{C[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}]}$ such that:

$$f^{\bullet}.C[\mathsf{r}_A] * (\alpha_{C[\mathsf{r}_A]}[f^{\bullet}] * \tilde{h}_c[f^{\bullet}]) = (\alpha_{C[\mathsf{r}_A]} * \tilde{h}_c) * f^{\bullet} = h_c * f^{\bullet}$$

it must be that $\alpha_{C[\mathsf{r}_A]}[f^{\bullet}] * \tilde{h}_c[f^{\bullet}] = h_c[f^{\bullet}]$, which implies that:

$$\tilde{h}_c[f^{\bullet}] = \tilde{h}_{c[f^{\bullet}]}$$

as $h_c[f^{\bullet}] = h_{c[f^{\bullet}]}$ and $\alpha_{C[\mathsf{r}_A]}[f^{\bullet}] = \alpha_{C[\mathsf{r}_A][f^{\bullet}]} = \alpha_{C[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}]}$.

Finally, referring to the diagram:

$$
\begin{tikzcd}
 & & {} \arrow[d, dashed, "\mathsf{H}_c"] \arrow[dr, "\tilde{h}_c"] & \\
{} \arrow[urr] \arrow[d, "\mathsf{H}_{c[f^\bullet]}"'] \arrow[dr, "\tilde{h}_{c[f^\bullet]}"] & & \mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c] \arrow[r] \arrow[dd, dashed] & C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown.\mathsf{Id}_{C[\mathsf{r}_A]} \arrow[dd] \\
\mathsf{Id}_{C'}[\mathsf{J}_{c[f^\bullet]}[\mathsf{r}_{A[f]}]; c[f^\bullet]] \arrow[r] \arrow[urr, dashed] \arrow[dd] & C'.C'^\blacktriangledown.\mathsf{Id}_{C'} \arrow[urr] \arrow[dd] & & \\
 & & {} \arrow[r, dashed, "\mathsf{J}_c[\mathsf{r}_A];c"'] & C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown \\
{} \arrow[urr] \arrow[r, "\mathsf{J}_{c[f^\bullet]}[\mathsf{r}_{A[f]}];c[f^\bullet]"'] & C'.C'^\blacktriangledown \arrow[urr] & &
\end{tikzcd}
$$

—where we omitted the context $\Gamma.A$ in the objects of the front side and the context $\Delta.A[f]$ in the ones of the back side, and where $C' := C[f^{\bullet\bullet\bullet}][\mathsf{r}_{A[f]}]$—a priori every atomic diagram commutes apart from:

$$
\begin{tikzcd}
\Delta.A[f] \arrow[r] \arrow[d, "\mathsf{H}_{c[f^\bullet]}"'] & \Gamma.A \arrow[d, dashed, "\mathsf{H}_c"] \\
\Delta.A[f].\mathsf{Id}_{C'}[\mathsf{J}_{c[f^\bullet]}[\mathsf{r}_{A[f]}]; c[f^\bullet]] \arrow[r, dashed] & \Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c]
\end{tikzcd}
\tag{10}
$$

and we are going to prove that it actually does. In fact, we observe that:

$$
\begin{aligned}
P_{\mathsf{Id}_{C[\mathsf{r}_A]}}(f^\bullet.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_C[\mathsf{r}_A]; c]\mathsf{H}_{c[f^\bullet]}) &= f^\bullet P_{\mathsf{Id}_{C'}[\mathsf{J}_{c[f^\bullet]}[\mathsf{r}_{A[f]}];c[f^\bullet]]} h_{c[f^\bullet]} \\
&= f^\bullet \\
&= P_{\mathsf{Id}_{C[\mathsf{r}_A]}}(\mathsf{H}_c f^\bullet)
\end{aligned}
$$

and:

$$
\begin{aligned}
(\mathsf{J}_c[\mathsf{r}_A]; c).\mathsf{Id}_{C[\mathsf{r}_A]}(f^\bullet.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_C[\mathsf{r}_A]; c]\mathsf{H}_{c[f^\bullet]}) &= f^\bullet.C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown.\mathsf{Id}_{C[\mathsf{r}_A]}\mathsf{J}_{c[f^\bullet]}[\mathsf{r}_{A[f]}; c[f^\bullet]].\mathsf{Id}_{C'}\mathsf{H}_{c[f^\bullet]} \\
&= f^\bullet.C[\mathsf{r}_A].C[\mathsf{r}_A]^\blacktriangledown.\mathsf{Id}_{C[\mathsf{r}_A]}\tilde{h}_{c[f.A]} \\
&= \tilde{h}_c f^\bullet \\
&= (\mathsf{J}_c[\mathsf{r}_A]; c).\mathsf{Id}_{C[\mathsf{r}_A]}(\mathsf{H}_c f^\bullet).
\end{aligned}
$$

hence, by the universal property of $\Gamma.A.\mathsf{Id}_{C[\mathsf{r}_A]}[\mathsf{J}_c[\mathsf{r}_A]; c]$, the diagram (10) commutes. Therefore:

$$
\mathsf{H}_c[f^\bullet] = \mathsf{H}_{c[f^\bullet]}
$$

and we are done.

Let $(\mathcal{C}, \mathcal{D})$ be a display map 2-category endowed with axiomatic =-types and axiomatic Π-types & axiomatic function extensionality. Let f be an arrow $\Delta \to \Gamma$, let P_A be a display map of codomain Γ and let P_B be a display map of codomain $\Gamma.A$.

Stability of Π. The first stability condition $\Pi_A^B[f] = \Pi_{A[f]}^{B[f^\bullet]}$ holds by definition.

Stability of ev. We verified the second stability condition $\mathsf{ev}^B_A[f^{\bullet\bullet}] = \mathsf{ev}^{B[f^\bullet]}_{A[f]}$ in (7) in the paragraph **Stability of app and λ under re-indexing.**

Stability of abst. Let υ be a section of P_B. Then:

$$\mathsf{abst}_\upsilon = \lambda^B_A \upsilon \quad \text{and} \quad \mathsf{abst}_{\upsilon[f^\bullet]} = \lambda^{B[f^\bullet]}_{A[f]}(\upsilon[f^\bullet])$$

by definition. Hence:

$$\mathsf{abst}_\upsilon[f] = (\lambda^B_A \upsilon)[f] \overset{(6)}{=} \lambda^{B[f^\bullet]}_{A[f]}(\upsilon[f^\bullet]) = \mathsf{abst}_{\upsilon[f^\bullet]}$$

and we are done.

Stability of β. By definition:

$$\beta_\upsilon = \{\beta^{A,B}_\upsilon\} \quad \text{and} \quad \beta_{\upsilon[f^\bullet]} = \{\beta^{A[f],B[f^\bullet]}_{\upsilon[f^\bullet]}\}$$

hence:

$$\begin{aligned}
\beta_\upsilon[f^\bullet] &= \{\beta^{A,B}_\upsilon\}[f^\bullet] \\
&= \{\beta^{A,B}_\upsilon[f^\bullet]\} \\
&= \{1; \beta^{A,B}_\upsilon f^\bullet\} \\
&= \{1; \beta^{A,B}_{\upsilon f^\bullet}\} \\
&= \{\beta^{A[f],B[f^\bullet]}_{1;\upsilon f^\bullet}\} \\
&= \{\beta^{A[f],B[f^\bullet]}_{\upsilon[f^\bullet]}\} \\
&= \beta_{\upsilon[f^\bullet]}
\end{aligned}$$

where the fourth and the fifth equalities hold by Definition 4.6. We are done.

D Further details on the content of Section 5

We provide further details on the structure that the display map 2-category $(\mathbf{Grpd}, \mathcal{D})$, defined in Section 5, is endowed with.

We stipulated that a display map $P_A : \Gamma.A \to \Gamma$ of the class $\mathcal{D}$ is obtained by applying the Grothendieck construction to a pseudofunctor $(A, \phi^A, \psi^A) : \Gamma \to \mathbf{Grpd}$, where Γ is a groupoid, and ϕ^A and ψ^A are the *coherent* families of natural isomorphisms $\phi^A_{p,q} : A_q A_p \to A_{qp}$ and $\psi^A_\gamma : A_{1_\gamma} \to 1_{A_\gamma}$—where γ is any object of Γ and p and q any composable arrows of Γ—making A into a pseudofunctor. We recall that the coherence laws are the equalities:

$$\phi^A_{p,rq}(\phi^A_{q,r} * A_p) = \phi^A_{qp,r}(A_r * \phi^A_{p,q})$$

$$A_p * \psi^A_\gamma = \phi^A_{1_\gamma,p} \qquad \psi^A_{\gamma'} * A_p = \phi^A_{p,1_{\gamma'}}$$

for every choice of arrows $p : \gamma \to \gamma'$, $q : \gamma' \to \gamma''$, and $r : \gamma'' \to \gamma'''$ in Γ.

This means that $\Gamma.A$ is the groupoid whose objects are the pairs (γ, x) where γ is an object of Γ and x is an object of A_γ, and the arrows (γ, x) to (γ', x') are the pairs (p_1, p_2) where p_1 is an arrow $\gamma \to \gamma'$ and p_2 is an arrow $A_{p_1} x \to x'$.

In detail, the composition of two arrows:

$$(\gamma, x) \xrightarrow{(p_1,p_2)} (\gamma', x') \xrightarrow{(q_1,q_2)} (\gamma'', x'')$$

is:

$$(q_1p_1, A_{q_1p_1}x \xrightarrow{(\phi^A_{p_1,q_1})^{-1}x} A_{q_1}A_{p_1}x \xrightarrow{A_{q_1}p_2} A_{q_1}x' \xrightarrow{q_2} x'')$$

the identity arrow of the object (γ, x) is $(1_\gamma, \psi^A_\gamma x)$, and the inverse of (p_1, p_2) exists and is the pair:

$$(p_1{}^{-1},$$

$$A_{p_1{}^{-1}}x' \xrightarrow{A_{p_1{}^{-1}}p_2{}^{-1}} A_{p_1{}^{-1}}A_{p_1}x \xrightarrow{\phi^A_{p_1,p_1{}^{-1}}x} A_{p_1{}^{-1}p_1}x =$$

$$= A_{1_\gamma}x \xrightarrow{\psi^A_\gamma x} x)$$

and associativity, unitalities, and invertibilities follow from the coherence laws of the families ϕ^A and ψ^A. Hence $\Gamma.A$ is a groupoid. The functor $\Gamma.A \to \Gamma$ maps (γ, x) to γ and (p_1, p_2) to p_1.

Regarding the re-indexing structure on display maps, we observed that, for every 1-cell $f : \Delta \to \Gamma$ in **Grpd** and every pseudofunctor $(A, \phi^A, \psi^A) : \Gamma \to \mathbf{Grpd}$, the triple:

$$(Af, \phi^A_{f-,f-}, \psi^A_{f-})$$

defines a pseudofunctor $\Delta \to \mathbf{Grpd}$. We stipulated that the associated Grothendieck construction:

$$P_{Af} : \Delta.Af \to \Delta$$

denoted as $P_{A[f]} : \Delta.A[f] \to \Delta$, is the re-indexing of P_A along f. The mappings:

$$(\ \delta : \Delta\ ,\ x : (Af)_\delta\ ) \mapsto (\ f\delta\ ,\ x\ ) \text{ and } (\ \delta \xrightarrow{p_1} \delta'\ ,\ (Af)_{p_1}x \xrightarrow{p_2} x') \mapsto (\ fp_1\ ,\ p_2\ )$$

define a functor $\Delta.A[f] \xrightarrow{f.A} \Gamma.A$ such that the square:

$$\begin{array}{ccc} \Delta.A[f] & \xrightarrow{f.A} & \Gamma.A \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

is verified to be a pullback and a 2-pullback. In fact, if we are given functors $g_1 : \Omega \to \Delta$ and $(g_2, g_3) : \Omega \to \Gamma.A$ in **Grpd** such that $fg_1 = g_2$ then the unique factorisation functor $\Omega \to \Delta.A[f]$ will be given by the pairing (g_1, g_3). Similarly one verifies the 2-dimensional factorisation property.

Moreover, as the triples:

$$(\ A(fg)\ ,\ \phi^A_{fg_-,fg_-}\ ,\ \psi^A_{fg_-}\ ) \text{ and } (\ (Af)g\ ,\ \phi'^{Af}_{g_-,g_-}\ ,\ \psi'^{Af}_{g_-}\ )$$

—where $\phi'^{Af} = \phi^A_{f_-,f_-}$ and $\psi'^{Af} = \psi^A_{f_-}$—coincide whenever f and g are composable 1-cells, the display maps $P_{A[fg]}$ and $P_{A[f][g]}$ coincide. Similarly, one verifies that $P_{A[1_\Gamma]}$ coincides with P_A, and that the cloven isofibration structure on display maps that we provided in Section 5 is compatible with this re-indexing choice in the sense of Definition 4.1.

Finally, we observed that the mappings:

$$(\gamma, x, y) \mapsto A_\gamma(x, y)$$

$$(p_1, p_2, p_3) \mapsto (p \mapsto (x' \xrightarrow{p_2^{-1}} A_{p_1}x \xrightarrow{A_{p_1}p} A_{p_1}y \xrightarrow{p_3} y'))$$

—where the homset $A_\gamma(x,y)$ is endowed with the trivial groupoid structure—define a pseudofunctor—in fact a strict functor—$\Gamma.A.A[P_A] \to \mathbf{Grpd}$ that we indicate as Id_A. Hence we obtained a display map $\Gamma.A.A[P_A].\mathsf{Id}_A \to \Gamma.A.A[P_A]$. Additionally, we stated that the 2-cell:

$$\begin{tikzcd}
\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A \arrow[r, shift left, "P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}"] \arrow[r, shift right, "P^{\bullet}_A P_{\mathsf{Id}_A}"'] \arrow[r, phantom, "\Downarrow \alpha_A"] \arrow[dr, bend right, "P_A P_{A^{\blacktriangledown}} P_{\mathsf{Id}_A}"'] & \Gamma.A \arrow[d, "P_A"] \\
& \Gamma
\end{tikzcd}$$

whose (γ,x,y,p)-component is the arrow:

$$(\ 1_\gamma\ ,\ A_{1_\gamma} x \xrightarrow{\psi^A} x \xrightarrow{p} y\ ) : (\gamma,x) \to (\gamma,y)$$

constitutes an arrow object, which can be verified to be compatible with the re-indexing choice in the sense of Definition 4.2.

We also mentioned that, following the construction at paragraphs **Elim Rule** and **Comp Axiom for =-types**, one can reconstruct the choice functions r, ϕ, and J in order to observe that $\mathsf{J}_c[\mathsf{r}_A]$ does not need to coincide with c, if c is a section of $P_{C[\mathsf{r}_A]}$ for a given display map P_C over $\Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$. In detail, the setions $\mathsf{J}_c[\mathsf{r}_A]$ and c will be different when P_C is not normal, i.e. when C is not a strict functor to **Grpd**.

For sake of completeness, we mention that the functor:

$$\mathsf{r}_A : \Gamma.A \to \Gamma.A.A^{\blacktriangledown}.\mathsf{Id}_A$$

acts *on objects* as the mapping:

$$(\gamma,x) \mapsto (\gamma,x,x,1_x)$$

and that the (γ,x,y,p)-component of φ_A is the arrow:

$$(1_\gamma, p\psi^A_\gamma x, \psi^A_\gamma x, 1_{1_y})$$

from (γ,x,y,p) to $(\gamma,y,y,1_y)$. This can be used to verify that, whenever the mapping $(\gamma,x) \mapsto (\gamma,x,c(\gamma,x))$ is the action on objects of a section c of $P_{C[\mathsf{r}_A]}$ for some display map P_C over $\Gamma.A.A[P_A].\mathsf{Id}_A$, then the functor:

$$\mathsf{J}_c : \Gamma.A.A[P_A].\mathsf{Id}_A \to \Gamma.A.A[P_A].\mathsf{Id}_A.C$$

acts on objects as the mapping:

$$(\gamma,x,y,p) \mapsto (\gamma,x,y,p,C_{\varphi_A^{-1}} c(\gamma,y))$$

i.e. as

$$(\gamma,x,y,p) \mapsto (\gamma,x,y,p,C_{(1_\gamma,p^{-1}\psi^A_\gamma y,\psi^A_\gamma y,1_p)} c(\gamma,y))$$

hence the functor $\mathsf{J}_c[\mathsf{r}_A] : \Gamma.A \to \Gamma.A.C[\mathsf{r}_A]$ acts on objects as the mapping:

$$(\gamma,x) \mapsto (\gamma,x,C_{1_{(\gamma,x,x,1_x)}} c(\gamma,x)).$$

Therefore, as we said, unless C was chosen to be a strict functor $\Gamma.A.A[P_A].\mathsf{Id}_A \to \mathbf{Grpd}$, in general $\mathsf{J}_c[\mathsf{r}_A]$ and c will not coincide.