Boundary representations of non-positively curved groups

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1 Background

The proposed research generally fits into the framework of Noncommutative Geometry, in particular research related to the Baum–Connes conjecture, analysis on groups, and representation theory. The Baum–Connes conjecture connects geometry, topology and algebra. From one point of view, it proposes a way to understand the algebraic topology (K-theory) of (a part of) the representation space of a group. While it is possible to effectively *describe* all the representations of (semisimple) Lie groups, this task is impossible for discrete groups in general. Here we propose to construct explicit families of representations for large classes of discrete groups, using geometry (non-positive curvature) and boundaries. They directly address important questions (Shalom's conjecture), relate to existing approaches to the Baum–Connes conjecture, and harmonic analysis on discrete groups. The proposed pathway combines ideas from analytic and geometric group theory, representation theory of Lie groups and random walks.

Concretely, the overarching goal is to construct and study representations of discrete groups, arising from their action on non-positively curved spaces. The focus here is on two particular instances: groups acting on finite dimensional CAT(0) cubical complexes, and Gromov hyperbolic groups.

The story begins, as so often, with free groups, where "analysis" can be so often replaced with "combinatorics on trees". The 1980s saw the development of "**harmonic analysis on free groups**" by Figà-Talamanca, Picardello, Mantero, Zappa and others [7]: essentially a computation of the Poisson transform between the Gromov boundary of a homogeneous tree and the tree itself. This was subsequently used to describe the eigenfunctions of the combinatorial Laplacian on the tree, and to construct a family of **uniformly bounded representations on the boundary**. There is an fascinating analogy between this picture and representations of semisimple Lie groups, $\mathrm{PSL}(2,\mathbb{R})$ in particular. The boundary (or "compact") picture for uniformly bounded representations was then complemented by the beautiful construction of Pytlik and Szwarc [20] on the tree itself (the "non-compact" picture, corresponding to the symmetric space for Lie groups), and the two constructions were later intertwined in [16].

Apart from the innate interest in uniformly bounded representations and harmonic analysis, in the late 1990s and early 2000s the developments surrounding the **Baum-Connes conjecture** have stimulated more interest in them. To prove the most general version of the Conjecture "with coefficients", Kasparov and Higson devised a powerful method, the so-called Dirac-dual Dirac method [10]. However, it does not apply for groups with Kazhdan's Property (T), while staying in the C*-algebraic context. Roughly speaking, one of the ingredients in this method is a path between the trivial and the left regular representations in the unitary dual; however Property (T) asserts that the trivial representation is isolated. The representation theory of rank one (i.e. hyperbolic) Lie groups $\mathrm{Sp}(n,1)$ suggested such a path may exist if one allows uniformly bounded representations, instead of just the unitary ones. This approach to the Baum-Connes conjecture was pioneered by Julg [12], and taken up again recently [14]. For completess, let us note that the approach of V. Lafforgue [15], generalising the KK framework to certain Banach algebras, proved more successful and was used to resolve the Conjecture for hyperbolic groups (with coefficients).

There is another important problem about uniformly bounded representations, which is sometimes referred to as **Shalom's conjecture** [21]: any hyperbolic group admits a uniformly bounded representation with a proper cocycle. To connect this statement with the above, let us recall Gromov's a-T-menability, also known as the Haagerup property: a group G is *a-T-menable*, if it admits a proper isometric action on a Hilbert space. Decomposing the action into the linear part and the cocycle part, one sees that this is equivalent to admitting a unitary representation with a proper cocycle (hence Shalom's conjecture is immediate for any a-T-menable hyperbolic group). The application of Schoenberg's Theorem turns this single representation with a cocycle into a path of unitary representations precisely of the sort that appears in the

Baum-Connes framework. In this spirit, a "uniformly bounded" version of Schoenberg's Theorem (unpublished) would produce, from Shalom's conjecture, a path of uniformly bounded representations. These could be potentially used to provide an alternative proof of the Baum-Connes conjecture for hyperbolic groups which stays within the C* world.

Apart from a-T-menable groups, Shalom's conjecture appears to be verified (to an extent), only for lattices in $\mathrm{Sp}(n,1)$, by virtue of the knowledge of the representation theory of $\mathrm{Sp}(n,1)$ [4]. The state-of-the-art is the result of Dooley [5], which achieves uniform bounds on the uniform boundedness constants of the path of Cowling, all the way to the trivial representation. By the converse of the Schoenberg's Theorem (essentially taking the derivative of the path of representations at the endpoint; unpublished), this is sufficient for Shalom's conjecture. In relation to the work of Julg, note that Dooley works in the "non-compact" picture. However Julg requires the representations in the "compact" picture; the connection is provided by the "generalised Cayley transform" studied in [1]. Unfortunately, their work does not yield a uniform bound on the uniform boundedness constant along the whole path.

There is a consequence of Shalom's conjecture that has been studied in its own right, under the heading of *weak amenability*. This is obtained by considering the coefficients of the uniformly bounded representations, which turn out to be *completely bounded multipliers* on the group. This should be compared to a perhaps more familiar context: a group is amenable iff there exists a sequence of finitely supported positive-definite functions on the group which tend pointwise to 1. One generalisation is a-T-menability, trading the finite support requirement for "tending to 0 at ∞". Generalising amenability in another direction, one relaxes positive-definiteness to complete boundedness, obtaining weak amenability. This *has* been proved for all countable hyperbolic groups [19]. However, unlike in the positive-definite setting, there is no uniformly/completely bounded analogue of the Gelfand-Naimark-Segal construction, hence weak amenability does not imply the existence of uniformly bounded representations.

Let us leave the hyperbolic strand now and come back to analysis on free groups. Instead of generalising "metrically" to hyperbolic groups, one can instead opt to keep the combinatorial nature, non-positive curvature, and allow higher dimensions: this leads to **CAT(0) cubical complexes**. Groups acting properly on these have been very intensively studied within Geometric Group Theory. That is not to say that they have been ignored in the Baum-Connes community: such groups are a-T-menable (so that the general Higson-Kasparov proof applies), but the combinatorial nature allows for a beautiful concrete construction of the KK-elements involved [3]. Further, known analytic properties include a-T-menability [17] and weak amenability [9]. The latter was done by constructing an analytic family of uniformly bounded representations and considering their coefficients, done on the complex itself (the "non-compact picture"). This was adapted to Pytlik-Szwarc-type construction by Joyce in his thesis [11].

We intend to develop the "compact picture" for the representations. The analysis will require describing measures on the boundaries. For CAT(0) cubical groups that entails considering the Roller boundary of the complexes. In this context, we highlight recent work [6], where the authors identify the Roller boundary as the Poisson boundary of random walks on the group, as well as describe the relation between the Roller, horofunction and visual boundaries. In the context of hyperbolic groups, we consider the boundary as a metric measure space; however this is better done by adjusting the word metric on the group itself to a *strongly hyperbolic* metric. This has been studied in our work with Nica [18], distilling the geometric requirements from the work of Lafforgue, and Mineyev and Yu. We also show that the Green metric [2] coming from a random walk provides such a metric. Irreducibility of the (unitary) boundary representations of hyperbolic groups on the boundary (as Poisson boundary) was studied in [8].

2 Research hypothesis and objectives

The philosophy of this project is to capitalise on, and further develop, connections between Geometric Group Theory and Analysis/Noncommutative Geometry. We propose to construct a "compact picture" for **(uniformly bounded) representations** of prominent classes of **non-positively curved groups**. First, we deal with the case where one can do **"combinatorial harmonic analysis"**, i.e. the case of groups acting properly on (finite dimensional) CAT(0) cube complexes. Second, we distill the main features of the construction and perform it with **hyperbolic groups**. We believe that the constructions that will be used to achieve the Objectives will also result in further developments in related areas of mathematics: computing the spectrum of the combinatorial Laplacian on sufficiently symmetric CAT(0) cubical complexes, irreducibility of the constructed representations, and linking the representations to random walks.

OBJECTIVE 1: HARMONIC ANALYSIS ON CAT(0) CUBICAL COMPLEXES

The goal is to construct, for groups acting properly on finite dimensional CAT(0) cubical complexes, an analytic family of uniformly bounded representations on the Roller boundary of the complex, and describe intertwiners that will match it with the family of Guentner–Higson–Joyce [9, 11]. This shall be achieved in three steps.

OBJECTIVE 2: HYPERBOLIC GROUPS: TOWARDS SHALOM'S CONJECTURE

The overall aim is to construct a family of uniformly bounded representations on the boundary of hyperbolic groups, tending to the trivial representation, and with uniform bound on the uniform boundedness constants. This would establish Shalom's conjecture via the uniformly bounded version of Schoenberg's Theorem.

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