

# GENG0002 Mathematics B lecture notes

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# Contents

<b>Front Matter</b>	<b>4</b>
<b>1 Week 1</b>	<b>5</b>
1.1 Limits . . . . .	5
1.2 Differentiation from 1st Principles . . . . .	7
1.3 Differentiation by Rule . . . . .	10
1.4 Standard Derivatives . . . . .	12
<b>Week 1 Exercises</b>	<b>13</b>
1.5 Sheet 1 Limits . . . . .	13
1.6 Sheet 2 Differentiation by first principles . . . . .	14
1.7 Sheet 3 Differentiation by rule . . . . .	14
<b>2 Week 2</b>	<b>16</b>
2.1 Tangents and Normals . . . . .	16
2.2 The Chain Rule . . . . .	19
2.3 The Product and Quotient Rules . . . . .	22
<b>Week 2 Exercises</b>	<b>26</b>
2.4 Sheet 4 Tangents and Normals . . . . .	26
2.5 Sheet 5 The Chain Rule . . . . .	26
2.6 Sheet 6 The Product Rule . . . . .	27
2.7 Sheet 7 The Quotient Rule . . . . .	28
<b>3 Week 3</b>	<b>29</b>
3.1 Successive Differentiation . . . . .	29
3.2 Maxima, Minima and points of inflection . . . . .	31
<b>Week 3 Exercises</b>	<b>38</b>

3.3	Sheet 8 Successive Differentiation . . . . .	38
3.4	Sheet 8B More Examples of Successive Differentiation . . . . .	39
3.5	Sheet 9 Maxima, minima and points of inflexion . . . . .	41
<b>4</b>	<b>Week 4</b>	<b>42</b>
4.1	Curve Sketching . . . . .	42
4.2	Engineering Problems with Max/Min . . . . .	46
4.3	Further forms of differentiation . . . . .	50
	<b>Week 4 Exercises</b>	<b>54</b>
4.4	Sheet 10 - Curve Sketching using differentiation . . . . .	54
4.5	Sheet 11 - Problems involving maxima & minima . . . . .	55
4.6	Sheet 11A - Logarithmic Differentiation . . . . .	56
4.7	Sheet 11B - Further Differentiation . . . . .	56
<b>5</b>	<b>Week 5</b>	<b>58</b>
5.1	Partial Differentiation . . . . .	58
5.2	Integration as the opposite of differentiation . . . . .	61
5.3	Indefinite Integration . . . . .	63
	<b>Week 5 Exercises</b>	<b>66</b>
5.4	Sheet 12 - Partial Differentiation . . . . .	66
5.5	Sheet 12A - Indefinite Integration . . . . .	66
<b>6</b>	<b>Week 6</b>	<b>68</b>
6.1	Integration by Substitution . . . . .	68
6.2	Integration by Parts . . . . .	70
	<b>Week 6 Exercises</b>	<b>74</b>
6.3	Sheet 13 Integration by substitution . . . . .	74
6.4	Sheet 14 - Integration by parts . . . . .	74
<b>7</b>	<b>Week 7</b>	<b>76</b>
7.1	Integration using partial fractions . . . . .	76
7.2	Definite Integrals . . . . .	78
7.3	Area between two curves and Parametric Integration . . . . .	83
	<b>Week 7 Exercises</b>	<b>86</b>
7.4	Sheet 15 - Integration using Partial Fractions . . . . .	86
7.5	Sheet 16 - Definite Integration . . . . .	87

<i>CONTENTS</i>	3
<b>8 Week 8</b>	<b>88</b>
8.1 Mid-ordinate Rule . . . . .	88
8.2 Trapezium rule . . . . .	91
8.3 Simpson's Rule . . . . .	92
<b>Week 8 Exercises</b>	<b>97</b>
8.4 Sheet 17 Approximate Integration . . . . .	97
<b>9 Week 9</b>	<b>100</b>
9.1 Mean and RMS Values . . . . .	100
<b>Week 9 Exercises</b>	<b>103</b>
9.2 Sheet 17 More problems on Numerical Integration . . . . .	103
9.3 Sheet 18 Mean and RMS Values . . . . .	104
<b>10 Week 10</b>	<b>106</b>
10.1 Centre of Gravity and Centroids . . . . .	106
10.2 Volume and Surface of Revolution . . . . .	110
<b>Week 10 Exercises</b>	<b>115</b>
10.3 Sheet 19 Centroids of area . . . . .	115
10.4 Sheet 20 Volumes of Revolution . . . . .	116
<b>11 Week 11</b>	<b>117</b>
11.1 Vectors I . . . . .	117
11.2 Vectors II . . . . .	121
11.3 Vectors III . . . . .	126
<b>Week 11 Exercises</b>	<b>132</b>
11.4 Sheet 21 Vectors I . . . . .	132
11.5 Sheet 22 Vectors II . . . . .	134
11.6 Sheet 23 Vectors III . . . . .	135

# Front Matter

These notes are based heavily on the notes provided over many years by a number of colleagues, notably A. Barney, V. Perisic, N. Petrosyan, and JH Renshaw.

I am still in the process of updating these and the online and PDF form of these notes may change in due course.

# 1

## Week 1

To start our exploration of the calculus, we start in lecture 1 with a look at the concept of limits of functions. We then consider the definition of the derivative of a function from first principles before looking at a few simple rules that enable us to evaluate some derivatives more easily than by first principles.

### 1.1 Limits

In some cases, the value of a function is undefined or cannot be determined. For example

$$f(x) = \frac{x^2 - 9}{x - 3} \quad (1)$$

has the value  $\frac{0}{0}$  for  $f(3)$ , which is undetermined. In such cases it is convenient to introduce the idea of a limit or a limiting value.

Informally, a *limit* is a value which  $f(x)$  gets closer and closer to, but may never actually reach.

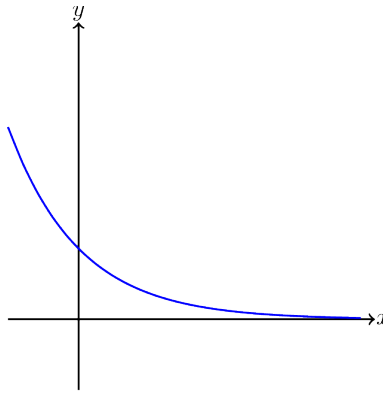
For example, the function  $f(x) = e^{-x}$  as it approaches the  $x$ -axis.

In function (1) above, we can try to determine the limit  $f(x)$  as  $x$  gets closer to 3.

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3.$$

Now if  $x$  gets closer to 3, then  $x + 3$  gets closer to 6. Formally we write

$$\lim_{x \rightarrow 3} f(x) = 6$$

Figure 1.1:  $e^{-x}$ 

and we say “in the limit, as  $x$  tends to 3 then  $f(x)$  tends to 6”.

We can approach this in a slightly different way, by first letting  $x = a + 3$  so that

$$f(x) = f(a + 3) = \frac{(a + 3)^2 - 9}{(a + 3) - 3} = \frac{a^2 + 6a + 9 - 9}{a} = \frac{a^2 + 6a}{a} = a + 6.$$

Now we can observe, that  $a$  gets closer to 0 if and only if  $x$  gets closer to 3. As  $a$  approaches 0 then  $f(a + 3)$  approaches 6. So

$$\lim_{x \rightarrow 3} f(x) = \lim_{a \rightarrow 0} f(a + 3) = 6.$$

Note that the limit is not the same thing as substituting the value of  $x = 3$  into the function  $f(x)$ , as  $f(x)$  is not defined at the value  $x = 3$ . We say that  $f$  is *discontinuous* at  $x = 3$ .

A function  $f$  of  $x$  is said to be *continuous* at the value  $x = a$  if the value of the limit of  $f(x)$  as  $x$  tends to  $a$  is equal to  $f(a)$ . In other words

$$\lim_{x \rightarrow a} f(x) = f(a).$$

**Example 1.1.** Determine the limit of

$$\frac{2x^2 - 5x - 3}{x - 3}$$

as  $x$  tends to 3.

*Solution.* Let  $x = a + 3$  so that as  $a$  tends to 0 then  $x$  tends to 3. Then

$$f(a+3) = \frac{2(a+3)^2 - 5(a+3) - 3}{(a+3) - 3} = \frac{2a^2 + 12a + 18 - 5a - 15 - 3}{a} = \frac{2a^2 + 7a}{a} = a + 7.$$

So

$$\lim_{x \rightarrow 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \rightarrow 3} f(x) = \lim_{a \rightarrow 0} f(a + 3) = \lim_{a \rightarrow 0} 2a + 7 = 7.$$

Alternatively, you may spot that  $2x^2 - 5x - 3 = (x - 3)(2x + 1)$  so that

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{(x - 3)(2x + 1)}{x - 3} = \lim_{x \rightarrow 3} 2x + 1 = 7.$$

## 1.2 Differentiation from 1st Principles

For a straight line we can consider the slope or *gradient*. This is a measure of how steep the line is

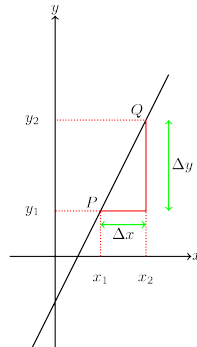


Figure 1.2: gradient of a line

The *gradient* is defined to be the number

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

For the line above, which has equation  $y = 2x - 1$  then

$$\frac{\Delta y}{\Delta x} = \frac{(2x_2 - 1) - (2x_1 - 1)}{x_2 - x_1} = \frac{2(x_2 - x_1) - 1 + 1}{x_2 - x_1} = \frac{2(x_2 - x_1)}{x_2 - x_1} = 2.$$

Suppose now we are given a curve with equation  $y = f(x)$  and want to find the gradient of a line segment joining two points on the curve.



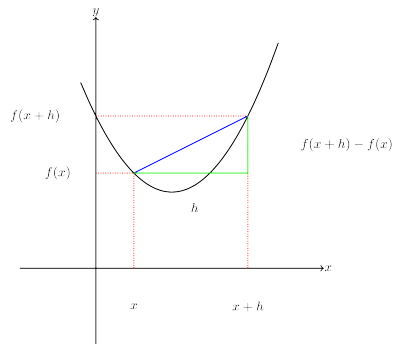


Figure 1.3: line segment on a curve

The gradient of this line segment is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}.$$

Notice that when  $h = 0$  then this would have the indeterminate value of  $0/0$ , and so the gradient is undefined when the two points on the line coincide. However, the limit of this expression, as  $h \rightarrow 0$ , may exist and it is instructive to imagine what the limit would represent geometrically.

With some thought, it should be clear that the line segment tends to a line that just meets the curve at the point  $(x, f(x))$ , and we call this line the *tangent* to the curve at the point  $(x, f(x))$ .

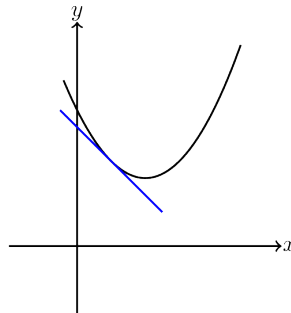


Figure 1.4: line segment on a curve

We call this value the *derivative* of the function  $y = f(x)$  at the point  $x$  and write

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

and we refer to the process of calculating this limit as *differentiation by 1st principles*.

**Example 1.2.** Find the derivative by 1st principles for the function  $f(x) = x^2$  at the point  $x$ .

*Solution.*

$$\begin{aligned}\frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x.\end{aligned}$$

**Example 1.3.** Find the derivative by 1st principles for the function  $f(x) = x^3$  at the point  $x$ .

*Solution.*

$$\begin{aligned}\frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2.\end{aligned}$$

**Example 1.4.** Find the derivative by 1st principles for the function  $f(x) = 1/(2x-1)$  at the point  $x$ .

*Solution.* Notice that

$$\begin{aligned}f(x+h) - f(x) &= \frac{1}{2(x+h)-1} - \frac{1}{2x-1} = \\ &= \frac{2x+1 - (2(x+h)-1)}{(2(x+h)-1)(2x-1)} = \frac{-2h}{(2x+2h-1)(2x-1)},\end{aligned}$$

and so

$$\frac{f(x+h) - f(x)}{h} = \frac{-2h}{h(2x+2h-1)(2x-1)} = \frac{-2}{(2x+2h-1)(2x-1)}.$$

Hence

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{-2}{(2x+2h-1)(2x-1)} = \frac{-2}{(2x-1)(2x-1)} = \frac{-2}{(2x-1)^2}.$$

### 1.3 Differentiation by Rule

We have discovered that functions may be differentiated from first principles by considering small changes in  $x$ , which lead to small changes in  $y$ . Now we would like to know if there is a rule so that we can differentiate a function without going through the process of finding a limit every time.

If the notation of functions is used so that  $y = f(x)$  then  $dy/dx$  is defined as

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

and  $dy/dx$  may also be written as  $f'(x)$ , where the apostrophe indicates that  $f(x)$  has been differentiated once. So  $dy/dx$  and  $f'(x)$  have the same meaning.

#### 1.3.1 Rule for differentiation

Consider the differentiation of  $y = x^n$  from first principles. We have  $f(x+h) = (x+h)^n$ .

Expanding the right-hand side using the binomial theorem gives

$$f(x+h) = x^n + nx^{(n-1)}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \dots + h^n$$

So that

$$f(x+h) - f(x) = xn^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \dots + h^n,$$

and thus

$$\frac{f(x+h) - f(x)}{h} = xn^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \dots + h^{n-1},$$

Now, taking limit we have

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{dy}{dx} = nx^{n-1}.$$

Therefore in general, we differentiate  $x^n$  for all values of  $n$  using the rule

$$\frac{dy}{dx} = \frac{d(x^n)}{dx} = nx^{n-1}.$$

This is known as the rule for differentiation.

**Example 1.5.** Find the derivative by rule for the function  $y = x^4$  at the point  $x$ .

*Solution.*

$$\frac{dy}{dx} = 4x^3.$$

In words, the operation is multiply the function by the index and then reduce the value of the index by one.

In fact this rule is also true for any real value of  $n$  not just when  $n$  is a positive integer.

### 1.3.2 Differentiation of $ax^n$ .

Where a power of  $x$  is multiplied by a constant, that constant remains unchanged by the process of differentiation. For example

$$y = ax^n \Rightarrow \frac{dy}{dx} = nax^{n-1}.$$

**Example 1.6.** Find the derivative by rule for the function  $y = 3x^4$  at the point  $x$ .

*Solution.*

$$\frac{dy}{dx} = 3 \times 4x^3 = 12x^3.$$

Note that the differential of  $ax$  and of  $a$  also follow the rule. For  $ax$  we may write  $ax = ax^1$ . So differentiating by rule

$$\frac{dy}{dx} = a \times 1x^0 = a.$$

For  $a$  we may write  $a = ax^0$  and hence differentiating by rule we have

$$\frac{dy}{dx} = a \times 0x^{-1} = 0.$$

That is to say, the differential of a constant value is always zero.

The above rule works in a more general situation. If  $f(x) = ag(x)$  where  $a$  is a constant and  $g(x)$  is function then

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{ag(x+h) - ag(x)}{h} = a \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = a \frac{dg}{dx}$$

### 1.3.3 Differentiation of a sum of terms

If we have to differentiate more terms added together, then we differentiate each term separately and add them together. This is illustrated with the following example.

**Example 1.7.** Find the derivative by rule for the function  $y = 4x^5 + 3x^2 - 6x$  at the point  $x$ .

*Solution.*

$$\frac{dy}{dx} = 4 \times 5x^4 + 3 \times 2x^1 - 6 \times 1 = 20x^4 + 6x - 6$$

So to summarise we have the following rules for differentiation:

$$\frac{d(x^n)}{dx} = nx^{n-1} \text{ for } n \in \mathbb{R} \quad (1.1)$$

$$\frac{d(ag(x))}{dx} = a \frac{dg}{dx} \quad (1.2)$$

$$\frac{d(f(x) + g(x))}{dx} = \frac{df}{dx} + \frac{dg}{dx}. \quad (1.3)$$

## 1.4 Standard Derivatives

The derivatives in the following table are well-known and may be quoted without having to explain where they come from.

f	df
$e^x$	$e^x$
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\operatorname{cosec}(x)$	$-\operatorname{cosec}(x) \cot(x)$
$\sec(x)$	$\sec(x) \tan(x)$
$\cot(x)$	$-\operatorname{cosec}^2(x)$

# Week 1 Exercises

## 1.5 Sheet 1 Limits

### Exercise 1

Show that the following curves are continuous at the given  $x$  values by demonstrating that

- a.  $\lim_{x \rightarrow 1} \frac{3x+2}{2x-1} = 5;$
- b.  $\lim_{x \rightarrow 0} \frac{x+3}{x-2} = -\frac{3}{2};$
- c.  $\lim_{x \rightarrow 3} \frac{x^3+27}{x+3} = 27;$

### Exercise 2

Determine the following limits:

- a.  $\lim_{x \rightarrow 1} \frac{x^2+x-2}{x-1}$
- b.  $\lim_{x \rightarrow 2} \frac{2x^2-x-6}{x-2}$
- c.  $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2}$
- d.  $\lim_{x \rightarrow 0} \frac{2x^2+3x}{x}$
- e.  $\lim_{x \rightarrow 2} \frac{x^2-x-2}{2x^2-3x-2}$
- f.  $\lim_{x \rightarrow \infty} \frac{x}{2x+1}$

### Exercise 3

If  $f(x) = 3x^2$ , determine  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ .

**Exercise 4**

If  $f(x) = x^2 - 2x$ , determine  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ .

[Solutions: 2. (a) 3, (b) 7, (c) 4, (d) 3, (e) 3/5, (f) 1/2; 3.  $6x$ ; 4.  $2x - 2$ ]

**1.6 Sheet 2 Differentiation by first principles**

For each of the following functions differentiate  $y$  with respect to  $x$  from first principles.

1.  $y = 4x^2$
2.  $y = 6x^3$
3.  $y = 3x^2 + 2x$
4.  $y = 6x^2 - 4$
5.  $y = 3x^3 + 4x - 5$
6.  $y = x^4 + x^2 - 3x$
7.  $y = \frac{1}{x^2}$
8.  $y = \frac{1}{5x+3}$
9.  $y = \frac{1}{(1+x)^2}$
10.  $y = x - \frac{2}{x}$

[Solutions: 1.  $y = 4x^2$ ; 2.  $y = 6x^3$ ; 3.  $y = 3x^2 + 2x$ ; 4.  $y = 6x^2 - 4$ ; 5.  $y = 3x^3 + 4x - 5$ ; 6.  $y = x^4 + x^2 - 3x$ ; 7.  $y = 1/x^2$ ; 8.  $y = 1/(5x + 3)$ ; 9.  $y = 1/(1 + x)^2$ ; 10.  $y = x - 2/x$ ]

**1.7 Sheet 3 Differentiation by rule**

Differentiate each of the following functions by rule:

1. $y = x^7$	2. $y = 6x^5$
3. $y = 4x^3$	4. $s = 0.5t^3$
5. $A = \pi r^2$	6. $y = x^{1/2}$
7. $s = 0.2t^2$	8. $y = 4x^{3/2}$
9. $y = 2\sqrt{x}$	10. $y = 3(\sqrt[3]{x^2})$
11. $y = \frac{1}{x^2}$	12. $y = \frac{1}{x}$
13. $y = \frac{3}{5x}$	14. $y = \frac{2}{x^3}$
15. $y = \frac{1}{\sqrt{x}}$	16. $y = \frac{2}{3\sqrt{x}}$
17. $y = \frac{5}{x\sqrt{x}}$	18. $s = 3\sqrt[5]{t}$
19. $k = \frac{0.01}{h}$	20. $y = \frac{5}{x}$
21. $y = 4x^2 - 3x + 2$	22. $s = 3t^3 - 2t^2 + 5t - 3$
23. $q = 2u^2 - u + 7$	24. $y = 5x^4 - 7x^3 + 3x^2 - 2x$
25. $s = 7t^5 - 3t^2 + 7$	26. $y = 3.1x^{1.5} - 2.4x^{0.6}$
27. $y = \frac{x+x^3}{\sqrt{x}}$	28. $y = \frac{3+x^2}{x}$
29. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$	30. $y = x^3 + \frac{3}{\sqrt{x}}$
31. $t^{1.3} - \frac{1}{4t^{2.3}}$	32. $y = \frac{3x^3}{5} - \frac{2x^2}{7} - \sqrt{x}$
33. $y = 0.008 + \frac{0.001}{x}$	34. $s = 10 - 6t + 7t^2 - 2t^3$

[Solutions: 1.  $dy/dx = 7x^6$ ; 2.  $dy/dx = 30x^4$ ; 3.  $dy/dx = 12x^2$ ; 4.  $ds/dt = 1.5t^2$ ; 5.  $dA/dr = 2\pi r$ ; 6.  $dy/dx = 1/(2x^{1/2})$ ; 7.  $ds/dt = 0.4t$ ; 8.  $dy/dx = 6x^{1/2}$ ; 9.  $dy/dx = 1/\sqrt{x}$ ; 10.  $dy/dx = 2/\sqrt[3]{x}$ ; 11.  $dy/dx = -2/x^3$ ; 12.  $dy/dx = -1/x^2$ ; 13.  $dy/dx = -3/(5x^2)$ ; 14.  $dy/dx = -6/x^4$ ; 15.  $dy/dx = -1/(2x^{3/2})$ ; 16.  $dy/dx = -1/(3x^{3/2})$ ; 17.  $dy/dx = -15/(2x^{5/2})$ ; 18.  $ds/dt = 3/(10\sqrt[5]{t})$ ; 19.  $dk/dh = -0.01/h^2$ ; 20.  $dy/dx = -5/x^2$ ; 21.  $dy/dx = 8x - 3$ ; 22.  $ds/dt = 9t^2 - 4t + 5$ ; 23.  $dq/du = 4u - 1$ ; 24.  $dy/dx = 20x^3 - 21x^2 + 6x - 2$ ; 25.  $ds/dt = 35t^4 - 6t$ ; 26.  $dy/dx = 4.65x^{0.5} - 1.44/x^{0.4}$ ; 27.  $dy/dx = 1/(2x^{1/2}) + 5x^{3/2}/2$ ; 28.  $dy/dx = -3/x^2 + 1$ ; 29.  $dy/dx = 1/(2\sqrt{x}) - 1/(2x\sqrt{x})$ ; 30.  $dy/dx = 3x^2 - 3/(2x^{3/2})$ ; 31.  $ds/dt = 1.3t^{0.3} + 0.575/t^{3.3}$ ; 32.  $dy/dx = 9x^2/5 - 4x/7 - 1/(2\sqrt{x})$ ; 33.  $dy/dx = -0.001/x^2$ ; 34.  $ds/dt = -6 + 14t - 6t^2$ ]



## 2

# Week 2

This week we will consider the equations of tangents and normals and then complete our rules for differentiation by looking at

1. Chain rule
2. Product rule
3. Quotient rule.

### 2.1 Tangents and Normals

Two lines are *parallel* if their gradients are identical. For example  $y = 2x$ ,  $y = 2x - 2$  and  $y = 2x + 3$  all have gradient of 4 and hence are all parallel.

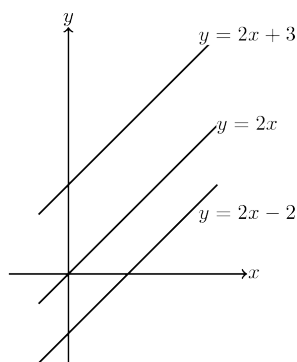


Figure 2.1: parallel lines

If two lines are perpendicular we may also find a relationship between their gra-

dients. Consider the figure below:

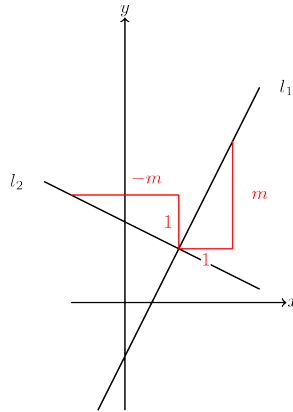


Figure 2.2: perpendicular lines

The gradient of the line  $l_1$  is given by the ratio

$$\frac{m}{1},$$

while the gradient of the line  $l_2$  is given by the ratio

$$\frac{1}{-m}.$$

So if a line has gradient  $m$  then the gradient of the line perpendicular to it has gradient  $-1/m$ , and so

$$\text{gradient of } (l_1) \times \text{gradient of } (l_2) = -1.$$

Now consider the figure below:

Here we have the tangent  $PT$  and the normal  $PN$  to a curve. The *normal* is the line at  $90^\circ$  to the tangent. We already know that the gradient of the tangent is given by

$$m = \frac{dy}{dx} \text{ at the point } P,$$

and so from the argument above, the gradient of the normal to the tangent is given by

$$\frac{-1}{m} = \frac{-1}{dy/dx}.$$

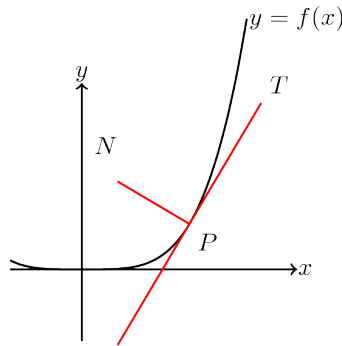


Figure 2.3: tangent and normal

Now, the equation of the line with gradient  $m$ , passing through the point  $P = (x_1, y_1)$  is given by

$$y - y_1 = m(x - x_1).$$

So the *equation of the tangent* is

$$y - y_1 = \frac{dy}{dx}(x - x_1),$$

while the *equation of the normal* to the curve is

$$y - y_1 = \frac{-1}{dy/dx}(x - x_1).$$

**Example 2.1.** Find the equation of the tangent and the normal to the curve  $y = x^2 + 5x$  at the point  $x = 2$ .

*Solution.* If  $y = x^2 + 5$  then  $dy/dx = 2x + 5$ .

At  $x_1 = 2$ ,  $y_1 = 2^2 + 5 \times 2 = 14$  and  $dy/dx = 2 \times 2 + 5 = 9$ . The equation of the tangent is therefore

$$y - 14 = 9(x - 2) = 9x - 18 \text{ and so } y = 9x - 4.$$

The equation of the normal is

$$y - 14 = \frac{-1}{9}(x - 2) = -\frac{1}{9}x + \frac{2}{9} \text{ and so } y = -\frac{x}{9} + \frac{128}{9}.$$

## 2.2 The Chain Rule

A function may be considered as an operation that has to be done to  $x$  to get  $y$ . For example if  $y = f(x) = x^2$  then the function (called  $f$  in this case) is: square the value of  $x$ .

For  $y = g(\theta) = \sin(\theta)$  the function (called  $g$  in this case) is: take the sine of  $\theta$ .

If we now look at  $(x^2 + 4)^3$  then we can consider this as one function described in the following way: take  $x$ , square it, add 4 and then cube the answer. So we can write

$$y = f(x) = (x^2 + 4)^3$$

or we can consider the function  $f$  as two functions:

$$\begin{aligned} u &= g(x) = (x^2 + 4) \\ y &= h(u) = u^3 \end{aligned}$$

or we can even consider it as 3 functions

$$\begin{aligned} u &= k(x) = x^2 \\ v &= m(u) = u + 4 \\ y &= n(v) = v^3. \end{aligned}$$

In the first case, the value of  $y$  is calculated all in one attempt. In the second case,  $u$  is calculated from  $x$  according to the rule of the function  $g$ , then  $y$  is calculated using  $u$  and the rule of the function  $h$ . In the third case  $u$  is calculated from  $x$  according to the rule of function  $k$ , then  $v$  is calculated from  $u$  according to the rule of function  $m$  and finally  $y$  is calculated from the value of  $v$  according to the rule of function  $n$ .

We shall consider the second case as this is most helpful for differentiation. In the second case

$$\begin{aligned} u &= g(x) = (x^2 + 4) \\ y &= h(u) = u^3 \end{aligned}$$

so by substituting for  $u$  we can write  $y = h(g(x))$  which means: to get  $y$  from  $x$ , first apply the rule  $g$  (get  $x$ , square it and add 4) and then apply the rule  $h$  (cube the answer to operation  $g$ ). When  $y$  is written in this form it is called a *function of a function* or a *composite function*.

### 2.2.1 Differentiation of a function of a function (the Chain Rule)

Let consider the function  $y = (x^2 + 4)^3$ . Up to now, if we wanted to differentiate  $y = (x^2 + 4)^3$ , we had to multiply out the brackets to get a string of things added together since so far the only things we can differentiate are sums of terms of standard functions (e.g.  $\sin(x)$ ,  $\cos(x)$ ,  $\ln(x)$ , etc).

The chain rule allows us to differentiate functions of functions without having to multiply out all the brackets.

We can consider  $y$  as a function of a function if we let  $u = x^2 + 4$ . In that case  $u$  is a function of  $x$  and  $y$  is a function of  $u$ , namely  $y = u^3$ . Therefore we can say that

$$\frac{dy}{du} = 3u^2$$

(using ordinary differentiation by rule).

However, we want to find

$$\frac{dy}{dx}.$$

The chain rule says that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This can be proved from first principles, but we shall not do that here.

Now since  $u = x^2 + 4$ , then

$$\frac{du}{dx} = 2x$$

(again using differentiation by rule). So

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times 2x = 6xu^2.$$

But since  $u = (x^2 + 4)$  then

$$\frac{dy}{dx} = 6x(x^2 + 4)^2.$$

The chain rule can be extended for any number of functions e.g.

$$\frac{dy}{dx} = \frac{dy}{da} \times \frac{da}{db} \times \frac{db}{dc} \times \frac{dc}{de} \times \frac{de}{df} \times \frac{df}{dx}.$$

**Example 2.2.** Differentiate  $s = (2t^2 - 4t)^5$  with respect to  $t$ .

*Solution.* Let  $u = 2t^2 - 4t$  and  $s = u^5$ . Then

$$\frac{du}{dt} = 4t - 4 \text{ and } \frac{ds}{du} = 5u^4$$

So

$$\frac{ds}{dt} = \frac{ds}{du} \times \frac{du}{dt} = 5u^4(4t - 4).$$

But since  $u = 2t^2 - 4t$  then

$$\frac{ds}{dt} = 5(2t^2 - 4t)^4(4t - 4).$$

**Example 2.3.** Differentiate  $y = 5 \cos(5\theta)$  with respect to  $\theta$ .

*Solution.* Let  $u = 5\theta$  and  $y = 5 \cos(u)$ . Then

$$\frac{du}{d\theta} = 5 \text{ and } \frac{dy}{du} = -5 \sin(u)$$

So

$$\frac{dy}{d\theta} = \frac{dy}{du} \times \frac{du}{d\theta} = -5 \sin(u) \times 5.$$

But since  $u = 5\theta$  then

$$\frac{dy}{d\theta} = -25 \sin(5\theta).$$

**Example 2.4.** Differentiate  $y = e^{2x+3}$  with respect to  $x$ .

*Solution.* Let  $u = 2x + 3$  and  $y = e^u$ . Then

$$\frac{du}{dx} = 2 \text{ and } \frac{dy}{du} = e^u$$

So

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2.$$

But since  $u = 2x + 3$  then

$$\frac{dy}{dx} = 2e^{2x+3}.$$

## 2.3 The Product and Quotient Rules

### 2.3.1 Differentiation of a product

So far, if we want to differentiate a function which is a product of two functions of  $x$ , like  $y = 4x^2(3x^3 - 6x)$ , we need to multiply out the brackets and differentiate term by term as follows;

$$y = 4x^2(3x^3 - 6x) = 12x^5 - 24x^3$$

$$\frac{dy}{dx} = 60x^4 - 72x^2 = 12x^2(5x^2 - 6)$$

For some products, multiplying terms out still doesn't help to differentiate them e.g.

$$y = \sqrt{x^2 - 2x} \sin(x).$$

This would require a binomial expansion for the first part and would still leave a set of terms that were products of a function of  $x$  and  $\sin(x)$ .

It would be helpful to find a way of differentiating a product of two functions without having to multiply out. From the first example above, it is clear that the differential of a product is not just the differential of each part (that would give  $8x(9x^2 - 6) = 72x^3 - 48x$ ).

Let  $u$  and  $v$  be functions of  $x$  so that  $u = u(x)$  and  $v = v(x)$ . Hence a small change  $h$  in  $x$  will give a small change in  $u$  and a small change in  $v$ .

If  $y = uv$  then  $y(x+h) = u(x+h)v(x+h)$  so  $y(x+h) - y(x) = u(x+h)v(x+h) - u(x)v(x)$ . Now by adding zero to the right side of the equation by adding and subtracting  $v(x)u(x+h)$  we get

$$y(x+h) - y(x) = u(x+h)v(x+h) - v(x)u(x+h) + v(x)u(x+h) - u(x)v(x).$$

Further, dividing both sides of equation by  $h$  gives

$$\frac{y(x+h) - y(x)}{h} = u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}.$$

As  $h$  tends to zero,  $u(x+h)$  tends to  $u(x)$  and in the limit we have

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}.$$

or, put another way,

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}.$$

This is called the *Product Rule*.

**Example 2.5.** Differentiate  $y = 4x^2(3x^3 - 6x)$  with respect to  $x$ .

*Solution.* Let  $u = 4x^2$  and  $v = 3x^3 - 6x$ . Then

$$\frac{du}{dx} = 8x \text{ and } \frac{dv}{dx} = 9x^2 - 6.$$

Hence the product rule gives

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (3x^3 - 6x) \times 8x + 4x^2 \times (9x^2 - 6) \\ &= 24x^4 - 48x^2 + 36x^4 - 24x^2 = 60x^4 - 72x^2 = 12x^2(5x^2 - 6). \end{aligned}$$

**Example 2.6.** Differentiate  $y = \sqrt{x^2 - 2x} \sin(x)$  with respect to  $x$ .

*Solution.* Let  $u = \sqrt{x^2 - 2x} = (x^2 - 2x)^{1/2}$  and  $v = \sin(x)$ . Then

$$\frac{du}{dx} = \frac{1}{2}(x^2 - 2x)^{-1/2}(2x - 2) = (x^2 - 2x)^{-1/2}(x - 1) \text{ and } \frac{dv}{dx} = \cos(x).$$

Hence the product rule gives

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} = (x^2 - 2x)^{1/2} \cos(x) + \sin(x)(x^2 - 2x)^{-1/2}(x - 1) \\ &= \frac{(x - 1) \sin(x)}{(x^2 - 2x)^{1/2}} + (x^2 - 2x)^{1/2} \cos(x). \end{aligned}$$

### 2.3.2 Differentiation of a quotient

Consider  $y$  to be a function consisting of the quotient of two functions of  $x$ ,  $u$  and  $v$ . So  $y$  may be written as

$$y = \frac{u(x)}{v(x)} = \frac{u}{v}.$$

Since  $u$  and  $v$  are both functions of  $x$ , then if  $x$  changes by a small amount to  $x + h$ , there will be a corresponding small change in  $u$  and  $v$  and hence will result in small change in  $y$ . So if

$$y(x) = \frac{u(x)}{v(x)} \text{ then } y(x + h) = \frac{u(x + h)}{v(x + h)}$$

Hence

$$\begin{aligned} y(x + h) - y(x) &= \frac{u(x + h)}{v(x + h)} - \frac{u(x)}{v(x)} \\ &= \frac{v(x)u(x + h) - u(x)v(x + h)}{v(x + h)v(x)} \\ &= \frac{v(x)u(x + h) - v(x)u(x) + v(x)u(x) - u(x)v(x + h)}{v(x + h)v(x)} \end{aligned}$$



Rearranging the numerator and dividing by  $h$  we get

$$\frac{y(x+h) - y(x)}{h} = \frac{v(x) \frac{u(x+h)-u(x)}{h} + u(x) \frac{v(x)-v(x+h)}{h}}{v(x+h)v(x)}$$

And in the limit as  $h$  tends to zero

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{y(x+h) - y(x)}{h} = \frac{v(x) \frac{du}{dx} - u(x) \frac{dv}{dx}}{v(x)^2}$$

This is the formula for the *differentiation of a quotient*.

Note that for the product formula it is up to you which part of the product you call  $u$  and which  $v$ , but that for the quotient formula, the denominator of the quotient must be  $v$  and the numerator must be  $u$ .

There is an alternative way to derive this formula. Since

$$y = \frac{u(x)}{v(x)} = u(x)v(x)^{-1}$$

and since  $v(x)^{-1}$  is a composite of the function  $v(x)$  and the function  $x^{-1}$ , we can use a combination of the product rule and the chain rule. So

$$\begin{aligned} \frac{d(uv^{-1})}{dx} &= u \frac{d(v^{-1})}{dx} + v^{-1} \frac{du}{dx} = u \left( -1 \times v^{-2} \frac{dv}{dx} \right) + v^{-1} \frac{du}{dx} \\ &= \frac{du}{dx} \frac{1}{v} - \frac{u}{v^2} \frac{dv}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \end{aligned}$$

**Example 2.7.** Differentiate  $y = \frac{3x^2+2}{(4x^2-4)^2}$  with respect to  $x$ .

*Solution.* Let  $u = 3x^2 + 2$  and  $v = (4x^2 - 4)^2$ . Then

$$\frac{du}{dx} = 6x \text{ and } \frac{dv}{dx} = 2(4x^2 - 4) \times 8x = 16x(4x^2 - 4)$$

and so

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(4x^2 - 4)^2 6x - (3x^2 + 2) 16x(4x^2 - 4)}{((4x^2 - 4)^2)^2} = \frac{6x(4x - 4) - 16x(3x^2 + 2)}{(4x^2 - 4)^3}.$$

**Example 2.8.** Differentiate  $y = \frac{1-\sin(x)}{1+\sin(x)}$  with respect to  $x$ .

*Solution.* Let  $u = 1 - \sin(x)$  and  $v = 1 + \sin(x)$ . Then

$$\frac{du}{dx} = -\cos(x) \text{ and } \frac{dv}{dx} = \cos(x)$$

and so

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{(1 + \sin(x))(-\cos(x)) - (1 - \sin(x))(\cos(x))}{(1 + \sin(x))^2} = \frac{-2\cos(x)}{(1 + \sin(x))^2}.$$

# Week 2 Exercises

## 2.4 Sheet 4 Tangents and Normals

### Exercise 1

In each of the examples below determine the equations of the tangent and normal at the point indicated.

- a.  $y = x^2$  at the point where  $x = 1$ ,
- b.  $y = 2x^2 - 3x$  at the point where  $x = 2$ ,
- c.  $y = 1 + x - x^2$  at the point where  $x = 2$ ,
- d.  $y = x^3 - 5x + 3$  at the point where  $x = 3$ ,
- e.  $y = 1/x$  at the point where  $x = 2$ ,
- f.  $y = (x - 3)^2$  at the point where  $x = 5$ .

### Exercise 2

Determine the gradient for  $y = 6x^3 - 9x^2 - 20x + 6$  and show that the slope is zero when  $x = 5/3$  and when  $x = -2/3$ . Establish the gradient at  $x = 1$  and then write the equations of the tangent and the normal to the curve at this point.

[Solutions. 1: (a)  $y = 2x - 1/y = -x/2 + 3/2$ ; (b)  $y = 5x - 8/y = -x/5 + 12/5$ ; (c)  $y = -3x + 5/y = x/3 - 5/3$ ; (d)  $y = 22x - 51/y = -x/22 + 333/22$ ; (e)  $y = -x/4 + 1/y = 4x - 15/2$ ; (f)  $y = 4x - 16/y = -x/4 + 21/4$

2:  $dy/dx = 18x^2 - 18x - 20 = 0$  gives  $x = 5/3$  and  $x = -2/3$ .  $dy/dx(1) = -20$   
Equation of tangent is  $y = -20x + 3$ , equation of normal is  $y = x/20 - 341/20$ .]

## 2.5 Sheet 5 The Chain Rule

Differentiate each of the following functions using the chain rule.

1.  $y = (2x + 5)^2$
2.  $y = (1 - 5x)^4$
3.  $s = (3t + t)^5$
4.  $p = (1 - 2q^2)^{11}$
5.  $y = (1 - 2x)^3$
6.  $y = (3x^2 - 5x + 4)^4$
7.  $s = (4t^3 + 3t - 2)^2$
8.  $y = \frac{1}{\sqrt{x^2 - 1}}$
9.  $y = a \sin(a\theta)$
10.  $y = \sin(\omega t)$
11.  $y = \cos(2\pi ft + \frac{\pi}{2})$
12.  $y = \tan(4x + 3)$
13.  $y = 6e^{5x}$
14.  $y = \ln(2x^5)$
15.  $y = \ln(\frac{1}{x})$

{Solutions. (1)  $dy/dx = 8x + 20$ ; (2)  $dy/dx = -20(1 - 5x)^3$ ; (3)  $ds/dt = 15(3t + 7)^4$ ; (4)  $dp/dq = -44q(1 - 2q^2)^{10}$ ; (5)  $dy/dx = -6(1 - 2x)^2$ ; (6)  $dy/dx = (24x - 20)(3x^2 - 5x + 4)^3$ ; (7)  $ds/dt = (24t^2 + 6)(4t^3 + 3t - 2)$ ; (8)  $dy/dx = -x/(x^2 - 1)^{3/2}$ ; (9)  $y/d\theta = a^2 \cos(a\theta)$ ; (10)  $dy/dt = \omega \cos(\omega t)$ ; (11)  $dy/dt = -2\pi f \sin(2\pi ft + \pi/2)$ ; (12)  $dy/dx = 4 \sec^2(4x + 3)$ ; (13)  $dy/dx = 30e^{5x}$ ; (14)  $dy/dx = 5/x$ ; (15)  $dy/dx = -1/x$ }

## 2.6 Sheet 6 The Product Rule

if  $y(x) = u(x)v(x)$  then  $dy/dx = u dv/dx + v du/dx$

Differentiate the following functions

1.  $y = (4x + 5)(x^2 - 2x)$  1.  $y = x^{5/3}(4x - 2)^2$  1.  $y = (x + 1)^2(1 - 2x)$  1.  $y = 4x\sqrt{5 - 2x^2}$  1.  $y = (4x^2 + 3x)(2x - 8)^5$  1.  $y = 8x^3 \sin(3x)$  1.  $y = 4x^2 \cos(x)$   
 1.  $y = (x^2 - 2x) \sin(x)$  1.  $y = e^{2x} \sin(x)$  1.  $y = x \ln(x)$  1.  $y = \sqrt{x^2 + 4} \cos(2x + 1)$   
 1.  $y = \sqrt{x^2 - 1} \ln(2x + 1)$  1.  $y = e^{x \ln(x)}$  1.  $y = x^x$

[Solutions. 1.  $(4x + 5)(2x - 2) + 4(x^2 - 2x)$ ; 2.  $8x^{5/3}(4x - 2) + 5x^{2/3}(4x - 2)^2/3$ ; 3.  $-2(x + 1)^2 + 2(1 - 2x)(x + 1)$ ; 4.  $-8x^2(5 - 2x^2)^{-1/2} + 4(5 - 2x^2)^{1/2}$ ; 5.  $10(4x^2 + 3x)(2x - 8)^4 + (8x + 3)(2x - 8)^5$ ; 6.  $24x^2 \sin(3x) + 24x^3 \cos(3x)$ ; 7.  $8x \cos 9x - 4x^2 \sin(x)$ ; 8.  $(2x - 2) \sin(x) + (x^2 - 2x) \cos(x)$ ; 9.  $2e^{2x} \sin(x) + e^{2x} \cos(x)$ ; 10.  $x(x^2 + 4)^{-1/2} \cos(2x + 1) - 2(x^2 + 4)^{1/2} \sin(2x + 1)$ ]

## 2.7 Sheet 7 The Quotient Rule

Differentiate the following functions with respect to  $x$  using the quotient rule

1.  $y = \frac{3x^2+2}{5x+4}$
2.  $y = \frac{(9x^2+3)^2}{2x-8}$
3.  $y = \frac{1+x}{1-x}$
4.  $y = \frac{2x+2}{(3x^3+2x)^2}$
5.  $y = 9x^2 - \frac{x^2+2x}{1-x}$
6.  $y = \frac{3x-4}{2x^2+1}$
7.  $y = \frac{\sin(3x)}{x+1}$
8.  $y = \frac{4x(3x-4)^2}{\sin(x)}$

[Solutions: 1.  $dy/dx = (15x^2 + 24x - 10)/(5x + 4)^2$ ; 2.  $dy/dx = 18(3x^2 + 1)(9x^2 - 48 - 1)/(2x - 8)^2$ ; 3.  $dy/dx = 2/(1 - x)^2$ ; 4.  $dy/dx = -2(3x^3 + 2x)(15x^3 + 18x^2 + 2x + 4)/(3x^3 + 2x)^4$ ; 5.  $dy/dx = 18x + (x^2 - 2x - 2)/(1 - x)^2$ ; 6.  $dy/dx = (-6x^2 + 16x + 3)/(2x^2 + 1)^2$ ; 7.  $dy/dx = (3 \cos(3x)(x + 1) - \sin(3x))/(x + 1)^2$ ; 8.  $dy/dx = 4(3x - 4)((9x - 4) \sin(x) - x(3x - 4) \cos(x))/\sin^2(x)$ ]

# 3

## Week 3

We start this week by looking at derivatives of derivatives and then move on to look at derivatives from the point of view of rates of change of the function with respect to the variable. This leads us to look at velocity and acceleration.

### 3.1 Successive Differentiation

So far we have only differentiated each function once, but there is no reason why a function may not be differentiated several times. Suppose  $y = x^3 + 2x^2$  then

$$\frac{dy}{dx} = 3x^2 + 4x.$$

We may then differentiate  $dy/dx$  as well and this may be written

$$\frac{d}{dx} \frac{dy}{dx} \text{ or more usually } \frac{d^2y}{dx^2}.$$

So in this case  $d^2y/dx^2 = 6x + 4$ .

This is called the *second derivative of y*.

$d^2y/dx^2$  may also be differentiated with respect to  $x$  to give

$$\frac{d^3y}{dx^3} = 6$$

and is called the *third derivative of y*, and  $d^3y/dx^3$  may be differentiated with respect to  $x$  to give

$$\frac{d^4y}{dx^4} = 0,$$

the *fourth derivative of  $y$* . Once the result of the successive differentiation operations is zero, no further differential coefficients exist.

**Example 3.1.** Find  $dy/dx$ ,  $d^2y/dx^2$ ,  $d^3y/dx^3$  for  $y = e^x \sin(x)$ .

*Solution.* We use the product rule. If

$$y = e^x \sin(x) \text{ then } \frac{dy}{dx} = e^x \cos(x) + e^x \sin(x).$$

Now

$$\frac{d^2y}{dx^2} = e^x(-\sin(x)) + e^x \cos(x) + e^x \cos(x) + e^x \sin(x) = 2e^x \cos(x),$$

and finally

$$\frac{d^3y}{dx^3} = 2(-e^x \sin(x) + e^x \cos(x)) = 2e^x(\cos(x) - \sin(x)).$$

### 3.1.1 Rates of Change

The derivative  $dy/dx$  may be considered as the rate at which  $y$  changes as a result of a change in  $x$ .

This is easiest to understand if we consider the distance  $s$  in metres, of a particle from an origin  $O$  at a time,  $t$  seconds.

Let us assume that  $s(t) = 3t^3 + 2t$ . This equation tells us that at time  $t = 0$ , the particle is at  $O$ , at time  $t = 1$ , the particle is 5 m away from  $O$ , at time  $t = 2$ , the particle is 28 m away from  $O$  etc.

If we differentiate  $s$ , we get

$$\frac{ds}{dt} = 9t^2 + 2.$$

This tells us that the slope of the graph of  $s$  against  $t$  is not constant but varies depending on the value of  $t$ . So when  $t = 0$ , the slope is 2, when  $t = 1$ , the slope is 11, etc.

$ds/dt$  is the rate at which  $s$  changes for a change in  $t$  or the rate at which distance changes with time, so  $ds/dt$  is the *velocity*,  $v$ , of the particle in  $ms^{-1}$ .

If we differentiate the velocity  $v$ , we get  $dv/dt = 18t$ . Also, since the derivative of the velocity is the second derivative of the distance, we have

$$\frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

$dv/dt$  is the rate of change of velocity with time or the *acceleration* of the particle.

Note that  $d^3s/dt^3$  is not zero for  $s = 3t^3 + 2t$ , so we could find it. The rate of change of acceleration has no physical meaning and so it is not useful for describing the behaviour of the particle.

**Example 3.2.** A body moves such that its displacement  $s$ , in metres, for any instant of time  $t$  seconds is given by

$$s(t) = 0.4t - 0.2t^2.$$

1. Show that its acceleration is constant;
2. determine its velocity after 3 seconds and its position after 2 seconds.

*Solution.*

1. Since  $s(t) = 0.4t - 0.2t^2$  then velocity  $= ds/dt = 0.4 - 0.4t$ ; acceleration  $= d^2s/dt^2 = -0.4$ . So the acceleration is always  $-0.4ms^{-2}$  which is a constant value (so does not depend on the value of  $t$ ).
2. At  $t = 3$ ,  $ds/dt = 0.4 - (0.4 \times 3) = -0.8$ . At  $t = 2$ ,  $s = 0.4 \times 2 - (0.2 \times 2^2) = 0.8 - 0.8 = 0$ . Therefore after 3 seconds the velocity is  $-0.8ms^{-1}$  and after 2 seconds the particle has a displacement (distance in metres from the origin  $O$ ) of  $0m$ .

### 3.2 Maxima, Minima and points of inflection

The derivative of a function at a point gives the gradient (the slope) of that curve at the point. The gradient may be positive, negative or zero.

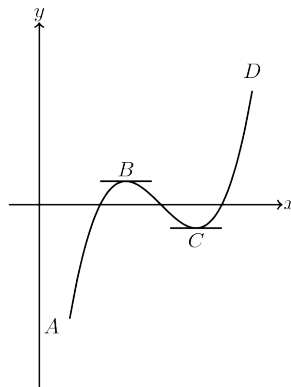


Figure 3.1: turning points



For the graph above:

- The slope from  $A$  to  $B$  is positive.
- The slope at  $B$  is zero.
- The slope from  $B$  to  $C$  is negative.
- The slope at  $C$  is zero.
- The slope from  $C$  to  $D$  is positive.

As  $x$  increases from just before  $B$  to just after, the slope of the graph changes from positive to negative. A point such as  $B$  is known as a **maximum** or **local maximum**. The word local is used to indicate that this is not the maximum  $y$  value that the curve can have, but is the maximum value in the region between  $A$  and  $C$ .

As  $x$  increases from just before  $C$  to just after, the slope of the graph changes from negative to positive. A point such as  $C$  is known as a **minimum** or **local minimum**.

Points such as  $B$  and  $C$  are also sometimes called **critical points** or **stationary points** or **turning points**. A function can have more than one maximum or minimum along its range of  $x$  values.

### 3.2.1 Points of inflection

At any point on a graph where a maximum or minimum occurs, the gradient or rate of change of the curve is zero i.e.

$$\frac{dy}{dx} = 0.$$

However, there is a further situation where the gradient of the graph may be zero, but where the sign of the slope of the graph does not change. Such a point is called a **point of inflection**.

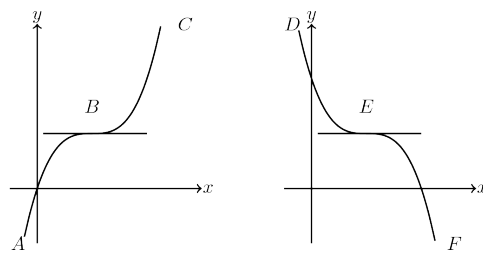


Figure 3.2: points of inflection

In the left hand graph above, the slope from  $A$  to just before  $B$  is positive, the slope at  $B$  is zero and the slope from just after  $B$  to  $C$  is negative.

In the right hand graph above, the slope from  $D$  to just before  $E$  is negative, the slope at  $E$  is zero and the slope from just after  $E$  to  $F$  is negative.

The points  $B$  and  $E$  are both points of inflection.

We would like to be able to find out how many maxima, minima and points of inflection a curve has without having to plot the graph. We can do this by examining the derivatives of the function.

Consider the curve of the function with the equation

$$y = x^3 - \frac{3}{2}x^2 - 18x + 10.$$

We know that at a stationary point,  $dy/dx = 0$ . So by finding  $dy/dx$  and setting it equal to zero, we can determine the  $x$  values of the stationary points as follows:

$$\frac{dy}{dx} = 3x^2 - 3x - 18 \quad (1)$$

When  $dy/dx = 0$ , then

$$3x^2 - 3x - 18 = 0 \quad (2)$$

The values of  $x$  where this equation is satisfied can be found by solving the quadratic equation (2) for  $x$ .

We find that the quadratic equation (2) has roots at  $x = 3$  and  $x = -2$ . So now we know that there are two turning points, one at  $x = 3$  and one at  $x = -2$ . We can find the  $y$  co-ordinates of these points by substituting each  $x$  value into equation (1). This gives  $y = -30.5$  when  $x = 3$  and  $y = 32$  when  $x = -2$ .

We now know the co-ordinates of the turning points, but we do not know whether they are maxima, minima or points of inflection. We can often establish the type of turning point by considering the second differential  $d^2y/dx^2$ .

Consider first a maximum.

In the left hand graph above, of the function  $y$ , we have a local maximum. The gradient of the curve,  $dy/dx$ , at point  $A$  will be a large positive value as the curve here is steeply sloped up from left to right. We can plot the value of the gradient at  $A$  on a graph (right hand graph above, of the function  $dy/dx$ ). At  $B$  the slope of graph  $y$  is still positive, but less steep, so the gradient at  $B$ , shown on the graph of  $dy/dx$  is still a positive value, but is smaller than at  $A$ .

At  $C$  the gradient of graph  $y$  is zero. This is the point where the curve in graph of  $dy/dx$  crosses the  $x$  axis. At  $D$  the gradient of graph  $y$  is now negative, but

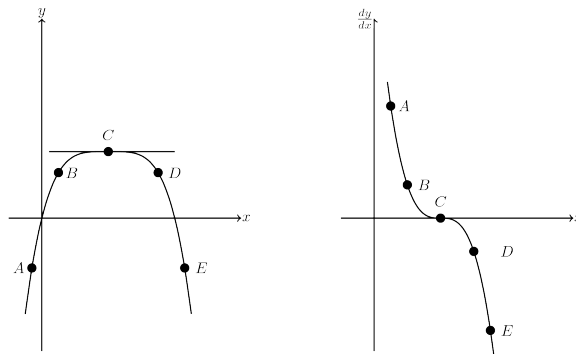


Figure 3.3: maximum turning point

not very steep, so on graph  $dy/dx$  the gradient at  $D$  is a small negative value. At  $E$  the gradient of graph  $y$  is steep and negative, so on graph  $dy/dx$  the point is marked as a large negative value.

Now consider the gradient of graph  $dy/dx$ . Graph  $dy/dx$  is a plot of  $dy/dx$  against  $x$ . The gradient of this graph is given by  $d^2y/dx^2$ . We can see from the curve in that graph that the gradient of the curve is negative at all points. That is  $d^2y/dx^2 < 0$  for all  $x$  values close to the local maximum. Therefore if  $d^2y/dx^2$  is a negative number for the  $x$  value of the turning point, we can be sure the turning point is a maximum.

By a similar argument and considering the graphs below, we can show that for any turning point where,  $d^2y/dx^2 > 0$  we have a local minimum.

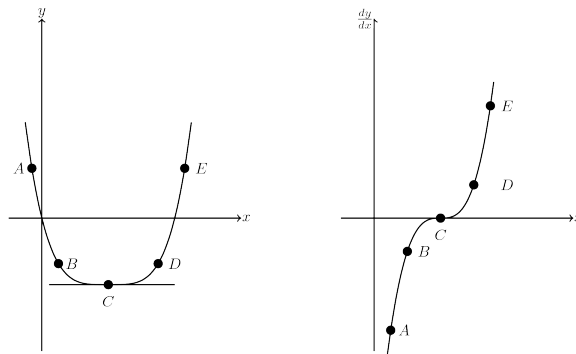


Figure 3.4: minimum turning point

Previously, we established that the curve  $y = x^3 - \frac{3}{2}x^2 - 18x + 10$  had turning points at  $x = 3$  and  $x = -2$ . We are now in a position to find out what kind of

turning points each of these is.

$$\frac{dy}{dx} = 3x^2 - 3x - 18 \text{ therefore } \frac{d^2y}{dx^2} = 6x - 3.$$

When  $x = 3$ ,  $d^2y/dx^2 = 15 > 0$ . So we deduce that the turning point at  $x = 3$  is therefore a local minimum.

When  $x = -2$ ,  $d^2y/dx^2 = -15 < 0$  and so we deduce that the turning point at  $x = -2$  is therefore a local maximum.

Note that the statement

$$\text{When } x = 3, \frac{d^2y}{dx^2} = 15$$

is often written as

$$\left. \frac{d^2y}{dx^2} \right|_{x=3} = 15$$

and similarly we can write for the other statement,

$$\left. \frac{d^2y}{dx^2} \right|_{x=-2} = -15.$$

There is a third possibility for the value of  $d^2y/dx^2$ . It may be zero.

When  $d^2y/dx^2 = 0$ , we may have a maximum, a minimum or a point of inflection at our turning point. We will need a further test to find out which. If we consider the points of inflection in Figure 3.2, we can see that for the left-hand graph, the slope,  $dy/dx$ , is positive on both sides of the point of inflection. For the right hand graph the slope is negative on both sides of the point of inflection. For a maximum the slope always changes from positive to negative as  $x$  increases and for a minimum the slope always changes from negative to positive as  $x$  increases. This suggests a way of finding out what kind of turning point we have when  $d^2y/dx^2$  turns out to be zero.

We can substitute an  $x$  value just smaller than the value at the turning point into our equation for  $dy/dx$  and find if  $dy/dx$  is positive or negative at this  $x$  value, then we can do the same for an  $x$  value just larger than the  $x$  value of the turning point and again determine the sign of the outcome.

Then the type of turning point can be established according to the following table:

slope before turning point	slope at turning point	slope after turning point	type of point
positive	zero	negative	maximum
negative	zero	positive	minimum
positive	zero	positive	inflection
negative	zero	negative	inflection

**Example 3.3.** Find and identify the critical or turning points of the function  $y = x^3$ .

*Solution.* For  $y = x^3$ ,  $dy/dx = 3x^2$ . At a turning point  $dy/dx = 0$  and so  $3x^2 = 0$  and so there is a turning point of some kind at  $x = 0$ .

$d^2y/dx^2 = 6x$  and so

$$\left. \frac{d^2y}{dx^2} \right|_{x=0} = 0.$$

Thus we cannot identify the type of turning point from the second derivative. However, we can consider the first derivative.

For  $x = -0.1$  (just less than 0)  $dy/dx = 3(-0.1)^2 = 0.03 > 0$ .

For  $x = 0.1$  (just greater than 0)  $dy/dx = 3(0.1)^2 = 0.03 > 0$ .

So from examining the table above we see that the critical point at  $x = 0$  must be a point of inflection.

### 3.2.2 Points of inflection

The points of inflection we have considered so far are a special case where the gradient is zero at some value of  $x$ . In fact any point where the curve changes from concave to convex is called a point of inflection regardless of whether the gradient becomes zero or not. Two points of this type are shown in the graphs below.

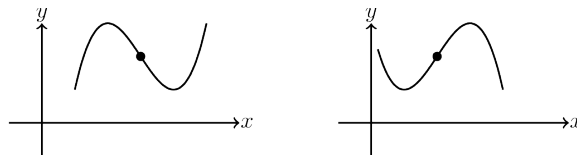


Figure 3.5: points of inflection

Here the graph changes from concave downwards to concave upwards as  $x$  increases in the first case and from concave upwards to concave downwards in the

second case. In each case the marked point is a point of inflection. But the gradient is clearly never zero!

For points such as these  $dy/dx$  is not zero, but  $d^2y/dx^2$  always is. We can find the position of such points by finding where  $d^2y/dx^2$  is zero. We can decide which way they bend by considering  $dy/dx$  as for the previous point of inflection example.

**Example 3.4.** Find the  $x$  coordinates of all the maxima, minima and points of inflection for the curve

$$y = 2x^3 - 9x^2 + 12x.$$

*Solution.* For  $y = 2x^3 - 9x^2 + 12x$ ,  $dy/dx = 6x^2 - 18x + 12$ . When  $dy/dx = 0$  then  $6x^2 - 18x + 12 = 0$  and so  $x = 2$  or  $x = 1$ .

Now  $d^2y/dx^2 = 12x - 18$ , so when  $x = 2$ ,  $d^2y/dx^2 > 0$  and the point is a minimum.

When  $x = 1$ ,  $d^2y/dx^2 < 0$  and the point is a maximum.

Also,  $d^2y/dx^2 = 0$  when  $x = 1.5$  and so at  $x = 1.5$  here is a point of inflection (but not of zero gradient).

From examining the value of  $dy/dx$  just before and just after  $x = 1.5$  we find that the gradient is negative on both sides of the point of inflection and thus the curves slopes down from left to right.

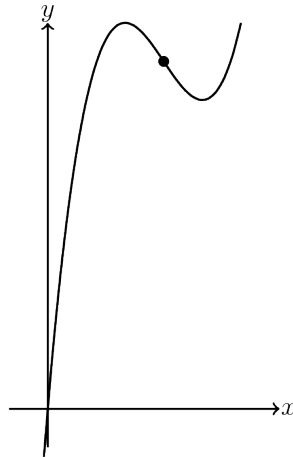


Figure 3.6:  $y = 2x^3 - 9x^2 + 12x$

# Week 3 Exercises

## 3.3 Sheet 8 Successive Differentiation

### Exercise 1

Determine the first four derivatives of the function

$$y = x^3 - 4x^{3/2}.$$

### Exercise 2

Show that  $y = 4x$  is a solution to the differential equation

$$2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0.$$

### Exercise 3

The displacement in metres for a particle moving along a line at time  $t$  seconds is given by the equation

$$s = 2t^2 - 10t + 4.$$

Determine the velocity at times of 1.5 and 10 seconds.

### Exercise 4

The velocity  $v$  of a particle moving along a line is given by

$$v = 33t^2 + 18t + 3$$

where  $t$  is measured in seconds and  $v$  in metres per second. Determine the acceleration after 2s and 4s.

**Exercise 5**

The displacement of a body, measured in meters, from a reference point after time  $t$  seconds is given by

$$s = 4t^2 - 3t + 2.$$

Show that the acceleration of the body is constant at  $8ms^{-2}$  and that the velocity is zero after  $375m$ .

**Exercise 6**

The equation of motion of a particle is given by

$$s = 3 + 5t - 6t^2$$

where  $s$  is the displacement in metres and  $t$  is the time in seconds.

- Determine its position, velocity and acceleration after a time of  $4s$ .
- Show that after  $415.66m$  the body is momentarily at rest.

[Solutions: 1.  $dy/dx = 3x^2 - 6x^{1/2}$ ,  $d^2y/dx^2 = 6x - 3x^{-1/2}$ ,  $d^3y/dx^3 = 6 + 3x^{-3/2}/2$ ,  $d^4y/dx^4 = -9x^{-5/2}/4$ ; 2.  $dy/dx = 4$ ,  $d^2y/dx^2 = 0$ ; 3.  $v(1.5) = -4 m/s$ ,  $v(10) = 30 m/s$ ; 4.  $a(2) = 6 m/s^2$ ,  $a(4) = -6 m/s^2$ ; 5.  $v = 8t - 3$ ,  $a = 8$ ,  $v(0.375) = 0 m/s$ ; 6.  $v = -12t + 5$ ,  $a = -12$ ,  $s(4) = -73 m$ ,  $v(4) = -43 m/s$ ,  $a(4) = -12 m/s^2$ .  $v(0.41666) = 0 m/s$ .]

### 3.4 Sheet 8B More Examples of Successive Differentiation

**Exercise 1**

If  $y(t) = a + bt^2 + ct^4$ ; where  $a$  and  $b$  are constants, find  $dy/dt$  and  $d^2y/dt^2$ .

**Exercise 2**

If  $y(t) = at - \frac{1}{2}gt^2$ , where  $a$  and  $g$  are constants find  $dy/dt$  and  $d^2y/dt^2$ .

**Exercise 3**

If a body moves according to the law  $s = 12 - 4.5t + 6.2t^2$  find its velocity in metres when  $t = 4$  seconds. What can you say about the variation of the acceleration with time?



**Exercise 4**

The angle  $\theta$  in radians turned through by a revolving wheel is connected by the time  $t$  (in seconds) that has elapsed since starting by the law

$$\theta = 2.1 - 3.2t + 4.8t^2.$$

Find the angular velocity (in radians per second) of that wheel when 1.5 seconds have elapsed. Find also its angular acceleration.

**Exercise 5**

A slider moves so that, during the first part of its motion, its distance  $s$  (in metres) from its starting point is given by the expression

$$s = 6.8t^3 - 10.8t$$

$t$  being in seconds. Find the expression for the velocity and acceleration at any time; and hence find the velocity and acceleration after 3 seconds.

**Exercise 6**

The motion of a rising balloon is such that its height  $h$  in miles is given at any instant by the expression

$$h = 0.5 + \frac{1}{10} \sqrt[3]{t - 125}$$

$t$  being in seconds. Find an expression for the velocity and acceleration at any time. Draw curves to show the variation of height, velocity and acceleration during the first ten minutes of the ascent.

**Exercise 7**

A body moves in such a way that the space described in the time  $t$  from starting is given by  $s = t^n$ , where  $n$  is a constant. Find the value of  $n$  when the velocity is doubled from the 5th to the 10th second; find it also when the velocity is numerically equal to the acceleration at the end of the 10th second.

[Solutions: 1.  $dy/dt = 2bt + 4ct^3, d^2y/dt^2 = 2b + 12ct^2$ ; 2.  $dy/dt = a - gt, d^2y/dt^2 = -g$ ; 3.  $v = 12.4t - 4.5, v(4) = 45.1 \text{ m/s}, a = 12.4$ ; 4.  $d\theta/dt(1.5) = 11.2 \text{ rad/s}, s^2\theta/dt = 9.6$ ; 5.  $v = 20.4t^2 - 10.8, a = 40.8t, v(3) = 172.8 \text{ m/s}, a(3) = 122.4 \text{ m/s}^2$ ; 6.  $v = (t-125)^{-2/3}/30, a = -(t-125)^{-5/3}/45$ ; 7.  $n = 2, n = 11$ ]

### 3.5 Sheet 9 Maxima, minima and points of inflexion

For each of the functions given below determine the  $x$  and  $y$  coordinates of any maxima, minima or points of inflexion. Find the coordinates of the points where the curves cross the  $x$  and  $y$  axes. Using all this information, sketch the graph of the function.

1.  $y = x^2 - 4x + 4$ ,
2.  $y = -2x^2 + 4x - 5$ ,
3.  $y = x^2 - x - 12$ ,
4.  $y = 2x^3 - 9x^2 + 12x + 4$ ,
5.  $y = x^3 + 2x^2 - 1$ ,
6.  $y = x^3 + 6x^2 - 15x + 3$ ,
7.  $y = x^3 - 9x^2 + 24x - 7$ ,
8.  $y = x^4 - 8x^2$ .

[Solutions: 1. minimum at  $(2, 0)$ ; 2. maximum at  $(1, -3)$ , passes through  $(0, -5)$ ; 3. minimum at  $(0.5, -12.25)$ , passes through  $(0, -12)$ ,  $(-3, 0)$ ,  $(4, 0)$ ; 4. maximum at  $(1, 9)$ , minimum at  $(2, 8)$ , point of inflection at  $(1.5, 8.5)$ , passes through  $(0, 4)$ ,  $(-0.274, 0)$ ; 5. minimum at  $(0, -1)$ , maximum at  $(-4/3, 5/27)$ , point of inflection at  $(-2/3, -11/27)$ , passes through  $(0, -1)$ ,  $(-1.618, 0)$ ,  $(-1, 0)$ ,  $(0.618, 0)$ ; 6. local maximum at  $(-5, 103)$ , minimum at  $(1, -5)$ , point of inflection at  $(-2, 49)$ , passes through  $(0, 3)$ ,  $(-7.937, 0)$ ,  $(0.22, 0)$ ,  $(1.717, 0)$ ; 7. maximum at  $(2, 13)$ , minimum at  $(4, 9)$ , point of inflection at  $(3, 11)$ , passes through  $(0, -7)$ ,  $(0.331, 0)$ ; 8. maximum at  $(0, 0)$ , minimum at  $(-2, -16)$ , minimum at  $(2, -16)$ , points of inflection at  $(-\sqrt{4/3}, -80/9)$  and  $(\sqrt{4/3}, -80/9)$ , passes through  $(0, 0)$ ,  $(-2\sqrt{2}, 0)$ ,  $(2\sqrt{2}, 0)$ ]

# 4

## Week 4

We complete the content of the last ‘section’ by learning about curve sketching before looking at specific problems from the world of Engineering relating to the computation of Maxima and Minima. Finally we look at implicit differentiation, parametric differentiation and logarithmic differentiation.

### 4.1 Curve Sketching

Given a function  $y = f(x)$  representing a curve in the plane, we wish to examine the function and its derivative(s) so as to be able to (roughly) sketch the graph of the function.

**Example 4.1.** Sketch the graph of the function

$$y = x^2 + 3x - 2.$$

We collect the following information

1.  $x$ —intercepts - when  $y = 0$ ;
2.  $y$ —intercept(s) - when  $x = 0$ ;
3. critical points - from  $dy/dx, d^2y/dx^2$  and their classifications;
4. points of inflection;
5. asymptotes.

$x$ —***intercepts***

Put  $y = 0$  and solve for  $x$ .

$$\begin{aligned}x^2 + 3x - 2 &= 0 \\x &= \frac{-3 \pm \sqrt{3^2 - 4(-2)}}{2} \\&= \frac{-3 \pm \sqrt{17}}{2}\end{aligned}$$

So the curve passes through the points  $((-3 - \sqrt{17})/2, 0)$  and  $((-3 + \sqrt{17})/2, 0)$

***y-intercept(s)***

Put  $x = 0$  to find  $y(0) = 0^2 + 3 \times 0 - 2 = -2$ .

So the curve passes through the point  $(0, -2)$ .

***critical points***

$$\begin{aligned}\frac{dy}{dx} &= 2x + 3 = 0 \Rightarrow x = -\frac{3}{2} \\y\left(-\frac{3}{2}\right) &= \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) - 2 = -\frac{17}{4} \\\frac{d^2y}{dx^2} &= 2 > 0 \Rightarrow x = -\frac{3}{2} \text{ is a local minimum.}\end{aligned}$$

***points of inflection***

Since  $d^2y/dx^2 = 2 \neq 0$  there are no points of inflection and the curve is always concave up.

**Example 4.2.** Sketch the graph of the function

$$y = x^3 + 2x + 3.$$

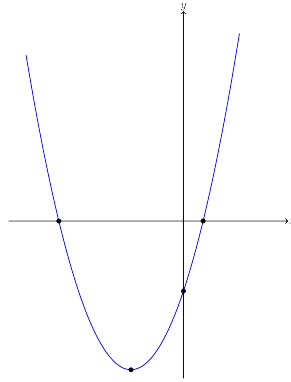
***x-intercepts***

Put  $y = 0$  and solve for  $x$ .

$$x^3 + 2x + 3 = 0.$$

Notice that when  $x = -1$  then  $y = 0$  and so  $x + 1$  is a factor of  $y$ . We can then find the other factor and see that

$$y = (x + 1)(x^2 - x + 3).$$

Figure 4.1:  $f(x) = x^2 + 3x - 2$ 

The second factor  $x^2 - x + 3$  then has roots

$$x = \frac{1 \pm \sqrt{1 - 4 \times 3}}{2}$$

which is not real and so there are no other  $x$ -intercepts, except  $x = -1$ .

Hence the curve passes through  $(-1, 0)$ .

***y-intercept(s)***

Put  $x = 0$  to find  $y(0) = 0^3 + 2 \times 0 + 3 = 3$ .

So the curve passes through the point  $(0, 3)$ .

***critical points***

$$\frac{dy}{dx} = 3x^2 + 2 = 0 \Rightarrow x^2 = -\frac{2}{3}.$$

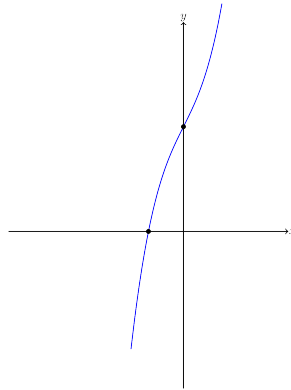
This clearly has no solutions and so there are no critical points.

***points of inflection***

Since  $d^2y/dx^2 = 6x$  then  $(0, 3)$  is a point of inflection. When  $x < 0$ ,  $d^2y/dx^2 < 0$  and so the curve is concave down here, while if  $x > 0$  then  $d^2y/dx^2 > 0$  and the curve is concave up here.

**Example 4.3.** Sketch the graph of the curve

$$y = \frac{1}{x+1}.$$

Figure 4.2:  $f(x) = x^3 + 2x + 3$ ***x-intercepts***

Put  $y = 1/(x + 1) = 0$  we see that there are no solutions and so the curve does not cross the  $x$ -axis.

***y-intercept(s)***

Put  $x = 0$  to find  $y(0) = 1/(0 + 1) = 1$  and so the curve passes through the point  $(0, 1)$ .

***critical points***

$$y = \frac{1}{x + 1} = (x + 1)^{-1}$$

$$\frac{dy}{dx} = -(x + 1)^{-2} = -\frac{1}{(x + 1)^2}.$$

This clearly has no solutions and so there are no critical points.

***points of inflection***

$$\frac{d^2y}{dx^2} = -(-2)(x + 1)^{-3} = \frac{2}{(x + 1)^3}.$$

This also has no solutions and so there are no points of inflection. Notice however that for  $x < -1$ ,  $d^2y/dx^2 < 0$  and so the function is concave down, while if  $x > -1$  then  $d^2y/dx^2 > 0$  and so the function is concave up.

***asymptotes***

An *asymptote* of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the  $x$  or  $y$  coordinates tends to infinity.

For the function

$$y = \frac{1}{x+1}$$

Notice that the function is not actually defined at the point  $x = -1$  but that as  $x \rightarrow -1$  then  $y \rightarrow \pm\infty$ . In fact if we approach the value of  $x = -1$  **from above** (i.e. we start with  $x > -1$  and reduce its value to approach  $-1$ ) then it is always the case that  $y > 0$  and so  $y \rightarrow +\infty$  in this case. On the other hand if we approach  $x = -1$  **from below** (i.e. start with  $x < -1$  and increase its value to approach  $-1$ ) then it is always the case that  $y < 0$  and so  $y \rightarrow -\infty$  in this case.

So the line  $x = -1$  is a *vertical asymptote*.

There is also a *horizontal asymptote*. Notice that as  $x \rightarrow +\infty$  then  $y \rightarrow 0$  (from above), and as  $x \rightarrow -\infty$  then  $y \rightarrow 0$  (from below). So the line  $y = 0$  is a horizontal asymptote.

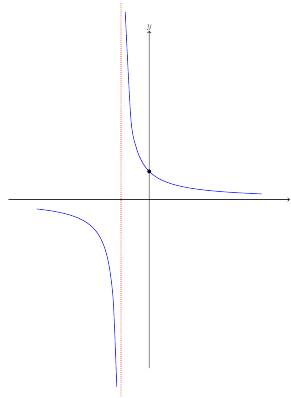


Figure 4.3:  $f(x) = 1/(x+1)$

## 4.2 Engineering Problems with Max/Min

Maxima and minima may be used in engineering to solve problems such as finding the minimum surface area required to make an object with a fixed volume.

All such problems are solved by forming an equation for the quantity to be minimised (or maximised), i.e. the surface area in the example above, and then using the first and second differentials of the equation to find the minimum (or maxi-

mum) values. However, some intuition is often required to decide how the equation should be formed from the available information.

**Example 4.4.** Metal boxes are to be made from sheets of steel 800 mm long by 400 mm wide, by cutting squares from the corners and bending the sides. Determine the size of the square to be cut from a sheet such that the volume of the box is maximum.

It is usually helpful to begin by drawing a diagram

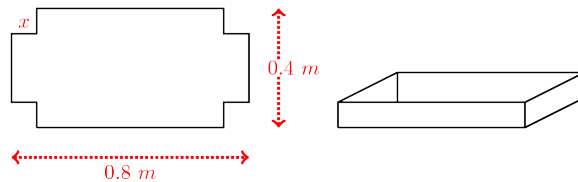


Figure 4.4: maximum volume

Let  $x$  be the length in metres of side of the square to be cut.

The volume of the box

$$V = \text{length} \times \text{breadth} \times \text{depth} \quad (1)$$

Equation (1) is the equation we want to maximise, but we need to put the equation in terms of a single variable before we can differentiate.

In terms of  $x$ ,

$$\text{length} = 0.8 - 2x, \text{ breadth} = 0.4 - 2x, \text{ depth} = x$$

So

$$V = (0.8 - 2x)(0.4 - 2x)x = (0.32 - 2.4x + 4x^2)x = 4x^3 - 2.4x^2 + 0.32x$$

Then

$$\frac{dV}{dx} = 12x^2 - 4.8x + 0.32$$

Maximum and minimum values occur when  $dV/dx = 0$

That is, when  $12x^2 - 4.8x + 0.32 = 0$

Solving  $12x^2 - 4.8x + 0.32 = 0$  for  $x$  using the quadratic formula gives:

$$x = 0.316 \text{ or } x = 0.085$$



Now we need to find which value of  $x$  gives the maximum volume.

$$\begin{aligned}\frac{d^2V}{dx^2} &= 24x - 4.8 \\ \left. \frac{d^2V}{dx^2} \right|_{x=0.316} &= 2.784 \\ \left. \frac{d^2V}{dx^2} \right|_{x=0.085} &= -2.772\end{aligned}$$

So  $x = 0.316$  gives a positive value for  $d^2V/dx^2$  and so represents the minimum volume,

and  $x = 0.085$  gives a negative value for  $d^2V/dx^2$  and so represents the maximum volume.

For maximum volume, the square to be cut out must have sides of 0.085 m, i.e. 85 mm

**Example 4.5.** A beam which is built into a wall at one end is simply supported by a force of 75 kN at the other end. The beam is loaded as shown below and the mass of the beam produces a force of 30kN/m. The beam is 4 m long, and the bending moment  $M$  at any distance  $x$  from the simply supported end is given by

$$M = 75x - 20(x - 1.25) - 30x\frac{x}{2}.$$

Determine the position of the maximum bending moment and its magnitude.

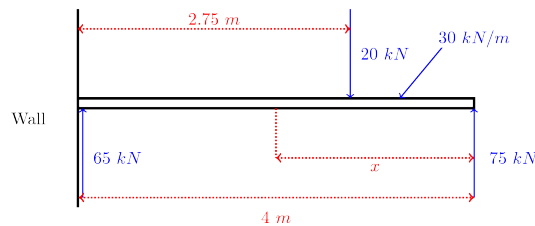


Figure 4.5: maximum bending moment

$$\begin{aligned}M &= 75x - 20(x - 1.25) - 30x\frac{x}{2} \\ \frac{dM}{dx} &= 75 - 20 - 30x = 55 - 30x\end{aligned}$$

For a turning point,  $dM/dx = 0$  so  $55 - 30x = 0$  and  $x = 1.83 \text{ m}$

$d^2M/dx^2 = -30$ , which is always negative, so the turning point is a maximum.

The bending moment  $M$  when  $x = 1.83$ , is

$$M = 75 \times 1.83 - 20(1.83 - 1.25) - \frac{30 \times 1.83^2}{2}$$

Or  $M = 75 \text{ kNm}$

The maximum bending moment occurs at  $1.83 \text{ m}$  from the simply supported end and has a value of  $75 \text{ kNm}$

**Example 4.6.** Given a cylindrical can with surface area  $S = 100 \text{ m}^2$ , height  $h$  and radius  $r$ ,

1. prove that the volume  $V = 50r - \pi r^3$ ;
2. find the value of  $r$  that maximises the volume.

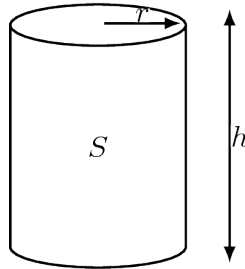


Figure 4.6: maximum volume

Let  $A = \pi r^2$  be the surface area of the top of the can.

Notice that  $V = Ah = \pi r^2 h$ . So

$$\begin{aligned} S &= 2A + \text{area of sides} \\ &= 2\pi r^2 + 2\pi r h = 100 \end{aligned}$$

$$h = \frac{100 - 2\pi r^2}{2\pi r} = \frac{50 - \pi r^2}{\pi r}$$

$$V = \pi r^2 \left( \frac{50 - \pi r^2}{\pi r} \right) = 50r - \pi r^3$$

We need to find the critical points and so we find  $dV/dr$  and equate to 0.

$$\frac{dV}{dr} = 50 - 3\pi r^2 = 0 \text{ giving } r = \sqrt{\frac{50}{3\pi}}.$$

Now

$$\frac{d^2V}{dr^2} = -6\pi r \quad \text{and so} \quad \left. \frac{d^2V}{dr^2} \right|_{r=\sqrt{50/3\pi}} = -6\pi\sqrt{\frac{50}{3\pi}} < 0.$$

By the 2nd derivative test we see that  $r = \sqrt{50/3\pi}$  is a local maximum.

### 4.3 Further forms of differentiation

#### 4.3.1 Differentiation of implicit functions

An implicit function is one of a form such as

$$y^2 + x^2 + 2xy + 2x = 0$$

where  $y = f(x)$ . Thus the whole expression is a function of  $x$  even though we don't know what function of  $x$   $y$  represents.

As the function  $f(x)$  is unknown, we cannot replace  $y$  by  $f(x)$  in order to differentiate. However, we can find the differential of the terms with  $y$  in by using the chain rule.

To differentiate  $y^2$  with respect to  $x$  we write

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 2y \frac{dy}{dx}.$$

To differentiate  $2xy$  we note that if  $f(x)$  was known, this would be a product of two functions of  $x$  so using the product rule

$$\frac{d}{dx}(2xy) = 2x \frac{d}{dx}(y) + 2y \frac{d}{dx}(x) = 2x \frac{dy}{dx} + 2y.$$

Hence if

$$y^2 + x^2 + 2xy + 2x = 0$$

then the differential with respect to  $x$  is

$$2y \frac{dy}{dx} + 2x + 2x \frac{dy}{dx} + 2y + 2 = 0.$$

Which may be rearranged so that

$$\begin{aligned} 2y \frac{dy}{dx} + 2x \frac{dy}{dx} &= -2x - 2y - 2 \\ \frac{dy}{dx}(2y + 2x) &= -2(x + y + 1) \\ \frac{dy}{dx} &= \frac{-(x + y + 1)}{x + y}. \end{aligned}$$

**Example 4.7.** Differentiate with respect to  $x$ ,  $10x^2 + 12xy + 10y^2 = 1$  where  $y = f(x)$ . Hence find the gradient of the tangents to the curve when  $x = 0.2$ .

*Solution.* This is actually the equation of an ellipse.

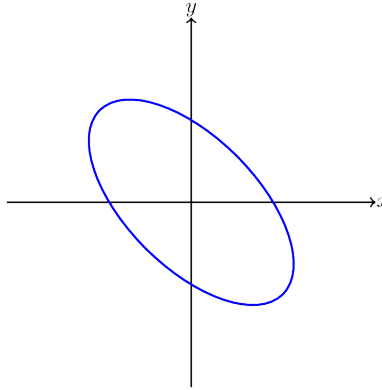


Figure 4.7: ellipse  $10x^2 + 12xy + 10y^2 = 1$

$$\begin{aligned}
 10x^2 + 12xy + 10y^2 &= 1 \\
 20x + 12x \frac{dy}{dx} + 12y + 20y \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= \frac{-20x - 12y}{12x + 20y} = -\frac{5x + 3y}{3x + 5y}.
 \end{aligned}$$

When  $x = 0.2$  then

$$10y^2 + \frac{12}{5}y - \frac{3}{5} = 0 \text{ and so } y = \frac{-\frac{12}{5} \pm \sqrt{\left(\frac{12}{5}\right)^2 + 4 \times 10 \times \frac{3}{5}}}{20} = 0.153 \text{ and } -0.393$$

So

$$\left. \frac{dy}{dx} \right|_{x=-0.153} = -\frac{5 \times 0.2 + 3 \times 0.153}{3 \times 0.2 + 5 \times 0.153} = -1.069$$

and

$$\left. \frac{dy}{dx} \right|_{x=-0.393} = -\frac{5 \times 0.2 - 3 \times 0.393}{3 \times 0.2 - 5 \times 0.393} = -0.091$$

### 4.3.2 Differentiation in parameters

If a function is expressed in parametric form it is convenient to be able to differentiate it in this form rather than having to convert to Cartesian form first.

A function expressed in parametric form has a third variable e.g.

$$y = f(t), x = g(t)$$

where  $t$  is a parameter.

Now using the chain rule again

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \text{ and so } \frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}.$$

Thus if we can find  $dy/dt$  and  $dx/dt$  we can establish  $dy/dx$ .

**Example 4.8.** A function is expressed in parametric form as  $y = 2t + t^2, x = t^2 - 1$ , where  $t$  is a parameter. Find  $dy/dx$ .

*Solution.*

$$\begin{aligned} \frac{dy}{dt} &= 2 + 2t \\ \frac{dx}{dt} &= 2t \\ \frac{dy}{dx} &= \frac{dy}{dt} / \frac{dx}{dt} = \frac{2 + 2t}{2t} = \frac{1 + t}{t}. \end{aligned}$$

### 4.3.3 Logarithmic differentiation

This is a convenient way to deal with the differential of a product or quotient with more than two terms.

Consider

$$y = \frac{(x-4)(x+5)}{(x+3)(x-2)}$$

Taking logs of both sides gives

$$\ln(y) = \ln(x-4) + \ln(x+5) - \ln(x+3) - \ln(x-2)$$

Differentiating and treating  $\ln(y)$  as a function of a function gives

$$\begin{aligned} \frac{1}{y} \times \frac{dy}{dx} &= \frac{1}{x-4} + \frac{1}{x+5} - \frac{1}{x+3} - \frac{1}{x-2} \\ \frac{dy}{dx} &= y \times \left( \frac{1}{x-4} + \frac{1}{x+5} - \frac{1}{x+3} - \frac{1}{x-2} \right) \\ &= \left( \frac{1}{x-4} + \frac{1}{x+5} - \frac{1}{x+3} - \frac{1}{x-2} \right) \frac{(x-4)(x+5)}{(x+3)(x-2)}. \end{aligned}$$

**Example 4.9.** Differentiate with respect to  $x$ ,  $y = x^x$ .

*Solution.*

$$\begin{aligned}\ln(y) &= x \ln(x) \\ \frac{1}{y} \times \frac{dy}{dx} &= \ln(x) + x \times \frac{1}{x} = \ln(x) + 1 \\ \frac{dy}{dx} &= y(\ln(x) + 1) = x^x(\ln(x) + 1).\end{aligned}$$

## Week 4 Exercises

### 4.4 Sheet 10 - Curve Sketching using differentiation

For each of the given functions find the  $y$ -intercept,  $x$ -intercepts, local maxima, local minima and points of inflection. Sketch the graph of the function clearly indicating these points.

1.  $y = x^2 + 3x - 2$ .
2.  $y = x^3 + 2x + 3$ .
3.  $f(x) = x^4 - 6x^2$ .
4.  $f(x) = x^5 - 5x^4$ .
5.  $g(x) = \sin(x)$ .
6.  $g(x) = \cos(x)$ .
7.  $y = 1/(x + 1)$ .
8.  $y = 1/(x^2 + 1)$ .
9.  $f(x) = e^x$ .
10.  $h(x) = \ln(x)$ .

[Solutions: 1.  $(0, -2), (-3.56, 0), (0.56, 0), (-3/2, -17, 4)$  is a minimum; 2.  $(0, 3), (-1, 0), (0, 3)$  is point of inflection; 3.  $(0, 0), (0, 0), (\pm\sqrt{6}, 0), (0, 0)$  is a maximum,  $(\pm\sqrt{3}, -9)$  are minima,  $(\pm 1, -5)$  are points of inflection; 4.  $(0, 0), (0, 0), (5, 0), (0, 0)$  is a maximum,  $(4, -256)$  is a minimum,  $(0, 0), (3, -162)$  are points of inflection; 5.  $(0, 0), (0, 0), (\pi, 0), (\pi/2, 1)$  is a maximum,  $(3\pi/2, -1)$  is a minimum; 6.  $(0, 1), (\pi/2, 0), 3\pi/2, 0), (0, 1)$  is a maximum,  $(\pi, -1)$  is a minimum; 7.  $(0, 1)$ ; 8.  $(0, 1), (0, 1)$  is a maximum,  $(\pm\sqrt{1/3}, 3/4)$  are points of inflection; 9.  $(0, 1)$ ; 10.  $(1, 0)$ ]

## 4.5 Sheet 11 - Problems involving maxima & minima

### Exercise 1

A beam 3 m long whose weight is 2 kN, is simply supported at both ends and the bending moment at a distance  $x$  m from one end is given by

$$m = 2(3x - x^2) \text{ kNm}$$

Prove that the bending moment is a maximum in the middle of the beam and calculate its value.

### Exercise 2

If 150 m of fencing is to be used to enclose three sides of a rectangular plot of land, whilst the remaining side is to be a river bank, determine the lengths of the sides such that the area enclosed will be a maximum.

### Exercise 3

A totally enclosed cylinder of radius  $r$ , has a surface area of  $100m^2$ . Prove that the volume of the shape,  $V$ , is given by  $V = 50r - \pi r^3$  and determine the value of  $r$  which maximize the volume.

### Exercise 4

Prove that a rectangle of fixed area has its minimum perimeter when it becomes a square.

### Exercise 5

An arch of a bridge is designed as a rectangle surmounted by a semicircle. If the area of the archway is  $100 m^2$ , determine the radius of the semicircle such that the perimeter of the edge has a minimum value.

### Exercise 6

Find the least area of metal required to make a cylindrical container from thin metal sheet in order that it might have a capacity of  $2000\pi cm^3$ .

[Solutions: 1.  $3/2, 18/4$ ]; 2.  $x = 37.5, y = 75$ ; 3.  $r = \sqrt{50/(3\pi)}$ ; 5.  $r = \sqrt{100/(2 + \pi/2)}$ ; 6.  $600\pi$ ]



## 4.6 Sheet 11A - Logarithmic Differentiation

Use logarithmic differentiation to find  $dy/dx$  when

1.  $y = (5x + 2)(3x - 7)$ .
2.  $y = \frac{3x+5}{(x-3)(x+4)}$ .
3.  $y = \frac{3x+5}{x^2+x-12}$ .
4.  $y = \frac{x^2-12x+26}{(x-2)(x-3)(x-4)}$ .
5.  $y = \frac{x+3}{(x+2)^2(x-1)}$ .
6.  $y = \frac{7x^2+9x-1}{(3x-2)^4}$ .
7.  $y = \frac{(x+2)^2(2x-5)}{(6x+5)^3(x^3+4)^2}$ .
8.  $y = x^{\sin(x)}$ .
9.  $y = x^x$ .
10.  $y = a^x$ .

[Solutions: 1.  $(5x + 2)(3x - 7) (5/(5x + 2) + 3/(3x - 7))$ ; 2.  $(3x + 5)/((x - 3)(x + 4)) (3/(3x + 5) - 1/(x - 3) - 1/(x + 4))$ ; 3.  $(3x + 5)/(x^2 + x - 12) (3/(3x + 5) - (2x + 1)/(x^2 + x - 12))$ ; 4.  $(x^2 - 13x + 26)/((x - 2)(x - 3)(x - 4)) ((2x - 13)/(x^2 - 13x + 26) - 1/(x - 2) - 1/(x - 3) - 1/(x - 4))$ ; 5.  $(x + 3)/((x + 2)^2(x - 1)) (1/(x + 3) - 2/(x + 2) - 1/(x - 1))$ ; 6.  $(7x^2 + 9x - 1)/(3x - 2)^4 ((14x + 9)/(7x^2 + 9x - 1) - 12/(3x - 2))$ ; 7.  $(x + 2)^2(2x - 5)/((6x + 5)^3(x^3 + 4)^2) (2/(x + 2) + 1/(2x - 5) - 18/(6x + 5) - 6x^2/(x^3 + 4))$ ; 8.  $x^{\sin(x)} (\cos(x) \ln(x) + \sin(x)/x)$ ; 9.  $x^x (\ln(x) + 1)$ ; 10.  $a^x \ln(a)$ ]

## 4.7 Sheet 11B - Further Differentiation

### Exercise 1

Differentiate the following implicit functions

1.  $x^2 + y^2 - 5x - 6y + 8 = 0$ ;
2.  $3x^2 + 2xy + 3y^2 = 0$ ;
3.  $x^3 + y^3 + 8x^2y = 25$ .

### Exercise 2

Find  $dy/dx$  for the following functions quoted in parametric form where  $t$  and  $\theta$  are parameters and  $a$  is a constant.

1.  $y = 2at, x = at^2$ ;
2.  $y = t^2, x = 2t$ ;

$$3. y = 3 \cos(\theta) - \sin^3(\theta), x = \cos^3(\theta).$$

### Exercise 3

Obtain  $dy/dx$  for each function by writing it in logarithmic form

1.  $y = \frac{x^2 \sin(x)}{\cos(2x)}$ ;
2.  $y = x^5 e^{3x} \tan(x)$ ;
3.  $y = \frac{(3x+2) \cos(2x)}{e^{2x}}$ .

[Solutions: 1(i)  $(2y - 6)dy/dx + 2x - 5 = 0$ , (ii)  $(2x + 6y)dy/dx + 6x + 2y = 0$ , (iii)  $(3y^2 + 8x^2)dy/dx + 3x^2 + 16xy = 0$ ; 2(i)  $dy/dx = 1/t$ , (ii)  $dy/dx = t$ , (iii)  $dy/dx = (1 + \sin(\theta) \cos(\theta))/\cos^2(\theta)$ ; 3 (i)  $dy/dx = x^2 \sin(x)/\cos(x) (2/x + \cos(x)/\sin(x) + 2 \sin(2x)/\cos(2x))$ , (ii)  $dy/dx = x^5 e^{3x} \tan(x) (5/x + 3 + \sec^2(x)/\tan(x))$ , (iii)  $dy/dx = ((3x + 2) \cos(2x))/e^{2x} (3/(3x + 2) - 2 \sin(2x)/\cos(2x) - 2)$ ]

# 5

## Week 5

We first of all consider functions that have more than one variable e.g.  $z = f(x, y)$  and look at differentiation with respect to either of the variables.

We then move on to look at the opposite process to differentiation, called integration. Given the derivative of a function, can we ‘reconstruct’ the function that gave rise to that derivative?

### 5.1 Partial Differentiation

If an expression is a function of two variables then it can have two differential coefficients.

For example, if

$$z(x, y) = x^2 + xy + y^2$$

then we can fix, then value of the variable  $y$  and consider this as a function of the variable  $x$  only.

$$z(x, 2) = x^2 + 2x + 4.$$

It then make sense to differentiate this resultant function with respect to  $x$  to get  $dz(x, 2)/dx = 2x + 2$ .

Alternatively, we can fix the value of the variable  $x$  and consider it as a function of  $y$  only.

$$z(-1, y) = 1 - y + y^2.$$

The derivative of this with respect to  $y$  is then  $dz(-1, y)/dy = -1 + 2y$ .

In general, if we think of  $y$  as being a fixed constant (let's not worry about the actual value) then we can differentiate  $z$  with respect to  $x$ . We use the following notation

$$\frac{\partial z}{\partial x} = 2x + y.$$

In the same way, thinking of  $x$  as being a fixed constant and differentiating with respect to  $y$  we write

$$\frac{\partial z}{\partial y} = x + 2y.$$

Consider the expression for the volume of a cylinder,  $V = \pi r^2 h$  where  $r$  is the radius of the cylinder and  $h$  is its height.

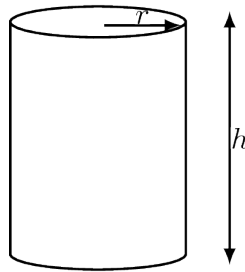


Figure 5.1: cylinder

The differential coefficients can be

$$\begin{aligned} \text{treating } h \text{ as a constant } \frac{\partial V}{\partial r} &= 2\pi r h \\ \text{treating } r \text{ as a constant } \frac{\partial V}{\partial h} &= \pi r^2 \end{aligned}$$

We use the symbol  $\partial$  in the differential coefficient to indicate that there is more than one variable by which the expression may be differentiated and that we have chosen to treat any other variables as constants while differentiating.

The first differential coefficient tells us how the volume would vary as the radius was varied for a cylinder of constant height. The second differential coefficient tells us how the volume would vary as the height varied for a cylinder of constant radius.

When a function of more than one variable is differentiated with respect to one variable only, this process is called *partial differentiation*.

**Example 5.1.** From the function  $z = x^3y^2$ , determine  $\partial z/\partial x$  and  $\partial z/\partial y$ .

*Solution.*

$$\frac{\partial z}{\partial x} = 3x^2y^2, \quad \frac{\partial z}{\partial y} = 2yx^3.$$

**Example 5.2.** From the function  $z = 3x^5 - 5y^3 + 6$ , determine  $\partial z/\partial x$  and  $\partial z/\partial y$ .

*Solution.*

$$\frac{\partial z}{\partial x} = 15x^4, \quad \frac{\partial z}{\partial y} = -15y^2.$$

### 5.1.1 Successive partial derivatives

If a second partial derivative is required and there are, say, two variables, then a second derivative may be made with each of these variables.

**Example 5.3.**

$$z = 3x^2 + 2x^2y^2 + 3y^2.$$

*Solution.*

$$\frac{\partial z}{\partial x} = 6x + 4xy^2, \quad \frac{\partial z}{\partial y} = 4x^2y + 6y.$$

So

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = 6 + 4y^2$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = 6 + 4x^2$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y} = 8xy$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x} = 8xy$$

**Example 5.4.**

$$z = x \cos(y) - y \cos(x)$$

*Solution.*

$$\frac{\partial z}{\partial x} = \cos(y) + y \sin(x), \quad \frac{\partial z}{\partial y} = -x \sin(y) - \cos(x).$$

So

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = y \cos(x)$$

$$\begin{aligned}\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) &= \frac{\partial^2 z}{\partial y^2} = -x \cos(y) \\ \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) &= \frac{\partial^2 z}{\partial x \partial y} = -\sin(y) + \sin(x) \\ \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) &= \frac{\partial^2 z}{\partial y \partial x} = -\sin(y) + \sin(x)\end{aligned}$$

## 5.2 Integration as the opposite of differentiation

Given a function  $f(x)$  we are used to thinking of the **inverse function**, that is a function  $g(x)$  such that  $f(g(x)) = x$ . For example if  $f(x) = x^2$  then  $g(x) = \sqrt{x}$  so that  $f(g(x)) = (\sqrt{x})^2 = x$ . The inverse ‘undoes’ what the original function does.

Integration is the converse of differentiation. That is, having been given the derivative of a function, we are asking; what was the original function that was differentiated to get that derivative?

The symbol for integration is  $\int$  which is the Old English symbol for “s”. The symbol is used to indicate that integration is a summation process. We shall see why later.

### 5.2.1 Integration by rule

Suppose  $dy/dx = 2x$ , then the instruction “integrate the function with respect to  $x$  is written

$$\int 2x dx$$

We know that to get

$$\frac{dy}{dx} = 2x$$

we could have started with  $y = x^2$  so this is one possible solution.

However, we could also have started with  $y = x^2 + 5$  or  $y = x^2 + 1$  or  $y = x^2 + 25$  or  $y = x^2 + 1245$ .

We can’t tell which because all information about the any constant value is lost when we differentiate. To make this clear we write

$$\int 2x dx = x^2 + c$$

where  $c$  is called an arbitrary constant.

A general rule for simple functions follows from the differentiation rules.

For differentiation we had if  $y = ax^n$  then  $dy/dx = anx^{n-1}$ .

For integration if  $dy/dx = ax^n$  then  $\int ax^n dx = y = ax^{n+1}/(n+1) + c, n \neq -1$

After a function has been integrated, the result of integration may be checked by differentiating the answer.

**Example 5.5.** Integrate the functions

1.  $y = x^3$ ,
2.  $y = 3x^2$ ,
3.  $y = 6x - 4$ ,
4.  $y = (8x^2 - 4x + 5)/x^4$ ,
5.  $y = x^{3/5}(2x - \sqrt{x})$ .

*Solution.*

1.  $\int x^3 dx = x^4/4 + c$ ,
2.  $\int 3x^2 dx = 3x^3/3 + c = x^3 + c$ ,
3.  $\int (6x - 4) dx = \int 6x dx - \int 4 dx = 6x^2/2 - 4x + c = 3x^2 - 4x + c$ ,
4.  $y = (8x^2 - 4x + 5)/x^4 = 8/x^2 - 4/x^3 + 5/x^4 = 8x^{-2} - 4x^{-3} + 5x^{-4}$  and so

$$\begin{aligned} \int y dx &= \int 8x^{-2} dx - \int 4x^{-3} dx + \int 5x^{-4} dx = \frac{8x^{-1}}{-1} - \frac{4x^{-2}}{-2} + \frac{5x^{-3}}{-3} + C \\ &= \frac{-1}{8x} + \frac{2}{x^2} - \frac{5}{3x^3} = \frac{-3x^2 + 48x - 40}{24x^3} + C, \end{aligned}$$

5.  $y = x^{3/5}(2x - \sqrt{x}) = 2x^{8/5} - x^{11/10}$  and so

$$\int y dx = \int 2x^{8/5} dx - \int x^{11/10} dx = \frac{2x^{13/5}}{13/5} - \frac{x^{21/10}}{21/10} + C = \frac{10}{13}x^{13/5} - \frac{10}{21}x^{21/10} + C.$$

Note that when a sum of several terms is integrated, as in example (3), the result is the sum of the integrals of each of the separate terms. In fact we have the following rules.

- Summation Rule:  $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$ .

- Coefficient Rule:  $\int af(x)dx = a \int f(x)dx.$
- Power Rule:  $\int x^n dx = x^{n+1}/(n+1) + c.$

### 5.3 Indefinite Integration

The result of an integration which has the arbitrary constant stated as  $c$  (as for the examples above) is called an *indefinite integration* because it does not tell us which of an infinite number of possible curves we are dealing with. If we want to know the value of  $c$  we need more information such as a pair of co-ordinates through which our particular curve passes. Such additional information is called a *boundary condition*.

**Example 5.6.** Find  $y$ , the integral of  $4x + 5$  and evaluate the arbitrary constant given that the boundary condition is: When  $x = 1, y = 3$ .

*Solution.*

$$y = \int (4x + 5)dx = \frac{4x^2}{2} + 5x + c = 2x^2 + 5x + c$$

Now we use the condition that when  $x = 1$  then  $y = 3$  so that

$$3 = 2 \times 1^2 + 5 \times 1 + c = 7 + c.$$

Hence  $c = 3 - 7 = -4$  and therefore

$$y = 2x^2 + 5x - 4.$$

**Example 5.7.** Find  $y$ , the integral of  $3\sqrt{x} - 2/x^2$  and evaluate the arbitrary constant given that the boundary condition is: When  $x = 1, y = 4$ .

*Solution.*

$$\begin{aligned} y &= \int (3\sqrt{x} - 2/x^2)dx = 3 \int x^{1/2}dx - 2 \int x^{-2}dx \\ &= \frac{3x^{3/2}}{3/2} - \frac{2x^{-1}}{-1} + C = 2x^{3/2} + \frac{2}{x} + C. \end{aligned}$$

Now the boundary condition tells us that when  $x = 1, y = 4$  and so  $4 = 2 \times 1^{3/2} + 2/1 + C = 2 + 2 + C$ . Hence  $C = 0$  and so

$$y = 2x^{3/2} + \frac{2}{x}.$$



### 5.3.1 Standard integrals

For differentiation we had some standard derivatives which could be quoted without explanation. The same is true for integration. A list is given below.

derivative	integral
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln(x) + c, x > 0$
$\frac{d}{dx}(e^x)dx = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}(e^{kx})dx = ke^{kx}$	$\int e^{kx} dx = \frac{e^{kx}}{k} + c$
$\frac{d}{dx}(a^x)dx = a^x \ln(a)$	$\int a^x dx = \frac{a^x}{\ln(a)} + c$
$\frac{d}{dx}(\sin(x))dx = \cos(x)$	$\int \sin(x)dx = -\cos(x) + c$
$\frac{d}{dx}(\cos(x))dx = -\sin(x)$	$\int \cos(x)dx = \sin(x) + c$
$\frac{d}{dx}(\tan(x))dx = \sec^2(x)$	$\int \sec^2(x)dx = \tan(x) + c$

**Example 5.8.** Find  $y$ , the integral of  $\cos(2x)$ .

*Solution.* Notice that

$$\frac{d}{dx} \sin(2x) = 2 \cos(2x)$$

and so

$$\int \cos(2x) dx = \frac{1}{2} \sin(2x).$$

In general if

$$\int f(x) dx = g(x) \text{ then } \int f(kx) dx = \frac{1}{k} g(kx).$$

**Example 5.9.** Find  $y$ , the integral of  $2 \sin(x) - \sec^2(x) + 1/\pi$  and evaluate the arbitrary constant given that the boundary condition is: when  $x = \pi/4, y = 0$ .

*Solution.*

$$\begin{aligned} y &= \int (2 \sin(x) - \sec^2(x) + \frac{1}{\pi}) dx = 2 \int \sin(x) dx - \int \sec^2(x) dx + \frac{1}{\pi} \int 1 dx \\ &= -2 \cos(x) - \tan(x) + \frac{x}{\pi} + C \end{aligned}$$

Now the boundary condition tells us that when  $x = \pi/4, y = 0$  and so  $0 = -2 \cos(\pi/4) - \tan(\pi/4) + \pi/(4\pi) + C = -\sqrt{2} - 1 + 1/4 + C$ . Hence  $C = \sqrt{2} + 3/4$  and so

$$y = -2 \cos(x) - \tan(x) + \frac{x}{\pi} + \sqrt{2} + \frac{3}{4}.$$

**Example 5.10.** Calculate  $\int \sin^2(x) dx$ .

*Solution.* This is not one of our standard integrals so at first sight it would appear we cannot integrate this function (as yet). However, recall that

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2 \sin^2(x).$$

Hence

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \text{ and so } \int \sin^2(x) dx = \frac{1}{2} \int (1 - \cos(2x)) dx$$

Now by Example 5.8 we know that  $\int \cos(2x) dx = \frac{1}{2} \sin(2x) + C$ . Hence

$$\int \sin^2(x) dx = \frac{1}{2} \int (1 - \cos(2x)) dx = \frac{1}{2} \left( x - \frac{1}{2} \sin(2x) \right) + C = \frac{2x - \sin(2x)}{4} + C.$$

Notice that we can check we are correct by differentiation:

$$\frac{d}{dx} \left( \frac{2x - \sin(2x)}{4} \right) = \frac{1}{4} (2 - 2 \cos(2x)) = \frac{1}{2} (1 - \cos(2x)) = \sin^2(x).$$

# Week 5 Exercises

## 5.4 Sheet 12 - Partial Differentiation

For each of the functions below, obtain where possible

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y}$$

1.  $z = 3x^2 + 4xy - 5y^2$ .
2.  $z = xy^2 + 1/y - y/x^2$ .
3.  $z = \sin(2x) \cos(2y)$ .
4.  $z = x \cos(y) - y \cos(x)$ .
5.  $z = \sin(3x + 2y)$ .

## 5.5 Sheet 12A - Indefinite Integration

Integrate the following functions with respect to the variable

1. $x^2 + 3 - 1/x$	2. $x + 1/\sqrt{x}$
3. $(4x^3 - 3x)/\sqrt{x}$	4. $1/x^3 + x^3$
5. $(x + 1)^2$	6. $(1 + \sqrt{x})^2$
7. $5x^5 - \sqrt{x}$	8. $e^{0.2x} + x$
9. $\cos(5x)$	10. $\sin(ax) + \cos(bx)$
11. $(t + 1)(3 + t)$	12. $\sqrt{r}(1 + r)$
13. $(2 + \sqrt{t})^2$	14. $\cos(3x) - \sin(2x)$
15. $\sin(100\pi t) - \cos(100\pi t)$	16. $\sec^2(x) - \sin(x)$
17. $\sec^2(\theta) + \cos(3\theta)$	18. $4\sec^2(x) - \operatorname{cosec}^2(x)$
19. $4/x - 1/x^2$	20. $e^{2x} - 1/x$
21. $\sin(x/2)$	22. $e^x + 2/x - 3/e^x$
23. $(t + 1)/t$	24. $(x + 1/x)^2$
25. $1/(2x)$	26. $(e^{3x} - e^{-3x})/2$
27. $\operatorname{cosec}^2(x) - \sin(x)$	28. $v^{-1.2} - v^{-1}$

## 6

# Week 6

We continue our study of integration by looking at two very important techniques to calculate the integral of certain functions that are difficult to integrate by other means - integration by substitution and integration by parts.

### 6.1 Integration by Substitution

So far we have learnt to integrate simple functions such as  $x^2 + 2x + 3$  and standard integrals such as  $\sin(x)$ ,  $e^x$  etc. If we want to integrate more complicated functions, more advanced techniques will be needed. One very useful method is *integration by substitution*. Consider

$$\int x\sqrt{x^2 + 1}dx. \quad (1)$$

We cannot integrate this directly, and if we try making a binomial expansion of the part under the square root sign and then multiplying by  $x$ , we will only get an approximate answer. However, if we make a substitution, we can make the integral look like something we can integrate.

Let  $u = x^2 + 1$  then (1) becomes

$$\int x\sqrt{u}dx = \int xu^{1/2}dx. \quad (2)$$

So far, this doesn't look much better, but if we now try to integrate with respect to  $u$  instead of  $x$ , we will find the substitution has helped. Before we can integrate with respect to  $u$ , we must replace  $dx$  with  $du$ .

If  $u = x^2 + 1$ , then  $du/dx = 2x$  and so  $du = 2xdx$  or  $du/2 = xdx$ . If we substitute for  $dx$  in (2) we get

$$\int u^{1/2} x dx = \int \frac{1}{2} u^{1/2} du.$$

Now our function is expressed entirely in terms of  $u$  and we can integrate with respect to  $u$  to give

$$\int \frac{1}{2} u^{1/2} du = \frac{1}{2} \frac{2}{3} u^{3/2} + c = \frac{1}{3} u^{3/2} + c.$$

Finally we can substitute for  $u$  to give the answer in terms of  $x$

$$\frac{1}{3} u^{3/2} + c = \frac{1}{3} (x^2 + 1)^{3/2} + c.$$

**Example 6.1.** Evaluate  $\int \sin^2(x) \cos(x) dx$ .

*Solution.* Let  $u = \sin(x)$  so that  $du/dx = \cos(x)$ . Then  $dx = du/\cos(x)$  and so

$$\int \sin^2(x) \cos(x) dx = \int u^2 du = \frac{1}{3} u^3 + c.$$

Hence

$$\int \sin^2(x) \cos(x) dx = \frac{1}{3} \sin^3(x) + c.$$

There are many functions where making a substitution helps us to integrate. In all cases, the method for finding the integral has the general form of the examples above. One common class of problems is those where we have a product and one half of the product is the differential of the other half. The examples above belong to this class. A second class is a quotient where the numerator is the differential of the denominator.

**Example 6.2.** Find

$$\int \frac{3x^2}{x^3 - 4} dx.$$

*Solution.* Let  $u = x^3 - 4$  so that  $du = 3x^2 dx$ . Then

$$\int \frac{3x^2}{x^3 - 4} dx = \int \frac{du}{u} = \ln |u| + c = \ln |x^3 - 4| + c.$$

For integrals of this class, the result is always the natural log of the denominator.

### 6.1.1 Trig substitutions

Often a difficult integral may be made simpler by the use of trig identities. Knowing which identities will help when is a matter of practise.

**Example 6.3.** Find

$$\int \sin^2(x) dx.$$

*Solution.* Using the trig identity for  $\cos(2x)$  we see that  $\sin^2(x) = (1 - \cos(2x))/2$ . So

$$\int \sin^2(x) dx = \int \frac{1}{2} - \int \frac{\cos(2x)}{2} = \frac{x}{2} - \frac{\sin(2x)}{4} + c.$$

**Example 6.4.** Find

$$\int \frac{dx}{\sqrt{a^2 - x^2}}.$$

*Solution.* Let  $x = a \sin(\theta)$  then  $dx = a \cos(\theta) d\theta$ . So  $x^2 = a^2 \sin^2(\theta)$  and  $\theta = \sin^{-1}(x/a)$ . Hence

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos(\theta) d\theta}{\sqrt{a^2 - a^2 \sin^2(\theta)}} = \int \frac{a \cos(\theta) d\theta}{a \sqrt{1 - \sin^2(\theta)}} = \int \frac{\cos(\theta) d\theta}{\cos(\theta)} = \int d\theta = \theta + c.$$

Hence

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c.$$

## 6.2 Integration by Parts

To integrate a product of two terms where one half is the differential of the other, we have seen that integration by substitution may be used. When we have a product where one half is not the differential of the other, integration by parts may be useful. The rule for integration by parts is derived from the rule for differentiating a product

If  $y = uv$ , where  $u$  and  $v$  are both functions of  $x$  then,

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \quad (1)$$

If we integrate both sides of equation (1) with respect to  $x$  then we have

$$\int \frac{dy}{dx} dx = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx \quad (2)$$

Now the integral of the left-hand side is given by

$$\int \frac{dy}{dx} dx = \int dy = y$$

and  $y = uv$  so

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx \quad (3)$$

and therefore, rearranging gives us

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \quad (4)$$

Equation (4) is the formula for **integration by parts** and can be used to integrate some types of product.

**Example 6.5.** Find  $\int x^3 \ln(x) dx$

*Solution.* This is a product but one half is not the derivative of the other, so integration by parts must be used.

First choose one part of the product and let it be  $u$ , then the other half is  $dv/dx$ .

So, let  $u = \ln(x)$  and  $dv/dx = x^3$ . Then  $du/dx = 1/x$  and  $v = \int x^3 dx = x^4/4$ .

Then substituting for  $u, du/dx, v$  and  $dv/dx$  in equation (4) we get

$$\begin{aligned} \int u \frac{dv}{dx} dx &= uv - \int v \frac{du}{dx} dx \\ \int x^3 \ln(x) dx &= \ln(x) \times \frac{x^4}{4} - \int \left( \frac{x^4}{4} \times \frac{1}{x} \right) dx \\ &= \left( \frac{x^4}{4} \ln(x) \right) - \int \frac{x^3}{4} dx = \left( \frac{x^4}{4} \ln(x) \right) - \left( \frac{x^4}{16} \right) + c \\ \int x^3 \ln(x) dx &= \frac{x^4}{4} \left( \ln(x) - \frac{1}{4} \right) + c. \end{aligned}$$

By choosing the terms we did for  $u$  and  $dv/dx$ , the formula helped us simplify what had to be integrated. We could have chosen  $u$  as  $x^3$  and  $dv/dx$  as  $\ln(x)$ , but if we had, we would have found that after substituting our terms into the formula, the integral was no more simple than before.

Knowing which term to choose for which part of the formula is on the whole a question of practice.



**Example 6.6.** Find  $\int x e^{-x} dx$

*Solution.* Let  $u = x$  and  $dv/dx = e^{-x}$  so that  $du/dx = 1$  and  $v = \int e^{-x} dx = -e^{-x}$ . So using integration by parts

$$\begin{aligned}\int u \frac{dv}{dx} dx &= uv - \int v \frac{du}{dx} dx \\ \int x e^{-x} dx &= x(-e^{-x}) - \int -e^{-x} dx \\ &= -x e^{-x} + \int e^{-x} dx \\ &= -x e^{-x} - e^{-x} + c.\end{aligned}$$

**Example 6.7.** Find  $\int \ln(x) dx$

*Solution.* At first sight this doesn't look as if integration by parts will be useful as it doesn't seem to involve a product. However we can think of it as  $\int 1 \times \ln(x) dx$ .

Let  $u = \ln(x)$  and  $dv/dx = 1$  then  $du/dx = 1/x$  and  $v = x$ . hence using integration by parts we see that

$$\begin{aligned}\int u \frac{dv}{dx} dx &= uv - \int v \frac{du}{dx} dx \\ \int \ln(x) dx &= x \ln(x) - \int x \times \frac{1}{x} dx \\ &= x \ln(x) - \int 1 dx \\ &= x \ln(x) - x + c.\end{aligned}$$

Sometimes it is necessary to apply the formula twice (or more).

**Example 6.8.** Find  $\int x^2 \sin(2x) dx$

*Solution.* Let  $u = x^2$  and let  $dv/dx = \sin(2x)$ . Then  $du/dx = 2x$  and  $v = \int \sin(2x) dx = -\cos(2x)/2$ .

Using integration by parts.

$$\begin{aligned}\int u \frac{dv}{dx} dx &= uv - \int v \frac{du}{dx} dx \\ \int x^2 \sin(2x) dx &= \left( x^2 \times \frac{-\cos(2x)}{2} \right) - \int \left( \frac{-\cos(2x)}{2} \times 2x \right) dx \\ &= \left( \frac{-x^2 \cos(2x)}{2} \right) + \int x \cos(2x) dx\end{aligned}\quad (5)$$

We still have to integrate the term under the integral sign on the right-hand side to get the answer, and that term is a product with neither part being the differential of the other, so integration by parts should help again. Considering just that term:-

We have  $\int x \cos(2x) dx$ . Let  $u = x$  and  $dv/dx = \cos(2x)$ . Then  $du/dx = 1$  and  $v = \sin(2x)/2$ .

$$\int x \cos(2x) dx = \left( x \times \frac{\sin(2x)}{2} \right) - \int \frac{\sin(2x)}{2} \times 1 dx = \left( \frac{x \sin(2x)}{2} \right) + \left( \frac{\cos(2x)}{4} \right). \quad (6)$$

Now substituting (6) into (5) we can obtain the answer to the original integral

$$\int x^2 \sin(2x) dx = -\frac{x^2 \cos(2x)}{2} + \frac{x \sin(2x)}{2} + \frac{\cos(2x)}{4} + c.$$

# Week 6 Exercises

## 6.3 Sheet 13 Integration by substitution

By using suitable substitution, integrate the following

1. $\int (x+4)^2 dx$	2. $\int \sqrt{(2x+3)} dx$
3. $\int (2-x)^3 dx$	4. $\int e^{3x} dx$
5. $\int 1/(2x+1) dx$	6. $\int 1/(2x+1)^3 dx$
7. $\int (x^2+1)^5 2x dx$	8. $\int (x^2+1)^3 dx$
9. $\int \sqrt{(x^2+x)}(2x+1) dx$	10. $\int (ax^2+bx)^4(2ax+b) dx$
11. $\int (2x+1) \cos(x^2+x) dx$	12. $\int e^{5x^2-x}(10x-1) dx$
13. $\int \ln(x)/x dx$	14. $\int e^e/(1+e^x) dx$
15. $\int \cos(x/2+1) dx$	16. $\int 1/(x-1)^2 dx$
17. $\int_0^1 (2x-1)^7 dx$	18. $\int_3^5 3x\sqrt{x^2-9} dx$
19. $\int \sin(6x) \cos(5x) dx$	20. $\int \cos(2x) \sin(5x) dx$
21. $\int \cos^4(\theta) \sin(\theta) d\theta$	22. $\int \cos(\theta) (\sin^4(\theta+2)) d\theta$
23. $\int \cos(x)/\sin(x) dx$	24. $\int \sin(\theta)/\cos(\theta) d\theta$
25. $\int 1/(1+x^2) dx$	26. $\int 1/\sqrt{4-x^2} dx$
27. $\int \tan(\theta) \sec^2(\theta) d\theta$	28. $\int (6x+2)(3x^2+2x-9) dx$
29. $\int 4x^3/(x^4-1) dx$	30. $\int \sec^2(x)/\tan(x) dx$
31. $\int 1/\sqrt{2-x^2} dx$	32. $\int 1/\sqrt{9-2x^2} dx$
33. $\int 1/(5+4x^2) dx$	34. $\int \sqrt{4-x^2} dx$

## 6.4 Sheet 14 - Integration by parts

Integrate the following integrals using integration by parts

1.  $\int x \cos(x) dx$

2.  $\int x \ln(x) \, dx$
3.  $\int x \sin(4x) \, dx$
4.  $\int x(1+x)^{10} \, dx$
5.  $\int x^7 \ln(x) \, dx$
6.  $\int \sqrt{x} \ln(x) \, dx$
7.  $\int x^2 e^x \, dx$
8.  $\int x^2 \cos(x) \, dx$
9.  $\int x^2 e^{2x} \, dx$
10.  $\int 2x^2 e^{3x} \, dx$
11.  $\int x^3 \cos(3x) \, dx$
12.  $\int x e^{-x} \, dx$
13.  $\int e^{3x} \sin(2x) \, dx$
14.  $\int \ln(x) \, dx$
15.  $\int \log(x)/x^3 \, dx$

# 7

## Week 7

We finish our study of indefinite integrals by looking at integration by converting functions using partial fractions. We then consider definite integrals and their connection to areas under curves.

### 7.1 Integration using partial fractions

Consider the integral

$$\int \frac{x}{(x-1)(x-2)} dx$$

This is not a standard integral. Also, if we multiply out the denominator we get  $x^2 - 3x + 2$ . The derivative of this denominator is  $2x - 3$ . Clearly the numerator is not the derivative of the denominator and so making a substitution won't help us to integrate. However, if we split the quotient into its partial fractions, we may be able to integrate these.

We know that

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

and hence

$$x = A(x-2) + B(x-1)$$

Let  $x = 2$ , so that  $2 = A \times 0 + B \times 1$  i.e.  $B = 2$ .

Let  $x = 1$ , so that  $1 = A \times -1 + B \times 0$  i.e.  $A = -1$ .

So

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}$$

and thus

$$\int \frac{x}{(x-1)(x-2)} dx = \int \frac{-1}{x-1} dx + \int \frac{2}{x-2} dx$$

Now the integrals can be arranged so that the numerator of each is the derivative of the denominator

$$\int \frac{-1}{x-1} dx + \int \frac{2}{x-2} dx = -\ln|x-1| + 2\ln|x-2| + c = \ln\left|\frac{(x-2)^2}{x-1}\right| + c.$$

**Example 7.1.** Evaluate  $\int \frac{x^2+2}{x-2} dx$

*Solution.* This is an improper fraction so we must make a long division first

$$\begin{array}{r} x+2 \\ x-2 \overline{) x^2 \phantom{+ 2x} + 2} \\ \underline{-(x^2 - 2x)} \phantom{+ 2} \\ 2x + 2 \\ \underline{-(2x - 4)} \\ 6 \end{array}$$

So

$$\frac{x^2+2}{x-2} = x+2 + \frac{6}{x-2}$$

and hence

$$\int \frac{x^2+2}{x-2} dx = \int (x+2) dx + \int \frac{6}{x-2} dx = \frac{x^2}{2} + 2x + 6\ln|x-2| + c$$

**Example 7.2.** Evaluate  $I = \int \frac{x^3-3x+3}{(x-1)(x-2)(x-3)} dx$

*Solution.*

$$\frac{x^3-3x+3}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

and so

$$x^3 - 3x + 3 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Let  $x = 1$  so that  $1 = 2A$  or  $A = 1/2$ .

Let  $x = 2$  so that  $1 = -B$  or  $B = -1$ .

Let  $x = 3$  so that  $3 = 2C$  or  $C = 3/2$ .

$$\begin{aligned} I &= \int \frac{1}{2(x-1)} dx - \int \frac{1}{x-2} dx + \int \frac{3}{2(x-3)} dx \\ &= \frac{1}{2} \ln|x-1| - \ln|x-2| + \frac{3}{2} \ln|x-3| + c \end{aligned}$$

**Example 7.3.** Evaluate  $I = \int \frac{5x+2}{x^2-4x+4} dx$

*Solution.*

$$\frac{5x+2}{x^2-4x+4} = \frac{5x+2}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

and so

$$5x+2 = A(x-2) + B$$

Let  $x = 2$  so that  $12 = B$ .

$A(x-2) + 12 = 5x+2$  and so  $Ax + 10 - 2A = 5x$ , so that  $A = 5$ .

$$\begin{aligned} I &= \int \frac{5x+2}{x^2-4x+4} dx \\ &= \int \frac{5}{x-2} dx + \int \frac{12}{(x-2)^2} dx \\ &= 5 \ln|x-2| - \frac{12}{x-2} + c \end{aligned}$$

## 7.2 Definite Integrals

Let  $F(x) = \int f(x) dx$  E.g.  $F(x) = \int x dx = x^2/2 + C$ .

Define

$$\int_a^b f(x) dx = F(b) - F(a).$$

**Example 7.4.**

$$\begin{aligned}
 \int_1^4 (x^2 - 2x)dx &= \left[ \frac{x^3}{3} - x^2 + C \right]_1^4 \\
 &= \left( \frac{4^3}{3} - 4^2 + C \right) - \left( \frac{1^3}{3} - 1^2 + C \right) \\
 &= \frac{64}{3} - 16 - \frac{1}{3} + 1^2 = 6.
 \end{aligned}$$

Note that during the subtraction process the arbitrary constant disappears. This will always happen for a definite integral since we will always get  $+c$  for the upper limit minus  $+c$  for the upper limit. Thus for a definite integral we need not bother to write an arbitrary constant in at all.

**Example 7.5.** Determine the value of

$$\int_0^{\pi/2} \cos(x)dx.$$

*Solution.*

$$\begin{aligned}
 \int_0^{\pi/2} \cos(x)dx &= [\sin(x)]_0^{\pi/2} \\
 &= \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.
 \end{aligned}$$

**7.2.1 Area under a curve**

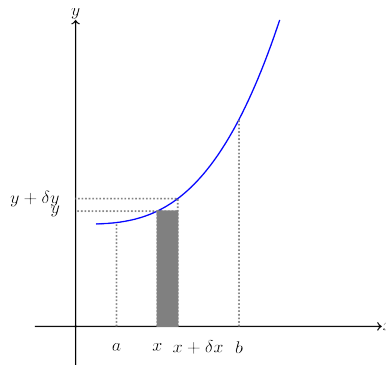
A most important application of the definite integral is for finding the area between a curve and an axis.

Consider the graph of  $y = f(x)$  and the area it encloses between the curve and the  $x$ -axis between the  $x$  values  $x = a$  and  $x = b$ .

At point  $x$ , the curve has a height  $y$  at point  $x + \delta x$  the curve has height  $y + \delta y$ .

The area of the shaded rectangle under the curve is  $y\delta x$ . As drawn, the area of the rectangle has a value quite close to the area under the curve between  $x$  and  $x + \delta x$ . If we chose  $\delta x$  to have a smaller value, then the area of the resulting rectangle would be nearer in size to the area between  $x$  and  $x + \delta x$ . The smaller we choose  $\delta x$  to be, the better the approximation to the area. In the limit as  $\delta x \rightarrow 0$ , the area of the rectangle is exactly the same as the area under the curve.



Figure 7.1:  $f(x)$ 

If we now consider the area between  $a$  and  $b$ , we can divide it up into lots of rectangles each of width  $\delta x$ . The total area,  $A$ , under the curve is the sum of the areas of all the rectangles and may be written as

$$A = \sum_{x=a}^{x=b} y \delta x.$$

where the sign  $\Sigma$  (sigma) means, work out the area of each rectangle in turn then add all the areas together. Now as  $\delta x \rightarrow 0$ , the total area becomes exactly the sum of all the small areas and we may replace the  $\delta$  with an integration sign and write

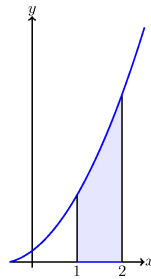
$$A = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} y \delta x = \int_a^b y \, dx.$$

This sometimes known as the *(Second) Fundamental Theorem of Calculus*.

**Example 7.6.** Determine the area between the curve  $y = 2x^2 + 3x + 1$  and the  $x$ -axis between the values of  $x = 1$  and  $x = 2$ .

*Solution.*

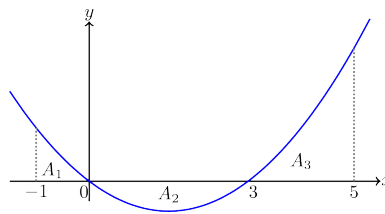
$$A = \int_1^2 (2x^2 + 3x + 1) dx = \left[ \frac{2x^3}{3} + \frac{3x^2}{2} + x \right]_1^2 = \left[ \frac{16}{3} + \frac{12}{2} + 2 \right] - \left[ \frac{2}{3} + \frac{3}{2} + 1 \right] = \frac{61}{6}.$$

Figure 7.2:  $2x^2 + 3x + 1$ 

### 7.2.2 Negative Areas

If the curve lies below the  $x$ -axis between  $a$  and  $b$ , then  $y$  is negative and  $y\delta x$  will have a negative value. As a result the area for regions below the  $x$ -axis will be negative. We are not suggesting that the area itself is negative, just that the presence of the minus sign after integration indicates an area below the  $x$ -axis. For this reason there is a small difference between the definite integral and the area under a curve.

Consider the curve  $y = x^2 - 3x$ . We would like to find the area enclosed between the  $x$ -axis and the values  $x = -1$  and  $x = 5$ .

Figure 7.3:  $x^2 - 3x$ 

The definite integral over this range is

$$\int_{-1}^5 (x^2 - 3x) dx = \left[ \frac{x^3}{3} - \frac{3x^2}{2} \right]_{-1}^5 = \left[ \frac{125}{3} - \frac{75}{2} \right] - \left[ \frac{-1}{3} - \frac{3}{2} \right] = 6.$$

However, this is not the area this is  $A_1 + A_2 + A_3$  where  $A_2$  is negative, so the value of the definite integral over this range is smaller than the actual area.

The area of each section is given by

$$\begin{aligned}
 A_1 &= \int_{-1}^0 (x^2 - 3x) dx = \left[ \frac{x^3}{3} - \frac{3x^2}{2} \right]_{-1}^0 = 1.83 \\
 A_2 &= \int_0^3 (x^2 - 3x) dx = \left[ \frac{x^3}{3} - \frac{3x^2}{2} \right]_0^3 = -4.5 \\
 A_3 &= \int_3^5 (x^2 - 3x) dx = \left[ \frac{x^3}{3} - \frac{3x^2}{2} \right]_3^5 = 8.67
 \end{aligned}$$

and the total area is given by  $A = A_1 + (-A_2) + A_3 = 1.83 + 4.5 + 8.67 = 15$ .

If we are asked to find the area under a curve for a certain range of  $x$ , we must first determine whether the curve crosses the axis in that range. If it does we need to be very careful about how we find the area, making sure we find the area for each section individually and then add all the results at the end.

**Example 7.7.** Determine the area enclosed between the curve  $y = x^3$  and the  $x$ -axis, between the points  $x = -2$  and  $x = 2$ .

*Solution.* First sketch the curve.

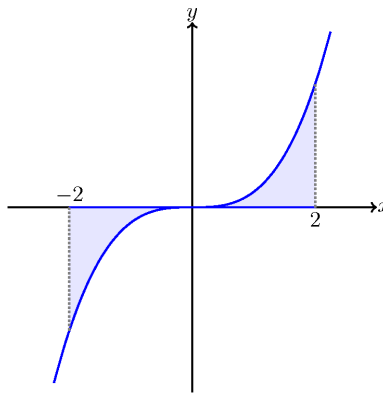


Figure 7.4:  $y = x^3$

We might expect the area to be given by

$$\int_{-2}^2 x^3 dx = \left[ \frac{x^4}{4} \right]_{-2}^2 = \frac{2^4}{4} - \frac{(-2)^4}{4} = 4 - 4 = 0.$$

This is clearly not the case. The area to the left of the  $y$ -axis clearly has a negative area while the area to the right has a positive area and these are cancelling one another out.

We actually need

$$\left| \int_{-2}^0 x^3 dx \right| + \left| \int_0^2 x^3 dx \right| = \left| \left[ \frac{x^4}{4} \right]_{-2}^0 \right| + \left| \left[ \frac{x^4}{4} \right]_0^2 \right| = \left| 0 - \frac{(-2)^4}{4} \right| + \left| \frac{2^4}{4} - 0 \right| = 4 + 4 = 8.$$

### 7.3 Area between two curves and Parametric Integration

We may use definite integration to find the area between two curves (or a curve and a straight line).

**Example 7.8.** Determine the area enclosed between the curve  $y = x^2 + 1$  and the straight line  $y = 7 - x$ .

*Solution.*

In this example we are not told the limits of integration. We must work them out. It is usually best to draw a sketch.

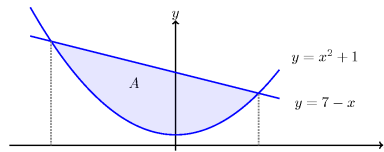


Figure 7.5:  $y = x^2 + 1$  and  $y = 7 - x$

We want to find the shaded area. The limits of integration will be the points of intersection of the two curves.

First we need to solve the equations simultaneously to find the  $x$ -co-ordinates of the points of intersection.

$$7 - x = x^2 + 1 \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x - 2)(x + 3) = 0$$

Hence we see that the curves intersect at  $x = -3$  and  $x = 2$ . These are the limits of integration.

From the sketch, the shaded area will be given by

area under straight line between limits - area under curve between limits.

$$\begin{aligned}
 A &= \int_{-3}^2 (7-x)dx - \int_{-3}^2 (x^2+1)dx = \left[7x - \frac{x^2}{2}\right]_{-3}^2 - \left[\frac{x^3}{3} + x\right]_{-3}^2 \\
 &= \left[7 \times 2 - \frac{4}{2}\right] - \left[7(-3) - \frac{9}{2}\right] - \left\{\left[\frac{8}{3} + 2\right] - \left[\frac{-27}{3} - 3\right]\right\} \\
 &= 14 - 2 + 21 + \frac{9}{2} - \frac{8}{3} - 2 - \frac{27}{3} - 3 = 20.83.
 \end{aligned}$$

The area is therefore 20.83 units<sup>2</sup>.

**Example 7.9.** Determine the area enclosed between the curve  $y = e^x$  and the curve  $y = e^{-x}$  and  $x = 0$  and  $x = 2$ .

*Solution.*

A quick sketch of the curves is:

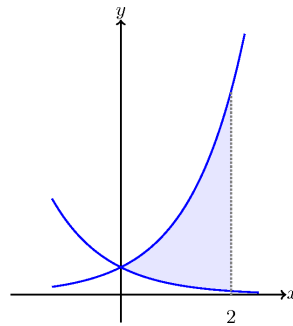


Figure 7.6:  $y = e^x$  and  $y = e^{-x}$

$$A = \int_0^2 (e^x - e^{-x})dx = [e^x + e^{-x}]_0^2 = (e^2 + e^{-2}) - (e^0 + e^0) = e^2 + e^{-2} - 2 = 5.52.$$

### 7.3.1 Parametric Integration

If a curve is defined parametrically such that  $x = f(t)$  and  $y = g(t)$  it can be integrated parametrically as follows

$$\int_a^b y \, dx = \int_{x=a}^{x=b} g(t) f'(t) dt.$$

Notice that  $dx/dt = f'(t)$  and so we write  $dx = f'(t)dt$ . It is normally best to convert the limits of integration into values of  $t$  rather than  $x$ . So we find  $a', b'$  such that  $a = f(a'), b = f(b')$  and then the integral becomes

$$\int_{a'}^{b'} g(t)f'(t)dt.$$

**Example 7.10.** A curve is described by the parametric equations  $x = at^2$  and  $y = 2at$  where  $t$  is a parameter and  $a$  is a constant. Determine the definite integral between  $t = 1$  and  $t = 2$ .

*Solution.* Here  $x = f(t) = at^2$  and  $y = g(t) = 2at$ . Hence  $f'(t) = 2at$  and the integral is

$$A = \int_a^{4a} y dx = \int_{t=1}^{t=2} y dx = \int_1^2 2at \times 2at dt = \int_1^2 4a^2 t^2 dt = 4a^2 \left[ \frac{t^3}{3} \right]_1^2 = 4a^2 \left[ \frac{8}{3} - \frac{1}{3} \right] = \frac{28}{3} a^2.$$

**Example 7.11.** A curve is described by the parametric equations  $x = \sin(t)$  and  $y = \cos(t)$  where  $t$  is a parameter. Determine the definite integral of  $y$  between  $t = 0$  and  $t = \pi/2$ .

*Solution.*

$$\begin{aligned} \int_{\sin(0)}^{\sin(\pi/2)} y dx &= \int_0^{\pi/2} \cos(t) \cos(t) dt = \int_0^{\pi/2} \cos^2(t) dt = \int_0^{\pi/2} \frac{\cos(2t) + 1}{2} dt \\ &= \frac{1}{2} \left[ \frac{\sin(2t)}{2} + t \right]_0^{\pi/2} = \frac{1}{2} [0 + \pi/2] - \frac{1}{2} [0] = \frac{\pi}{4}. \end{aligned}$$

## Week 7 Exercises

### 7.4 Sheet 15 - Integration using Partial Fractions

1.  $\int \frac{2x + 3}{x^2 + x - 30} dx$
2.  $\int \frac{17 - 7x}{(3 - x)(2 - x)} dx$
3.  $\int \frac{5}{x^2 - 1} dx$
4.  $\int \frac{x^2 - 3x + 3}{(x - 1)(x - 2)(x - 3)} dx$
5.  $\int \frac{2x^2 - 2x - 2}{x^3 - 2x^2 - x + 2} dx$
6.  $\int \frac{7}{x^2 - 4} dx$
7.  $\int \frac{5x + 2}{x^2 - 4x + 4} dx$
8.  $\int \frac{1}{4x^2 - 9} dx$
9.  $\int \frac{1}{x^2 - 4x - 8} dx$
10.  $\int \frac{1}{x^2 - 4x - 12} dx$
11.  $\int \frac{\cos(x)}{\sin(x)(1 + \sin(x))} dx$
12.  $\int \frac{\sin(x)}{1 - 4\cos^2(x)} dx$
13.  $\int \frac{e^x}{1 - e^{2x}} dx$
14.  $\int \frac{1}{x^2 - 2x - 1} dx$

## 7.5 Sheet 16 - Definite Integration

### Exercise 1

Determine the values of the following definite integrals

1. $\int_0^2 x^2 dx$	2. $\int_0^3 (4-x)^2 dx$
3. $\int_0^4 3\sqrt{x} dx$	4. $\int_1^2 (t^2 + 3t) dt$
5. $\int_0^{\pi/2} 4\cos(\theta) d\theta$	6. $\int_1^2 \cos(\theta) - \sin(\theta) d\theta$
7. $\int_1^9 1/\sqrt{p} dp$	8. $\int_1^2 r^3 - 1/r dr$
9. $\int_2^3 1/x dx$	10. $\int_1^4 (x+1)/(2\sqrt{x}) dx$
11. $\int_2^8 3/x dx$	12. $\int_{0.5}^1 (e^t + e^{-t})/2 dt$
13. $\int_1^4 \sqrt{t}(1+t)^2 dt$	14. $\int_{0.2}^{0.5} \cos(3x) dx$
15. $\int_3^4 (x+1)(2-x) dx$	16. $\int_{\pi/6}^{\pi/4} \sec^2(\theta) d\theta$

### Exercise 2

Evaluate the area bounded by the curve, the x axis and the given ordinates for the following:

1.  $y = x^2 - 2x, x = 0$  and  $x = 2$
2.  $y = 3x + x^2, x = 1$  and  $x = 3$
3.  $y = \sin(x), x = 0$  and  $x = \pi/3$
4.  $y = 3\cos(3x), x = \pi/6$  and  $x = \pi/3$
5.  $xy = 4, x = 3$  and  $x = 6$
6.  $y = \sin(x) + \cos(x), x = \pi/6$  and  $x = 1$
7.  $y = 1/e^{2x}, x = 0$  and  $x = 2$
8.  $y = (e^x - e^{-x})/2, x = 0$  and  $x = 2$
9.  $y = 4at, t = 0$  and  $t = 1$  where  $a$  is a constant.



## 8

# Week 8

Some function can't be, or are difficult to, integrate analytically. In these cases we need to rely on numerical techniques to approximate the value of a definite integral. We shall consider three common methods during this week.

### 8.1 Mid-ordinate Rule

To evaluate a definite integral  $\int_a^b f(x)dx$  we must find an antiderivative of the function  $f(x)$ . If such a function cannot be found then we must use numerical methods to approximate the integral to a desired degree of accuracy. To do this, we make use of the fact that the integral  $\int_a^b f(x)dx$  represents the area beneath the curve and the  $x$ -axis.

We shall consider three methods

1. Mid-ordinate rule;
2. Trapezium rule;
3. Simpson's rule.

With the *mid-ordinate rule*, the curve for which we want to know the area is divided into strips of equal width  $s$ .

Considering each strip in turn, we measure the height of the strip,  $h$ , along the centre line or mid-ordinate. The area of the strip is then taken to be the area of a rectangle with height  $h$  and width  $s$ .

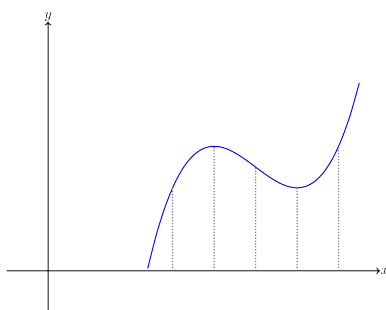
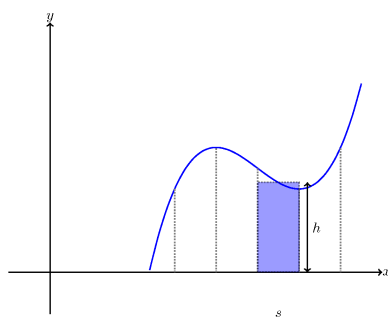
Figure 8.1:  $f(x)$ 

Figure 8.2: mid-ordinate rule

The area is  $h \times s$ , the area of the shaded rectangle. Note that the approximate area slightly over estimates the area to the right of the mid-ordinate, but slightly under estimates it by an approximately similar amount on the left of the mid-ordinate. As a result, the shaded area is a good approximation to the total area.

The approximate total area under the curve is the sum of the approximate areas of all the strips so

$$\text{Total area} = sh_1 + sh_2 + \dots + sh_n = s(h_1 + h_2 + \dots + h_n) = s \sum h$$

where  $\sum h$  is shorthand for the sum of all the values of  $h$ . So the total area = width of strip  $\times$  sum of the mid-ordinates.

**Example 8.1.** Use the mid-ordinate approximation to calculate the integral

$$\int_1^4 x^2 dx$$

by taking 1, 2 or 3 strips. Compare the values with the correct value.

*Solution.* First we know that  $\int_1^4 x^2 dx = [x^3/3]_1^4 = 4^3/3 - 1/3 = 63/3 = 21$ .

Using one strip,  $s = 3$ , the mid-point is  $x = 2.5$  and  $h = 2.5^2 = 6.25$ . Hence the mid-ordinate approximation gives

$$\int_1^4 x^2 dx \sim 3 \times 6.25 = 18.75.$$

Using two strips,  $s = 1.5$ , the two mid-points are  $x_1 = 1.75$  and  $x_2 = 3.25$  with heights  $h_1 = 1.75^2 = 3.0625$ ,  $h_2 = 3.25^2 = 10.5625$ . Hence the mid-ordinate approximation gives

$$\int_1^4 x^2 dx \sim 1.5(3.0625 + 10.5625) = 1.5 \times 13.625 = 20.4375.$$

Using three strips,  $s = 1$ , the three mid-points are  $x_1 = 1.5$ ,  $x_2 = 2.5$  and  $x_3 = 3.5$  with heights  $h_1 = 1.5^2 = 2.25$ ,  $h_2 = 2.5^2 = 6.25$ ,  $h_3 = 3.5^2 = 12.25$ . Hence the mid-ordinate approximation gives

$$\int_1^4 x^2 dx \sim 1 \times (2.25 + 6.25 + 12.25) = 1 \times 20.75 = 20.75.$$

## 8.2 Trapezium rule

For the *trapezium rule*, the curve is again divided into strips of equal width  $s$ , but this time, the area of each strip is approximated by a trapezium instead of a rectangle.

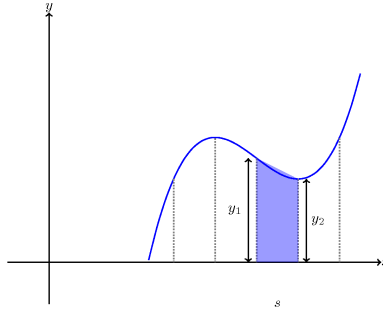


Figure 8.3: trapezium rule

For a trapezium, the area is given by

$$A = \frac{1}{2}(\text{sum of lengths of parallel sides}) \times (\text{perpendicular distance between parallel sides})$$

So for each strip  $\delta A = s(y_1 + y_2)/2$ . The total area is then, for  $n$  strips

$$\begin{aligned} A &= \frac{s}{2}(y_1 + y_2) + \frac{s}{2}(y_2 + y_3) + \dots + \frac{s}{2}(y_n + y_{n+1}) \\ &= \frac{s}{2}(y_1 + 2y_2 + 2y_3 + \dots + 2y_n + y_{n+1}). \end{aligned}$$

**Example 8.2.** Use the trapezium rule to approximate the integral

$$\int_1^4 x^2 dx$$

by taking 1, 2 or 3 strips.

*Solution.* If there is one strip then  $s = 3$ ,  $x_1 = 1$ ,  $x_2 = 4$ ,  $y_1 = 1^2 = 1$ ,  $y_2 = 4^2 = 16$ . Hence the trapezium rule gives

$$\int_1^4 x^2 dx \sim \frac{3}{2}(1 + 16) = \frac{51}{2} = 25.5$$

With two strips then  $s = 1.5, x_1 = 1, x_2 = 2.5, x_3 = 4, y_1 = 1^2 = 1, y_2 = 2.5^2 = 6.25, y_3 = 4^2 = 16$ . Hence the trapezium rule gives

$$\int_1^4 x^2 dx \sim \frac{1.5}{2} (1 + 2 \times 6.25 + 16) = \frac{354}{16} = 22.125$$

With three strips then  $s = 1, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, y_1 = 1^2 = 1, y_2 = 2^2 = 4, y_3 = 3^2 = 9, y_4 = 4^2 = 16$ . Hence the trapezium rule gives

$$\int_1^4 x^2 dx \sim \frac{1}{2} (1 + 2 \times (4 + 9) + 16) = \frac{43}{2} = 21.5$$

### 8.3 Simpson's Rule

The idea behind *Simpson's rule* is that instead of approximating the curve using a series of trapeziums, we approximate by using a series of quadratic polynomials.

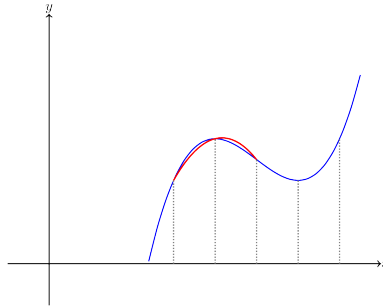


Figure 8.4: Simpson's rule

Given 3 equally spaced points on the curve  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ , we construct a quadratic polynomial  $y = ax^2 + bx + c$  which passes through all three points. In order to simplify the calculations, we can 'move' the curve so that the point  $x_2$  coincides with the origin. In other words assume that  $x_2 = 0$ . We also let  $x_3 - x_2 = x_2 - x_1 = s$  and so  $x_3 = s, x_1 = -s$ .

for this to be the case we then need

$$y_1 = a(-s)^2 + b(-s) + c$$

$$y_2 = a0^2 + b0 + c$$

$$y_3 = as^2 + bs + c.$$

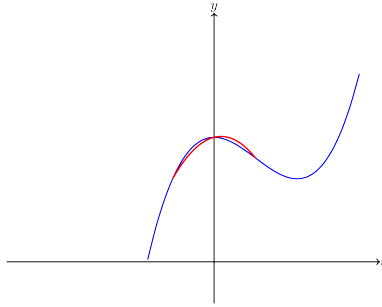


Figure 8.5: Simpson's rule

Solving these gives

$$\begin{aligned} a &= \frac{y_1 + y_3 - 2y_2}{2s^2} \\ b &= \frac{y_3 - y_1}{2s} \\ c &= y_2. \end{aligned}$$

The area under that quadratic will then be

$$\int_{-s}^s (ax^2 + bx + c)dx = \left[ a\frac{x^3}{3} + b\frac{x^2}{2} + cx \right]_{-s}^s = \frac{s}{3} [y_1 + 4y_2 + y_3].$$

Notice that the answer is not dependent on the values of  $x_1, x_2$  or  $x_3$ . Notice also that we are using 2 intervals of length  $s$  to construct the quadratic.

If we now split out interval over which we are integrating into an even number, say  $2n$ , of intervals of length  $s$ , we can then construct a quadratic using  $x_1, x_2, x_3$  then another quadratic using  $x_3, x_4, x_5$ , and so on until we get a final quadratic using  $x_{2n-1}, x_{2n}, x_{2n+1}$  and the sum of the areas under those quadratic will be an approximation to the area under the curve. This gives us

$$\begin{aligned} \int_{x_1}^{x_{2n+1}} y \, dx &= \frac{s}{3} [y_1 + 4y_2 + y_3] + \frac{s}{3} [y_3 + 4y_4 + y_5] + \dots + \frac{s}{3} [y_{2n-1} + 4y_{2n} + y_{2n+1}] \\ &= \frac{s}{3} [y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \dots + 2y_{2n-1} + 4y_{2n} + y_{2n+1}]. \end{aligned}$$

This is known as *Simpson's rule*.

**Example 8.3.** Use Simpson's rule to approximate the integral

$$\int_1^4 x^3 dx$$

by taking 2 and 4 strips.

*Solution.* First notice that

$$\int_1^4 x^3 dx = \left[ \frac{x^4}{4} \right]_1^4 = \frac{4^4}{4} - \frac{1}{4} = \frac{255}{4} = 63.75$$

In our above formula we let  $n = 1$  and then 2.

For  $n = 1$  we have  $s = 3/2, x_1 = 1, x_2 = 2.5, x_3 = 4$  with  $y_1 = 1, y_2 = 15.625, y_3 = 64$ .

$$\int_1^4 x^3 dx \sim \frac{3}{2} \frac{1}{3} [1 + 4 \times 15.625 + 64] = 63.75.$$

For  $n = 2$  we have  $s = 0.75, x_1 = 1, x_2 = 1.75, x_3 = 2.5, x_4 = 3.25, x_5 = 4$  with  $y_1 = 1, y_2 = 1.75^3 = 5.36, y_3 = 2.5^3 = 15.625, y_4 = 3.25^3 = 34.33, y_5 = 64$ .

$$\int_1^4 x^3 dx \sim \frac{3}{4} \frac{1}{3} [1 + 4 \times 5.36 + 2 \times 15.625 + 4 \times 34.33 + 64] = 63.75.$$

**Example 8.4.** Use Simpson's rule to approximate the integral

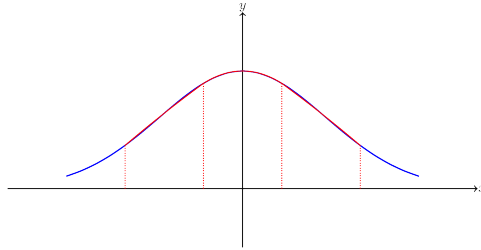
$$\int_{-1}^1 e^{-x^2} dx$$

by taking 6 strips.

*Solution.* This is an example of a function that does not have a suitable antiderivative, so we need to use numerical techniques.

With 6 strips,  $s = 1/3$  so we have

$x_i$	$y_i$
-1	0.368
-2/3	0.641
-1/3	0.895
0	1
1/3	0.895
2/3	0.641
1	0.368

Figure 8.6:  $e^{-x^2}$ 

Hence using Simpson's rule

$$\int_{-1}^1 e^{-x^2} dx \sim \frac{1}{9} [0.368 + 4 \times 0.641 + 2 \times 0.895 + 4 \times 1 + 2 \times 0.895 + 4 \times 0.641 + 0.368] = 1.494.$$

**Example 8.5.** Use Simpson's rule to approximate the integral

$$\int_0^2 \frac{1}{1+x^2} dx$$

by taking 10 strips and calculate the error involved.

*Solution.* With 10 strips,  $s = 2/10 = 0.2$  so we have

$x_1$	$y_1$
0	1
0.2	0.96154
0.4	0.86207
0.6	0.73529
0.8	0.60976
1	0.5
1.1	0.40984
1.4	0.33784
1.6	0.28090
1.8	0.23585
2	0.2



Hence using Simpson's rule

$$\begin{aligned}\int_0^2 \frac{dx}{1+x^2} &\sim \frac{0.2}{3} [1 + 0.2] + \\ &\quad \frac{0.2}{3} \times 4 [0.96154 + 0.73529 + 0.5 + 0.33784 + 0.23585] + \\ &\quad \frac{0.2}{3} \times 2 [0.86207 + 0.60976 + 0.40984 + 0.28090] \\ &= 1.10715.\end{aligned}$$

The actual value of the integral is

$$\int_0^2 \frac{dx}{1+x^2} = [\tan^{-1}(x)]_0^2 = 1.106587$$

and so the error is 0.000563.

# Week 8 Exercises

## 8.4 Sheet 17 Approximate Integration

### Exercise 1

An irregular area has a horizontal base of length 84 mm. The vertical heights measured at regular intervals of 12 mm are, 25.0, 24.5, 23.5, 21.0, 18.0, 14.0, 9.0 mm. Using the midordinate rule, calculate the area of the shape.

### Exercise 2

An indicator diagram for a steam engine is 90 mm long. Seven evenly spaced ordinates, including the end ordinates are measured with the following results:-

5.10, 4.60, 3.20, 2.70, 2.32, 2.18, 2.06, mm

Determine the mean pressure in the engine cylinder using the trapezium rule. The pressure scale is 10 kPa to 1 mm. The mean pressure is given by the ratio of the diagram area to the base length.

### Exercise 3

By drawing a semi-circle of radius 100 mm, determine the percentage error to two decimal places in the area calculated using i) the mid-ordinate rule ii) the trapezium rule and iii) Simpson's rule. The ordinate spacing is to be taken as 20 mm.

### Exercise 4

A vehicle starts from rest and its velocity is measured every second for sixty seconds, the results being:-

Velocity (m/s)	Time (s)
0	0
1.2	1.0
2.4	2.0
3.7	3.0
5.2	4.0
6.0	5.0
9.2	6.0

Determine the distance travelled in the six seconds and the average speed over this time using Simpson's rule.

### Exercise 5

Evaluate by Simpson's rule using 9 ordinates

$$\int_0^2 \frac{dx}{4+x^2}$$

### Exercise 6

The table below relates the values of y at regular intervals of x

x	y
1.00	2.45
1.25	2.80
1.50	3.44
1.75	4.20
2.00	4.33
2.25	3.97
2.50	3.12
2.75	2.38
3.00	1.80

Using Simpson's rule with 8 intervals evaluate  $\int y \, dx$ .

### Exercise 7

Determine  $\int_0^{\pi/2} \sqrt{\cos(\theta)} \, d\theta$  using 6 intervals.

**Exercise 8**

A pin moves along a straight guide such that its velocity  $v$  ( $m.s^{-1}$ ) when it is at a distance  $s$  ( $m$ ) from the start after time  $t$  ( $s$ ) is as given below

$v$ (m/s)	$t$ (s)
0	0
4.00	0.50
7.94	1.00
11.68	1.50
14.97	2.00
17.39	2.50
18.25	3.00
16.08	3.50
0	4.00

Apply Simpson's rule with 8 intervals to determine the approximate distance travelled by the pin for the time interval  $t = 0$  to  $t = 4$  s

**Exercise 9**

Evaluate correct to three decimal places

- $\int_0^1 \sqrt{x} \cos(x) dx,$
- $\int_0^1 \sqrt{x} \sin(x) dx.$

**Exercise 10**

By using Simpson's rule with six intervals, determine the approximate value of

$$\int_0^{\pi/2} \sqrt{(2.5 - 1.5 \cos(2\theta))} d\theta.$$

**Exercise 11**

By using Simpson's rule with 8 intervals, establish the approximate value of

$$\int_0^1 \sqrt{4 + x^4} dx$$

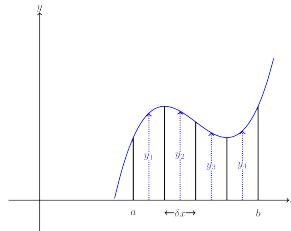
# 9

## Week 9

In this section, we consider the mean and root mean square values of a function, which is important in studying sinusoidal type functions.

### 9.1 Mean and RMS Values

The *mean* value for a function  $y = f(x)$  for the range  $a$  to  $b$  is the mean value of all the ordinates in that range. Consider a function  $y = f(x)$  for the values  $x = a$  to  $x = b$ , divided into a number of strips of equal width  $\delta x$ . Let the mid-ordinate for each strip be  $y_1, y_2, y_3, \dots, y_n$ .



The mean or average value of the mid-ordinates is

$$\bar{y} = \frac{y_1 + \dots + y_n}{n}$$

Multiplying the numerator and denominator by  $\delta x$  gives

$$\bar{y} = \frac{y_1 \delta x + \dots + y_n \delta x}{n \delta x}$$

and since  $n\delta x = b - a$  we have

$$\bar{y} = \frac{\sum y\delta x}{b - a}.$$

When the number of strips increases indefinitely such that  $\delta x \rightarrow 0$

$$\bar{y} = \frac{1}{b - a} \int_a^b y \, dx.$$

That is, the mean value of a function between two limits is the area under the curve between those limits divided by the range of  $x$  between those limits.

**Example 9.1.** If the velocity  $v$  in  $m/s$  of a body is given by  $v = 4t + 3$  where  $t$  is in seconds, determine the mean velocity of the body from  $t = 2$  s to  $t = 6$  s.

*Solution.*

$$\begin{aligned} \text{Mean velocity} &= \frac{1}{6 - 2} \int_2^6 (4t + 3) \, dt = \frac{1}{4} [2t^2 + 3t]_2^6 \\ &= \frac{1}{4} [(2 \times 36 + 3 \times 6) - (2 \times 4 + 3 \times 2)] = \frac{76}{4} = 19 \, m/s. \end{aligned}$$

### 9.1.1 RMS Value

The **RMS** (or **root mean square**) value is obtained by taking  $f(x)$ , squaring it, finding the mean value and then taking the square root.

Its importance in engineering is demonstrated when a sinusoidal function is investigated, for example when the average value of the current or voltage in a circuit is to be found. For all sine or cosine functions, the mean value over a complete cycle is zero because the curve is equally distributed above and below the  $x$ -axis. However, if the ordinates are squared first, there are then no negative ordinates and an average may be found. Ammeters and voltmeters for a.c. circuits are usually calibrated to find the RMS value.

The RMS value is obtained from

$$\text{RMS} = \sqrt{\frac{1}{b - a} \int_a^b y^2 \, dx}.$$

**Example 9.2.** Determine the RMS value of the current in an a.c. circuit given by  $i = 20 + 100 \sin(100\pi t)$  between  $t = 0$  and  $t = 0.02$ s.

*Solution.*

$$\text{RMS} = \sqrt{\frac{1}{b-a} \int_a^b i^2 dt}$$

$$i = 20 + 100 \sin(100\pi t)$$

$$i^2 = 400 + 4000 \sin(100\pi t) + 10000 \sin^2(100\pi t)$$

$$= 400 + 40000 \sin(100\pi t) + 10000 \times \frac{1}{2}(1 - \cos(200\pi t))$$

$$\text{RMS}^2 = \frac{1}{0.02 - 0} \int_0^{0.02} (400 + 40000 \sin(100\pi t) + 10000 \times \frac{1}{2}(1 - \cos(200\pi t))) dt$$

$$= \frac{1}{0.02} \left[ 400t - \frac{40000 \cos(100\pi t)}{100\pi} + 5000t - \frac{5000 \sin(200\pi t)}{200\pi} \right]_0^{0.02}$$

$$= \frac{1}{0.02} [(8 - 12.732 + 100 - 0) - (0 - 12.732 + 0 - 0)] = \frac{1}{0.02} \times 108$$

$$\text{RMS} = \sqrt{\frac{108}{0.02}} = 73.5 \text{ A}$$

# Week 9 Exercises

## 9.2 Sheet 17 More problems on Numerical Integration

### Exercise 1

(Exam 2001/2002)

Use the Simpson's rule with five ordinates to find an approximate value for

$$\int_0^{\pi/4} x^2 \sin(2x) dx,$$

giving your answer correct to four decimal place

### Exercise 2

(Exam 2000/2001)

By using the Simpson's rule with 8 equal intervals, determine

$$\int_0^4 \frac{x+3}{x^2+3x+2} dx,$$

stating the answer to four decimal places.

### Exercise 3

(Exam 1999/2000)

Using the Simpson's rule with 5 ordinates, find an approximate value for

$$\int_4^6 \frac{12}{(x-3)(x+1)} dx,$$

giving your answer correct to four decimal places.



### 9.3 Sheet 18 Mean and RMS Values

#### Exercise 1

Determine the mean and RMS values for the following functions

1.  $x^2 - x$  from  $x = 1$  to  $x = 3$ ;
2.  $x(3 - x)$  from  $x = 0$  to  $x = 3$ ;
3.  $3 \sin(2t)$  from  $t = 0$  to  $t = \pi/2$ ;
4.  $2 + 3 \sin(\theta)$  from  $\theta = 0$  to  $\theta = \pi$ ;
5.  $\sin(x)$  from  $x = 0$  to  $x = \pi$  and from  $x = 0$  to  $x = 2\pi$ ;
6.  $\cos^2(x)$  from  $x = 0$  to  $x = 2\pi$ ;
7.  $\sin^2(x)$  from  $x = 0$  to  $x = \pi/6$ .

#### Exercise 2

Determine the average value of  $y = \sin(3x) + \cos(x)$  for values of  $x$  ranging from 0 to  $\pi/6$ .

#### Exercise 3

Evaluate the mean values for

1.  $1/\sqrt{16 - 2x^2}$  from  $x = 0$  to  $x = 2$ ;
2.  $1/(1 + x^2)$  from  $x = 0$  to  $x = 1$ .

#### Exercise 4

Determine the RMS values for the following functions

1.  $x(2 - x)$  from  $x = 0$  to  $x = 2$ ;
2.  $1 - \sin(x)$  from  $x = 0$  to  $x = 2\pi$ ;
3.  $3 + 2 \cos(x)$  from  $x = 0$  to  $x = 2\pi$ ;
4.  $100 \sin(pt) + 60 \sin(3pt)$  from  $t = 0$  to  $t = \pi/p$ .

#### Exercise 5

The instantaneous voltage in an a.c.(alternating current) circuit is given by  $v = 6 + 8 \cos(\omega t)$ . Determine the RMS value for the time period  $t = 0$  to  $t = 2\pi/\omega$ .

#### Exercise 6

An alternating current is given by  $i = a \sin(\theta)$ . Determine the RMS value of the current over a half wave and show that the ratio of the RMS value to the mean

value is approximately 1.11.

# 10

## Week 10

We look at the important topics of centroids and centers of gravity, before a brief look at volumes and surfaces of revolution.

### 10.1 Centre of Gravity and Centroids

Many structures and mechanical systems act as if their masses were concentrated at a single point called the *center of mass* or *centre of gravity*. We wish to find out how to locate this point. First let us consider the situation in 1-dimension and then move to consider it in 2-dimensions.

#### 10.1.1 Masses along a line

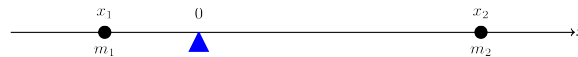


Figure 10.1: mass along a line

Suppose that the  $x$ -axis is supported by a fulcrum at the origin and we have 2 masses,  $m_1$  and  $m_2$ , situated at positions  $x_1$  and  $x_2$  from the origin. This is similar to a 'see-saw'. Does the system balance, or not? The answer, from experience, depends on both the values of the masses and the positions of the objects. The quantity  $m_1x_1$  is referred to as the *moment* of the mass  $m_1$  about the origin. Notice that in our figure,  $m_1x_1 < 0$  and  $m_2x_2 > 0$ . The *principle of moments* says that the system will balance providing

$$m_1x_1 + m_2x_2 = 0.$$

If we are given a system with more than 2 masses

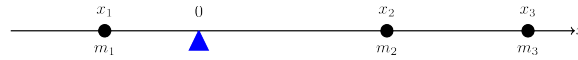


Figure 10.2: mass along a line - many bodies

the principle says that the system will balance providing

$$\sum m_i x_i = 0.$$

In principle, we should use the force acting on the body, which is  $m_i g$  where  $g$  is the gravitational acceleration. However the constant  $g$  will cancel out. Suppose however, the system does not balance. Can we find a point  $\bar{x}$  such that if we were to move the fulcrum to that position, the system would balance?

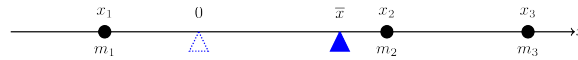


Figure 10.3: center of mass

This would require that

$$\begin{aligned} \sum (x_i - \bar{x}) m_i &= 0 \\ \sum x_i m_i - \sum \bar{x} m_i &= 0 \\ \sum m_i x_i &= \bar{x} \sum m_i \\ \bar{x} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{\text{system moment about } O}{\text{system mass}}. \end{aligned}$$

The point  $\bar{x}$  is called the *center of mass* or *center of gravity*.

In order to generalise this situation to 2-dimensions and to consider an ‘infinite number of points’, we need to use integration, rather than a finite sum.

Consider the 2-dimensional strip below, and suppose that the *density* at the point  $x$  is given by  $\rho(x)$ .

The density is defined as being

$$\text{density} = \frac{\text{mass}}{\text{volume}}.$$

In our case, as we are working in 2-dimensions, the volume becomes the ‘surface area’. The mass of the shaded region, of area  $\delta A$  (where the width  $\delta x$  is small), is then

$$\rho(x) \delta A$$

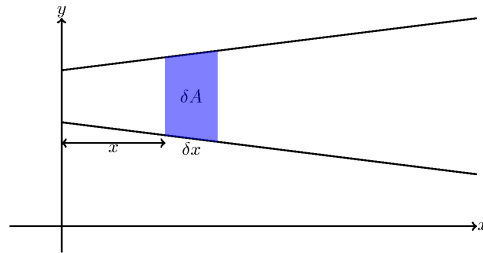


Figure 10.4: 2-dimensional rod

Summing over all these ‘small areas’ gives

$$\text{mass} = \int \rho(x) dA.$$

**Example 10.1.** Suppose that the rod above is 5 m long with mass per unit length at the point  $x$  given by  $2 + x^2/2 \text{ Kg/m}$ . Find the total mass of the rod.

*Solution.*

Take a small rectangular strip of width  $\delta x$  at the point  $x$  so that the mass of this rectangle is

$$\left(2 + \frac{x^2}{2}\right) \delta x.$$

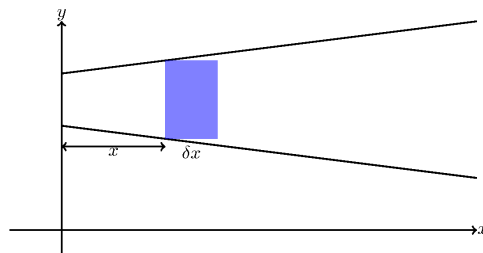


Figure 10.5: 2-dimensional rod

Now sum over all such rectangles and allow  $\delta x \rightarrow 0$  (in other words compute the integral) to get the mass

$$\text{mass} = \int_0^5 \left(2 + \frac{x^2}{2}\right) dx = \left[2x + \frac{x^3}{6}\right]_0^5 = 30.8 \text{ Kg}.$$

Suppose we wish to find the centre of mass of the rod in Figure 10.5. We use the same principle as in the discrete case when considering a finite number of points except we use an integral rather than a sum. The moment of the small rectangle of width  $\delta x$  is  $x(2 + x^2/2)\delta x$ . So

$$\bar{x} = \frac{\text{moment}}{\text{mass}} = \frac{1}{30.8} \int_0^5 x(2 + \frac{x^2}{2}) dx = \frac{1}{30.8} \left[ x^2 + \frac{x^4}{8} \right]_0^5 = 3.35.$$

### 10.1.2 Centroid of a uniform lamina

A *lamina* is a solid body in the form of a thin sheet of uniform thickness, with uniform density. As a result, the weight is proportional to the area and the formula for the centre of gravity can have the  $m$  values replaced by  $a$  values where  $a$  represents the area of an element and  $A$  is the total area of the lamina. So

$$\bar{x} = \frac{\int xa \, dx}{A}, \quad \bar{y} = \frac{\int ya \, dy}{A}$$

Then  $\bar{x}$  and  $\bar{y}$  are the coordinates of the *centroid of the area* or the *centroid*.

Suppose that the centroid of the area bounded by the curve  $y = f(x)$ , the  $x$ -axis and the lines  $x = a$  and  $x = b$  is required.

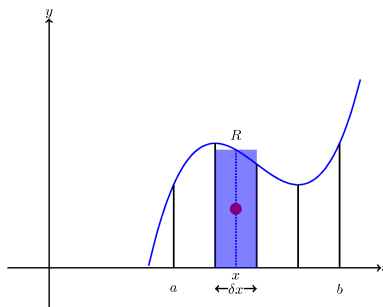


Figure 10.6: lamina

Consider a narrow strip of width  $\delta x$ . The strip can be considered as approximately a rectangle with height  $y$  and area  $y\delta x$ . The centre of the area is at the midpoint with coordinate  $(x, y/2)$ . By summing the moments of the centre of area for each strip we can find the centroid.

Taking moments about the  $y$ -axis

$$\bar{x} = \frac{\int_a^b xy \, dx}{\int_a^b y \, dx}.$$

Similarly taking moments about the  $x$ -axis

$$\bar{y} = \frac{\int_a^b \frac{y^2}{2} \, dx}{\int_a^b y \, dx}.$$

**Example 10.2.** Determine the centroid of area under the curve  $y = x^2$  between the coordinates  $x = 1$  and  $x = 3$ .

*Solution.*

$$\begin{aligned}\bar{x} &= \frac{\int_1^3 xy \, dx}{\int_1^3 y \, dx} = \frac{\int_1^3 x^3 \, dx}{\int_1^3 x^2 \, dx} = \frac{\left[\frac{x^4}{4}\right]_1^3}{\left[\frac{x^3}{3}\right]_1^3} = \frac{20}{8.67} = 2.31 \\ \bar{y} &= \frac{\int_1^3 \frac{y^2}{2} \, dx}{\int_1^3 y \, dx} = \frac{\int_1^3 \frac{x^4}{2} \, dx}{\int_1^3 x^2 \, dx} = \frac{\left[\frac{x^5}{10}\right]_1^3}{8.67} = \frac{24.1}{8.67} = 2.78\end{aligned}$$

The centroid therefore has coordinates  $(2.31, 2.78)$ .

## 10.2 Volume and Surface of Revolution

Consider a curve  $y = f(x)$  between the ordinates  $x = a$  and  $x = b$ . The volume of revolution is the volume of the equivalent solid body obtained by rotating the curve around the  $x$  or  $y$  axes.

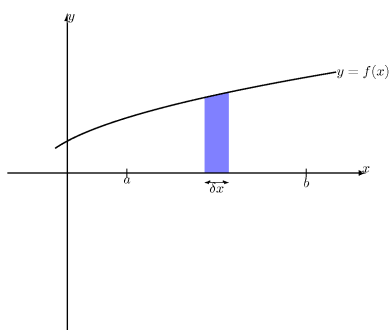


Figure 10.7: Volume of revolution

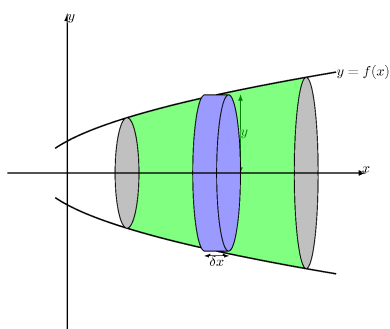


Figure 10.8: Volume of revolution



### 10.2.1 Rotation about the $x$ -axis

If the shaded strip in the above figure is rotated about the  $x$ -axis then, provided  $\delta x$  is small, it will trace out a disc with thickness  $\delta x$  and volume  $\delta V = \pi y^2 \delta x$ , where  $y$  is the radius of the disc.

We may sum the volumes generated by all the strips between  $x = a$  and  $x = b$  to obtain the volume of the solid of revolution for the section of curve about the  $x$  axis. Thus the total volume  $V$  is given by

$$V = \int_a^b \pi y^2 dx$$

### 10.2.2 Rotation about the $y$ -axis

By a similar method it can be shown that for a rotation about the  $y$ -axis, the volume the solid of revolution will be

$$V = 2\pi \int_a^b xy dx.$$

### 10.2.3 Surface area of revolution

The surface area of revolution is the surface area of the body generated by rotating a curve about the  $x$  or  $y$  axes. For a strip as shown in the figure above, the length of arc is approximately  $\delta x$ . If the strip is rotated about the  $x$  axis, the surface area generated will be approximately  $\delta S = 2\pi y \delta x$ . The surface area of the whole solid generated by rotating the curve around the  $x$  axis between  $x = a$  and  $x = b$  is obtained by summing the contributions to the surface area from each of the elementary strips and we can eventually deduce that

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

#### Example 10.3.

The straight line  $y = mx$  between  $x = 0$  and  $x = h$  is rotated about the  $x$ -axis thus generating a right circular cone. Calculate its volume in terms of  $h$  and  $R$  where  $R$  is the  $y$  value at  $x = h$ . Find also the surface area of the resultant cone.

*Solution.* The volume of the solid of revolution for  $y = mx$  is given by

$$V = \pi \int_0^h y^2 dx = \pi \int_0^h (mx)^2 dx = m^2 \pi \int_0^h x^2 dx = m^2 \pi \left[ \frac{x^3}{3} \right]_0^h = m^2 \pi \frac{h^3}{3}.$$

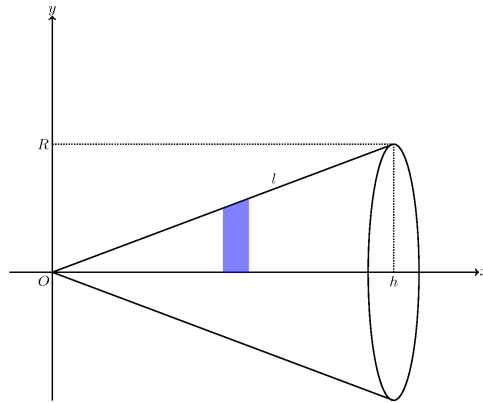


Figure 10.9: right circular cone

Now  $m = R/h$  and so

$$V = \frac{1}{3} \frac{R^2}{h^2} h^3 = \frac{1}{3} \pi R^2 h.$$

The surface area is given by

$$\begin{aligned} S &= 2\pi \int_0^h y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_0^h mx \sqrt{1 + m^2} dx \\ &= 2m\sqrt{1 + m^2} \pi \left[ \frac{x^2}{2} \right]_0^h = m\sqrt{1 + m^2} \pi h^2 = \pi R h \sqrt{1 + m^2} \\ &= \pi R l. \end{aligned}$$

where  $l$  is the length of the inclined line.

**Example 10.4.** A curve is given parametrically as  $x = 4t^2$ ,  $y = 6t - t^2$ . Determine the volume generated when the figure bounded by the curve, the  $x$ -axis and the ordinates corresponding to  $t = 0$  and  $t = 2$  is rotated about the  $x$ -axis.

*Solution.*

$$V = \pi \int_a^b y^2 dx.$$

Since  $x = 4t^2$  then  $dx/dt = 8t$  and so we can write  $dx = 8t \, dt$ . Hence

$$\begin{aligned} V &= \pi \int_0^t (6t - t^2)^2 8t \, dt = 8\pi \int_0^2 (36t^3 - 12t^4 + t^5) \, dt \\ &= 8\pi \left[ 9t^4 - \frac{12t^5}{5} + \frac{t^6}{6} \right]_0^2 \\ &= 8\pi \times 77.87 = 1957 \text{ units}^3. \end{aligned}$$

# Week 10 Exercises

## 10.3 Sheet 19 Centroids of area

### Exercise 1

Determine the positionns of the centroids of the uniform laminae represented by the following curves and indicated boundaries.

1.  $y = x^2$ , the  $x$ -axis and  $x = 0, x = 2$ ;
2.  $y = 2x^3$ , the  $x$ -axis and  $x = 1, x = 2$ ;
3. The portion of  $y = 2x - x^2$  above the  $x$ -axis.

### Exercise 2

Determine the position of the centroid of the figure bounded by  $y = e^{2x}$ , the  $x$ -axis, the  $y$ -axis and the ordinate at  $x = 2$ .

### Exercise 3

Determine the position of the centroid of the figure bounded by the curve  $y = 5 \sin(2x)$ , the  $x$ -axis and the ordinates at  $x = 0$  and  $x = \pi/6$ .

### Exercise 4

A curve is given parametrically as  $x = at^2, y = 2at$  where  $t$  is a parameter and  $a$  is a constant. Determine the position of the centroid about the for figure bounded by the values  $t = 1$  and  $t = 2$ .

## 10.4 Sheet 20 Volumes of Revolution

### Exercise 1

Determine the volume of the solid of revolution obtained by rotating the curve  $y = x^2$  about

1. The  $x$ -axis between the limits  $x = 1$  and  $x = 3$ ;
2. The  $y$ -axis between the same limits.

### Exercise 2

A cone is generated by rotating the line  $y = x/2$  about the  $x$ -axis from  $x = 0$  to  $x = 10$ . Determine the volume of the cone.

### Exercise 3

Determine the volumes of revolution formed by rotating about the  $x$ -axis the areas between the curves of

1.  $y = x^2 - 2x$  and the  $x$ -axis;
2.  $y = 3x - x^2$  and the  $x$ -axis.

### Exercise 4

Determine the volume generated when the plane figure bounded by  $y = 5 \cos(2x)$ , the  $x$ -axis and the ordinates at values  $x = 0$  and  $x = \pi/4$  rotates about the  $x$ -axis for a complete revolution.

### Exercise 5

The parametric equations of a curve are given as  $x = 6t^2, y = 2t - t^2$ . Determine the volume generated when the plane figure bounded by the curve, the  $x$ -axis and the ordinates corresponding to  $t = 0$  and  $t = 2$  rotates about the  $x$ -axis

# 11

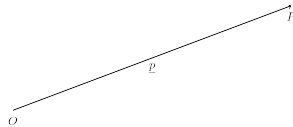
## Week 11

### 11.1 Vectors I

A *scalar* quantity has only size or magnitude. Examples of scalars are mass, time, speed ...

A *vector* quantity has magnitude and direction, examples are force, velocity acceleration, moment, ...

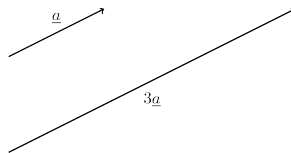
A vector is represented graphically by a straight line with an arrowhead showing its direction. Often the line is drawn to scale so that the magnitude of the vector may be determined by measuring its length.



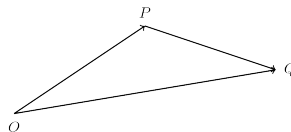
The diagram above shows two ways that vectors are often drawn. To refer to the vector we give it a name. There are many different methods used to indicate that we are referring to a vector quantity. The vector above will often be referred to in books as  $\underline{OP}$  or  $\underline{p}$  or  $\vec{p}$ . The point at which a vector starts is called the initial point. The point at which it ends is called the final point.

Sometimes we are only interested in the magnitude of the vector. In this case we show that we are considering the magnitude only by writing, for example  $|\underline{OP}|$ . The vertical lines are called modulus lines and can be used with any of the ways of the different ways of writing a vector. So the vector  $\underline{OP}$  has a length (or magnitude)  $|\underline{OP}|$ .

We can multiply a vector by a positive scalar (or ordinary number). This has the effect of leaving the direction the same, but changing the magnitude. For example, below we have the vectors  $\underline{a}$  and  $3\underline{a}$ . Note that  $3\underline{a}$  is parallel to  $\underline{a}$  (has the same direction) but is 3 times as long.



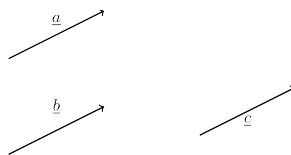
Multiplication by a negative scalar turns the vector around through  $180^\circ$ . A position vector indicates the distance and direction that we have to travel from the origin to get to a point. In the diagram below, if  $O$  is the origin then  $\underline{OP}$  is the position vector of the point called  $P$ .  $\underline{PQ}$  is called a displacement vector. It shows the distance and direction that we have to travel from  $P$  to get to  $Q$ .  $\underline{OQ}$  is the position vector of the point  $Q$ . Position vectors always refer to the position of a point relative to the origin. Displacement vectors refer to the position of any point relative to any other.



### 11.1.1 Vector Algebra

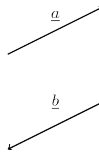
We have already seen that a vector may be *multiplied by a scalar*. If the scalar is positive this changes the magnitude but not the direction of the vector. If the scalar is negative the vector is turned around through  $180^\circ$  and the magnitude is changed.

Two vectors  $\underline{a}$  and  $\underline{b}$  are equal if they have the same magnitude and the same direction. They do not actually have to start and finish at the same point.



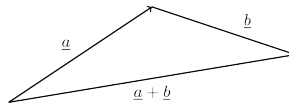
In the above diagram,  $\underline{a} = \underline{b} = \underline{c}$ .

If a vector  $\underline{b}$  has the same magnitude as vector  $\underline{a}$ , but points in the opposite direction then  $\underline{b} = -\underline{a}$ .



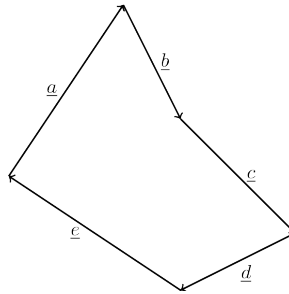
Again it doesn't matter whether they start and finish at the same point or not.

The **sum** (or **resultant**) of two vectors is obtained by placing the initial point of the second vector in the sum at the final point of the first vector in the sum. The sum is then the vector that completes the triangle (i.e. runs from the initial point of the first to the final point of the second). This is called the triangle law of vector addition.



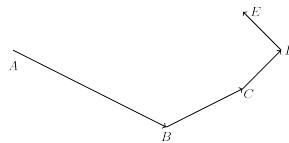
The difference of two vectors is given by  $\underline{a} + (-\underline{b})$ , where  $-\underline{b}$  points in the opposite direction to  $\underline{b}$ . The sum of  $\underline{a} + (-\underline{b})$  is then found from the triangle law above. Hence we write  $\underline{a} - \underline{b} = \underline{a} + (-\underline{b})$ .

The sum of a set of vectors that form a closed loop is always zero.



$$\underline{a} + \underline{b} + \underline{c} + \underline{d} + \underline{e} = \underline{0}.$$

**Example 11.1.** For the diagram below, state the magnitude of the sum of the vectors  $\underline{AB}$ ,  $\underline{BC}$ ,  $\underline{CD}$  and  $\underline{DE}$ .

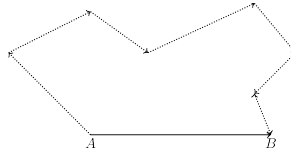


**Solution.** The sum of all the vectors starts at the initial point of the first vector and ends at the final point of the last vector. This is the vector  $\underline{AE}$ . Its magnitude



may be written  $|\underline{AE}|$ .

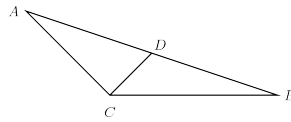
If a vector joins two points  $A$  and  $B$ , then it is equivalent to any other system of vectors that form a path from  $A$  to  $B$ . For example the vector  $\underline{AB}$  shown as a solid line below, is equivalent to the sum of the vectors shown dotted.



This fact may be used to solve geometric problems using vectors.

**Example 11.2.** For the triangle  $ABC$ , if  $D$  is the mid-point of  $AB$  show that

$$2\underline{AB} + 3\underline{BC} + \underline{CA} = 2\underline{DC}. \quad (1)$$



*Solution.* From the triangle

$$\underline{AB} = 2\underline{AD} \quad (2)$$

$$\underline{BC} = \underline{BD} + \underline{DC} \quad (3)$$

$$\underline{CA} = \underline{CD} + \underline{DA} \quad (4)$$

Substituting the equations (2), (3) and (4) for the left-hand side of (1) gives

$$2\underline{AB} + 3\underline{BC} + \underline{CA} = 4\underline{AD} + 3\underline{BD} + 3\underline{DC} + \underline{CD} + \underline{DA}. \quad (5)$$

Now

$$\underline{BD} = -\underline{AD} \quad (6)$$

$$\underline{CD} = -\underline{DC} \quad (7)$$

$$\underline{DA} = -\underline{AD} \quad (8)$$

Substituting these into the right-hand side of equation (5) gives

$$2\underline{AB} + 3\underline{BC} + \underline{CA} = 4\underline{AD} - 3\underline{AD} + 3\underline{DC} - \underline{DC} - \underline{AD}.$$

So

$$2\underline{AB} + 3\underline{BC} + \underline{CA} = 2\underline{DC}.$$

## 11.2 Vectors II

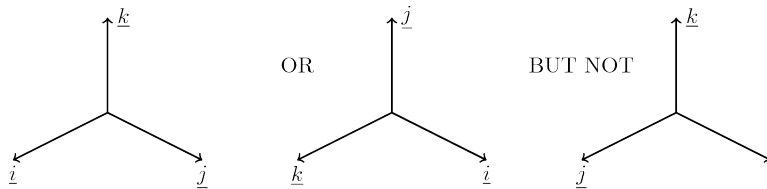
### 11.2.1 Unit Vectors

A **unit vector** is a vector with magnitude of 1.

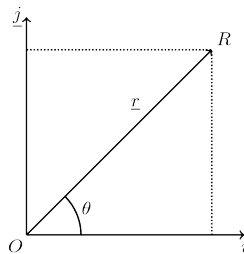
If  $\underline{a}$  is a vector with magnitude  $|\underline{a}|$ , then a unit vector with the same direction is written  $\hat{\underline{a}}$  where  $|\hat{\underline{a}}| = 1$ . Notice that

$$\hat{\underline{a}} = \frac{1}{|\underline{a}|} \underline{a}.$$

For vector systems, there are three special unit vectors  $\underline{i}$ ,  $\underline{j}$  and  $\underline{k}$ . These vectors form a set of axes for a vector system and are perpendicular to each other (like  $x$ ,  $y$  and  $z$  in Cartesian notation). Strictly, they should be written  $\hat{\underline{i}}$ ,  $\hat{\underline{j}}$  and  $\hat{\underline{k}}$  to indicate that they are unit vectors, but they are used so often that the circumflex,  $\hat{\phantom{x}}$ , tends to get left off. The axis vectors are arranged as a righthand system. This means that a screw turned from  $\underline{i}$  to  $\underline{j}$  should travel up the  $\underline{k}$  axis.



A vector may be resolved in terms of its components in the direction of the unit axis vectors. We shall start by considering a two dimensional vector. For this we need only the  $\underline{i}$  and  $\underline{j}$  axes.



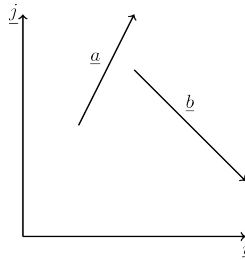
The vector  $\underline{r}$  makes an angle  $\theta$  with the  $\underline{i}$  axis and has magnitude  $|\underline{r}| = r$ . We may say that to get to the point  $R$ , we have to start at the  $\underline{i}$  axis, turn through an angle  $\theta$  and then travel a distance  $r$  from the origin. Alternatively, we may get to  $R$ , by travelling a distance along the  $\underline{i}$  axis and then a distance up in the direction of the  $\underline{j}$  axis. The distance to be travelled along the  $\underline{i}$  axis is  $r \cos(\theta)$  and the distance to

be travelled up in the direction of the  $\underline{j}$  axis is  $r \sin(\theta)$ . The vector  $\underline{r}$  may thus be written as  $\underline{r} = r \cos(\theta)\underline{i} + r \sin(\theta)\underline{j}$ .

When written in this form  $\underline{r}$  is said to be resolved into its **component vectors**.

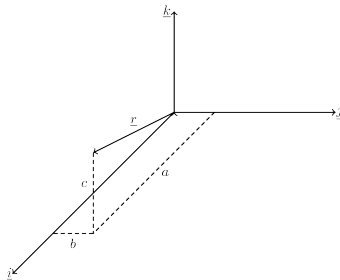
A 2D vector may be given in its component form as, for example  $\underline{r} = 6\underline{i} + 5\underline{j}$ .

If we have two 2D vectors lying in the same plane, then they are said to be co-planar vectors.



The vectors  $\underline{a}$  and  $\underline{b}$  are co-planar.

We may also write 3D vectors in terms of their unit vectors, but this time we need all three axes.



The vector  $\underline{r}$  projects a length  $a$  onto the  $\underline{i}$  axis, a length  $b$  onto the  $\underline{j}$  axis and a length  $c$  onto the  $\underline{k}$  axis, so  $\underline{r}$  may be written  $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ .

### 11.2.2 Vector algebra with component vectors

When a vector is written in terms of its components, then the sum of two vectors is the sum of the components for each direction. e.g.

$$\underline{a} = 6\underline{i} + 2\underline{j}, \underline{b} = 5\underline{i} - \underline{j} \text{ so } \underline{a} + \underline{b} = (6 + 5)\underline{i} + (2 + (-1))\underline{j} = 11\underline{i} + \underline{j}.$$

Note that  $1\underline{i}$ ,  $1\underline{j}$ , or  $1\underline{k}$  are usually written as just  $\underline{i}$ ,  $\underline{j}$  or  $\underline{k}$ .

Similarly

$$\underline{a} = 5\underline{i} + 4\underline{j} - \underline{k}, \underline{b} = 3\underline{i} - 7\underline{j} + 8\underline{k} \text{ so } \underline{a} + \underline{b} = (5+3)\underline{i} + (4-7)\underline{j} + (-1+8)\underline{k} = 8\underline{i} - 3\underline{j} + 7\underline{k}.$$

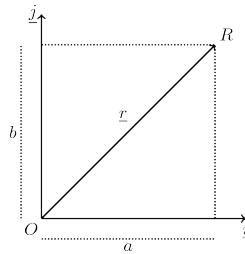
When a vector is written in terms of components, the difference of the two vectors is the difference of the components for each direction. e.g.

$$\underline{a} = 5\underline{i} + 4\underline{j} - \underline{k}, \underline{b} = 3\underline{i} - 7\underline{j} + 8\underline{k} \text{ so } \underline{a} - \underline{b} = (5-3)\underline{i} + (4-(-7))\underline{j} + ((-1)-8)\underline{k} = 2\underline{i} + 11\underline{j} - 9\underline{k}.$$

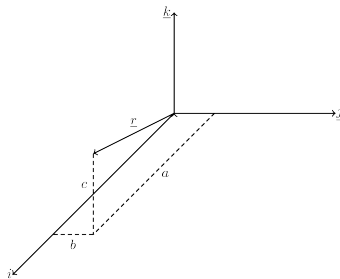
When a vector written in terms of components is multiplied by a scalar, each component is multiplied by that scalar e.g.

$$\underline{a} = 5\underline{i} + 4\underline{j} - \underline{k} \text{ so } 3\underline{a} = 15\underline{i} + 12\underline{j} - 3\underline{k}.$$

### 11.2.3 The magnitude of a vector



The vector  $\underline{r} = a\underline{i} + b\underline{j}$ . From Pythagoras' theorem, the magnitude of vector  $|\underline{r}| = \sqrt{a^2 + b^2}$ . In three dimensions, if  $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ , then  $|\underline{r}| = \sqrt{a^2 + b^2 + c^2}$ .



### 11.2.4 Finding a unit vector

If  $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ , then its magnitude is  $|\underline{r}| = \sqrt{a^2 + b^2 + c^2}$ . That is the length of the vector is  $\sqrt{a^2 + b^2 + c^2}$ . If we want to find a vector that has the same

direction as  $\underline{r}$ , but has length of 1, i.e. the unit vector  $\hat{\underline{r}}$ , then we must divide  $\underline{r}$  by its own magnitude. So if  $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ , then

$$\hat{\underline{r}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}}(a\underline{i} + b\underline{j} + c\underline{k})$$

which can be written

$$\hat{\underline{r}} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}\underline{i} + \frac{b}{\sqrt{a^2 + b^2 + c^2}}\underline{j} + \frac{c}{\sqrt{a^2 + b^2 + c^2}}\underline{k}$$

or

$$\hat{\underline{r}} = \frac{a}{|\underline{r}|}\underline{i} + \frac{b}{|\underline{r}|}\underline{j} + \frac{c}{|\underline{r}|}\underline{k}$$

**Example 11.3.** If  $\underline{r} = 5\underline{i} + 2\underline{j} - \underline{k}$ , find a unit vector that is

- parallel to  $\underline{r}$ ,
- has the opposite direction to  $\underline{r}$ .

*Solution.*  $|\underline{r}| = \sqrt{5^2 + 2^2 + 1^2} = \sqrt{25 + 4 + 1} = \sqrt{30}$ .

- The unit vector in the direction of  $\underline{r}$  is

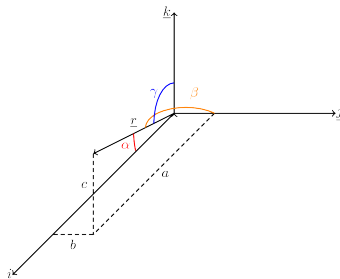
$$\hat{\underline{r}} = \frac{5}{\sqrt{30}}\underline{i} + \frac{2}{\sqrt{30}}\underline{j} - \frac{1}{\sqrt{30}}\underline{k}.$$

- The unit vector in the opposite direction is

$$-\hat{\underline{r}} = -\left(\frac{5}{\sqrt{30}}\underline{i} + \frac{2}{\sqrt{30}}\underline{j} - \frac{1}{\sqrt{30}}\underline{k}\right) = -\frac{5}{\sqrt{30}}\underline{i} - \frac{2}{\sqrt{30}}\underline{j} + \frac{1}{\sqrt{30}}\underline{k}$$

### 11.2.5 Direction Cosines

The *direction cosines* of a vector are the cosines of the angles that that vector makes with each of the three axes. The angles are marked  $\alpha$ ,  $\beta$  and  $\gamma$  in the figure below



The direction cosines are

$$l = \cos(\alpha) = \frac{a}{|\underline{r}|}, \quad m = \cos(\beta) = \frac{b}{|\underline{r}|}, \quad n = \cos(\gamma) = \frac{c}{|\underline{r}|}$$

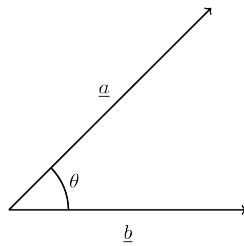
**Example 11.4.** Find the direction cosines of the vector  $\underline{r} = 3\underline{i} + 4\underline{j} - 5\underline{k}$ .

*Solution.* The magnitude of  $\underline{r}$  is  $|\underline{r}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50}$ .

The direction cosines of  $\underline{r}$  are then

$$l = \frac{3}{\sqrt{50}}, \quad m = \frac{4}{\sqrt{50}}, \quad n = \frac{-5}{\sqrt{50}}$$

### 11.2.6 The angle between two vectors



If we have two vectors  $\underline{a}$  and  $\underline{b}$  where  $\underline{a}$  has direction cosines  $l_a, m_a, n_a$  and  $\underline{b}$  has direction cosines  $l_b, m_b, n_b$ , then the cosine of the angle  $\theta$  between the vectors is given by:

$$\cos(\theta) = l_a l_b + m_a m_b + n_a n_b.$$

**Example 11.5.** Determine the angle between the two vectors and  $\underline{a} = 2\underline{i} + 3\underline{j} + 4\underline{k}$  and  $\underline{b} = 4\underline{i} - 3\underline{j} + 2\underline{k}$ .

*Solution.*

$$|\underline{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}, \quad |\underline{b}| = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}.$$

So  $\underline{a}$  has direction cosines

$$l_a = \frac{2}{\sqrt{29}}, \quad m_a = \frac{3}{\sqrt{29}}, \quad n_a = \frac{4}{\sqrt{29}}$$

and  $\underline{b}$  has direction cosines

$$l_b = \frac{4}{\sqrt{29}}, \quad m_b = \frac{-3}{\sqrt{29}}, \quad n_b = \frac{2}{\sqrt{29}}$$

The cosine of the angle  $\theta$  between the two vectors is given by

$$\cos(\theta) = \frac{2}{\sqrt{29}} \frac{4}{\sqrt{29}} + \frac{3}{\sqrt{29}} \frac{-3}{\sqrt{29}} + \frac{4}{\sqrt{29}} \frac{2}{\sqrt{29}} = \frac{8}{29} - \frac{9}{29} + \frac{8}{29} = \frac{7}{29}.$$

Hence the angle is  $\theta = \cos^{-1}(7/29) = 76^\circ$ .

## 11.3 Vectors III

### 11.3.1 Multiplication of vectors

We know from studying mechanical systems that there are two possible situations when two vectors are multiplied together. A vector multiplied by a vector can result in a scalar quantity, for example, a force (vector) multiplied by the distance (vector) an object moves as a result of that force gives the amount of work (scalar) done by that force. A vector multiplied by a vector can also result in a vector quantity, for example a force (vector) multiplied by the perpendicular distance (vector) of that force from a given point, gives the moment (vector) of that force about the point. Since there may be two different forms multiplying vectors together, we can deduce that there are two different types of vector multiplication; one that results in a scalar quantity and one that results in a vector quantity.

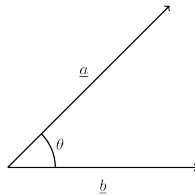
### 11.3.2 The scalar or dot product

The scalar product of two vectors has a scalar quantity as its result.

The scalar product of two vectors  $\underline{a}$  and  $\underline{b}$  is defined as

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos(\theta)$$

where  $\theta$  is the angle between the two vectors as shown below



If  $\underline{a}$  and  $\underline{b}$  are parallel then  $\theta = 0^\circ$  and  $\cos(\theta) = 1$ , so  $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|$ .

If  $\underline{a}$  and  $\underline{b}$  are perpendicular then  $\theta = 90^\circ$  and  $\cos(\theta) = 0$  so  $\underline{a} \cdot \underline{b} = 0$ .

If  $\underline{a}$  and  $\underline{b}$  are given in terms of component vectors as  $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$  and  $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$  then

$$\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

**Example 11.6.** Find the scalar product of  $\underline{a} = 3\underline{i} + 2\underline{j} - 6\underline{k}$  and  $\underline{b} = -\underline{i} + 5\underline{j} - \underline{k}$ .

*Solution.*

$$\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3 = (3)(-1) + (2)(5) + (-6)(-1) = -3 + 10 + 6 = 13.$$

The scalar product of the vectors is 13.

Now since the scalar product is defined as

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos(\theta) \quad (1)$$

it gives us another way of finding the angle  $\theta$  between two vectors. We can rearrange equation (1) to give

$$\cos(\theta) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}. \quad (2)$$

**Example 11.7.** Using the vectors from the previous example,  $\underline{a} = 3\underline{i} + 2\underline{j} - 6\underline{k}$  and  $\underline{b} = -\underline{i} + 5\underline{j} - \underline{k}$ , we found that  $\underline{a} \cdot \underline{b} = 13$ . Also,  $|\underline{a}| = \sqrt{3^2 + 2^2 + (-6)^2} = \sqrt{49} = 7$  and  $|\underline{b}| = \sqrt{(-1)^2 + 5^2 + (-1)^2} = \sqrt{27}$  so from equation (2)

$$\cos(\theta) = \frac{13}{7\sqrt{27}} = 0.3574$$

and thus  $\theta = \cos^{-1}(0.3574) = 69^\circ$ .

**Example 11.8.** Find the unit vector perpendicular to the vector  $4\underline{i} + 4\underline{j} - 7\underline{k}$  and to the vector  $3\underline{i} - 2\underline{j} + \underline{k}$

*Solution.* Let the vector which is perpendicular to the two given vectors be  $p\underline{i} + q\underline{j} + r\underline{k}$ .

If the vectors are perpendicular then the scalar product is zero since the angle between them is  $90^\circ$  so

$$(4\underline{i} + 4\underline{j} - 7\underline{k}) \cdot (p\underline{i} + q\underline{j} + r\underline{k}) = 0 \quad (1)$$

and

$$(3\underline{i} - 2\underline{j} + \underline{k}) \cdot (p\underline{i} + q\underline{j} + r\underline{k}) = 0. \quad (2)$$



Evaluating the scalar products (1) and (2) gives

$$4p + 4q - 7r = 0 \quad (3)$$

$$3p - 2q + r = 0 \quad (4)$$

We now solve equations (3) and (4) simultaneously

$$4p + 4q - 7r = 0 \quad (3)$$

$$6p - 4q + 2r = 0 \quad (4 \times 2)$$

therefore by adding  $10p - 5r = 0$  so

$$r = 2p. \quad (5)$$

Substituting into (4) gives  $3p - 2q + 2p = 0$  so

$$q = \frac{5}{2}p. \quad (6)$$

We may choose any value for  $p$  and, provided  $q$  and  $r$  have the relationships (5) and (6) with our chosen value, the vector will be perpendicular to the given vectors. We will choose  $p = 2$ , then  $r = 4$  and  $q = 5$  so

$$p\underline{i} + q\underline{j} + r\underline{k} = 2\underline{i} + 5\underline{j} + 4\underline{k}.$$

We were asked to find the unit vector. The magnitude of  $2\underline{i} + 5\underline{j} + 4\underline{k} = \sqrt{45} = 3\sqrt{5}$  so the unit vector perpendicular to the given vectors is

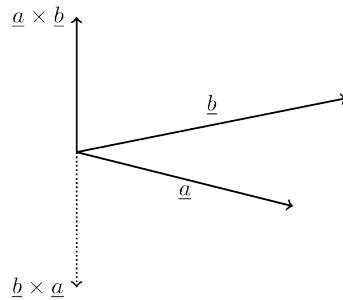
$$\frac{2}{3\sqrt{5}}\underline{i} + \frac{5}{3\sqrt{5}}\underline{j} + \frac{4}{3\sqrt{5}}\underline{k}.$$

### 11.3.3 The vector or cross product

The vector product of two vectors has as its result a vector.

The vector product is defined as  $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin(\theta)\hat{n}$ .

That is to say that the magnitude of  $\underline{a} \times \underline{b}$  is  $|\underline{a}||\underline{b}| \sin(\theta)$  where  $\theta$  is the angle between  $\underline{a}$  and  $\underline{b}$  and, since it is a vector, it must also have a direction so  $\hat{n}$  indicates the direction. For a vector product,  $\hat{n}$  is always perpendicular to both  $\underline{a}$  and  $\underline{b}$ . For  $\hat{n}$  to be perpendicular to  $\underline{a}$  and  $\underline{b}$ , it could either point in the direction shown below, or in the opposite direction, however for a vector product,  $\hat{n}$  always forms a right-hand set of axes (like  $\underline{i}$ ,  $\underline{j}$  and  $\underline{k}$ ). The vector in the opposite direction (shown dotted) would be obtained from  $\underline{b} \times \underline{a}$ .



If  $\underline{a}$  and  $\underline{b}$  are parallel (i.e.  $\theta = 0$ ), then since  $\sin(\theta) = 0$ ,  $\underline{a} \times \underline{b} = \underline{0}$ .

If  $\underline{a}$  and  $\underline{b}$  are perpendicular (i.e.  $\theta = 90^\circ$ ) then since  $\sin(90^\circ) = 1$ ,  $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}|\hat{n}$

If  $\underline{a}$  and  $\underline{b}$  are specified in terms of their component vectors, so that  $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$  and  $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$  then

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

**Example 11.9.** Find the vector product of  $\underline{a} = 3\underline{i} + 2\underline{j} - 6\underline{k}$  and  $\underline{b} = -\underline{i} + 5\underline{j} - \underline{k}$ .

*Solution.*

$$\begin{aligned} \underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -6 \\ -1 & 5 & -1 \end{vmatrix} \\ &= \underline{i}((2)(-1) - (-6)(5)) - \underline{j}((3)(-1) - (-6)(-1)) + \underline{k}((3)(5) - (2)(-1)) \\ &= 28\underline{i} + 9\underline{j} + 17\underline{k}. \end{aligned}$$

**Example 11.10.** Find a unit vector which is perpendicular to the vector  $4\underline{i} + 4\underline{j} - 7\underline{k}$  and to the vector  $3\underline{i} - 2\underline{j} + \underline{k}$ .

*Solution.* From the definition of the vector product, a vector perpendicular to both

these vectors is given by

$$\begin{aligned}
 (4\underline{i} + 4\underline{j} - 7\underline{k}) \times (3\underline{i} - 2\underline{j} + \underline{k}) &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 4 & -7 \\ 3 & -2 & 1 \end{vmatrix} \\
 &= \underline{i}((4)(1) - (-7)(-2)) \\
 &\quad - \underline{j}((4)(1) - (-7)(3)) \\
 &\quad + \underline{k}((4)(-2) - (4)(3)) \\
 &= -10\underline{i} - 25\underline{j} - 20\underline{k}.
 \end{aligned}$$

So a **unit vector** perpendicular to both given vectors is

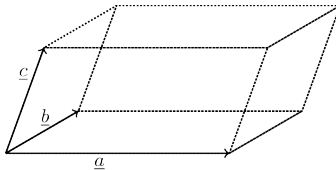
$$\frac{-10}{15\sqrt{5}}\underline{i} - \frac{25}{15\sqrt{5}}\underline{j} - \frac{20}{15\sqrt{5}}\underline{k} = \frac{-2}{3\sqrt{5}}\underline{i} - \frac{5}{3\sqrt{5}}\underline{j} - \frac{4}{3\sqrt{5}}\underline{k}.$$

### 11.3.4 The scalar triple product

If we have three vectors  $\underline{a}$ ,  $\underline{b}$  and  $\underline{c}$  then the scalar triple product of those vectors is defined as  $\underline{a} \cdot (\underline{b} \times \underline{c})$ .

Note that the vector product must be evaluated first, then the scalar product.

The scalar triple product gives the volume of a parallelepiped with edges defined by the vectors  $\underline{a}$ ,  $\underline{b}$  and  $\underline{c}$ .



**Example 11.11.** Find the volume of a parallelepiped with edges  $\underline{a} = 2\underline{i} - 3\underline{j} + 4\underline{k}$ ,  $\underline{b} = \underline{i} + 2\underline{j} - \underline{k}$  and  $\underline{c} = 3\underline{i} - \underline{j} + 2\underline{k}$ .

*Solution.* The volume is given by  $\underline{a} \cdot (\underline{b} \times \underline{c})$ . First we calculate the vector product

$$\underline{b} \times \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 3\underline{i} - 5\underline{j} - 7\underline{k}.$$

and now

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = (2\underline{i} - 3\underline{j} + 4\underline{k}) \cdot (3\underline{i} - 5\underline{j} - 7\underline{k}) = -7.$$

The volume is therefore 7 units (and the -ve sign tells us that the system was not right handed).

### 11.3.5 The vector triple product

If we have three vectors  $\underline{a}$ ,  $\underline{b}$  and  $\underline{c}$  then the vector triple product of those vectors is defined as

$$\underline{a} \times (\underline{b} \times \underline{c})$$

**Example 11.12.** Find the vector triple product  $\underline{a} \times (\underline{b} \times \underline{c})$  where  $\underline{a} = 2\underline{i} - 3\underline{j} + 4\underline{k}$ ,  $\underline{b} = \underline{i} + 2\underline{j} - \underline{k}$  and  $\underline{c} = 3\underline{i} - \underline{j} + 2\underline{k}$ .

*Solution.* From the previous example,  $\underline{b} \times \underline{c} = 3\underline{i} - 5\underline{j} - 7\underline{k}$ .

$$\underline{a} \times (\underline{b} \times \underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 4 \\ 3 & -5 & 7 \end{vmatrix} = 41\underline{i} + 26\underline{j} - \underline{k}$$

# Week 11 Exercises

## 11.4 Sheet 21 Vectors I

### Exercise 1

Two vectors  $\underline{A}$  and  $\underline{B}$  are at right angles. Find

1. the magnitude of the vector  $\underline{A} + \underline{B}$  if  $|\underline{A}| = 1$  and  $|\underline{B}| = 2$ .
2. the magnitude of the vector  $\underline{A} + \underline{B}$  if  $|\underline{A}| = 20$  and  $|\underline{B}| = 21$ .

### Exercise 2

Determine the resultant of the vectors  $\underline{A}$ ,  $\underline{B}$ ,  $\underline{C}$  and  $\underline{D}$  given that  $\underline{A} = 20 \text{ km } 30^\circ \text{ S of E}$ ,  $\underline{B} = 50 \text{ km due W}$ ,  $\underline{C} = 40 \text{ km NE}$  and  $\underline{D} = 30 \text{ km SW}$ .

### Exercise 3

An object  $P$  is acted on by three forces as shown below. All forces lie in the same plane (are *co-planar*). Determine the required force and its direction so as to prevent  $P$  from moving.

### Exercise 4

For the triangle shown, prove by vectors that the line  $DE$ , where  $D$  and  $E$  are the mid-points of  $AB$  and  $AC$  respectively, is half the length of the side  $BC$ .

### Exercise 5

The quadrilateral  $ABCD$  shown below has the mid-points of the diagonals as  $P$  and  $Q$ . Show that  $\underline{AB} + \underline{AD} + \underline{CB} + \underline{CD} = 4\underline{PQ}$

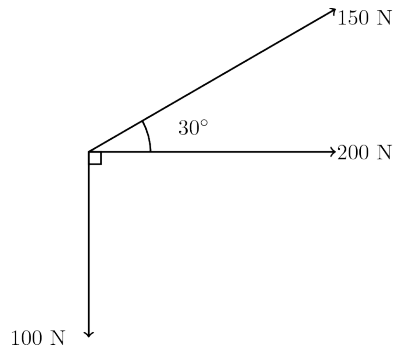


Figure 11.1: resultant force

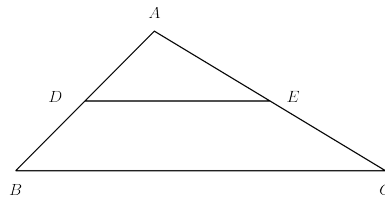


Figure 11.2: triangles

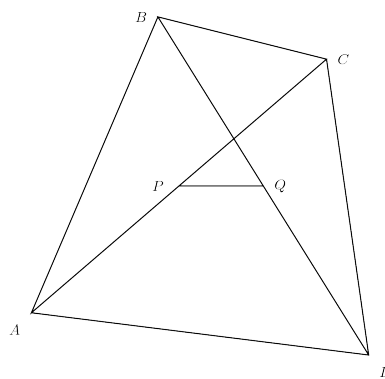


Figure 11.3: quadrilateral

## 11.5 Sheet 22 Vectors II

### Exercise 1

If  $\underline{z}_1 = 5\underline{i} - 2\underline{j}$ ,  $\underline{z}_2 = 3\underline{i} + 3\underline{j}$  and  $\underline{z}_3 = 4\underline{i} - \underline{j}$ , determine  $\underline{z}_1 + \underline{z}_2 + \underline{z}_3$  and  $\underline{z}_1 - \underline{z}_2 - \underline{z}_3$ .

### Exercise 2

If  $\underline{OA} = 4\underline{i} + 3\underline{j}$  and  $\underline{OB} = 5\underline{i} - 2\underline{j}$  determine  $\underline{AB}$ .

### Exercise 3

If  $\underline{OA} = 4\underline{i} + 3\underline{j}$ ,  $\underline{OB} = 6\underline{i} - 2\underline{j}$  and  $\underline{OC} = 2\underline{i} - \underline{j}$  determine  $\underline{AB}$ ,  $\underline{BC}$  and  $\underline{CA}$  and establish the lengths of the sides of the triangle  $ABC$ .

### Exercise 4

Determine the magnitude and direction cosines of each of the vectors  $3\underline{i} + 7\underline{j} - 4\underline{k}$ ,  $\underline{i} - 5\underline{j} - 8\underline{k}$  and  $6\underline{i} - 2\underline{j} + 12\underline{k}$  and then determine the magnitude and direction cosines of their sums.

### Exercise 5

If  $\underline{A} = 2\underline{i} + 4\underline{j} - 3\underline{k}$  and  $\underline{B} = \underline{i} + 3\underline{j} + 2\underline{k}$  establish the value of the angle between  $\underline{A}$  and  $\underline{B}$ .

### Exercise 6

If  $\underline{OA} = 2\underline{i} + 3\underline{j} - \underline{k}$  and  $\underline{OB} = \underline{i} - 2\underline{j} + 3\underline{k}$  establish the value of the angle between  $\underline{OA}$  and  $\underline{OB}$ .

### Exercise 7

Given that the position vectors describing points  $P$  and  $Q$  are  $\underline{i} + 3\underline{j} - 7\underline{k}$  and  $5\underline{i} - 2\underline{j} + 4\underline{k}$  respectively, find the vector  $\underline{PQ}$  and determine its direction cosines.

### Exercise 8

If  $\vec{a} = \frac{2}{11}\underline{i} + p\underline{j} + \frac{9}{11}\underline{k}$  is a unit vector, what is the value of  $p$ ?

**Exercise 9**

Find the unit vector  $\vec{a}$  parallel to the vector  $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$ .

**11.6 Sheet 23 Vectors III****Exercise 1**

If  $\underline{a} = 2\underline{i} + 4\underline{j} - 3\underline{k}$  and  $\underline{b} = \underline{i} + 3\underline{j} + 2\underline{k}$ , calculate the scalar and vector product and the angle between the vectors  $\underline{a}$  and  $\underline{b}$ .

**Exercise 2**

Given that  $\underline{OA} = 2\underline{i} + 3\underline{j} - \underline{k}$  and  $\underline{OB} = \underline{i} - 2\underline{j} + 3\underline{k}$  determine

1. the scalar product  $\underline{OA} \cdot \underline{OB}$ ,
2. the vector product  $\underline{OA} \times \underline{OB}$ ,
3. the angle between  $\underline{OA}$  and  $\underline{OB}$ .

**Exercise 3**

Determine the scalar and vector product of  $\underline{a}$  and  $\underline{b}$  when

1.  $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$  and  $\underline{b} = 2\underline{i} + 3\underline{j} + \underline{k}$ ,
2.  $\underline{a} = 2\underline{i} + 3\underline{j} + 4\underline{k}$  and  $\underline{b} = 5\underline{i} - 2\underline{j} + \underline{k}$ .

**Exercise 4**

Use the scalar product to find the angle between the vectors  $\underline{a} = 2\underline{i} + 3\underline{j} - \underline{k}$  and  $\underline{b} = 3\underline{i} - 5\underline{j} + 2\underline{k}$ .

**Exercise 5**

If  $\underline{a} = 3\underline{i} - \underline{j} + 2\underline{k}$  and  $\underline{b} = \underline{i} + 3\underline{j} - 2\underline{k}$ , determine the magnitude and direction cosines of the vector product  $\underline{a} \times \underline{b}$ .

**Exercise 6**

If  $\underline{a} = 5\underline{i} + 4\underline{j} + 2\underline{k}$ ,  $\underline{b} = 4\underline{i} - 5\underline{j} + 3\underline{k}$  and  $\underline{c} = 2\underline{i} - \underline{j} - 2\underline{k}$  determine

1. the value of  $\underline{a} \cdot \underline{b}$  and the angle between the vectors  $\underline{a}$  and  $\underline{b}$ ,
2. the magnitude and direction cosines of the vector product  $\underline{a} \times \underline{b}$ ,
3. the angle that  $\underline{a} \times \underline{b}$  makes with  $\underline{c}$ .



**Exercise 7**

If position vectors  $\underline{OA}$ ,  $\underline{OB}$  and  $\underline{OC}$  are given as  $2\underline{i} - \underline{j} + 3\underline{k}$ ,  $3\underline{i} + 2\underline{j} - 4\underline{k}$  and  $-\underline{i} + 2\underline{j} - 4\underline{k}$  respectively. Determine

1. the vector  $\underline{AB}$ ,
2. the vector  $\underline{BC}$ ,
3. the vector product  $\underline{OA} \times \underline{OB}$ ,
4. the area of the parallelogram with sides  $\underline{OA}$  and  $\underline{OB}$ ,
5. the area of the triangle with sides  $\underline{OA}$  and  $\underline{OB}$ ,
6. the angle between the vectors  $\underline{AB}$  and  $\underline{BC}$ .

**Exercise 8**

If  $\underline{a} = 2\underline{i} - 3\underline{j} - \underline{k}$  and  $\underline{b} = \underline{i} + 4\underline{j} - 2\underline{k}$ , determine

1.  $\underline{a} \times \underline{b}$ ,
2.  $\underline{b} \times \underline{a}$ ,
3.  $(\underline{a} + \underline{b}) \times (\underline{a} - \underline{b})$ .

**Exercise 9**

If  $\underline{a} = 3\underline{i} - \underline{j} + 2\underline{k}$ ,  $\underline{b} = 2\underline{i} + \underline{j} - \underline{k}$  and  $\underline{c} = \underline{i} - 2\underline{j} + 2\underline{k}$ , determine

1.  $(\underline{a} \times \underline{b}) \times \underline{c}$ ,
2.  $\underline{a} \times (\underline{b} \times \underline{c})$ ,
3.  $5(\underline{a} \times \underline{b})$ ,  $(5\underline{a}) \times \underline{b}$ ,  $\underline{a} \times (5\underline{b})$ .

**Exercise 10**

(optional) Prove the sine rule using vectors.

**Exercise 11**

(optional) Prove the cosine rule using vectors.

**Exercise 12**

Show that the three vectors  $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$ ,  $\underline{b} = \underline{i} - 3\underline{j} + 5\underline{k}$  and  $\underline{c} = 2\underline{i} + \underline{j} - 4\underline{k}$  form a right angled triangle.

**Exercise 13**

Determine the value of  $\alpha$  which will make the two vectors  $\underline{a} = 2\underline{i} + \alpha\underline{j} + \underline{k}$  and  $\underline{b} = 4\underline{i} - 2\underline{j} - 2\underline{k}$  be perpendicular to one another.

**Exercise 14**

Determine the area of the triangle having vertices at  $A = (1, 3, 2)$ ,  $B = (2, -1, 1)$  and  $C = (-1, 2, 3)$ .

**Exercise 15**

If  $\underline{a} = 3\underline{i} + 2\underline{j} - 4\underline{k}$ ,  $\underline{b} = 4\underline{i} - 3\underline{j} + 2\underline{k}$  and  $\underline{c} = 2\underline{i} - 4\underline{j} - 3\underline{k}$  determine

1.  $\underline{a} \cdot (\underline{b} \times \underline{c})$ ,
2.  $\underline{b} \cdot (\underline{c} \times \underline{a})$ ,
3.  $\underline{c} \cdot (\underline{a} \times \underline{b})$ ,
4.  $\underline{a} \times (\underline{b} \times \underline{c})$ ,
5.  $\underline{c} \times (\underline{a} \times \underline{b})$ ,
6.  $\underline{b} \times (\underline{c} \times \underline{a})$ .

**Exercise 16**

Given that a parallelepiped has its edges represented by the vectors  $\underline{a} = 4\underline{i} - 2\underline{j} - 3\underline{k}$ ,  $\underline{b} = \underline{i} + 2\underline{j} + \underline{k}$  and  $\underline{c} = 3\underline{i} - \underline{j} + 2\underline{k}$  determine the volume of the parallelepiped.

**Exercise 17**

Determine  $x$  so that the vectors  $\underline{a} = x\underline{i} - 2x\underline{j} - 3\underline{k}$ ,  $\underline{b} = 2\underline{i} + \underline{j} - \underline{k}$  and  $\underline{c} = \underline{i} - \underline{j} + 14\underline{k}$  are coplanar.

# Index

- (Second) Fundamental Theorem of Calculus*, 80
- acceleration*, 31
- asymptote*, 46
- boundary condition*, 63
- center of gravity*, 107
- center of mass*, 106, 107
- centre of gravity*, 106
- centroid*, 109
- centroid of the area*, 109
- component vectors*, 122
- composite function*, 19
- continuous*, 6
- critical points*, 32
- density*, 107
- derivative*, 8
- differentiation by 1st principles*, 9
- differentiation of a quotient*, 24
- direction cosines*, 124
- discontinuous*, 6
- equation of the normal*, 18
- equation of the tangent*, 18
- fourth derivative of  $y$* , 30
- function of a function*, 19
- gradient*, 7
- horizontal asymptote*, 46
- indefinite integration*, 63
- integration by parts*, 71
- integration by substitution*, 68
- lamina*, 109
- limit*, 5
- local maximum*, 32
- local minimum*, 32
- maximum*, 32
- mean*, 100
- mid-ordinate rule*, 88
- minimum*, 32
- moment*, 106
- multiplied by a scalar*, 118
- normal*, 17
- parallel*, 16
- partial differentiation*, 59
- point of inflection*, 32
- principle of moments*, 106
- Product Rule*, 22
- resultant*, 119
- RMS*, 101
- root mean square*, 101
- scalar*, 117
- second derivative of  $y$* , 29
- Simpson's rule*, 92, 93
- stationary points*, 32
- sum*, 119

*tangent*, 8

*third derivative of  $y$* , 29

*trapezium rule*, 91

*turning points*, 32

*unit vector*, 121

*vector*, 117

*velocity*, 30

*vertical asymptote*, 46