UNIVERSITY OF LEEDS

A gentle introduction to the study of mathematical logic via doctrines

Matteo Spadetto

Content

These slides contain a very brief and informal description of the intuition that brings one to the notion of a doctrine.

The prerequisites are a general knowledge of:

- basic category theoretic notions;
- basic first-order logic and its Tarski's semantics.

Content

- Regular logic: syntax and semantics.
- The role of the power set functor $\mathcal{P}\colon \mathbf{Set}^{op} \to \mathbf{Pos}$.
- How to abstract and isolate this role: doctrines.
- Bonus: the existential and the universal completions.

Regular logic: syntax and semantics.

Regular logic

We are interested in *statements* of the form $\phi \vdash_\Gamma \psi$, called sequents.

A regular language L is made of:

1. sorts $A, B, C, ...$
2. variables $x_1, x_2, ... : A$, $y_1, y_2, ... : B$, $z_1, z_2, ... : C$, ...
3. function symbols $f\colon A, C \to B \quad g\colon A \to B$
   $h\colon A_2, A_1, ..., A_n, B \to C \quad c\colon 1 \to A$
4. relation symbols $R\colon A, B \quad S\colon A_1, B, A_3, A_2, ..., A_n, B \quad T\colon A, A, A$

Then we can define the L-terms and the L-formulas.

L-terms and L-formulas

The L-terms (in context) are inductively defined as:

1. Any variable $x\colon A$ is an L-term of sort A in context $x\colon A$
2. Whenever $t_1\colon A_1 \quad t_2\colon A_2 \quad \ldots \quad t_n\colon A_n$ in a given context Γ and $f\colon A_1, A_2, \ldots, A_n \to B$ is a function symbol of L, then $f(t_1, \ldots, t_n)\colon B$ in context Γ.

The L-formulas (in context) are inductively defined as follows:

1. $\top$ is a formula (in any context).
2. If $t_1, t_2 : A$ are L-terms in context Γ then $t_1 = t_2$ is a formula in context Γ.
3. If $R\colon A_1, A_2, \ldots, A_n$ is a relation symbol and $t_1\colon A_1 \quad t_2\colon A_2 \quad \ldots$ $t_n\colon A_n$ are L-terms in context Γ then $R(t_1, \ldots, t_n)$ is a formula in context Γ.
4. If ϕ and ψ are formulas in context Γ then $\phi \wedge \psi$ is a formula in context Γ and $(\exists x\colon A)\phi$ is a formula in context $\Gamma \setminus [x\colon A]$.

(Regular) deduction rules for sequents

Some examples:

- if $\phi \vdash_\Gamma \psi$ and $\phi \vdash_\Gamma \omega$, then $\phi \vdash_\Gamma \psi \wedge \omega$
- if $\phi(y) \vdash_{\Gamma \cup \{y\}} \psi$ then $(\exists y)\phi(y) \vdash_\Gamma \psi$
- if ... then $\phi \vdash T$

A (regular) L-theory T is a set of L-sequents. We say that T proves a sequent if it can be deduced from the elements of T by using the (regular) deduction rules for sequents.

Semantics

An L-structure M is given by:

- a set A^M for any sort A of L,
- a function $f^M\colon A_1^M \times \dots \times A_n^M \to B^M$ for any function symbol $f\colon A_1, \dots, A_n \to B$ of L,
- a subset $R^M \subset A_1^M \times \dots \times A_n^M$ for any relation symbol $R\colon A_1, \dots, A_n$ of L.

Given an L-structure M, we can interpret L-terms and L-formulas in M. If Γ is the context $x_1 : A_1, \dots, x_n : A_n$ then we define Γ^M as the cartesian product $A_1^M \times \dots \times A_n^M$.

Terms $t\colon A$ in context Γ are interpreted as functions:

$$t^M\colon \Gamma^M \to A^M.$$

Formulas $\phi(\vec{y})$ in context $\vec{y}\colon \Gamma$ are interpreted as subsets:

$$\phi^M \subseteq \Gamma^M.$$

The function t^M and the subset ϕ^M are defined inductively as follows.

Definition of $t^M : \Gamma^M \to A^M$

1. If t is a varialbe $x \colon A$ then t^M is the identity function $A^M \to A^M$.
2. If t is $f(t_1, ..., t_n)$ where $t_1 \colon A_1 \quad t_2 \colon A_2 \quad ... \quad t_n \colon A_n$ in a given context Γ and $f \colon A_1, A_2, ..., A_n \to B$ is a function symbol of L, then t^M is the function:

$$\Gamma^A \xrightarrow{\langle t_1^M, ..., t_n^M \rangle} A_1^M \times ... \times A_n^M \xrightarrow{f^M} B^M .$$

Definition of $\phi^M \subseteq \Gamma^M$

1. If ϕ is $\top$ and Γ is a context then ϕ^M is Γ^M.
2. If ϕ is $t_1 = t_2$ where $t_1, t_2 : A$ are L-terms in context Γ, then ϕ^M is:
$$\{\vec{a} \in \Gamma^M : t_1^M(\vec{a}) = t_2^M(\vec{a})\}.$$
3. If ϕ is $R(t_1, ..., t_n)$ where $R\colon A_1, A_2, ..., A_n$ is a relation symbol and $t_1\colon A_1 \quad t_2\colon A_2 \quad ... \quad t_n\colon A_n$ are L-terms in context Γ, then ϕ^M is:
$$\langle t_1^M, ..., t_n^M\rangle^{-1} R^M = \{\vec{a} \in \Gamma^M : (t_1^M(\vec{a}), ..., t_n^M(\vec{a})) \in R^M\}.$$
4. If ϕ and ψ are formulas in context Γ then $(\phi \wedge \psi)^M$ is:
$$\phi^M \cap \psi^M$$
and $((\exists x\colon A)\phi)^M$ is:
$$\{\vec{a} \in (\Gamma \setminus [x\colon A])^M : \text{there is } a \in A^M \text{ such that } (\vec{a}, a) \in \phi^M\}.$$

Notion of semantics

We say that M models an L-sequent $\phi \vdash_{\Gamma} \psi$ if $\phi^M \subseteq \psi^M$. We say that M is a model of an L-theory T if M models every sequent of T.

This defines a sound semantics (called *Tarski's semantics* for regular logic). This semantics is also complete if, in place of regular logic, we include the whole classic first-order logic.

Example

L

$A \quad f\colon A, A \to A \quad c\colon 1 \to A$

An L-structure M is a set A^M together with a binary function $f^M\colon A^M \times A^M \to A^M$ and a constant $c^M \in A^M$.

T

$\top \vdash_{x,y,z\colon A} f(f(x,y),z) = f(x,f(y,z)) \quad \top \vdash_{x\colon A} f(x,c) = x = f(c,x)$

The L-structure M is a model of T precisely when M is a monoid.

The role of the power set functor
$\mathcal{P}\colon \mathbf{Set}^{op} \to \mathbf{Pos}$.

The (contravariant) power set functor

We remind that a *category* $\mathcal{C}$ is a collection of composable arrows $A \xrightarrow{f} B$. A *functor* $\mathcal{C} \to \mathcal{D}$ is a mapping between categories that preserves compositions and identities.

We are focusing on the functor:

$$\mathcal{P} \colon \mathbf{Set}^{op} \to \mathbf{Pos}$$
$$(A \xleftarrow{f} B) \mapsto (\mathcal{P}A \xrightarrow{f^{-1}} \mathcal{P}B)$$

in order to analyse Tarski's semantics.

Interpretation of $\wedge$ and $\top$

Let us consider a sort A (for simplicity) of the language L and let $x : A$. Let $\phi(x)$ and $\psi(x)$ be L-formulas in the context $x\colon A$. Let us assume that we already know $\phi^M, \psi^M \subseteq A^M$ (for a given L-structure M). Then:

$$(\phi \wedge \psi)^M = \phi^M \cap \psi^M \subseteq A^M$$

Moreover $\top^M = A^M$ (keep in mind also the interpretation of a sequent of the form $\top \vdash_{x\colon A} \phi(x)$).

We understand that the reason why we can interpret $\wedge$ and $\top$ is precisely the fact that the images $\mathcal{P}A^M$ of the sets A^M through $\mathcal{P}$ are inf-semilattices.

Interpretation of $\exists$

We remind that, given a functor $G\colon \mathcal{C} \to \mathcal{D}$, a *left adjoint* $F\colon \mathcal{D} \to \mathcal{C}$ is a functor such that there is a (natural) bijection between the arrows in $\mathcal{C}$ of the form $FD \to C$ and the arrows in $\mathcal{D}$ of the form $D \to GC$, whenever $C\colon \mathcal{C}$ and $D\colon \mathcal{D}$. A left adjoint is essentially unique (hence unique if $\mathcal{C}$ is a poset).

Given a projection $\pi\colon A^M \times B^M \to A^M$, the functor:

$$\mathcal{P}A^M \xrightarrow{\pi^{-1}} \mathcal{P}(A^M \times B^M)$$

has a left adjoint $\mathcal{P}(A^M \times B^M) \xrightarrow{\exists_\pi} \mathcal{P}A^M$ i.e. $\exists_\pi S \subseteq T$ if and only if $S \subseteq \pi^{-1}T$ for any $S \subseteq A^M \times B^M$ and $T \subseteq A^M$.

In fact, if:

$$\exists_\pi S := \{a \in A : \text{there is } b \in B^M \text{ such that } (a,b) \in S\} \subseteq T$$

and $(a,b) \in S$, then $a \in T$, hence $(a,b) \in \pi^{-1}T$. Vice versa, if $S \subset \pi^{-1}T$ and $a \in \exists_\pi S$, then there is $b \in B^M$ such that $(a,b) \in S \subset \pi^{-1}T$, hence $a \in T$. But why do we care of this fact?

Interpretation of $\exists$

Suppose that $S \subseteq A^M \times B^M$ is the interpretation of some formula $\phi(x, y)$ in the context $x\colon A, y\colon B$. I.e. $S = \phi(x, y)^M$ Then:

$$\begin{aligned}
\exists_\pi(\phi(x,y)^M) &= \exists_\pi S \\
&= \{a \in A : \text{there is } b \in B^M \text{ such that } (a,b) \in S\} \\
&= \{a \in A : \text{there is } b \in B^M \text{ such that } (a,b) \in \phi(x,y)^M\} \\
&= (\,(\exists y)\phi(x,y)\,)^M .
\end{aligned}$$

We undestand that the interpretation of $\exists$ according to Tarski's semantics is completely characterised by the left adjoints to the inverse image functors along the projections.

Interpretation of =

Given a diagonal $\Delta \colon A^M \to A^M \times A^M$, the functor:

$$\mathcal{P}(A^M \times A^M) \xrightarrow{\Delta^{-1}} \mathcal{P}A^M$$

has a left adjoint $\mathcal{P}A^M \xrightarrow{=_A} \mathcal{P}(A^M \times A^M)$ i.e. $(=_A)(S) \subset T$ if and only if $S \subset \Delta^{-1}T$ for any $S \subset A^M$ and any $T \subset A^M \times A^M$.

In fact one can check that:

$$(=_A)(S) := \{(a,b) \in S \times S : a = b\} = \{(a,a) : a \in S\}$$

works (a pleasant exercise!).

But, again, why do we care of this?

Interpretation of $=$

Let $t_1 \colon A$, $t_2 \colon A$ be terms in context Γ. Then $t_1^M, t_2^M \colon \Gamma^M \to A^M$ and then:

$$\begin{aligned}(t_1 = t_2)^M &= \{\vec{a} \in \Gamma^M : t_1^M(\vec{a}) = t_2^M(\vec{a})\} \\ &= \{\vec{a} \in \Gamma^M : (t_1^M(\vec{a}), t_2^M(\vec{a})) \in (=_A)(A)\} \\ &= \langle t_1^M, t_2^M \rangle^{-1}((=_A)(A)).\end{aligned}$$

We understand that the interpretation of $=$ can be characterised in terms of the left adjoints to the inverse image functors along the diagonals.

Summary

We characterised the interpretation of the regular logic symbols according to Tarski's semantics in terms of the following categorical properties of the contravariant power set functor $\mathcal{P}$:

1. The functor $\mathcal{P}$ factors through **InfSL** ($\wedge$ and T);
2. The functor $\mathcal{P}$ has the left adjoints to the inverse images along the product projcections ($\exists$).
3. The functor $\mathcal{P}$ has the left adjoints to the inverse images along the diagonals ($=$).

But then we do not really need $\mathcal{P}$. We only need a functor enojoying the same categorical properties as $\mathcal{P}$.

How to abstract and isolate this role: doctrines.

Doctrines of posets

A doctrine is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, where $\mathcal{C}$ is a category with finite products, and $\mathbf{Pos}$ is the category of posets. The functors Pf, where f is an arrow of $\mathcal{C}$, are usually called pullback functors (as an abstraction of the following example), re-indexing functors (as an abstraction of some syntax) or inverse image functors (as an abstraction of the power set functor).

We think of the base category $\mathcal{C}$ of a doctrine as a category of *contexts* where, for every object A of $\mathcal{C}$, we have a poset $P(A) = (P(A), \vdash)$ of *predicates* in context A.

Doctrines of posets

Example

Let $\mathcal{C}$ be a category with finite limits. The functor:

$$\mathrm{Sub}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$$

assigning to an object A in $\mathcal{C}$ the poset $\mathrm{Sub}_{\mathcal{C}}(A)$ of subobjects of A and such that for an arrow:

$$B \xrightarrow{f} A$$

the morphism $\mathrm{Sub}_{\mathcal{C}}(f) \colon \mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f, is a doctrine.

Primary, existential and elementary doctrines

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is:

1. primary if P factors through $\mathbf{InfSL}$;
2. existential if, for every A_1 and A_2 in $\mathcal{C}$ and every projection $\pi_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor:
$$P_{\pi_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$$
has a left adjoint $\exists_{\pi_i}$;
3. elementary if, for every A in $\mathcal{C}$, the functor:
$$P_{\Delta}\colon P(A \times A) \longrightarrow P(A)$$
has a left adjoint $=_A$.

We understand that a primary existential and elementary doctrine is what we need in order to define a generalised Tarski's semantics and allow generalised models of a regular theory.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a primary existential and elementary doctrine.

An L-structure M in P is given by: an object A^M of $\mathcal{C}$ for any sort A of L, an arrow $f^M\colon A_1^M \times ... \times A_n^M \to B^M$ of $\mathcal{C}$ for any function symbol $f\colon A_1, ..., A_n \to B$ of L, and an element $R^M \in P(A_1^M \times ... \times A_n^M)$ for any relation symbol $R\colon A_1, ..., A_n$ of L.

Given an L-structure M in P, we can interpret L-terms and L-formulas in M *formally as we did for the ordinary Tarski's semantics, taking advantage of its categorical characterisation*. Hence, terms $t\colon A$ in context Γ are interpreted as arrows:

$$t^M\colon \Gamma^M \to A^M$$

of $\mathcal{C}$.

Let us go through the details for the formulas.

Interpretation of formulas in M

- If A is any sort, then $\top^M := \top_A \in P(A^M)$.
- If $t_1, t_2 : A$ are L-terms in context Γ, then:

$$(t_1 = t_2)^M := P_{\langle t_1^M, t_2^M \rangle}((=_A)(\top_A)) \in P(\Gamma^M).$$

- If $R\colon A_1, A_2, ..., A_n$ is a relation symbol and $t_1\colon A_1 \quad t_2\colon A_2 \quad ...$ $t_n\colon A_n$ are L-terms in context Γ, then:

$$R(t_1, ..., t_n)^M := P_{\langle t_1^A, ..., t_n^A \rangle}(R^M) \in P(\Gamma^M).$$

- If ϕ and ψ are formulas in context Γ, then:

$$(\phi \wedge \psi)^M := \phi^M \wedge_{P(\Gamma^M)} \psi^M \in P(\Gamma^M).$$

- If ϕ is a formula in context $A \times B$, then:

$$((\exists y\colon A)\phi)^M := \exists_\pi(\phi^M) \in P(A^M)$$

where π is the projection $A \times B \to A$.

Soundness and completeness

Finally, we say that M models an L-sequent $\phi \vdash_{\Gamma} \psi$ if $\phi^M \leq \psi^M$ in $P(\Gamma^M)$. We say that M is a model of an L-theory T if M models all the sequents of T.

Theorem

This generalised Tarski's semantics *is sound and complete.*

In other words, this doctrine-valued semantics manages to restore the completeness property also for the fragments of classic first-order logic (e.g. the regular one that we considered during these slides).

To deepen the notion of a doctrine

- 1969. Lawvere. *Adjointness in foundations.*
- 1970. Lawvere. *Equality in hyperdoctrines and comprehension schema as an adjoint functor.*
- 2019. Trotta. *Existential completion and pseudo-distributive laws: an algebraic approach to the completion of doctrines.*

Bonus: the existential
and the universal completions.

We saw that primary existential and elementary doctrines define a notion of semantics for regular logic.

In general, depending on the categorical structure available in a given doctrine, one can define models of different fragments of first-order and higher-order logic.

As it might be expected, the current trend among scholars of doctrine theory is to determine whether it is possible to freely add a given piece of structure (which allows one to interpret a given piece of logic) to a given doctrine.

Category theoretically, such a construction, called completion, amounts to build a 2-left adjoint functor to the 2-inclusion of the doctrines with that piece of structure into the ones that do not necessarily have it. Once this is done, one can try determining whether the given piece is a property or simple structure of a doctrine, by looking at the associated monad.

In this last section, we provide two instances of completion, namely the existential and the universal completions. Secondly, we provide some references to deepen the study of the subject of doctrine completions.

Existential and universal completions

Let P be a doctrine $\mathcal{C}^{op} \longrightarrow \mathbf{Pos}$.

The existential completion $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ of P is a doctrine such that, for every object A of $\mathcal{C}$, the poset $P^{\exists}(A)$ is defined as follows:

- *Objects.* Triples (A, B, α), where A and B are objects of $\mathcal{C}$ and $\alpha \in P(A \times B)$.
- *Order.* $(A, B, \alpha) \leq (A, C, \beta)$ if there exists an arrow:
$$f\colon A \times B \longrightarrow C$$
of $\mathcal{C}$ such that $\alpha \leq P_{\langle \pi_A, f\rangle}\beta$ where $\pi_A\colon A \times B \longrightarrow A$ is the product projection.

Whenever $g\colon A \longrightarrow C$ is an arrow of $\mathcal{C}$, the functor $P^{\exists}_g\colon P^{\exists}(C) \longrightarrow P^{\exists}(A)$ sends an object (C, D, γ) of $P^{\exists}(C)$ to the object $(A, D, P_{\langle g\pi_A, \pi_D\rangle}\gamma)$ of $P^{\exists}(A)$.

One can show that $P^{\exists}$ is a doctrine (exercise). The logical intuition is that an element (A, B, α) of the fibre $P^{\exists}(A)$ represents a predicate $(\exists b : B)\alpha(a, b)$.

Existential and universal completions

The universal completion $P^{\forall}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ of P is a doctrine such that, for every object A of $\mathcal{C}$, the poset $P^{\forall}(A)$ is defined as follows:

- *Objects.* Triples (A, B, α), where A and B are objects of $\mathcal{C}$ and $\alpha \in P(A \times B)$.
- *Order.* $(A, B, \alpha) \leq (A, C, \beta)$ if there exists an arrow $g\colon A \times C \longrightarrow B$ of $\mathcal{C}$ such that:

$$P_{\langle \pi_A, g \rangle}(\alpha) \leq \beta$$

  where $\pi_A\colon A \times C \longrightarrow A$ is the product projection.

Whenever $f\colon A \longrightarrow C$ is an arrow of $\mathcal{C}$, the functor $P^{\forall}_f\colon P^{\forall}(C) \longrightarrow P^{\forall}(A)$ is defined as for the existential completion.

Existential and universal completions

- Whenever $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is a doctrine, it is the case that $P^{\forall} \cong (-)^{\mathrm{op}}((-)^{\mathrm{op}}P)^{\mathrm{ex}}$.
- The existential (resp. universal) completion produces existential (resp. universal) doctrines i.e. doctrines with left (resp. right) adjoints to the inverse image functors along the product projections.
- There is a 2-category of doctrines.
- It restricts to a 2-category of the existential ones and to a 2-category of the universal ones.
- The existential and universal completions are 2-left and 2-right adjoint to the respective forgetful functor.
- An existential (resp. universal) doctrine is an instance of existential (resp. universal) completion if and only if it has enough $\exists$-free objects (resp. $\forall$-free objects).

Preservation of logical structures

The following preservation result holds:

Proposition

Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a primary doctrine (i.e. P factors through the category **InfSL***). Then the existential doctrine $P^{\exists} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is primary as well.*

and, by the result $P^{\forall} \cong (-)^{\mathrm{op}}((-)^{\mathrm{op}}P)^{\mathrm{ex}}$, the dual result for the universal completion holds as well:

Proposition

Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a co-primary doctrine (i.e. P factors through the category **SupSL***). Then the universal doctrine $P^{\forall} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is co-primary as well.*

Preservation of logical structures

Definition

If $\mathcal{C}$ is distributive, a lat-doctrine is a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ that factors through the category **Lat** of lattices (i.e. the finitely complete and finitely cocomplete posets) and finite sup&inf-preserving maps (i.e. finite limit and finite colimit preserving functors).

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a lat-doctrine such that the images through P of the injections $j_A\colon A \longrightarrow A+B$ have left adjoints $\exists_{j_A}$. Then the following properties hold.

1. *The doctrine $P^{\exists}$ is a lat-doctrine.*
2. *Suppose that there is an arrow $c\colon 1 \longrightarrow C$ for every non-initial object C of $\mathcal{C}$ ($\mathcal{C}$ has points).*
   Then the images through $P^{\exists}$ of the injections j_A have left adjoints $\exists^{\exists}_{j_A}$.
3. *Suppose that $\mathcal{C}$ has points and that the images through P of the injections j_A have right adjoints $\forall_{j_A}$.*
   Then the images through $P^{\exists}$ of the injections j_A have right adjoints $\forall^{\exists}_{j_A}$.

Preservation of logical structures

Proof.

1. Whenever (A, B, α) and (A, C, β) are two elements of $P^{\exists}(A)$, it is the case that:

$$(A, B, \alpha) \vee (A, C, \beta) := (A, B + C, P_{\theta^{-1}}(\exists_{j_{A\times B}}\alpha \vee \exists_{j_{A\times C}}\beta))$$

   is their coproduct.

2. Let $(A + B, C, \alpha) \in P^{\exists}(A + B)$ and let $(A, D, \delta) \in P^{\exists}(A)$. We define $\exists^{\exists}_{j_A}(A, D, \delta)$ to be the object:

$$(A + B, D, P_{\theta^{-1}}\exists_{j_{A\times D}}\delta)$$

   being θ the isomorphism $(A \times D) + (B \times D) \to (A + B) \times D$ and $j_{A\times D}$ the injection $A \times D \to (A \times D) + (B \times D)$.

3. Analogously. $\square$

Preservation of logical structures

By the result $P^{\forall} \cong (-)^{\mathrm{op}}((-)^{\mathrm{op}}P)^{\mathrm{ex}}$, the dual result for the universal competion holds:

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a lat-doctrine such that the images through P of the injections $j_A\colon A \longrightarrow A+B$ have right adjoints $\forall_{j_A}$. Then the following properties hold.

1. *The doctrine $P^{\forall}$ is a lat-doctrine.*
2. *Suppose that $\mathcal{C}$ has points.*
   Then the images through $P^{\forall}$ of the injections j_A have right adjoints $\forall^{\forall}_{j_A}$.
3. *Suppose that $\mathcal{C}$ has points and that the images through P of the injections j_A have left adjoints $\exists_{j_A}$.*
   Then the images through $P^{\forall}$ of the injections j_A have left adjoints $\exists^{\forall}_{j_A}$.

2011. Hofstra. *The dialectica monad and its cousins.*

contains the following result:

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ be a universal doctrine and suppose that $\mathcal{C}$ has exponents. Then $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is existential and universal, i.e. the existential completion preserves the universal structure.

Proof.

Let A_1, A_2 be objects of $\mathcal{C}$, and let $\mathrm{pr}_{A_1} : A_1 \times A_2 \longrightarrow A_1$ be the first projection. Let $\forall^{\exists}_{\mathrm{pr}_{A_1}} : P^{\exists}(A_1 \times A_2) \longrightarrow P^{\exists}(A_1)$ be defined by:

$$(A_1 \times A_2, B, \alpha) \mapsto (A_1, B^{A_2}, \forall_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle} P_{\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{ev}\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle\rangle} \alpha)$$

where pr_i are the projections from $A_1 \times A_2 \times B^{A_2}$ and ev is the evaluation map $A_2 \times B^{A_2} \to B$.
The intuition is that the right adjoints act by mapping:

$$\exists b : B \alpha(a_1, a_2, b) \mapsto \exists f : B^{A_2} \forall a_2 : A_2 \alpha(a_1, a_2, f(a_2)).$$

$\square$

How to combine our results

Hence we deduce the following:

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ *be a lat-doctrine such that:*

- *the category* $\mathcal{C}$ *has points;*
- *the category* $\mathcal{C}$ *has exponents;*
- *the images* P_{j_A} *of the injections* $j_A\colon A \longrightarrow A+B$ *have left and right adjoints* $\exists_{j_A} \dashv P_{j_A} \dashv \forall_{j_A}$.

Then $(P^{\forall})^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Lat}$ *is an existential and universal lat-doctrine and the images* $(P^{\forall})^{\exists}_{j_A}$ *of the injection* j_A *have left and right adjoints:*

$$(\exists^{\forall})^{\exists}_{j_A} \dashv (P^{\forall})^{\exists}_{j_A} \dashv (\forall^{\forall})^{\exists}_{j_A}.$$

Other results

- The 2-monad associated to the existential (resp. universal) completion is lax-idempotent (resp. colax-idempotent). Hence, it is property-like, i.e. having existential (resp. universal) quantifiers is a property, not just a structure, of a theory.
- Some relevant choice rules and principles are enjoyed by the doctrines that happen to be existential, universal, or existential & universal completions. E.g. the rule of choice; the counterexample property; the principle of skolemisation; the principle of the prenex normal form.
- If A is a complete distributive lattice and A is a supercoherent suplattice whose subset B of supercompact elements of A is a super*co*coherent inflattice, then the presheaf over **Set** represented by A is the existential completion of the universal completion of the presheaf over **Set** represented by the subset of B of super*co*compact elements of B.

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