Lecture notes - Maths A - Foundation year

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These lecture notes are based on original design by Dr. Bernhard Koeck, building on previous iterations by Prof. Anna Barney and Dr. Vesna Perišić. The exercises were designed by Prof. Anna Barney.

Please let the author know if you find any errors, misprints and/or inconsistencies.

Chapter 1

Simultaneous equations

1.1 Linear equations

A **linear equation in one variable (or unknown)** is an equation of the form

$$ax = b,$$

where a and b are constants and $a \neq 0$.

Example 1.1.

$$9x = 12$$

To solve it (meaning to find all numbers which, if substituted for x, yield a true statement), we divide both sides by 9: $x = \frac{12}{9} = \frac{4}{3} = 1.\overline{3} \approx 1.33$ (2 d.p.).

In general, the **solution** of the equation $ax = b$ is $\boxed{x = \frac{b}{a}}$.

A **linear equation in two variables x and y** is an equation of the form

$$ax + by = c,$$

where a, b and c are constants, and both a and b are not zero. (Note that if one of them would be zero, then the equation has effectively only one variable.). If $c = 0$, then the equation is called **homogeneous**.

Example 1.2.

$$4x - 3y = 2$$

We can make x the subject: $x = \frac{2+3y}{4} = \frac{1}{2} + \frac{3}{4}y$.

We can make y the subject: $y = \frac{2-4x}{-3} = -\frac{2}{3} + \frac{4}{3}x$.

For each value of x we get one value of y so that the pair (x, y) satisfies the equation. Also vice versa, for each value of y we get one value of x.

So the equation $4x - 3y = 2$ has infinitely many solutions. Table 1.1 shows some of them.

Graphing these values (Figure 1.1) we see that they appear to lie on a line.

The graph of the set of solutions of a linear equation in two variables is always a straight line.

Table 1.1: Some solutions of $4x - 3y = 2$.

x	y
-2	-3.3333333
-1	-2.0000000
0	-0.6666667
1	0.6666667
2	2.0000000

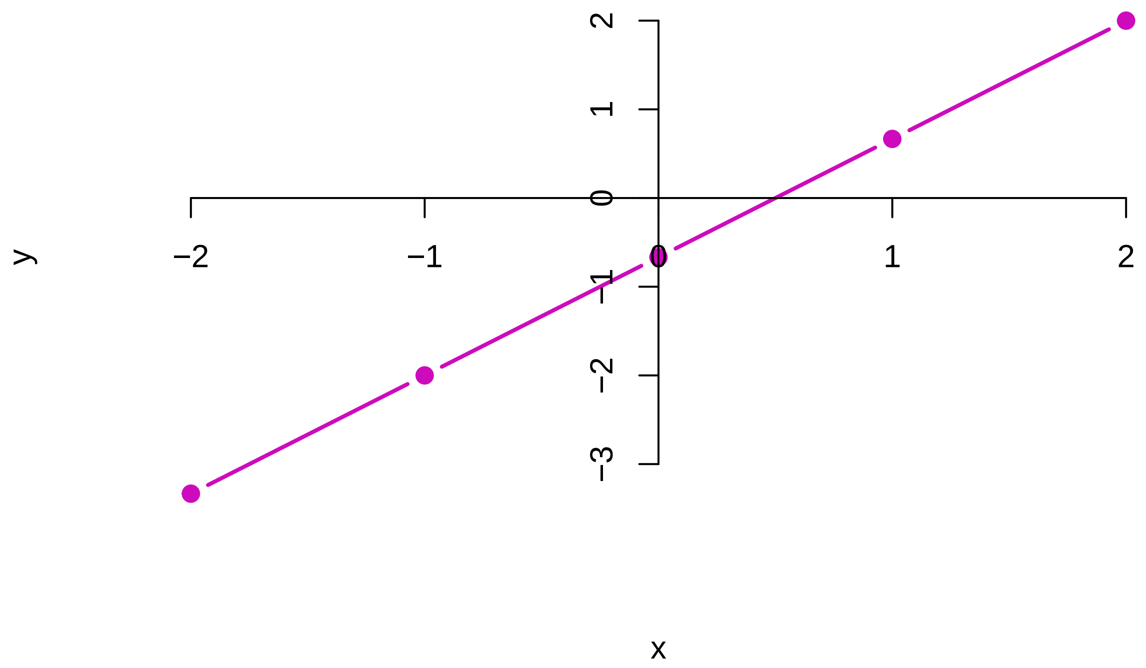


Figure 1.1: Graphing some solutions of $4x - 3y = 2$.

1.2 Solving a pair of linear equations in two variables

Example 1.3.

$$x + 2y = -2 \tag{A}$$
$$2x + 3y = 1 \tag{B}$$

1.2.1 By substitution

1	Express x from (A):	$x = -2 - 2y$
2	Substitute into (B):	$2(-2 - 2y) + 3y = 1$
3	Multiply out:	$-4 - 4y + 3y = 1$
4	Rearrange:	$y = -5$
5	Substitute back into (A):	$x + 2(-5) = -2$
6	Rearrange:	$x = 8$

1.2.2 By elimination

1	Multiply (A) by 2:	$2x + 4y = -4$	Call this (A')
2	Subtract (A') from (B):	$-y = 5$	
3	Rearrange:	$y = -5$	
4	Substitute into (A) (as above)	$x = 8$	

Extra Note (non-examinable): For "2-by-2" systems like these, the two methods are about the same complexity. However for larger systems of linear equations, the latter leads to a quite effective method called *Gaussian elimination*, which is also amenable to be implemented as a computer algorithm.

1.2.3 Graphical illustration

The solutions of the equation (A) *alone* lie on some straight line. Likewise for the set of solutions of the equation (B). The solutions of the system of equations, i.e. both equations simultaneously, are exactly the point(s) where do these two lines intersect. For a sketch for the system (A),(B) above, see Figure 1.2.

Extra Note (non-examinable): This is a general principle. Two lines in a plane can intersect either:

a) at a single point, or
b) at a line, or
c) not at all.

These are exactly the options for the set of solution of a pair of linear equations in two variables. Such a system can have:

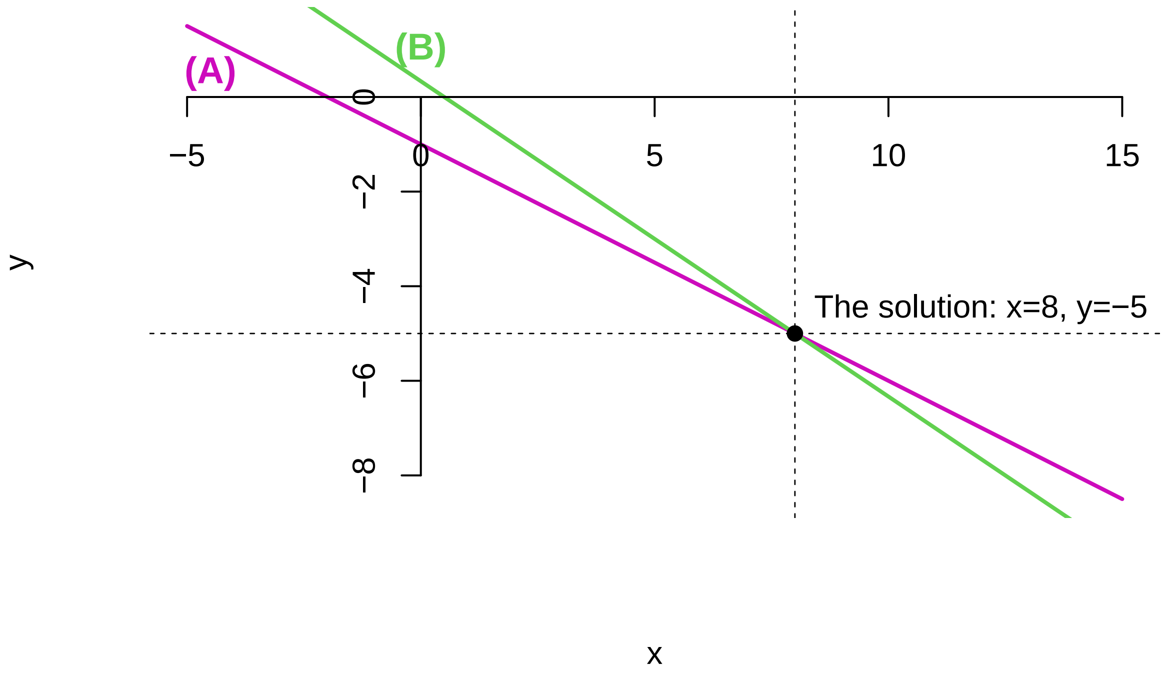


Figure 1.2: Solutions of (A) and (B) graphically

a) a unique solution, or
b) infinitely many solutions, arranged on a line, or
c) no solutions.

Going to higher dimensions (more equations and more unknowns), the sets of solutions of *linear* equations are always "flats" in a higher dimensional space (the dimension corresponds to the number of unknowns).

Exercises 1 (Simultaneous Equations)

Solve the following equations by elimination and check your results:

1.

$$\begin{aligned} 3x + 4y &= 11 \\ x + 7y &= 15 \end{aligned}$$

2.

$$\begin{aligned} 2x + 3y &= 16 \\ 3x + 2y &= 14 \end{aligned}$$

Solve the following equations by substitution and check your results:

3.

$$\begin{aligned} 5x + 3y &= 29 \\ \frac{3x}{4} - \frac{2y}{5} &= \frac{3}{10} \end{aligned}$$

4.

$$\begin{aligned} \frac{2x}{3} - \frac{y}{4} &= \frac{7}{12} \\ 4x + 7y &= 37 \end{aligned}$$

5.

$$\begin{aligned} \frac{x}{4} + \frac{y}{5} &= \frac{3}{2} \\ \frac{2x}{7} - \frac{y}{4} &= \frac{5}{14} \end{aligned}$$

6.

$$\begin{aligned} \frac{x}{2} + \frac{y}{3} &= \frac{13}{6} \\ 2x + 3y &= 19 \end{aligned}$$

7. During a lab experiment on force resolution, the following equations were found:

$$9F_H - 1.5F_V = 7.5$$
$$6.25F_H - 2.5F_V = 8.75$$

   Determine the values of both the horizontal and the vertical force components and check your results.

8. A weight being moved against a frictional force is related by the law

$$F = aW + b,$$

   where a and b are constants. When $F = 6$, $W = 7.5$, and when $F = 2.7$, $W = 2$. Determine the value of the constants a and b and check your results.

Solve by substitution:

9.

$$y = 2x$$
$$3x + 2y = 21$$

10.

$$y = 3x - 7$$
$$5x - 3y = 1$$

11.

$$x = 5y - 3$$
$$3x - 8y = 12$$

12.

$$2x - y = 10$$
$$3x + 2y = 29$$

13.

$$\frac{y}{2} - x = 2$$
$$6x - \frac{3y}{2} = 3$$

14.

$$\frac{x}{2} - \frac{y}{3} = \frac{1}{6}$$
$$\frac{y}{2} - \frac{x}{6} = 5$$

15. The cost of 4 ties and 6 pairs of socks was £68, while that of 5 ties and 8 pairs of socks was £87.40. What were the prices of a tie and a pair of socks, respectively?

16. The bill for the telephone for a quarter can be expressed in the form

$$C = a + \frac{nb}{100},$$

where C is the total cost in pounds, a is the fixed charge, n is the number of calls, and b is the price of each call in pence. When 104 calls were made, the bill was £58.30, and when 67 calls were made, the bill was £50.90. Find the fixed charge and the cost of each call.

Solutions: **1.** $x = 1, y = 2$; **2.** $x = 2, y = 4$; **3.** $x = 2.941, y = 4.765$; **4.** $x = 2.353, y = 3.941$; **5.** $x = 3.731, y = 2.836$; **6.** $x = 0.2, y = 6.2$; **7.** $F_H = 0.43, F_V = -2.43$; **8.** $a = 0.6, b = 1.5$; **9.** $x = 3, y = 6$; **10.** $x = 5, y = 8$; **11.** $x = 12, y = 3$; **12.** $x = 7, y = 4$; **13.** $x = 3, y = 10$; **14.** $x = 9, y = 13$; **15.** tie £9.80, socks £4.80; **16.** fixed charge £37.50, call 20p;

Chapter 2

Rectangular (Cartesian) coordinates

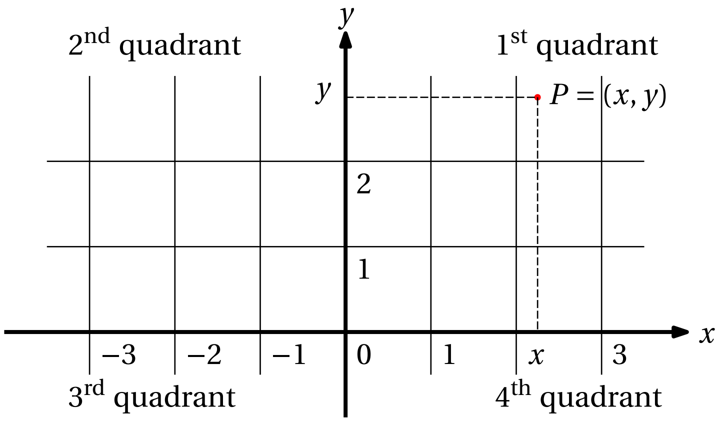


Figure 2.1: Cartesian coordinate system

x and y are called the **Cartesian** (or **rectangular**) coordinates of P.

The whole plane is split by the coordinate axes into four regions called **quadrants**.

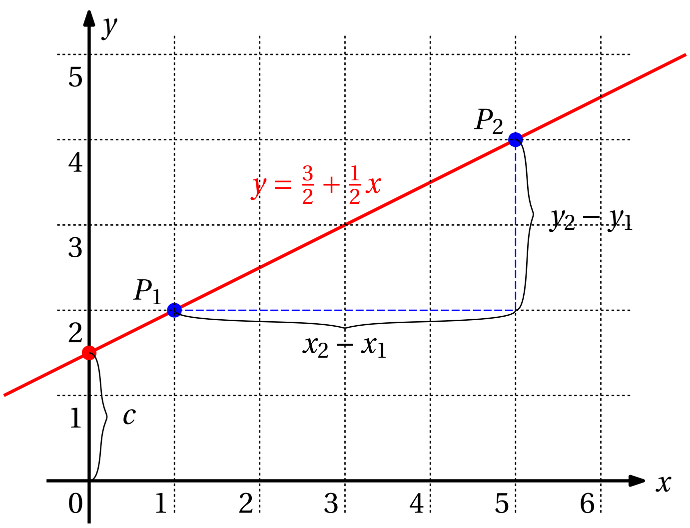


Figure 2.2: Line graph

Table 2.1: Measured data: v against t

t (s)	v (m/s)
1	7.7
2	10.5
3	13.3
4	15.5
5	16.3
6	20.5
7	23.0

The graph of a function of the form $f(x) = mx + c$ is a line; m is called the **slope**; and c is the **intercept** of the line. Note: $c = f(0)$.

If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are any two distinct points on the line, then $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

2.1 Guessing a law from an experiment

Experimental data for two (physical) variables x, y can be plotted as points in the (x, y)-plane. Sometimes they may look like they roughly lie on a line. This would indicate a linear law between x and y, i.e. $y = mx + c$ for some numbers m and c. The constants can be measured from the plot (explained in a video).

For an example, see Table 2.1 and Figure 2.3.

Note: sometimes the plotted data may exhibit a different "shape". It is useful to learn to "recognise" at least quadratics, exponentials and sine/cosine.

Extra Note (non-examinable): Nowadays one would use statistics/linear algebra to find the "best line" that fits the measured data. The most commonly used method for this is called linear regression (this is what the blue line in Figure 2.3 actually represents).

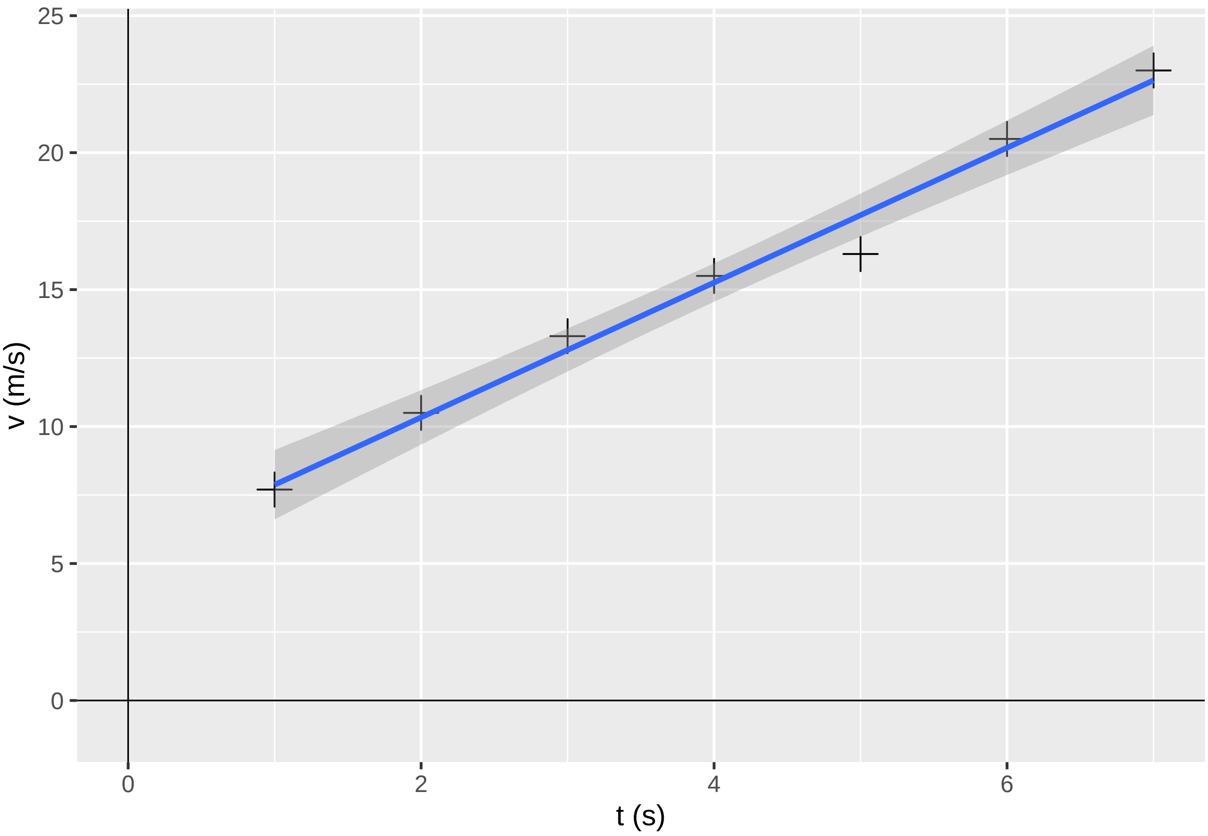


Figure 2.3: Plotted v against t

Exercises 2 (Plotting in rectangular coordinates)

1. The strain ε induced in a wire when subjected to a series of stress σ values produced the following results:

σ (M Pa)	10.8	21.6	33.3	37.8	45.9
ε ($\times 10^{-5}$)	12	24	37	42	51

Show that the stress is related to the strain by a law of the form $\sigma = E\varepsilon$, where E is a constant. Determine the law for the wire under test.

2. During a test on a simple lifting machine, the following results were obtained showing the applied force, F, for the load, L, lifted:

F (N)	19	37	50	93	125	149
L (N)	40	120	230	410	540	680

It is thought that the equation relating F and L is of the form $F = kL + c$ where both k and c are constants. Assuming that the law holds true, find the force necessary to lift a load of 1 kN.

3. The variation in pressure, p, within a vessel at a temperature, T, follows a law of the form $p = aT + b$. Verify that the data below relates the data by this law and determine the law.

p (kPa)	248	253	257	262	266	270
T (K)	273	278	283	288	293	298

4. Determine graphically the solution to the simultaneous equations:

$$2.5x + 0.45 - 3y = 0$$
$$1.6x + 0.8y - 0.8 = 0$$

5. (After Chapter 4: Quadratic equations) Plot graphs of:

   i) $y = 2x^2$
   ii) $y = 2x^2 - 4$
   iii) $y = 2x^2 - 2x + 0.5$

iv) $y = 2x^2 + x - 6$

6. (After Chapter 9: Long division and factorisation) Plot the graph of $y = 4x^3 - 4x^2 - 15x + 18$ for values of x from -3 to $+3$ and using the graph, determine the roots of the polynomial.

7. (After Chapter 8: Exponential functions) On the same axes and to the same scale plot the equations $y = 1.5e^{-1.18x}$ and $y = 1.1(1 - e^{-2.3x})$. Determine the solution to the equations from your graph.

Chapter 3

Factorising

3.1 Multiplying out brackets

$$\begin{gathered}3\cdot(2+4)=3\cdot 6=18\\ =\\ 3\cdot 2+3\cdot 4=6+12=18\end{gathered}$$

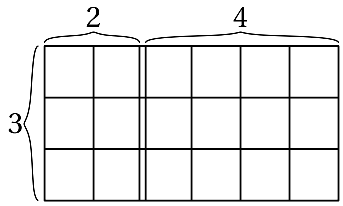


Figure 3.1: Illustrate distributivity with a rectangle area

General rules: $a(b+c)=ab+ac$ and $a(b-c)=ab-ac$.

Example 3.1. $4(6x-2)=4\cdot 6x-4\cdot 2=24x-12$.

Example 3.2. $(x+2y)(y-1)=(x+2y)\cdot y-(x+2y)\cdot 1=xy+2y\cdot y-x-2y$ $=xy-2y^2-x-2y$.

Example 3.3. $(a-b+3)(4a+2)=(a-b+3)\cdot 4a+(a-b+3)\cdot 2=a\cdot 4a-b\cdot 4a+3\cdot 4a+a\cdot 2-b\cdot 2+3\cdot 2=4a^2-4ab+14a-2b+6$.

Example 3.4. $(3x+1)(y-2)(x+y)=(3x+1)\cdot(y-2)\cdot x+(3x+1)\cdot(y-2)\cdot y$ $=(3x+1)\cdot(yx-2x)+(3x+1)\cdot(y^2-2y)=(3x+1)\cdot yx-(3x+1)\cdot 2x+(3x+1)\cdot y^2-(3x+1)\cdot 2y$ $=3x^2y+xy-6x^2-2x+3xy^2+y^2-6xy-2y$.

3.2 Factorising

Factors are the individual constituents of a *product* expression.

For example: $1\cdot 2\cdot 2\cdot 3,\quad x(y+z),\quad (a+b)(b+3)$

Factorising is writing something as a product.

For example: $12 = 1 \cdot 2 \cdot 2 \cdot 3$.

For numbers: a whole number $a > 0$ is **a factor of a number** b, if $\frac{b}{a}$ is also a whole number.

For example, the factors of 24 are $1, 2, 3, 6, 8, 12, 24$.

$24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6 = 2 \cdot 2 \cdot 6 = 3 \cdot 2 \cdot 4 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$.

Factors of 29 are 1 and 29.

Factors of a whole number a always include 1 and a.

A positive whole number $p \neq 1$ is called a **prime number** if it does not have factors other than 1 and p.

First prime numbers: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$

Example 3.5. Factorise $27 + 81$.

Solution. $27 + 81 = 27(1 + 3) = 27 \cdot 4 = 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2$.

Example 3.6. Factorise $x^3 - 3x^2y$.

Solution. $x^3 - 3x^2y = x^2(x - 3y)$.

Example 3.7. Factorise $6ax + 3ay + 2bx + by$.

Solution. $(6ax + 3ay) + (2bx + by) = 3a(2x + y) + b(2x + y) = (3a + b)(2x + y)$.

Example 3.8. Factorise $3x^2 - 5xy - 2y^2$.

Solution. Note: $-5 = 1 \cdot 1 - 2 \cdot 3$. So $3x^2 - 5xy - 2y^2 = 3x^2 - 6xy + xy - 2y^2 = 3x(x - 2y) + y(x - 2y) = (3x + y)(x - 2y)$.

Example 3.9. Factorise $25x^2y^2 - 30xy + 9$.

Solution. $25x^2y^2 - 30xy + 9 = 25x^2y^2 - 15xy - 15xy + 9 = 5xy(5xy - 3) - 3(5xy - 3)$ $= (5xy - 3)(5xy - 3)$.

Example 3.10. Factorise $10a^2 + ab - 21b^2$.

Solution. $10a^2 + ab - 21b^2 = 10a^2 + 15ab - 14ab - 21b^2 = (2a + 3b)(5a - 7b)$.

Notes:

- It is not always possible to sensibly factorise a given expression.
- Positive whole numbers can be always factorised (uniquely in the appropriate sense) as a product of prime numbers.
- Useful formula: $\boxed{a^2 - b^2 = (a - b)(a + b)}$.

Exercises 3 (Factorising)

Multiply out of following brackets and show that:

1. $(x+2)(x+4) = x^2 + 6x + 8$
2. $(-x-7)(2-3x) = 3x^2 + 19x - 14$
3. $(7t+6)(5t+8) = 35t^2 + 86t + 48$
4. $(s-5)(s+6) = s^2 + s - 30$
5. $(2q+3)(3q-5) = 6q^2 - q - 15$
6. $(x-4)(3x-1) = 3x^2 - 13x + 4$
7. $(x-2)(x+2) = x^2 - 4$
8. $(2x-1)(2x-1) = 4x^2 - 4x + 1$
9. $(x+4)^2 = x^2 + 8x + 16$
10. $(3x+5)^2 = 9x^2 + 30x + 25$

Factorise the following quadratic polynomials:

11. $x^2 + 8x + 15$
12. $x^2 + 11x + 28$
13. $x^2 + 7x + 6$
14. $x^2 - 10x + 9$
15. $x^2 - 6x + 9$
16. $x^2 + 5x - 14$
17. $x^2 - 4x - 5$
18. $x^2 - 10x - 24$
19. $x^2 - 1$
20. $x^2 - 16$
21. $4 + 5x + x^2$
22. $2x^2 - 3x + 1$
23. $9x^2 - 6x + 1$

24. $9 + 6x + x^2$
25. $x^2 + 2ax + a^2$
26. $4x^2 - 9$
27. $6x^2 + x - 12$
28. $4x^2 - 11x + 6$
29. $4x^2 + 3x - 1$
30. $3x^2 - 17x + 10$
31. $25x^2 - 16$
32. $3 - 2x - x^2$
33. $x^2 + 2xy + y^2$
34. $9 - 4x^2$
35. $x^2 - y^2$
36. $81x^2 - 36xy + 4y^2$
37. $49 - 84x + 36x^2$
38. $36x^2 + 60xy + 25y^2$
39. $4x^2 - 4xy - 3y^2$
40. $49p^2q^2 - 28pq + 4$

Solutions: **11.** $(x+3)(x+5)$ **12.** $(x+7)(x+4)$ **13.** $(x+6)(x+1)$ **14.** $(x-9)(x-1)$ **15.** $(x-3)(x-3)$ **16.** $(x+7)(x-2)$ **17.** $(x-5)(x+1)$ **18.** $(x-12)(x+2)$ **19.** $(x+1)(x-1)$ **20.** $(x+4)(x-4)$ **21.** $(x+1)(x+4)$ **22.** $(2x-1)(x-1)$ **23.** $(3x-1)(3x-1)$ **24.** $(x+3)(x+3)$ **25.** $(x+a)(x+a)$ **26.** $(2x+3)(2x-3)$ **27.** $(2x+3)(3x-4)$ **28.** $(x-2)(4x-3)$ **29.** $(4x-1)(x+1)$ **30.** $(3x-2)(x-5)$ **31.** $(5x-4)(5x+4)$ **32.** $(3+x)(1-x)$ **33.** $(x+y)(x+y)$ **34.** $(3-2x)(3+2x)$ **35.** $(x-y)(x+y)$ **36.** $(9x-2y)(9x-2y)$ **37.** $(7-6x)(7-6x)$ **38.** $(6x+5y)(6x+5y)$ **39.** $(2x-3y)(2x+y)$ **40.** $(7pq-2)(7pq-2)$

Chapter 4

Quadratic equations

A **quadratic polynomial** is an expression of the form ax^2+bx+c, where a, b, c are constants and $a \neq 0$. Any value of x that makes the equation $ax^2 + bx + c = 0$ hold true is called a **root** of the polynomial $ax^2 + bx + c$. The **graph** of the polynomial $f(x) = ax^2 + bx + c$ consists of all points (x, y) in the plane that satisfy $y = f(x)$.

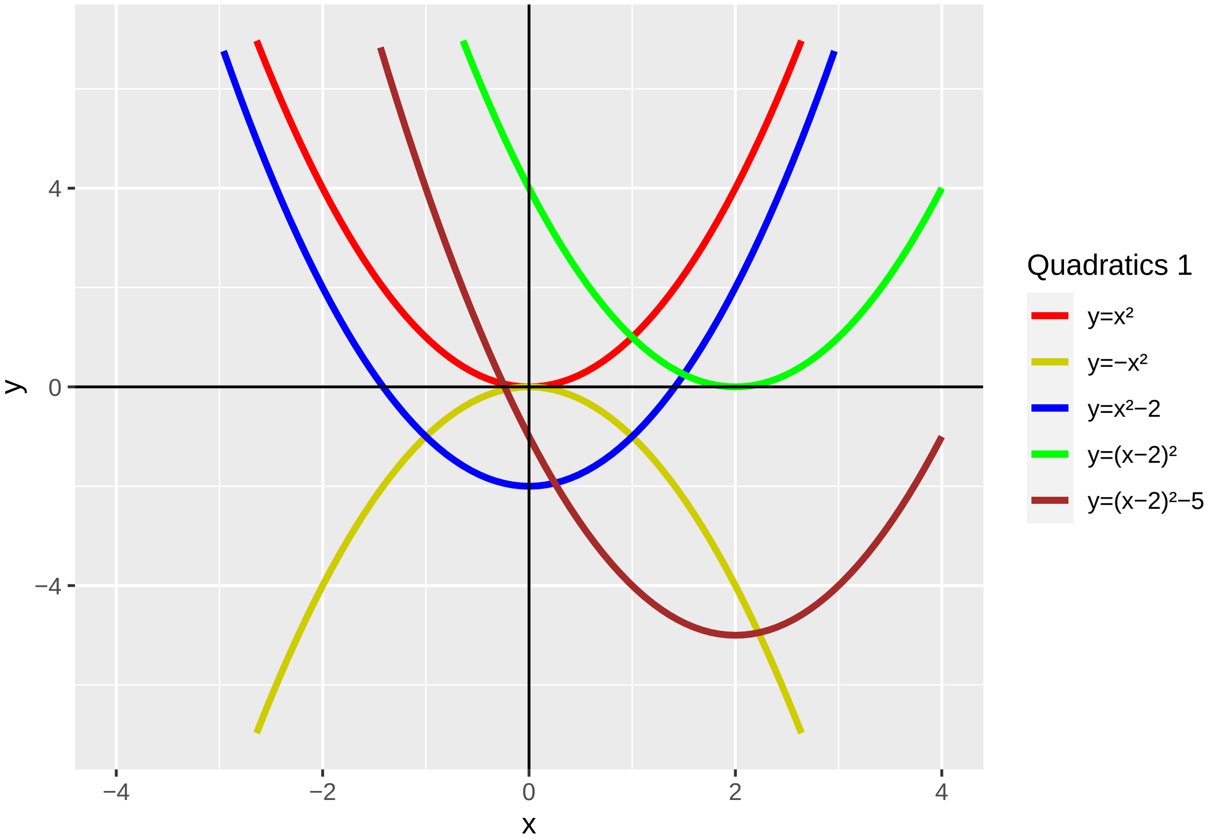


Figure 4.1: Examples of graphs of quadratic functions 1

Example 4.1. Find the roots of $2x^2 - x - 6$.

Solution. First factorise: $2x^2 - x - 6 = (2x + 3)(x - 2)$.
Thus the roots are $x_1 = -\frac{3}{2}$ and $x_2 = 2$.

Example 4.2. Find the roots of $4x^2 + 4x + 1$.

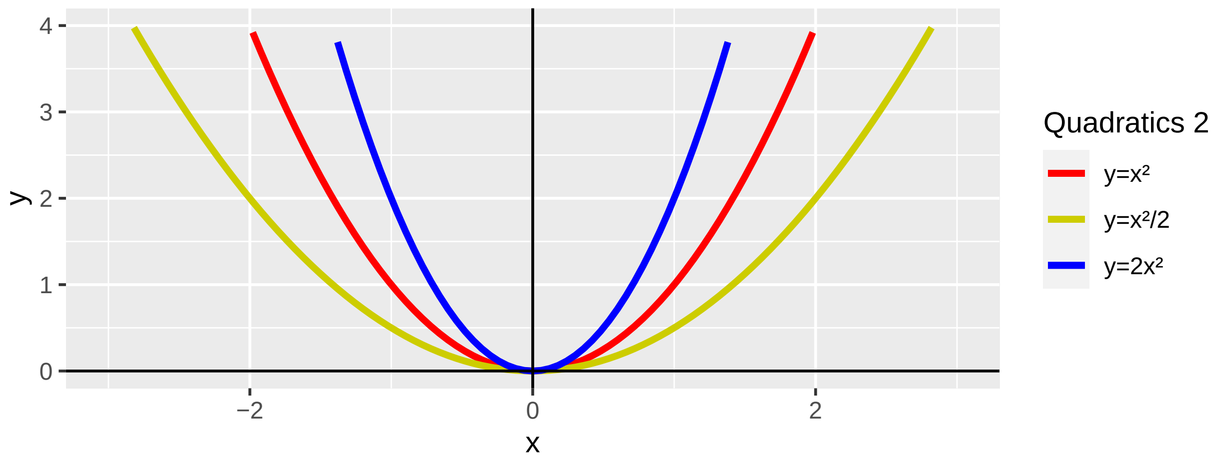


Figure 4.2: Examples of graphs of quadratic functions 2

Solution. Factorise: $4x^2 + 4x + 1 = (2x + 1)(2x + 1) = (2x + 1)^2$.
Hence the only root is $x_1 = -\frac{1}{2}$.

Example 4.3. Find the roots of $x^2 - 2x + 2$.

Solution. We can rewrite $x^2 - 2x + 2 = (x - 1)^2 + 1$.
This is positive for all real numbers x, thus there are no roots.

4.1 General formula for roots of $ax^2 + bx + c$

1. Case: $b^2 - 4ac < 0$: no roots
2. Case: $b^2 - 4ac = 0$: exactly one root $x_1 = -\frac{b}{2a}$.
3. Case: $b^2 - 4ac > 0$: two roots $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Learn this by heart!

Remarks:

- The number $b^2 - 4ac$ is sometimes referred to as the *discriminant* of the quadratic.
- A derivation of this formula in included at the end of the the chapter.

Example 4.4. Find the roots of $x^2 + 3x + 1$.

Solution. $x_{1,2} = \dfrac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \dfrac{-3 \pm \sqrt{9 - 4}}{2} = -\dfrac{3}{2} \pm \dfrac{1}{2}\sqrt{5}$.

Example 4.5. Find the roots of $x^2 - 9$.

Solution. $x_{1,2} = \dfrac{-0 \pm \sqrt{0 - 4 \cdot 1 \cdot (-9)}}{2 \cdot 1} = \pm\dfrac{36}{2} = \pm 3$.

Example 4.6. Find the roots of $4x^2 - 4x + 1$.

Solution. $x_{1,2} = \dfrac{-4 \pm \sqrt{(-4)^2 - 4 \cdot 4 \cdot 1}}{2 \cdot 8} = \dfrac{4 \pm \sqrt{0}}{8} = \dfrac{1}{2}$ (only one solution).

Example 4.7. Solve the simultaneous equations

$$y = -x^2 + 2x + 2 \tag{A}$$
$$x + y - 2 = 0 \tag{B}$$

Solution. Make x the subject of (B): $x = -y + 2$.
Substitute into (A): $y = -(-y + 2)^2 + 2(-y + 2) + 2$.
Rearrange: $y^2 - y - 2 = 0$.
Solve the quadratic: $y_{1,2} = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} = \dfrac{1 \pm \sqrt{9}}{2}$
$\implies \quad y_1 = 2, \quad y_2 = -1.$
Substitute back into (B): $\quad x_1 = 0, \quad x_2 = 3.$

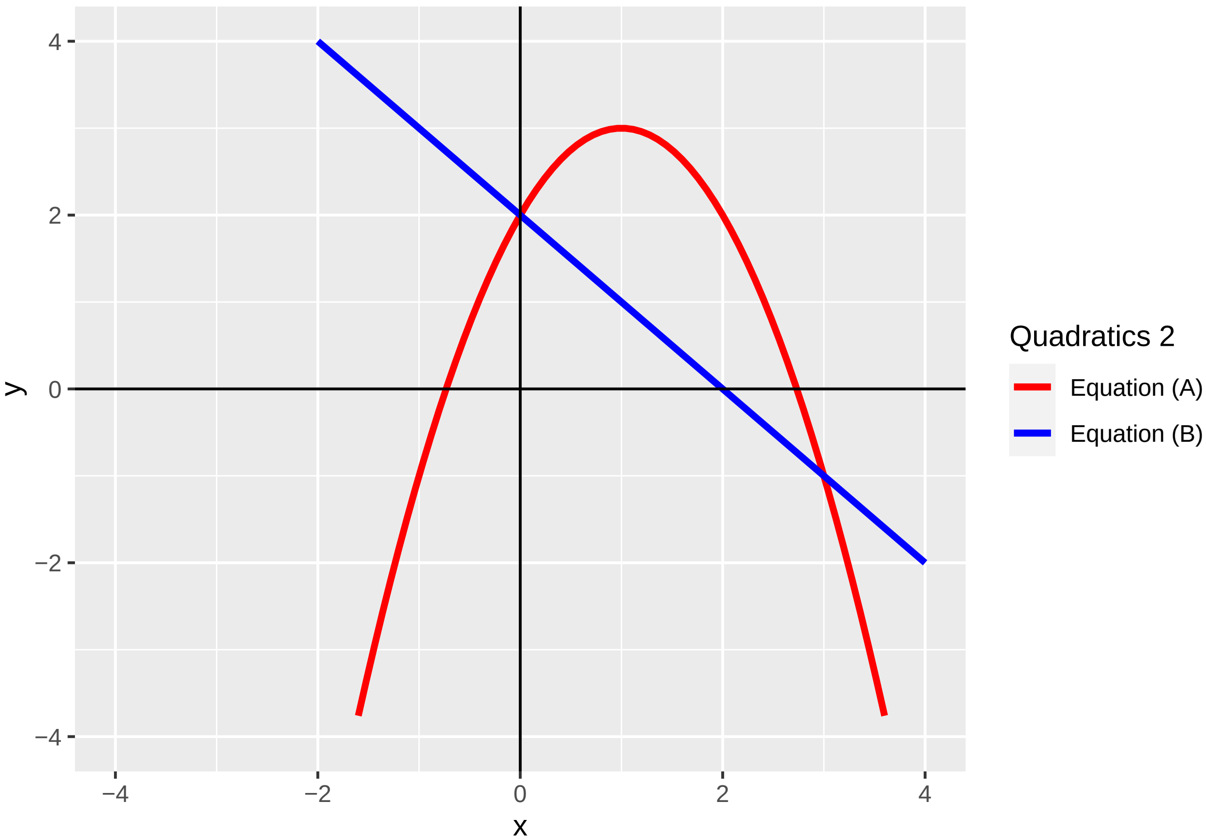


Figure 4.3: Graphical illustration of the solutions of simultaneous linear+quadratic system

4.2 Deriving the quadratic formula

This material is not examinable.

Why does the quadratic formula "work"? In other words, a "proof".

The following computation shows how to derive the general formula for the roots of a quadratic polynomial.

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
ax^2 + bx &= -c \\
x^2 + \tfrac{b}{a}x &= -\tfrac{c}{a} \quad \text{(using } a \neq 0\text{)} \\
x^2 + \tfrac{b}{a}x + \left(\tfrac{b}{2a}\right)^2 &= -\tfrac{c}{a} + \left(\tfrac{b}{2a}\right)^2
\end{aligned}$$

This trick (adding $\left(\frac{b}{2a}\right)^2$ to both sides) is called "completing the square".

$$\begin{aligned}
\left(x + \tfrac{b}{2a}\right)^2 &= \tfrac{b^2-4ac}{(2a)^2} \\
x + \tfrac{b}{2a} &= \frac{\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

The last step is only valid when $b^2 - 4ac \geq 0$!
One should be careful when taking square roots on both sides of an equation — which is why there is "$\pm$".

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Extra Note (non-examinable): For quadratic equations, there are formulas (depending only on the numbers appearing in the equation itself) which allow us to very easily determine how many solutions does the equation have, and what are they. This situation is rather rare – for example, if we stay with equations with a single variable x but allow the power of x to be higher, we very quickly run into issues: There is a general formula for degree 3, but it is rather unpleasantly complicated. From degree 5 onwards, it is possible to *prove* that no such formula can exist in general. That is not to say that we can't solve *some* high-degree equations, but there is no general formula which is guaranteed to always work. This wikipedia page goes in that direction.

Note - here we are talking about *exact* solutions. In practice, problems are almost always solved numerically, obtaining *approximate* solutions.

Exercises 4 (Quadratic equations)

1. Draw the graphs of the quadratic functions given below and indicate clearly where the curves intersect the x and y axes.

   i) $y = (x+1)^2 - 1$
   ii) $y = -(x+1)^2 - 1$
   iii) $y = (x+2)^2 - 3$
   iv) $y = (x+2)^2 + 3$
   v) $y = -(x+2)^2 + 3$

2. Solve the following quadratic equations by the method of factorisation:

   i) $x^2 - x - 6 = 0$
   ii) $x^2 - 16 = 0$
   iii) $x^2 - 2x = 0$
   iv) $x^2 - 6x + 9 = 0$
   v) $6x^2 + 18x + 12 = 0$
   vi) $6p^2 - 31p + 35 = 0$
   vii) $6x^2 - 11x - 7 = 0$
   viii) $-3r^2 - 14r + 5 = 0$
   ix) $14x^2 = 29x - 12$

3. Solve the following quadratic equations using the formula:

   i) $3x^2 - 8x + 2 = 0$
   ii) $-2x^2 + 3x + 7 = 0$
   iii) $4x^2 - 3x - 2 = 0$
   iv) $7r^2 + 8r - 2 = 0$
   v) $x^2 + x + \frac{1}{4} = \frac{1}{9}$
   vi) $5x^2 - 4x - 1 = 0$
   vii) $2a^2 - 5.3a + 1.25 = 0$
   viii) $x(x+4) + 2x(x+3) = 5$
   ix) $\frac{3}{2x-3} - \frac{2}{x+1} = 5$
   x) $\frac{2}{x+2} + \frac{3}{x+1} = 5$

4. Solve the following non-linear simultaneous equations:

   i) $p - 2q = 1, \quad p^2 - 3pq + 4q^2 = 11$
   ii) $x + y = 1, \quad 3x^2 - xy + y^2 = 37$
   iii) $x - y = 2, \quad x^3 - y^3 = 152$
   iv) $y = x^2 + 5x - 3, \quad y = 3x - 2$
   v) $2a^2 + ab - b^2 = 8, \quad 3a + 2b = 5$

Problems leading to quadratic equations

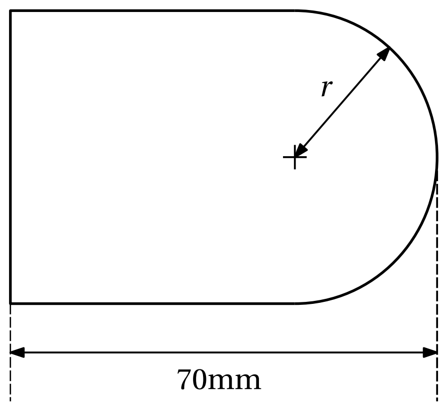
5. The angle in radians turned through by a shaft in t seconds is given by

$$\theta = \omega t + \frac{\alpha t^2}{2}.$$

   Determine the time taken to complete five revolutions given that $\omega = 2.7\,\text{rad/s}$ and $\alpha = 0.8\,\text{rad/s}^2$.

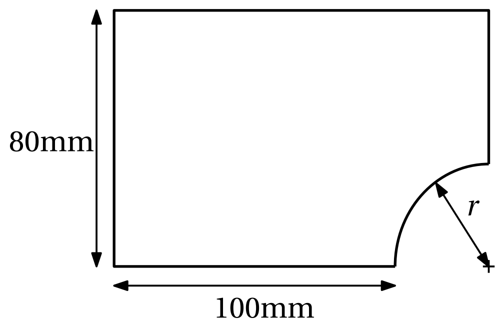
6. In a right angled triangle the hypotenuse is twice as long as one of the sides forming the right angle. The remaining side is 80 mm long. Calculate the area of the triangle.

7. The shape shown below has an area of $600\,\text{mm}^2$. Determine the radius r.



8. The total surface area of a cylinder whose ends are enclosed is $0.29\,\text{m}^2$. If the height of the cylinder is 75 mm, determine the radius.

9. If the area of the shape below is $9693\,\text{mm}^2$, determine the radius r.



10. A motorist travels 84 km from one city to another. On the return journey, the average speed was increased by 4 km/h and the journey took 30 minutes less. What was the average speed for the first part of the trip and how long did it take for the double journey?

Solutions: **2.** (i) $x = -2, x = 3$; (ii) $x = 4, x = -4$; (iii) $x = 0, x = 2$; (iv) $x = 3$ (double); (v) $x = -1, x = -2$; (vi) $p = 7/2, p = 5/3$; (vii) $x = 7/3, x = -1/2$; (viii) $r = 1/3, r = -5$; (ix) $x = 4/7, x = 3/2$

3. (i) $x = 2.39, x = 0.28$; (ii) $x = -1.26, x = 2.76$; (iii) $x = 1.17, x = -0.42$; (iv) $x = 0.21, x = -1.35$; (v) $x = -1/6, x = -5/6$; (vi) $x = 1, x = -1/5$; (vii) $a = 2.39, a = 0.26$; (viii) $x = 0.44, x = -3.77$; (ix) $x = 1.76, x = -1.36$; (x) $x = -0.22, x = -1.77$

4. (i) $p = -4, q = -5/2$ or $p = 5, q = 2$; (ii) $x = 3, y = -2$ or $x = -12/5, y = 17/5$; (iii) $x = 6, y = 4$ or $x = -4, y = -6$; (iv) $x = 0.41, y = -0.77$ or $x = -2.41, y = -9.23$; (v) $a = 3, b = -2$ or $a = 2.7, b = -1.55$

5. $t = 6.1$ s; **6.** $1848\,\text{mm}^2$; **7.** $r = 4.36\,\text{mm}$; **8.** $r = 180\,\text{mm}$; **9.** $r = 71.86\,\text{mm}$ or $r = 30\,\text{mm}$; **10.** 6.5 hours

Chapter 5

Inequalities

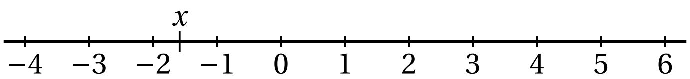
For two numbers a and b:

- $a < b$ means that a is less than b
- $a > b$ means that a is greater than b
- $a \leq b$ means that a is less than or equal to b
- $a \geq b$ means that a is greater than or equal to b

For example: $1 < 2,\ -10 < -5,\ -3 \leq -3,\ -2 \geq -4.$

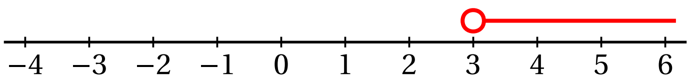
Note: $a > 0$ means the same as "a is positive".

If x is given on the number line as

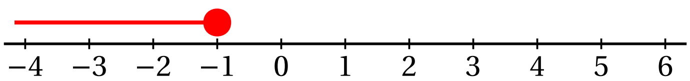


then the following hold: $-2 < x,\ -1 > x,\ x < 0,\ -5 < x,\ x < 3.$

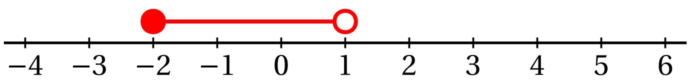
The notation "$\{x : x > 3\}$" means "the set of all numbers x that are greater than 3".



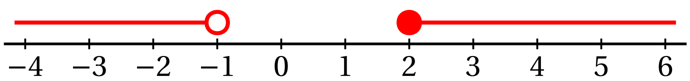
Similarly, $\{x : x \leq -1\}$ is the set of numbers that can be represented on the number line as



The notation $\{x : -2 \leq x < 1\}$ can be depicted as



The notation $\{x : x < -1 \text{ or } x \geq 2\}$ can be depicted as



Axioms of inequalities

1. If $a < b$ then $a + c < b + c$.

2. If $a < b$ then $a - c < b - c$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.

For example: $2 < 3 \iff -2 > -3$.

Example 5.1. Solve $3x + 5 > 17$.

Solution. $3x + 5 > 17$
$\iff 3x > 12$ (subtract 5)
$\iff x > 4$ (multiply by $1/3$)

Example 5.2. Determine the set $\{x : 2x - x^2 < 3x - 12\}$.

Solution. $2x - x^2 < 3x - 12$
$\iff 2x - x^2 - 3x + 12 < 0$ (subtract $3x - 12$)
$\iff -x^2 - x + 12 < 0$ (rearrange LHS)
$\iff x^2 + x - 12 > 0$ (multiply by -1)
$\iff (x + 4)(x - 3) > 0$ (factorise LHS)
$\iff (x + 4 > 0 \text{ and } x - 3 > 0) \text{ or } (x + 4 < 0 \text{ and } x - 3 < 0)$
$\iff (x > -4 \text{ and } x > 3) \text{ or } (x < -4 \text{ and } x < 3)$
$\iff x > 3 \text{ or } x < -4$
Answer: $\{x : x > 3 \text{ or } x < -4\}$.

Example 5.3. Solve $\dfrac{x-1}{x-2} < 0$.

Solution. $\frac{x-1}{x-2} < 0$
$\iff (x - 1)(x - 2) < 0$ (multiply by $(x - 1)^2 \geq 0$)
$\iff (x < 1 \text{ and } x > 2) \text{ or } (x > 1 \text{ and } x < 2)$
$\iff 1 < x < 2$

Example 5.4. Solve $\dfrac{3x-2}{1+x} \leq 1$.

Solution. $\frac{3x-2}{1+x} \leq 1$
$\iff \frac{(3x-2)-(1+x)}{1+x} \leq 0$
$\iff \frac{2x-3}{1+x} \leq 0$
$\iff (x \leq \frac{3}{2} \text{ and } x > -1) \text{ or } (x \geq \frac{3}{2} \text{ and } x < -1)$
$\iff -1 < x \leq \frac{3}{2}$

Exercises 5 (Inequalities)

Solve the following and illustrate the solution on the number line:

1. $2x + 7 > 23$
2. $\{x : 6x - 4 > x\}$
3. $\{x : 2 - 3x \leq 2x + 3\}$
4. $4x^2 - 9 < 0$
5. $x^2 + 3x + 2 \leq 5(x + 2)$
6. $6x + 5 \geq x - 5$
7. $|4x - 5| < 1$
8. $| - x + 3| > 2$
9. $(3x - 2) \div (1 + x) < 1$
10. $(x + 3)(x - 4) < 0$

Solutions: **1.** $x > 8$; **2.** $x > 4/5$; **3.** $x \geq -1/5$; **4.** $-3/2 < x < 3/2$; **5.** $-2 \leq x \leq 4$; **6.** $x \geq -2$; **7.** $1 < x < 3/2$; **8.** $x < 1$ or $x > 5$; **9.** $-1 < x < 3/2$; **10.** $-3 < x < 4$

Chapter 6

Indices

6.1 Exponents

Subscripts and superscripts are also called **indices** (singular **index**). Subscripts are usually used to denote families of variables, for example $a_1, a_2, a_3, \ldots$ (instead of a, b, c).

A superscript usually denotes an **exponent**, i.e. it represents the power to which a given number is raised.

Examples:

- $4^3 = \underbrace{4 \cdot 4 \cdot 4}_{3 \text{ times}} = 64$
- $2^5 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{5 \text{ times}} = 32$
- $x^4 = \underbrace{x \cdot x \cdot x \cdot x}_{4 \text{ times}}$

In general, we **define**:

- $x^n := \underbrace{x \cdot x \cdots x}_{n \text{ times}}$, when $n > 0$,
- $x^0 := 1$,
- $x^{-n} := \dfrac{1}{\underbrace{x \cdot x \cdots x}_{n \text{ times}}}$, when $n > 0$.

Further examples:

- $x^{\frac{1}{2}} := \sqrt{x} \quad$ for $x \geq 0$,
- $x^{\frac{1}{3}} := \sqrt[3]{x}$.

In general, we **define**:

- $x^{\frac{1}{n}} := \sqrt[n]{x} \quad$ for $x \geq 0$ and $n > 0$,
- $x^{\frac{m}{n}} := \sqrt[n]{x^m} \quad$ for $x \geq 0$ and $m, n > 0$,
- $x^{-\frac{m}{n}} := \dfrac{1}{\sqrt[n]{x^m}} \quad$ for $x \geq 0$ and $m, n > 0$.

6.2 Laws of exponents

The basic ones are:

1. $\boxed{a^r \cdot a^s = a^{r+s}}$
2. $\boxed{(a \cdot b)^r = a^r \cdot b^r}$
3. $\boxed{(a^r)^s = a^{r \cdot s}}$

From these, one can derive the following two:

4. $\boxed{\frac{a^r}{a^s} = a^{r-s}}$
5. $\boxed{\left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}}$

Deriving 4: $\frac{a^r}{a^s} = a^r \cdot \frac{1}{a^s} = a^r \cdot a^{-s} \overset{1.}{=} a^{r-s}$.

Deriving 5: $\left(\sqrt[n]{a}\right)^m = \left(a^{\frac{1}{n}}\right)^m \overset{3.}{=} a^{\frac{1}{n} \cdot m} = a^{\frac{m}{n}}$.

Example 6.1. Write the following expressions using only positive indices: x^{-4}, $3x^{-4}$, $(3x)^{-4}$ and $\frac{2}{y^{-3}}$.

Solution. Calculate

- $x^{-4} = \frac{1}{x^4}$
- $3x^{-4} = 3 \cdot \frac{1}{x^4} = \frac{3}{x^4}$
- $(3x)^{-4} = \frac{1}{(3x)^4} = \frac{1}{3^4 \cdot x^4} = \frac{1}{81x^4}$
- $\frac{2}{y^{-3}} = \frac{2}{\frac{1}{y^3}} = 2y^3$

Example 6.2. Write the following expressions using only single index: $\left(x^2\right)^{-3}$, $\left((-x)^{-2}\right)^{-3}$, $\left(\frac{t}{t^{-2}}\right)^4$.

Solution. Calculate

- $\left(x^2\right)^{-3} = x^{2\cdot(-3)} = x^{-6}$
- $\left((-x)^{-2}\right)^{-3} = (-x)^{(-2)\cdot(-3)} = (-x)^6 = \left((-1) \cdot x\right)^6 = (-1)^6 \cdot x^6 = x^6$
- $\left(\frac{t}{t^{-2}}\right)^4 = \frac{t^4}{(t^{-2})^4} = \frac{t^4}{t^{-8}} = t^{4-(-8)} = t^{12}$

Example 6.3. Simplify the expression $A = \dfrac{x^{-6}y^{3/2}w^2}{x^{-3}w^3} \div \dfrac{x^{\frac{1}{2}}\sqrt[3]{y}}{(xy)^3(\sqrt[4]{w})^{-2}}$ so that it uses only positive indices.

Solution.

$$\begin{aligned}
A &= \frac{x^{-6} y^{\frac{3}{2}} w^2 \, (xy)^3 (\sqrt[4]{w})^{-2}}{x^{-3} w^3 \, x^{\frac{1}{2}} \, \sqrt[3]{y}} \\
&= \frac{x^{-6} y^{\frac{3}{2}} w^2 x^3 y^3 w^{-\frac{1}{2}}}{x^{-3} w^3 x^{\frac{1}{2}} y^{\frac{1}{3}}} \\
&= x^{-6+3-(-3)-\frac{1}{2}} \, y^{\frac{3}{2}+3-\frac{1}{3}} \, w^{2-\frac{1}{2}-3} \\
&= x^{-\frac{1}{2}} \, y^{\frac{25}{6}} \, w^{-\frac{3}{2}} \\
&= \frac{y^{\frac{25}{6}}}{x^{\frac{1}{2}} \, w^{\frac{3}{2}}}.
\end{aligned}$$

6.3 Scientific notation of numbers

Standard (or **normalised**) scientific notation of number is:

$a \cdot 10^n$, where $1 \leq |a| < 10$ and n is an integer

Engineering scientific notation is:

$a \cdot 1000^n$, where $1 \leq |a| < 1000$ and n is a multiple of 3

The "a" is called **mantissa**, the "n" is called **exponent**.

Decimal expansion	Standard notation	Engineering notation
17	$1.7 \cdot 10^1$	17
620	$6.2 \cdot 10^2$	620
342567	$3.42567 \cdot 10^5$	$342.567 \cdot 10^3$
0.0001	$1 \cdot 10^{-4}$	$100 \cdot 10^{-6}$
0.00005	$5 \cdot 10^{-5}$	$50 \cdot 10^{-6}$

Exercises 6 (Indices)

1. Using the rules of indices put the following expressions in their simplest form:

   i) $\dfrac{9^2 \times 9 \times 9^5}{3^9 \times 3}$

   ii) $\dfrac{7^6 \times 7^3 \times 4^8}{7^8 \times 4^2 \times 4^3}$

2. Simplify the following expressions without using a calculator:

   i) $\dfrac{(7.5 \times 10^2)(1.2 \times 10^3)}{4}$

   ii) $\dfrac{(4.5 \times 10^{-7})(1.2 \times 10^9)}{9}$

   iii) $\dfrac{(3.3 \times 10^{-5})(4.2 \times 10^6)}{(1.1 \times 10^{-2})(2.1 \times 10^3)}$

   iv) $\dfrac{4^{\frac{1}{2}} \times 64^{\frac{2}{3}} \times 32^{\frac{1}{5}}}{16^{\frac{3}{2}} \times 81^{-\frac{3}{4}}}$

3. Simplify the following:

   i) $(a^2x^3y^{-2})^3 \times (a^{-3}xy^3)^{1/2} \div (axy^{-3})^{5/2}$

   ii) $\dfrac{\sqrt[3]{a}b^{-\frac{1}{2}}}{\sqrt{a^5}\sqrt{b}} \div \dfrac{ab^2c^{-\frac{3}{2}}}{\sqrt{a^2b^3}c^{\frac{5}{2}}}$

4. The *e.m.f.* induced in a circuit when N lines of induction are cut in a time t seconds is given by:
$$e.m.f. = \frac{N}{t \times 10^8}.$$
   Determine the *e.m.f.* induced when $N = 36 \times 10^8$ and $t = 30\,\text{ms}$.

5. Without using a calculator, determine the value of $\left(\dfrac{81^{\frac{1}{4}} \times 9^{\frac{1}{2}}}{3^2 \times 27^2}\right)^{-1}$.

6. Simplify:

   a) $5^7 \times 5^{13}$

   b) $9^8 \times 9^5$

   c) $11^2 \times 11^3 \times 11^4$

   d) $\dfrac{15^3}{15^2}$

   e) $\dfrac{4^{18}}{4^9}$

f) $\frac{5^{20}}{5^{19}}$
g) a^7a^3
h) a^4a^5
i) $b^{11}b^{10}b$
j) $x^7 \times x^8$
k) $y^4 \times y^8 \times y^9$

7. Simplify:

a) $(7^3)^2$
b) $(4^2)^8$
c) $(7^9)^2$
d) $\frac{1}{(5^3)^8}$
e) $(x^2y^3)(x^3y^2)$
f) $(a^2bc^2)(b^2ca)$

8. Remove the brackets from:

a) $(x^2y^4)^5$
b) $(9x^3)^2$
c) $(-3x)^3$
d) $(-x^2y^3)^4$

9. Simplify:

a) $\frac{(z^2)^3}{z^3}$
b) $\frac{(y^3)^2}{(y^2)^2}$
c) $\frac{(x^3)^2}{(x^2)^3}$

10. Write each of the following using only a positive power:

a) x^{-4}
b) $\frac{1}{x^{-5}}$
c) x^{-7}
d) y^{-2}
e) $\frac{1}{y^{-1}}$

11. Simplify the following and write your results using only positive powers:

a) $x^{-1}x^{-2}$
b) $x^{-3}x^{-2}$
c) x^3x^{-4}
d) $x^{-4}x^9$
e) $\frac{x^{-2}}{x^{11}}$
f) $(x^{-4})^2$
g) $(x^{-3})^3$

h) $(x^2)^{-2}$

12. Simplify:

a) $a^{13}a^{-2}$
b) $x^{-9}x^{-7}$
c) $x^{-21}x^2x$

13. Evaluate:

a) 10^{-3}
b) 10^{-4}
c) 10^{-5}
d) $\dfrac{4^{-8}}{4^{-6}}$
e) $\dfrac{3^{-5}}{3^{-8}}$

14. Simplify, then evaluate:

a) $64^{\frac{1}{3}}$
b) $144^{\frac{1}{2}}$
c) $16^{-\frac{1}{4}}$
d) $25^{-\frac{1}{2}}$
e) $\left(3^{-\frac{1}{2}}\right)^4$
f) $\left(8^{\frac{1}{3}}\right)^{-1}$

Solutions: **1.** (i) 3^6; (ii) 7×4^3; **2.** (i) 2.25×10^5; (ii) 60; (iii) 6; (iv) 27; **3.** (i) $a^2x^7y^3$; (ii) $\frac{c^4}{a^{13/6}b^{3/2}}$; **4.** 1.2×10^3; **5.** 729; **6.** (a) 5^{20}; (b) 9^{13}; (c) 11^9; (d) 15; (e) 4^9; (f) 5; (g) a^{10}; (h) a^9; (i) b^{22}; (j) x^{15}; (k) y^{21} **7.** (a) 7^6; (b) 4^{16}; (c) 7^{18}; (d) 5^{-24}; (e) x^5y^5; (f) $a^3b^3c^3 = (abc)^3$; **8.** (a) $x^{10}y^{20}$; (b) $81x^6$; (c) $-27x^3$; (d) x^8y^{12}; **9.** (a) z^3; (b) y^2; (c) 1; **10.** (a) $\frac{1}{x^4}$; (b) x^5; (c) $\frac{1}{x^7}$; (d) $\frac{1}{y^2}$; (e) y; **11.** (a) $\frac{1}{x^3}$; (b) $\frac{1}{x^5}$; (c) $\frac{1}{x}$; (d) x^5; (e) $\frac{1}{x^{13}}$; (f) $\frac{1}{x^8}$; (g) $\frac{1}{x^9}$; (h) $\frac{1}{x^4}$; **12.** (a) a^{11}; (b) $\frac{1}{x^{16}}$; (c) $\frac{1}{x^{18}}$; **13.** (a) 0.001; (b) 0.0001; (c) 0.00001; (d) $1/16$; (e) 27; **14.** (a) 4; (b) 12; (c) $1/2$; (d) $1/5$; (e) $1/9$; (f) $1/2$

Chapter 7

Logarithms

If $a = b^x$, then x is called the **logarithm of a to the base b**. In symbols $x = \log_b a$.

For example, $\log_3 9 = 2, \quad \log_2 32 = 5, \quad \log_{10} 1000 = 3.$

We *require* the base b to be a positive number, not equal to 1.

Notation for two most commonly used bases:

- $\log(a) := \log_{10}(a)$ (used in Engineering)
- $\ln(a) := \log_e(a)$ where e is the Euler's number, $e \approx 2.7183$ (to 4 d.p.)
- $\ln(a)$ is also called the **natural logarithm** of a.

Note:: $\log_{10}(a) \begin{cases} = 0 & \text{for } a = 1 \\ > 0 & \text{for } a > 1 \\ < 0 & \text{for } 0 < a < 1 \\ \text{does not exist} & \text{for } a \leq 0. \end{cases}$

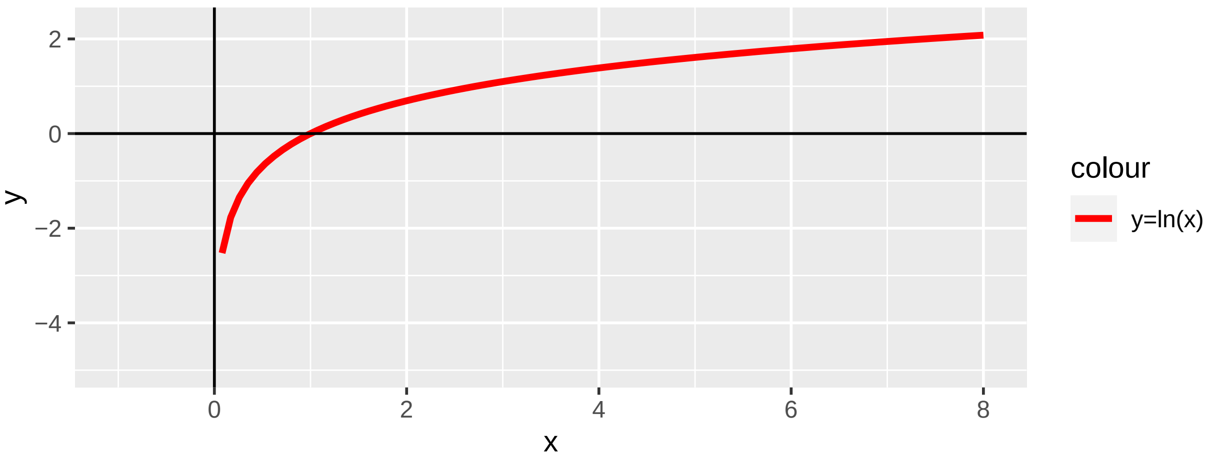


Figure 7.1: Graph of the natural logarithm function

7.1 Laws of logarithms

1. $\boxed{\log_b(M \cdot N) = \log_b(M) + \log_b(N)}$

2. $\boxed{\log_b(N^a) = a \cdot \log_b(N)}$

Proof (non-examinable):

1. $\log_b(M \cdot N)$ is defined by the equation $b^{\log_b(M \cdot N)} = M \cdot N$.
   So, we only need to verify that $b^{\log_b(M)+\log_b(N)} = M \cdot N$.
   But by laws of exponents and the definition of logarithm, we have LHS $= b^{\log_b(M)} \cdot b^{\log_b(N)} = M \cdot N$, so we are done.

2. $\log_b(N^a)$ is defined by the equation $b^{\log_b(N^a)} = N^a$.
   So, we only need to verify that $b^{a \cdot \log_b(N)} = N^a$.
   But by laws of exponents and the definition of logarithm, we have LHS $= \left(b^{\log_b(N)}\right)^a = N^a$, so we are done.

From 1. and 2. we can derive two more laws:

3. $\boxed{\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)}$

4. $\boxed{\log_b(N) = \dfrac{\log_{10}(N)}{\log_{10}(b)}} = \dfrac{\ln(N)}{\ln(b)} = \dfrac{\log_c(N)}{\log_c(b)}$ for any base c.

Proof (non-examinable):

3. LHS $= \log_b(M \cdot N^{-1}) \overset{1.}{=} \log_b(M) + \log_b(N^{-1}) \overset{2.}{=} \log_b(M) - \log_b(N) =$ RHS

4. We apply $\log_c$ to both sides of the equation $b^{\log_b(N)} = N$:

$$\begin{aligned} \log_c\left(b^{\log_b(N)}\right) &= \log_c(N) \\ \log_b(N) \cdot \log_c(b) &= \log_c(N) && \text{(by 2.)} \\ \log_b(N) &= \frac{\log_c(N)}{\log_c(b)} && \text{(dividing by } \log_c(b) \neq 0) \end{aligned}$$

Example 7.1. $\log_5(2.623) \overset{4.}{=} \dfrac{\log_{10}(2.623)}{\log_{10}(5)} \approx \dfrac{0.4188}{0.6990} \approx 0.5991$

Example 7.2. Solve $2^x = 5$.

Solution. $x = \log_2(5) = \dfrac{\log_{10}(5)}{\log_{10}(2)} \approx \dfrac{0.6990}{0.3010} \approx 2.322$

Example 7.3. Solve $3^{x+1} = 2^{2x-3}$.

Solution. Applying $\log_{10}$ on both sides: $\log_{10}(3^{x+1}) = \log_{10}(2^{2x-3})$.
$\implies (x+1)\log_{10}(3) = (2x-3)\log_{10}(2)$
$\implies x\left(\log_{10}(3) - 2\log_{10}(2)\right) = -3\log_{10}(2) - \log_{10}(3)$
$\implies x = \dfrac{-3\log_{10}(2) - \log_{10}(3)}{\log_{10}(3) - 2\log_{10}(2)} \approx 11.0471$

Example 7.4. Solve $2^{2x} - 2^{x+1} - 15 = 0$.

Solution. $2^{2x} - 2^{x+1} - 15 = 0$
$\iff (2^x)^2 - 2 \cdot (2^x) - 15 = 0$ (by laws of exponents)

$\iff y^2 - 2y - 15 = 0$ (where $y = 2^x$)
$\iff (y-5)(y+3) = 0$ (factorising)
$\iff y = 5$ or $y = -3$
$\iff x = \log_2(5) \approx 2.322$,
and $2^x = -3$ is not possible (so no solution)

Exercises 7 (Logarithms)

1. Determine the values for the following logarithms:

   i) $\log_2 98.5$
   ii) $\log_4 22.86$
   iii) $\log_7 1050$
   iv) $\log_8 211.746$

2. The formula $\dfrac{T_1}{T_2} = e^{\mu\theta}$ shows the relationship between the tension in a flat driving belt (T_1 being on the tight or driving side) whilst θ is the angle of contact around the pulley and μ is the coefficient of friction.

   i) Show that $\log T_1 = \log T_2 + \mu\theta \log e$
   ii) Determine T_1 using logs if $T_2 = 150$, $\mu = 0.25$, $\theta = 3$ and $e = 2.718$.

3. Solve for x:

   i) $5^{2x} = 0.5$
   ii) $3^{2x} + 3^x = 12$
   iii) $3^{2x} - 3^{x+1} + 2 = 0$

4. If $10^x + 10^{-x} = 4$, show that $x = \log_{10}(2 \pm \sqrt{3})$.

5. Express in terms of $\log p$, $\log q$ and $\log r$:

   a) $\log pq$
   b) $\log pqr$
   c) $\log pq/r$
   d) $\log p/qr$
   e) $\log p^2 q$
   f) $\log q/r^2$
   g) $\log p^2 q^3/r$
   h) $\log p^n q^m$
   i) $\log 2pq$
   j) $\log 2pq^2$

6. Simplify:

   a) $\log p + \log q$
   b) $2 \log p + \log q$
   c) $\log q - \log r$
   d) $3 \log q + 4 \log p$
   e) $\log p + 2 \log q - 3 \log r$

f) $\log p - \log 2$
g) $2 \log p - p \log 2$
h) $\log(p + 2) - \log(q - 2)$

Solutions: **1.** (i) 6.62; (ii) 2.26; (iii) 3.57; (iv) 2.58; **2.** (ii) $T_1 = 317$; **3.** (i) $x = -0.22$; (ii) $x = 1$; (iii) $x = 0.63$ or $x = 0$; **5.** (a) $\log p + \log q$; (b) $\log p + \log q + \log r$; (c) $\log p + \log q - \log r$; (d) $\log p - \log q - \log r$; (e) $2 \log p + \log q$; (f) $\log q - 2 \log r$; (g) $2 \log p + 3 \log q - \log r$; (h) $n \log p + m \log q$; (i) $\log 2 + \log p + \log q$; (j) $\log 2 + \log p + 2 \log q$; **6.** (a) $\log pq$; (b) $\log p^2 q$; (c) $\log q/r$; (d) $\log q^3 p^4$; (e) $\log pq^2/r^3$; (f) $\log p/2$; (g) $\log p^2/2^p$; (h) $\log(p + 2)/(q - 2)$

Chapter 8

Exponential functions

If $f(x) = b^x$ for some b, then $f(x)$ is called an **exponential function**.

The notation for the most often encountered exponential function is $\exp(x) := e^x$, where $e = 2.7128$ is the Euler's number.

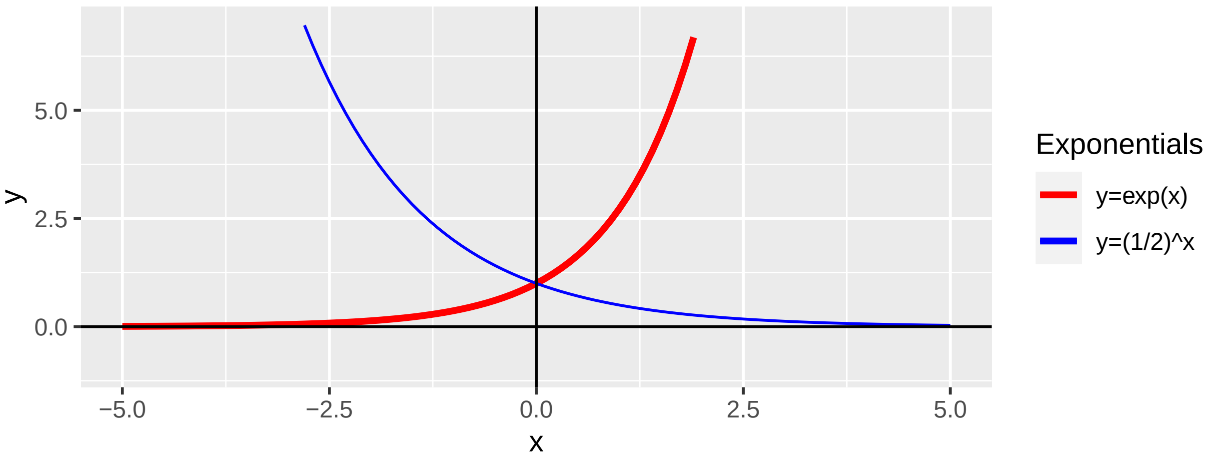


Figure 8.1: Graph of the natural logarithm function

Example 8.1. In a circuit containing a capacitor, the *instantaneous voltage* across the capacitor is given by the equation

$$\nu = V\left(1 - e^{\frac{-t}{RC}}\right),$$

where V is the initial supply voltage, R is the resistance, C is the capacitance and t is the time since the initial connection of the supply voltage. If $V = 200\,\mathrm{V}$, $R = 10\,\mathrm{k}\Omega$ and $C = 20\,\mu\mathrm{F}$, calculate the time when the voltage reaches $\nu = 100\,\mathrm{V}$.

Solution. We want to make t the subject of the formula above:

$$\begin{aligned}
\frac{\nu}{V} &= 1 - e^{\frac{-t}{RC}} \\
e^{\frac{-t}{RC}} &= 1 - \frac{\nu}{V} \\
-\frac{t}{RC} &= \ln\left(1 - \frac{\nu}{V}\right) \\
t &= -RC \ln\left(1 - \frac{\nu}{V}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
t &= -10\,\mathrm{k}\Omega \cdot 20\,\mu\mathrm{F} \cdot \ln\left(1 - \frac{100\,\mathrm{V}}{200\,\mathrm{V}}\right) \\
&\approx 138.6 \cdot 10^3 \cdot 10^{-6}\,\Omega\mathrm{F} \\
&= 138.6\,\mathrm{ms}
\end{aligned}$$

(since $\Omega\mathrm{F} = \mathrm{s}$).

Exercises 8 (Exponential functions)

1. Evaluate the following:

   i) $y = 200e^{1.75}$
   ii) $y = 40.8 + e^2$
   iii) $s = 3e^{5.1} - 49.8$
   iv) $x = (150e^{-1.34}) - (3.4e^{0.445})$
   v) $x = 3.2e^3 - (6 - e^{-0.8})$

2. Calculate the value of x in each case:

   i) $25 = e^x$
   ii) $34 = 3e^x$
   iii) $268 = e^{1.4x}$
   iv) $94.2 = e^{-3.2x}$
   v) $0.72 = e^{-0.08x}$

3. If $T = R\ln\left(\frac{a}{a-b}\right)$, calculate a when $T = 25$, $R = 28$ and $b = 3$.

4. In the formula $i = Ie^{-\frac{Rt}{L}}$, $i = 50\,\text{mA}$, $I = 150\,\text{mA}$ (units: miliAmperes), $R = 60\,\Omega$ (units: Ohm) and $L = 0.3\,\text{H}$ (units: Henry). Determine the corresponding value of t. (Recall that for units, we have $\text{H}/\Omega = \text{s}$. Optional bit: This formula describes current through the inductor when discharging in an RL-circuit.)

5. The instantaneous charge in a capacitive circuit is given by $q = Q\left(1 - e^{-\frac{t}{RC}}\right)$. Calculate the value of t (time) when $q = 0.01\,\text{C}$, $Q = 0.015\,\text{C}$ (units: Coulomb), $C = 0.0001\,\text{F}$ (units: Farad) and $R = 7000\,\Omega$ (units: Ohm). (Recall that for units, we have $\Omega \cdot \text{F} = \text{s}$.)

6. From the formula $v = V\left(1 - e^{-\frac{t}{RC}}\right)$, calculate the value of C (capacitance, units: Farad) when $v = 130\,\text{V}$, $V = 440\,\text{V}$ (units: Volt), $t = 0.156\,\text{s}$ (units: second) and $R = 44000\Omega$ (units: Ohm). (Recall that for units, we have $\Omega \cdot \text{F} = \text{s}$. Optional bit: This formula describes the instantaneous voltage over a capacitor when charging in an RC-circuit.)

7. Plot the function $y = 3e^{2x}$ over the range $x = -3$ to $x = 3$ and from the graph determine the value of y when $x = 1.7$ and the value of x when $y = 3.3$.

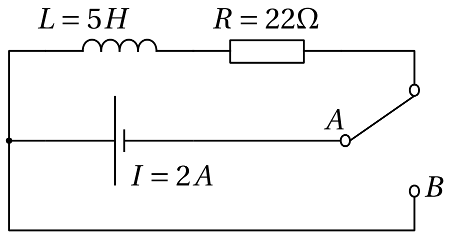
8. For values of x from -0.5 to 1.5, plot the graph represented by the equation $y = 10e^{2x}$.

9. Given the formula $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$, plot the curve of i against t when $E = 300$, $R = 30$ and $L = 5$ for the range of t from 0 to 0.8. From the graph, estimate the value of t when $i = 3.2$ and also calculate the value of t using the formula to check the accuracy of the graph.

10. The formula $i = 2(1 - e^{-10t})$ represents the relationship between the instantaneous current i (measured in Amps) and the time t (measured in seconds) in an inductive circuit. Plot a graph of i against t taking values of t from 0 to 0.3 at intervals of 0.05. From the graph determine the time taken for the current to increase from 1.0 to 1.6 A and check this value by calculation.

11. A coil has an inductance value of $L = 2.2\,\mathrm{H}$ and a resistance $R = 15\,\Omega$. It is connected to a voltage supply with $E = 12\,\mathrm{V}$. After connection, the current i is given by $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$. Draw a graph of the current plotted against time from the moment of connection and for the first 0.8 seconds. From the graph establish the time it will take for the current to reach 50% of its final value and check this value by calculation.

12. A coil of inductance L and resistance R are connected as shown below.



The switch is moved from contact A to contact B with the result that the current i decreases according to the equation $i = I\left(1 - e^{-\frac{Rt}{L}}\right)$. Draw the graph for this decrease plotting i against t for a time of 300 ms after the switch is moved. From the plot estimate the current flowing 158 ms after switching.

Solutions: **1.** (i) 1151; (ii) 48.2; (iii) 442; (iv) 33.97; (v) 58.7; **2.** (i) 3.2; (ii) 2.43; (iii) 3.99; (iv) -1.4; (v) 4.12; **3.** 5.1; **4.** 5.5 ms; **5.** 769 ms; **6.** 0.00001 F; **9.** $t = 64 \times 10^{-3}$; **10.** 91.9×10^{-3} s; **11.** 0.1 s; **12.** 1 A

Chapter 9

Long division and factorisation

9.1 Polynomials and long division

A **polynomial** is an expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, \ldots, a_n$ are constants, also called **coefficients**. Its **degree** (or **order**) is the highest occurring power of x; in other words if $a_n \neq 0$, then the degree of $p(x)$ above is n.

The low–degree polynomials have special names:

example polynomial	its degree	also called
$x - 7$	1	linear polynomial
$3x^2 + 2$	2	quadratic polynomial
$4x^3 + 5x^2 - 3x$	3	cubic polynomial
$-x^4 + x^2 - 2$	4	quartic polynomial

Recall long division of numbers:

$$\begin{array}{r l} \underline{205} & \text{r. } 18 \\ 21\overline{)4323} & \\ \underline{-42} & \\ 123 & \\ \underline{-105} & \\ 18 & \end{array} \qquad \text{or} \qquad \begin{array}{r l} 4323 & \div\, 21 = 205 \text{ r. } 18 \\ \underline{-42} & \\ 123 & \\ \underline{-105} & \\ 18 & \end{array}$$

We can write this as

$$\frac{4323}{21} = 205 + \frac{18}{21} \qquad \text{or} \qquad 4323 = 205 \cdot 21 + 18.$$

In general:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

or

$$\text{dividend} = \text{quotient} \cdot \text{divisor} + \text{remainder}.$$

Long division of polynomials works very similarly:

$$\begin{array}{rl}
(x^4 - x^3 + x^2 - 3x + 5) & \div\ (x-1) = x^3 + x - 2 \quad \text{r. } 3 \\
\underline{-(x^4 - x^3)} & \\
0 + x^2 - 3x & \\
\underline{-(x^2 - x)} & \\
-2x + 5 & \\
\underline{-(-2x + 2)} & \\
3 &
\end{array}$$

If we write $p(x) := x^4 - x^3 + x^2 - 3x + 5$ (for the dividend) and $q(x) := x^3 + x - 2$ (for the quotient), we can write

$$p(x) = q(x) \cdot (x - 1) + 3.$$

This holds for any x, so in particular for $x = 1$ we get $p(1) = q(1) \cdot 0 + 3 = 3$.

Theorem 9.1 (Remainder Theorem)**.** *The remainder left when a polynomial $p(x)$ is divided by $x - a$ is equal to $p(a)$. In particular, $x - a$ is a factor of $p(x)$ if an only if $p(a) = 0$.*

Example 9.1. Determine the remainder when $p(x) = 5x^3 + 2x^2 - 6$ is divided by $x - 2$.

Solution. The remainder is $p(2) = 5 \cdot 8 + 2 \cdot 4 - 6 = 42$.

Example 9.2. Divide $3x^3 - 16x^2 + 15x + 18$ by $x - 3$:

$$\begin{array}{rl}
(3x^3 - 16x^2 + 15x + 18) & \div\ (x-3) = 3x^2 - 7x - 6 \\
\underline{-(3x^3 - 9x^2)} & \\
-7x^2 + 15x & \\
\underline{-(-7x^2 + 21x)} & \\
-6x + 18 & \\
\underline{-(-6x + 18)} & \\
0 &
\end{array}$$

Thus $3x^3 - 16x^2 + 15x + 18 = (3x^2 - 7x - 6)(x - 3)$.
Using a quadratic formula or by inspection, we can further factorise the quadratic as $3x^2 - 7x - 6 = (x - 3)(3x + 2)$.
Altogether, we obtain complete factorisation

$$3x^3 - 16x^2 + 15x + 18 = (x - 3)^2(3x + 2).$$

9.2 Obtaining the factorisation of a polynomial

We start with a polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. We wish to factorise it as much as possible, ideally into linear factors.

(Note that this is a heuristic algorithm, i.e. it may not always work. The point is that getting a factorisation of a polynomial in the end amounts to finding its roots — and Abel's impossibility theorem asserts that there is no algebraic formula that would yield the solutions of an equation $p(x) = 0$ when degree of the polynomial $p(x)$ is 5 or higher.)

Step 1. ("Guess a root.") Calculate $p(b)$ for divisors b of a_0. When we find b such that $p(b) = 0$, we know that $x - b$ is a factor of $p(x)$, and we perform...

Step 2. ("Divide.") Calculate $q(x) := p(x) \div (x-b)$ using long division.

Repeat. Apply Steps 1. and 2. to the "new" polynomial $q(x)$ (in place of $p(x)$.\ Note that $q(x)$ has degree one less than the degree of $p(x)$, so eventually we end up with a linear polynomial, and so we are done.

Example 9.3. Factorise $p(x) = x^4 - x^3 + x^2 - 3x + 2$.

Solution. $p(1) = 1-1+1-3+2 = 0 \implies x-1$ is a factor of $p(x)$. Next, we perform long division:

$$\begin{array}{l}
\quad (x^4 - x^3 + x^2 - 3x + 2) \quad \div \; (x-1) = x^3 + x - 2 =: q(x) \\
\underline{-(x^4 - x^3)} \\
\qquad\qquad 0 + x^2 - 3x \\
\qquad\qquad \underline{-(x^2 - x)} \\
\qquad\qquad\qquad -2x + 2 \\
\qquad\qquad\qquad \underline{-(-2x + 2)} \\
\qquad\qquad\qquad\qquad 0
\end{array}$$

Next, we now look for a root of $q(x)$.
$q(1) = 1+1-2 = 0 \implies x-1$ is a factor of $q(x)$. We continue by performing long division:

$$\begin{array}{l}
\quad (x^3 \qquad + x - 2) \quad \div \; (x-1) = x^2 + x + 1 \\
\underline{-(x^3 - x^2)} \\
\quad 0 + x^2 + x \\
\quad \underline{-(x^2 - x)} \\
\qquad\qquad 2x - 2 \\
\qquad\qquad \underline{-(2x - 2)} \\
\qquad\qquad\qquad 0
\end{array}$$

We now try to continue by further factorising $x^2 + x + 1$. However the discriminant of this quadratic is $1 - 4 \cdot 2 = -7 < 0$, so it can not be further factorised.
Answer: $p(x) = (x-1)^2(x^2 + x + 1)$.

Example 9.4.

$$\begin{array}{l}
\quad (x^3 + 5x^2 - 7x + 3) \quad \div \; (x^2 + 2x + 1) = x + 3 \quad \text{rem.} \; -14x \\
\underline{-(x^3 + 2x^2 + \;\; x)} \\
\qquad\quad 3x^2 - 8x + 3 \\
\qquad \underline{-(3x^2 + 6x + 3)} \\
\qquad\qquad -14x
\end{array}$$

Example 9.5. The polynomial $p(x) = 6x^3 - 23x^2 + ax + b$ leaves remainder 11 when divided by $x - 3$ and remainder -21 when divided by $x + 1$.

(i) Show that a and b satisfy the equations $3a + b = 56$, $-a + b = 8$.
(ii) Solve the simultaneous equations above.
(iii) Show that $x - 2$ is a factor of $p(x)$.
(iv) Factorise $p(x)$.
(v) Determine the roots of $p(x)$.

(i) Deriving the equations:

- $p(x)$ leaves a remainder of 11 when divided by $x-3$
  $\iff p(3) = 11$ (by Remainder Theorem)
  $\iff 6 \cdot 27 - 23 \cdot 9 + 3a + b = 11$
  $\iff 3a + b = 56$ (A)
- $p(x)$ leaves a remainder of -21 when divided by $x+1$
  $\iff p(-1) = -21$ (by Remainder Theorem)
  $\iff -6 - 23 - a + b = -21$
  $\iff -a + b = 8$ (B)

(ii) From (A): $b = 56 - 3a$.
Substitute into (B): $-a + 56 - 3a = 8$
$\iff 4a = 48$
$\iff a = 12$. Substitute back into (B): $-12 + b = 8$
$\iff b = 20$.

(iii) From (ii): $p(x) = 6x^3 - 23x^2 + 12x + 20$.
$p(2) = 6 \cdot 8 - 23 \cdot 4 + 12 \cdot 2 + 20 = 48 - 92 + 24 + 20 = 0$, so $x-2$ is a factor of $p(x)$ by Remainder Theorem.

(iv) We start by long division by the factor we already know from (iii).

$$\begin{array}{rl}
(6x^3 - 23x^2 + 12x + 20) & \div\ (x-2) = 6x^2 - 11x - 10 \\
\underline{-(6x^3 - 12x^2)} \qquad\qquad\quad & \\
-11x^2 + 12x \quad\;\; & \\
\underline{-(-11x^2 + 22x)} \quad\;\; & \\
-10x + 20 & \\
\underline{-(-10x + 20)} & \\
0 &
\end{array}$$

By inspection: $6x^2 - 11x - 10 = (2x-5)(3x+2)$. So altogether $p(x) = (x-2)(2x-5)(3x+2)$.

(v) From (iv): the roots of $p(x)$ are $x = 2$, $x = \frac{5}{2}$ and $x = -\frac{2}{3}$.

Exercises 9 (Long division and the Remainder theorem)

1. Show that $x - 3$ is a factor of $6x^3 - 19x^2 + x + 6$ and hence determine the other factors.
2. Given that $x = 0.25$ is one root which satisfies the equation $4x^3 + 3x^2 - 5x + 1 = 0$. Determine the other roots.
3. Establish the remainder when:
   i) $x - 4$ is divided into $24x^4 - 5x^3 + 3x$
   ii) $x + 5$ is divided into $4x^3 + 25x^2 + 20x - 25$
   iii) $x - 2$ is divided into $12x^3 + x^2 - 38x - 24$
4. Determine the quotient and remainder when:
   i) $x - 2$ is divided into $x^3 + 14x^2 + 56x + 14$
   ii) $x + 2$ is divided into $4x^4 + 3x^2 - 21x + 4$
5. Solve for x: $2x^3 - 5x^2 - x + 6 = 0$.
6. The power of a flat belt is given by the formula $P = Av + Bv^3$, where v is the linear velocity of the belt and A and B are constants. Given that $A = 12$ and $B = -1$, determine the value of the velocity when $P = 16$.
7. The value of the magnetic field at a given point due to the presence of two magnets of equal moment, M, is given by

$$F = \frac{2M}{d_1^3} - \frac{2M}{d_2^3}, \tag{1}$$

   where d_1 and d_2 are the distances of the magnets from the given point. Show that the field may be expressed as

$$F = 2M \left(\frac{(d_2 - d_1)(d_2^2 + d_1 d_2 + d_1^2)}{d_1^3 d_2^3} \right). \tag{2}$$

   (Hint: Starting with the equation (1), add the fractions and then find a factor for the numerator.)
8. Use algebraic long division to find the quotient and the remainder for the following:
   a) $x^3 + 2x^2 – x – 2$ divided by $x – 1$
   b) $2x^3 + 9x^2 – 4x – 21$ divided by $2x – 3$
   c) $x^4 + x^3 + 7x – 3$ divided by $x^2 – x + 3$

d) $6x^4 + 14x^3 – 9x^2 – 7x + 3$ divided by $2x^2 – 1$

9. Find the quotient and remainder for:

a) $\dfrac{x^2 + 6x - 2}{x^2 + 4x + 1}$
b) $\dfrac{2x^2 + 5}{x^2 + 1}$
c) $\dfrac{5x^2 + 2x - 11}{x^2 + x - 2}$
d) $\dfrac{x^3 - 5x^2 + 9x - 7}{x^2 - 2x + 8}$

10. Use the remainder theorem to find the remainder when:

a) $6x^3 + 7x^2 - 15x + 4$ is divided by $(x - 1)$
b) $2x^3 - 3x^2 + 5x + 4$ is divided by $(x + 1)$
c) $x^3 - 7x^2 + 6x + 1$ is divided by $(x - 3)$
d) $5 + 6x + 7x^2 - x^3$ is divided by $(x + 2)$
e) $x^4 - 3x^3 + 2x^2 + 5$ is divided by $(x - 1)$

11. Factorise:

a) $x^3 - 2x^2 - 5x + 6$
b) $2x^3 + 7x^2 - 7x - 12$
c) $2x^3 + 3x^2 - 17x + 12$
d) $6x^3 - 5x^2 - 17x + 6$
e) $2x^4 + 7x^3 - 17x^2 - 7x + 15$
f) $6x^4 + 31x^3 + 57x^2 + 44x + 12$

12. Given that $x + 2$ is a factor of $2x^3 + 6x^2 + bx - 5$, find the value of b and then find the remainder when the expression is divided by $(2x - 1)$.

13. The expression $3x^3 + 2x^2 - bx + a$ is exactly divisible by $(x - 1)$, but leaves a remainder of 10 when divided by $(x + 1)$. Find the values of a and b.

14. The expression $8x^3 - 4x^2 + ax + b$ gives a remainder of -19 when divided by $(x + 1)$ and a remainder of 2 when divided by $(2x - 1)$. Find the values of a and b.

Solutions: **1.** $(x - 3)(3x - 2)(2x + 1)$; **2.** -1.618, 0.618; **3.** (i) 5836; (ii) 0; (iii) 0; **4.** (i) $Q = x^2 + 16x + 88$, $r = 190$; (ii) $Q = 4x^3 - 8x^2 + 19x - 59$, $r = 122$; **5.** $2, 3/2, -1$; **6.** $v = 2$ (twice), or $v = -4$; **8.** (a) $x^2 + 3x + 2$ rem 0; (b) $x^2 + 6x + 7$ rem 0; (c) $x^2 + 2x - 1$ rem 0; (d) $3x^2 + 7x - 3$ rem 0; **9.** (a) 1 rem $2x - 3$; (b) 2 rem 3; (c) 5 rem $-3x - 1$; (d) $x - 3$ rem $-5x + 17$; **10.** (a) 2; (b) -6; (c) -17; (d) 29; (e) 5; **11.** (a) $(x - 1)(x - 3)(x + 2)$; (b) $(2x - 3)(x + 4)(x + 1)$; (c) $(2x - 3)(x + 4)(x - 1)$; (d) $(3x - 1)(2x + 3)(x - 2)$; (e) $(x + 1)(x - 1)(2x - 3)(x + 5)$; (f) $(x + 1)(x + 2)(3x + 2)(2x + 3)$; **12.** $b = 3/2$ and the reminder when divided by $(2x - 1)$ is $-5/2$; **13.** $a = 3, b = 8$; **14.** $a = 6, b = -1$

Chapter 10

Partial fractions

A **rational function** is a quotient of two polynomials, i.e. an expression of the form $\frac{p_1(x)}{p_2(x)}$, where both $p_1(x)$ and $p_2(x)$ are polynomials, and $p_2(x) \neq 0$.

For example: $\frac{5x+7}{3x^2+5}, \frac{5x^2-4x+3}{x-7}, \dots$

Long division of polynomials allows us to write $\frac{p_1(x)}{p_2(x)}$ in the form

$$\frac{p_1(x)}{p_2(x)} = q(x) + \frac{r(x)}{p_2(x)},$$

where $q(x)$ and $r(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $p_2(x)$.

The aim of "partial fractions": simplify $\frac{r(x)}{p_2(x)}$ further.

First factorise the denominator $p_2(x)$ (completely[1]).

Then, every rational function of the type as in the left–hand column can be split into **partial fractions** as given in the right–hand column:

Type I	$\frac{Px+Q}{(ax+b)(cx+d)}$	=	$\frac{A}{ax+b} + \frac{B}{cx+d}$
Type II	$\frac{Px+Q}{(ax+b)^2}$	=	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
Type III	$\frac{Px^2+Qx+R}{(ax+b)^2(cx+d)}$	=	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{cx+d}$
Type IV	$\frac{Px^2+Qx+R}{(ax^2+bx+c)(dx+e)}$	=	$\frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$

Note: It is a Theorem that the above actually works — these are equations which work for all values of x simultaneously! It is not proved here though.

[1]When the coefficients of a polynomial are real numbers, it is always possible to factorise it into a product of factors, each of which is either linear, or an irreducible quadratic. (This is a consequence of the Fundamental Theorem of Algebra.)

Example 10.1. Split $\dfrac{-4x+15}{(x+2)(x-1)}$ into partial fractions.

Solution. First, note that the degree of the numerator is less than the degree of the denominator, so we do not need to perform any long division.

Second, the denominator is already factored as much as possible (into linear factors). We see that it is "Type I", so we want to find A and B such that

$$\frac{-4x+15}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Multiplying both sides by $(x+2)(x-1)$, we arrive at
$-4x+15 = A(x-1) + B(x+2)$.
Let $x = 1$. Then $-4+15 = A \cdot 0 + B \cdot 3 \implies B = \frac{11}{3}$.
Let $x = -2$. Then $8+15 = A(-3) + B \cdot 0 \implies A = -\frac{23}{3}$.
Answer: $\dfrac{-4x+15}{(x+2)(x-1)} = -\dfrac{23}{3(x+2)} + \dfrac{11}{3(x-1)}$.

Example 10.2. Split $\dfrac{5x^2-3x+2}{(x-3)^2(x+2)}$ into partial fractions.

Solution. (Type III.) We want to find A, B, C such that

$$\frac{5x^2-3x+2}{(x-3)^2(x+2)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+2}.$$

$\iff 5x^2 - 3x + 2 = A(x-3)(x+2) + B(x+2) + C(x-3)^2$.
Let $x = -2$. Then $5 \cdot 4 - 3(-2) + 2 = C \cdot (-5)^2 \implies C = \frac{28}{25}$.
Let $x = 3$. Then $5 \cdot 9 - 3 \cdot 3 + 2 = B \cdot 5 \implies B = \frac{38}{5}$.
Equating coefficients of x^2 gives $5 = A + C \implies A = 5 - C = 5 - \frac{28}{25} = \frac{97}{25}$.
Answer: $\dfrac{5x^2-3x+2}{(x-3)^2(x+2)} = \dfrac{97}{25(x-3)} + \dfrac{38}{5(x-3)^2} + \dfrac{28}{25(x+2)}$.

Example 10.3. Split $\dfrac{2x+5}{(x^2+x+1)(x+3)}$ into partial fractions.

Solution. (Type IV.) We want to find A, B, C such that

$$\frac{2x+5}{(x^2+x+1)(x+3)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x+3}.$$

$\iff 2x + 5 = (Ax+B)(x+3) + C(x^2+x+1)$.
Let $x = -3$. Then $2 \cdot (-3) + 5 = C \cdot (9 - 3 + 1) \implies C = -\frac{1}{7}$.
Equating coefficients of x^2 gives $0 = A + C \implies A = \frac{1}{7}$.
Equating constant coefficients gives $5 = 3B + C \implies B = \frac{5-C}{3} = \frac{12}{7}$.
Answer: $\dfrac{2x+5}{(x^2+x+1)(x+3)} = \dfrac{x+12}{7(x^2+x+1)} - \dfrac{1}{7(x+3)}$.

Exercises 10 (Partial fractions)

Express the following fractions in terms of partial fractions:

i) $\dfrac{x-4}{(x-2)(x+1)}$

ii) $\dfrac{3x+2}{(x+1)(x-5)}$

iii) $\dfrac{4x^2-2}{(x^2+3)(x-2)}$

iv) $\dfrac{3x^2+1}{(x^2+4)(3x-1)}$

v) $\dfrac{4x^2-3x+2}{(x-2)(2+x^2)}$

vi) $\dfrac{x-6}{(x-3)^2(x+1)}$

vii) $\dfrac{6x-7}{(x-5)(x-4)^2}$

viii) $\dfrac{2x^2+5x-3}{(x-1)(x-3)}$

ix) $\dfrac{3x^3+x^2+2x-4}{(x-2)^2(x-1)}$

x) $\dfrac{2x^2+5x-3}{(x+1)(2x-3)(x-4)}$

Solutions:

i) $\dfrac{-2}{3(x-2)}+\dfrac{5}{3(x+1)}$

ii) $\dfrac{1}{6(x+1)}+\dfrac{17}{6(x-5)}$

iii) $\dfrac{2x+4}{x^2+3}+\dfrac{2}{x-2}$

iv) $\dfrac{33x+11}{37(x^2+4)}+\dfrac{12}{37(3x-1)}$

v) $\dfrac{2x+1}{2+x^2}+\dfrac{2}{x-2}$

vi) $\dfrac{-3}{4(x-3)^2}+\dfrac{7}{16(x-3)}-\dfrac{7}{16(x+1)}$

vii) $\dfrac{-17}{(x-4)^2}-\dfrac{23}{x-4}+\dfrac{23}{x-5}$

viii) $2 - \dfrac{2}{x-1} + \dfrac{15}{x-3}$

ix) $3 + \dfrac{28}{(x-2)^2} + \dfrac{14}{x-2} + \dfrac{2}{x-1}$

x) $\dfrac{-6}{25(x+1)} - \dfrac{36}{25(2x-3)} + \dfrac{49}{25(x-4)}$

Chapter 11

Arithmetic and geometric progressions

11.1 Arithmetic progressions

An **arithmetic progression (AP)** is a (finite or infinite) sequence of numbers

$$a_1, a_2, a_3, \ldots$$

such that the difference between consecutive terms is a constant (also called the **common difference**) d. In other words,

$$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots .$$

For example:

- $1, 3, 5, 7, \ldots \quad (d = 2)$
- $3, 1, -1, -3, -5, \ldots \quad (d = -2)$

We have

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d = a_1 + 2d \\ a_4 &= a_3 + d = a_1 + 3d \end{aligned}$$

In general: $\boxed{a_n = a + (n-1)d}$ where $a = a_1$.

Theorem 11.1. *The sum S_n of the first n terms of an AP with the first term a and the common difference d is given by*

$$\boxed{S_n = \frac{n}{2}\left(2a + (n-1)d\right)}$$

Proof (non-examinable):

$$\begin{array}{rcccccccc} S_n & = & a & + & (a+d) & + & \ldots & + & (a+(n-1)d) \\ S_n & = & (a+(n-1)d) & + & (a+(n-2)d) & + & \ldots & + & a \\ \Longrightarrow & & & & & & & & \\ 2S_n & = & (2a+(n-1)d) & + & (2a+(n-1)d) & + & \ldots & + & (2a+(n-1)d) \\ & = & n(2a+(n-1)d) & & & & & & \end{array}$$

The " $\Longrightarrow$ " step is by summing the two previous equations "vertically". The last step holds as there are n summands, each of them equal to $2a + (n-1)d$).

Hence $S_n = \frac{n}{2}(2a + (n-1)d)$.

Example 11.1. Determine the sum S_{20} of the first 20 terms of the arithmetic progression: $10, 6, 2, -2, \dots$.

Solution. $a = 10, d = -4, n = 20$
$\Longrightarrow\ S_{20} = \frac{20}{2}(2 \cdot 10 + 19 \cdot (-4)) = 10(20 - 76) = -560.$

Example 11.2. The sum S_8 of the first 8 terms of an AP is 90, and its first term is 6. What is the common difference?

Solution. $90 = S_8 = \frac{8}{2}(2 \cdot 6 + 7 \cdot d)$
$\Longrightarrow\ 90 = 4(12 + 7d)$
$\Longrightarrow\ \frac{90}{4} - 12 = 7d$
$\Longrightarrow\ d = \frac{42}{28} = \frac{3}{2} = 1.5.$

Example 11.3. How many terms of the AP $3, 6, 9, \dots$ must be taken so that their sum is 135?

Solution. $a = 3, d = 3, S_n = 135, n = ?$
$\Longrightarrow\ 135 = S_n = \frac{n}{2}(2 \cdot 3 + (n-1) \cdot 3)$
$\Longrightarrow\ 270 = 3n^2 + 3n$
$\Longrightarrow\ n^2 + n - 90 = 0$
$\Longrightarrow\ (n + 10)(n - 9) = 0$
$\Longrightarrow\ n = 9$ (since n must be positive).

11.2 Geometric progressions

A **geometric progression (GP)** is a (finite or infinite) sequence of numbers

$$a_1, a_2, a_3, \dots$$

such that the quotient of the consecutive terms is a constant (also called the **common ratio**) r. In other words,

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \dots$$

For example:

- $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \quad (r = \frac{1}{2})$
- $2, -6, 18, -54, \dots \quad (r = -3)$

We have: $a_2 = a_1 r,\ \ a_3 = a_2 r = a_1 r^2,\ \ a_4 = a_3 r = a_1 r^3, \dots$

In general: $\boxed{a_n = a \cdot r^{n-1}}$ where $a = a_1$.

Theorem 11.2. *The sum S_n of the first n terms of a GP with the first term a and the common ratio r is given by*

$$\boxed{S_n = a \cdot \frac{1 - r^n}{1 - r}}$$

*In particular, if $-1 < r < 1$, then the sum S_∞ of all (infinitely many) terms is $S_\infty = a \cdot \dfrac{1}{1-r}$. (The sum of an infinite GP is also called **geometric series**.)*

Proof (non–examinable):

$$\begin{aligned}(1-r)S_n &= (1-r)(a+ar+\dots+ar^{n-1})\\ &= (a+ar+\dots+ar^{n-1})-(ar+ar^2+\dots+ar^n)\\ &= a-ar^n = a(1-r^n) \qquad \text{(everything else cancels)}\end{aligned}$$

Thus $S_n = a\frac{1-r^n}{1-r}$.
Now if $-1 < r < 1$, then $\lim_{n\to\infty} r^n = 0$. So $S_\infty = \lim_{n\to\infty} S_n = a\frac{1}{1-r}$.

Example 11.4. Determine the sum S_7 of the first 7 terms of the geometric progession $4, -8, 16, \dots$

Solution. $a = 4, r = -2, n = 7$
$\implies S_7 = 4\dfrac{1-(-2)^7}{1-(-2)} = 4\cdot\dfrac{129}{3} = 172.$

Example 11.5. A GP is given by $\dfrac{1}{4}, \dfrac{1}{16}, \dfrac{1}{64}, \dots$. Determine S_∞.

Solution. $a = \frac{1}{4}, r = \frac{1}{4}$
$\implies S_\infty = \dfrac{1}{4}\cdot\dfrac{1}{1-\frac{1}{4}} = \dfrac{1}{4}\cdot\dfrac{1}{\frac{3}{4}} = \dfrac{1}{3}.$

Exercises 11 (Arithmetic and geometric progressions)

1. For the two progressions below, determine the 20^{th} term:

   i) $3,\ 7.5,\ 18.75, \ldots$
   ii) $6,\ 18,\ 30, \ldots$

2. Determine the common difference for an AP where the sum for the first 10 terms of the progression is 233.75, given that the first term is 6.5.

3. How many terms of the progression $4, 10, 16, \ldots$ must be taken so that the sum is equal to 602?

4. A machine is required to have 5 speeds, the lowest being 60 rev/min and the highest 680 rev/min. State the complete range of speeds i) in AP and ii) in GP.

5. A drilling machine has 10 speeds arranged in GP and operates at a surface cutting speed of $15\,\mathrm{m/s}$. The smallest drill bit has diameter $6\,\mathrm{mm}$ and the largest drill bit has diameter $18\,\mathrm{mm}$. Determine the complete range of speeds. (Speed in $\mathrm{rev} \cdot \mathrm{s}^{-1}$ = surface cutting speed in $\mathrm{m} \cdot \mathrm{s}^{-1}$/drill bit circumference in m.)

6. A tie rod $5\,\mathrm{m}$ long is made such that the cross-sectional areas at equal distances are a geometric progression. The area at the smaller end is $10\,\mathrm{mm}^2$ whilst the area one tenth of the way down the rod is $25\,\mathrm{mm}^2$. Calculate the area of the cross section at the larger end.

7. A body falling freely falls $4.9\,\mathrm{m}$ in the 1^{st} second, $14.7\,\mathrm{m}$ in the 2^{nd} second, $24.5\,\mathrm{m}$ in the 3^{nd} second and so on. Determine:

   a) How far it falls in the 10^{th} second.
   b) The total distance fallen in 10 seconds.

8. If £50 is saved in a certain year and each year thereafter £5 more is saved than in the previous year, after how many years will the total equal £1950, excluding any interest?

Solutions: **1.** (i) 109×10^6; (ii) 234; **2.** $d = 3.75$; **3.** 14; **4.** (i) $60, 215, 370, 525, 680$; (ii) $60, 110, 202, 371, 680$; **5.** $265, 299, 338, 382, 432, 488, 552, 623, 704, 796$; **6.** $95 \times 10^3\,\mathrm{mm}^2$; **7.** (a) $93.1\,\mathrm{m}$; (b) $490\,\mathrm{m}$; **8.** 20 years

Chapter 12

Binomial Theorem

12.1 Pascal's triangle

Binomial Theorem gives a formula for $(a+b)^n$. First few:

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)^3 = (a^2 + 2ab + b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a+b)^4 = \cdots = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

In general:

$$(a+b)^n = C_{n,0} \cdot a^n + C_{n,1} \cdot a^{n-1}b + C_{n,2} \cdot a^{n-2}b^2 + \cdots + C_{n,n-1} \cdot ab^{n-1} + C_{n,n} \cdot b^n$$

for some coefficients $C_{n,0}, C_{n,1}, \ldots, C_{n,n}$. "Extrapolating" from the examples above, we can "guess":

$$\begin{aligned} C_{n,0} &= C_{n,n} = 1 \\ C_{n,1} &= C_{n,n-1} = n \\ C_{n,k} &= C_{n-1,k-1} + C_{n-1,k} \end{aligned}$$

The two rules: $C_{n,0} = C_{n,n} = 1$ and $C_{n,k} = C_{n-1,k-1} + C_{n-1,k}$ are the building laws for **Pascal's triangle**:

$$\begin{array}{ccccccccccccc} &&&&&& 1 &&&&&& \\ &&&&& 1 && 1 &&&&& \\ &&&& 1 && 2 && 1 &&&& \\ &&& 1 && 3 && 3 && 1 &&& \\ && 1 && 4 && 6 && 4 && 1 && \\ & 1 && 5 && 10 && 10 && 5 && 1 & \\ 1 && 6 && 15 && 20 && 15 && 6 && 1 \\ &&&&&& \vdots &&&&&& \end{array}$$

::: {.example} $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. ::: ::: {.example}

$$\begin{aligned} (a+2x)^4 &= a^4 + 4a^3(2x) + 6a^2(2x)^2 + 4a(2x)^3 + (2x)^4 \\ &= a^4 + 8a^3x + 24a^2x^2 + 32ax^3 + 16x^4. \end{aligned}$$

:::

Example 12.1. Expand $(1.005)^4$ to four decimal places (4 d.p).

Solution.

$$\begin{aligned}(1.005)^4 &= (1+0.005)^4 \\ &= 1 + 4\cdot 0.005 + 6\cdot(0.005)^2 + \underbrace{4\cdot(0.005)^3 + (0.005)^4}_{\text{do not contribute to 4 d.p.}} \\ &= 1 + 0.02 + 0.00015 + \ldots \\ &\approx 1.0202 \qquad \text{(to 4 d.p.)}\end{aligned}$$

12.2 Binomial coefficients

For a positive integer m, define m **factorial**, denoted "$m!$", as $m! = 1\cdot 2\cdot 3\cdots(m-1)\cdot m$; and declare that $0! = 1$.

Theorem 12.1. $\boxed{C_{n,k} = \dfrac{n!}{k!\cdot(n-k)!}}$

Note: The number $C_{n,k}$ is also denoted by $\binom{n}{k}$, read "n choose k' '[1].

Proof (non–examinable): To argue that the formula "works correctly", it suffices to check that the number above satisfies the laws defining Pascal's triangle.

That $C_{n,0} = C_{n,n} = 1$ is clear.

Now checking that $C_{n,k} = C_{n-1,k-1} + C_{n-1,k}$:

$$\begin{array}{ccccc} \dfrac{n!}{k!(n-k)!} & \stackrel{?}{=} & \dfrac{(n-1)!}{(k-1)!(n-k)!} & + & \dfrac{(n-1)!}{k!(n-1-k)!} \\ & & = & & = \\ & & \dfrac{k}{n}\cdot\dfrac{n!}{k!(n-k)!} & + & \dfrac{n-k}{n}\cdot\dfrac{n!}{k!(n-k)!} \end{array}$$

This finishes the proof.

Let us analyse the formula: observe that $C_{n,k} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}$.

We will use this expression as a *definition* for $C_{n,k}$ when n is negative or a fraction.

Theorem 12.2. *If* $-1 < x < 1$, *then the right hand side of*

$$\begin{aligned}(1+x)^n &= C_{n,0} + C_{n,1}x + C_{n,2}x^2 + C_{n,3}x^3 + \ldots \\ &= 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \ldots\end{aligned}$$

converges[2] *and this equality is valid also when* n *is negative or a fraction.*

[1] This is because it also computes the number of ways one can choose k objects out of n.

[2] "Converges" is a mathematical term that you will learn more precisely later on; the meaning here is roughly that "for each x, the infinite sum *does* describe a well-defined and unique number, and we get better and better approximations to it by adding up more and more terms of the infinite series".

Example 12.2. $\dfrac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \mp \cdots$

Example 12.3. $\dfrac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

Example 12.4. Expand $\dfrac{1}{(1+x)^3}$ up to the term in x^4.

Solution.

$$\begin{aligned}\frac{1}{(1+x)^3} &= (1+x)^{-3}\\ &= 1 + (-3)x + \frac{(-3)(-4)}{1\cdot 2}x^2 + \frac{(-3)(-4)(-5)}{1\cdot 2\cdot 3}x^3 + \frac{(-3)(-4)(-5)(-6)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots\\ &= 1 - 3x + 6x^2 - 10x^3 + 15x^4 \mp \cdots\end{aligned}$$

Example 12.5. Expand $(3-x)^{-4}$ up to the term in x^3.

Solution.

$$\begin{aligned}(3-x)^{-4} &= 3^{-4}\left(1 - \frac{x}{3}\right)^{-4}\\ &= 3^{-4}\left(1 + (-4)\left(-\frac{x}{3}\right) + \frac{(-4)(-5)}{1\cdot 2}\left(-\frac{x}{3}\right)^2 + \frac{(-4)(-5)(-6)}{1\cdot 2\cdot 3}\left(-\frac{x}{3}\right)^3 + \cdots\right)\\ &= \frac{1}{81}\left(1 + \frac{4}{3}x + \frac{10}{9}x^2 + \frac{20}{27}x^3 + \cdots\right)\end{aligned}$$

converges and equality is valid for $-1 < \frac{x}{3} < 1 \iff -3 < x < 3$.

12.3 Binomial approximation

Binomial approximation: If $-1 < x < 1$, then $(1+x)^n \approx 1 + nx$ (for arbitrary n).

Example 12.6. Approximate $\sqrt{1.05}$.

Solution. $\sqrt{1.05} = (1+0.05)^{\frac{1}{2}} \approx 1 + \frac{1}{2}\cdot 0.05 = 1.025$.

Exercises 12 (Binomial expansion)

1. Expand the following using the binomial expansion:
   i) $(a+2x)^5$
   ii) $\left(3a-\dfrac{x}{2}\right)^3$
   iii) $\left(\dfrac{x}{4}-a\right)^4$
2. If x is so small that terms in x^5 and higher may be neglected, show that
$$(x-3)^2(1+x)^9 \approx 666x^4+549x^3+271x^2+75x+9.$$
3. Expand the following to the fourth term using the binomial expansion:
   i) $(1+6x)^3$
   ii) $\left(1-\dfrac{5x}{2}\right)^{-3}$
   iii) $\dfrac{1}{(1+3x)^3}$
   iv) $\sqrt{1-2x}$
4. Assuming that x is so small that terms in x^3 and higher may be ignored, show that
$$\frac{1-\frac{x}{2}}{\sqrt{1+\frac{x}{2}}} \approx 1-\frac{3x}{4}+\frac{7x^2}{32}.$$
5. Assuming x is small, expand $\dfrac{\sqrt{1-x}}{\sqrt{1+2x}}$ up to and including the term in x^2.
6. Using the binomial expansion, evaluate to three decimal places:
   i) $\sqrt{1.01}$
   ii) $\sqrt[3]{27.3}$
7. Using the binomial approximation, simplify:
   i) $\dfrac{\sqrt{1+2x}}{(12+4x)^2}$
   ii) $\dfrac{1}{(3x-2)(1+3x)^{-\frac{1}{2}}}$
   iii) $\dfrac{1-6x+9x^2}{\sqrt{1+6x}}$

8. Expand the following in:

   i) $(3+x)^3$
   ii) $(5+2x)^3$
   iii) $(2+x)^4$
   iv) $(2-x)^4$
   v) $(2y+x)^5$
   vi) $(2x-3y)^5$
   vii) $\left(x-\dfrac{1}{x}\right)^4$
   viii) $\left(x-\dfrac{2}{x}\right)^5$

9. Expand $(2+x)^5$ and use your expansion to find a) $(2.1)^5$ and b) $(1.9)^5$.

10. Expand each of the following in ascending powers of x up to and including the term in x^3:

    i) $(1+2x)(1-x)^{10}$
    ii) $(1-3x)(1+x)^6$
    iii) $(1+x^2)(1+2x)^8$

Solutions: **1.** (i) $a^5+10a^4x+40a^3x^2+80a^2x^3+80ax^4+32x^5$; (ii) $27a^3-\frac{27a^2x}{2}+\frac{9ax^2}{4}-\frac{x^3}{8}$; (iii) $\frac{x^4}{256}-\frac{x^3a}{16}+\frac{3x^2a^2}{8}-xa^3+a^4$; **3.** (i) $1+18x+108x^2+216x^3$; (ii) $1+\frac{15x}{2}+\frac{75x^2}{2}+\frac{625x^3}{4}$; (iii) $1-9x+54x2-270x^3$; iv) $1-x-\frac{x^2}{2}-\frac{x^3}{2}$; **5.** $1-\frac{3x}{2}+\frac{15x^2}{8}$; **6.** (i) 1.005; (ii) 3.011; **7.** (i) $\frac{1}{144}(1+\frac{x}{3})$; (ii) $-\frac{1}{2}(1+3x)$; (iii) $1-9x$; **8.** (i) $27+27x+9x^2+x^3$; (ii) $125+150x+60x^2+8x^3$; (iii) $16+32x+24x^2+8x^3+x^4$; (iv) $16-32x+24x^2-8x^3+x^4$; (v) $32y^5+80y^4x+80y^3x^2+40y^2x^3+10yx^4+x^5$; (vi) $32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5$; (vii) $x^4-4x^2+6-\frac{4}{x^2}+\frac{1}{x^4}$; (viii) $x^5-10x^3+40x-\frac{80}{x}+\frac{80}{x^3}-\frac{32}{x^5}$; **9.** $32+80x+80x^2+40x^3+10x^4+x^5$; (a) 40.84101; (b) 24.76099; **10.** (i) $1-8x+25x^2-30x^3$; (ii) $1+3x-3x^2-25x^3$; (iii) $1+16x+113x^2+464x^3$

Chapter 13

Matrices and Determinants

Matrices and determinants provide a convenient way to solve systems of linear equations (in any number of variables). Consider the system

$$\begin{array}{ccccccc} 4x & + & y & & & = & 1 \\ x & + & 2y & + & z & = & 2 \\ 3x & - & y & + & 2z & = & -1 \end{array}$$

Lined up like this, we can record the coefficients on the LHS into a “matrix”:

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}.$$

13.1 Terminology

More precisely, an $m \times n$ (read “m by n”) **matrix** A is a rectangular array of numbers, with m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Shorthand notation: $A = (a_{ij})_{i=1,\ldots,m;j=1,\ldots,n}$, or just $A = (a_{ij})$.

The number a_{ij} is called the **entry** (or **element**) of A in the i^{th} row and j^{th} column. This can be also referred to as (i, j)-**entry**.

Note: in $m \times n$, the first number (m) refers to *rows*, and the second (n) to *columns*.

The “$m \times n$” is called the **size** (or **order**) of the matrix A.

If the size of a matrix is $n \times n$, we call such a matrix a **square matrix**.

If every entry of a matrix is 0, we call that matrix a **zero matrix**.

Example 13.1. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The **identity** or **unit matrix**, denoted I_n, is the $n \times n$ square matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal, if:

(a) they have the same size, and
(b) $a_{ij} = b_{ij}$ for all i and j.

The **transpose** of the $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix $A^T = (a_{ji})$. In other words, transposing turns rows into columns and vice versa.

Example 13.2. $\begin{pmatrix} 1 & 2 & 5 \\ 0 & -6 & 7 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & -6 \\ 5 & 7 \end{pmatrix}$.

The **sum** (resp. **difference**) of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the *same size* $m \times n$ is the $m \times n$ matrix $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$ (resp. $c_{ij} = a_{ij} - b_{ij}$).

Example 13.3. $\begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 5 \\ 7 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{pmatrix}$,
$\begin{pmatrix} 3 & 5 \\ 2 & 4 \\ -1 & 8 \end{pmatrix} - \begin{pmatrix} 7 & 2 \\ 6 & -9 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} 3-7 & 5-2 \\ 2-6 & 4+9 \\ -1-3 & 8-8 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -4 & 13 \\ -4 & 0 \end{pmatrix}$.

Note: The sum nor the difference of matrices of different sizes is *not defined.*

The **product** λA of a number λ with a matrix $A = (a_{ij})$ is the matrix $B = (b_{ij})$ given by $b_{ij} = \lambda a_{ij}$.

Example 13.4. $2 \cdot \begin{pmatrix} 3 & 4 \\ 1 & 3 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 & 2 \cdot 4 \\ 2 \cdot 1 & 2 \cdot 3 \\ 2 \cdot 2 & 2 \cdot (-2) \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 2 & 6 \\ 4 & -4 \end{pmatrix}$.

13.1.1 Multiplying matrices

First, the product of a $1 \times n$ matrix (also called a **row vector**) with an $n \times 1$ matrix (also called a **column vector**) is the following number (or a 1×1 matrix):

$$(a_1, a_2, \dots, a_n) \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} := a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Example 13.5. $(2, 3) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \cdot 1 + 3 \cdot (-1) = 2 - 3 = -1$.

$(2, 4, -1) \cdot \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} = 2 \cdot 0 + 4 \cdot 2 + (-1) \cdot (-4) = 0 + 8 + 4 = 12$.

In general: let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix. Then their **product** $A \cdot B$ is the $m \times p$ matrix whose (i, j)-entry is defined to be the product of the i^{th} row of A with the j^{th} column of B.

Example 13.6. $\begin{pmatrix} 2 & 4 \\ 9 & 1 \end{pmatrix} \cdot \begin{pmatrix} 8 & 7 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 8 + 4 \cdot 6 & 2 \cdot 7 + 4 \cdot 3 \\ 9 \cdot 8 + 1 \cdot 6 & 9 \cdot 7 + 1 \cdot 3 \end{pmatrix} = \begin{pmatrix} 40 & 26 \\ 78 & 66 \end{pmatrix}$.

Example 13.7. $\begin{pmatrix} 1 & 3 & 2 \\ 6 & 0 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \cdot 1 + 3 \cdot 0 + 2 \cdot 0 & 1 \cdot 0 + 3 \cdot 1 + 2 \cdot 1 & 1 \cdot 1 + 3 \cdot 0 + 2 \cdot 1 \\ 6 \cdot 1 + 0 \cdot 0 + (-4) \cdot 0 & 6 \cdot 0 + 0 \cdot 1 + (-4) \cdot 1 & 6 \cdot 1 + 0 \cdot 0 + (-4) \cdot 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 5 & 3 \\ 6 & -4 & 2 \end{pmatrix}$.

Notes:

(i) If the number of columns of A does not match the number of rows of B, then the product AB is *not defined.*

(ii) In general $AB \neq BA$ even if both sides are defined.

13.2 Determinants

The determinant of a *square* matrix $A = (a_{ij})$ is a certain number that will be explained in the following subsections — first special cases for small matrices, and subsequently in general.

We will write

$$\det(A) = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

for that number.

13.2.1 Determinants: formulas for small sizes

- $\det(a_{11}) := a_{11}$
- $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc,$ for example $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 2 \cdot 5 - 1 \cdot 3 = 7.$
- $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} := a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$

Example 13.8. $\begin{vmatrix} 2 & 0 & 3 \\ 4 & 1 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 2 \cdot 1 \cdot 3 + 0 \cdot 2 \cdot 1 + 3 \cdot 4 \cdot 0 - 3 \cdot 1 \cdot 1 - 2 \cdot 2 \cdot 0 - 0 \cdot 4 \cdot 3 = 6 - 3 = 3.$

13.2.2 Determinants: terminology required for the general formula

We associate a **sign** to positions in a matrix: the (i, j)-position gets $(-1)^{i+j}$. Schematically:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \text{(chessboard pattern)}$$

Next, the (i, j)-**minor** of a square matrix $A = (a_{ij})$ is the determinant of the square matrix that is left when we remove the i^{th} row and j^{th} column of A.

Example 13.9. The $(2, 2)$-minor of $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ is 3.

Example 13.10. The $(2, 1)$-minor of $\begin{pmatrix} 2 & 3 & 4 \\ -1 & 0 & 2 \\ 3 & 4 & 1 \end{pmatrix}$ is $\begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix} = 3 - 16 = -13.$

The (i, j)-**cofactor** of a square matrix $A = (a_{ij})$, denoted A_{ij}, is the (i, j)-minor multiplied by the sign of the (i, j)-position.

Example 13.11. The $(1, 2)$-cofactor of $\begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$ is -1.

Example 13.12. The $(2, 3)$-cofactor of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}$ is $-\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = -(1\cdot 0 - 2\cdot(-1)) = -2.$

13.2.3 Determinants: general definition

The **determinant** of the square matrix $A = (a_{ij})$ is

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.$$

Example 13.13. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + b(-c) = ad - bc.$

Example 13.14. $\begin{vmatrix} 2 & 0 & 3 \\ 4 & 1 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 2\cdot \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} - 0\cdot \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + 3\cdot \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} = 2\cdot(1\cdot 3 - 2\cdot 0) + 3(4\cdot 0 - 1\cdot 1) = 6 - 3 = 3.$

Note: This kind of "expansion' ' works not just along the first row, but along any row or any column, and always gives the same number! (This is a Theorem, and we will not prove it here.)

For example, using the second column: $\begin{vmatrix} 2 & 0 & 3 \\ 4 & 1 & 2 \\ 1 & 0 & 3 \end{vmatrix} = -0 \cdot \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} = 2 \cdot 3 - 3 \cdot 1 = 3.$

Exercises 13 (Matrices and determinants)

1. The matrices A, B and C are given by:

$$A = \begin{pmatrix} 2 & 9 \\ 6 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 9 \\ 2 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 6 & -8 \\ 5 & -2 \end{pmatrix}.$$

   Determine:

   i) $A + C$
   ii) $B + C$
   iii) $B - A$
   iv) $A - C$
   v) $C \cdot B$
   vi) $B \cdot C$

2. Calculate:

   i) $\begin{pmatrix} 7 & 3 \\ 1 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 5 \\ 4 & 7 \end{pmatrix}$
   ii) $\begin{pmatrix} 5 & -2 \\ 7 & 9 \end{pmatrix} - \begin{pmatrix} 5 & -4 \\ 8 & -1 \end{pmatrix}$
   iii) $\begin{pmatrix} 12 & 7 \\ 9 & 3 \end{pmatrix} \cdot \begin{pmatrix} 6 & 2 \\ 5 & -3 \end{pmatrix}$

3. For the matrices shown below, determine $A \cdot B$ and $B \cdot A$ when possible:

$$A = \begin{pmatrix} 5 & 2 & 9 \\ 3 & 1 & 4 \\ 6 & 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix}.$$

4. Evaluate the following determinants:

   i) $\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix}$
   ii) $\begin{vmatrix} 2 & -3 \\ 6 & 8 \end{vmatrix}$
   iii) $\begin{vmatrix} x & 2x \\ x^2 & -5x \end{vmatrix}$
   iv) $\begin{vmatrix} 1 & -5 & 4 \\ 6 & 2 & 8 \\ 1 & -3 & 5 \end{vmatrix}$

v) $\begin{vmatrix} 3 & 2 & 2 \\ 3 & -8 & 2 \\ 3 & 9 & 2 \end{vmatrix}$

Solutions: **1.** (i) $\begin{pmatrix} 8 & 1 \\ 11 & -1 \end{pmatrix}$; (ii) $\begin{pmatrix} 11 & 1 \\ 7 & 2 \end{pmatrix}$; (iii) $\begin{pmatrix} 3 & 0 \\ -4 & 3 \end{pmatrix}$; (iv) $\begin{pmatrix} -4 & 17 \\ 1 & 3 \end{pmatrix}$; (v) $\begin{pmatrix} 14 & 22 \\ 21 & 37 \end{pmatrix}$; (vi) $\begin{pmatrix} 75 & -58 \\ 32 & -24 \end{pmatrix}$; **2.** (i) $\begin{pmatrix} 9 & 8 \\ 5 & 13 \end{pmatrix}$; (ii) $\begin{pmatrix} 0 & 2 \\ -1 & 10 \end{pmatrix}$; (iii) $\begin{pmatrix} 107 & 3 \\ 69 & 9 \end{pmatrix}$; **3.** (i) undefined; (ii) $\begin{pmatrix} 29 & 10 & 26 \\ 41 & 15 & 34 \end{pmatrix}$; **4.** (i) -7; (ii) 34; (iii) $-5x^2 - 2x^3$; (iv) 64; (v) 0;

Chapter 14

Inverse matrix method and Cramer's rule

14.1 Inverse matrices

We can multiply matrices. Can we also "divide by matrices' '?

Recall that for numbers, $\frac{b}{a} = b \cdot a^{-1}$ (where a^{-1} is the number which "solves" the equation $a \cdot x = 1$, if such exists).

We can try to do a similar thing for matrices: Can we "solve" the equation $A \cdot X = I_n$ for a given fixed matrix A?

Answer: well, sometimes we can, sometimes we can't.

When we can, we call such a matrix A **invertible**, and we call the solution to $A \cdot X = I_n$ the **inverse** of A.

A note about I_n: we use this in place of "1", because it is *the* matrix that satisfies $B \cdot I_n = B$ and $I_n \cdot B = B$ whenever the multiplication makes sense (i.e. when B has the correct size).

Definition 14.1. A square $n \times n$ matrix A is called **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. (If such a matrix B exists, then it is automatically unique.) We denote this B by A^{-1}. So:

$$AA^{-1} = I_n = A^{-1}A.$$

Theorem 14.1. *A square $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. In this case*

$$\boxed{A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)},$$

*where $\operatorname{adj}(A)$ is called the **adjoint matrix** (the transpose of the matrix of cofactors):*

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{n2} & \cdots & A_{nn} \end{pmatrix}.$$

This Theorem is somewhat tricky to prove; you can learn the proof in MATH1048 Linear Algebra I module.

Example 14.1. Is $A = \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$ invertible? If so, determine A^{-1}.

Solution. $\det(A) = \begin{vmatrix} 5 & -3 \\ 2 & 1 \end{vmatrix} = 5 \cdot 1 + 3 \cdot 2 = 11 \neq 0$

A is invertible, and

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{11} & \frac{3}{11} \\ \frac{-2}{11} & \frac{5}{11} \end{pmatrix}.$$

We can write the general formula for 2×2 matrices, which you may want to remember:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{if} \quad ad - bc \neq 0.$$

Example 14.2. Determine the matrix A^{-1} of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}$ if it exists.

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + 1 \cdot \cdot \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \\
&= 5 + 1 - 7 = -1.
\end{aligned}$$

$\implies$ A^{-1} exists and

$$\begin{aligned}
A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) \\
&= \frac{1}{-1} \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{pmatrix}^T \\
&= -\begin{pmatrix} 5 & 1 & -7 \\ -3 & -1 & 4 \\ -1 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} -5 & -1 & 7 \\ 3 & 1 & -4 \\ 1 & 0 & -1 \end{pmatrix}^T \\
&= \begin{pmatrix} -5 & 3 & 1 \\ -1 & 1 & 0 \\ 7 & -4 & -1 \end{pmatrix}.
\end{aligned}$$

14.2 Inverse matrix method for simultaneous equations

Given a system of linear equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}, \tag{$*$}$$

let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Then (*) is equivalent to the following matrix equation:

$$A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem 14.2. *Consider a system of linear equations*

$$A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

If $\det(A) \neq 0$, *then it has exactly one solution given by:*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Sketch of Proof (non–examinable):

$$A \cdot A^{-1} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = I_n \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Example 14.3. Solve $\begin{array}{rcl} 4x + 2y & = & 17 \\ 5x - y & = & 7.25 \end{array}$ using the inverse matrix method.

Solution. Let $A = \begin{pmatrix} 4 & 2 \\ 5 & -1 \end{pmatrix}$

$\implies \det(A) = -4 - 10 = -14$ and $A^{-1} = \dfrac{1}{-14} \begin{pmatrix} -1 & -2 \\ -5 & 4 \end{pmatrix}$

$$\begin{aligned} \implies \qquad \begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1} \cdot \begin{pmatrix} 17 \\ 7.25 \end{pmatrix} \\ &= -\frac{1}{14} \cdot \begin{pmatrix} -1 & -2 \\ -5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 17 \\ 7.25 \end{pmatrix} \\ &= -\frac{1}{14} \cdot \begin{pmatrix} -17 - 14.5 \\ -85 + 29 \end{pmatrix} = -\frac{1}{14} \cdot \begin{pmatrix} -31.5 \\ -56 \end{pmatrix} \\ &= \begin{pmatrix} 2.25 \\ 4 \end{pmatrix}. \end{aligned}$$

$\implies x = 2.25$ and $y = 4$.

Example 14.4. Using the inverse matrix method, solve the following system of equations for a, b, c:

$$\begin{array}{ccccccc} 4a & + & 2b & + & c & = & 12 \\ a & - & b & + & c & = & -6 \\ a & + & b & + & c & = & 4. \end{array}$$

Solution. Let $A = \begin{pmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\implies \qquad \det(A) = 4 \cdot \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$
$$= 4 \cdot (-2) - 2 \cdot 0 + 2 = -6$$

$$\text{and} \qquad A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$
$$= \frac{1}{-6} \begin{pmatrix} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 2 \\ 1 & -1 \end{vmatrix} \end{pmatrix}^T$$
$$= -\frac{1}{6} \begin{pmatrix} -2 & 0 & 2 \\ -1 & 3 & -2 \\ 3 & -3 & -6 \end{pmatrix}^T = -\frac{1}{6} \begin{pmatrix} -2 & -1 & 3 \\ 0 & 3 & -3 \\ 2 & -2 & -6 \end{pmatrix}$$

$$\implies \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} 12 \\ -6 \\ 4 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} -2 & -1 & 3 \\ 0 & 3 & -3 \\ 2 & -2 & -6 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ -6 \\ 4 \end{pmatrix}$$
$$= -\frac{1}{6} \begin{pmatrix} -24 + 6 + 12 \\ -18 - 12 \\ 24 + 12 - 24 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} -6 \\ -30 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$$

$\implies a = 1, \quad b = 5, \quad c = -2.$

14.3 Cramer's rule (explained for 2×2 only)

Say we want to solve the following simultaneous equations (assuming that $ac - bd \neq 0$):

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array} \qquad (**)$$

Denote:

- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- $D_x := \begin{vmatrix} e & b \\ f & d \end{vmatrix}$ (the 1st column of A is replaced by the RHS of (**))
- $D_y := \begin{vmatrix} a & e \\ c & f \end{vmatrix}$ (the 2nd column of A is replaced by the RHS of (**))

Then $x = \dfrac{D_x}{\det(A)}$ and $y = \dfrac{D_y}{\det(A)}$ solves (**).

Example 14.5. Using Cramer's rule, solve: $\begin{array}{rcl} 2x + 4y & = & 16 \\ x + 3y & = & 11 \end{array}.$

Solution. Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \implies \det(A) = 6 - 4 = 2$

$\implies D_x = \begin{vmatrix} 16 & 4 \\ 11 & 3 \end{vmatrix} = 4$ and $D_y = \begin{vmatrix} 2 & 16 \\ 1 & 11 \end{vmatrix} = 6$

$\implies x = \frac{4}{2} = 2$ and $y = \frac{6}{2} = 3.$

Exercises 14 (Inverse matrix method and Cramer's rule)

1. Solve the first two pairs of equations using Cramer's rule and use a matrix method for the third:

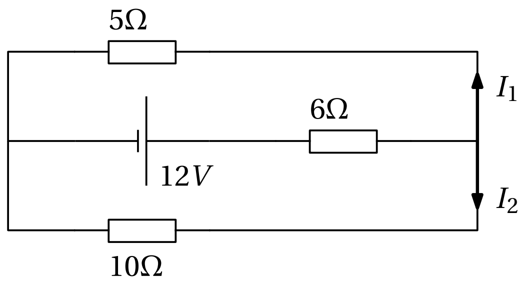
   i) $\begin{cases} 3x - 7y = -2 \\ 4x - 3y = 7 \end{cases}$

   ii) $\begin{cases} 4x - 3y = 18 \\ \ \ x + 2y = -1 \end{cases}$

   iii) $\begin{cases} 11x - 10y = 30 \\ 21y - 20x = -40 \end{cases}$

2. The equations of two straight lines are $6x–8.5y = 10$ and $2x - 4y = 8$. By using a determinant method, establish the coordinates where the two lines cross.

3. The following circuit yields a pair of simultaneous equations:



$$\begin{aligned} 5I_1 - 10I_2 &= 0 \\ 6(I_1 + I_2) + 5I_2 &= 12 \end{aligned}$$

Determine the values of I_1 and I_2 using a matrix method.

4. Solve the equations using a determinant method:

$$\begin{aligned} \frac{3}{x} - \frac{2}{y} &= 0.5 \\ \frac{5}{x} - \frac{3}{y} &= 2.57 \end{aligned}$$

5. A vector system to determine the shortest distance between two moving bodies is analysed and produces the following equations:

$$11s_1 - 10s_2 = 30$$
$$21s_2 - 20s_1 = -40$$

Using Cramer's rule solve for s_1 and s_2.

6. The law connecting friction, F, and load, L, for an experiment to establish the friction force between two surfaces is of the form $F = aL + b$, where both a and b are constants. When $F = 6$, $L = 7.5$ and when $F = 2.7$, $L = 2$, determine the values of a and b using a matrix method.

Solutions: **1.** (i) $x = 55/19$, $y = 29/19$; (ii) $x = 3$, $y = -2$; (iii) $x = 230/31$, $y = 160/31$; **2.** $x = -4$, $y = -4$; **3.** $I_1 = 24/23$, $I_2 = 12/23$; **4.** $x = 0.27$, $y = 0.19$; **5.** $s_1 = 230/31$, $s_2 = 160/31$; **6.** $a = 0.6$, $b = 1.5$

Chapter 15

Circular measure

15.1 Radians and degrees

Angles are normally measured in **degrees**, e.g. a right angle has 90°. In mathematics and engineering it is more natural and convenient to measure angles in **radians**: the length of an arc of a unit circle is equal to the measurement, *in radians*, of the angle that it subtends.

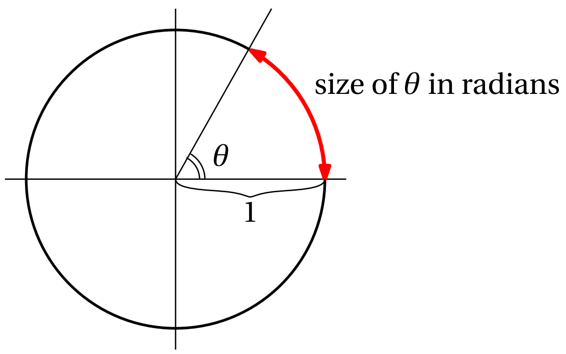


Figure 15.1: Unit circle and radians

degrees	360	180	90	60	45	30	0
radians	2π	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0

15.1.1 Converting degrees to radians

$$\theta^\circ = \left(\frac{2\pi}{360} \cdot \theta\right) \text{ rad}$$

Example 15.1. Find the value of 23° in radians.

Solution. $23^\circ = \left(\frac{2\pi}{360} \cdot 23\right) \text{ rad} \approx 0.4 \text{ rad}.$

15.1.2 Converting radians to degrees

$$\theta \text{ rad} = \left(\frac{360}{2\pi} \cdot \theta\right)^\circ$$

Example 15.2. Find the value of 1.5 rad in degrees.

Solution. $1.5\,\mathrm{rad} = \left(\dfrac{360}{2\pi} \cdot 1.5\right)^{\circ} = \left(\dfrac{270}{\pi}\right)^{\circ} \approx 85.9^{\circ}$.

15.2 Arc and sector

15.2.1 Length of an arc

The **length of an arc** of a circle is given by $s = r \cdot \theta$, where r is the radius and θ is the angle subtended in radians.

Example 15.3. Determine the length s of an arc of a circle with radius $45mm$ when the angle subtended is 85°.

Solution. $s = r\theta = 45\,\mathrm{mm} \cdot \left(\dfrac{2\pi}{360} \cdot 85\right) \approx 66.8\,\mathrm{mm}$.

15.2.2 Area of a sector

The **area of a sector** of a disc is $A = \left(\dfrac{\theta}{2\pi} \cdot \pi \cdot r^2\right) = \dfrac{1}{2} \cdot r^2 \cdot \theta$, where r is the radius of θ is the angle subtended in radians.

Example 15.4. A light source spreads illumination through an angle 130° to a distance of $35m$. Determine the illuminated area A.

Solution. $A = \frac{1}{2}r^2\theta = \frac{1}{2}\left(35\,\mathrm{m}\right)^2 \cdot \left(\dfrac{2\pi}{360} \cdot 130\right) \approx 1390\,\mathrm{m}^2$.

Exercises 15 (Length of an arc and area of a sector)

1. Convert the following to radians:

   i) $65°$
   ii) $125°$
   iii) $420°$
   iv) $1120°$

2. Convert the following to degrees

   i) 0.94 rad
   ii) 1.72 rad
   iii) 7.325 rad
   iv) 12.5 rad

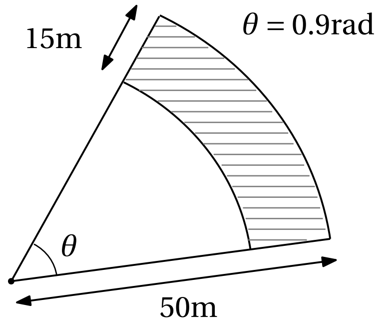
3. Calculate the length of an arc of a circle whose radius is 1.15 m when the angle subtended at the centre is $160°$.

4. An arc subtends an angle of 1.732 rad at the centre of a circle whilst the length of the arc is 258 mm. Determine the circle's diameter.

5. In a belt drive system, 300 mm are in contact with a pulley of 400 mm diameter. Determine the angle of lap in both degrees and radians.

6. Calculate the diameter and circumference of a circle if the area of a sector which subtends an angle of 1.45 rad is $620\,\text{mm}^2$.

7. Calculate the area of the shaded potion of the sketch given below and its percentage compared to the complete sector.



8. The exhaust ports of an engine consist of 4 ring sectors (i.e. each part has a shape like the shaded area in the above picture) with outer radii 50 mm, inner radii 22 mm and

angle 30°. Determine the total area of the ports.

Solutions: **1.** (i) 1.13; (ii) 2.18; (iii) 7.33; (iv) 19.55; **2.** (i) 53.86°; (ii) 98.55°; (iii) 419.69°; (iv) 716.20°; **3.** $3.2\,\mathrm{m}$; **4.** $d = 298\,\mathrm{mm}$; **5.** $1.5\,\mathrm{rad}$, 85.94°; **6.** $58.5\,\mathrm{mm}$, $183.7\,\mathrm{mm}$; **7.** $573.75\,\mathrm{m}^2$, 51% **8.** $2100\,\mathrm{mm}^2$;

Chapter 16

Trigonometric functions

16.1 From right triangles

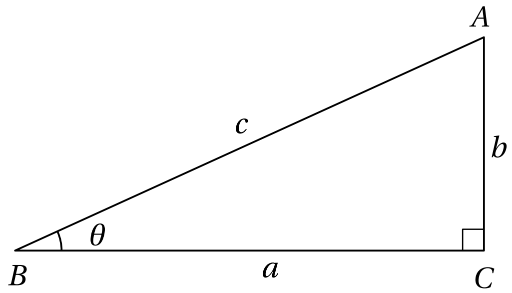
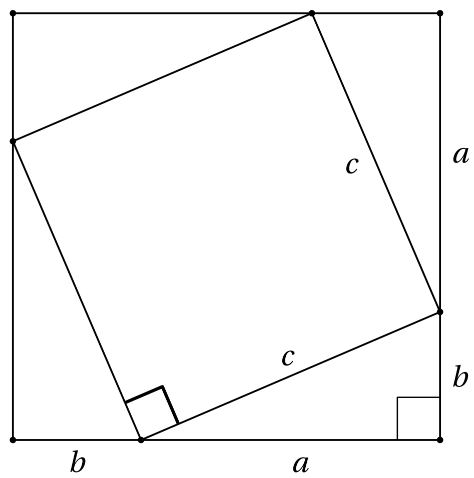


Figure 16.1: A right triangle

Theorem 16.1 (Pythagoras's Theorem). $a^2 + b^2 = c^2$

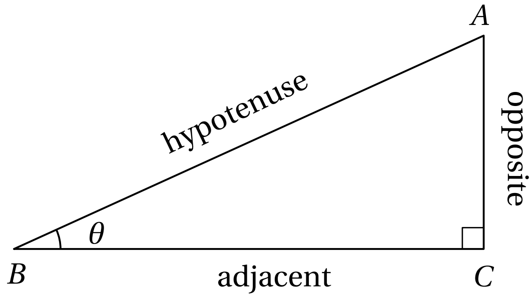
Proof (non-examinable):



$$\begin{aligned} & (a+b)^2 = \text{area of the big square} = 4\left(\frac{1}{2}ab\right) + c^2 \\ \implies \quad & a^2 + 2ab + b^2 = 2ab + c^2 \\ \implies \quad & a^2 + b^2 = c^2 \end{aligned}$$

Returning to the figure of a right triangle above:

- AB is called the **hypotenuse**.
- AC is the side **opposite to** θ
- BC is the side **adjacent to** θ



The *definitions* of the values of the trigonometric functions for θ between 0 and $\frac{\pi}{2}$:

- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{AC}{AB}$ (**sine of** θ)
- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{BC}{AB}$ (**cosine of** θ)
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{AC}{BC}$ (**tangent of** θ)
- $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{AB}{BC}$ (**secant of** θ)
- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{AB}{AC}$ (**cosecant of** θ)
- $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{BC}{AC}$ (**cotangent of** θ)

Theorem 16.2 (Pythagoras's identity). $\sin^2\theta + \cos^2\theta = 1$

Proof: $BC^2 + AC^2 = AB^2$ (by Pythagoras's Theorem)

$\implies \left(\dfrac{BC}{AB}\right)^2 + \left(\dfrac{AC}{AB}\right)^2 = 1$ (dividing by AB^2)

$\implies \cos^2\theta + \sin^2\theta = 1$ (by the definitions above)

Example 16.1. Prove that $\dfrac{1+\tan^2\theta}{1+\cot^2\theta} = \tan^2\theta$.

Solution.

$$\begin{aligned}\text{LHS} &= \frac{1+\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\cos^2\theta}{\sin^2\theta}}\\ &= \frac{\sin^2\theta\,(\cos^2\theta+\sin^2\theta)}{\cos^2\theta\,(\sin^2\theta+\cos^2\theta)}\\ &= \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta = \text{RHS}\end{aligned}$$

Example 16.2. Prove that $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$.

Solution.

$$\begin{aligned}
\text{RHS} &= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta}.\\
\text{LHS} &= \sqrt{\frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}}\\
&= \frac{\sqrt{(1-\cos\theta)^2}}{\sqrt{1-\cos^2\theta}} = \frac{1-\cos\theta}{\sqrt{\sin^2\theta}}\\
&= \frac{1-\cos\theta}{\sin\theta} = \text{LHS}.
\end{aligned}$$

A table of very commonly used values:

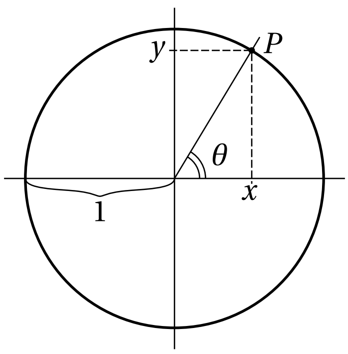
θ :—————	0 :—:	$\frac{\pi}{6}$:———————:	$\frac{\pi}{4}$:———————:	$\frac{\pi}{3}$:———————:	$\frac{\pi}{2}$:——————:
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

How to derive some of these?

- $\theta = \frac{\pi}{4}$ implies $AC = BC$, so $\sin\theta = \cos\theta$.
  So from Pythagoras' Identity "$\sin^2\theta + \cos^2\theta = 1$", we get
  $2\sin^2\theta = 1 \implies \sin\theta = 1/\sqrt{2}$.
- $\theta = \frac{\pi}{6}$: by doubling the triangle we get an equilateral one, thus
  $AC = \frac{1}{2}AB$. Hence $\sin\theta = \frac{AC}{AB} = \frac{1}{2}$, and also
  $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

16.2 As functions

Given an angle θ, let $P = (x, y)$ be the intersection point of the unit circle with the half-line through the origin making an angle θ with the positive half of the x-axis.



Then, by the definition of trig functions above:

$$\begin{aligned}
x &= \cos\theta,\\
y &= \sin\theta.
\end{aligned}$$

We can take *this* as the *definition* of sin and cos (and consequently of tan, ...) if θ is arbitrary (e.g. negative, or larger than $\frac{\pi}{2}$).

With this, we get **basic trigonometric identities**:

$$\begin{aligned} \cos(-\theta) &= \cos\theta, & \sin(-\theta) &= -\sin\theta \\ \cos(\theta+2\pi) &= \cos\theta & \sin(\theta+2\pi) &= \sin\theta \\ \cos(\theta+\pi) &= -\cos\theta & \sin(\theta+\pi) &= -\sin\theta \\ \tan(\theta+\pi) &= \tan\theta & \tan(-\theta) &= -\tan\theta \end{aligned}$$

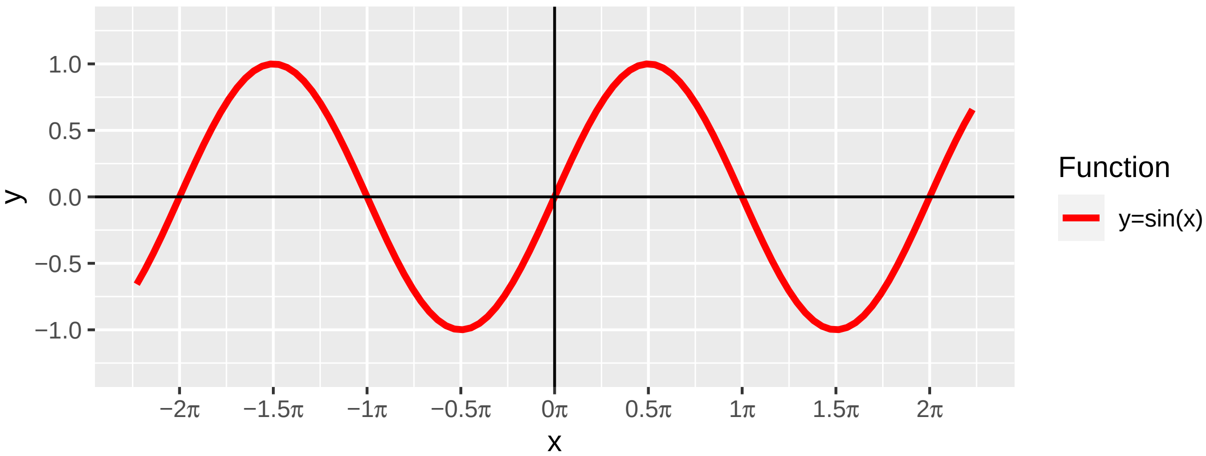


Figure 16.2: Graph of sine

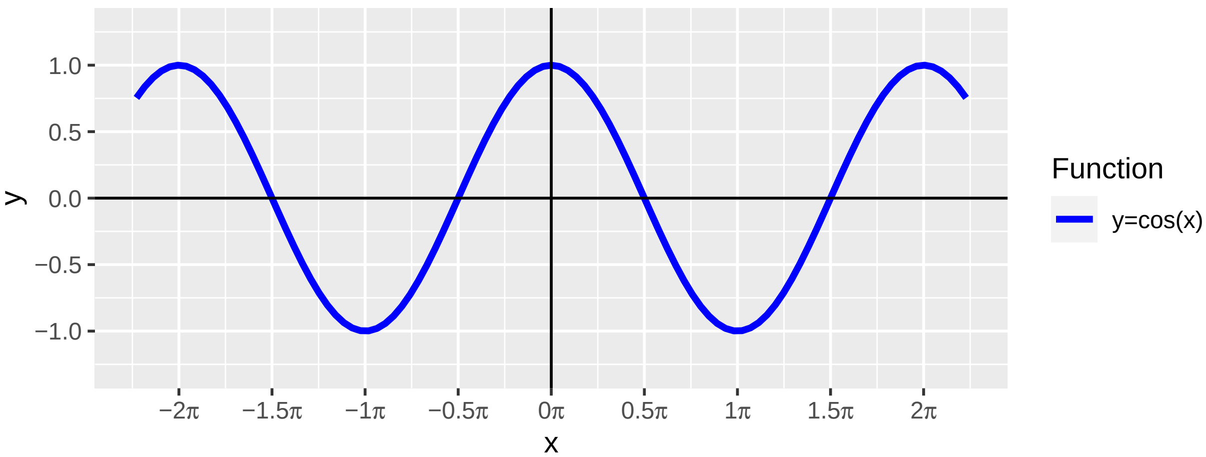


Figure 16.3: Graph of cosine

16.3 Inverse trigonometric functions

Given $-1 \leq x \leq 1$, we let $\arcsin x = \theta$, such that $\sin\theta = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Given $-1 \leq x \leq 1$, we let $\arccos x = \theta$, such that $\cos\theta = x$ and $0 \leq \theta \leq \pi$.

Given any x, we let $\arctan x = \theta$, such that $\tan\theta = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

- $\arcsin\frac{1}{2} = \frac{\pi}{6}$

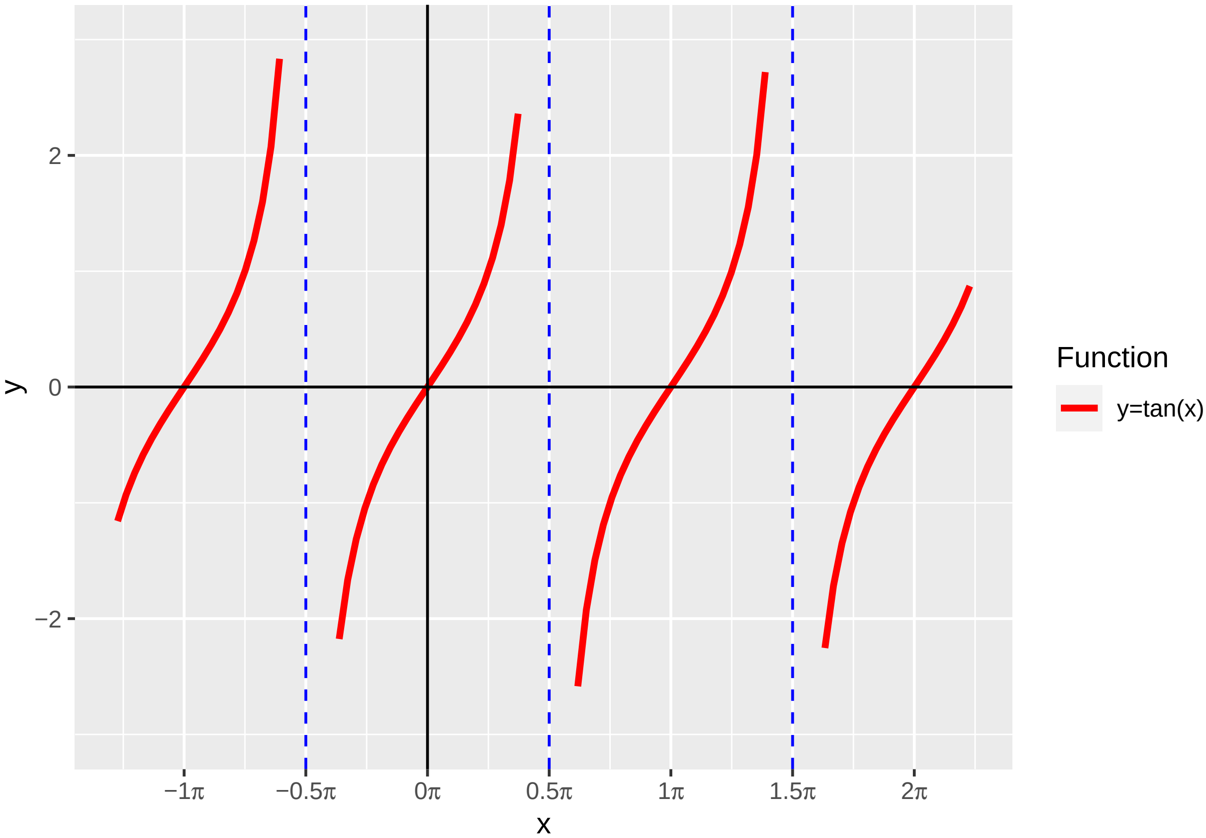


Figure 16.4: Graph of tangent

- $\arctan 1 = \frac{\pi}{4}$
- $\arcsin -\frac{1}{2} = -\frac{\pi}{6}$
- $\arccos \frac{1}{2} = \frac{\pi}{3}$
- $\arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4} = \arccos \frac{1}{\sqrt{2}}$
- $\arccos -\frac{1}{2} = \frac{2\pi}{3}$

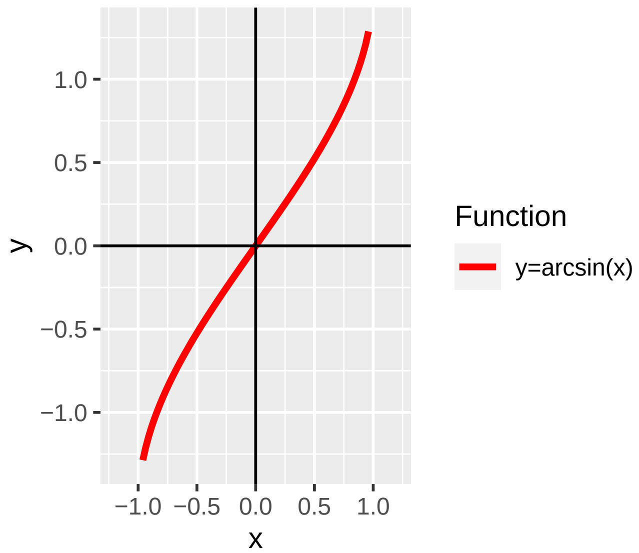


Figure 16.5: Graph of arcsin

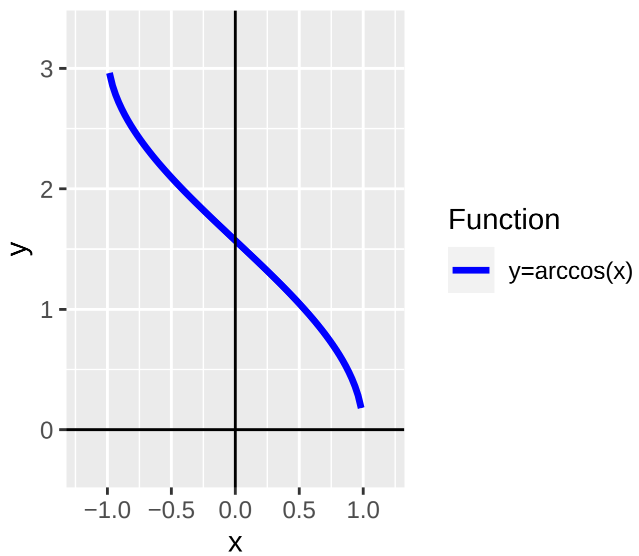


Figure 16.6: Graph of arccos

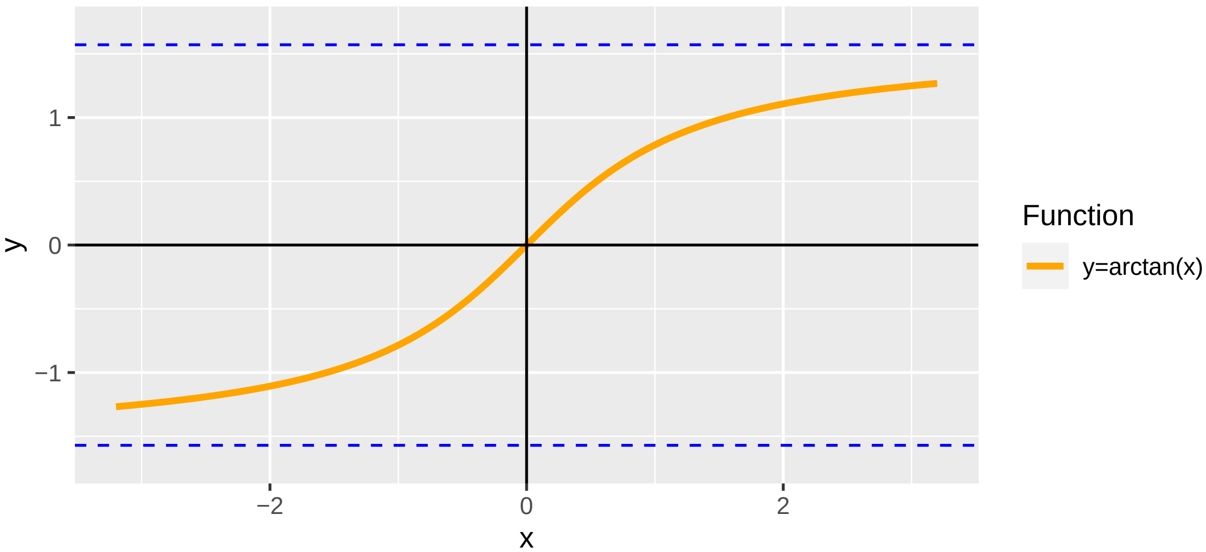


Figure 16.7: Graph of arctan

Exercises 16 (Trigonometric functions and equations)

1. Determine all the angles between 0° and 360° whose
   i) sine is -0.4848
   ii) cosine is 0.8361
   iii) tangent is -1.832
   iv) secant is 1.392
   v) cotangent is 0.6848
2. Evaluate to 4 significant figures:
$$\sec 286.08^\circ – 3.26 \operatorname{cosec} 146.72^\circ + 9 \cot 312.25^\circ.$$
3. Evaluate the following:
   i) $\sin \frac{3\pi}{8}$
   ii) $\cos \frac{5\pi}{9}$
   iii) $\tan \frac{5\pi}{16}$
   iv) $\sec \frac{7\pi}{12}$
   v) $\operatorname{cosec} 4.72\pi$
   vi) $\cot \frac{4\pi}{9}$
4. Given that $A = 32.9^\circ$ and $B = 63.48^\circ$, determine to four significant figures:
   i) $2 \sec A \cot B$
   ii) $\dfrac{\operatorname{cosec} A + \sec B}{1 - \tan A \cos B}$
   iii) $\dfrac{5 \cot B}{4 \sin A \operatorname{cosec} B}$
5. Prove the following:
   i) $\tan^2 \theta \, (\operatorname{cosec}^2\theta - 1) = 1$
   ii) $\tan \theta = \sqrt{\dfrac{1 - \cos^2 \theta}{\cos^2 \theta}}$
   iii) $\dfrac{\operatorname{cosec} \theta}{\sec \theta} - \dfrac{\sec \theta}{\operatorname{cosec} \theta} = (\cos^2 \theta - \sin^2 \theta) \sec \theta \operatorname{cosec} \theta$
   iv) $\sin \theta - \sin^3 \theta = \dfrac{\sin \theta}{\sec^2 \theta}$
6. Solve the equation $8 \sin^2 \theta + 2 \cos \theta = 5$, stating all the values of θ between 0° and 360°.

7. Solve the following equations for all values of x from 0° to 360°:

   a) $\sin x = 0.3$
   b) $\cos x = -0.7$
   c) $\tan x = -0.75$
   d) $2\sin x = 3\cos x$
   e) $4\sin x\cos x = 3\cos x$
   f) $4\cos^2 x + \cos x = 0$
   g) $2\sin^2 x - \sin x - 1 = 0$
   h) $\sin x - 2\cos^2 x + 1 = 0$

8. Solve the following equations for all values of x from -180° to 180°:

   a) $\cos^2 x = 0.75$
   b) $\sin 2x = 2\cos 2x$
   c) $3\sin^2 x = 2\sin x\cos x$
   d) $2\cos^2 x - 5\cos x + 2 = 0$
   e) $\sin^2 x + \cos x + 1 = 0$
   f) $\sin^2 x + 5\cos^2 x = 3$

Solutions: **1.** (i) 209°, 331°; (ii) 33.3°, 326.7°; (iii) 118.6°, 298.6°; (iv) 44.1°, 315.9°; (v) 55.6°, $235, 6^\circ$; **2.** -10.51; **3.** (i) 0.92; (ii) -0.17; (iii) 1.497; (iv) -3.86 (v) 1.298; (vi) 0.18; **4.** (i) 1.189; (ii) 5.738; (iii) 1.027; **6.** 41.9°, 318.6°, or 120°, 240°; **7.** (a) 17.5°, 162.5°; (b) 134.4°, 225.6°; (c) 143.1°, 323.1°; (d) 56.3°, 236.3°; (e) 48.6°, 90°, 131.4°, 270°; (f) 90°, 104.5°, 255.5°, 270°; (g) 90°, 210°, 330°; (h) 30°, 150°, 270°; **8.** (a) $\pm 30^\circ$, $\pm 150^\circ$; (b) -148.3°, -58.3°, 31.7°, 121.7°; (c) 0°, 33.7°, -146.3°, $\pm 180^\circ$; (d) $\pm 60^\circ$; (e) $\pm 180^\circ$; (f) $\pm 45^\circ$, $\pm 135^\circ$

Chapter 17

Parametric representation

Sometimes the variables x and y are themselves functions of another variable, called **parameter**, say $x = x(t)$ and $y = y(t)$. We may be able to express y in terms of x by eliminating t.

Example 17.1. $x = a\cos\theta,\ \ y = a\sin\theta$, where a is a positive constant and θ is the parameter.

We aim to eliminate θ and express y in terms of x.
We square both equations and add them: $x^2 = a^2\sin^2\theta,\ \ y^2 = a^2\cos^2\theta$
$\implies\ x^2 + y^2 = a^2(\cos^2\theta + \sin^2\theta) = a^2$
$\implies\ y = \pm\sqrt{a^2 - x^2}$ (thus in this case, there are two “branches” of the function: one with the “+” and one with the “-”; see Figure 17.1).

Example 17.2. A curve is given by $x = t^2$ and $y = 2t$. Find its Cartesian equation.

Solution. Calculate

1. $y = 2t \implies t^2 = \frac{y^2}{4}$
2. $x = t^2$

Putting 1. and 2. together, we get $x = \frac{y^2}{4} \iff y = \pm 2\sqrt{x}$. See Figure 17.2.

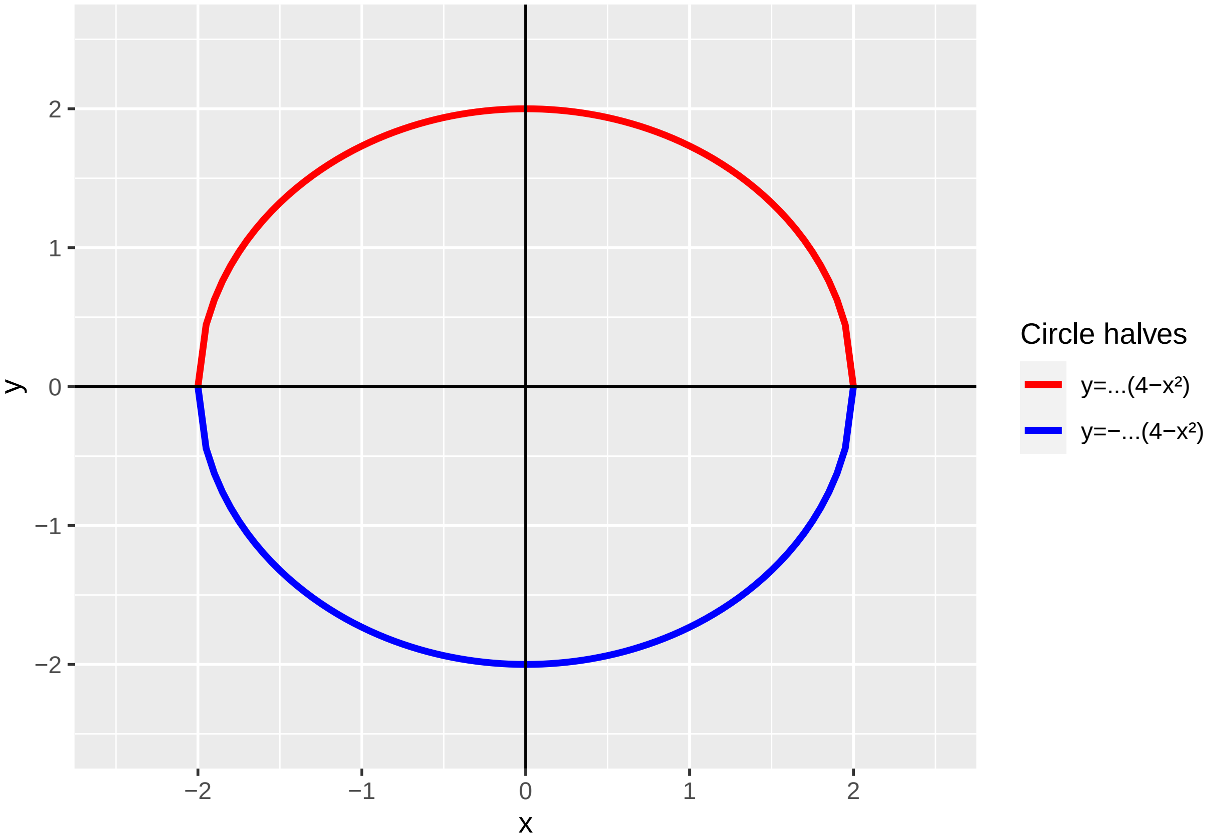


Figure 17.1: Graph of ±√(4-x²)

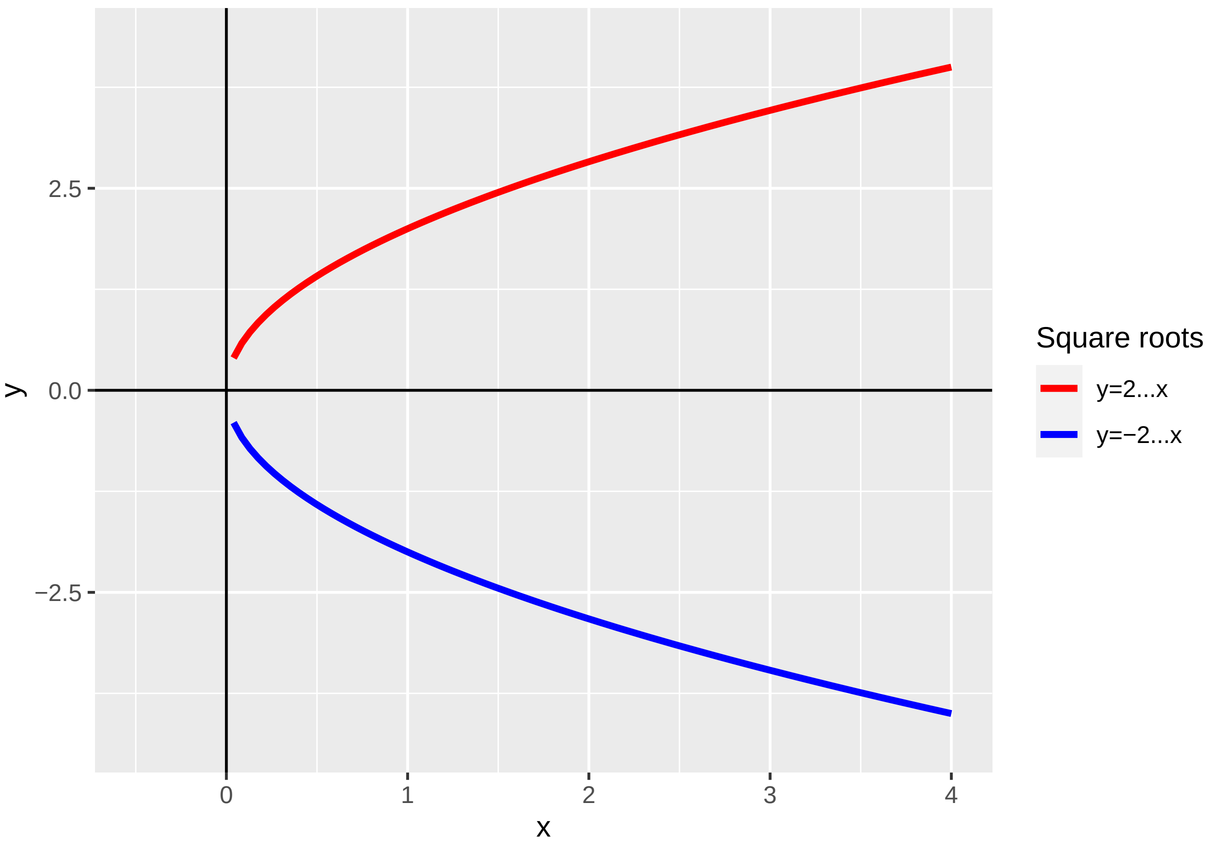


Figure 17.2: Graph of ±2√x

Exercises 17 (Parametric representation)

1. Plot the functions given below in parametric form for the range 0 to 2π rad:

   i) $x = 10\cos\theta$, $y = 10\sin\theta$
   ii) $x = 10\cos\theta$, $y = 5\sin\theta$

2. Let t vary from -4 to $+4$ for $x = 1 - t$, $y = t$ and plot the function. Also eliminate the variable t to express x in terms of y.

3. For the equations below, stated in parametric form, eliminate the parameter t to obtain the Cartesian form. Assume a and c are constants.

   i) $x = at^2, \quad y = 2at$
   ii) $x = ct, \quad y = \frac{c}{t}$
   iii) $x = 2t + 1, \quad y = 2t(t - 1)$

Solutions: **2.** $y = 1 - x$; **3.** (i) $y^2 = 4ax$; (ii) $y = \frac{c^2}{x}$; (iii) $y = \frac{x^2-4x+3}{2}$

Chapter 18

Polar coordinates

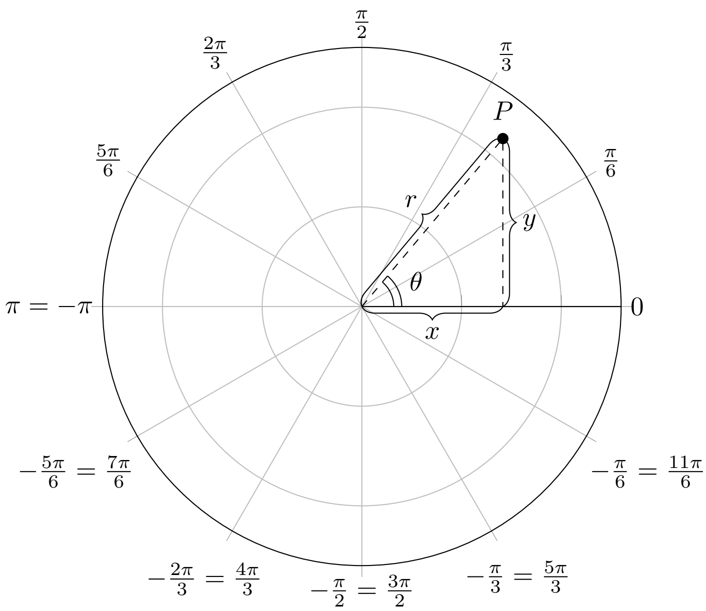


Figure 18.1: Polar coordinates chart

Given a point P in the plane as in Figure 18.1:

- r and θ are called the **polar coordinates** of P, also written as $r\angle\theta$ or $[r\angle\theta]$.
- The **principal angle convention** is that $-\pi < \theta \leq \pi$. This makes θ uniquely determined by P.
- The convention $0 \leq \theta < 2\pi$ is often used as well.

18.1 Converting polar to Cartesian coordinates

Given a point $P = [r\angle\theta]$ as in Figure 18.2, its Cartesian coordinates are given by the following formulas:

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta}$$

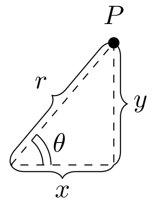


Figure 18.2: Conversion triangle

Example 18.1. $[2\sqrt{2}\angle\frac{\pi}{4}] = (2\sqrt{2}\cdot\frac{1}{\sqrt{2}}, 2\sqrt{2}\cdot\frac{1}{\sqrt{2}}) = (2,2)$.

Example 18.2. $[6.8\angle 55^\circ] = (6.8\cos 55^\circ, 6.8\sin 55^\circ) \approx (3.9, 5.6)$.

18.2 Converting Cartesian to polar coordinates

Given a point $P = (x,y)$ as in Figure 18.2, its polar coordinates are given by the following formulas:

$$\boxed{r = \sqrt{x^2+y^2}}$$

$$\boxed{\theta = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}}$$

Note: The value of "arctan(…)" is a quantity that describes *size of an angle.* The formula above assumes that it is in *radians* (hence the appearance of π). Some care is required when using a calculator which can be set to work in *degrees* instead, in which case the above formulas need to be adjusted by replacing every π by 180°.

Example 18.3.

$$\begin{aligned}(-67.4, 20.31) &= \left[\sqrt{(-67.4)^2 + (20.31)^2}\angle\left(\arctan\left(\frac{20.31}{-67.4}\right) + \pi\right)\cdot\frac{180^\circ}{\pi}\right] \\ &\approx [70.39\angle 163.23^\circ]\end{aligned}$$

Example 18.4. Six holes in the plane are given, each in polar coordinates relative to the preceding one: see Table 18.1. Sketch the system and determine the coordinates of hole 6 relative to hole 1 in rectangular and polar coordinates.

Table 18.1: Relative polar coordinates of holes in the plane

Hole	Relative polar coordinates
Hole 1	$90\angle 120^\circ$
Hole 2	$40\angle 90^\circ$
Hole 3	$55\angle 60^\circ$
Hole 4	$25\angle 45^\circ$
Hole 5	$20\angle -30^\circ$
Hole 6	$150\angle -150^\circ$

Solution. For a sketch, see Figure 18.3.

Use the above formulas to convert *relative polar* coordinates to *relative Cartesian* coordinates: see Table 18.2. Cartesian coordinates of hole 6 relative to hole 1 are the sum of the rectangular coordinates of holes 2 to 6. Executing this we arrive at: $(-67.4, 20.31)$.

We convert back to polar coordinates (see Example 18.3) and obtain the polar coordinates of hole 6 relative to hole 1 as: $70.39\angle 163.23^\circ$.

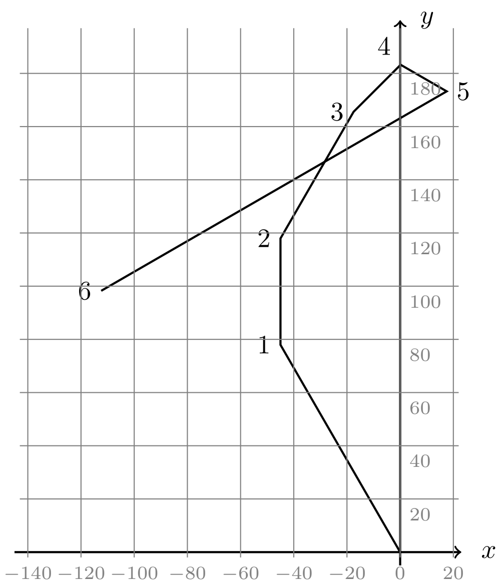
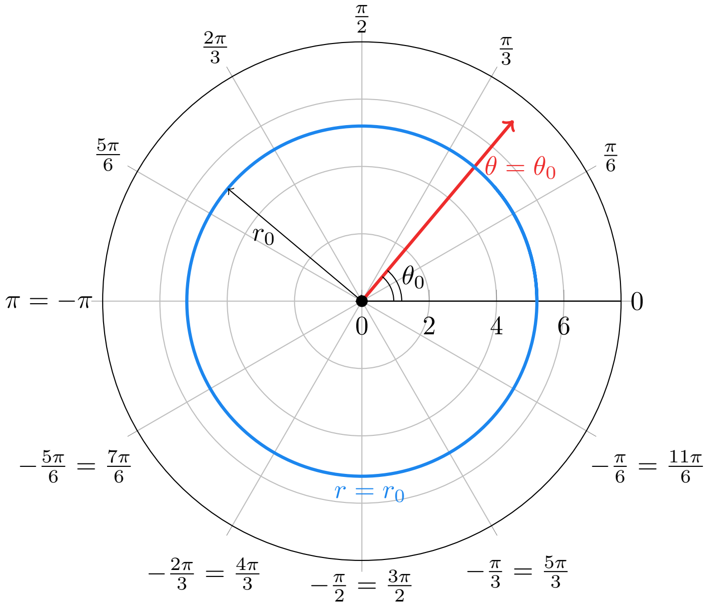


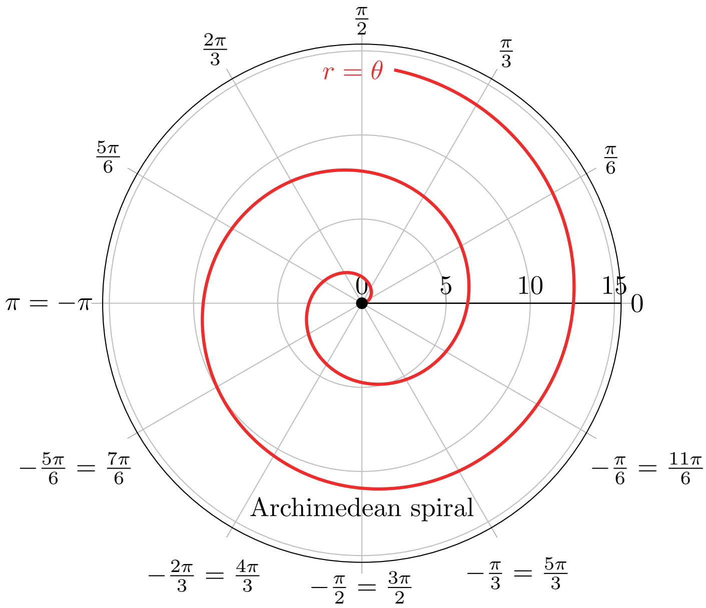
Figure 18.3: A diagram for the example of holes in the plane

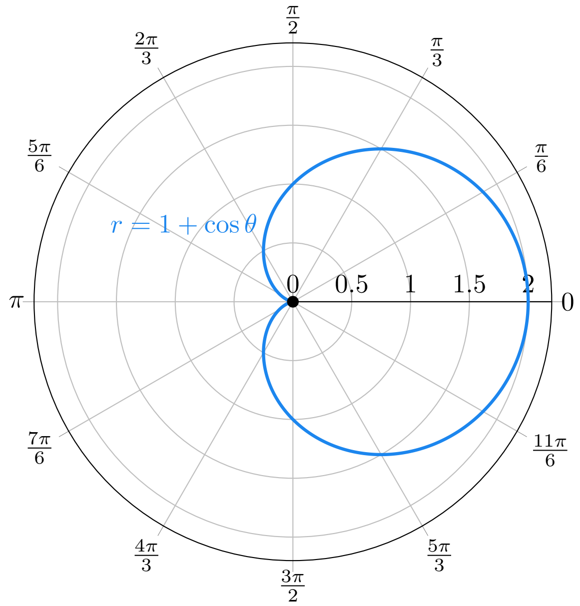
Table 18.2: Relative polar and Cartesian coordinates of holes

Hole	Relative polar coords	Relative Cartesian coords
Hole 1	$90\angle 120^\circ$	$(-45, 77.94)$
Hole 2	$40\angle 90^\circ$	$(0, 40)$
Hole 3	$55\angle 60^\circ$	$(27.5, 47.63)$
Hole 4	$25\angle 45^\circ$	$(17.68, 17.68)$
Hole 5	$20\angle -30^\circ$	$(17.32, -10)$
Hole 6	$150\angle -150^\circ$	$(-129.9, -75)$

18.3 Curves in polar coordinates






$\frac{\pi}{2}$
$\frac{2\pi}{3}$
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$\frac{5\pi}{6}$
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$r = 1 + \cos\theta$
π
0
0.5
1
1.5
2
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$\frac{7\pi}{6}$
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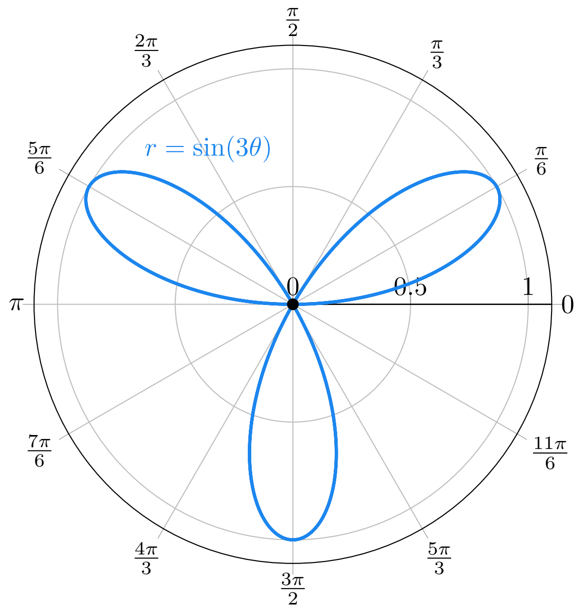
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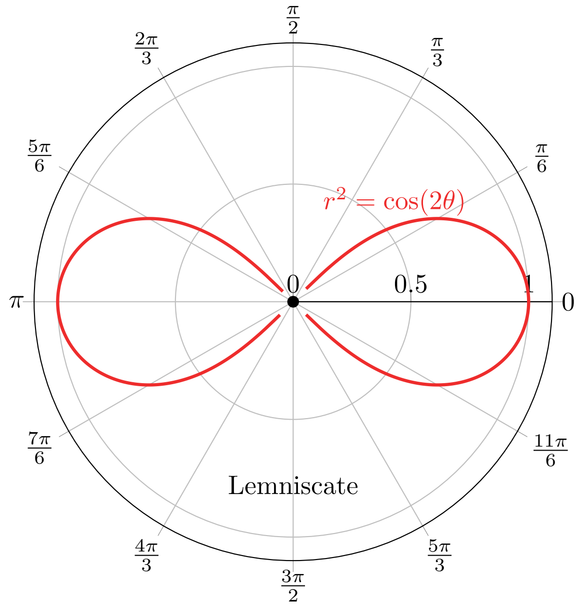
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$\frac{5\pi}{3}$
$\frac{3\pi}{2}$

Exercises 18 (Polar coordinates)

1. Express in polar form:

   i) $(5, 2)$
   ii) $(3, 7)$
   iii) $(-5, 2)$
   iv) $(-8, -9)$
   v) $(17, -12)$

2. Express in rectangular form:

   i) $5\angle 65°$
   ii) $3.8\angle 124°$
   iii) $7.2\angle -56°$
   iv) $15\angle -138°$

3. Six holes are marked out in polar coordinates as:

Hole 1	$65\angle 65°$
Hole 2	$55\angle 95°$
Hole 3	$45\angle -20°$
Hole 4	$80\angle 55°$
Hole 5	$160\angle -170°$
Hole 6	$95\angle -80°$

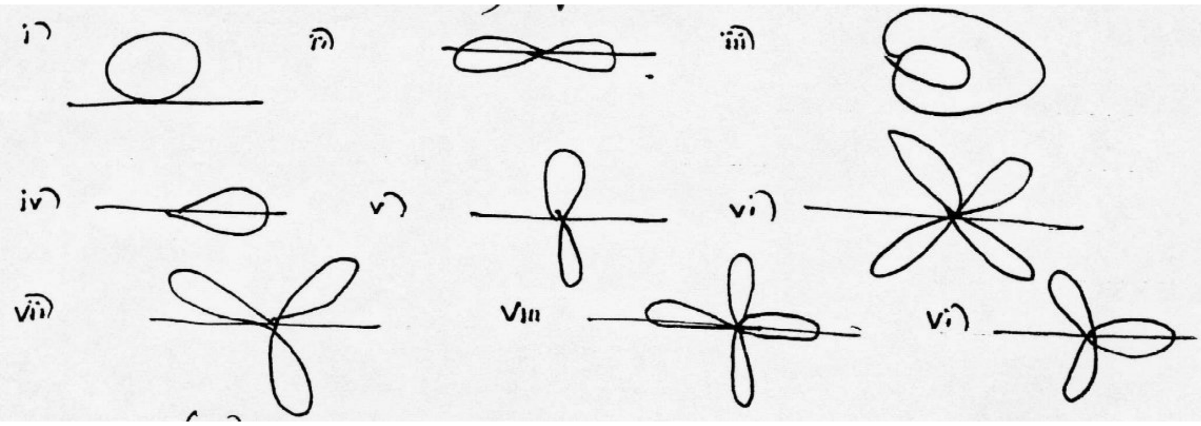
Sketch the system and determine the position of Hole 1 relative to Hole 6 in rectangular coordinates. The coordinates for each hole are given relative to the previous hole.

4. For values of θ in the range 0 to 2π, plot at intervals of $\pi/6$ radians the functions:

   i) $r = 2\sin\theta$
   ii) $r = 2\cos^2\theta$
   iii) $r = a(1 + 2\cos\theta)$
   iv) $r = a\cos\theta$
   v) $r = a\sin^2\theta$
   vi) $r = a\sin 2\theta$
   vii) $r = a\sin 3\theta$
   viii) $r = a\cos 2\theta$
   ix) $r = a\cos 3\theta$

Assume a is a constant.

Solutions: **1.** (i) $5.38\angle 21.8°$; (ii) $7.61\angle 66.8°$; (iii) $5.38\angle 158.2°$; (iv) $12.04\angle -131.6°$; (v) $20.8\angle -35.23°$; **2.** (i) $(2.1, 4.3)$; (ii) $(-2.1, 3.15)$; (iii) $(4.03, -5.97)$; (iv) $(-11.1, -10.0)$; **3.** $(57.7, 16.4)$; **4.**



Chapter 19

Sine and cosine rules

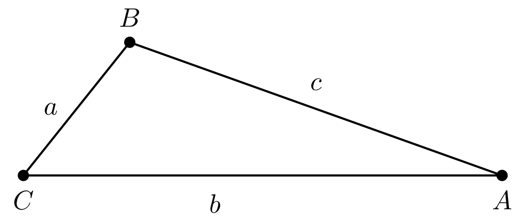


Figure 19.1: Triangle ABC

Theorem 19.1. *Given an arbitrary triangle (marked as on Figure 19.1), we have:*

1. ***Sine Rule***: $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$ *= diameter of the circumscribed circle.*

2. ***Cosine Rule***: $\boxed{a^2 + b^2 = c^2 + 2ab\cos C}$
and variants:
$b^2 + c^2 = a^2 + 2bc\cos A$
$a^2 + c^2 = b^2 + 2ac\cos B$

3. $\boxed{\text{Area} = \dfrac{1}{2}bh = \dfrac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)}}$ *where h is the* height *of the triangle (see Figure 19.2) and $s = \dfrac{a+b+c}{2}$.*
(The formula with s is called Heron's formula.*)*

Proofs (non-examinable): Start by further marking the triangle as in Figure 19.2.

1. (Sine rule) By definition of sine in the triangles CHB and HAB, we have $a\sin C = h = c\sin A$.
$\implies \dfrac{a}{\sin A} = \dfrac{c}{\sin C}$.
The other equalities can be derived similarly.

2. (Cosine rule) We begin by using Pythagoras' Theorem for the triangles CHB and HAB:

$$a^2 - x^2 = h^2 = c^2 - (b-x)^2$$
$$a^2 - x^2 = c^2 - b^2 + 2bx - x^2$$
$$a^2 + b^2 = c^2 + 2bx$$

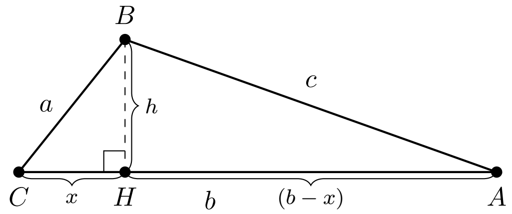


Figure 19.2: Triangle ABC with markings

From the definition of cosine for the triangle CHB, we have $x = a\cos C$, which leads to

$$a^2 + b^2 = c^2 - 2ab\cos C.$$

3. (Area of the triangle)

The area of the triangle ABC is the sum of the areas of the two right triangles CHB and HAB, each of which is a half of the area of a rectangle. Thus

$$\text{Area} = \frac{1}{2}xh + \frac{1}{2}(b-x)h = \frac{1}{2}bh.$$

Using the definition of sine for the triangle CHB, we get $h = a\sin C$, hence we get further

$$\text{Area} = \frac{1}{2}ab\sin C.$$

Deriving Heron's formula is a little more algebraically involved. Recall from above that

$$\begin{aligned} a^2 + b^2 &= c^2 + 2bx \\ \implies \qquad x &= \frac{a^2+b^2-c^2}{2b} \end{aligned}$$

Starting now with the Pythagoras's Theorem for triangle CHB, we obtain

$$\begin{aligned} h^2 = a^2 - x^2 = a^2 - &\left(\frac{a^2+b^2-c^2}{2b}\right)^2 \\ = &\frac{4a^2b^2 - (a^2+b^2-c^2)^2}{4b^2} \\ = &\frac{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)}{4b^2} && \text{NB: } X^2 - Y^2 = (X+Y)(X-Y) \\ = &\frac{((a+b)^2-c^2)\cdot(c^2-(a+b)^2)}{4b^2} \\ = &\frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4b^2} && \text{NB: same trick as above} \\ = &\frac{2s\cdot 2(s-c)\cdot 2(s-b)\cdot 2(s-a)}{4b^2} = 4\cdot\frac{s(s-a)(s-b)(s-c)}{b^2}. \end{aligned}$$

Consequently,

$$\begin{aligned} \text{Area} &= \frac{1}{2}bh \\ &= \frac{1}{2}b\sqrt{4\frac{s(s-a)(s-b)(s-c)}{b^2}} \\ &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned}$$

To find all the sides and angles of a triangle, you may want to use the following approach:

given:	use:
one side and two angles	sine rule
two sides and an angle (not between them)	sine rule
two sides and the angle between them	cosine rule
three sides	cosine rule

Example 19.1. Calculate the angles of a triangle with sides of length $a = 80\,\text{mm}$, $b = 70\,\text{mm}$, $c = 50\,\text{mm}$.

Solution. Using the cosine rule, we calculate

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{70^2 + 50^2 - 80^2}{2 \cdot 70 \cdot 50} = \frac{1}{7} \approx 0.1429\end{aligned}$$

$\implies \angle A = \arccos 0.1429 \approx 81.78°$.

Using the sine rule, we calculate

$$\begin{aligned}\sin B &= b \cdot \frac{\sin A}{a} \\ &= 70 \cdot \frac{\sin 81.78°}{80} \approx 0.866\end{aligned}$$

$\implies \angle B = \arcsin 0.866 \approx 60°$
$\implies \angle C = 180° - \angle A - \angle B = 38.22°$.

Example 19.2. Determine the area of a triangle with $a = 6\,\text{m}$, $b = 5\,\text{m}$, $c = 9\,\text{m}$.

Solution. We use Heron's formula:

$$\begin{aligned} s &= \frac{a+b+c+}{2} = \frac{6\,\text{m} + 5\,\text{m} + 9\,\text{m}}{2} = 10\,\text{m} \\ \implies \qquad \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{10\,\text{m} \cdot 4\,\text{m} \cdot 5\,\text{m} \cdot 1\,\text{m}} \\ &= \sqrt{200\,\text{m}^4} \approx 14.14\,\text{m}^2. \end{aligned}$$

Example 19.3. Determine all sides, angles and area of a triangle with $b = 10\,\text{m}$, $c = 5\,\text{m}$ and $\angle A = 120°$.

Solution. First, we use cosine rule to determine a:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 100\,\text{m}^2 + 25\,\text{m}^2 - 2 \cdot 10\,\text{m} \cdot 5\,\text{m} \cdot \cos 120° \\ &= 125\,\text{m}^2 + 50\,\text{m}^2 = 175\,\text{m}^2 \\ \implies \qquad a &= \sqrt{175\,\text{m}^2} = 5\sqrt{7}\,\text{m} \approx 13.23\,\text{m}. \end{aligned}$$

Second, we use sine rule to calculate $\angle B$:

$$
\begin{aligned}
&& \sin B &= b \cdot \frac{\sin A}{a} \\
&& &= 10\,\mathrm{m} \cdot \frac{\sin 120^\circ}{5\sqrt{7}\,\mathrm{m}} = \frac{10 \cdot \frac{\sqrt{3}}{2}}{5\sqrt{7}} = \frac{\sqrt{3}}{\sqrt{7}} \approx 0.6547 \\
\implies && \angle B &= \arcsin 0.6547 \approx 40.9^\circ.
\end{aligned}
$$

Third, we calculate

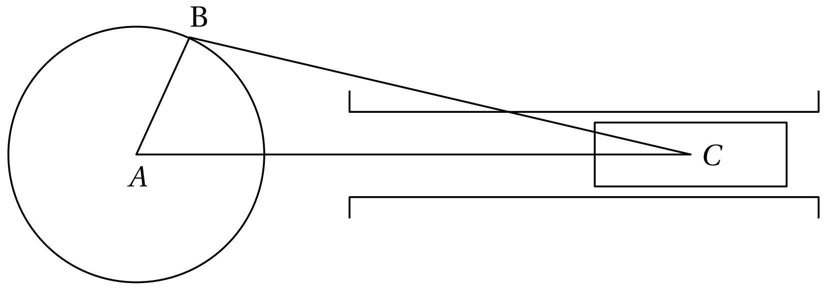
$$\angle C = 180^\circ - \angle A - \angle B = 180^\circ - 120^\circ - 40.9^\circ = 19.1^\circ.$$

Finally, we can use (for example) Heron's formula to obtain the area:

$$
\begin{aligned}
&& s &= \frac{a+b+c}{2} = \frac{10+5+5\sqrt{7}}{2}\,\mathrm{m} \approx 14.12\,\mathrm{m} \\
\implies && \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\
&& &= \sqrt{14.12 \cdot 0.89 \cdot 4.12 \cdot 9.12}\,\mathrm{m}^2 \approx 21.73\,\mathrm{m}^2.
\end{aligned}
$$

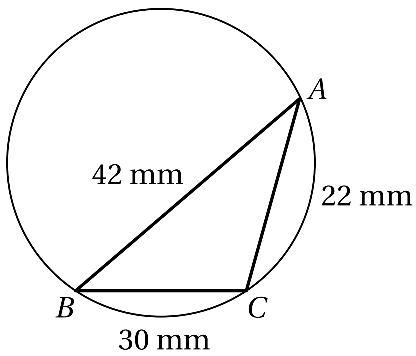
Exercises 19 (Sine and cosine rules)

1. For the simple crank mechanism ABC, given that $AB = 60\,\text{mm}$ and $BC = 170\,\text{mm}$, calculate the distance AC when the angle at B is $140°$.

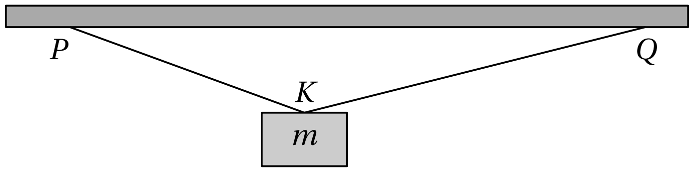


2. A scalene triangle ABC has the lengths of the sides set out as $AB = 65\,\text{mm}$, $AC = 95\,\text{mm}$ and $BC = 80\,\text{mm}$. Determine to the nearest degree the value of each angle.

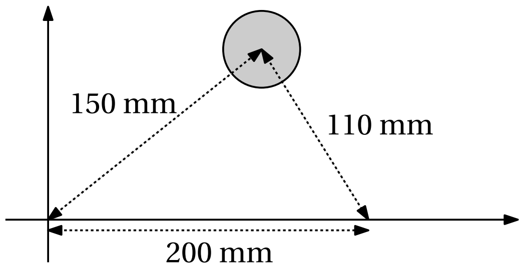
3. Three holes are marked on a pitch circle with chordal distances as shown below. Determine the values for the angles BAC and ABC and the pitch circle diameter.



4. A mass m is suspended by two wires from a horizontal beam. If $PK = 3.3\,\text{m}$, $KQ = 4.7\,\text{m}$ and $PQ = 5\,\text{m}$, determine the angle PQK and the vertical distance of K below the beam.



5. Calculate the coordinates x and y of the centre of the hole relative to the two axes shown.



Solutions: **1.** $219.4\,\mathrm{mm}$; **2.** $56.3°$, $42.6°$, $81.1°$; **3.** $43.16°$, $30.1°$, $43.87\,\mathrm{mm}$; **4.** $39.6°$, $3.00\,\mathrm{m}$; **5.** $(126, 81.4)$

Chapter 20

Compound and multiple angles

Theorem 20.1 (Compound angle formulas for sine and cosine)**.**

$$\boxed{\begin{array}{rcl} \sin(\alpha+\beta) & = & \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) & = & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{array}}$$

Proof of Theorem 20.1 (non-examinable):

First, we will arrange points in the plane to arrive at Figure 20.1.

- Define E by $|AE| = 1$.
- Define C so that $\angle ACE = 90°$.
- Define B so that $\angle ABC = 90°$.
- Form rectangle $ABDF$, so that E lies on DF.

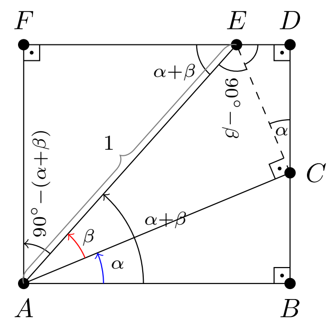


Figure 20.1: Diagram for the proof of compound angle formulas

Then:

$$|CE| = \sin\beta, \qquad |AC| = \cos\beta \qquad (\triangle ACE)$$
$$|EF| = \cos(\alpha+\beta), \qquad |AF| = \sin(\alpha+\beta) \qquad (\triangle AEF)$$

This implies that

$$|AB| = \cos\alpha\cos\beta, \qquad |BC| = \sin\alpha\cos\beta \qquad (\triangle ABC)$$
$$|CD| = \cos\alpha\sin\beta, \qquad |DE| = \sin\alpha\sin\beta \qquad (\triangle CDE)$$

Hence

$$\begin{aligned}\sin(\alpha+\beta) &= |AF| = |BD| = |BC| + |CD| = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= |EF| = |DF| - |DE| = \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

This finishes the proof.

Corollary 20.1 (Compound angle formula for tangent).

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

Proof of Corollary 20.1 using Theorem 20.1:

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} && \text{(by Theorem)} \\ &= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.\end{aligned}$$

(For the penultimate step, we divide both top and bottom by $\cos\alpha\cos\beta$.)

Corollary 20.2 (Double angle formulas).

$$\boxed{\begin{array}{rcl}\sin(2\alpha) &=& 2\sin\alpha\cos\alpha \\ \cos(2\alpha) &=& 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \\ \tan(2\alpha) &=& \dfrac{2\tan\alpha}{1-\tan^2\alpha}\end{array}}$$

Proof of double angle cosine formula (Corollary 20.2):

$$\begin{aligned}\cos(2\alpha) &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha && \text{(by Theorem)} \\ &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.\end{aligned}$$

(For the last step, we use Pythagoras's formula: $\sin^2\alpha + \cos^2\alpha = 1$.)

Example 20.1. Express $\cos(3\alpha)$ in terms of $\cos\alpha$.

Solution.

$$\begin{aligned}\cos(3\alpha) &= \cos(2\alpha+\alpha) \\ &= \cos(2\alpha)\cos\alpha - \sin(2\alpha)\sin\alpha \\ &= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha \\ &= 2\cos^3\alpha - \cos\alpha - 2(1-\cos^2\alpha)\cos\alpha \\ &= 4\cos^3\alpha - 3\cos\alpha.\end{aligned}$$

Example 20.2. Express $\tan(3\alpha)$ in terms of $\tan\alpha$.

Solution.

$$\begin{aligned}\tan(3\alpha) &= \tan(2\alpha+\alpha)\\ &= \frac{\tan(2\alpha)+\tan\alpha}{1-\tan(2\alpha)\tan\alpha}\\ &= \frac{\frac{2\tan\alpha}{1-\tan^2\alpha}+\tan\alpha}{1-\frac{2\tan\alpha}{1-\tan^2\alpha}\cdot\tan\alpha}\\ &= \frac{2\tan\alpha+\tan\alpha(1-\tan^2\alpha)}{1-\tan^2\alpha-2\tan\alpha\cdot\tan\alpha}\\ &= \frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}.\end{aligned}$$

Corollary 20.3 (Changing sums of (co)sines into products)**.**

$$\boxed{\begin{array}{rcl}\sin\alpha+\sin\beta &=& 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\\ \sin\alpha-\sin\beta &=& 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\\ \cos\alpha+\cos\beta &=& 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\\ \cos\alpha-\cos\beta &=& -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\end{array}}$$

Proof of Corollary 20.3 (non-examinable):

Note that $\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}=\alpha$ and $\frac{\alpha+\beta}{2}-\frac{\alpha+\beta}{2}=\beta$. Then

$$\begin{aligned}\sin\alpha+\sin\beta &= \sin\left(\tfrac{\alpha+\beta}{2}+\tfrac{\alpha-\beta}{2}\right)+\sin\left(\tfrac{\alpha+\beta}{2}-\tfrac{\alpha-\beta}{2}\right)\\ &= \sin\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}+\cos\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}\\ &\quad+\sin\tfrac{\alpha+\beta}{2}\cos\left(-\tfrac{\alpha-\beta}{2}\right)+\cos\tfrac{\alpha+\beta}{2}\sin\left(-\tfrac{\alpha-\beta}{2}\right)\\ &= 2\sin\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2},\end{aligned}$$

since $\cos(-x)=\cos(x)$ and $\sin(-x)=-\sin(x)$. The rest are similar.

Example 20.3. Prove that $\dfrac{\sin(7\theta)+\sin(5\theta)}{\cos(7\theta)-\cos(5\theta)}=-\cot\theta$.

Solution.

$$\begin{aligned}\text{LHS} &= \frac{2\sin\frac{7\theta+5\theta}{2}\cos\frac{7\theta-5\theta}{2}}{-2\sin\frac{7\theta+5\theta}{2}\sin\frac{7\theta-5\theta}{2}}\\ &= \frac{\sin 6\theta\cos\theta}{-\sin 6\theta\sin\theta}=\text{RHS}.\end{aligned}$$

Example 20.4. Express $\sin 50^\circ+\sin 30^\circ$ as a product of (co)sines.

Solution. $\sin 50^\circ+\sin 30^\circ = 2\sin\dfrac{50^\circ+30^\circ}{2}\cos\dfrac{50^\circ-30^\circ}{2}=2\sin 40^\circ\cos 10^\circ$.

Example 20.5. Without a calculator, evaluate $16\cdot\sin 75^\circ\cdot\cos 15^\circ$.

Solution. First, we want to find α and β so that

$$75^\circ = \frac{\alpha+\beta}{2}, \qquad 15^\circ = \frac{\alpha-\beta}{2}.$$

Solving the equations, we get $\alpha = 90^\circ$ and $\beta = 60^\circ$. Hence

$$\begin{aligned} 16\sin 75^\circ \cos 15^\circ &= 16 \sin\frac{90^\circ + 60^\circ}{2} \cos\frac{90^\circ - 60^\circ}{2} \\ &= 8(\sin 90^\circ + \sin 60^\circ) \\ &= 8\left(1 + \frac{\sqrt{3}}{2}\right) = 8 + 4\sqrt{3}. \end{aligned}$$

Example 20.6. Prove that $\dfrac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$.

Solution.

$$\begin{aligned} \text{LHS} &= \frac{2\sin\alpha\cos\alpha}{1 + 2\cos^2\alpha - 1} \\ &= \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha} = \frac{\sin\alpha}{\cos\alpha} = \text{RHS}. \end{aligned}$$

Exercises 20 (Compound and multiple angles)

1. Without using a calculator find:
   i) $\sin 15^\circ$
   ii) $\cos 15^\circ$
   iii) $\tan 15^\circ$
   iv) $\sin 105^\circ$
   v) $\cos 105^\circ$
2. If $\cos A = 0.6$ and $\cos B = 0.8$, determine without using a calculator the values of $\sin(A+B)$ and $\cos(A+B)$.
3. Given that $\cos A = 0.2$ and $\cos B = 0.5$, find without using a calculator, the values of $\sin(A+B)$ and $\cos(A-B)$.
4. Prove that $\sin(\theta+45^\circ) = \frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$.
5. If $\tan(A+B) = 0.75$ and $\tan A = 2$, determine B.
6. Show that $\sin(\theta+90^\circ)+\cos(\theta-180^\circ) = 0$.
7. Determine the acute angle θ, which satisfies the equation $3\cos(\theta-14^\circ) = 4\sin\theta$.
8. Prove that:
   i) $\dfrac{\cos 2\theta}{\cos\theta+\sin\theta} = \cos\theta-\sin\theta$
   ii) $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$
9. Show that $\sin 3A = 3\sin A - 4\sin^3 A$.
10. Express as a sum or difference:
    i) $2\cos 5\theta\cos 3\theta$
    ii) $\sin 25^\circ\cos 65^\circ$
    iii) $\cos 3x\cos x$
    iv) $\sin 5x\cos x$
    v) $2\sin(x+y)\sin(x-y)$
11. Show that $\sin(\theta+45^\circ)\sin(45^\circ-\theta) = -\frac{1}{2}\cos 2\theta$.
12. Prove that $\dfrac{\sin A+\sin 3A+\sin 5A}{\cos A+\cos 3A+\cos 5A} = \tan 3A$.

13. Prove that $\dfrac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A$.

14. (* difficult) Show that in any triangle $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$.

Solutions: **1.** (i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$; (ii) $\frac{\sqrt{3}+1}{2\sqrt{2}}$; (iii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$; (iv) $\frac{\sqrt{3}+1}{2\sqrt{2}}$; (v) $\frac{1-\sqrt{3}}{2\sqrt{2}}$; **2.** $\sin(A+B) = 1, \cos(A+B) = 0$; **3.** $\sin(A+B) = 0.6631$, $\cos(A-B) = 0.7485$; **5.** $B = -26.6°$; **7.** $\theta = 41.6°$; **10.** (i) $\cos 8\theta + \cos 2\theta$; (ii) $\frac{1}{2}(1 - \sin 40°)$; (iii) $\frac{1}{2}(\cos 4x + \cos 2x)$; (iv) $\frac{1}{2}(\sin 6x + \sin 4x)$; (v) $\cos 2y - \cos 2x$