OPTIMAL CONTROL THEORY AND APPLICATIONS IN INFECTIOUS DISEASE MODELING

by

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THESIS
Presented to the Graduate Faculty of
The University of Texas at San Antonio
In Partial Fulfillment
Of the Requirements
For the Degree of

MASTERS OF SCIENCE IN MATHEMATICS

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May 2024

DEDICATION

I dedicate this thesis to my loving family who have been so supportive throughout the duration of this project.

ACKNOWLEDGEMENTS

First, I wish to express my sincere gratitude to the Lutcher Brown Fellowship for their support, and to thank Mostafa Fazly for his continuous encouragement, invaluable ideas, and guidance in the production of this thesis.

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May 2024

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We explore the fundamentals of optimal control theory and the work of Lev Pontryagin's Maximum Principle [39]. The general minimization functional is defined below

$$\min_u J(u) = \min_u \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt,$$

$$\text{subject to} \quad \dot{x}(t) = g(t, x(t), u(t)),$$

$$x(t_0) = x_0 \text{ and } x(t_1) \text{ free.}$$

where $x(t)$ denote the state of the system at time t, and $u(t)$ represent the control, where t spans the interval $[t_0, t_f]$

In our first application, We use optimal control to analyze the Susceptible-Exposed-Infected-Recovered (SEIR) epidemic model, offering a comparative analysis of minimization and maximization control strategies. This approach aims to minimize the number of infected people and the overall cost of vaccines over a fixed time period T compared to maximizing the overall healthy population $N(t)$.

Our second application focuses on two different compartmental models: the Susceptible-Vaccinated-Infected-Recovered (SVIR) and the Susceptible-Infected-Treated-Recovered (SITR). By comparing and contrasting the outcomes of vaccination and treatment as control strategies, we are able to highlight the effectiveness of each approach within the models. The comparative analysis is extended to different scenarios to simulate different stages of an epidemic.

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CHAPTER 1: INTRODUCTION TO OPTIMAL CONTROL THEORY

Optimal control theory, an extension of the calculus of variations that emerged largely from the work of Lev Pontryagin in the 1950s, employs the maximum principle to deal with finding control policies that minimize or maximize some measure of performance [10]. This involves developing a mathematical model to represent a dynamical system and then applying optimization algorithms to find the best control policy. This has many applications in economics, engineering, and biology. Specifically, I will focus on different aspects of biological systems throughout the sections. But, to start with, the systems that are typically modeled in optimal control theory can often be described using various forms of differential equations. Let $x(t)$ denote the state of the system at time t, and $u(t)$ the control. The dynamics of the system are governed by a differential equation of the form

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where t ranges over some interval $[t_0, t_f]$. The goal is to find a piecewise continuous control $u(t)$ that maximizes an objective functional $J(u)$, defined as the integral of a function $f(t, x(t), u(t))$ from t_0 to t_1. A functional refers to a map from a certain set of functions to the real numbers, which in this case will be an integral, as shown below in [eq:1.2](1.2). In general, the objective functional depends on one or more of the state and the control variables [32]. Below, the objective is to maximize

$$J(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt, \tag{1.2}$$

where $u(t)$ is the control variable, $x(t)$ is the state variable evolving according to some differential equations, and f is a function that quantifies the cost associated with the state and control variables at time t. The objective is to find a control policy $u^*(t)$ such that the cost $J(u)$ is minimized. This involves solving the Hamiltonian equation, a function that characterizes the total energy within a system. It aims to integrate the system's dynamics with the cost function to be optimized.

$$H = f(t, x(t), u(t)) + \lambda^T g(t, x(t), u(t)), \tag{1.3}$$

where term λ^T is the transpose of the adjoint vector, reflecting the system's sensitivity to state changes and $g(t, x(t), u(t))$ describes the system's dynamics under the influence of the control.

1.1 Optimal Control Preliminaries

Before jumping into deeper problems and discussions, in this section, we will go over key definitions and theorems from analysis and calculus which will be used to frame the optimal control problem and much-needed background

Definition 1.1.1. Let $I \subseteq \mathbb{R}$ be an interval (finite or infinite). We say a finite-valued function $u : I \to \mathbb{R}$ is *piecewise continuous* if it is continuous at each $t \in I$, with the possible exception of at most a finite number of t, and if u is equal to either its left or right limit at every $t \in I$.

The concept of piecewise continuous functions, as defined above, is pivotal in optimal control theory. In practical scenarios, control functions often exhibit discontinuities at certain points in time due to sudden changes in control strategy or external influences. However, ensuring that these functions are piecewise continuous, as opposed to being arbitrarily discontinuous allows for the application of various optimization techniques and theoretical analyses, ensuring that the solutions to the optimal control problems remain well-defined and manageable, especially when dealing with complex dynamical systems.

Definition 1.1.2. Let $x : I \to \mathbb{R}$ be continuous on I and differentiable at all but finitely points of I. Further, suppose that x' is continuous wherever it is defined. Then, we say x is *piecewise differentiable*.

In the optimal control problem, the concept of a function being piecewise differentiable is useful for both the state and control variables. This ensures that state variables, which represent the system's evolving conditions over time, have a well-defined rate of change at almost every point. This characteristic is crucial because optimal control problems often require analyzing how small changes in control strategies impact the system's state. Specifically, the Hamiltonian formulation and the derivation of the necessary conditions for optimality. For a control system [eq:1.1](1.1) piecewise differentiability of $x(t)$ ensures the well-definedness of the adjoint equation.

Definition 1.1.3. A function $k(t)$ is said to be concave on $[a, b]$ if

$$\alpha k(t_1) + (1-\alpha)k(t_2) \leq k(\alpha t_1 + (1-\alpha)t_2)$$

for all $0 \leq \alpha \leq 1$ and for any $a \leq t_1, t_2 \leq b$.

In the context of convex functions, a function k is considered convex over an interval $[a, b]$ if it satisfies the reverse inequality, equivalent to stating that the negation of k is concave. For functions that are twice differentiable and concave, a key characteristic is that their second derivative will not exceed zero. This concept is consistent with calculus terminology, where a function is described as 'concave down' if it is concave and 'concave up' if it is convex. When a function k is concave and differentiable, it manifests a notable property concerning tangent lines. Specifically, for any t_1, t_2 in the interval $[a, b]$, the function satisfies

$$k(t_2) - k(t_1) \geq (t_2 - t_1)k'(t_2) \tag{1.4}$$

This implies that the slope of the secant line between any two points on the function is less steep than the slope of the tangent at the left point and steeper than that at the right point.

Definition 1.1.4. A function k is called Lipschitz if there exists a constant c (particular to k) such that $|k(t_1) - k(t_2)| \leq c|t_1 - t_2|$ for all points t_1, t_2 in the domain of k. The constant c is called the Lipschitz constant of k.

The existence of a Lipschitz constant c for a function k implies a bound on how rapidly k can change, which is essential in controlling systems where sudden large variations could lead to instability or unpredictable behaviors. This property is particularly important when dealing with differential equations that describe system dynamics, as it ensures the solutions to these equations are stable and robust.

Theorem 1.1.1 (Rolle's Theorem)**.** *Let h be a function that satisfies the following three conditions:*

1. *h is continuous on the closed interval $[a, b]$,*

2. *h is differentiable on the open interval (a, b),*

3. *$h(a) = h(b)$.*

Then there exists at least one number c in the open interval (a, b) such that $f'(c) = 0$.

Theorem 1.1.2 (Mean Value Theorem)**.** *Let k be continuous on $[a, b]$ and differentiable on (a, b). Then, there is some $x_0 \in (a, b)$ such that $k(b) - k(a) = k'(x_0)(b - a)$.*

Proof. Define

$$h(x) = k(x) - k(a) - \frac{k(b) - k(a)}{b - a}(x - a) \tag{1.5}$$

Notice that h is continuous on $[a, b]$ and differentiable on (a, b), and $h(a) = h(b) = 0$. By Rolle's Theorem, there exists a $c \in (a, b)$ such that $h'(c) = 0$. Now,

$$h'(x) = k'(x) - \frac{k(b) - k(a)}{b - a} \tag{1.6}$$

Substituting c for x, we get

$$h'(c) = k'(c) - \frac{k(b) - k(a)}{b - a} = 0 \tag{1.7}$$

Therefore,

$$k'(c) = \frac{k(b) - k(a)}{b - a} \tag{1.8}$$

Hence, there exists an $x_0 = c \in (a, b)$ such that $k(b) - k(a) = k'(x_0)(b - a)$. □

This implies that for a given system's trajectory or cost function k that is continuous over a time interval and differentiable within it, there exists at least one instant where the rate of change (velocity or gradient of cost) matches the average rate of change over the entire period. This can be used to predict system behavior or optimize performance, ensuring that the control policies being implemented are efficient and effective at some point within the interval.

1.2 The Basic Problem and Necessary Conditions

Each optimal control problem is unique, implying that the control varies from one problem to another, thereby influencing the solution. This suggests that the relationship can be viewed as a mapping, denoted as $u(t) \mapsto x = x(u)$. The objective is to find a piecewise continuous control, $u(t)$, and the corresponding state, $x(t)$, to maximize:

$$\begin{aligned} &\max_{u} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \\ &\text{subject to} \quad \dot{x}(t) = g(t, x(t), u(t)), \\ &\qquad\qquad\quad x(t_0) = x_0 \text{ and } x(t_1) \text{ free.} \end{aligned} \tag{1.9}$$

To find a solution to the optimal control problem, certain conditions need to be met. These are called "necessary" conditions and are those that must be satisfied for a control to be optimal. In optimal control theory, these conditions are often derived from Pontryagin's Maximum Principle. The first necessary condition would be the transversality condition, which are boundary conditions in optimal control problems, particularly when the endpoint of the control interval is not fixed or when there are state constraints at the endpoints. These conditions are crucial for determining the optimal trajectory and the optimal control. If the endpoints in time and state are fixed, the transversality conditions can be straightforward, often leading to specific values for the adjoint variables at these endpoints. In cases where the terminal time is free, the transversality condition typically involves the Hamiltonian and states that the derivative of the Hamiltonian with respect to time must be zero at the terminal time. When the terminal state is not fixed, the transversality condition often relates to the gradient of the Hamiltonian with respect to the state variables at the terminal time. The optimality condition in optimal control theory is primarily derived from Pontryagin's

Maximum Principle. This principle provides a set of conditions that must be satisfied for a control to be considered optimal. The Hamiltonian is constructed, which incorporates the dynamics of the system, the adjoint variables (Lagrange multipliers), and the control variables. According to Pontryagin's Maximum Principle, for a control to be optimal, the Hamiltonian must be maximized (or minimized, depending on the problem formulation) with respect to the control variables at every point in time. The adjoint equations, which are part of the Hamiltonian system, must be satisfied. These equations describe how the adjoints evolve over time and are typically derived from the partial derivatives of the Hamiltonian with respect to the state variables. In addition to the transversality conditions, there are specific boundary conditions for the adjoint variables that must be satisfied, depending on the initial and terminal constraints of the problem. We can show this problem in the context of our general optimal control problem from [32].

Let us consider that an optimal control in a piecewise continuous form, denoted as $u^*(t)$ and a state $x^*(t)$. We have $J(u) \leq J(u^*) \leq \infty$ for all u. For a given piecewise continuous variation function $h(t)$ and a scalar $\epsilon \in \mathbb{R}$ which is a constant, we then define a new control:

$$u^\epsilon(t) = u^*(t) + \epsilon h(t) \tag{1.10}$$

which is also piecewise continuous. Let $x^\epsilon(t)$ be the state trajectory corresponding to the control $u^\epsilon(t)$, that is, $x^\epsilon(t)$ satisfies the differential equation

$$\frac{dx^\epsilon(t)}{dt} = g(t, x^\epsilon(t), u^\epsilon(t)) \tag{1.11}$$

where $u^\epsilon(t)$ is continuous. Given that all trajectories commence from the same initial state, we set $x^\epsilon(t_0) = x_0$. Then $u^\epsilon(t) \to u^*(t)$ as ϵ approaches zero. Moreover, for all t, the following holds true:

$$\left.\frac{\partial u^\epsilon(t)}{\partial \epsilon}\right|_{\epsilon=0} = h(t). \tag{1.12}$$

A similar argument applies to x^ϵ due to the assumptions made on g, implying that:

$$x^\epsilon(t) \to x^*(t) \tag{1.13}$$

for each fixed t. Further, the derivative

$$\left.\frac{\partial}{\partial \epsilon} x^\epsilon(t)\right|_{\epsilon=0} \tag{1.14}$$

exists for each t. The objective functional at u^ϵ is

$$J(u^\epsilon) = \int_{t_0}^{t_1} f(t, x^\epsilon(t), u^\epsilon(t))\, dt. \tag{1.15}$$

Introducing the adjoint function or variable λ. Let $\lambda(t)$ be a piecewise differentiable function on $[t_0, t_1]$ to be determined. By the Fundamental Theorem of Calculus,

$$\int_{t_0}^{t_1} \frac{d}{dt}[\lambda(t) x^\epsilon(t)]\, dt = \lambda(t_1) x^\epsilon(t_1) - \lambda(t_0) x^\epsilon(t_0), \tag{1.16}$$

which implies

$$\int_{t_0}^{t_1} \frac{d}{dt}[\lambda(t) x^\epsilon(t)]\, dt + \lambda(t_0) x_0 - \lambda(t_1) x^\epsilon(t_1) = 0. \tag{1.17}$$

Adding this 0 expression to our $J(u^\epsilon)$ gives

$$\begin{aligned} J(u^\epsilon) = &\int_{t_0}^{t_1} \left[f(t, x^\epsilon(t), u^\epsilon(t)) + \frac{d}{dt}[\lambda(t) x^\epsilon(t)] \right] dt \qquad (1.18) \\ &+ \lambda(t_0) x_0 - \lambda(t_1) x^\epsilon(t_1) \\ = &\int_{t_0}^{t_1} \left[f(t, x^\epsilon(t), u^\epsilon(t)) + \dot{\lambda}(t) x^\epsilon(t) + \lambda(t) g(t, x^\epsilon(t), u^\epsilon(t)) \right] dt \qquad (1.19) \\ &+ \lambda(t_0) x_0 - \lambda(t_1) x^\epsilon(t_1), \end{aligned}$$

where we used the product rule and the fact that $g(t, x^*, u^*) = \frac{dx^*}{dt}$ at all but finitely many points. Since the maximum of J with respect to the control u occurs at u^*, the derivative of $J(u^*)$ with respect to ϵ (in the direction h) is zero, i.e.,

$$0 = \frac{d}{d\epsilon} J(u^\epsilon)\bigg|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{J(u^\epsilon) - J(u^*)}{\epsilon}. \tag{1.20}$$

which implies

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} J(u^\epsilon)\bigg|_{\epsilon=0} \\ &= \int_{t_0}^{t_1} \frac{\partial}{\partial \epsilon} \left[f(t, x^\epsilon(t), u^\epsilon(t)) + \dot{\lambda}(t) x^\epsilon(t) + \lambda(t) g(t, x^\epsilon(t), u^\epsilon(t) dt \right] \bigg|_{\epsilon=0} \\ &\quad - \frac{\partial}{\partial \epsilon} \lambda(t_1) x^\epsilon(t_1) \bigg|_{\epsilon=0}. \end{aligned} \tag{1.21}$$

Applying the chain rule to f and g, it follows

$$\begin{aligned} 0 = &\int_{t_0}^{t_1} \left[f_x \frac{\partial x^\epsilon}{\partial \epsilon} + f_u \frac{\partial u^\epsilon}{\partial \epsilon} + \dot{\lambda}(t) \frac{\partial x^\epsilon}{\partial \epsilon} + \lambda(t) \left(g_x \frac{\partial x^\epsilon}{\partial \epsilon} + g_u \frac{\partial u^\epsilon}{\partial \epsilon} \right) \right] \bigg|_{\epsilon=0} dt \\ &- \lambda(t_1) \frac{\partial x^\epsilon}{\partial \epsilon}(t) \bigg|_{\epsilon=0} \end{aligned} \tag{1.22}$$

where the arguments of the f_x, f_u, g_x, and g_u terms are $(t, x^*(t), u^*(t))$. Rearranging the terms in [eq:1.28](1.28) gives

$$0 = \int_{t_0}^{t_1} \left[\left(f_x + \lambda(t) g_x + \dot{\lambda}(t) \right) \frac{\partial x^*}{\partial \epsilon}(t) \bigg|_{\epsilon=0} + \left(f_u + \lambda(t) g_u \right) h(t) \right] dt - \lambda(t_1) \frac{\partial x^\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} \tag{1.23}$$

We want to choose the adjoint function to simplify [eq:1.29](1.29) by making the coefficients of

$$\frac{\partial x^\epsilon}{\partial \epsilon}(t) \bigg|_{\epsilon=0} \tag{1.24}$$

vanish. Thus, we choose the adjoint function $\lambda(t)$ to satisfy

$$\dot{\lambda}(t) = -\left[f_x(t, x^*(t), u^*(t)) + \lambda(t) g_x(t, x^*(t), u^*(t)) \right] \quad \text{(adjoint equation)} \tag{1.25}$$

and the boundary condition

$$\lambda(t_1) = 0 \quad \text{(transversality condition)} \tag{1.26}$$

Now [eq:1.29](1.29) reduces to

$$0 = \int_{t_0}^{t_1} \left(f_u(t, x^*(t), u^*(t)) + \lambda(t) g_u(t, x^*(t), u^*(t))\right) h(t) dt. \tag{1.27}$$

As this holds for any piecewise continuous variation function $h(t)$, it holds for

$$h(t) = f_u(t, x^*(t), u^*(t)) + \lambda(t) g_u(t, x^*(t), u^*(t)). \tag{1.28}$$

In this case

$$0 = \int_{t_0}^{t_1} \left(f_u(t, x^*(t), u^*(t)) + \lambda(t) g_u(t, x^*(t), u^*(t))\right)^2 dt, \tag{1.29}$$

which implies the optimality condition

$$f_u(t, x^*(t), u^*(t)) + \lambda(t) g_u(t, x^*(t), u^*(t)) = 0 \quad \text{for all } t_0 \leq t \leq t_1. \tag{1.30}$$

To find the optimal controls, we typically set up the Hamiltonian and solve the resulting two-point boundary value problem, considering both the state and adjoint equations along with the transversality conditions. Below, we let H be the Hamiltonian, which is defined as such

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u) \tag{1.31}$$

where $f(t, x, u)$ is the integrand and $\lambda g(t, x, u)$ is the adjoint multiplied by the right-hand side of the differential equation. We are maximizing H with respect to u at u^*, and the above conditions can be written in terms of the Hamiltonian:

$$\frac{\partial H}{\partial u} = 0 \text{ at } u^* \implies f_u + \lambda g_u = 0 \quad \text{(optimality condition)}, \tag{1.32}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \implies \dot{\lambda} = -(f_x + \lambda g_x) \quad \text{(adjoint equation)}, \tag{1.33}$$

$$\lambda(t_1) = 0 \quad \text{(transversality condition)}. \tag{1.34}$$

We are given the dynamics of the state equation:

$$\dot{x} = g(t, x, u) = \frac{\partial H}{\partial \lambda}, \quad x(t_0) = x_0. \tag{1.35}$$

In summary, the necessary conditions for an optimal control problem-the transversality and optimality conditions-establish the foundation upon which we can determine the characteristics of an optimal solution. In our future sections, they will ensure that our control strategies are not only feasible but optimal with given constraints.

1.3 Pontryagin's Maximum Principle with Related Examples

Before delving into examples using Pontryagin's Maximum Principle I will introduce a couple of theorems relating to the original problem [eq:1.15](1.15) and the Hamiltonian.

Theorem 1.3.1. *If $u^*(t)$ and $x^*(t)$ are optimal for [eq:1.15](1.15), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t))$$

for all controls u at each time t, where the Hamiltonian H is

$$H = f(t, x(t), u(t)) + \lambda(t) g(t, x(t), u(t)),$$

and

$$\dot{\lambda}(t) = -\frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x}, \quad \lambda(t_1) = 0.$$

We make this statement at the end of [sec:The Basic Problem and Necessary Conditions]Section

1.2, but the proof is quite complicated and done by Pontryagin himself in [39].

Theorem 1.3.2. *Suppose that $f(t,x,u)$ and $g(t,x,u)$ are both continuously differentiable functions in their three arguments and concave in u. Suppose $u^*(t)$ is an optimal control for [eq:1.15](1.15), with the associated state x^*, and $\lambda(t)$ a piecewise differentiable function with $\lambda(t) \geq 0$ for all t. Suppose for all $t_0 \leq t \leq t_1$ that*

$$0 = H_u(t, x^*(t), u^*(t), \lambda(t)).$$

Then for all controls u and each $t_0 \leq t \leq t_1$, we have

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)).$$

Proof. Fix a control u and a point in time $t_0 \leq t \leq t_1$. Then,

$$\begin{aligned}
H(t, x^*(t), u^*(t), \lambda(t)) - H(t, x^*(t), u(t), \lambda(t)) \\
&= [f(t, x^*(t), u^*(t)) + \lambda(t) g(t, x^*(t), u^*(t))] \\
&\quad - [f(t, x^*(t), u(t)) + \lambda(t) g(t, x^*(t), u(t))] \\
&= [f(t, x^*(t), u^*(t)) - f(t, x^*(t), u(t))] \\
&\quad + \lambda(t) [g(t, x^*(t), u^*(t)) - g(t, x^*(t), u(t))] \\
&\geq (u^*(t) - u(t)) f_u(t, x^*(t), u^*(t)) \\
&\quad + \lambda(t)(u^*(t) - u(t)) g_u(t, x^*(t), u^*(t)) \\
&= (u^*(t) - u(t)) H_u(t, x^*(t), u^*(t), \lambda(t)) = 0.
\end{aligned}$$

□

Similarly, if we are trying to minimize an optimal control problem the following still holds from [thm:hamiltonian-max]Theorem 4

$$H(t, x^*(t), u(t), \lambda(t)) \geq H(t, x^*(t), u^*(t), \lambda(t)) \tag{1.36}$$

To determine whether a given control problem involves maximizing or minimizing the objective functional, we can examine the concavity conditions by using the second partial derivatives of the Hamiltonian.

$$\frac{\partial^2 H}{\partial u^2} < 0 \text{ at } u^*, \tag{1.37}$$

then the problem is maximization, while

$$\frac{\partial^2 H}{\partial u^2} > 0 \text{ at } u^*, \tag{1.38}$$

is minimization.

The process of solving optimal control problems typically involves the maximization (or minimization) of an objective functional. The procedure for addressing even the simplest optimal control problems can be broadly outlined, as first we will formulate the Hamiltonian for the problem. Then derive the adjoint differential equation and establish the transversality boundary conditions, alongside the optimality condition. This expands the set of unknowns to include the control u^*, state x^*, and the adjoint variable λ. Now we attempt to eliminate u^* by employing the optimality condition $H_u = 0$, thereby solving for u^* in terms of x^* and λ. Finally, we can solve the coupled system of differential equations for x^* and λ subject to the appropriate boundary conditions, substituting the expression for the optimal control obtained in the previous step. Once the optimal state and adjoint are determined, solve for the optimal control by inverting the Hamiltonian's relation to the control variable [32].

Now, we have all of the information and background needed in order to solve simple optimal control problems using our general formulation from the introduction of this chapter.

Example 1.3.1. *Minimize the following optimal control problem with respect to the boundary* *conditions:*

$$\min_u \int_0^1 u(t)^2\,dt$$

$$\textit{subject to} \quad \dot{x}(t) = x(t) + u(t),$$

$$x(0) = 1,$$

$$x(1)\,\textit{free}.$$

We will begin by formulating the Hamiltonian H

$$H = u^2 + \lambda(x + u)$$

The optimality condition is

$$0 = \frac{\partial H}{\partial u} = 2u + \lambda \text{ at } u^* \implies u^* = -\frac{1}{2}\lambda$$

We see the problem is indeed minimization as

$$\frac{\partial^2 H}{\partial u^2} = 2 > 0$$

The adjoint equation is given by

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\lambda \Rightarrow \lambda(t) = ce^{-t}$$

for some constant c. But, the transversality condition is

$$\lambda(1) = 0 \Rightarrow ce^{-1} = 0 \Rightarrow c = 0$$

Thus, $\lambda = 0$, so that $u^* = -\frac{1}{2}\lambda = 0$. So, x^* satisfies $\dot{x} = x$ and $x(0) = 1$. Hence, the optimal solutions are

$$\lambda = 0, \quad u^* = 0, \quad x^*(t) = e^t$$

Example 1.3.2. *Solve*

$$\max_u \int_1^5 u(t)x(t) - u(t)^2 - x(t)^2 \, dt$$
$$\textit{subject to} \quad \dot{x}(t) = x(t) + u(t), \quad x(1) = 2$$

The Hamiltonian H for this problem is given by

$$H = ux - u^2 - x^2 + \lambda(x + u).$$

The optimality condition, which comes from setting $\frac{\partial H}{\partial u} = 0$, is

$$0 = x - 2u + \lambda \implies u = \frac{x + \lambda}{2}.$$

To confirm that this maximizes the Hamiltonian, we check the second derivative:

$$\frac{\partial^2 H}{\partial u^2} = -2 < 0.$$

The adjoint equation is given by

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -u + x - \lambda,$$

which simplifies to

$$\dot{\lambda} = -\frac{2x}{3} + \frac{\lambda}{3}.$$

The transversality condition for this problem, since $x(5)$ is free, is

$$\lambda(5) \text{ is free.}$$

We have two coupled first-order differential equations:

$$\dot{x} = x + \frac{x + \lambda}{2}, \qquad \dot{\lambda} = \frac{\lambda - x}{2}.$$

Solving these equations with the initial condition $x(1) = 2$, we find the optimal state and control. To solve these ODEs numerically, I used the Runge-Kutta method using MATLAB.

Example 1.3.3. *Formulate an optimal control problem for a population with an Allee effect growth term, in which the control is the proportion of the population to be harvested. This means that differential equation has an Allee effect term. Choose an objective functional which maximizes revenue from the harvesting while minimizing the cost of harvesting. The revenue is the integral of amount harvested per time. Assume the cost of harvesting has a quadratic format.*

We investigate the dynamics of the scalar delay-differential equation governing a single-species population with an Allee effect [33]:

$$\dot{x}(t) = -u(t)x(t) + f(x(t - \tau)), \tag{1.39}$$

where $u > 0$ represents the harvesting rate, $f : [0, \infty) \to [0, \infty)$ is the recruitment function showing a strong Allee effect, and $\tau > 0$ is a constant delay in the birth process. The objective functional aims to maximize the revenue from harvesting while minimizing the cost. The revenue at any time t can be modeled as $pu(t)x(t)$, where p is the price per unit of the population harvested. The cost of harvesting is assumed to have a quadratic form, represented by $cu(t)^2$, where c is the cost coefficient that reflects the effort and resources expended in harvesting. Thus, the objective functional J to be maximized is:

$$J(u) = \int_0^T pu(t)x(t) - cu(t)^2 dt \tag{1.40}$$

where T is the terminal time of the management period. The optimal control problem is to find a harvesting strategy $u^*(t)$ that maximizes the objective functional J subject to the population dynamics with the Allee effect. The problem is subject to the constraints $0 \leq u(t) \leq 1$, $x(0)$ given an initial population size, and $x(T)$ free.

1.4 Existence, Uniqueness, and Other Fundamental Properties

In this section, we discuss the existence and uniqueness of solutions to optimal control problems that directly impact the solvability and reliability of solutions. This section builds on the necessary conditions for optimal control problems established in the previous sections. We discuss the possibility of these conditions leading to multiple solution sets, among which only a few might be truly optimal. The section also addresses the critical issue of the existence of an optimal control, acknowledging that there are cases where, despite the solvability of necessary conditions, the original control problem might not have a viable solution. This is particularly relevant when the objective functional, evaluated at the optimal state and control, results in ∞ or $-\infty$, indicating the absence of a solution. Further, we examine the potential unboundedness in optimal controls and states, especially due to quadratic nonlinearities in differential equations. We state some simple results for existence from [7] [34] [39].

Theorem 1.4.1. *Consider the optimal control problem defined by the functional*

$$J(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

subject to $x'(t) = g(t, x(t), u(t))$, with initial condition $x(t_0) = x_0$. If $f(t, x, u)$ and $g(t, x, u)$ are continuously differentiable and are concave in x and u, and if we have a control u^*, associated state x^*, and a piecewise differentiable function λ satisfying the following conditions on $t_0 \leq t \leq t_1$:

$$f_u + \lambda g_u = 0,$$

$$\lambda' = -(f_x + \lambda g_x),$$

$$\lambda(t_1) = 0,$$

$$\lambda(t) \geq 0,$$

then for all controls u*,* $J(u^*) \geq J(u)$.

Proof. Note by the tangent line property this gives us

$$f(t,x^*,u^*) - f(t,x,u) \geq (x^* - x)f_x(t,x^*,u^*) + (u^* - u)f_u(t,x^*,u^*).$$

This gives

$$J(u^*) - J(u) = \int_{t_0}^{t_1} f(t,x^*,u^*) - f(t,x,u)\,dt$$

which by the fundamental lemma of calculus of variations implies

$$J(u^*) - J(u) \geq \int_{t_0}^{t_1} (x^*(t) - x(t))f_x(t,x^*,u^*) + (u^*(t) - u(t))f_u(t,x^*,u^*)\,dt. \tag{1.41}$$

Substituting

$$\begin{aligned} f_x(t,x^*,u^*) &= -\dot{\lambda}(t) - \lambda(t)g_x(t,x^*,u^*), \\ f_u(t,x^*,u^*) &= -\lambda(t)g_u(t,x^*,u^*), \end{aligned}$$

and as given by the hypothesis, the last term in [eq:1.47](1.47) becomes

$$\int_{t_0}^{t_1} (x^*(t) - x(t))(-\lambda(t)g_x(t,x^*,u^*) - \dot{\lambda}(t)) + (u^*(t) - u(t))(-\lambda(t)g_u(t,x^*,u^*))\,dt. \tag{1.42}$$

Using integration by parts, and recalling $\lambda(t_1) = 0$ and $x(t_0) = x^*(t_0)$, we see

$$\begin{aligned} \int_{t_0}^{t_1} -\dot{\lambda}(t)(x^*(t) - x(t))\,dt &= \int_{t_0}^{t_1} \lambda(t)(x(t) - x^*(t))'\,dt \\ &= \int_{t_0}^{t_1} \lambda(t)(g(t,x(t)^*,u(t)^*) - g(t,x(t),u(t)))\,dt. \end{aligned}$$

Making this substitution,

$$J(u^*) - J(u) \geq \int_{t_0}^{t_1} \lambda(t)\left[g(t,x^*,u^*) - g(t,x,u)\right]\left[(x^* - x)g_x(t,x^*,u^*) - (u^* - u)g_u(t,x^*,u^*)\right]\,dt.$$

Taking into account $\lambda(t) \geq 0$ and that g is concave in both x and u, this gives the desired result $J(u^*) - J(u) \geq 0$. $\square$

The proof establishes that, under the given conditions, the chosen control strategy, denoted as u^*, is indeed optimal. It assures us that this strategy won't be outperformed by any other, guaranteeing the lowest possible cost in the context of the system's dynamics.

Theorem 1.4.2. *Let the set of controls be Lebesgue integrable functions on $t_0 \leq t \leq t_1$. If $f(t,x,u)$ is convex in u, and if there exist constants $C_1, C_2, C_3 > 0$, C_4, and $\beta > 1$ such that:*

$$g(t,x,u) = \alpha(t,x) + \beta(t,x)u,$$

$$|g(t,x,u)| \leq C_1(1 + |x| + |u|),$$

$$|g(t,x_1,u) - g(t,x,u)| \leq C_2|x_1 - x|(1 + |u|),$$

$$f(t,x,u) \geq C_3|u|^\beta - C_4,$$

then an optimal control u^* maximizing $J(u)$ exists with $J(u^*)$ finite.

Proof. Consider a sequence of controls $\{u_n\}$ such that $J(u_n)$ approaches $\sup J(u)$ as n approaches infinity. Given the coercivity condition $f(t,x,u) \geq C_3|u|^\beta - C_4$, the sequence $\{u_n\}$ must be bounded in $L^\beta([t_0,t_1])$ to prevent the functional $J(u)$ from diverging negatively, ensuring the existence of a supremum. By the reflexivity of $L^\beta([t_0,t_1])$, there exists a weakly convergent subsequence u_{n_k} that converges weakly to some u^* in L^β. The convexity of $f(t,x,u)$ in u implies that $J(u)$ is weakly lower semicontinuous, which in turn ensures that $\liminf_{k\to\infty} J(u_{n_k}) \geq J(u^*)$. Furthermore, the supremum definition ensures that for any positive ϵ, there is a k such that $J(u^*) \geq J(u_{n_k}) - \epsilon$. As k goes to infinity, the weak lower semicontinuity of J allows us to conclude that $J(u^*) \geq \sup J(u)$, thus establishing the optimality of u^*. Finally, the coercivity and boundedness of $J(u)$ combined with the weak lower semicontinuity provide that $J(u^*)$ is finite. Hence, u^* is the optimal control we sought to prove exists. $\square$

The concept of uniqueness is also significant. Assuming the existence of an optimal control, denoted by u^*, which satisfies $J(u) \leq J(u^*) < \infty$ for all possible controls u in the context of maximization, we consider u^* to be unique if, for any control u yielding $J(u^*) = J(u)$, it follows that u^* and u differ at most at a finite number of points. In such cases, the associated state trajectories are identical, and we refer to this state as x^*, the unique optimal state. Uniqueness of the solution to the optimality system typically implies the uniqueness of the optimal control, provided one exists. Uniqueness can often be demonstrated for solutions of the optimality system over short time intervals, primarily because the state and adjoint equations evolve in opposite temporal directions-the state equation progresses forward in time from an initial condition, whereas the adjoint equation works backward from a final condition. It is important to note, however, that the uniqueness of an optimal control does not inherently imply the uniqueness of the entire optimality system. Proving the uniqueness of the optimal control directly would require the strict concavity of the objective functional $J(u, x(u))$. We will not delve into direct uniqueness results here, as they are complex and not necessary for our discussion. Nevertheless, if the functions f, g, and the adjoint equation's right-hand side are Lipschitz continuous with respect to state and adjoint variables, then uniqueness for the optimality system is assured for sufficiently small intervals. Additionally, if the solutions to the optimality system are bounded, the Lipschitz condition and the consequent uniqueness typically follow with ease. [32].

Another fundamental property in optimal control and dynamic programming is the principle of optimality. This principle is vital for understanding how the optimal control strategy over a smaller subinterval of time relates to the optimal control over the entire time span. It essentially states that an optimal policy, when viewed over a shorter timeframe, must itself be optimal for that subinterval. Which allows for the decomposition of a complex problem into smaller segments and thus more manageable.

Theorem 1.4.3. *Let u^* be an optimal control, and x^* the resulting state, for the problem*

$$\begin{aligned} &\max_u J(u) = \max_u \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \\ &\textit{subject to} \quad \dot{x}(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0. \end{aligned} \tag{1.43}$$

Let $\hat{t}$ be a fixed point in time such that $t_0 < \hat{t} < t_1$. Then, the restricted functions $\hat{u}^* = u^*|_{[\hat{t},t_1]}$, $\hat{x}^* = x^*|_{[\hat{t},t_1]}$ form an optimal pair for the restricted problem

$$\begin{aligned} \max_u \hat{J}(\hat{u}) &= \max_u \int_{\hat{t}}^{t_1} f(t, x(t), u(t))\, dt \\ \textit{subject to} \quad \dot{x}(t) &= g(t, x(t), u(t)), \quad x(\hat{t}) = x^*(\hat{t}). \end{aligned} \tag{1.44}$$

Further, if u^* is the unique optimal control for [eq:1.49](1.49), then $\hat{u}^*$ is the unique optimal control for [eq:1.50](1.50).

Proof. We argue by contradiction. Assume that $\hat{u}^*$ is not an optimal control; that is, there is some control $\hat{u}_1$ on the interval $[\hat{t}, t_1]$ such that $J(\hat{u}_1) > J(\hat{u}^*)$. Define a new control u_1 over the interval $[t_0, t_1]$ by

$$u_1(t) = \begin{cases} u^*(t) & \text{for } t_0 \leq t \leq \hat{t}, \\ \hat{u}_1(t) & \text{for } \hat{t} < t \leq t_1. \end{cases}$$

Let x_1 be the state trajectory corresponding to u_1. Since u_1 and u^* coincide on $[t_0, t]$, the states x_1 and x^* also match on this interval. Consequently, we have

$$\begin{aligned} J(u_1) - J(u^*) &= \left(\int_{t_0}^{\hat{t}} f(t, x_1, u_1)\, dt + \hat{J}(\hat{u}_1) \right) - \left(\int_{t_0}^{\hat{t}} f(t, x^*, u^*)\, dt + \hat{J}(\hat{u}^*) \right) \\ &= J(\hat{u}_1) - J(\hat{u}^*) \\ &> 0. \end{aligned}$$

This contradicts the initial assumption that u^* was the optimal control for the problem [eq:1.49](1.49), implying no such $\hat{u}_1$ can exist and $\hat{u}^*$ is indeed optimal for [eq:1.50](1.50). □

This theorem is applicable to both maximization and minimization problems. The intuition behind the theorem is straightforward: if we have an optimal control and state pair, u^* and x^*, for a given optimal control problem, and we consider the system at a later time $\hat{t}$, then continuing with the previously determined optimal trajectory seems naturally to be an optimal strategy for the remaining duration.

Example 1.4.1. *Consider the optimization problem from [32] given by:*

$$\min_u \int_0^2 \left(x(t) + \frac{1}{2}u(t)^2 \right) dt$$

subject to the differential equation

$$\dot{x}(t) = x(t) + u(t), \quad x(0) = \frac{1}{2}e^2 - 1.$$

First, we will solve this example on $[0, 2]$, then solve the same problem on a smaller interval $[1, 2]$. The Hamiltonian in this example is

$$H = x + \frac{1}{2}u^2 + \lambda x + \lambda u.$$

The adjoint equation and transversality condition give

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -1 - \lambda, \quad \lambda(2) = 0 \quad \Rightarrow \quad \lambda(t) = e^{2-t} - 1,$$

and the optimality condition leads to

$$0 = \frac{\partial H}{\partial u} = u + \lambda \quad \Rightarrow \quad u^*(t) = -\lambda(t) = 1 - e^{2-t}.$$

Finally, from the state equation, the associated state is

$$x^*(t) = \frac{1}{2}e^{2-t} - 1.$$

Now, consider the same problem, except on the interval $[1, 2]$

$$\min_u \int_1^2 x(t) + \frac{1}{2}u(t)^2 \, dt$$

subject to

$$\dot{x}(t) = x(t) + u(t), \quad x(1) = \frac{1}{2}e - 1$$

By the Principle of Optimality we can find an optimal pair immediately. The problem above has the same optimal control as the one with the new interval and initial conditions. The adjoint equation remains unchanged as

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -1 - \lambda, \quad \lambda(2) = 0 \quad \Rightarrow \quad \lambda(t) = e^{2-t} - 1,$$

while the optimality condition is also unchanged,

$$0 = \frac{\partial H}{\partial u} = u + \lambda \quad \Rightarrow \quad u^*(t) = -\lambda(t) = 1 - e^{2-t}.$$

Using the new initial condition $x(1) = \frac{1}{2}e^{-1}$, we find the corresponding state

$$x^*(t) = \frac{1}{2}e^{2-t} - 1.$$

Which gives us the same optimal control pair as the previous conditions thanks to the Principle of Optimality.

Transitioning from the Principle of Optimality, we now shift our focus to the Hamiltonian, denoted as $H(t, x, u, \lambda)$, which is a function defined by four variables. However, the variable t is the underlying variable since the state x, the control u, and the adjoint λ are all functions that depend on time. Thus, the Hamiltonian is implicitly a function of time. Given that the state and adjoint are continuous and the control is piecewise continuous, the Hamiltonian inherits these properties, making it piecewise continuous with respect to time. Moreover, for the optimal control u^* and the corresponding optimal state x^*, the Hamiltonian exhibits a much stronger continuity property.

Theorem 1.4.4. *The Hamiltonian is a Lipschitz continuous function of time t on the optimal path.*

Proof. Let u^*, x^* be an optimal pair for [eq:1.15](1.15), and λ the associated adjoint. For $t \in$

$[t_0, t_1]$, write

$$M(t) = H(t, x^*(t), u^*(t), \lambda(t)).$$

As u^* is piecewise continuous on a compact interval, there is some bounded interval P such that $u^*(t) \in P$ for all $t \in [t_0, t_1]$. Similarly, there exist bounded intervals Q and R such that $x^*(t) \in Q$ and $\lambda(t) \in R$ for all $t \in [t_0, t_1]$. Consider the Hamiltonian as a function of four variables $H(t, x, u, \lambda)$, where we think of x, u, λ as only numbers for a moment. By the original choices of f and g, H is continuously differentiable in all four arguments. Therefore, it is possible to choose a constant K_1 such that

$$|H_t(t, x, u, \lambda)| \leq K_1, \quad |H_x(t, x, u, \lambda)| \leq K_1, \quad \text{and} \quad |H_\lambda(t, x, u, \lambda)| \leq K_1,$$

for all tuples (t, x, u, λ) in the compact set $[t_0, t_1] \times P \times Q \times R$. Fix $s, t \in [t_0, t_1]$. For convenience, write $x_t = x^*(t)$ and $x_s = x^*(s)$. Define $u_t, u_s, \lambda_t, \lambda_s$ similarly. Let $\tau \in P$. By a few applications of the mean value theorem, we have

$$\begin{aligned} |H(t, x_t, \tau, \lambda_t) - H(s, x_s, \tau, \lambda_t)| &\leq |H_t(c_1, x_t, \tau, \lambda_t)|\,|t - s| + |H_x(s, c_2, \tau, \lambda_t)|\,|(x_t - x_s)| \\ &\quad + |H_\lambda(s, x_s, \tau, c_3)|\,|(\lambda_t - \lambda_s)| \\ &\leq K_1|t - s| + K_1|x_t - x_s| + K_1|\lambda_t - \lambda_s|, \end{aligned}$$

for some intermediary points $c_1 \in [t_0, t_1]$, $c_2 \in Q$, and $c_3 \in R$. On the other hand, x^* and λ are piecewise differentiable on a compact interval, thus Lipschitz continuous. Let K_2 be the maximum of the two Lipschitz constants. Then we have

$$\begin{aligned} |H(t, x_t, \tau, \lambda_t) - H(s, x_s, \tau, \lambda_t)| &\leq K_1\,|t - s| + K_1\,|x_t - x_s| + K_1\,|\lambda_t - \lambda_s| \\ &\leq (K_1 + 2K_1K_2)\,|t - s|\,. \end{aligned} \tag{1.45}$$

Set $K = K_1 + 2K_2K_1$ and note this holds for all $\tau \in P$. Now, $M(t) = H(t, x_t, u_t, \lambda_t)$ is similar for s. By [thm:3]Theorem 3, the Hamiltonian is maximized pointwise by u^*, so

$$H(t, x_t, u_s, \lambda_t) \leq H(t, x_t, u_t, \lambda_t) \quad \text{and} \quad H(s, x_s, u_t, \lambda_s) \leq H(s, x_s, u_s, \lambda_s). \tag{1.46}$$

Applying [eq:1.51](1.51) for $\tau = u_s$ and $\tau = u_t$, and combining with [eq:1.52](1.52), we see

$$\begin{aligned}
-K|t-s| &\leq H(t, x_t, u_s, \lambda_t) - H(s, x_s, u_s, \lambda_s) \\
&\leq H(t, x_t, u_t, \lambda_t) - H(s, x_s, u_s, \lambda_s) \\
&= M(t) - M(s) \\
&\leq H(t, x_t, u_t, \lambda_t) - H(s, x_s, u_t, \lambda_s) \\
&\leq K|t-s|.
\end{aligned}$$

Namely, $|M(t) - M(s)| \leq K|t-s|$. As t, s are arbitrary, M is Lipschitz continuous. □

The theorem we have established demonstrates that the Hamiltonian, within the context of an optimal control problem, satisfies Lipschitz continuity with respect to time along the optimal trajectory. This characteristic is important for optimal control problems because it implies stability and predictability of the system's behavior under optimal control. In practical terms, the Lipschitz condition on the Hamiltonian ensures that small changes in time do not lead to disproportionately large variations in the Hamiltonian's value.

Theorem 1.4.5. *If an optimal control problem is autonomous, then the Hamiltonian is a constant function of time along the optimal path.*

Proof. Let u^*, x^* be the optimal pair for the control problem

$$\begin{aligned}
&\max_u \int_{t_0}^{t_1} f(x(t), u(t))\, dt \\
&\text{subject to} \quad \dot{x}(t) = g(x(t), u(t)), x(0) = x_0.
\end{aligned} \tag{1.47}$$

and λ the associated adjoint. Let $M(t) = H(x^*(t), u^*(t), \lambda(t))$ be defined as in the proof of [thm:8]Theorem 8, except now H is only a function of three variables. As M is Lipschitz continuous, we have from measure theory that M is differentiable almost everywhere, with respect to Lebesgue measure [41]. Let $\bar{t} \in (t_0, t_1)$ be any point where $\dot{M}$ exists. Denote $u^*(\bar{t}) = \tau$. Note, for small enough $\delta > 0$ so that $\bar{t} + \delta \in [t_0, t_1]$, the Maximum Principle gives $M(\bar{t} + \delta) \geq H(x^*(\bar{t} + \delta), u^*(\bar{t}), \lambda(\bar{t} + \delta)) = H(x^*(\bar{t} + \delta), \tau, \lambda(\bar{t} + \delta))$. So,

$$M(\bar{t} + \delta) - M(\bar{t}) \geq H(x^*(\bar{t} + \delta), \tau, \lambda(\bar{t} + \delta)) - H(x^*(\bar{t}), \tau, \lambda(\bar{t})).$$

Divide by δ and then let $\delta \to 0$. This shows

$$\begin{aligned}
\dot{M}(\bar{t}) &\geq \frac{d}{dt} H(x^*(t), \tau, \lambda(t)) \bigg|_{t=\bar{t}} \\
&= H_x(x^*(\bar{t}), \tau, \lambda(\bar{t}))(\dot{x}^*)(\bar{t}) + H_\lambda(x^*(\bar{t}), \tau, \lambda(\bar{t}))(\dot{\lambda})(\bar{t}) \\
&= -(\dot{\lambda})(\bar{t})(\dot{x}^*)(\bar{t}) + (\dot{x}^*)(\bar{t})(\dot{\lambda})(\bar{t}) \\
&= 0.
\end{aligned}$$

By the same argument,

$$M(\bar{t}) - M(\bar{t} - \delta) \leq H(x^*(\bar{t}), \tau, \lambda(\bar{t})) - H(x^*(\bar{t} - \delta), \tau, \lambda(\bar{t} - \delta)).$$

Dividing by δ and letting $\delta \to 0$, we see $\dot{M}(\bar{t}) \leq 0$. Hence, $\dot{M} = 0$ almost everywhere. Combined with the fact that M is continuous, we see M is constant. □

1.5 Payoff Terms and States with Fixed Endpoints

In optimal control theory, the concept of 'Payoff Terms' can be useful when our objective extends beyond the simple maximization or minimization of a cumulative function over time. Instead, we may be particularly interested in the value of a function at a specific point in time-most commonly, the end of the considered time interval. An example would include the number of infected individuals at the final time in an epidemic model. This is a scenario that I could potentially explore when

I minimize the number of infectious persons and the overall cost of the vaccine during a fixed time period concerning a micro-parasitic infectious disease. Our general formulation for the problem will change from [eq:1.15](1.15) and will be structured as shown below

$$\max_u \left[\phi(x(t_1)) + \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt\right] \tag{1.48}$$

subject to

$$\dot{x} = g(t, x(t), u(t)), \quad x(t_0) = x_0,$$

where $\phi(x(t_1))$ is a goal with respect to the final position or population level, $x(t_1)$. We call $\phi(x(t_1))$ a *payoff term*. It is sometimes referred to as the salvage term. Consider the resulting change in the derivation of the necessary conditions. Our objective functional becomes

$$J(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt + \phi(x(t_1)) \tag{1.49}$$

In the calculation of

$$0 = \lim_{\epsilon \to 0} \frac{J(u^\epsilon) - J(u^*)}{\epsilon},$$

the only change occurs in the conditions at the final time,

$$0 = \int_{t_0}^{t_1} \left[\left(f_x + \lambda g_x + \dot{\lambda}\right) \frac{dx^\epsilon}{d\epsilon}\bigg|_{\epsilon=0} + (f_u + \lambda g_u)h\right] dt - \left(\lambda(t_1) - \dot{\phi}(x(t_1))\right) \frac{\partial x^\epsilon(t_1)}{\partial \epsilon}\bigg|_{\epsilon=0} \tag{1.50}$$

we make the choice of the adjoint variable λ to satisfy the previously stated adjoint equation and also

$$\dot{\lambda}(t) = -f_x(t, x^*, u^*) - \lambda(t) g_x(t, x^*, u^*), \quad \lambda(t_1) = \dot{\phi}(x^*(t_1)) \tag{1.51}$$

then [eq:1.56](1.56) simplifies to

$$0 = \int_{t_0}^{t_1} (f_u + \lambda g_u) h\, dt, \tag{1.52}$$

and the optimality condition

$$f_u(t, x^*, u^*) + \lambda g_u(t, x^*, u^*) = 0 \tag{1.53}$$

follows as before. Hence, the only change in the necessary conditions lies in the transversality condition

$$\lambda(t_1) = \dot{\phi}(x^*(t_1)). \tag{1.54}$$

We will now solve an example below including the addition of the payoff term.

Example 1.5.1. *Consider the optimal control problem given by*

$$\min_u \frac{1}{2}\int_0^1 u(t)^2 dt + x(1)^2$$

subject to

$$\dot{x}(t) = x(t) + u(t), \quad x(0) = 1.$$

Our goal includes minimizing the term $x(1)^2$, in addition to the square integral of the control. We can view this as minimizing a population, with exponential growth, at the end of a time frame. We should expect u to be negative, in order to decrease x, but $|u|$ cannot be too large because of the integral. In this example, the Hamiltonian is

$$H = \frac{1}{2}u^2 + \lambda x + \lambda u.$$

The optimality condition gives

$$0 = \frac{\partial H}{\partial u} = u + \lambda \Rightarrow u^*(t) = -\lambda(t).$$

Also, the adjoint equation is

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -\lambda \Rightarrow \lambda(t) = Ce^{-t},$$

for some constant C. Hence,

$$u^*(t) = -\lambda(t) = -Ce^{-t}.$$

So,

$$x^*(t) = x - Ce^{-t}, \quad x(0) = 1,$$

which gives

$$x^*(t) = \frac{C}{2}e^{-t} + Ke^{t},$$

where K is a constant. Recall, the transversality condition here is

$$\lambda(1) = \dot{\phi}(x(1)) = (\dot{x}^2(1)) = 2x(1).$$

We have the system of linear equations

$$1 = x(0) = \frac{C}{2} + K$$

$$Ce^{-1} = \lambda(1) = 2x(1) = 2\left(\frac{C}{2}e^{-1} + Ke^{1}\right),$$

which can be solved to give $C = 2, K = 0$. Thus,

$$x^*(t) = e^{-t}, u^*(t) = -2e^{-t}$$

and $u*$ is negative as we expect.

Next, we will discover the possibility of fixing the position of the state whether it is at the beginning of the time interval, at the end of the time interval, or both. The objective functional could depend on the final or initial position. We will formulate the general form as shown below

$$\max_{u} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt + \phi(x(t_0))$$

subject to the dynamics

$$\dot{x}(t) = g(t, x(t), u(t)), \quad x(t_0) \text{ free}, \quad x(t_1) = x_1 \text{ fixed}.$$

This is different from the problems we have been examining, as the state is fixed at the end of the time interval, not at the beginning. Also, similar to past sections, our necessary conditions for the optimal pair u^* and x^* will stay the same, but our transversality condition will change to

$$\lambda(t_0) = \dot{\phi}(x(t_0)). \tag{1.55}$$

Consider the problem below, where the state is fixed at both the beginning and end of the time interval,

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \tag{1.56}$$

subject to

$$x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0, \quad x(t_1) = x_1 \text{ both fixed}.$$

In this scenario, we aim to identify the optimal control strategy from a set of admissible controls that comply with all given constraints. Specifically, for the problem at hand (1.56), we seek controls that can successfully navigate the system from a predetermined starting state to a prespecified ending state. To address this problem, we need to refine the standard necessary conditions to accommodate the fixed endpoint requirements. The theorems below will clarify these refined conditions.

Theorem 1.5.1. *If $u^*(t)$ and $x^*(t)$ are optimal for problem* (1.56)*, then there exists a piecewise differentiable adjoint variable $\lambda(t)$ and a constant λ_0, equal to either 0 or 1, such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t))$$

for all admissible controls u at each time t, where the Hamiltonian H is

$$H = \lambda_0 f(t, x(t), u(t)) + \lambda(t) g(t, x(t), u(t))$$

and

$$\dot{\lambda}(t) = -\frac{\partial H(t, x^*(t), u^*(t))}{\partial x}$$

This formulation is important because it addresses a common class of problems in optimal control where the initial and final states are predetermined. The optimality of $u^*(t)$ and $x^*(t)$ is not just about guiding the system along a desired trajectory; it's also about doing so in the most efficient way possible under the given dynamics of g and the performance criteria f. The presence of $\lambda(t)$ and λ_0 is representative of the system's sensitivity to changes in the state and control, and they play a central role in characterizing the optimal trajectory.

We can solve an example below to demonstrate a computational approach towards states with a fixed endpoint.

Example 1.5.2. *Consider the optimal control problem [32]:*

$$\min_u \int_0^4 u(t)^2 + x(t)\, dt$$

subject to

$$\dot{x}(t) = u(t), \quad x(0) = 0, \quad x(4) = 1.$$

We begin by forming the Hamiltonian

$$H = u^2 + x + \lambda u.$$

There is no transversality condition, as x has both boundary conditions, but we make use of the adjoint condition,

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -1 \Rightarrow \lambda(t) = k - t$$

for some constant k. Then, the optimality condition gives

$$0 = \frac{\partial H}{\partial u} = 2u + \lambda \Rightarrow u^* = -\frac{\lambda}{2} = \frac{t-k}{2}.$$

Solving the state equation with this control gives

$$x^*(t) = \frac{t^2}{4} - \frac{kt}{2} + c$$

for some constant c. Using the boundary conditions, $x(0) = 0$ implies $c = 0$, and $x(4) = 1$ gives $k = \frac{3}{2}$. So,

$$u^*(t) = \frac{2t-3}{4} \quad \text{and} \quad x^*(t) = \frac{t^2 - 3t}{4}.$$

The main reason why I included this section is due to the importance of epidemic control problems where payoff terms and fixed endpoint conditions allow us to not only minimize the spread of an infectious disease over a given timeframe but can ensure that the number of infections is reduced to a specified level by the end of the outbreak or intervention period. As I have mentioned previously, my next chapter will cover a specific epidemic control problem where I will minimize the number of infectious persons and the overall cost of the vaccine during a fixed time period concerning a micro-parasitic infectious disease.

CHAPTER 2: OPTIMAL CONTROL THEORTY IN MODELING AND ANALYSIS OF MICRO-PARASITIC DISEASE DYNAMICS

2.1 Introduction

Micro-parasitic pathogens have historically played a pivotal role in human history, causing devastating pandemics and significantly influencing public health. The Influenza A pandemic in 1914, the Plague in the 13th century, and the Rubella outbreak in the 1960s in the United States are testament to the severe impact of these pathogens, which are characterized by their small size, rapid multiplication within hosts, and the ability to cause either transient or chronic infections. Their nature, including short generation times, lack of specialized infective stages, and potential to lead to host immunity or death, underlines the critical importance of understanding their dynamics for effective disease surveillance, prevention, and control in public health and epidemiology [44].

Micro-parasites, including viruses, bacteria, protozoa, and fungi, differ from larger macro-parasites like worms and arthropods in their degree of within-host replication, ability to generate a lasting host immune response, and their quantification in natural populations [14]. The Susceptible-Infected-Recovered (SIR) model, developed by Kermack and others, plays a crucial role in epidemiology, especially in analyzing the dynamics of micro-parasitic diseases [26]. These models categorize the host population into susceptible, infected, and recovered compartments, providing insights into the spread and control of diseases. Diseases characterized by high transmission rates, low severity, and low host recovery rates are most likely to become established in host populations. Furthermore, micro-parasitic diseases can regulate or depress the overall host population size through effects on host survival or reproductive capacity, highlighting the intricate relationship between pathogens and their hosts. Micro-parasites are less likely to cause problems in low-density natural populations because infected hosts may die before transmitting the disease, influencing strategies for vaccination and disease eradication [5]. The role of pathogens in ecological communities is also crucial; they can influence species coexistence, maintain biodiversity, and drive successional dynamics. The stability of ecological communities often depends on the number of species and the strength of their interactions, including those with pathogens. Pathogens can also alter community structures significantly, as seen in examples like the myxoma virus epidemic in rabbits in England, which led to major changes in plant and animal communities [27]. These are just the many examples of micro-parasitic pathogens and the harm they can cause to communities and highlight the necessity of research and modeling this epidemic. Below, we discuss the basic preliminaries, the optimal control problem, and the necessary conditions to define the problem [32]. Following this, we expand on the SIR model and extend our analysis to the Susceptible-Exposed-Infected-Recovered (SEIR) model. In this context, we apply optimal control strategies as opposed to the traditional methods typically used for solving SIR/SEIR models. Optimal control theory has been applied to modeling various amounts of disease in the field of HIV/AIDS by Okosun et al. [37], Karrakchou et al. [25], Agusto and Adekunle [4] and Wang and Li [45]. In the area of vector-borne diseases, Lashari and Zaman [31], and Graesboll et al. [21]. Other diseases, such as SARS-CoV-2, have been addressed by Yan and Zou [47], Zaman et al. [48], Lai et al. [30], and Abadias et al. [1]. General epidemiological and disease control research has seen contributions from Adnaoui and El Alami Laaroussi [2], Elwefati et al. [15], Bassey and Henry [6], Shang [42], Chen et al. [12], Hindes et al. [22], and Bolzoni et al. [8], Agusto [3].

The results in this chapter have been developed into a research article, currently submitted and under review [20].

2.2 Basic optimal control problem

As stated in the previous chapter, optimal control theory, an extension of the calculus of variations that emerged largely from the work of Lev Pontryagin in the 1950s, employs the maximum principle to deal with finding control policies that minimize or maximize some measure of performance [10]. This involves developing a mathematical model to represent a dynamical system and then applying optimization algorithms to find the best control policy. This has many applications in economics, engineering, and biology. Specifically, we focus on applying this problem to micro-parasitic diseases and minimizing the number of infectious persons and the overall cost of the vaccine during a fixed time period. In the context of our problem, we define the general minimization functional

$$\begin{aligned} &\min_u J(u) = \min_u \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt, \\ &\text{subject to} \quad \dot{x}(t) = g(t, x(t), u(t)), \\ &\qquad\qquad x(t_0) = x_0 \text{ and } x(t_1) \text{ free.} \end{aligned} \tag{2.1}$$

Let $x(t)$ denote the state of the system at time t, and $u(t)$ represent the control, where t spans the interval $[t_0, t_f]$. The objective is to identify a piecewise continuous control policy $u^*(t)$ that minimizes the objective functional $J(u)$. A functional, in this context, is a mapping from a specific set of functions to real numbers, here represented as an integral, as illustrated in (2.1). In general, the objective functional depends on one or more of the state and the control variables [32]. This process also involves solving the Hamiltonian equation, a function that characterizes the total energy within a system. It aims to integrate the system's dynamics with the cost function to be optimized.

$$H = f(t, x(t), u(t)) + \lambda g(t, x(t), u(t)), \tag{2.2}$$

where the term λ represents the adjoint variable, reflecting the system's sensitivity to state changes, and $g(t, x(t), u(t))$ describes the system's dynamics under the influence of the control.

To find a solution to the optimal control problem, certain conditions need to be met. These are called "necessary" conditions and are those that must be satisfied for a control to be optimal. In optimal control theory, these conditions are often derived from Pontryagin's Maximum Principle [39]. The first necessary condition would be the transversality condition, which are boundary conditions in optimal control problems, particularly when the endpoint of the control interval is not fixed or when there are state constraints at the endpoints. These conditions are crucial for determining the optimal trajectory and the optimal control. If the endpoints in time and state are fixed, the transversality conditions can be straightforward, often leading to specific values for the adjoint variables at these endpoints. In cases where the terminal time is free, the transversality condition typically involves the Hamiltonian and states that the derivative of the Hamiltonian with respect to time must be zero at the terminal time. When the terminal state is not fixed, the transversality condition often relates to the gradient of the Hamiltonian with respect to the state variables at the terminal time. The optimality condition in optimal control theory is primarily derived from Pontryagin's Maximum Principle. This principle provides a set of conditions that must be satisfied for a control to be considered optimal. The Hamiltonian is constructed, which incorporates the dynamics of the system, the adjoint variables (Lagrange multipliers), and the control variables. According to Pontryagin, for a control to be optimal, the Hamiltonian must be maximized (or minimized, depending on the problem formulation) with respect to the control variables at every point in time. The adjoint equations, which are part of the Hamiltonian system, must be satisfied. These equations describe how the adjoints evolve over time and are typically derived from the partial derivatives of the Hamiltonian with respect to the state variables. In addition to the transversality conditions, there are specific boundary conditions for the adjoint variables that must be satisfied, depending on the initial and terminal constraints of the problem.

The model (2.1) will be central to our forthcoming sections on optimal control problems. Unlike traditional methods, such as basic reproduction number analysis (R_0) and sensitivity analysis, we use optimal control to specifically target a particular compartment in the SEIR model-be it Susceptible, Exposed, Infected, or Recovered. This focused approach allows us to better identify which segment of the population requires attention to prevent an epidemic. Traditional methods for solving SEIR models often involve examining the model's response to variations in its parameters. This examination can reveal critical thresholds where the system's behavior undergoes significant changes, such as the onset of an epidemic. However, optimal control differs by actively seeking strategies to manipulate the model parameters, aiming for the most favorable outcome, such as minimizing infection rates or maximizing the total healthy population. Optimal control theory has been extensively utilized in epidemiological models, as demonstrated in various studies such as those by Gaff and Schaefer [18] and Buonomo et al. [11], highlighting its effectiveness in addressing public health challenges and disease management strategies.

For an in-depth overview of optimal control theory and notation, refer to Lenhart and Workman [32], Bressan and Piccoli [9] and Evans [16]. When using optimal control theory in epidemiology, most models depend on Pontryagin's Maximum Principle [17, 24, 39]. This principle confirms the necessary conditions. It is presented below, but to start, we can define the control set and let $a, b, T > 0$.

$$U = \{u(t) : a \leq u(t) \leq b,\ 0 \leq t \leq T,\ u(t) \text{ is Lebesgue measurable}\}. \tag{2.3}$$

Theorem 2.2.1. *Let $u^* \in U$ be an optimal control. Then there exists an adjoint function $\lambda : \mathbb{R} \to \mathbb{R}^n$ such that $x(t, u^*), u^*, \lambda$ satisfy the state system*

$$\begin{cases} \dot{x}(t) = g(t, x, u^*), \\ x(0) = x_0, \end{cases} \tag{2.4}$$

and

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)), \tag{2.5}$$

where the Hamiltonian H is given by:

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u). \tag{2.6}$$

The adjoint system

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -(f_x + \lambda g_x), \\ \lambda(T) = 0,\ \textit{transversality condition}. \end{cases} \tag{2.7}$$

In optimal control theory, once a model is formulated and an appropriate objective function is established for the scenario, there are several key problems to address. These include proving the existence of an optimal control, characterizing what this optimal control looks like, confirming its uniqueness, computing it numerically, and investigating how it varies with different parameters in the model. Pontryagin's Maximum Principle gives our necessary conditions for the existence of an optimal solution, for sufficient conditions we state a theorem from Joshi et al. [23] to satisfy existence, and we also mention Lipschitz for uniqueness.

Theorem 2.2.2. *Consider the optimal control problem (2.1) on a fixed interval* $[0, T]$*. Assume that*

1. *There exists an* $M > 0$*, such that* $\|x(t, u)\| \leq M$ *for all* $u \in U$ *and* $0 \leq t \leq T$*;*

2. f *is lower semicontinuous;*

3. $D^+ = \{(y^0, y) : \exists v \in U, y = g(t, x, v), y^0 \geq f(t, x, v)\}$ *is convex for* $(t, x) \in [0, T] \times \{x : \|x\| \leq M\}$*.*

Then there exists an optimal control $u^* \in U$*.*

To confirm that there is only one optimal control, there needs to be only one solution to the optimality system. The system is described as having the Lipschitz property if, within a certain domain, the rate of change of the function does not vary too rapidly with respect to the dependent variable. An ODE $y' = f(x, y)$ satisfies a Lipschitz condition in a domain D if there exists a constant L, known as the Lipschitz constant, such that for any two points (x, y_1) and (x, y_2) in D, the inequality $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ holds. According to the Picard-Lindelof theorem, if an ODE satisfies the Lipschitz condition, then for any given initial condition, there exists a unique solution to the ODE in the neighborhood of the initial point. For extra information about the property within the context of our problem, see [23]

2.3 SEIR model for micro-parasitic diseases

In this section, we have two different optimal control problems to tackle an infectious disease. The first is a minimization approach that aims to decrease the number of infectious people, $I(t)$, and to cut down on vaccination costs, using a control variable $u(t)$ to manage vaccinations. This strategy is focused on quickly reducing the spread of the disease. The second approach is about maximization, which aims to keep as many people in the population, $N(t)$, healthy as possible while also considering the cost of doing so. Unlike the first method, this problem aims for the long-term health of the whole community, not just those currently sick. Each method has different goals: one to control the immediate crisis, and the other to ensure ongoing health and well-being.

Now, we expand upon the basic SIR model by incorporating an "Exposed" category, denoting individuals who have been infected but are not yet infectious due to the disease's incubation period. Below, we have provided a flowchart from [32] and our ODE system.

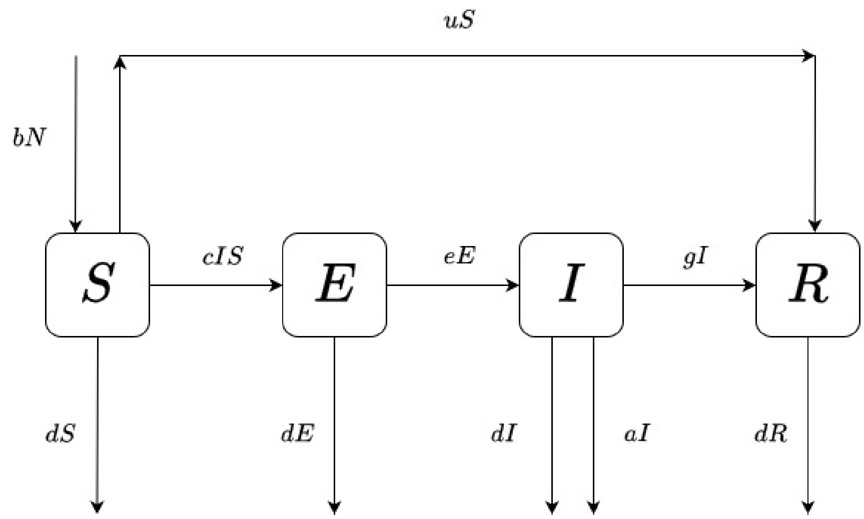


Figure 2.1: Flowchart of the SEIR model

$$
\begin{aligned}
\dot{S}(t) &= bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), && S(0) = S_0 \geq 0,\\
\dot{E}(t) &= cS(t)I(t) - (e+d)E(t), && E(0) = E_0 \geq 0,\\
\dot{I}(t) &= eE(t) - (g+a+d)I(t), && I(0) = I_0 \geq 0,\\
\dot{R}(t) &= gI(t) - dR(t) + u(t)S(t), && R(0) = R_0 \geq 0,\\
\dot{N}(t) &= (b-d)N(t) - aI(t), && N(0) = N_0.
\end{aligned} \tag{2.8}
$$

In this model, $S(t)$ represents the susceptible individuals, $E(t)$ the exposed, $I(t)$ the infectious, and $R(t)$ the recovered or immune individuals at any given time t. The total population $N(t)$ is the sum of these groups: $N(t) = S(t) + E(t) + I(t) + R(t)$ [13]. Let $u(t)$, the control, be the percentage of susceptible individuals being vaccinated per unit of time. As vaccination of the entire susceptible population is impossible, we bound the control with $0 \leq u(t) \leq 0.9$. Let b be the natural exponential birth rate of the population and d the natural exponential death rate.

The transmission of the disease is described by the term $cS(t)I(t)$. The parameter e is the rate at which the exposed individuals become infectious, and g is the rate at which infectious individuals recover. Therefore, $\frac{1}{e}$ is the mean latent period, and $\frac{1}{g}$ is the mean infectious period before recovery, if recovery occurs. The death rate due to the disease in infectious individuals is a.

2.3.1 Minimization of infectious cases and vaccination costs

Now we present the first optimal control problem from [32]. This approach aims to minimize the number of infected people and the overall cost of vaccines over a fixed time period T. The objective is to find the optimal control strategy that can effectively reduce the infection rates and prevent the spread of the disease. By implementing this strategy, it is possible to quickly eliminate the disease and save costs associated with vaccination. The optimization problem considers various parameters and constraints to determine the best course of action. The goal is to protect public health and minimize the economic burden caused by infectious diseases.

Our first model, the objective cost functional, is minimized below

$$J(u) = \int_0^T AI(t) + u^2(t)\,dt. \tag{2.9}$$

For (2.9), we let $A > 0$ where A represents the "weight" on the cost and our term $AI(t)$ denotes the cost of infection. We need to minimize our infectious group I while keeping our vaccination cost $u^2(t)$ low. Also, since our optimal functional is convex, we can assume the cost of control to be nonlinear. The goal is to find an optimal control $u^*(t)$, such that

$$J(u^*) = \min_{U_1} J(u), \tag{2.10}$$

where

$$U_1 = \{u(t) : 0 \leq u(t) \leq 0.9, \text{Lebesgue measurable}\}. \tag{2.11}$$

Theorem 2.3.1. *There exists an optimal control $u^*(t)$ and the corresponding solutions ($S(t)$, $E(t)$,*

$I(t)$, and $N(t)$) of the system (2.8), that minimizes (2.9) over U_1. Furthermore, there exists adjoint functions, $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)$, such that

$$\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S} = \lambda_1(d + cI + u) - \lambda_2(cI), \\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial E} = \lambda_2(e + d) - \lambda_3(e), \\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial I} = -A + \lambda_1(cS) - \lambda_2(cS) + \lambda_3(g + a + d) + \lambda_4(a), \\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial N} = -\lambda_1(b) - \lambda_4(b - d),
\end{aligned} \tag{2.12}$$

with the transversality conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0, \tag{2.13}$$

and the control $u^*(t)$ satisfies the optimality condition

$$u^*(t) = \max\left\{0, \min\left\{\frac{1}{2}S\lambda_1, 0.9\right\}\right\}. \tag{2.14}$$

Proof. Upon applying Theorem 2.2.1 and verifying the necessary conditions presented in Section 2.2, we proceed with the computation of the Hamiltonian, denoted as

$$H = AI(t) + u^2(t) + \sum_{i=1}^{4} \lambda_i f_i, \tag{2.15}$$

giving us,

$$\begin{aligned}
H = \;& AI(t) + u^2(t) + \lambda_1(bN(t) - dS(t) - cS(t)I(t) - u(t)S(t)) \\
& + \lambda_2(cS(t)I(t) - (e + d)E(t)) \\
& + \lambda_3(eE(t) - (g + a + d)I(t)) \\
& + \lambda_4((b - d)N(t) - aI(t)).
\end{aligned} \tag{2.16}$$

Now, we use the variables S, E, I, N and $R = N - S - E - I$. Should we choose to use the variable R, the adjoint function would turn out to be negative. Let $\lambda_1(t), \lambda_2(t), \lambda_3(t)$, and $\lambda_4(t)$ be piecewise differentiable functions. Thus,

$$
\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S}, \lambda_1(T) = 0, \\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial E}, \lambda_2(T) = 0, \\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial I}, \lambda_3(T) = 0, \\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial N}, \lambda_4(T) = 0.
\end{aligned} \tag{2.17}
$$

Evaluating (2.17) at the optimal and corresponding states will give (2.12) and (2.13). Lastly, the optimality condition,

$$\frac{\partial H}{\partial u} = 2u - \lambda_1 S, \tag{2.18}$$

$$\frac{\partial H}{\partial u} > 0 \Rightarrow u^*(t) = 0 \Rightarrow 2u - \lambda_1 S > 0 \Rightarrow S\lambda_1 < 0 \Rightarrow \frac{1}{2}S\lambda_1 < 0, \tag{2.19}$$

$$\frac{\partial H}{\partial u} < 0 \Rightarrow u^*(t) = 0.9 \Rightarrow 2u - \lambda_1 S < 0 \Rightarrow 1.8 < S\lambda_1 \Rightarrow \frac{1}{2}S\lambda_1 > 0.9, \tag{2.20}$$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow 0 \leq u^*(t) \leq 0.9 \Rightarrow 2u - \lambda_1 S = 0 \Rightarrow u^*(t) = \frac{1}{2}S\lambda_1 \Rightarrow 0 \leq \frac{1}{2}S\lambda_1 \leq 0.9. \tag{2.21}$$

Therefore, giving us (2.14) and completing the proof. □

It can be shown that the state $S(t), E(t), I(t), N(t)$ and the adjoint functions $\lambda_1(t), \ldots, \lambda_4(t)$ are all bounded. Furthermore, by the Lipschitz property of the ODE system, a unique optimal control $u^*(t)$ is obtained for small T. The uniqueness of the optimal control follows from the uniqueness of the optimality system, which consists of (2.8), (2.12) and (2.13) with the characterizations (2.14). For more information relating to uniqueness within optimal control problems refer to [23, 32].

2.3.2 Maximization of total population health with cost-effective control

Now, we present an alternative optimal control problem that shifts the focus to a broader public health strategy. This alternative approach is designed to maximize the overall healthy population $N(t)$, while managing the costs of control measures, over a fixed time period T. This shift in focus reflects a transition from an emergency response strategy, which is crucial during the acute phases of an epidemic for rapid containment, to a more sustainable public health policy. While the first scenario concentrates on the immediate reduction of the infection rates, balancing the immediate needs against available resources, the alternative scenario considers the long-term health of the entire population, including susceptible, exposed, and recovered groups, and weighs the economic implications of ongoing control measures. This dual perspective allows us to explore the trade-offs between aggressive, short-term measures aimed at immediate disease containment and more balanced, long-term strategies that seek to maintain overall population health within the constraints of public health budgets. By comparing these two models, insights can be gained into how different objectives influence resource allocation and health policy formulation, offering a comprehensive view of public health management that ranges from crisis management to sustainable disease control strategies.

Our second model, the objective cost functional, is maximized below

$$J(u) = \int_0^T AN(t) - u^2(t)\, dt. \tag{2.22}$$

For (2.22), we let $A > 0$ where A represents the "weight" on the cost and our term $AN(t)$ denotes the cost of the total population. We need to maximize our total healthy population N while keeping our vaccination cost $u^2(t)$ low. Also, since our optimal functional is concave, we can assume the cost of control to be nonlinear. The goal is to find an optimal control $u^*(t)$, such that

$$J(u^*) = \max_{U_1} J(u), \tag{2.23}$$

where

$$U_1 = \{u(t) : 0 \leq u(t) \leq 0.9, \text{Lebesgue measurable}\}. \tag{2.24}$$

Theorem 2.3.2. *There exists an optimal control $u^*(t)$ and the corresponding solutions ($S(t)$, $E(t)$, $I(t)$, and $N(t)$) of the system (2.8), that maximizes (2.22) over U_2. Furthermore, there exists adjoint functions, $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)$, such that*

$$\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S} = \lambda_1(d + cI + u) - \lambda_2(cI), \\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial E} = \lambda_2(e + d) - \lambda_3(e), \\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial I} = \lambda_1(cS) - \lambda_2(cS) + \lambda_3(g + a + d) + \lambda_4(a), \\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial N} = -A - \lambda_1(b) - \lambda_4(b - d),
\end{aligned} \tag{2.25}$$

with the transversality conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0, \tag{2.26}$$

and the control $u^*(t)$ satisfies the optimality condition

$$u^*(t) = \max\left\{0, \min\left\{-\frac{1}{2}S\lambda_1, 0.9\right\}\right\}. \tag{2.27}$$

Proof. Upon applying Theorem 2.2.1 and verifying the necessary conditions presented in Section 2.2, we proceed with the computation of the Hamiltonian, denoted as

$$H = AN(t) - u^2(t) + \sum_{i=1}^{4} \lambda_i f_i, \tag{2.28}$$

giving us,

$$
\begin{aligned}
H = \; & AN(t) - u^2(t) + \lambda_1(bN(t) - dS(t) - cS(t)I(t) - u(t)S(t)) \\
& + \lambda_2(cS(t)I(t) - (e+d)E(t)) \\
& + \lambda_3(eE(t) - (g+a+d)I(t)) \\
& + \lambda_4((b-d)N(t) - aI(t)).
\end{aligned} \tag{2.29}
$$

Now, we use the variables S, E, I, N and $R = N - S - E - I$. Should we choose to use the variable R, the adjoint function would turn out to be negative. Let $\lambda_1(t), \lambda_2(t), \lambda_3(t)$, and $\lambda_4(t)$ be piecewise differentiable functions. Thus,

$$
\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S}, \lambda_1(T) = 0, \\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial E}, \lambda_2(T) = 0, \\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial I}, \lambda_3(T) = 0, \\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial N}, \lambda_4(T) = 0.
\end{aligned} \tag{2.30}
$$

Evaluating (2.30) at the optimal and corresponding states will give (2.25) and (2.26). Lastly, the optimality condition,

$$\frac{\partial H}{\partial u} = -2u - \lambda_1(S), \tag{2.31}$$

$$\frac{\partial H}{\partial u} > 0 \Rightarrow u^*(t) = 0 \Rightarrow -2u - \lambda_1(S) > 0 \Rightarrow -2u > \lambda_1(S) \Rightarrow u < -\frac{1}{2}\lambda_1(S), \tag{2.32}$$

$$\frac{\partial H}{\partial u} < 0 \Rightarrow u^*(t) = 0.9 \Rightarrow -2u - \lambda_1(S) < 0 \Rightarrow -1.8 > \lambda_1(S) \Rightarrow u > -\frac{1}{2}\lambda_1(S) + 0.9, \tag{2.33}$$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow 0 \leq u^*(t) \leq 0.9 \Rightarrow -2u - \lambda_1(S) = 0 \Rightarrow u^*(t) = -\frac{1}{2}\lambda_1(S). \tag{2.34}$$

Therefore, giving us (2.27) and completing the proof. □

It can be shown that the state $S(t), E(t), I(t), N(t)$ and the adjoint functions $\lambda_1(t), \ldots, \lambda_4(t)$ are all bounded. Furthermore, by the Lipschitz property of the ODEs, a unique optimal control $u^*(t)$ is obtained for small T. The uniqueness of the optimal control follows from the uniqueness of the optimality system, which consists of (2.8), (2.25), and (2.26) with the characterizations (2.27). For more information relating to uniqueness within optimal control problems refer to [32].

2.4 Simulations

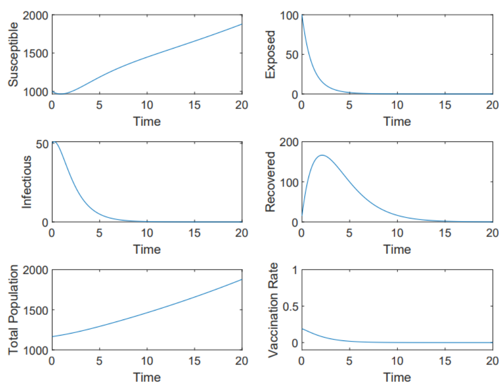
We now continue to run a series of simulations, each differentiated by distinct parameter values, to demonstrate the impact of varying rates on the epidemiological dynamics within our proposed population model. The results are visualized through MATLAB, we provide a collection of figures, and we display each optimal control problem side-by-side. As previously mentioned, this study focuses on two primary objectives: firstly, minimizing the count of infectious individuals alongside the aggregate vaccine cost within a predetermined timeframe, and secondly, maximizing the total vaccinated population in correlation with the overall vaccine expenditure during the same time period.

For our first simulation, our parameters are shown below in the table.

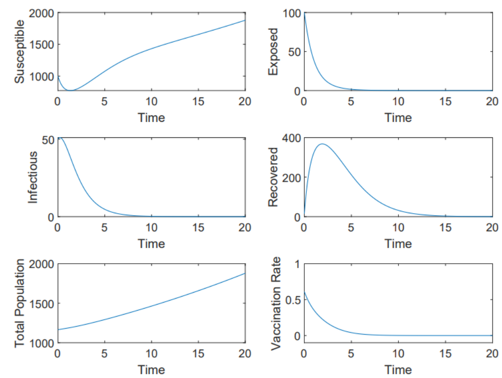
Table 2.1: SEIR - Simulation 1

Parameter	Description	Value
b	Natural Birth Rate	0.525
d	Natural Death Rate	0.5
c	Transmission Rate	0.0001
e	Exposure Rate	0.5
g	Recovery Rate	0.1
a	Death Rate of disease	0.2
S_0	Initial Susceptible	1000
E_0	Initial Exposed	100
I_0	Initial Infected	50
R_0	Initial Recovered	15
A	Weight	0.1
T	Final time (yrs)	20

The simulation spans 20 years, providing an opportunity to assess the long-term effects of our strategies.

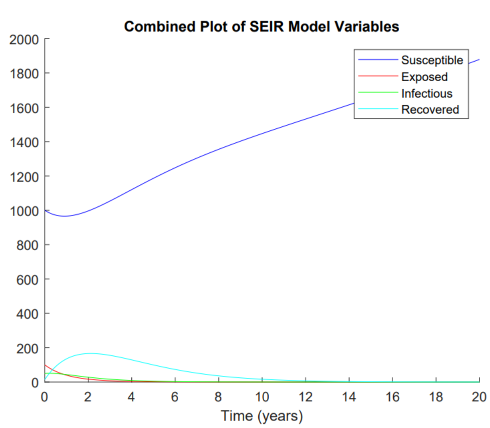


(a)

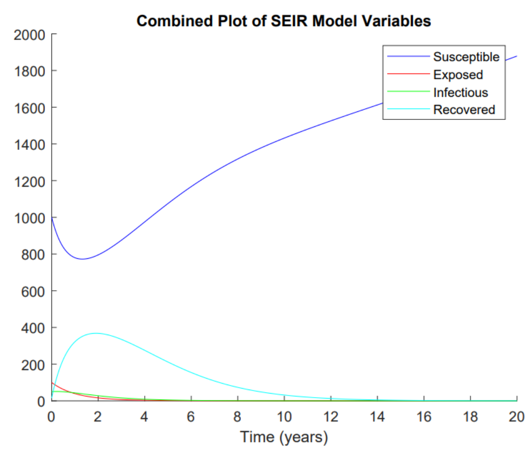


(b)

Figure 2.2: (a) Simulation 1 - 20 year simulation minimizing the number of infectious persons (b) Simulation 1 - 20 year simulation maximizing the total population that remains healthy



(a)



(b)

Figure 2.3: (a) Simulation 1 - SEIR model over a 20-year period minimizing the number of infectious persons (b) Simulation 1 - SEIR model over a 20-year period while maximizing the total population that remains healthy

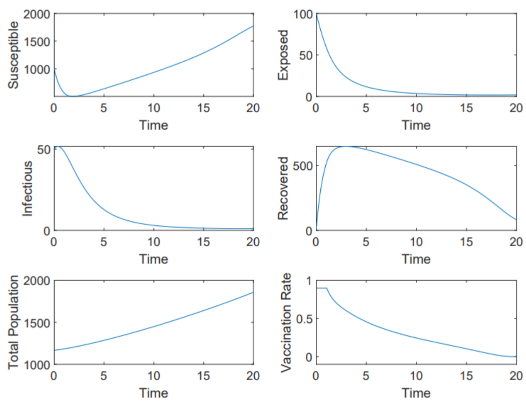
Analyzing the two simulation outcomes, it's clear that there is a significant difference in the recovered population's trajectory when the model's objective shifts from minimizing infectious individuals to maximizing the total population. Notably, there is a substantial increase in the recovered count during the second year in the maximization scenario. This increase implies that a larger fraction of the population has gained immunity, through recovery from the infection and an effective vaccination strategy. This is shown by the significantly higher vaccination rate in the model aiming to maximize the total population, which more than doubled compared to the model that minimizes infectious individuals. The rate increased from around 0.25 to 0.5, which contributed heavily to the success of the second optimal control scenario. Not only were the recovered and vaccination compartments affected, but the susceptible individuals also diminished rapidly, leading to a quicker and more pronounced decline in disease prevalence. This observation suggests that the model focusing on maximizing the total population is more effective in achieving a widespread immune response, thereby reducing the transmission potential of the disease. It also indicates a more aggressive and proactive approach to vaccination, leading to higher herd immunity levels and a more resilient population against the spread of the infection.

In the upcoming simulation, we modify the transmission rate, denoted by c, from 0.0001 to 0.001.

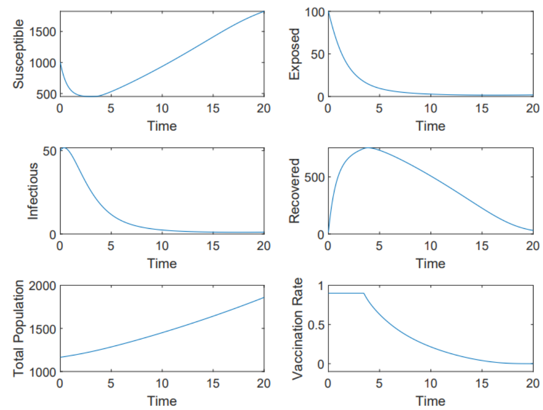
Table 2.2: SEIR - Simulation 2

Parameter	Description	Value
b	Natural Birth Rate	0.525
d	Natural Death Rate	0.5
c	Transmission Rate	0.001
e	Exposure Rate	0.5
g	Recovery Rate	0.1
a	Death Rate of disease	0.2
S_0	Initial Susceptible	1000
E_0	Initial Exposed	100
I_0	Initial Infected	50
R_0	Initial Recovered	15
A	Weight	0.1
T	Final time (yrs)	20

This change is expected to accelerate the disease's spread within the population. By increasing the transmission rate, we simulate a scenario where the disease is slightly more contagious, representing a situation where the pathogen has either evolved to be more transmissible or where social factors have led to increased contact rates among individuals. While maintaining the other parameters constant, we expect the disease to persist for a slightly longer duration due to the increased proportion of individuals susceptible to the infection. Despite this increase in transmissibility, we expect the population to stabilize over time, with both the exposed and infectious groups diminishing to zero.

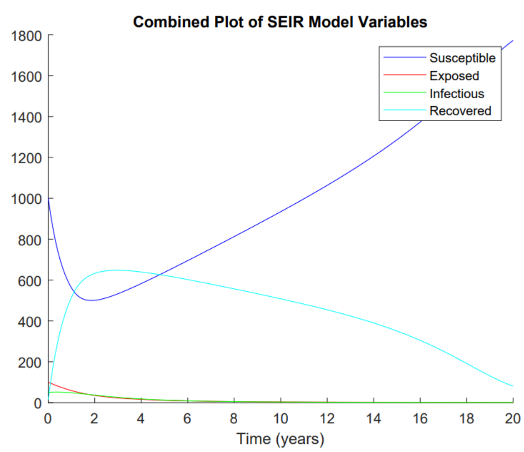


(a)

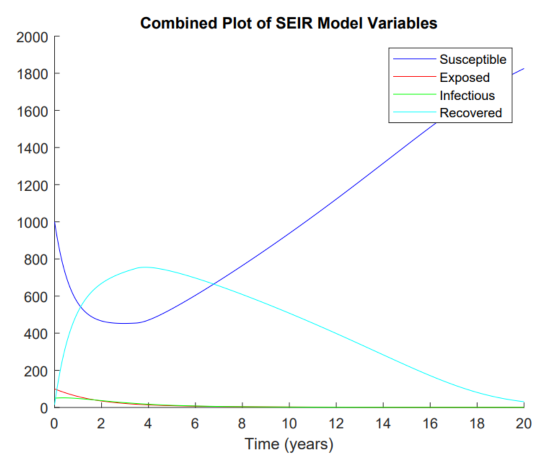


(b)

Figure 2.4: (a) Simulation 2 - 20 year simulation minimizing the number of infectious persons (b) Simulation 2 - 20 year simulation maximizing the total population that remains healthy



(a)



(b)

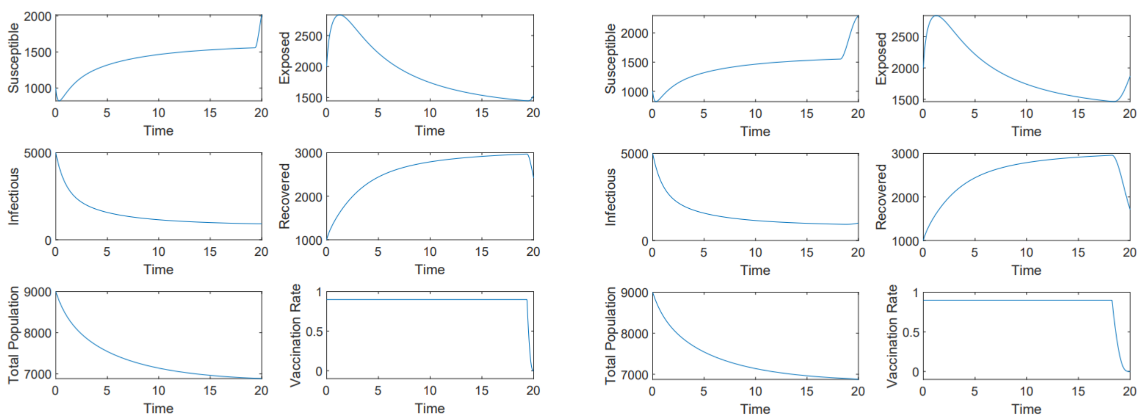
Figure 2.5: (a) Simulation 2 - SEIR model over a 20-year period minimizing the number of infectious persons (b) Simulation 2 - SEIR model over a 20-year period while maximizing the total population that remains healthy

These changes led to a more rapid spread of the disease compared to our first run, but the population soon stabilized due to effective vaccination efforts. Once again, the most significant change between the two optimal control problems is the higher number of recoveries and the dip in susceptible individuals shown in both figures. When maximizing the total healthy population with a low starting infected population, the second scenario will always be more effective than the first since there is already little infection to kill off. From Figure 4(b), we can see that the vaccination rate is at a higher rate for a longer period of time, which most likely is helping more individuals recover and thus fewer are susceptible.

For the final simulation, we modify the parameters to achieve an endemic equilibrium. The adjustments include altering our initial numbers of exposed, infectious, and recovered individuals to 2000, 5000, and 1000, respectively.

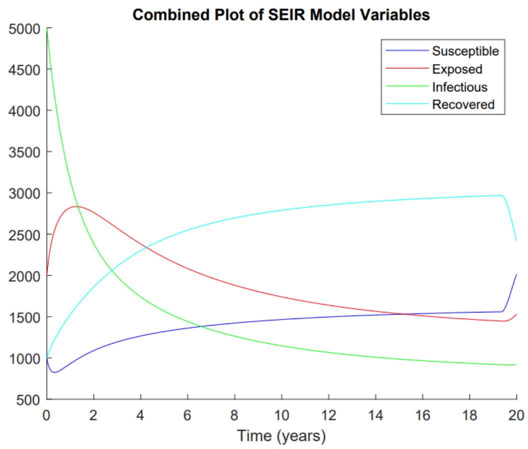
Table 2.3: SEIR - Simulation 3

Parameter	Description	Value
b	Natural Birth Rate	0.525
d	Natural Death Rate	0.5
c	Transmission Rate	0.001
e	Exposure Rate	0.5
g	Recovery Rate	0.1
a	Death Rate of disease	0.2
S_0	Initial Susceptible	1000
E_0	Initial Exposed	2000
I_0	Initial Infected	5000
R_0	Initial Recovered	1000
A	Weight	0.1
T	Final time (yrs)	20

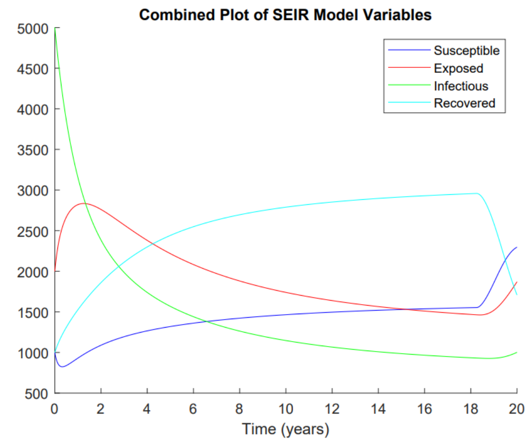


(a) (b)

Figure 2.6: (a) Simulation 3 - 20 year simulation minimizing the number of infectious persons (b) Simulation 3 - 20 year simulation maximizing the total population that remains healthy



(a)



(b)

Figure 2.7: (a) Simulation 3 - SEIR model over a 20-year period minimizing the number of infectious persons (b) Simulation 3 - SEIR model over a 20-year period while maximizing the total population that remains healthy

In this last simulation, we see the infection persist, and we reach an endemic equilibrium. The total population decreased significantly throughout the 20 years, while the infected compartment decreases but then stabilizes around decade mark. This is the result of the population starting vaccinations too late, which led to a gradual decrease in the overall population, despite nearly two decades of sustained, maximum vaccination efforts. The constant decline in population size clearly shows that by the time the vaccinations were in full swing, the infection had already spread too far. This emphasizes a crucial point for public health policy: the timing of vaccinations is key. Vaccinations are most effective when administered early in an epidemic; if the infection has already gathered pace, high rates of vaccination may not suffice to curb the spread. Comparing strategies, in this scenario, it appears that targeting the reduction of infection cases is marginally more effective than aiming to maximize the overall number of healthy individuals. This makes sense, given that we started with a higher number of infected and exposed individuals compared to a relatively healthy population. In the scenario which minimizes infected individuals, we see by the end of the time period, there is a slightly smaller number of infected and exposed individuals, with more people having recovered. Also, the vaccination rate lasts longer, unlike in the first two simulations where it was the opposite. This change happens because of the change in populations dynamics. The second optimal control problem is not equipped to handle the parameters in the third simulation compared to the first two simulations. While the first optimal control problem does not fully prevent the epidemic, it delays it further and even sees lower exposed and infected numbers compared to the second optimal control problem.

2.5 Conclusions

We established a solid foundation in the basic principles of optimal control theory and solving the necessary conditions using Pontryagin's Maximum Principle. This theoretical groundwork was crucial in framing our approach, allowing us to tailor the control problem to address the specific challenges posed by micro-parasitic infections. Optimal control in epidemic modeling can be better than traditional methods by dynamically adapting strategies based on evolving epidemic conditions and incorporating real-world constraints like resource limitations. This approach enhances the effectiveness and cost-efficiency of epidemic management strategies, making it more applicable to varying scenarios. We have also shown that we can focus on different compartments of the model to give us a better insight into which optimal control problem is best fit for the current situation based on the stage of the disease. Below, we provide a table combining (2.1), (2.2), and (2.3) that displays the changes in parameters for our different simulation for each of our optimal control scenarios:

Table 2.4: Comparison of Parameters Across Three Tables

Parameter	Description	Simulation 1	Simulation 2	Simulation 3
b	Natural Birth Rate	0.525	0.525	0.525
d	Natural Death Rate	0.5	0.5	0.5
c	Transmission Rate	0.0001	0.001	0.001
e	Exposure Rate	0.5	0.5	0.5
g	Recovery Rate	0.1	0.1	0.1
a	Death Rate of disease	0.2	0.2	0.2
S_0	Initial Susceptible	1000	1000	1000
E_0	Initial Exposed	100	100	2000
I_0	Initial Infected	50	50	5000
R_0	Initial Recovered	15	15	1000
A	Weight	0.1	0.1	0.1
T	Final time (yrs)	20	20	20

In our first simulation, with a low disease transmission rate ($c = 0.0001$), we found that starting vaccinations early and aggressively was very effective and even more effective with our optimal control problem relating to maximizing the total healthy population. There was almost double the amount of recovered people by the second year compared to minimizing infectious people. So, our second strategy quickly reduced the number of exposed and infected individuals, showing the importance of acting fast to control an epidemic. In our second simulation, we increased the transmission rate (c) to 0.001 and saw a change in how the disease affected the population. Initially, infections slightly rose, but over time, things balanced out. Yet, once again, maximizing the total healthy population proved more effective for individuals to recover, almost 200 more people recovered by the second year compared to our minimization problem and the disease died out quicker. The most significant changes came from our last simulation, where we delayed the start of vaccinations. Even though we vaccinated heavily for the majority of two decades, the population kept decreasing while the infection leveled out and persisted. This showed us how critical it is to start vaccinations early. When we waited too long, the disease spread too far, and even intense vaccination could not turn things around. In this final case, minimizing infectious individuals was more effective compared to maximizing the total healthy population. This makes sense because we started with more individuals in the infected compartment compared to our other simulations which makes minimizing the infected people more efficient in preventing the spread of the disease. Our maximization problem in this scenario has a clear increase in infected people by the end of the two decades as well as more exposed and susceptible people compared to our minimization problem.

We highlight the importance of public health, noting that the timing of interventions is as crucial as the amount of effort invested. The results show that when we start interventions, especially vaccinations, is key in determining how well we can manage an epidemic. Also, we can compare different compartments within the optimal control problem to find the most efficient way to stop an epidemic. This study establishes a foundation for future research, where similar methodologies could be tailored to specific diseases, whether bacterial, viral, fungal, or protozoan. By customizing our approach to particular pathogens, we can develop more effective strategies for disease prevention and control. For further exploration of utilizing optimal control theory to different epidemic models please see [23], [28], and [43].

CHAPTER 3: EVALUATING TREATMENT VERSUS VACCINATION THROUGH OPTIMAL CONTROL THEORY

3.1 Introduction

In the face of rising infectious diseases worldwide, it's increasingly crucial to find effective ways to control epidemics. Mathematical models play a key role in public health strategies by predicting how diseases spread and assessing the impact of interventions like vaccination and treatment. This chapter focuses on using optimal control theory-a method to determine the best strategies for controlling an epidemic. We specifically look at two models, SVIR (Susceptible-Vaccinated-Infected-Recovered) and SITR (Susceptible-Infected-Treated-Recovered), to understand how different approaches to vaccination and treatment can affect disease outcomes.

Our study begins with setting up an optimal control problem, aiming to minimize the costs related to disease spread and control efforts, guided by a set of differential equations that model the epidemic's dynamics. We employ Pontryagin's Maximum Principle to find the conditions that make a control strategy optimal. Through numerical simulations, we illustrate our theoretical findings, providing a practical perspective on managing disease spread.

Moreover, we explore an alternate scenario with high initial numbers of infections, offering insights into how the effectiveness of control measures might change under severe epidemic conditions. This approach reveals the adaptability of the different epidemic models with optimal control to address complex health challenges and the considerations needed when choosing one strategy over another.

SVIR and SITR epidemic models have scarcely been applied to optimal control, but here are some papers by Ramponi and Tessitore [40], Oke et al. [36], Mahrouf et al. [35], Witbooi et al. [46], Pan et al. [38], and Khatun et al. [29].

The results presented in this chapter are written as a research article that is currently being expanded to increase its novelty before submission to a journal [19].

3.2 Basic optimal control problem

In the context of our problem, we define the general minimization functional

$$\begin{aligned}
&\min_u J(u) = \min_u \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt, \\
&\text{subject to} \quad \dot{x}(t) = g(t, x(t), u(t)), \\
&\qquad\qquad\quad x(t_0) = x_0 \text{ and } x(t_1) \text{ free.}
\end{aligned} \tag{3.1}$$

Let $x(t)$ denote the state of the system at time t, and $u(t)$ represent the control, where t spans the interval $[t_0, t_f]$. The objective is to identify a piecewise continuous control policy $u^*(t)$ that minimizes the objective functional $J(u)$. A functional, in this context, is a mapping from a specific set of functions to real numbers, here represented as an integral, as illustrated in (3.1). In general, the objective functional depends on one or more of the state and the control variables [32]. This process also involves solving the Hamiltonian equation, a function that characterizes the total energy within a system. It aims to integrate the system's dynamics with the cost function to be optimized.

$$H = f(t, x(t), u(t)) + \lambda g(t, x(t), u(t)), \tag{3.2}$$

where the term λ represents the adjoint variable, reflecting the system's sensitivity to state changes, and $g(t, x(t), u(t))$ describes the system's dynamics under the influence of the control.

As we have talked about previously, a key component of optimal control theory is Pontryagin's Maximum Principle, which offers a first-order necessary condition for optimality. This principle is concisely reproduced here,

Theorem 3.2.1. *Let $u^* \in U$ be an optimal control. Then there exists an adjoint function $\lambda : \mathbb{R} \to \mathbb{R}^n$ such that $x(t, u^*), u^*, \lambda$ satisfy the state system*

$$\begin{cases}
\dot{x}(t) = g(t, x, u^*), \\
x(0) = x_0,
\end{cases} \tag{3.3}$$

and

$$H(t, x^*(t), u(t), \lambda(t)) \geq H(t, x^*(t), u^*(t), \lambda(t)), \tag{3.4}$$

where the Hamiltonian H is given by:

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u). \tag{3.5}$$

The adjoint system

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -(f_x + \lambda g_x), \\ \lambda(T) = 0, \text{ \textit{transversality condition}}. \end{cases} \tag{3.6}$$

To establish the uniqueness of an optimal control, it is necessary that the corresponding optimality system admits a unique solution. This system is said to exhibit the Lipschitz property when, within a specified domain, the function's rate of change with respect to its arguments does not exceed a certain threshold. Mathematically, an ordinary differential equation (ODE) $y' = f(x, y)$ satisfies the Lipschitz condition in a domain D if there exists a Lipschitz constant L such that, for every pair of points (x, y_1), (x, y_2) in D, the inequality $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ is satisfied. Utilizing the Picard-Lindelof theorem, the fulfillment of the Lipschitz condition ensures that for any given initial condition, there exists a unique solution to the ODE for any initial value.

3.3 SVIR vs. SITR: Optimizing epidemic control strategies

In this section, we explore two different epidemic models and optimal control problems: the SVIR and SITR models, where V stands for vaccination and T for treatment. Both models are framed as minimization problems aimed at reducing the number of infectious individuals. In the SVIR model, the control function $u(t)$, bounded by $0 \leq u(t) \leq 1$, represents the fraction of susceptible individuals that are vaccinated. In the SITR model, the control function $u(t)$, also within $0 \leq u(t) \leq 1$, denotes the fraction of infected and recovered individuals who are identified and will not only be treated but also strengthen those treatment strategies to decrease the potential for further infections. This study will investigate the effectiveness of each scenario-vaccination versus treatment-towards controlling the spread of infectious diseases, aiming to discern which strategy more effectively preserves the long-term health and well-being of the community.

3.3.1 Optimizing vaccination in the SVIR model

Now, we expand upon the basic SIR model by incorporating a "Vaccinated" category, denoting individuals who have been vaccinated but can still get infected by the disease at a much lower rate. Below, is a flowchart and our ODE system,

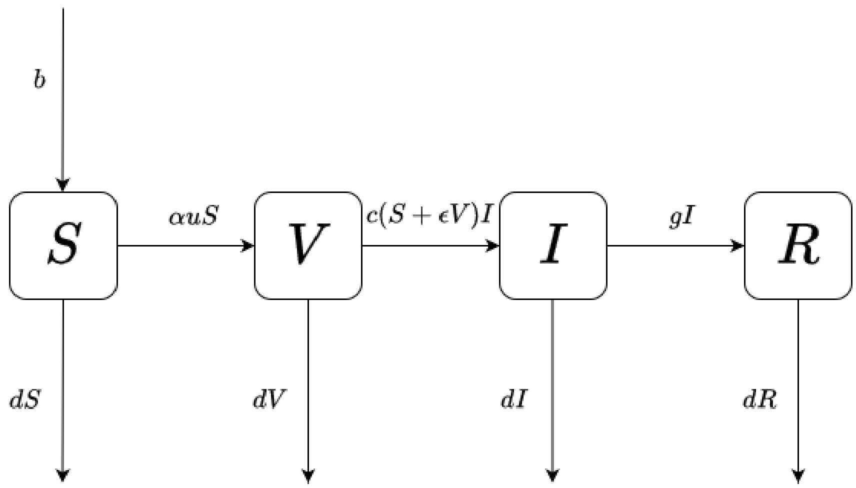


Figure 3.1: Flowchart of the SVIR model

$$
\begin{aligned}
\dot{S}(t) &= b - dS(t) - cS(t)I(t) - \alpha u(t)S(t), && S(0) = S_0 \geq 0,\\
\dot{V}(t) &= \alpha u(t)S(t) - \epsilon c V(t)I(t) - dV(t), && V(0) = V_0 \geq 0,\\
\dot{I}(t) &= c(S(t) + \epsilon V(t))I(t) - (g+d)I(t), && I(0) = I_0 \geq 0,\\
\dot{R}(t) &= gI(t) - dR(t), && R(0) = R_0 \geq 0.
\end{aligned}
\tag{3.7}
$$

In this mathematical model for studying disease spread, we define $S(t)$ as the number of susceptible people, $V(t)$ as those who have been vaccinated, $I(t)$ as individuals currently infected, and $R(t)$ as those who have either recovered or are naturally immune at any given time t. The term $u(t)$, referred to as the control, represents the rate at which susceptible people are vaccinated over time. We limit our control to - $0 \leq u(t) \leq 1$. We also consider b as the natural birth rate and d as the death rate of the population. The spread of the disease through contact between susceptible and infected individuals is captured by the term $cS(t)I(t)$. Moreover, we introduce parameters α, ϵ, and g to represent the rates of vaccination among susceptible people, the chance of vaccinated people becoming infected, and the recovery rate from the disease, respectively.

Our goal is to solve a problem that minimizes both the number of infections and the costs associated with vaccination over a specified time period, T. This involves finding the most efficient strategy for using vaccines to slow down and prevent the spread of the disease. By identifying and implementing this optimal strategy, we aim to control the disease more effectively and reduce the cost of vaccination. This optimization problem takes into account various factors and constraints to determine the best approach for protecting public health and reducing the economic impact of infectious diseases.

Our first model, the objective cost functional, is minimized below

$$J(u) = \int_0^T A_0 I(t) + \frac{A_1}{2} u^2(t)\, dt. \tag{3.8}$$

For (3.8), we let $A_0 > 0$ and $A_1 > 0$ where A_0, A_1 represents the "weight" on the cost and our term $A_0 I(t)$ denotes the cost of infection. Our goal is to minimize the number of infected individuals I while also keeping the cost of vaccination, represented by $u^2(t)$, as low as possible. The fact that our optimal function is convex allows us to consider the cost of implementing control measures as nonlinear. We aim to identify an optimal control strategy $u^*(t)$, such that

$$J(u^*) = \min_{U_1} J(u), \tag{3.9}$$

where

$$U_1 = \{u(t) : 0 \leq u(t) \leq 1, \text{Lebesgue measurable}\}. \tag{3.10}$$

Theorem 3.3.1. *There exists an optimal control $u^*(t)$ and the corresponding solutions ($S(t)$, $V(t)$, $I(t)$, and $R(t)$) of the system (3.7), that minimizes (3.8) over U_1. Furthermore, there exists adjoint* *functions, $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)$, such that*

$$\begin{aligned}
\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial S} = \lambda_1(d + cI + \alpha u) - \lambda_2(\alpha u) - \lambda_3(cI), \\
\dot{\lambda}_2(t) &= -\frac{\partial H}{\partial V} = \lambda_2(c\epsilon I + d) - \lambda_3(c\epsilon I), \\
\dot{\lambda}_3(t) &= -\frac{\partial H}{\partial I} = -A_0 + \lambda_1(cS) + \lambda_2(c\epsilon V) - \lambda_3(c(S + \epsilon V) - (g + d)) + \lambda_4(g), \\
\dot{\lambda}_4(t) &= -\frac{\partial H}{\partial R} = \lambda_4(d).
\end{aligned} \tag{3.11}$$

with the transversality conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0, \tag{3.12}$$

and the control $u^*(t)$ satisfies the optimality condition

$$u^*(t) = \max\left\{0, \min\left\{\frac{\alpha S(\lambda_1 - \lambda_2)}{A_1}, 1\right\}\right\}. \tag{3.13}$$

Proof. Upon applying Theorem 3.2.1 and verifying the necessary conditions presented in Section 3.2, we proceed with the computation of the Hamiltonian, denoted as

$$H = A_0 I(t) + \frac{A_1}{2} u^2(t) + \sum_{i=1}^{4} \lambda_i f_i, \tag{3.14}$$

giving us,

$$\begin{aligned}
H = \ & AI(t) + u^2(t) + \lambda_1(b - dS(t) - cS(t)I(t) - \alpha u(t)S(t)) \\
& + \lambda_2(\alpha u(t)S(t) - \epsilon cV(t)I(t) - dV(t)) \\
& + \lambda_3(c(S(t) + \epsilon V(t))I(t) - (g + d)I(t)) \\
& + \lambda_4(gI(t) - dR(t)).
\end{aligned} \tag{3.15}$$

Now, we use the variables S, V, I, R and $N = S - V - I - R$. Let $\lambda_1(t), \lambda_2(t), \lambda_3(t)$, and $\lambda_4(t)$

be piecewise differentiable functions. Thus,

$$\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S}, \lambda_1(T) = 0,\\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial V}, \lambda_2(T) = 0,\\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial I}, \lambda_3(T) = 0,\\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial R}, \lambda_4(T) = 0.
\end{aligned} \tag{3.16}$$

Evaluating (3.16) at the optimal and corresponding states will give (3.11) and (3.12). Lastly, the optimality condition,

$$\frac{\partial H}{\partial u} = A_1 u - \alpha(\lambda_1 - \lambda_2)S, \tag{3.17}$$

$$\begin{aligned}
\frac{\partial H}{\partial u} &= 0 \Rightarrow 0 \leq u^* \leq 1\\
&\Rightarrow A_1 u - \alpha(\lambda_1 - \lambda_2)S = 0\\
&\Rightarrow u^* = \frac{\alpha S(\lambda_1 - \lambda_2)}{A_1}\\
&\Rightarrow 0 \leq \frac{\alpha S(\lambda_1 - \lambda_2)}{A_1} \leq 1.
\end{aligned} \tag{3.18}$$

Therefore, giving us (3.13) and completing the proof. □

It can be demonstrated that the variables representing the state of the system-$S(t)$, $V(t)$, $I(t)$, $R(t)$-as well as the adjoint variables $\lambda_1(t)$ through $\lambda_4(t)$, are all bounded. Moreover, the uniqueness of the optimal control $u^*(t)$ for a small T is guaranteed by the Lipschitz continuity of the differential equations governing the system. The uniqueness of the optimal control follows from the uniqueness of the optimality system, which consists of (3.7), (3.11) and (3.12) with the characterizations (3.13). For more information relating to uniqueness within optimal control problems, refer to [23, 32].

3.3.2 Optimizing treatment in the SITR model

Next, we introduce our second epidemic model which also expands upon the basic SIR model by incorporating a "Treated" category, denoting individuals who receive treatment to recover from infection compared to getting a vaccine to prevent infection. Below, is a flowchart and our ODE system,

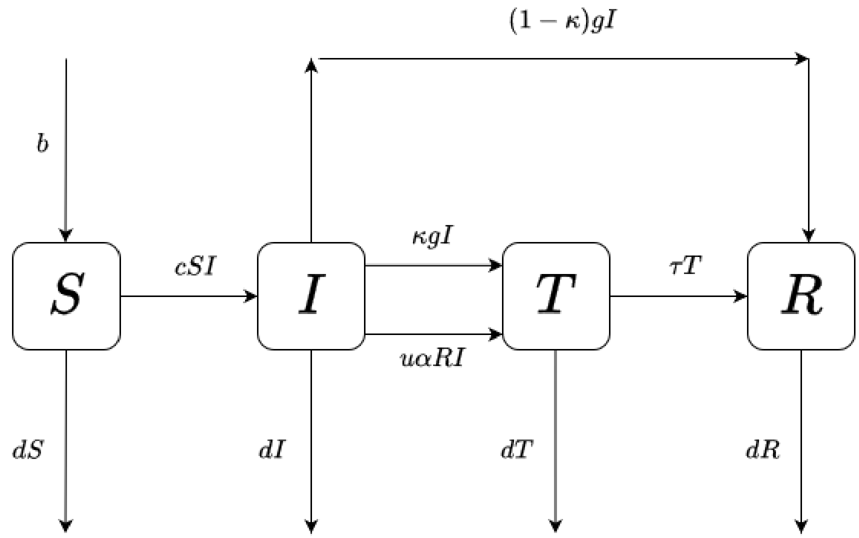


Figure 3.2: Flowchart of the SITR model

$$
\begin{aligned}
\dot{S}(t) &= b - dS(t) - cS(t)I(t), & S(0) = S_0 \geq 0,\\
\dot{I}(t) &= cS(t)I(t) - (g + d + u(t)\alpha R(t))I(t), & I(0) = I_0 \geq 0,\\
\dot{T}(t) &= (\kappa g + u(t)\alpha R(t))I(t) - \tau T(t) - dT(t), & T(0) = T_0 \geq 0,\\
\dot{R}(t) &= (1-\kappa)gI(t) - dR(t) + \tau T(t), & R(0) = R_0 \geq 0.
\end{aligned}
\tag{3.19}
$$

In this model for understanding how diseases spread, we use $S(t)$ to represent the number of susceptible people, $I(t)$ for those who are currently infected, and $R(t)$ for those who have recovered or are immune at any given time t. We've added $T(t)$ to show people who are being treated. The control variable $u(t)$ measures the rate at which the recovered population influences the treatment of the infected population, bound $0 \leq u(t) \leq 1$. We also have b for the birth rate and d for the death rate in the population. The term $cS(t)I(t)$ models the disease's spread from infected to susceptible individuals. Instead of using ϵ, this model uses α as the influence rate, κ to indicate the rate at which infected people begin treatment, and τ for how fast those being treated recover, along with g for the recovery rate of those not receiving treatment.

The main goal is to develop a treatment strategy that lowers both the infection rate and the costs associated with treatment over a certain period, T. We aim to find the best way to increase or strengthen different treatment approaches, $u^*(t)$, that effectively slows the disease's spread while being cost-effective. By identifying and implementing this optimal strategy, we want to improve disease control and minimize the expenses related to treatment. This process involves analyzing various factors and limitations to find the most effective way to protect public health and reduce the financial impact of infectious diseases.

Our second model, the objective cost functional, is minimized below

$$J(u) = \int_0^T A_0 I(t) + \frac{A_1}{2} u^2(t)\, dt. \tag{3.20}$$

For (3.20), we let $A_0 > 0$ and $A_1 > 0$ where A_0, A_1 represents the "weight" on the cost and our term $A_0 I(t)$ denotes the cost of infection. Our goal is to minimize the number of infected individuals I while also keeping the cost of strengthening treatment, represented by $u^2(t)$, as low as possible. The fact that our optimal function is convex allows us to consider the cost of implementing control measures as nonlinear. We aim to identify an optimal control strategy $u^*(t)$, such that

$$J(u^*) = \max_{U_1} J(u), \tag{3.21}$$

where

$$U_1 = \{u(t) : 0 \leq u(t) \leq 1, \text{Lebesgue measurable}\}. \tag{3.22}$$

Theorem 3.3.2. *There exists an optimal control $u^*(t)$ and the corresponding solutions ($S(t)$, $I(t)$, $T(t)$, and $R(t)$) of the system (3.19), that maximizes (3.20) over U_2. Furthermore, there exists* *adjoint functions,* $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)$*, such that*

$$\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S} = \lambda_1(d + cI) - \lambda_2(cI), \\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial I} = -A_0 + \lambda_1(cS) - \lambda_2(cS - g - d - u\alpha R) - \lambda_3(u\alpha R + \kappa g) + \lambda_4((1-\kappa)g), \\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial T} = \lambda_3(\tau + d) - \lambda_4(\tau), \\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial R} = \lambda_2(u\alpha I) - \lambda_3(u\alpha I) + \lambda_4(d),
\end{aligned} \tag{3.23}$$

with the transversality conditions

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0, \tag{3.24}$$

and the control $u^*(t)$ *satisfies the optimality condition*

$$u^*(t) = \max\left\{0, \min\left\{\frac{\lambda_2(\alpha RI) - \lambda_3(\alpha RI)}{A_1}, 1\right\}\right\}. \tag{3.25}$$

Proof. Upon applying Theorem 3.2.1 and verifying the necessary conditions presented in Section 3.2, we proceed with the computation of the Hamiltonian, denoted as

$$H = A_0 I(t) + \frac{A_1}{2}u^2(t) + \sum_{i=1}^{4} \lambda_i f_i, \tag{3.26}$$

giving us,

$$\begin{aligned}
H = \ & A_0 I(t) + \frac{A_1}{2}u^2(t) + \lambda_1(b - dS(t) - cS(t)I(t)) \\
& \qquad + \lambda_2(cS(t)I(t) - (g + d + u(t)\alpha R(t))I(t)) \\
& \qquad + \lambda_3((\kappa g + u(t)\alpha R(t))I(t) - \tau T(t) - dT(t)) \\
& \qquad + \lambda_4((1-\kappa)gI(t) - dR(t) + \tau T(t)).
\end{aligned} \tag{3.27}$$

Now, we use the variables S, I, T, R and $N = S - I - T - R$. Let $\lambda_1(t), \lambda_2(t), \lambda_3(t)$, and $\lambda_4(t)$

be piecewise differentiable functions. Thus,

$$\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S}, \lambda_1(T) = 0,\\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial I}, \lambda_2(T) = 0,\\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial T}, \lambda_3(T) = 0,\\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial R}, \lambda_4(T) = 0.
\end{aligned} \tag{3.28}$$

Evaluating (3.28) at the optimal and corresponding states will give (3.23) and (3.24). Lastly, the optimality condition,

$$\frac{\partial H}{\partial u} = A_1 u + \lambda_3(\alpha RI) - \lambda_2(\alpha RI), \tag{3.29}$$

$$\begin{aligned}
\frac{\partial H}{\partial u} &= 0 \Rightarrow 0 \leq u^* \leq 1\\
&\Rightarrow A_1 u + \lambda_3(\alpha RI) - \lambda_2(\kappa g I) = 0\\
&\Rightarrow u^* = \frac{\lambda_2(\alpha RI) - \lambda_3(\alpha RI)}{A_1}\\
&\Rightarrow 0 \leq \frac{\lambda_2(\alpha RI) - \lambda_3(\alpha RI)}{A_1} \leq 1.
\end{aligned} \tag{3.30}$$

Therefore, giving us (3.25) and completing the proof. □

It can be demonstrated that the variables representing the state of the system-$S(t)$, $I(t)$, $T(t)$, $R(t)$-as well as the adjoint variables $\lambda_1(t)$ through $\lambda_4(t)$, are all bounded. Moreover, the uniqueness of the optimal control $u^*(t)$ for a small T is guaranteed by the Lipschitz continuity of the differential equations governing the system. The uniqueness of the optimal control follows from the uniqueness of the optimality system, which consists of (3.19), (3.23) and (3.24) with the characterizations (3.25). For more information relating to uniqueness within optimal control problems, refer to [32].

3.4 Simulations

We move forward by carrying out various simulations, each with its own set of parameters, to investigate how different rates affect the spread of disease in our model of the population. These simulations are graphically represented using MATLAB, allowing us to visually compare different scenarios. In this part of our research, we shift our focus to a specific comparison: evaluating vaccination against treatment strategies in terms of their efficiency in minimizing the number of infectious people. This comparison is framed within the same optimal control problem, aiming to decrease infection rates. However, we also assess the cost-effectiveness and outcomes of choosing vaccination over treatment within the same period.

3.4.1 Impact of control measures

For our first simulation, our parameters are shown below in the table.

Table 3.1: SVIR - Simulation 1

Parameter	Description	Value
b	Natural birth rate	0.525
d	Natural death rate	0.5
c	Transmission rate	0.001
α	Vaccination rate	0.6
ϵ	Vaccinated transmission rate	0.3
g	Recovery rate	0.1
a	Death rate of disease	0.2
S_0	Initial susceptible	1000
V_0	Initial vaccinated	100
I_0	Initial infected	50
R_0	Initial recovered	15
A	Weight	1
T	Final time (yrs)	20

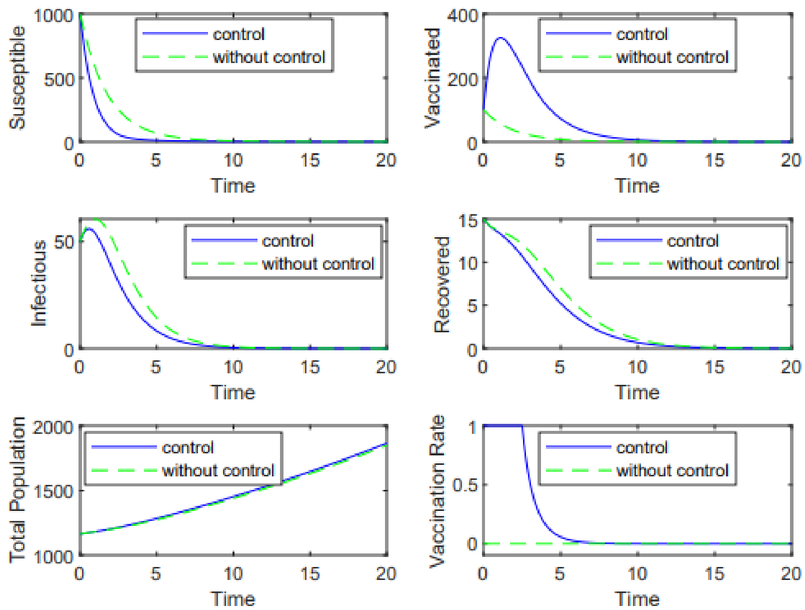


Figure 3.3: Simulation 1 - Comparing the model with control versus no control

The SVIR model results demonstrate the effectiveness of control measures in managing the spread of an infectious disease. Control strategies, influencing vaccination, show a significant reduction in the rate of susceptible individuals becoming infected, compared to the scenario without control. This suggests that interventions are effectively slowing the transmission. The increase in the vaccinated compartment under control measures is rapid and levels off, indicating a higher amount of the population getting vaccinated. The number of infectious cases has lowered when control measures are in place, peaking earlier and at a lower level, pointing to efficient containment of the disease. The graph for recovered individuals aligns with this, showing that less of the population needs to recover from the disease with control measures. In contrast, without control measures, the population experiences a higher peak of infections showing that more of the population needs to recover from the disease without control measures, implying that more individuals are being infected. The total population remains constant across both scenarios, as expected in a closed model. The steady vaccination rate under control measures highlights its pivotal role in disease control. Overall, the data supports the use of control measures as a critical component in reducing the spread of infections and aiding quicker recovery within the population.

Table 3.2: SITR - Simulation 1

Parameter	Description	Value
b	Natural birth rate	0.525
d	Natural death rate	0.5
c	Transmission rate	0.001
α	Influence Rate	0.01
κ	Treatment rate	0.4
τ	Treated recovery rate	0.8
g	Recovery rate	0.1
a	Death rate of disease	0.2
S_0	Initial susceptible	1000
I_0	Initial infected	50
T_0	Initial treated	100
R_0	Initial recovered	15
A	Weight	1
T	Final time (yrs)	20

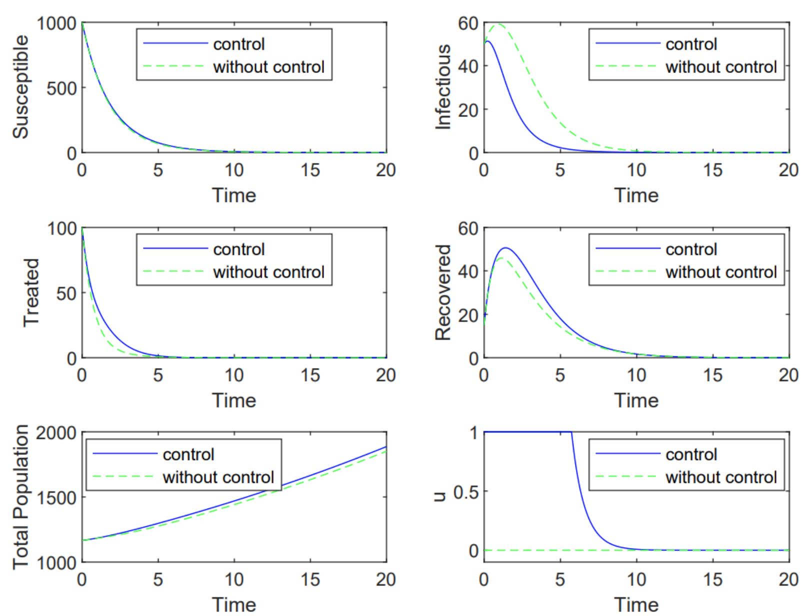


Figure 3.4: Simulation 1 - Comparing the model with control versus no control

In our first simulation using the SITR model, we find that control measures significantly affect the management of infectious diseases, proving to be more effective than the SVIR model. These measures, which focus on treatment over vaccination, lead to noticeable changes when implemented. We observe slight increases in the numbers of susceptible, treated, and total populations, highlighting the effectiveness of the control measures. More importantly, the number of infected individuals decreases quicker than the SVIR model, with the infection being eradicated within approximately five years. During this time, there is also a peak of recovered individuals, indicating a widespread recovery due to the treatment strategies that have developed over time. This shows that the control measures not only accelerate recovery, but also substantially alter the overall dynamics of the model.

The comparison between the SVIR and SITR models demonstrates the effectiveness of each in controlling infectious diseases, with the SITR model showing particularly strong results in reducing the number of infections. The SVIR model, which focuses on vaccination, effectively decreases the number of susceptible and infected individuals, highlighting the value of prevention. However, the SITR model, concentrating on treatment, significantly enhances recovery rates and lowers infection levels to a greater degree. This success is largely due to a specific control variable, alpha, that optimizes treatment strategies. This allows for faster treatment and leads to a large increase in recovered individuals, which helps to eliminate the disease more quickly than in the SVIR model. Therefore, while both models can effective, the SITR model's approach to rapid treatment, facilitated by strategic control over infections, proves to be a more efficient method for disease management, emphasizing the importance of targeted treatment strategies.

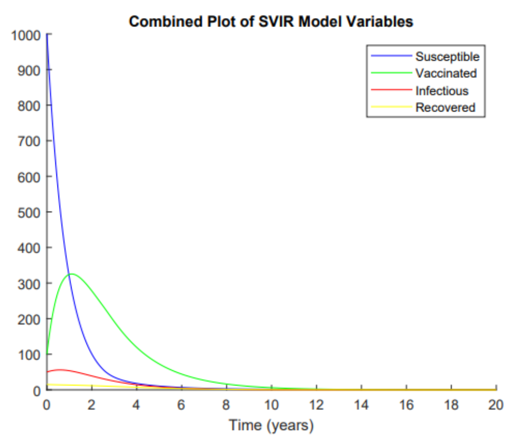


Figure 3.5: Simulation 1 - SVIR model over a 20-year period minimizing the number of infectious persons

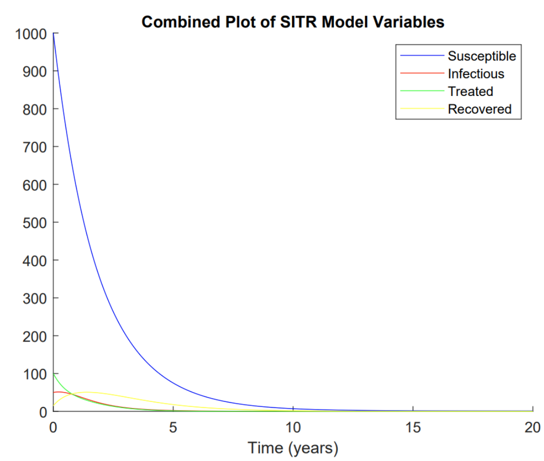


Figure 3.6: Simulation 1 - SITR model over a 20-year period minimizing the number of infectious persons

The graphs for the SVIR and SITR models display notable differences in the dynamics of disease progression and control. In the SVIR model, there is a pronounced peak in the vaccinated population, which correlates with a lower peak in the number of infected individuals. This indicates that the vaccination campaign was successful in preemptively reducing the spread of the infection, leading to fewer people needing to recover from the disease.

The SITR model demonstrates a different trend: rather than showing a significant peak, the number of treated individuals decreases steadily over time. This indicates that the implemented treatment and recovery strategies effectively reduced the spread of the infection, leading to fewer individuals requiring treatment. As a result, the SITR model exhibits fewer infected individuals and, consequently, a substantial increase in recovered individuals, attributed to the rapid decline of the disease.

While both the SVIR and SITR models offer effective approaches to disease management, the SITR model, with its focused treatment and recovery strategies, demonstrates a notable advantage in controlling and ultimately eliminating the disease. The model's ability to quickly adapt treatment protocols leads to a significant decrease in infections and a rapid increase in recoveries, outperforming the SVIR model in terms of reducing disease prevalence. This suggests that in scenarios where effective treatment can be rapidly deployed, the SITR model's approach of directly addressing infections can be more effective than preventative vaccination strategies. Therefore, for diseases where quick and effective treatment options are available, integrating the SITR model's treatment-based control strategies has proven to be more successful for eliminating the disease, optimizing health outcomes more effectively than vaccination alone.

3.4.2 Model responses to high initial infections

In this section, we analyze the SVIR and SITR models when faced with a high number of initial infections. This involves adjusting the initial conditions to include a significant number of initial infections, equal to the susceptible population, alongside 250 individuals who are either treated (in the SITR model) or vaccinated (in the SVIR model), and 500 recovered individuals. This setup is designed to mirror the later stages of an epidemic, providing insights into how vaccination (SVIR) and treatment (SITR) strategies perform under significant epidemic pressure. By comparing the outcomes with and without control measures in both models, we aim to assess their effectiveness and identify potential challenges in managing a widespread disease outbreak. In addition to the adjustments already outlined for the high infection scenario, this analysis will further involve lowering the influence level (α) in the SITR model to account for the observation that control measures in the SITR model exhibit a more pronounced impact than those in the SVIR model.

Table 3.3: SVIR - Simulation 2

Parameter	Description	Value
b	Natural birth rate	0.525
d	Natural death rate	0.5
c	Transmission rate	0.001
α	Vaccination rate	0.6
ϵ	Vaccinated transmission rate	0.3
g	Recovery rate	0.1
a	Death rate of disease	0.2
S_0	Initial susceptible	1000
V_0	Initial vaccinated	250
I_0	Initial infected	1000
R_0	Initial recovered	500
A	Weight	1
T	Final time (yrs)	20

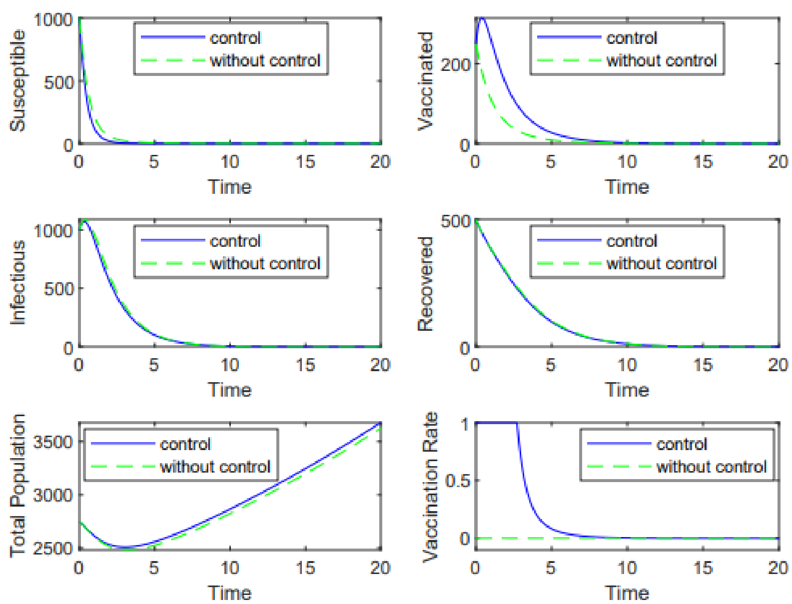


Figure 3.7: Simulation 2 - Comparing the model with control versus no control

The SVIR model outcomes, when simulated with a high initial number of infections, provide insights into the control measures relative to the high infection. In this second simulation, unlike the previous scenario, the total population experiences a slight decline, attributable to the high initial infection count. This indicates a more pronounced impact of the disease on overall population numbers. Despite control efforts, the decrease in susceptible and infected compartments is not significant as seen with lower initial infections, suggesting a dampened effectiveness of the control measures in this more challenging scenario. The vaccinated group reaches a higher peak more rapidly under control measures than without, showing an aggressive response to the heightened risk. However, the number of recovered individuals does not significantly diverge from the control and non-control scenarios initially, reflecting the struggle to contain the spread among a highly infected population. Over time, the population does recover and stabilizes. This stabilization hints at a delayed, yet eventual, equilibrium brought about by persistent vaccination efforts, despite the initial setbacks in controlling the spread among susceptible and infected individuals.

Table 3.4: SITR - Simulation 2

Parameter	Description	Value
b	Natural birth rate	0.525
d	Natural death rate	0.5
c	Transmission rate	0.001
α	Influence Rate	0.001
κ	Treatment rate	0.4
τ	Treated recovery rate	0.8
g	Recovery rate	0.1
a	Death rate of disease	0.2
S_0	Initial susceptible	1000
I_0	Initial infected	1000
T_0	Initial treated	250
R_0	Initial recovered	500
A	Weight	1
T	Final time (yrs)	20

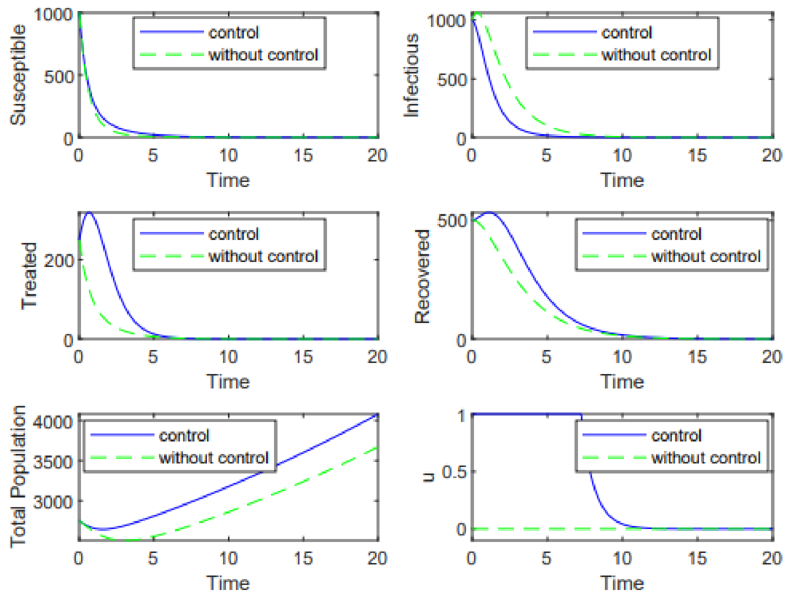


Figure 3.8: Simulation 2 - Comparing the model with control versus no control

In the updated simulation using the SITR model, the control measures' influence on disease management is stronger in contrast to the SVIR model, especially in a scenario with a high initial number of infections. Notably, the total population initially dips slightly under control measures but then makes a strong recovery, more pronounced than that observed in the SVIR model. The treated population experiences a substantial peak, reflecting an intense and effective treatment campaign that surpasses the impact of vaccination strategies in the SVIR model. Furthermore, there is a significant increase in the recovered population, which is thanks to the control which increases both the treatment and thus more recovered. While the infection takes longer to die out compared to the first simulation, the control measures help assist the population to a disease-free state more effectively than in the SVIR model. This highlights that, despite a longer duration of infection presence, the control strategies in the SITR model result in a stronger containment and eventual elimination of the disease compared to that of the SVIR model which struggled to adapt more.

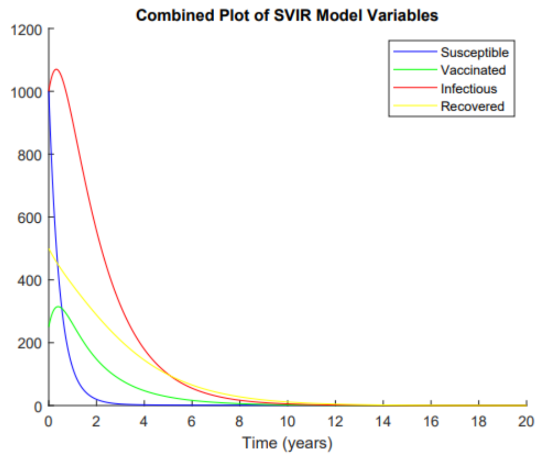


Figure 3.9: Simulation 2 - SVIR model over a 20-year period minimizing the number of infectious persons

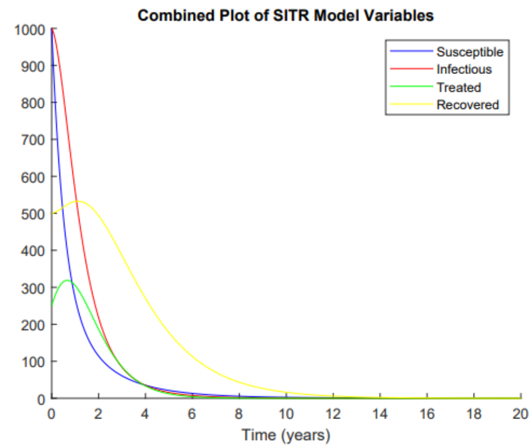


Figure 3.10: Simulation 2 - SITR model over a 20-year period minimizing the number of infectious persons

In the comparative analysis of the second simulation for the SVIR and SITR models, the graphs reveal different trends and outcomes. For the SVIR model, we observe a higher number of susceptible individuals, which is consistent with a larger infected population compared to the SITR model. Notably, the SVIR model exhibits a peak in infections that is not present in the SITR model, and the infection persists for an additional four years before dying out.

On the other hand, the SITR model shows a pronounced peak in the treated population, which directly aligns with a substantial increase in recovered individuals-a reflection of the control that affects both compartments. This peak, mirrored in both the treated and vaccinated populations of the respective models, highlights a key difference: in the SITR model, the treated group is closely associated with the recovered group, resulting in a more significant recovery outcome.

The SVIR model, while showing effectiveness in its preventative strategy, lags slightly in comparison to the SITR model, where the infection control is more assertive, leading to quicker recovery rates. The SITR model, with its direct treatment approach, demonstrates a notable advantage in reducing the number of susceptible and infected individuals and facilitating a stronger recovery response. This suggests that despite identical peaks in control measures (vaccinated in SVIR and treated in SITR), the treatment-based strategies in the SITR model lead to a more pronounced impact on recovery figures, showcasing its robustness in managing the epidemic, particularly in scenarios where a high number of infections have already occurred.

3.5 Conclusions

In this chapter, we explored how optimal control theory can help manage infectious diseases, focusing on two models: SVIR and SITR. These models allowed us to compare different strategies for vaccination and treatment, aiming to find a balance between reducing infections and the costs involved. By applying mathematical principles and running simulations, we gained insights into how effective these strategies could be under various conditions, including when facing a high number of initial infections. These simulations help determine which model had more effectiveness with control depending on the parameters used. Below, we provide two tables combining (3.1) and (3.2), then (3.3) and (3.4) that displays the changes in parameters for our different simulation for each of our optimal control scenarios:

Table 3.5: SVIR combined parameters

Parameter	Description	Simulation 1	Simulation 2
b	Natural birth rate	0.525	0.525
d	Natural death rate	0.5	0.5
c	Transmission rate	0.001	0.001
α	Vaccination rate	0.6	0.6
ϵ	Vaccinated transmission rate	0.3	0.3
g	Recovery rate	0.1	0.1
a	Death rate of disease	0.2	0.2
S_0	Initial susceptible	1000	1000
V_0	Initial vaccinated	100	250
I_0	Initial infected	50	1000
R_0	Initial recovered	15	500
A	Weight	1	1
T	Final time (yrs)	20	20

Table 3.6: SITR combined parameters

Parameter	Description	Simulation 1	Simulation 2
b	Natural birth rate	0.525	0.525
d	Natural death rate	0.5	0.5
c	Transmission rate	0.001	0.001
α	Influence Rate	0.01	0.01
κ	Treatment rate	0.4	0.4
τ	Treated recovery rate	0.8	0.8
g	Recovery rate	0.1	0.1
a	Death rate of disease	0.2	0.2
S_0	Initial susceptible	1000	1000
I_0	Initial infected	50	1000
T_0	Initial treated	100	250
R_0	Initial recovered	15	500
A	Weight	1	1
T	Final time (yrs)	20	20

Our study delved into how infectious diseases can be managed through the SVIR and SITR models, focusing on the roles of vaccination and treatment. The SVIR model highlights the effectiveness of vaccinations in preventing disease spread. It shows a notable decrease in transmission rates among those susceptible when a vaccination strategy is deployed, leading to a quick increase in the number of vaccinated individuals. This not only slows down the disease spread but also lessens the need for treatments and recoveries, as fewer people get infected. This outcome highlights vaccination as a crucial strategy for controlling disease outbreaks efficiently, reducing the overall burden on healthcare systems.

On the other hand, the SITR model puts a spotlight on treatment strategies. Here, direct interventions to treat infected individuals significantly cut down the infection rates and speed up recovery times. This model changes the course of the epidemic, suggesting that focusing on treating the infected can control and even stop the spread of the disease more swiftly than vaccination strategies, as seen in our simulation outcomes. This direct approach not only aids in faster recovery for the infected but also stabilizes the health of the population more broadly. The strategic application of treatment, particularly through optimizing control measures, shows that focusing on treatment can be a powerful method to improve recovery rates and lower infection levels, offering an effective alternative to prevention through vaccination.

When comparing both models under high initial infection scenarios, we gain further insights. Adjusting the models to simulate a significant initial infection load offers a closer look at how each strategy performs under pressure. The SVIR model faces challenges in controlling the disease spread in a highly infected population, slightly reducing the total population. Meanwhile, the SITR model's focus on treatment showcases a robust response, with the population making a notable recovery. This demonstrates the treatment strategy's effectiveness, particularly when facing a large number of initial infections, highlighting its potential to more efficiently manage disease outbreaks compared to vaccination efforts in the SVIR model.

Our analysis reveals the differences and potential of vaccination and treatment in disease management. Each approach has its strengths, influenced by the epidemic's context, such as the initial number of infections and available interventions. Our findings suggest that treatment-based strategies might offer a quicker and more efficient way to control and eradicate diseases when rapid treatment options are available. Conversely, vaccination remains crucial for preventing disease spread, especially in scenarios aiming to prevent widespread infection.

In conclusion, this study shows the importance of different approaches in combating infectious diseases. By understanding the specific conditions of an epidemic and effectively utilizing both vaccination and treatment strategies, public health policies can be better designed to manage and ultimately eliminate infectious diseases, ensuring the health and safety of populations against the threat of future outbreaks.

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VITA

Spalding Garakani was born and raised in San Antonio, TX. After graduating from Winston Churchill High school in 2019, Spalding attended the University of Texas at San Antonio to study mathematics. In 2022, Spalding graduated with a Bachelor of Science in Mathematics, earning Summa Cum Laude honors. Subsequently, he enrolled in the Master of Science in Mathematics program at the University of Texas at San Antonio. As a graduate student, Spalding focused his studies on optimal control and modeling infectious diseases. Moreover, Spalding has recently accepted an offer of admission from Texas A&M University doctoral program in mathematics where he intends to start in the Fall of 2024 and focus on optimal control, dynamical systems, and modeling infectious diseaes.