



UNIVERSITY OF LEEDS

A gentle introduction to the study of mathematical logic via doctrines

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Content

These slides contain a very brief and informal description of the intuition that brings one to the notion of a [doctrine](#).

The prerequisites are a general knowledge of:

- ▶ basic category theoretic notions;
- ▶ basic first-order logic and its Tarski's semantics.

Content

- ▶ Regular logic: syntax and semantics.
- ▶ The role of the power set functor $\mathcal{P}: \mathbf{Set}^{op} \rightarrow \mathbf{Pos}$.
- ▶ How to abstract and isolate this role: doctrines.
- ▶ Bonus: the existential and the universal completions.

Regular logic: syntax and semantics.

Regular logic

We are interested in *statements* of the form $\phi \vdash_{\Gamma} \psi$, called **sequents**.

A **regular language** L is made of:

1. sorts A, B, C, \dots
2. variables $x_1, x_2, \dots : A, y_1, y_2, \dots : B, z_1, z_2, \dots : C, \dots$
3. function symbols $f: A, C \rightarrow B \quad g: A \rightarrow B$
 $h: A_2, A_1, \dots, A_n, B \rightarrow C \quad c: 1 \rightarrow A$
4. relation symbols $R: A, B \quad S: A_1, B, A_3, A_2, \dots, A_n, B \quad T: A, A, A$

Then we can define the L -terms and the L -formulas.

L -terms and L -formulas

The L -terms (in context) are inductively defined as:

1. Any variable $x: A$ is an L -term of sort A in context $x: A$
2. Whenever $t_1: A_1 \quad t_2: A_2 \quad \dots \quad t_n: A_n$ in a given context Γ and $f: A_1, A_2, \dots, A_n \rightarrow B$ is a function symbol of L , then $f(t_1, \dots, t_n): B$ in context Γ .

The L -formulas (in context) are inductively defined as follows:

1. \top is a formula (in any context).
2. If $t_1, t_2: A$ are L -terms in context Γ then $t_1 = t_2$ is a formula in context Γ .
3. If $R: A_1, A_2, \dots, A_n$ is a relation symbol and $t_1: A_1 \quad t_2: A_2 \quad \dots \quad t_n: A_n$ are L -terms in context Γ then $R(t_1, \dots, t_n)$ is a formula in context Γ .
4. If ϕ and ψ are formulas in context Γ then $\phi \wedge \psi$ is a formula in context Γ and $(\exists x: A)\phi$ is a formula in context $\Gamma \setminus [x: A]$.

(Regular) deduction rules for sequents

Some examples:

- ▶ if $\phi \vdash_{\Gamma} \psi$ and $\phi \vdash_{\Gamma} \omega$, then $\phi \vdash_{\Gamma} \psi \wedge \omega$
- ▶ if $\phi(y) \vdash_{\Gamma \cup \{y\}} \psi$ then $(\exists y)\phi(y) \vdash_{\Gamma} \psi$
- ▶ if ... then $\phi \vdash T$

A (regular) ***L*-theory** T is a set of L -sequents. We say that T proves a sequent if it can be deduced from the elements of T by using the (regular) deduction rules for sequents.

Semantics

An L -structure M is given by:

- ▶ a set A^M for any sort A of L ,
- ▶ a function $f^M: A_1^M \times \dots \times A_n^M \rightarrow B^M$ for any function symbol $f: A_1, \dots, A_n \rightarrow B$ of L ,
- ▶ a subset $R^M \subset A_1^M \times \dots \times A_n^M$ for any relation symbol $R: A_1, \dots, A_n$ of L .

Given an L -structure M , we can interpret L -terms and L -formulas in M . If Γ is the context $x_1: A_1, \dots, x_n: A_n$ then we define Γ^M as the cartesian product $A_1^M \times \dots \times A_n^M$.

Terms $t: A$ in context Γ are interpreted as functions:

$$t^M: \Gamma^M \rightarrow A^M.$$

Formulas $\phi(\vec{y})$ in context $\vec{y}: \Gamma$ are interpreted as subsets:

$$\phi^M \subseteq \Gamma^M.$$

The function t^M and the subset ϕ^M are defined **inductively** as follows.

Definition of $t^M: \Gamma^M \rightarrow A^M$

1. If t is a variable $x: A$ then t^M is the identity function $A^M \rightarrow A^M$.
2. If t is $f(t_1, \dots, t_n)$ where $t_1: A_1 \quad t_2: A_2 \quad \dots \quad t_n: A_n$ in a given context Γ and $f: A_1, A_2, \dots, A_n \rightarrow B$ is a function symbol of L , then t^M is the function:

$$\Gamma^A \xrightarrow{\langle t_1^M, \dots, t_n^M \rangle} A_1^M \times \dots \times A_n^M \xrightarrow{f^M} B^M.$$

Definition of $\phi^M \subseteq \Gamma^M$

1. If ϕ is \top and Γ is a context then ϕ^M is Γ^M .
2. If ϕ is $t_1 = t_2$ where $t_1, t_2 : A$ are L -terms in context Γ , then ϕ^M is:

$$\{\vec{a} \in \Gamma^M : t_1^M(\vec{a}) = t_2^M(\vec{a})\}.$$

3. If ϕ is $R(t_1, \dots, t_n)$ where $R : A_1, A_2, \dots, A_n$ is a relation symbol and $t_1 : A_1 \quad t_2 : A_2 \quad \dots \quad t_n : A_n$ are L -terms in context Γ , then ϕ^M is:

$$\langle t_1^M, \dots, t_n^M \rangle^{-1} R^M = \{\vec{a} \in \Gamma^M : (t_1^M(\vec{a}), \dots, t_n^M(\vec{a})) \in R^M\}.$$

4. If ϕ and ψ are formulas in context Γ then $(\phi \wedge \psi)^M$ is:

$$\phi^M \cap \psi^M$$

and $((\exists x : A)\phi)^M$ is:

$$\{\vec{a} \in (\Gamma \setminus [x : A])^M : \text{there is } a \in A^M \text{ such that } (\vec{a}, a) \in \phi^M\}.$$

Notion of semantics

We say that M models an L -sequent $\phi \vdash_{\Gamma} \psi$ if $\phi^M \subseteq \psi^M$. We say that M is a model of an L -theory T if M models every sequent of T .

This defines a sound and complete semantics (called *Tarski's semantics* for regular logic).

Example

$$\begin{array}{l} L \\ A \quad f: A, A \rightarrow A \quad c: 1 \rightarrow A \end{array}$$

An L -structure M is a set A^M together with a binary function $f^M: A^M \times A^M \rightarrow A^M$ and a constant $c^M \in A^M$.

$$\begin{array}{l} T \\ \top \vdash_{x,y,z: A} f(f(x,y),z) = f(x,f(y,z)) \quad \top \vdash_{x: A} f(x,c) = x = f(c,x) \end{array}$$

The L -structure M is a model of T precisely when M is a monoid.

The role of the power set functor
 $\mathcal{P}: \mathbf{Set}^{op} \rightarrow \mathbf{Pos}.$

The (contravariant) power set functor

We remind that a *category* \mathcal{C} is a collection of composable arrows $A \xrightarrow{f} B$. A *functor* $\mathcal{C} \rightarrow \mathcal{D}$ is a mapping between categories that preserves compositions and identities.

We are focusing on the functor:

$$\mathcal{P}: \mathbf{Set}^{op} \rightarrow \mathbf{Pos}$$

$$(A \xleftarrow{f} B) \mapsto (\mathcal{P}A \xrightarrow{f^{-1}} \mathcal{P}B)$$

in order to analyse Tarski's semantics.

Interpretation of \wedge and \top

Let us consider a sort A (for simplicity) of the language L and let $x : A$. Let $\phi(x)$ and $\psi(x)$ be L -formulas in the context $x : A$. Let us assume that we already know $\phi^M, \psi^M \subseteq A^M$ (for a given L -structure M). Then:

$$(\phi \wedge \psi)^M = \phi^M \cap \psi^M \subseteq A^M$$

Moreover $\top^M = A^M$ (keep in mind also the interpretation of a sequent of the form $\top \vdash_{x:A} \phi(x)$).

We understand that the reason why we can interpret \wedge and \top is precisely the fact that the images $\mathcal{P}A^M$ of the sets A^M through \mathcal{P} are inf-semilattices.

Interpretation of \exists

We remind that, given a functor $G: \mathcal{C} \rightarrow \mathcal{D}$, a *left adjoint* $F: \mathcal{D} \rightarrow \mathcal{C}$ is a functor such that there is a (natural) bijection between the arrows in \mathcal{C} of the form $FD \rightarrow C$ and the arrows in \mathcal{D} of the form $D \rightarrow GC$, whenever $C: \mathcal{C}$ and $D: \mathcal{D}$. A left adjoint is essentially unique (hence unique if \mathcal{C} is a poset).

Given a projection $\pi: A^M \times B^M \rightarrow A^M$, the functor:

$$\mathcal{P}A^M \xrightarrow{\pi^{-1}} \mathcal{P}(A^M \times B^M)$$

has a left adjoint $\mathcal{P}(A^M \times B^M) \xrightarrow{\exists_\pi} \mathcal{P}A^M$ i.e. $\exists_\pi S \subseteq T$ if and only if $S \subseteq \pi^{-1}T$ for any $S \subseteq A^M \times B^M$ and $T \subseteq A^M$.

In fact, if:

$$\exists_\pi S := \{a \in A : \text{there is } b \in B^M \text{ such that } (a, b) \in S\} \subseteq T$$

and $(a, b) \in S$, then $a \in T$, hence $(a, b) \in \pi^{-1}T$. Vice versa, if $S \subseteq \pi^{-1}T$ and $a \in \exists_\pi S$, then there is $b \in B^M$ such that $(a, b) \in S \subseteq \pi^{-1}T$, hence $a \in T$. But why do we care of this fact?

Interpretation of \exists

Suppose that $S \subseteq A^M \times B^M$ is the interpretation of some formula $\phi(x, y)$ in the context $x: A, y: B$. I.e. $S = \phi(x, y)^M$ Then:

$$\begin{aligned}\exists_{\pi}(\phi(x, y)^M) &= \exists_{\pi} S \\ &= \{a \in A : \text{there is } b \in B^M \text{ such that } (a, b) \in S\} \\ &= \{a \in A : \text{there is } b \in B^M \text{ such that } (a, b) \in \phi(x, y)^M\} \\ &= ((\exists y)\phi(x, y))^M.\end{aligned}$$

We understand that the interpretation of \exists according to Tarski's semantics is completely characterised by the [left adjoints to the inverse image functors along the projections](#).

Interpretation of =

Given a diagonal $\Delta: A^M \rightarrow A^M \times A^M$, the functor:

$$\mathcal{P}(A^M \times A^M) \xrightarrow{\Delta^{-1}} \mathcal{P}A^M$$

has a left adjoint $\mathcal{P}A^M \xrightarrow{=A} \mathcal{P}(A^M \times A^M)$ i.e. $(=A)(S) \subset T$ if and only if $S \subset \Delta^{-1}T$ for any $S \subset A^M$ and any $T \subset A^M \times A^M$.

In fact one can check that:

$$(=A)(S) := \{(a, b) \in S \times S : a = b\} = \{(a, a) : a \in S\}$$

works (a pleasant exercise!).

But, again, why do we care of this?

Interpretation of $=$

Let $t_1: A, t_2: A$ be terms in context Γ . Then $t_1^M, t_2^M: \Gamma^M \rightarrow A^M$ and then:

$$\begin{aligned}(t_1 = t_2)^M &= \{\vec{a} \in \Gamma^M : t_1^M(\vec{a}) = t_2^M(\vec{a})\} \\ &= \{\vec{a} \in \Gamma^M : (t_1^M(\vec{a}), t_2^M(\vec{a})) \in (=_A)(A)\} \\ &= \langle t_1^M, t_2^M \rangle^{-1}((=_A)(A)).\end{aligned}$$

*We understand that the interpretation of $=$ can be characterised in terms of the **left adjoints to the inverse image functors along the diagonals**.*

Summary

We characterised the interpretation of the regular logic symbols according to Tarski's semantics in terms of the following **categorical properties** of the contravariant power set functor \mathcal{P} :

1. The functor \mathcal{P} factors through **InfSL** (\wedge and T);
2. The functor \mathcal{P} has the left adjoints to the inverse images along the product projections (\exists).
3. The functor \mathcal{P} has the left adjoints to the inverse images along the diagonals ($=$).

But then we do not really need \mathcal{P} . We only need a functor enjoying the same categorical properties as \mathcal{P} .

How to abstract and isolate this role: doctrines.

Doctrines of posets

A **doctrine** is a functor $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Pos}$, where \mathcal{C} is a category with finite products, and \mathbf{Pos} is the category of posets. The functors Pf , where f is an arrow of \mathcal{C} , are usually called **pullback functors** (as an abstraction of the following example), **re-indexing functors** (as an abstraction of some syntax) or **inverse image functors** (as an abstraction of the power set functor).

We think of the base category \mathcal{C} of a doctrine as a category of *contexts* where, for every object A of \mathcal{C} , we have a poset $P(A) = (P(A), \vdash)$ of *predicates* in context A .

Doctrines of posets

Example

Let \mathcal{C} be a category with finite limits. The functor:

$$\mathrm{Sub}_{\mathcal{C}}: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$$

assigning to an object A in \mathcal{C} the poset $\mathrm{Sub}_{\mathcal{C}}(A)$ of subobjects of A and such that for an arrow:

$$B \xrightarrow{f} A$$

the morphism $\mathrm{Sub}_{\mathcal{C}}(f): \mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f , is a doctrine.

Primary, existential and elementary doctrines

A doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is:

1. **primary** if P factors through **InfSL**;
2. **existential** if, for every A_1 and A_2 in \mathcal{C} and every projection $\pi_i: A_1 \times A_2 \rightarrow A_i$, $i = 1, 2$, the functor:

$$P_{\pi_i}: P(A_i) \rightarrow P(A_1 \times A_2)$$

has a left adjoint \exists_{π_i} ;

3. **elementary** if, for every A in \mathcal{C} , the functor:

$$P_{\Delta}: P(A \times A) \rightarrow P(A)$$

has a left adjoint $=_A$.

*We understand that a primary existential and elementary doctrine is what we need in order to define a generalised Tarski's semantics and allow **generalised models of a regular theory**.*

Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Pos}$ be a primary existential and elementary doctrine.

An L -structure M in P is given by: an object A^M of \mathcal{C} for any sort A of L , an arrow $f^M: A_1^M \times \dots \times A_n^M \rightarrow B^M$ of \mathcal{C} for any function symbol $f: A_1, \dots, A_n \rightarrow B$ of L , and an element $R^M \in P(A_1^M \times \dots \times A_n^M)$ for any relation symbol $R: A_1, \dots, A_n$ of L .

Given an L -structure M in P , we can interpret L -terms and L -formulas in M *formally as we did for the ordinary Tarski's semantics, taking advantage of its categorical characterisation*. Hence, terms $t: A$ in context Γ are interpreted as arrows:

$$t^M: \Gamma^M \rightarrow A^M$$

of \mathcal{C} .

Let us go through the details for the formulas.

Interpretation of formulas in M

- ▶ If A is any sort, then $\top^M := \top_A \in P(A^M)$.
- ▶ If $t_1, t_2 : A$ are L -terms in context Γ , then:

$$(t_1 = t_2)^M := P_{\langle t_1^M, t_2^M \rangle}((=_A)(\top_A)) \in P(\Gamma^M).$$

- ▶ If $R : A_1, A_2, \dots, A_n$ is a relation symbol and $t_1 : A_1 \quad t_2 : A_2 \quad \dots \quad t_n : A_n$ are L -terms in context Γ , then:

$$R(t_1, \dots, t_n)^M := P_{\langle t_1^A, \dots, t_n^A \rangle}(R^M) \in P(\Gamma^M).$$

- ▶ If ϕ and ψ are formulas in context Γ , then:

$$(\phi \wedge \psi)^M := \phi^M \wedge_{P(\Gamma^M)} \psi^M \in P(\Gamma^M).$$

- ▶ If ϕ is a formula in context $A \times B$, then:

$$((\exists y : A)\phi)^M := \exists_{\pi}(\phi^M) \in P(A^M)$$

where π is the projection $A \times B \rightarrow A$.




Soundness and completeness

Finally, we say that M models an L -sequent $\phi \vdash_{\Gamma} \psi$ if $\phi^M \leq \psi^M$ in $P(\Gamma^M)$. We say that M is a model of an L -theory T if M models all the sequents of T .

Theorem

This generalised Tarski's semantics is sound and complete.

To deepen the notion of a doctrine

-  1969. Lawvere. *Adjointness in foundations*.
-  1970. Lawvere. *Equality in hyperdoctrines and comprehension schema as an adjoint functor*.
-  2019. Trotta. *Existential completion and pseudo-distributive laws: an algebraic approach to the completion of doctrines*.

Bonus: the existential
and the universal completions.

We saw that primary existential and elementary doctrines define a notion of semantics for regular logic.

In general, depending on the categorical structure available in a given doctrine, one can define models of [different fragments](#) of first-order and higher-order logic.

As it might be expected, the current trend among scholars of doctrine theory is to determine whether it is possible to [freely add](#) a given piece of structure (which allows one to interpret a given piece of logic) to a given doctrine.

Category theoretically, such a construction, called [completion](#), amounts to build a 2-left adjoint functor to the 2-inclusion of the doctrines with that piece of structure into the ones that do not necessarily have it. Once this is done, one can try determining whether the given piece is a [property](#) or simple structure of a doctrine, by looking at the associated monad.

In this last section, we provide two instances of completion, namely [the existential and the universal completions](#). Secondly, we provide some references to deepen the study of the subject of doctrine completions.

Existential and universal completions

Let P be a doctrine $\mathcal{C}^{op} \rightarrow \mathbf{Pos}$.

The **existential completion** $P^\exists: \mathcal{C}^{op} \rightarrow \mathbf{Pos}$ of P is a doctrine such that, for every object A of \mathcal{C} , the poset $P^\exists(A)$ is defined as follows:

- *Objects.* Triples (A, B, α) , where A and B are objects of \mathcal{C} and $\alpha \in P(A \times B)$.
- *Order.* $(A, B, \alpha) \leq (A, C, \beta)$ if there exists an arrow:

$$f: A \times B \rightarrow C$$

of \mathcal{C} such that $\alpha \leq P_{\langle \pi_A, f \rangle} \beta$ where $\pi_A: A \times B \rightarrow A$ is the product projection.

Whenever $g: A \rightarrow C$ is an arrow of \mathcal{C} , the functor $P_g^\exists: P^\exists(C) \rightarrow P^\exists(A)$ sends an object (C, D, γ) of $P^\exists(C)$ to the object $(A, D, P_{\langle g\pi_A, \pi_D \rangle} \gamma)$ of $P^\exists(A)$.

One can show that P^\exists is a doctrine (exercise). The logical intuition is that **an element** (A, B, α) of the fibre $P^\exists(A)$ represents a predicate $(\exists b : B)\alpha(a, b)$.

Existential and universal completions

The **universal completion** $P^\vee: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ of P is a doctrine such that, for every object A of \mathcal{C} , the poset $P^\vee(A)$ is defined as follows:

- *Objects.* Triples (A, B, α) , where A and B are objects of \mathcal{C} and $\alpha \in P(A \times B)$.
- *Order.* $(A, B, \alpha) \leq (A, C, \beta)$ if there exists an arrow $g: A \times C \rightarrow B$ of \mathcal{C} such that:

$$P_{\langle \pi_A, g \rangle}(\alpha) \leq \beta$$

where $\pi_A: A \times C \rightarrow A$ is the product projection.

Whenever $f: A \rightarrow C$ is an arrow of \mathcal{C} , the functor $P_f^\vee: P^\vee(C) \rightarrow P^\vee(A)$ is defined as for the existential completion.

Existential and universal completions

- ▶ Whenever $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is a doctrine, it is the case that $P^{\forall} \cong (-)^{\text{op}}((-)^{\text{op}}P)^{\text{ex}}$.
- ▶ The existential (resp. universal) completion produces existential (resp. universal) doctrines i.e. doctrines with left (resp. right) adjoints to the inverse image functors along the product projections.
- ▶ There is a 2-category of doctrines.
- ▶ It restricts to a 2-category of the existential ones and to a 2-category of the universal ones.
- ▶ The existential and universal completions are 2-left and 2-right adjoint to the respective forgetful functor.
- ▶ An existential (resp. universal) doctrine is an instance of existential (resp. universal) completion if and only if it has enough \exists -free objects (resp. \forall -free objects).

Preservation of logical structures

The following preservation result holds:

Proposition

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ be a primary doctrine (i.e. P factors through the category \mathbf{InfSL}). Then the existential doctrine $P^{\exists}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is primary as well.

and, by the result $P^{\forall} \cong (-)^{\text{op}}((-)^{\text{op}}P)^{\text{ex}}$, the dual result for the universal completion holds as well:

Proposition

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ be a co-primary doctrine (i.e. P factors through the category \mathbf{SupSL}). Then the universal doctrine $P^{\forall}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is co-primary as well.

Preservation of logical structures

Definition

If \mathcal{C} is distributive, a **lat-doctrine** is a doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ that factors through the category **Lat** of lattices (i.e. the finitely complete and finitely cocomplete posets) and finite sup&inf-preserving maps (i.e. finite limit and finite colimit preserving functors).

Theorem

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ be a lat-doctrine such that the images through P of the injections $j_A: A \rightarrow A + B$ have left adjoints \exists_{j_A} . Then the following properties hold.

- 1. The doctrine P^\exists is a lat-doctrine.*
- 2. Suppose that there is an arrow $c: 1 \rightarrow C$ for every non-initial object C of \mathcal{C} (\mathcal{C} has points).
Then the images through P^\exists of the injections j_A have left adjoints $\exists_{j_A}^\exists$.*
- 3. Suppose that \mathcal{C} has points and that the images through P of the injections j_A have right adjoints \forall_{j_A} .
Then the images through P^\exists of the injections j_A have right adjoints $\forall_{j_A}^\exists$.*

Preservation of logical structures

Proof.

1. Whenever (A, B, α) and (A, C, β) are two elements of $P^\exists(A)$, it is the case that:

$$(A, B, \alpha) \vee (A, C, \beta) := (A, B + C, P_{\theta^{-1}}(\exists_{j_{A \times B}} \alpha \vee \exists_{j_{A \times C}} \beta))$$

is their coproduct.

2. Let $(A + B, C, \alpha) \in P^\exists(A + B)$ and let $(A, D, \delta) \in P^\exists(A)$. We define $\exists_{j_A}^\exists(A, D, \delta)$ to be the object:

$$(A + B, D, P_{\theta^{-1}} \exists_{j_{A \times D}} \delta)$$

being θ the isomorphism $(A \times D) + (B \times D) \rightarrow (A + B) \times D$ and $j_{A \times D}$ the injection $A \times D \rightarrow (A \times D) + (B \times D)$.

3. Analogously. □

Preservation of logical structures

By the result $P^\vee \cong (-)^{\text{op}}((-)^{\text{op}}P)^{\text{ex}}$, the dual result for the universal completion holds:

Theorem

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ be a lat-doctrine such that the images through P of the injections $j_A: A \rightarrow A + B$ have right adjoints \forall_{j_A} . Then the following properties hold.

- 1. The doctrine P^\vee is a lat-doctrine.*
- 2. Suppose that \mathcal{C} has points.
Then the images through P^\vee of the injections j_A have right adjoints $\forall_{j_A}^\vee$.*
- 3. Suppose that \mathcal{C} has points and that the images through P of the injections j_A have left adjoints \exists_{j_A} .
Then the images through P^\vee of the injections j_A have left adjoints $\exists_{j_A}^\vee$.*



2011. Hofstra. *The dialectica monad and its cousins*.

contains the following result:

Theorem

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ be a universal doctrine and suppose that \mathcal{C} has exponents. Then $P^{\exists}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is existential and universal, i.e. the existential completion preserves the universal structure.

Proof.

Let A_1, A_2 be objects of \mathcal{C} , and let $\text{pr}_{A_1}: A_1 \times A_2 \rightarrow A_1$ be the first projection. Let $\forall_{\text{pr}_{A_1}}^{\exists}: P^{\exists}(A_1 \times A_2) \rightarrow P^{\exists}(A_1)$ be defined by:

$$(A_1 \times A_2, B, \alpha) \mapsto (A_1, B^{A_2}, \forall_{\langle \text{pr}_1, \text{pr}_3 \rangle} P_{\langle \text{pr}_1, \text{pr}_2, \text{ev}_{\langle \text{pr}_2, \text{pr}_3 \rangle}} \alpha)$$

where pr_i are the projections from $A_1 \times A_2 \times B^{A_2}$ and ev is the evaluation map $A_2 \times B^{A_2} \rightarrow B$.

The intuition is that the right adjoints act by mapping:

$$\exists b: B\alpha(a_1, a_2, b) \mapsto \exists f: B^{A_2} \forall a_2: A_2\alpha(a_1, a_2, f(a_2)).$$



How to combine our results

Hence we deduce the following:

Theorem

Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Pos}$ be a lat-doctrine such that:

- ▶ the category \mathcal{C} has points;
- ▶ the category \mathcal{C} has exponents;
- ▶ the images P_{j_A} of the injections $j_A: A \longrightarrow A + B$ have left and right adjoints $\exists_{j_A} \dashv P_{j_A} \dashv \forall_{j_A}$.

Then $(P^\forall)^\exists: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Lat}$ is an existential and universal lat-doctrine and the images $(P^\forall)^\exists_{j_A}$ of the injection j_A have left and right adjoints:

$$(\exists^\forall)^\exists_{j_A} \dashv (P^\forall)^\exists_{j_A} \dashv (\forall^\forall)^\exists_{j_A}.$$

Other results

- ▶ The 2-monad associated to the existential (resp. universal) completion is lax-idempotent (resp. colax-idempotent). Hence, it is property-like, i.e. having existential (resp. universal) quantifiers is a **property**, not just a structure, of a theory.
- ▶ Some relevant choice rules and principles are enjoyed by the doctrines that happen to be existential, universal, or existential & universal completions. E.g. the **rule of choice**; the **counterexample property**; the **principle of skolemisation**; the **principle of the prenex normal form**.
- ▶ If A is a complete distributive lattice and A is a supercoherent suplattice whose subset B of supercompact elements of A is a supercocoherent inflattice, then the presheaf over **Set** represented by A is the **existential completion of the universal completion** of the presheaf over **Set** represented by the subset of B of supercocompact elements of B .

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