Research statement

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A brief description of the area that I work on. Throughout my doctoral studies, I specialised in mathematical logic and categorical logic, areas employing algebraic, topological, and categorical tools to study properties of logical systems. If I had to summarise my research interests in one sentence, I would say that, in essence, my research focuses on exploring diverse *calculi* (or systems of rules to make formal deductions) within the realm of mathematical logic by means of their (category theory based) *semantics*. A semantics for a given calculus can be seen as the collection of all potential "copies" of this calculus, copies which in essence are defined as particular (substantially) algebraic structures. One hopes to define this family of copies in such a way that the class of the properties verified *simultaneously by all of them* is precisely the class of the theorems that the calculus can deduce. Once a semantics is enstablished with this feature (usually called completeness), it is understood that the problem of studying the deductive power of a calculus (such as ZFC) is reduced to a matter of algebraic and combinatorial nature.

In detail, so far I have worked on (categorical semantics of) *dependent type theories* and fragments of many-sorted *intuitionistic first-order logic*:

- First-order logic is one of the most classical calculi where many important theories of ordinary mathematics (the theory of groups, the theory of fields, the theory ZFC of sets) are defined. Its fragments, whose study is fundamental for the investigation of many other algebraic theories, are obtained by removing some of its deduction rules (e.g. the law of the excluded middle, or the rules involving the universal quantification). When one does this, the primary challenge arises from the loss of the property of completeness that we discussed in the previous paragraph, which is enjoyed by the full first-order logic and its semantics. To address this, it becomes necessary to extend this notion of semantics by means of category-theoretic structures, called doctrines or Grothendieck fibrations. These extensions manage to restore the fundamental feature of completeness also for these fragments of first-order logic.

- Dependent type theories are calculi emerged as powerful tools that can be used to provide constructive foundations of mathematics. The concept of "dependent types", i.e. types varying over the terms of other types, allows for precise and expressive specifications of mathematical statements and proofs, together with a formal language for reasoning about them [CH88]. Recently, the field has seen significant developments, based on the numerous insights provided by the foundational work of Vladimir Voevodsky in univalent foundations, and of Jacob Lurie in infinity categories. This includes the emergence of Homotopy Type Theory [TUFP13], which provides a new understanding of the concept of equality.

Reaching the end of my PhD journey, I bring along several results, as well as additional research projects, in both of these areas. Some very general keywords that appear through my research are the following: (hyper)doctrines; quantifier completions of doctrines; fibrations; simple (co)product completions of fibrations; gödel doctrines/fibrations; categorified quantifier-freeness; categorical semantics for (fragments of) first order logic; dialectica categories; dialectica construction; dialectica completion; existence property; choice principles; propositional type theory; propositional Σ and Π types; propositional η and β computation rules; syntactic categories; categorical semantics for dependent type theories; display map classes; bicategorical semantics; bicategories with a display map classes; categories with attributes; comprehension categories; propositional versus judgemental in dependent type theories; conservativity; path categories; coherence; weak adjunctions; property-like 2-monads.

Essentially, my research so far has focused on: various formulations of the usual type constructors (e.g. depending on the strength of their elimination and computation rules) from both a syntactic and a semantic point of view; propositional dependent type theories (i.e. with propositional computation rules); 2/3-dimensional notions of semantics for dependent type theories; conservativity between different dependent type theories; various notions of semantics for various notions of dependent type theories; the property of coherence for notions of semantics; quantifier completion of fibrations; the notion of dialectica completion and an internal characterisation of this notion in term of the existential and the universal quantification; an analysis of the fragment of first-order logic that this completion preserves; an analysis of the fragment of first-order logic that this completion constitutes a complete semantics of; applications of the dialectica completion in dependent type theory.

Among my future research plans that I describe below, the ones related to dependent type theories are: a coherence-related study for the semantics of propositional type theories (a calculus that has strong ties to cubical type theory -see [CCHM18]-) and its presumed property of homotopy canonicity; a generalisation of methods for inferring conservativity properties between type theories; a categorical characterisation of the various notions of dependent sum types -depending on their strength- which would lead to a better understanding of this fundamental notion for the type-theoretical formulation of mathematics. Below I also discuss my research plans in many-sorted first-order logic, with connections and applications to dependent type theory.

1 Previous research in dependent type theory

Regarding dependent type theories, I have been particularly interested in *propositional* ones. A dependent type theory is said to have *propositional identity types* if it is endowed with a type constructor satisfying the usual formation, introduction and elimination rules of intensional identity types, but not the corresponding computation rule, which is only required to hold in a weakened form, called *propositional form.* In detail, as illustrated in [vdB18], whenever we are given judgements:

$$\lfloor\, x, y : A,\ p : x = y \,\rfloor \quad C(x, y, p) : \textsc{Type}$$
$$\lfloor\, x : A \,\rfloor \quad q(x) : C(x, x, r(x))$$

in place of asking that the judgemental equality $\lfloor\, x : A \,\rfloor \quad J(x, x, r(x), q) \equiv q(x)$ holds, we only ask that it holds propositionally, i.e. that the judgement:

$$\lfloor\, x : A \,\rfloor \quad H(x, q) : J(x, x, r(x), q) = q(x)$$

holds (here J denotes the identity type eliminator). See [vdB18, vdBM18] for more details.

One might consider the same form of weakening for the computation rule of dependent sum types and dependent product types: these type constructors satisfying a propositional computation rule will be called *propositional dependent sum types* and *propositional dependent product types* respectively. We call *propositional* a dependent type theory with these inference rules.

In extensional type theory, Martin Löf dependent sum types enjoy an equivalent formulation as a negative type (i.e. as two projection rules for the elimination, together with computation rules): my fascination for propositional type theories started from the desire to find such a characterisation of (some notion of) dependent sum types also within an intensional type theoretic framework. In effect, in presence of intensional identity types -as well as in the weaker case of propositional identity types-, I was able to prove that **the strongest notion of dependent sum type enjoying such a characterisation is in fact the one of propositional dependent sum type**. The proof of this result is completely syntactical and exploits the general results on half-adjoint equivalences presented in [TUFP13]. Apart from this well-behaved property enjoyed by the propositional dependent type theory, other motivations for working in such a calculus are that its semantics admits a wider class of models than the corresponding one of an intensional type theory, and that the types-in-context are homotopy invariant.

Motivated by these considerations developed for propositional type theories, my next goal was to propose a **3-dimensional notion of sound and complete semantics for these calculi**. This involves a 3-dimensional formulation of the notion of *display map category* and a characterisation, in 3-categorical terms, of the properties that rephrase the calculus of a propositional type theory once we organise its syntax as such a *display map 3-category*: a model of a propositional type theory according to this notion of semantics will be a display map 3-category satisfying these properties. E.g. such a structure is a model of propositional dependent sum types as long as the pullback 2-functor of display maps along a given display map has a locally adjointly 2-equivalent 3-left adjoint. This notion of semantics can be equivalently reformulated in 2-dimensional categorical terms, but still the 3-dimensional case is not obsolete as it provides further insights on the nature of a propositional type theory. An application of this semantics is the identification of new models (based on [Pal03]) of a given propositional type theory that are not models of the corresponding intensional one.

My most fulfilling achievement lies in establishing **a conservativity property of extensional type theories over their propositional counterparts**. It involves the analysis of an inductively generated class of types-in-context of a given propositional type theory that are called *h-elementary.* Informally, this result asserts that, for judgements essentially concerning h-sets, reasoning with extensional or propositional type theories is equivalent. In other words, by "reinforcing" a propositional type theory T with the rules of the corresponding extensional type theory $T_{\mathtt{ext}}$, one does not risk to modify - namely increase - the deductive power of T on the class of *h-elementary* types-in-context. The proof of this result adapts the one contained in [Hof96], and entails a purely syntactic aspect, wherein we inductively define a family of *canonical homotopy equivalences* between contexts and types-in-context of the theory T. By establishing specific properties of this family, one deduces that the identification of the h-elementary contexts, types and terms of T via these canonical equivalences makes the syntax of the T into a model M of the corresponding $T_{\mathtt{ext}}$. The conservativity property is then deduced by means of the soundness of the semantics of dependent type theories -that we opportunely generalise to propositional ones- as well as some specific properties of M. In this case, we phrase the semantics via the notion of *category with attributes* (see [Spa23] for more details). This positively addresses a conjecture of [vd21] stating that a propositional type theory is sufficient for performing all of constructive mathematics and formalising most of the HoTT book.

1.1 Current/future work in dependent type theory

Coherence in the class of path categories. As part of a joint work in progress together with Benno van den Berg and Daniël Otten (ILLC), I am working on determining in what sense path categories provide a notion of semantics for propositional type theories, as well as comparing this with the known notion of semantics for propositional and strong type theories, i.e. the one based on the notion of category with attributes (or category with families, or comprehension category, or natural model). This work involves seeking a coherence theorem for path categories that repairs the issue of the substitution, when this is not strictly functorial (this problem is similar to the one for locally cartesian closed categories, for which Martin Hofmann produced a coherence theorem). In detail, comparing the notion of path category to the one of comprehension category, we showed that a path category can be naturally characterised as a comprehension category with three properties: it has propositional identity types; it has strong dependent sum types; it is democratic (i.e. contexts are represented by types in the empty context). This shows that a path category is a (pseudo) model of a dependent type theory T with propositional identity types and strong dependent sum types. A pseudo model, because the obtained comprehension category in general is non split and the type constructors are only weakly stable. However, since it is well-known that any non-split comprehension category with weakly stable type constructors is equivalent to a split one with stable type constructors (see [LW15] and [Boc22]) we can infer that every pseudo model of the theory T that we determined is equivalent to an actual (strict) model. This is precisely the statement of **a coherence result, that any path category admits a split replacement**.

We are investigating the categories of the models of a given propositional type theory according to both the notions of semantics, working on proving that the correspondence between path categories and comprehension categories is a **2-equivalence**. We believe that it **generalises the 2-equivalence by Clairambault and Dybier (see [CD14]) between finitely complete categories and democratic comprehension categories with extensional identities and strong sigmas**. In general, we are working on **proposing a unified notion of category theory-based semantics for any (generalised form of) dependent type theory**. This will involve building on the insights gained from the comparison of the different notions semantics for weak and strong type theories, and investigating the nature of the substitution for the unified semantics. This project needs to pass through a weakening of the notion of path category to model fully propositional type theories, i.e. propositional identites, propositional sigmas (in place of strong sigmas), and propositional pis.

The problem of the coherence for the semantics of propositional type theories via path categories is the first step through the composition of a proof of van den Berg's conjecture: the homotopy canonicity for propositional type theories via an *à la Freyd gluing* argument (see [vdB23]).

Additional conservativity results. I aim to expand and generalise the proof of the coherence result for extensional type theories over propositional ones (see [Spa23]), looking for new discoveries regarding possible coherences between (generalised) dependent type theories. E.g. taking advantage of the work by [AGS17], I plan to include *propositional w-types* in the propositional type theory and analyse whether a coherence result of the corresponding extensional type theory over it still holds. Secondly, as the restriction -in the proof of our coherence result- from general contexts to contexts with h-propositional identities and the one from contexts with h-propositional identities to h-elementary contexts are both marginal parts of the proof, it would be worthwhile to consider what can be learned by replicating our argument without these restrictions. In detail, I plan to study whether the argument can be extended, without restrictions on the family of canonical equivalences, in order to deduce a conservativity result for those generalised type theories that are modelled by the categorical structure obtained -by quotienting- from the general family of canonical equivalences, without any restriction. More generally, I am interested in looking for **other ways of applying Hofmann's argument** (see [Hof96]) in order to get similar results or generalised versions, depending e.g. on the strictness of the notion of substitution that one allows in a theory of dependent types. Finally, taking advantage of [Boc20], I aim to explore whether this non-constructive argument and its generalisations admit an internalisation in a dependent type theoretic metatheory.

2-dimensional coherence. I aim to study the relation between my notion of 2/3-dimensional semantics for propositional type theories to the ordinary semantics provided by 1-dimensional comprehension categories, as well as the 2-dimensional one introduced in [Gar09]. This represents a first step through an answer to the **coherence conjecture** that Garner formulates in [Gar09], stating that every 2-dimensional comprehension category admits a split replacement. Addressing this question involves the use of results contained in [Pow07] on the notion of 2-fibration presented in [Her99].

Various notions of dependent sums as various notions of algebras. One of my long-term goals is to provide a thorough description of the various notions of Dependent Sum Type appearing in the literature from an algebraic point of view, i.e. by looking at (the notions of) Sigmas as additional structure that a theory of dependent types might have. This project starts from the identification of the pseudo-dependent type theories (type theory with a non strict -explicit- notion of substitution) as the comprehension categories (CCs), looking at them as objects of a 2-category whose 1-cells represent pseudo-interpretations (i.e. interpretations non strictly preserving the substitution and the context extension) or pseudo-models and whose 2-cells represent morphisms of pseudo-models (see [CD14]). We also define the various subcategories of (pseudo) dependent type theories with (pseudo-stable or weakly-stable) dependent sum types, depending on the strength of the elimination and the computation rules (i.e. whether the elimination is or is not dependent, whether the computation rule is judgemental or propositional, whether an expansion rule is or is not assumed).

We show how to build a CC with strong dependent sums from a given CC and show how this construction is the *free one*, as it extends to a left bi-adjoint to the inclusion of CCs with dependent sums into CCs. Depending on the domain category of this inclusion, we aim to determine in what cases the induced bi-adjunction is monadic and in what cases it is property-like, in order to **clarify which notions of Dependent Sum are an additional structure and which ones constitute a property of a given dependent type theory**. As a consequence of the fact that the dependent pairing morphisms induced in a comprehension category A by a pseudo-algebra structure $TA \to A$ over A w.r.t. the associated pseudo-monad T -where A is a comprehension category- is defined by means of the *context extension preservation morphism* appearing in the definition of the given 1-cell $TA \to A$, this classification problem amounts to characterise the pseudo and lax algebras of T depending on the strength of the context extension preservation of the 1-cells in the 2-categories involved in the bi-adjunction.

Additionally, we aim to study the distributivity between T and the left adjoint and right adjoint splittings of comprehension categories (depending on whether the various notions of Dependent sums are assumed to be weakly-stable or pseudo-stable, respectively). This involves the use of results contained in [LW15, GL23].

2 Previous research in many-sorted intuitionistic first-order logic

Grothendieck fibration, and particularly Lawvere doctrines (i.e. posetal fibrations), provide a notion of generalised Tarski's semantics for fragments of many-sorted first-order logic, respectively in a proof-relevant and in a proof-irrelevant setting. The advantage of this approach is that it allows one to recover the completeness property for every such fragment

In a joint work together with Davide Trotta (University of Pisa) and Valeria de Paiva (Topos Institute), we introduced the notion(s) of *Gödel fibration* (and Gödel doctrine) as a model of the Skolemisation and the existence of a prenex normal form for its predicates. Taking advantage of Hofstra's recent work relating the dialectica construction to existential and universal completions of a given fibration, we characterised the instances of the Dialectica completion, showing that these are precisely the Gödel fibrations (see [TSdP21b]). This provides an i**nternal characterisation of de Paiva's notion of Dialectica construction**.

Secondly, we look at natural requests on a fibration to make some fragment of intuitionistic many sorted first-order logic in its predicative part be preserved by the Dialectica pseudomonad. Hence we determined **what fragment of first-order logic behaves well with the Dialectica completion** (see [TS20, TSdP21a]). It turns out that the Dialectica completion preserves all the propositional connectives with their inference rules, possibly apart from the implication.

Finally, we showed that the fragment of classic first-order logic modelled by a Gödel hyperdoctrine contains (the whole intuitionistic first-order logic together with) the Principle of Independence of Premise, the Modified Markov

Principle and the Principle of Skolemisation. Hence **this fragment is right in-between the intuitionistic first order logic and the classical first-order logic** (see [TSdP22, TSdP23]).

2.1 Current/future work in many-sorted intuitionistic first-order logic

Exlpoiting our results on the principles modelled by a Gödel fibration, I plan to provide a **complete characterisation of the internal logic** of the fibrations obtained as Dialectica competions. I also plan to **generalise this construction** to the dependent case, by applying the construction that freely add to a fibration the existential and universal quantifiers, to a general family of display maps in the base category, in place of the one of product project. Finally, I aim to study **whether this generalised completion is compatible with the structure of a comprehension category**, and study the peculiarities of the dependent type theory modelled by a comprehension category obtained from this construction, from both a syntactic and a semantic point of view.

References

[AGS17] Steve Awodey, Nicola Gambino, and Kristina Sojakova. Homotopy-initial algebras in type theory. *J. ACM*, 63(6):Art. 51, 45, 2017.

[Boc20] Rafaël Bocquet. Coherence of strict equalities in dependent type theories. arXiv:2010.14166, 2020.

[Boc22] Rafaël Bocquet. Strictification of weakly stable type-theoretic structures using generic contexts. In *27th International Conference on Types for Proofs and Programs*, volume 239 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 3, 23. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2022.

[CCHM18] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: a constructive interpretation of the univalence axiom. In *21st International Conference on Types for Proofs and Programs*, volume 69 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 5, 34. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.

[CD14] Pierre Clairambault and Peter Dybjer. The biequivalence of locally cartesian closed categories and Martin-Löf type theories. *Math. Structures Comput. Sci.*, 24(6):e240606, 54, 2014.

[CH88] Thierry Coquand and Gérard Huet. The calculus of constructions. *Information and Computation*, 76(2):95–120, 1988.

[Gar09] Richard Garner. Two-dimensional models of type theory. *Math. Structures Comput. Sci.*, 19(4):687–736, 2009.

[GL23] Nicola Gambino and Marco Federico Larrea. Models of Martin-Löf type theory from algebraic weak factorisation systems. *J. Symb. Log.*, 88(1):242–289, 2023.

[Her99] Claudio Hermida. Some properties of **Fib** as a fibred 2-category. *J. Pure Appl. Algebra*, 134(1):83–109, 1999.

[Hof96] Martin Hofmann. Conservativity of equality reflection over intensional type theory. In Stefano Berardi and Mario Coppo, editors, *Types for Proofs and Programs*, pages 153–164, Berlin, Heidelberg, 1996. Springer Berlin Heidelberg.

[LW15] Peter LeFanu Lumsdaine and Michael A. Warren. The local universes model: an overlooked coherence construction for dependent type theories. *ACM Trans. Comput. Log.*, 16(3):Art. 23, 31, 2015.

[Pal03] Erik Palmgren. Groupoids and local cartesian closure. Link to the preprint here, 2003.

[Pow07] John Power. Three dimensional monad theory. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 405–426. Amer. Math. Soc., Providence, RI, 2007.

[Spa23] Matteo Spadetto. A conservativity result for homotopy elementary types in dependent type theory. arXiv:2303.05623, 2023.

[TS20] Davide Trotta and Matteo Spadetto. Quantifier completions, choice principles and applications. Link arXiv here. 2020.

[TSdP21a] Davide Trotta, Matteo Spadetto, and Valeria de Paiva. The Gödel fibration (long version). Link arXiv here. Reduced version to be presented in *46th International Symposium on Mathematical Foundations of Computer Science* and published in the conference proceedings *volume 171 of LIPIcs*, 2021.

[TSdP21b] Davide Trotta, Matteo Spadetto, and Valeria de Paiva. The Gödel Fibration. In *46th International Symposium on Mathematical Foundations of Computer Science (2021)*, volume 202 of *LIPIcs*, pages 87:1–87:16, 2021.

[TSdP22] Davide Trotta, Matteo Spadetto, and Valeria de Paiva. Dialectica logical principles. In S. Artemov and A. Nerode, editors, *Logical Foundations of Computer Science*, pages 346–363, Cham, 2022. Springer International Publishing. Link arXiv here.

[TSdP23] Davide Trotta, Matteo Spadetto, and Valeria de Paiva. Dialectica principles via Gödel doctrines. *Theoret. Comput. Sci.*, 947:Paper No. 113692, 25, 2023.

[TUFP13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[vd21] Benno van den Berg and Martijn den Besten. Quadratic type checking for objective type theory. *arXiv e-prints*, page arXiv:2102.00905, February 2021.

[vdB18] Benno van den Berg. Path categories and propositional identity types. *ACM Trans. Comput. Logic*, 19(2), jun 2018.

[vdB23] Benno van den Berg. Towards homotopy canonicity for propositional type theory (talk). Slides of the talk here, 2023.

[vdBM18] Benno van den Berg and Ieke Moerdijk. Exact completion of path categories and algebraic set theory: Part i: Exact completion of path categories. *Journal of Pure and Applied Algebra*, 222(10):3137–3181, 2018.