1 Linear Approximation

The linear approximation of $f(x)$ at a point a is the linear function:

$$L(x) = f(a) + f'(a)(x-a).$$

The linear approximation of $f(x,y)$ at (a,b) is the linear function:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

the linearization can be written more compactly using the **gradient** as:

$$L(\tilde{\mathbf{x}}) = f(\tilde{\mathbf{a}}) + \nabla f(\tilde{\mathbf{a}}) \cdot (\tilde{\mathbf{x}} - \tilde{\mathbf{a}}).$$

1.1 Chain Rule

The single variable chain rule is given by:

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

For a multivariate function, the chain rule is:

$$\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x_i} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x_i} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x_i} + \cdots$$

The matrix form of the multivariate chain rule is expressed using Jacobian matrices:

$$D(\mathbf{y})(\mathbf{x}) = Df(\mathbf{u})(\mathbf{x}) \cdot Dg(\mathbf{x})$$

where $Df(\mathbf{u})$ and $Dg(\mathbf{x})$ are the Jacobian matrices of f and g respectively.

1.2 Tangent line, tangent plane & tangent hyperplane

The equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is:

$$y = f(a) + f'(a)(x-a)$$

The equation of the tangent plane to the surface $z = f(x,y)$ at the point $(a, b, f(a,b))$ is:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Here, f_x and f_y denote the partial derivatives of f with respect to x and y, respectively.

For a function $f(x_1, x_2, \ldots, x_n)$ at a point $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, the tangent hyperplane is given by:

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

where $\nabla f(\mathbf{a})$ is the gradient of f at $\mathbf{a}$ and $\mathbf{x}$ is the vector of variables.

1.3 Derivatives of Vector Functions

Let $F : \mathbb{R}^n \to \mathbb{R}^m, F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}$ be a vector function of the variables x $= (x_1, ..., x_n)$.

Recall that the derivative of the vector function F with respect to the vector of variables $\tilde{x}$ is defined as

$$\frac{\partial \tilde{F}}{\partial \tilde{x}} = J_F(\tilde{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\tilde{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\tilde{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\tilde{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\tilde{x}) \end{bmatrix}$$

The second derivative of the function $f : \mathbb{R}^n \to \mathbb{R}$ (here m $= 1$) is given as

$$\frac{\partial^2 f}{\partial \tilde{x}^2} = \frac{\partial}{\partial \tilde{x}} \left(\frac{\partial f}{\partial \tilde{x}} \right)^T$$

.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex on D, if

$$f(t\tilde{x} + (1-t)\tilde{y}) \le t f(\tilde{x}) + (1-t) f(\tilde{y})$$

for all $\tilde{x}, \tilde{y} \in D$ and for all $t \in [0,1]$. The function f is concave on D, if the function $-f$ is convex on D.

1.4 Rules for Differentiating Vector Functions

1. $\frac{\partial \tilde{x}}{\partial \tilde{x}} = I_n$
2. If $A \in \mathbb{R}^{m \times n}$, then $\frac{\partial A\tilde{x}}{\partial \tilde{x}} = A$.
3. If $\tilde{a} \in \mathbb{R}^n$, then $\frac{\partial \tilde{a}^T \tilde{x}}{\partial \tilde{x}} = \tilde{a}^T$.
4. If $A \in \mathbb{R}^{n \times n}$, then $\frac{\partial (\tilde{x}^T A \tilde{x})}{\partial \tilde{x}} = \tilde{x}^T (A + A^T)$.
5. If $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then $\frac{\partial (\tilde{x}^T A \tilde{x})}{\partial \tilde{x}} = 2\tilde{x}^T A$.
6. $\frac{\partial \|\tilde{x}\|^2}{\partial \tilde{x}} = 2\tilde{x}^T$.
7. If $\tilde{z} = \tilde{z}(\tilde{x})$ and $\tilde{y} = \tilde{y}(\tilde{x})$, then $\frac{\partial (\tilde{y}^T \tilde{z})}{\partial \tilde{x}} = \tilde{y}^T \frac{\partial \tilde{z}}{\partial \tilde{x}} + \tilde{z}^T \frac{\partial \tilde{y}}{\partial \tilde{x}}$.
8. If $G : D_G \subseteq \mathbb{R}^m \to \mathbb{R}^n$ and $F : D_F \subseteq \mathbb{R}^n \to \mathbb{R}^p$ and $H = F \circ G$, then $\frac{\partial H}{\partial \tilde{x}} = \frac{\partial F}{\partial G}(\tilde{G}(\tilde{x})) \frac{\partial G}{\partial \tilde{x}}$.

1.5 Linear Approximation of Vector Functions

$$f(X(p_1, p_2, \ldots, p_m), S(p_1, p_2, \ldots, p_m)) \approx k_n(p_1, p_2, \ldots, p_m)$$

$$X = \begin{bmatrix} X_1(p_1, \ldots, p_m) \\ \vdots \\ X_n(p_1, \ldots, p_m) \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix}$$

$$f(X(p_0 + \delta)) \approx f(X(p_0)) + df$$

$$df = \frac{\partial f}{\partial X} \frac{\partial X}{\partial p} \delta = J_f \delta, \quad \text{take expansion of } \frac{\partial X}{\partial p} \text{ around } p_0$$

$$dX = \begin{bmatrix} \frac{\partial X_1}{\partial p_1} & \frac{\partial X_1}{\partial p_2} & \cdots & \frac{\partial X_1}{\partial p_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial p_1} & \frac{\partial X_n}{\partial p_2} & \cdots & \frac{\partial X_n}{\partial p_m} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_m \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{m} \frac{\partial X_1}{\partial p_j} \delta_j \\ \vdots \\ \sum_{j=1}^{m} \frac{\partial X_n}{\partial p_j} \delta_j \end{bmatrix}$$

$$J_f = \frac{\partial f}{\partial X} \frac{\partial X}{\partial p}$$

$$f(X(p_0 + \delta)) \approx f(X(p_0)) + J_f \delta$$

2 Optical Flow

2.1 Lucas-Kanade Method

$$\delta_x = \frac{-\left(\sum I_y^2\right)\left(\sum I_x I_t\right) + \left(\sum I_x I_y\right)\left(\sum I_y I_t\right)}{\left(\sum I_x^2\right)\left(\sum I_y^2\right) - \left(\sum I_x I_y\right)^2}$$

$$\delta_y = \frac{\left(\sum I_x I_y\right)\left(\sum I_x I_t\right) - \left(\sum I_x^2\right)\left(\sum I_y I_t\right)}{\left(\sum I_x^2\right)\left(\sum I_y^2\right) - \left(\sum I_x I_y\right)^2}$$

where:

$$I_x = \frac{\partial I}{\partial x}, I_y = \frac{\partial I}{\partial y}, I_t = \frac{\partial I}{\partial t},$$

and

$$\left(\sum I_x^2\right)\left(\sum I_y^2\right) - \left(\sum I_x I_y\right)^2$$

is the determinant of covariance matrix.
To further clarify the system, it can be written in matrix form as:

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} = -\begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{A}\mathbf{d} = \mathbf{A}^T\mathbf{b}$$

where $\mathbf{A}$ and $\mathbf{b}$ are matrices containing the sums of gradients, and $\mathbf{d}$ is the displacement vector.

And $\mathbf{A}^T\mathbf{A}$ is a covariance matrix of local gradients.

When is this system solvable?

- $\mathbf{A}^T\mathbf{A}$ must not be singular (cannot invert it otherwise)
  - Eigenvalues λ_1 and λ_2 of $\mathbf{A}^T\mathbf{A}$ must not be too small
- $\mathbf{A}^T\mathbf{A}$ has to be well conditioned
  - Ratio $\frac{\lambda_1}{\lambda_2}$ must not be too large
  - (λ_1 is the larger eigenvalue)

Consequences:

- large λ_1, small λ_2 $\rightarrow$ large gradient in one direction.
- small λ_1 & λ_2 $\rightarrow$ small gradient in both directions.
- large λ_1 & λ_2 $\rightarrow$ large gradient in both directions.

2.2 Horn-Schunck Method