I outline the argument of Ostrovskii [Ost12] that extends our [AGŠ12] construction of a non-A coarsely embeddable example to produce one which is non-A and embeds into ℓ^1 in a bilipschitz way. Ostrovskii uses notation very different from ours, hence this note that builds on our terminology.

First, summarise our construction: Given a graph G, we construct a graph $C(G)$ which we call a $\mathbb{Z}/2$-homology cover of G in the paper. $C(G)$ is endowed with two metrics: the graph metric d_G and a wall metric d_W.

What we do in the paper is that we start with a "flower" graph (one vertex plus a couple of loops), denote it by G_0, and consider the family $G_0, C(G_0), C(C(G_0)), \ldots,$ where on each successive graph we consider the wall metric coming from "it being C(previous graph)".

The main point in our paper is that on the scale of girth(G), the two metrics d_G and d_W on $C(G)$ are the same. This alone is enough to show that the family endowed with the graph metric is coarsely equivalent to the (same underlying) family endowed with the wall metrics. Note that anything endowed with a wall metric embeds isometrically into ℓ^1 (hence coarsely into a Hilbert space).

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The first upgrade is due to Ana Khukhro: The point is that one can start with any family of graphs $G_1, G_2, \ldots$ with large girth. Then the family $C(G_1), C(G_2), \ldots$ will have large girth (so non-A by Rufus Willett's argument) and the graph metric will be still coarsely equivalent to the wall metric (so coarsely embeddable). This doesn't require any extra computations than those in our paper.

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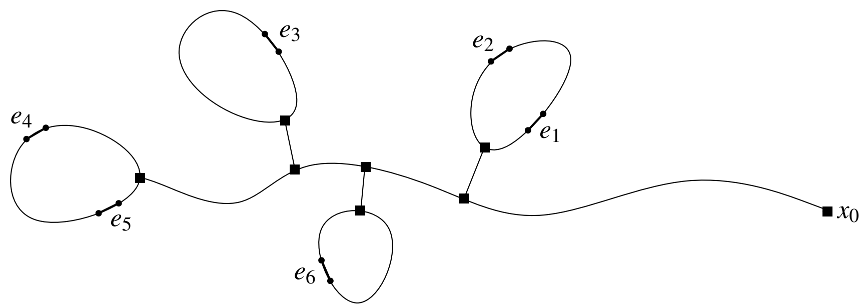
The upgrade of Ostrovskii is that with a good choice of $G_1, G_2, \ldots,$ one may get a *bilipschitz* equivalence of the graph and wall metrics on $C(G_1), C(G_2), \ldots,$ so that this family now bilipschitzly embeds into ℓ^1.

The choice of $G_1, G_2, \ldots$ that he needs is that $\mathrm{girth}(G_n)/\operatorname{diam}(G_n)$ is bounded away from 0. The known examples of families with large girth and this requirement are Ramanujan graphs (special expanders).

Here's the idea of his argument: Recall little bits of the construction of $C(G)$. We proved that paths in $C(G)$ cover some special paths in G and that shortest paths in $C(G)$ map down to paths in G that look like this one:



One should think of it as a path from x_0 back to x_0, where we travel "left", then branch to go around a loop, then again further "left", etc, until we go around all the loops and then return back to x_0. The loops are there, because the e_i's are "prescribed" edges that we must use odd number of times (hence exactly once).

Now the graph length (d_G) in $C(G)$ that we cover is just the total length of our travel, while the wall length (d_W) that we cover only counts the edges that we use only once, i.e. precisely the loops (all the other edges cancel out). [Note that in general the path can start and end at different vertices, so there can be more edges that we use only once, but this doesn't destroy the argument.] So the difference between the two metrics comes precisely from the edges of the tree that we get if we throw out the loops in the above picture, so that's what we need to estimate.

Denote now the number of loops by C. Observations:

- Thinking of the tree that we get by throwing out the loops, let's call the portions between the vertices marked by squares in the picture by *pieces*. Each piece is a shortest path between two vertices in G, so its length is at most $\operatorname{diam}(G)$.
- By an easy induction one sees that the number of pieces is at most $2C+1$ (throwing away one "leaf" square vertex contributes to the number of pieces by 1, and we get another 1 if we need to merge the two pieces at the place where it "branches").
- Each loop has length at least $\operatorname{girth}(G)$.

Putting these facts together, we see that

$$\begin{aligned}
&\frac{\text{length traveled in graph metric}}{\text{length traveled in wall metric}} = \\
&\qquad = \frac{\text{length of all the loops} + 2\cdot\text{length of all the pieces}}{\text{length of all the loops}} = \\
&\qquad = 1 + 2\cdot\frac{\text{length of all the pieces}}{\text{length of all the loops}} \leq \\
&\qquad \leq 1 + 2\cdot\frac{(2C+1)\operatorname{diam}(G)}{C\operatorname{girth}(G)} \leq \text{constant } C'.
\end{aligned}$$

Thus $1/C'\cdot d_G \leq d_W \leq d_G$ and we are done.

References

[AGŠ12] Goulnara Arzhantseva, Erik Guentner, and Ján Špakula, *Coarse non-amenability and coarse embeddings*, Geom. Funct. Anal. **22** (2012), no. 1, 22–36. MR 2899681

[Ost12] Mikhail I. Ostrovskii, *Low-distortion embeddings of graphs with large girth*, J. Funct. Anal. **262** (2012), no. 8, 3548–3555. MR 2889167