

Compatible Growth and Yield Models For Loblolly Pine

BY
JEROME L. CLUTTER

Abstract. Past studies in loblolly pine mensuration have resulted in the development of numerous analytic models for growth and yield. Most have treated growth and yield as essentially independent phenomena with no attempt to develop models that possess the logical compatibility which must exist between growth and yield observations. The present study develops models intended to express realistically the relationships between growth and yield. As an initial step, previously-used basal area and cubic-foot yield models were examined and screened for use in the analysis. The yield models selected for use were differentiated with respect to age to produce models for cubic-foot and basal area growth. The resulting growth models, together with the selected model for cubic-foot yield, were then fitted to data obtained in the five-and-ten-year remeasurements of 102 permanent sample plots located in three states. Equations were developed for cubic-foot yield, basal area growth, cubic-foot growth, basal area projection, and cubic-foot volume projection. Three of these equations can be used to predict total per acre production for various rotation ages and thinning regimes.

THE MOST IMPORTANT species in the forest industries of the South is loblolly pine (*Pinus taeda* L.), with a total growing stock volume of nearly 22 billion cubic feet and a range extending through the Piedmont and the Atlantic and Gulf Coastal Plains from Maryland to eastern Texas (Wahlenberg 1960). Throughout this area, loblolly pine is highly favored as a valuable species well suited to intensive forest management.

As might be expected from its commercial importance, considerable attention has been devoted to the development of methods for predicting growth and yield in loblolly pine. Although most of the growth and yield tables published for the species have been based on measurements of yield on temporary plots, a few studies have used remeasured sample plots. Data for the present analyses were obtained from one of the larger of these permanent plot installations—a loblolly pine stand density study installed by the Southeastern Forest Experi-

ment Station during the period 1948 to 1950. This study consists of 152 permanent sample plots in loblolly pine stands on the Hitchiti, Camp, Santee, and Westvaco Experimental Forests in Georgia, Virginia, and South Carolina. (Not all of these plots were used in the analysis reported here).

Two five-year remeasurements of the stand density study plots have now been

The author, formerly on the staff of the Southeastern Forest Expt. Sta., Asheville, N. C., is now employed by the Research Analysis Corp., Bethesda, Md.

This paper is part of a dissertation submitted in partial fulfillment of the requirements for the D. F. degree, School of Forestry, Duke Univ., Durham, N. C. The cooperation of the Georgia Kraft Company, the Union Bag-Camp Paper Company, and the West Virginia Pulp and Paper Company is gratefully acknowledged. Cooperators' assistance in providing study areas, locating plots, and in professional aid contributed greatly to the success of the study. Manuscript received May 2, 1962.

completed. An analysis of the data obtained at the first remeasurement was reported by Wenger *et al.* (1958) and partial results from the second remeasurement analysis have been given by Nelson (1960).

In neither of these papers, nor in any other published analysis of permanent plot growth data for loblolly pine, has there been any attempt to relate the models used for growth analysis to the previously accepted yield models for the species. The present study was begun to develop compatible analytic models for total-stand cubic-foot growth and yield in loblolly pine. Such models are here defined as compatible when the yield model can be obtained by summation of the predicted growth through the appropriate growth periods or, more precisely, when the algebraic form of the yield model can be derived by mathematical integration of the growth model. It is recognized that total-stand cubic-foot volume estimates are probably not as useful as estimates of merchantable cubic-foot volume or board-foot volume. However, as Bruce and Schumacher (1950) point out, the introduction of merchantability limits can generally be expected to increase the difficulties of prediction. Total stand volume has therefore been chosen for this study as the proper point of beginning with the hope that future investigations will extend any usable results to other volume measures.

Several benefits should result from use of such compatible models. Perhaps the most obvious is the logical consistency that would be obtained in separate growth and yield analyses. The user of growth and yield information is confused when confronted by a growth analysis that, upon summation, produces an entirely different yield surface from that obtained in an independent yield analysis.

A second advantage of compatible models would be realized through the use of existing knowledge of yield models to suggest appropriate growth analysis models. Many years of experience have been accumulated in yield analysis techniques, whereas the statistical analysis of permanent plot

growth data is a relatively new area of investigation.

Finally, as experience with compatible models accumulates, it should be possible to develop considerably more efficient estimating techniques for growth and yield than those presently available. Such new techniques are a distinct possibility where growth and yield observations are not considered as independent phenomena but are analyzed as mathematically dependent quantities.

Previous Work

Mensurational studies of loblolly pine began with the construction of normal yield tables. In the preparation of such tables, plot selection was restricted to stands of so-called "normal" stocking¹ and the yields of these normally stocked stands were analyzed by graphic techniques to determine their relationship to site and age. The first widely used normal yield tables for loblolly pine were contained in Miscellaneous Publication 50 produced by the U. S. Forest Service (1929). Similar tables, based on a considerably more restricted sample, were also prepared by Meyer (1942).

Difficulties in the application of the normal yield tables became apparent soon after their introduction. These difficulties stemmed from attempts to use the tables for growth projection purposes. Instructions contained in Miscellaneous Publication 50 for the prediction of future yields called for the reduction of the normal yield at the prediction age by the percentage that the present stand differed from normal stocking. Such a static concept of stocking was felt to be erroneous by many foresters and attempts were undertaken to clarify the situation.

Gevorkiantz (1934) discussed the ques-

¹Editor's note: For an understanding of the early standards in use for the collection of data and construction of yield tables in the U. S. see "Methods of preparing volume and yield tables." J. For. 24:653-666. 1926.

tion of the approach toward normality of understocked stands and suggested the use of a formula introduced by Gerhardt to estimate the trend of such stocking changes, and the definite existence of a stocking trend toward normality was demonstrated by Simmons and Schnur (1937) in their analysis of remeasured loblolly pine plots in Maryland. This analysis showed significant correlations between percentage growth of basal area and age, stand density as measured by stand density index (Reineke 1933), and initial basal area.

In 1939, a significant generalization of the normal yield table procedure was achieved with the development by MacKinney and Chaiken (1939) of empirical yield tables for loblolly pine. These tables were developed from sample plots with a wide range of stocking, using multiple regression techniques in place of the earlier graphic methods. Stand density was utilized as a third independent variable in addition to age and site. Data from this study were also used by Chaiken (1939) to estimate the trend toward normality of stand density. These estimates were obtained by sorting the densities obtained from the temporary sample plots into age groups and plotting means for stocking sextiles over age. Connection of the corresponding sextile means across ages produced the stocking trend curves. A similar procedure was employed by Wellwood (1943) to re-analyze the same data using regression techniques rather than graphics.

A more recent example of empirical yield table methods is offered by the loblolly pine yield tables prepared by Schumacher and Coile (1960). These tables are based on the development of equations describing the relation of various stand parameters, such as basal area, number of trees, and height to age, site index, and the stocking percentage that is a modification of the earlier concept of tree-area ratio (Chisman and Schumacher 1940). Volume estimates are obtained through the relationship of volume to height and average diameter. Growth projection with the

Schumacher and Coile tables is accomplished by the use of formulas that predict future height, stocking percentage, and number of trees from present age, future age, present stocking percentage, and site index. Although the formula for trend of stocking percentage with stand age was verified from records of 55 permanent plots, the majority of the results are based upon temporary plot data.

The studies utilizing data from permanent plots for growth and yield estimation in loblolly pine are of particular pertinence to the problems being explored in this investigation. Two earlier studies which involved permanent plot data were those of Simmons and Schnur (1937) and Meyer (1942).

In a more recent study, Wenger *et al.* (1958) analyzed the five-year remeasurement data from the Southeastern Forest Experiment Station's loblolly pine stand density study. The prediction equations developed for merchantable cubic-foot growth were:

For thinned stands

$$PAG = 272.08 - 2.47S - 5.72D + 0.074DS$$

For unthinned stands

$$PAG = -91.36 + 4908.0A^{-1} - 0.075S - 51.79DA^{-1} + .0289DS$$

where

PAG = periodic five-year annual growth in cubic feet per acre of trees 4.6 inches dbh and larger

A = initial stand age

S = site index

D = density percent according to methods given by Stahelin (1949).

Neither of these equations accounted for more than 40 percent of the variation in growth.

The ten-year remeasurement of this same study contained an increased amount of data from thinned plots and resulted in the development of a considerably more

precise estimating equation. The regression developed by Nelson (1960) for merchantable cubic-foot growth in thinned stands was

$$PAG = 205.0 + 2600.5A^{-1} \\ - 8.11990S - .03087D^2 \\ + .05824DS + .05328S^2$$

which accounted for over 78 percent of the variation in growth.

An analysis similar to that performed with the stand density study data has been reported by Brender (1960) who worked with remeasurements of 202 loblolly pine stands on the Hitchiti Experimental Forest in Georgia. Using expressions involving age, site, and basal area as independent variables, Brender accounted for 68 percent of the variation in merchantable cubic-foot growth with the regression

$$PAG = -20.748 + 6.64535B \\ - .02529B^2 \\ + 50260.6A^{-1}S^{-1} \\ - 235.034BS^{-1}$$

where

PAG = periodic annual growth of trees 5.6 inches dbh and larger

B = basal area per acre of trees 5.6 inches dbh and larger

S = Site index

A = age.

Consideration of this previous experience in loblolly pine mensuration suggests two points of particular importance to the present study. The first of these is the lack of logical connection between growth formulas developed from permanent plot studies and yield models used in the previous analyses of temporary plot data. A second point of interest is that of the loblolly pine mensurational studies already mentioned, only the permanent plot analysis by Brender (1960) measured stand density on an absolute scale. Analyses have been performed by Nelson and Brender (1963) to investigate the effect of substituting basal area for density percent in the growth

model used in the analysis of the stand density study ten-year remeasurement data. However, the results of these analyses were not available at the time this paper was being prepared.

In the remaining studies, density has been expressed by comparison to some standard (usually as a percent of either normal stocking or average stocking). Whether it is necessary, or even desirable, to express density on some relative scale appears to be an open question (Bickford *et al.* 1957). In the present study, stand density has been consistently expressed on absolute rather than relative scales. The success or failure of the present analysis should, therefore, provide useful information regarding the utility of absolute scales for the measurement of density.

Data Collection and Summarization

Basic data for the analyses described in this study were obtained from the five- and ten-year remeasurements on 102 thinned permanent plots in the loblolly pine stand density study. In approximately half of these plots, cutting was begun at the time of plot installation, while the remaining stands received initial thinning at the time of the five-year remeasurement. Cutting specifications called for thinning from below but permitted the removal of defective and deformed trees from all crown classes. Cutting in the upper crown classes was also permissible where necessary to achieve a uniform spacing of the residual stand.

The study plots, .25 acre in size surrounded by .5-chain isolation strips, were selectively located to provide uniform sampling intensity throughout the ranges of age, site, and density. The 102 plots ranged from 21 to 69 years of age at the time of the five-year remeasurements. Site index varied from 53 to 110; basal area at the beginning of the second five-year growth period ranged from 30 to 154 square feet per acre. Plot installation specifications required that the selected plots contain essentially pure stands of even-

aged, uniformly spaced, insect- and disease-free loblolly pine.

At each measurement time, the diameters of all trees 1.0 inch dbh and larger were recorded, together with total height and age data for at least ten sample trees on each plot. Summarization of these data produced the following plot parameters:

1. Average age during the growth period—obtained from increment borings of dominant and codominant sample trees and computed as the average of the initial and final ages.

2. Site index—obtained from average age and average total height of dominant and codominant sample trees using site index curves published by Coile (1952).

3. Average basal area during the growth period—computed as the average of the initial and final basal area per acre for all stems 1.0 inch dbh and larger.

4. Average cubic-foot volume during the growth period—computed as the average of the initial and final total cubic-foot volumes per acre (inside bark) of all stems 1.0 inch dbh and larger. The initial and final per acre volumes were obtained by fitting the sample tree data separately by plots and measurement times to height-diameter regressions of the form

$$H = a + b_1d + b_2d^2 + b_3d^3$$

where

H = total height

d = dbh.

Substitution of the measured dbh values into the appropriate height-diameter regressions produced total height estimates for every tree at each measurement time. Inside-bark cubic-foot volumes for the entire stem were then obtained from the appropriate volume table given by MacKinney and Chaiken (1939). Accumulation and expansion of the individual tree volumes produced the total cubic-foot volume values for each measurement time.

5. Net annual per acre basal area growth—obtained as the difference between initial and final basal area per acre divided by five (the number of years in the growth period).

6. Net annual per acre cubic-foot growth—obtained as the difference between initial and final cubic-foot volume per acre divided by five.

Average age, basal area, and volume for the growth period were used throughout the analysis in place of the more commonly encountered initial age, initial basal area, and initial volume. The adoption of such an approach permitted the use of differential calculus methods for the derivation of growth models from yield expressions. Growth models so obtained are appropriate for expressing the growth rate at a given age (instantaneous rate of growth) but are not applicable, without further manipulation, for estimating the average rate of growth that would occur in a given number of years beginning at some initial age.

The use of the stand density study data with the calculus-derived growth models assumes that the periodic annual growth may be used as the instantaneous growth rate at the average growth period values for age, basal area, and volume. It should be noted that this assumption will introduce inaccuracies somewhat analogous to those that might be caused by measurement errors since the relationships between basal area, age, volume, and growth are almost certainly non-linear. With growth periods of only five years, however, it is felt that the errors introduced by this approximation are inconsequential.

The Cubic-Foot Yield Equation

The first phase of the analysis involved the development of a regression equation to predict per acre cubic-foot yields from age, site, and basal area. The yield model selected is a modification of an equation form originally suggested by Schumacher (1939) and first applied to loblolly pine by MacKinney and Chaiken (1939). Since the introduction of this model, it has been successfully utilized in various forms for many mensurational studies of the southern pines. As used in the present study, the model is

$$\ln V = a + b_1S + b_2(\ln B) + b_3A^{-1} \quad (1)$$

where

$\ln V$ = the logarithm to the base "e" of inside-bark cubic-foot volume per acre of all pine stems including stump and top (Base "e" logarithms are used throughout this study to simplify various mathematical derivations.)

A = stand age in years

S = site index in feet

B = basal area per acre in square feet.

The principal advantages of this model are:

1. The mathematical form of the variates implies relationships which agree with our biological concepts of even-aged stand development (Schumacher 1939).

2. The use of $\ln V$ as the dependent variable rather than V will generally be more compatible with the statistical assumptions customarily made in regression analysis (linearity, normality, additivity, and homogeneity of variance).

3. The use of $\ln V$ as the dependent variate is a convenient way to mathematically express the interaction of the independent variables in their effect on V . (For example, the change in expected volume occurring as a result of a change in site index from 60 to 70 would depend on the associated values of $\ln B$ and A^{-1} .)

In order to determine the adequacy of the model (Eq. 1), the three possible two-variable interactions were added to it to form the model

$$\ln V = a + b_1 S + b_2 (\ln B) + b_3 A^{-1} + b_4 S (\ln B) + b_5 S A^{-1} + b_6 A^{-1} (\ln B) \quad (2)$$

which was fitted to the average yields for the second growth period of the stand density study. The resulting analysis of variance (Table 1) indicates that the three independent variables in the first model (Eq. 1) are highly significant and that none of the interaction variables contributes a significant reduction in the residual sum of squares. The final regression equation was calculated as

$$\ln V = 2.8076 + .015108S + .95931(\ln B) - 21.863A^{-1} \quad (3)$$

This equation accounts for 99.32 percent of the variation about mean $\ln V$, and results in a standard error of .00442 (on the logarithmic scale).

Derivation of the Growth Equations

As a result of the cubic-foot yield analysis, the model

$$\ln V = b_0 + b_1 S + b_2 (\ln B) + b_3 A^{-1} \quad (4)$$

was accepted as the cubic-foot yield expression. Differentiating this expression with respect to age (noting that basal area B is a function of age) yields the total derivative

$$V^{-1}(dV/dA) = b_2 B^{-1}(dB/dA) - b_3 A^{-2} \quad (5)$$

or

$$(dV/dA) = b_2 V B^{-1}(dB/dA) - b_3 V A^{-2} \quad (6)$$

TABLE 1. Analysis of variance for per acre cubic-foot volume.

Source of variation	DF	Sum of squares	Mean square
Effect of S	1	13.517522	13.517522**
Added effect of B	1	12.581230	12.581230**
Added effect of A^{-1}	1	2.031274	2.031274**
Added effect of $S \times \ln B$	1	.000919	.000919
Added effect of $(\ln B) \times A^{-1}$	1	.000518	.000518
Added effect of $S \times A^{-1}$	1	.002685	.002685
Residuals	95	.187096	.001969
Total	101	28.321244	

**Significant at the 1 percent level.

where

(dV/dA) = rate of change of cubic-foot volume with respect to age or instantaneous rate of cubic-foot growth

(dB/dA) = rate of change of basal area with respect to age or instantaneous rate of basal area growth

Although Eq. 6 provides a form for the cubic-foot growth rate, the formula is not suitable for practical application because it involves the basal area growth rate not ordinarily available.

In order to modify Eq. 6 so that cubic-foot growth can be obtained from age, site, and basal area, it becomes necessary to specify the relationship between basal area growth and age, site, and basal area.

A search of the available literature on loblolly pine mensuration failed to produce any suitable analytic models previously applied to the prediction of basal area growth or yield. Lacking such models, the loblolly pine basal area yield curves presented by Schumacher and Coile (1960) were examined. These curves—prepared by means of stand projection methods based on height-age curves and a formula for trend through time of the stocking percentage—show basal area yield as a function of stand age, site, and basal area of the stand at age 20. Because the formulas used by Schumacher and Coile cannot be combined into a single linear equation giving basal area yield as a function of age, site, and basal area at age 20, a graphic analysis of the curves was undertaken to see if a linear equation could be developed that would produce curves similar to those in the Schumacher and Coile publication. The graphic analysis yielded the following results:

1. For a given site and initial basal area (basal area at age 20), the logarithm of basal area plotted linearly over the reciprocal of age.

2. The slopes of the $\ln B$ over A^{-1} lines plotted linearly over site by initial

density groups. The intercepts for the various initial density group lines plotted linearly over the logarithm of initial density.

3. The logarithms of the upper asymptotes for basal area on each site plotted linearly over site.

A mathematical interpretation of the plottings dictates the following basal area yield equation:

$$\ln B = c_0 + c_1 S + c_2 A^{-1} + c_3 (\ln D) A^{-1} + c_4 S A^{-1} \quad (7)$$

where

D = basal area at age 20

and

c_0, c_1, \dots, c_4 are regression coefficients. Differentiating this expression with respect to age yields

$$B^{-1} (dB/dA) = -A^{-1} (c_2 A^{-1} + c_3 (\ln D) A^{-1} + c_4 S A^{-1}) \quad (8)$$

but from Eq. 7 above

$$\ln B - c_0 - c_1 S = c_2 A^{-1} + c_3 (\ln D) A^{-1} + c_4 S A^{-1}$$

Hence,

$$B^{-1} dB/dA = -A^{-1} (\ln B - c_0 - c_1 S) \quad (9)$$

or

$$dB/dA = -B (\ln B) A^{-1} + c_0 A^{-1} B + c_1 B S A^{-1} \quad (10)$$

Since Eq. 10 relates basal area growth to functions of age, site, and basal area, it was accepted as a feasible model from the standpoint of practicality. To evaluate its statistical adequacy, Eq. 10 was fitted to the basal area growth data from the stand density study using multiple regression methods. For this regression analysis, the dependent variable was defined as $(dB/dA + B (\ln B) A^{-1})$ and the two independent variables were of the form $A^{-1} B$, and $B S A^{-1}$. The analysis of variance (Table 2) shows that both independent variables were highly significant. The final regression equation

$$dB/dA = -B (\ln B) A^{-1} + 4.6012 A^{-1} B + .013597 B S A^{-1} \quad (11)$$

accounts for 65.3 percent of the variation

TABLE 2. Analysis of variance for annual per acre basal area growth.

Source of variation	DF	Sum of squares	Mean square
The conditioning factor ($B(\ln B)A^{-1}$)		20,888.9890	
Effect of BSA^{-1}	1	365.7602	365.7602**
Added effect of BA^{-1}	1	399.4272	399.4272**
Residuals	100	59.1726	.591726
Total (of $dB/dA + B(\ln B)A^{-1}$)	102	21,713.3490	

**Significant at the 1 percent level.

about the mean annual basal area growth, with a standard error of .77 square foot per acre per year. (Mean annual basal area growth for the study plots was 2.53 square feet per acre per year.) Calculation and plotting of the residuals (actual growth minus expected growth) over age, site, and basal area failed to produce any indication that the data did not conform to the model in Eq. 10. This model was therefore adopted for the expression of basal area growth.

With the specification of the basal area growth model, further development of the cubic-foot growth expression becomes possible. Substitution of the right-hand side of Eq. 11 for the quantity dB/dA in Eq. 6 produces

$$dV/dA = b_2VB^{-1}(-B(\ln B)A^{-1} + c_0A^{-1}B + c_1BSA^{-1}) - b_3VA^{-2} \quad (12)$$

or

$$dV/dA = -b_2V(\ln B)A^{-1} + b_2c_0VA^{-1} + b_2c_1VSA^{-1} - b_3VA^{-2} \quad (13)$$

Since V can be obtained from Eq. 3 as a function of age, site, and basal area, Eq. 13 presents a feasible regression model for the prediction of cubic-foot growth from age, site, and basal area. The independent variables for this model are $V^*(\ln B)A^{-1}$, V^*A^{-1} , V^*SA^{-1} , and V^*A^{-2} . The notation V^* is used to signify estimated rather than actual volumes, where the estimated volumes are obtained from the regression

$$\ln V^* = 2.8076 + .015108S + .95931(\ln B) - 21.863A^{-1} \quad (14)$$

Using this equation, predicted volumes were computed for each plot in the stand density study. These V^* values were then used in conjunction with the age, site, and basal area data to form the independent variables specified in Eq. 13. Fitting the cubic-foot growth data with this model resulted in the regression equation

$$dV/dA = 5.7907V^*A^{-2} - .78166V^*(\ln B)A^{-1} + 3.6562V^*A^{-1} + .017410V^*SA^{-1} \quad (15)$$

which accounts for 69.7 percent of the variation about the mean annual per acre cubic-foot growth, with a standard error of 40.96 cubic feet per acre per year. (Mean annual cubic-foot growth was 105.66 cubic feet per acre per year.) As shown in the analysis of variance presented in Table 3, all four independent variables in Eq. 13 were found to be highly significant during the fitting of Eq. 15. Since plotting of the residuals from Eq. 15 over age, site, and basal area gave no indication that the errors of prediction could be reduced by modification of the form of the independent variables, Eq. 13 was accepted as the model for cubic-foot growth.

Projection Formulas for Basal Area and Cubic-Foot Volume

The regressions for basal area growth

$$dB/dA = -B(\ln B)A^{-1} + 4.6012A^{-1}B + .013597BSA^{-1} \quad (16)$$

and cubic-foot growth

$$dV/dA = 5.7907VA^{-2} - .78166V(\ln B)A^{-1} + 3.6562VA^{-1} + .017410VSA^{-1} \quad (17)$$

were used to obtain prediction equations for

TABLE 3. Analysis of variance for annual per acre cubic-foot growth.

Source of variation	DF	Sum of squares	Mean square
Effect of $V \cdot A^{-2}$	1	1,372,397.15	1,372,397.15**
Added effect of $V \cdot (\ln B) A^{-1}$	1	49,491.71	49,491.71**
Added effect of $V \cdot A^{-1}$	1	16,621.82	16,621.82**
Added effect of $V \cdot S A^{-1}$	1	23,734.54	23,734.54**
Residuals	98	164,411.78	1,677.67
Total	102	1,626,657.00	

**Significant at the 1 percent level.

future basal area and cubic-foot volume of a given present stand. To develop such equations, Eq. 16 and 17 were interpreted as differential equations and solved for B and V , respectively. Solution of the basal area differential equation was obtained by rewriting Eq. 16 as

$$dB/dA = B(4.6012 + .013597S - \ln B)A^{-1} \quad (18)$$

or

$$B^{-1}(4.6012 + .013597S - \ln B)^{-1}dB = A^{-1}dA \quad (19)$$

Integrating on the left from B_o (initial basal area) to B_p (projected basal area) and on the right from A_o (initial age) to A_p (projection age) yields the expression

$$\begin{aligned} & -\ln(4.6012 + .013597S - \ln B_p) \\ & + \ln(4.6012 + .013597S - \ln B_o) \\ & = \ln A_p - \ln A_o + C \end{aligned} \quad (20)$$

Noting that C is zero, since $B_p = B_o$ when $A_p = A_o$, and taking antilogs on both sides of the equation

$$\begin{aligned} & (4.6012 + .013597S - \ln B_o)(4.6012 \\ & + .013597S - \ln B_p)^{-1} = A_p A_o^{-1} \end{aligned} \quad (21)$$

or

$$\begin{aligned} & \ln B_p = 4.6012 + .013597S \\ & - A_o A_p^{-1}(4.6012 + .013597S - \ln B_o) \end{aligned} \quad (22)$$

To derive a solution for Eq. 17, it must be remembered that the term $\ln B$ is a function of A where the form of the function is specified by Eq. 22. After the substitution of Eq. 22 (using the notation B and A rather than B_p and A_p) in Eq. 17 and some

algebraic manipulation, Eq. 17 was reduced to

$$\begin{aligned} \frac{1}{V} dV = & (5.7907A^{-2} + .059626A^{-1} \\ & + .0067818A^{-1}S \\ & + 3.5966A_oA^{-2} \\ & + .010628A_oA^{-2}S \\ & - .78166A_oA^{-2} \ln B_o) dA \end{aligned} \quad (23)$$

Integrating on the left from V_o (initial volume) to V_p (projected volume) and on the right from A_o (initial age) to A_p (projection age) and combining terms resulted in

$$\begin{aligned} \ln V_p = & \ln V_o - 5.7907(A_p^{-1} - A_o^{-1}) \\ & + (.059626 + .0067818S) \\ & (\ln A_p - \ln A_o) - A_o(3.5966 \\ & + .010628S - .78166 \ln B_o) \\ & (A_p^{-1} - A_o^{-1}) + C \end{aligned} \quad (24)$$

where, since $V_p = V_o$ when $A_p = A_o$, C is zero.

Discussion

Through the use of deductive methods to obtain model forms, and through the application of statistical least-squares techniques to estimate coefficients in the models, five equations have been developed to describe the growth and yield relationships present in data from the first and second remeasurements of the loblolly pine stand density study plots. These equations are:

Cubic-foot yield

$$\begin{aligned} \ln V = & 2.8076 + .015108S \\ & + .95931(\ln B) \\ & - 21.863A^{-1} \end{aligned} \quad (25)$$

Basal area growth rate

$$dB/dA = B(4.6012 + .013597S - \ln B)A^{-1} \quad (26)$$

Cubic-foot growth rate

$$dV/dA = V*(5.7907A^{-1} - .78166(\ln B) + 3.6562 + .01741S)A^{-1} \quad (27)$$

Projected basal area

$$\ln B = 4.6012 + .013597S - A_o(4.6012 + .013597S - \ln B_o)A^{-1} \quad (28)$$

Projected cubic-foot volume

$$\ln V = \ln V_o - 5.7907(A^{-1} - A_o^{-1}) + (.059626 - .006781S)(\ln A - \ln A_o) - A_o(3.5966 + .010628S - .78166\ln B_o)(A^{-1} - A_o^{-1}) \quad (29)$$

These equations have been developed from models satisfying the conditions for compatibility between growth and yield models previously described. Furthermore, analyses obtained from the fitting of the models to the loblolly pine stand density study strongly supported the validity of the models, since all included independent variables proved to be highly significant and no significant trends of growth or yield residuals with age, site, or basal area were discernible.

Graphs of the five final equations have been prepared (Figs. 1-5). Figure 1 shows solutions of Eq. 25 for various ages and basal areas and site indices of 70 and 90. This figure is essentially a conventional empirical yield table from which present estimated stand volume can be obtained for a given combination of age, site index, and basal area. Figures 2 and 3 present solutions of Eq. 26 and 27 and show the annual per acre basal area and cubic-foot growth rates that can be expected on sites 70 and 90 at various ages and basal area levels.

Solutions of the basal area projection equation (Eq. 28) are charted in Figure 4. These curves show the expected development of basal area in stands with given initial basal areas and subsequent basal area growth as shown in Figure 2. Similar

curves for the projected cubic-foot volume equation (Eq. 29) are given in Figure 5. This figure shows the change in cubic-foot volume with increasing age for various initial basal areas on sites 70 and 90. The initial volumes (volumes at age 25) are obtainable from Figure 1. Subsequent volume values—actually obtained by integrating the cubic-foot growth equation (Eq. 29)—are approximately equal to volumes obtained by adding successive annual cubic-foot growth values from Figure 3 to the initial volume. In determining the cubic-foot growth for successive ages, the increase in basal area with age (Fig. 4) must be considered.

Through the use of Figures 4 and 5, the effect on total cubic-foot production of varying thinning regimes, initial densities, and rotation ages can be easily estimated. For example, consider a comparison between the two following management alternatives:

- (a) Site index 90 stands with basal area at age 25 of 120 square feet per acre will not receive any cutting until age 50 at which time the stands will be clear cut.
- (b) Site index 90 stands with basal area at age 25 of 120 square feet per acre will be thinned back at subsequent five-year intervals to a basal area of 120 square feet. At age 50 the stands will be clear cut.

As shown in Figure 6, alternative (a) would result in a stand at age 50 containing 202 square feet of basal area and a volume of 7,125 cubic feet.² Since there are no intermediate cuts under this alternative, the total production during the life of the

²It should be noted that long projections or uncut stands may be somewhat unrealistic, because the stands on which the growth equations were based had all received cutting at least once during the five years preceding the growth measurement period. The extent to which cut and uncut stands of the same age, site, and basal area differ in growth was not investigated.

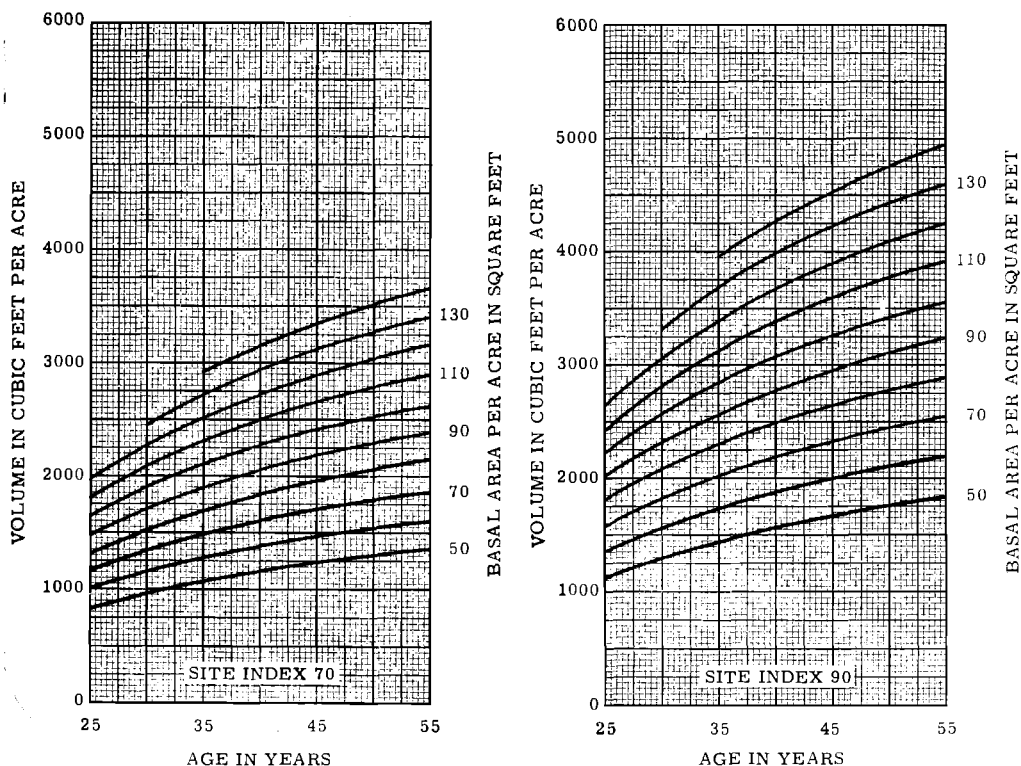


FIGURE 1. Expected total cubic-foot yields at various ages and densities for site indices 70 and 90.

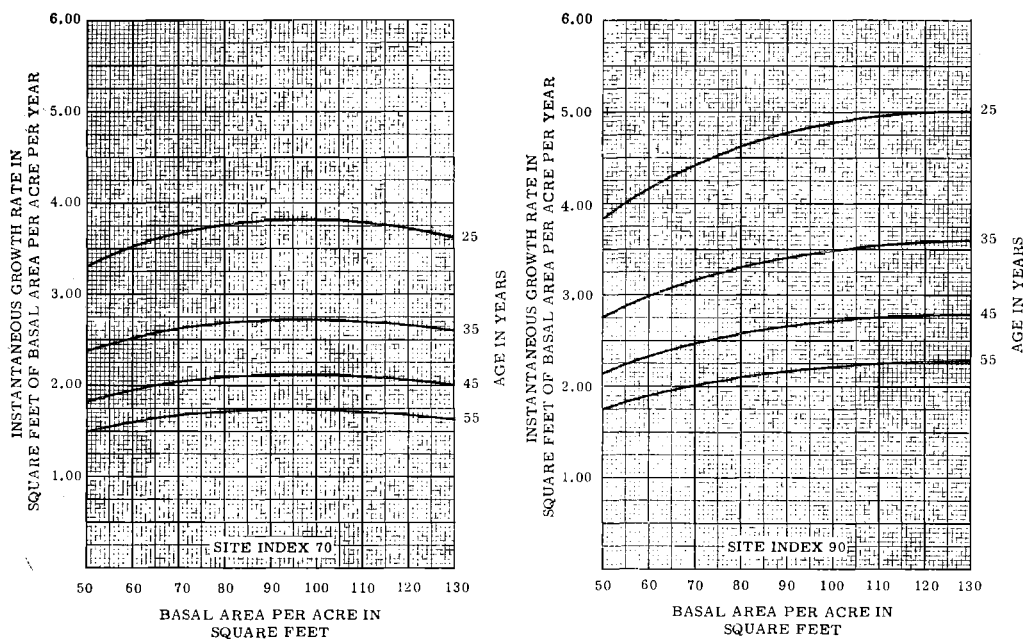


FIGURE 2. Expected basal area growth rates at various ages and densities for site indices 70 and 90.

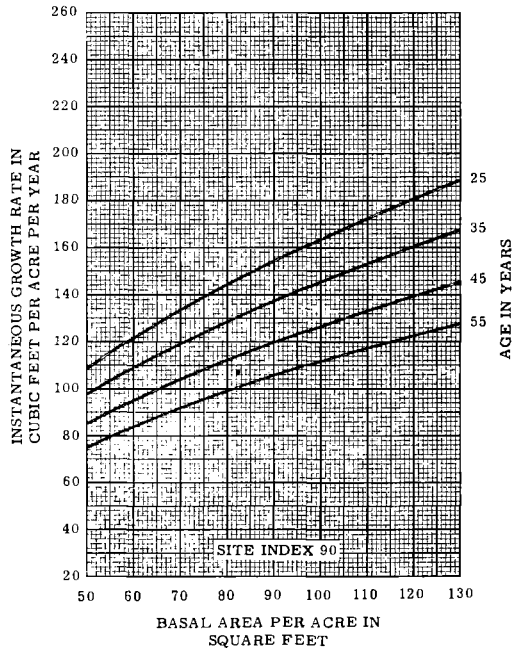
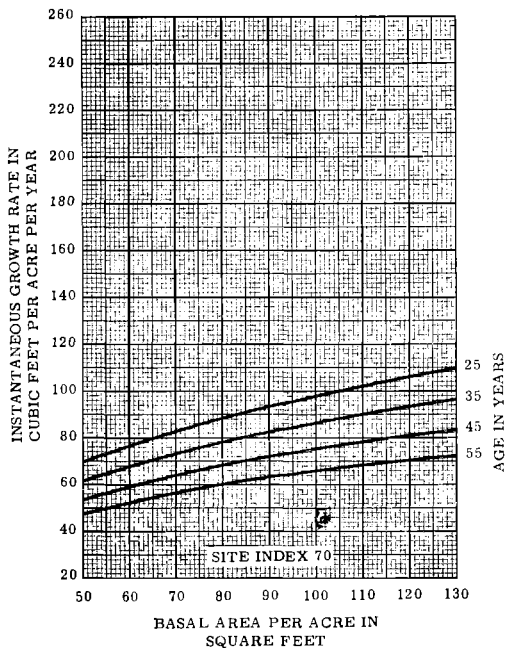


FIGURE 3. Expected cubic-foot growth rates at various ages and densities for site indices 70 and 90.

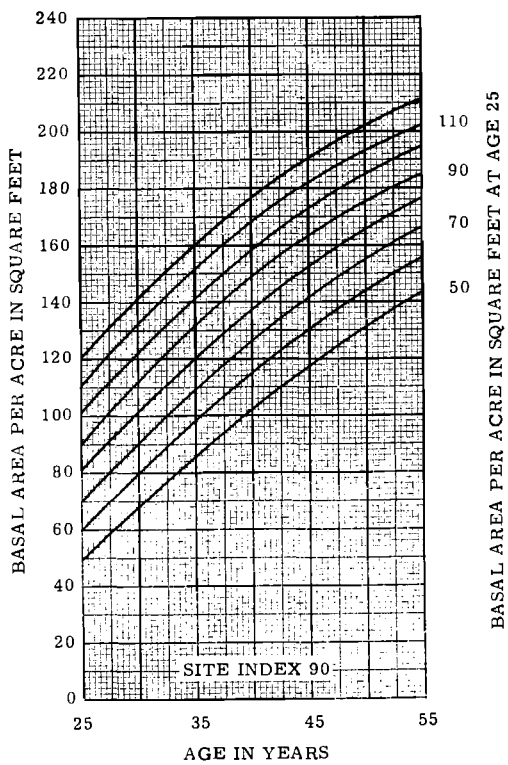
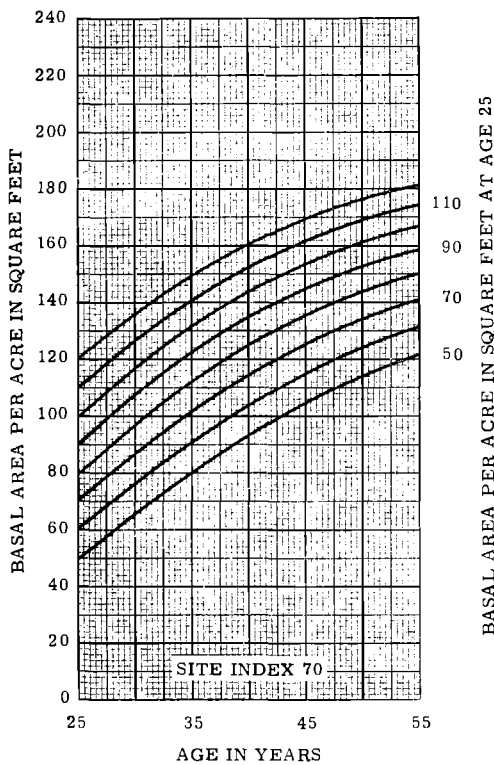


FIGURE 4. Projected per acre basal area at various ages and initial densities for site indices 70 and 90.

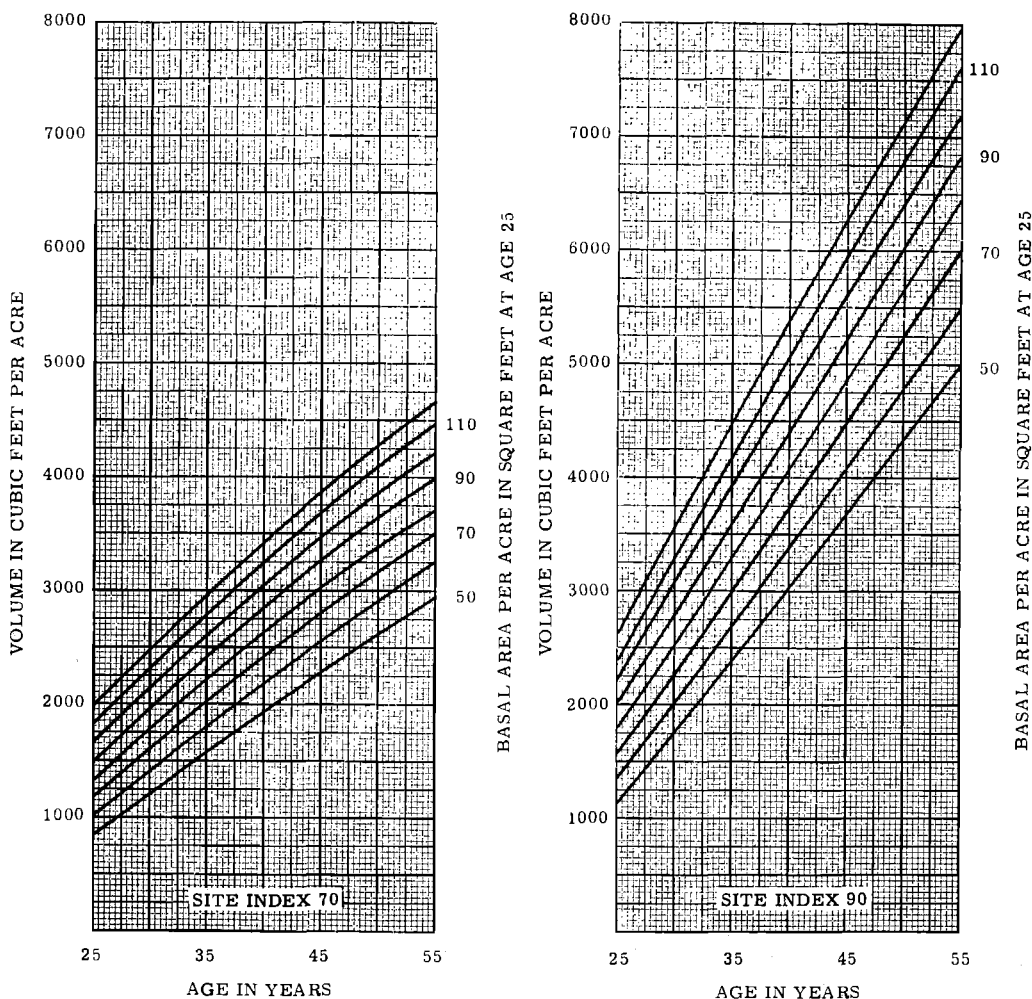


FIGURE 5. Projected per acre cubic-foot volume at various ages and initial densities for indices 70 and 90.

stand is 7,125 cubic feet. The development of stands receiving the management regime described in alternative (b) is shown in Figure 7. Table 4 presents the necessary computations for determining the total production under alternative (b). As indicated in Table 4, the estimated yields obtained under the thinning regime described in (b) are as shown in the tabulation at the right.

The cubic-foot volumes given above are total inside-bark volume including stump and top of all stems .5-inch dbh and above.

Age	Yield in cubic feet
30	500
35	625
40	525
45	425
50	4,450
Total	6,525

If merchantable cubic-foot volume were used, the comparison would probably be considerably different. It is hoped that compatible models can be adapted for use with merchantable volume data. If so, it

TABLE 4. Computation of total stand production for stands of site index 90 and initial basal area of 120 square feet per acre under management alternative (b).

Cut	Cutting age	Basal area		Initial density class of cut stand ¹	Volume		Yield
		Before cutting	After cutting		Before cutting	After cutting	
	Years	Square feet			Cubic feet		
1	30	141	120	100	3,575	3,075	500
2	35	141	120	80	3,925	3,300	625
3	40	137	120	64	4,075	3,550	525
4	45	135	120	53	4,225	3,800	425
5	50	134	0	---	4,450	0	4,450
							6,525

¹Initial density class of cut stand is defined as the stand basal area at age 25 which, on the same site index and in the absence of cutting, could be expected to produce a stand at the given cutting age with the specified basal area after cutting. These values are readily determined by reference to the basal area projection curves shown in Figure 7.

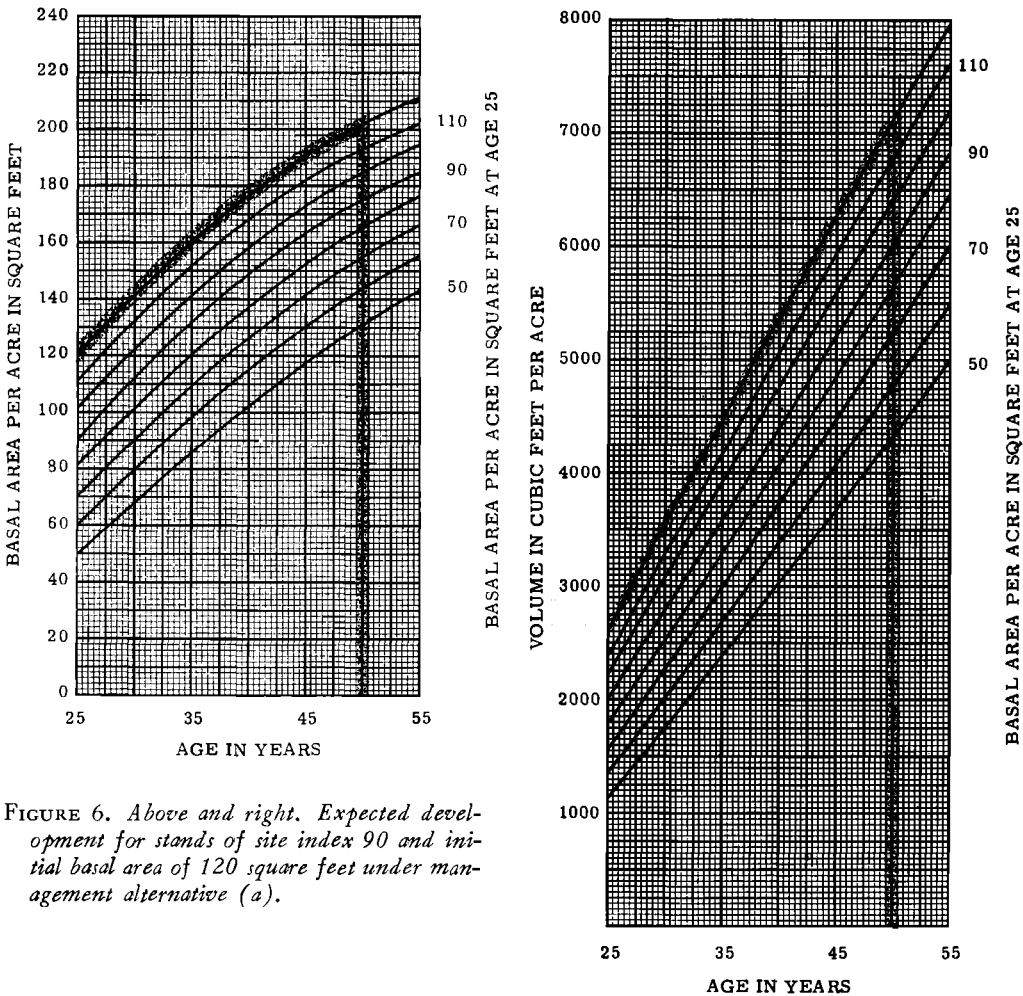


FIGURE 6. Above and right. Expected development for stands of site index 90 and initial basal area of 120 square feet under management alternative (a).

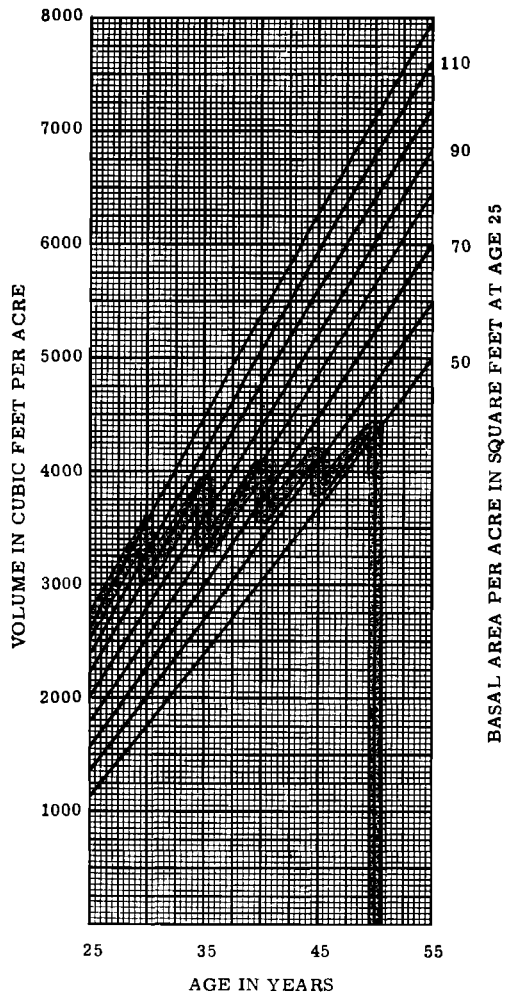
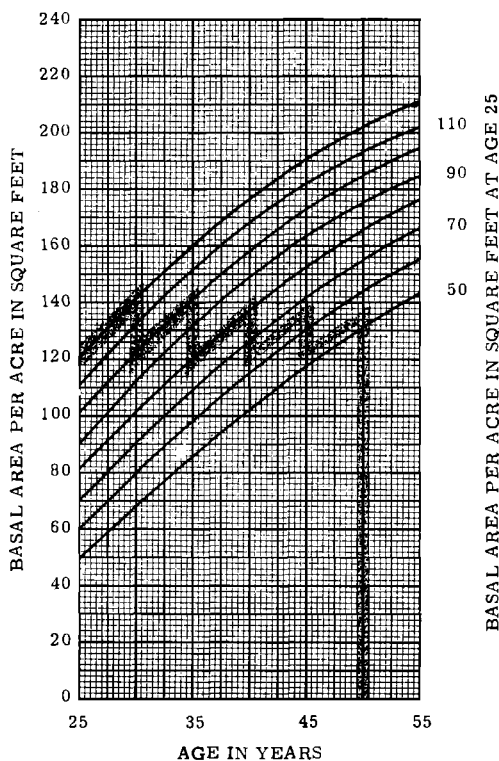


FIGURE 7. Above and right. Expected development for stands of site index 90 and initial basal area of 120 square feet under management alternative (b).

should be possible to provide the forest manager with merchantable volume production curves (similar to the total volume curves of Fig. 5) that would permit the quick and accurate estimation of total production under various management regimes that might be under consideration.

Conclusion

A few of the more important methodology considerations involved in the study should be noted. First of all, it must be emphasized that the system of models developed here is not presented as necessarily applicable to all species, management uses, or

units of measure. It is certainly not the only set of consistent models which could be constructed for similar application with loblolly pine and may, indeed, not be as satisfactory as one or more of the other model systems which have been or could be developed. It is to be hoped, however, that the present system will, at the least, be useful as an initial hypothesis which can be tested and improved by other mensurationists working in the field of growth and yield prediction.

This testing and further development can be implemented in two ways. Where actual growth or yield data are currently

available, empirical tests can be made to check the validity of the actual equations (if the available data are from loblolly pine stands of suitable description) or new equations can be calculated to determine the applicability of the models to management systems other than that considered here. In the absence of actual data, model improvement is still possible through a detailed consideration of the biological implications that are inherent in the current system.

For example, the present models could be used to give answers to such questions as the following: (1) How is the density at which maximum growth occurs related to age and site? (2) Does the model allow for the possibility that, with sufficiently high density, net basal area or volume growth rates may become negative and, if so, what is the implied effect of site and age upon this density? (3) How is the rotation age of maximum mean annual increment related to site, initial density, and thinning regime?

If the solutions provided to such questions are internally contradictory or biologically unreasonable, some revision of the model system is obviously necessary. It is of the utmost importance, however, to note that indicated revisions cannot generally be introduced into one portion of the system without affecting its logical compatibility with other components of the set. For example, any evidence refuting the form of the basal area growth model would almost certainly imply inaccuracy or inadequacy in the cubic-foot growth and yield models.

In view of the prevalence in the growth and yield literature of relative measures of density, some mensurationists will, no doubt, wish to consider and, perhaps, strictly adhere to models that avoid absolute measures of stand density. It is, of course, impossible to make general inferences, on the basis of this study, concerning the adequacy of possible compatible models systems which might use relative scales for the expression of stand density. However, it can be noted that the present system of models compares

favorably in predictive ability with the various southern pine growth or yield regressions that have used relative density measurements and for which some measure of predictive efficiency is available. In addition, it is difficult to imagine the development of a complete model system with relative density expression that would not be considerably more complex conceptually and analytically than the set of models developed here.

Finally, a comment may be pertinent as to the necessity for compatibility in growth and yield models for it may well be that many mensurationists will question whether a search for consistency will justify the additional model construction and computational effort that is necessary beyond that involved in the simple fitting of independent growth and yield surfaces.

It would seem that at least two rather cogent arguments can be advanced to defend this additional labor. First, when growth and yield surfaces for the same stands are not compatible, at least one of them must be inadequate or inaccurate and the situation then certainly suggests that a gain in predictive efficiency can be realized by revising one or both of the models. Hence, when mensurationists propose non-compatible growth and yield models they are simultaneously presenting *prima facie* evidence that their predictors are not as good as they might be. Second, the briefest consideration of past and present scientific thought suggests that the development of logical but admittedly approximate relationships or predictors can be justifiably recognized as scientific achievements of competence and importance.

However, it is difficult to point to any recognized contributions of importance or competence among such developments that involved the propounding of relationships that were, regardless of their predictive efficiency, obviously self-contradictory. One might speculate, for example, on the success which Newton might have achieved if he had attacked the problem of the proverbial aggressive apple by simply fitting

unrelated regression equations to predict (1) the speed of the apple during its fall and (2) the time required for it to fall a given distance. His resulting predictive ability would, no doubt, have been quite impressive and he would have nicely avoided the mental effort involved in grappling with such a nasty mathematical concept as differential equations. In spite of these apparent advantages, one cannot help but feel that the resulting system might not have been an unbounded success. Who knows, perhaps a search for consistency is occasionally justified just for the sake of being consistent.

Summary

A review of previous work in loblolly pine mensuration failed to produce any reference to the mathematical dependency that should logically exist between regression models for growth and yield. Working from applicable yield models, tentative cubic-foot and basal area growth models were developed by differentiation of the yield models with respect to age. The growth models were then fitted, by multiple regression techniques, to data from the Southeastern Forest Experiment Station's loblolly pine stand density study. All independent variables included in the models proved to be highly significant. Because plotting of the residuals failed to suggest any discrepancies between the models and the actual data, the tentative models were accepted as realistic growth models that were logically consistent in form with commonly accepted yield models. Cubic-foot and basal area production formulas were then obtained by integration of the corresponding growth equations with respect to age. Using these production formulas, total cubic-foot production can be estimated for any desired combination of initial density, site index, thinning regime, and rotation age.

Literature Cited

- BICKFORD, C. A., F. S. BAKER, and FRED G. WILSON. 1957. Stocking, normality, and measurement of stand density. *J. For.* 55: 99-104.

- BRENDER, E. V. 1960. Growth predictions for natural stands of loblolly pine in the lower Piedmont. Ga. Forest Res. Council Rpt. 6. 7 pp.
- BRUCE, DONALD, and F. X. SCHUMACHER. 1950. Forest mensuration. McGraw-Hill Book Company, New York. 483 pp.
- CHAIKEN, L. E. 1939. The approach of loblolly and Virginia pine stands toward normal stocking. *J. For.* 37:866-871.
- CHISMAN, H. H., and F. X. SCHUMACHER. 1940. On the tree-area ratio and certain of its applications. *J. For.* 38:311-317.
- COILE, T. S. 1952. Soil productivity for southern pines. Part I. Shortleaf and loblolly pines. *Forest Farmer* 11(7):10-11, 13.
- GEVORKIANTZ, S. R. 1934. The approach of under-stocked stands to normality. *J. For.* 32:487-488.
- MACKINNEY, A. L., and L. E. CHAIKEN. 1939. Volume, yield and growth of loblolly pine in the mid-Atlantic coastal region. Appalachian Forest Expt. Sta. Tech. Note 33. 30 pp.
- MEYER, WALTER H. 1942. Yields of even-aged stands of loblolly pine in northern Louisiana. *Yale School Forestry Bul.* 54. 58 pp.
- NELSON, T. C. 1960. Current research in the South. Fourth Conf. South. Indus. Forest Mangt. Proc., pp. 52-60.
- _____, and ERNST V. BRENDER. 1963. Comparison of stand density measures for loblolly pine cubic-foot growth prediction. *For. Sci.* 9:8-14.
- REINEKE, L. H. 1933. Perfecting a stand-density index for even-aged forests. *J. Agric. Res.* 46:627-638.
- SCHUMACHER, F. X. 1939. A new growth curve and its application to timber-yield studies. *J. For.* 37:819-820.
- SCHUMACHER, F. X., and T. S. COILE. 1960. Growth and yield of natural stands of the southern pines. T. S. Coile, Inc., Durham, N. C. 115 pp.
- SIMMONS, E. M., and G. L. SCHNUR. 1937. Effect of stand density on mortality and growth of loblolly pine. *J. Agric. Res.* 54: 47-58.
- STAHELIN, R. 1949. Thinning even-aged loblolly and slash pine stands to specified densities. *J. For.* 47:538-540.
- U. S. FOREST SERVICE. 1929. Volume, yield, and stand tables for second-growth southern pines. U. S. Dept. Agric. Misc. Pub. 50.

202 pp.

WAHLENBERG, W. G. 1960. Loblolly pine.
Duke Univ. School Forestry, Durham, N. C.
603 pp.

WELLWOOD, R. W. 1943. Trend towards
normality of stocking for second-growth

loblolly pine stands. J. For. 41:202-209.
WENGER, K. F., T. C. EVANS, T. LOTTI, R.
W. COOPER, and E. V. BRENDER. 1958.
The relation of growth to stand density
in natural loblolly pine stands. U. S. For-
est Serv. Southeast. Forest Expt. Sta. Paper
97. 10 pp.

Commonwealth Forestry Bureau

Oxford, England

Forestry Abstracts

This presents the essence of current world literature extracted by an experienced permanent staff from publications in over 30 languages (nearly 550 periodicals, 850 serials and innumerable miscellaneous papers, books, etc.) and now totalling over 5,500 abstracts each year. Special features include: the abstracting at length of literature published in the more unfamiliar languages (e.g. Slav, Hungarian and Oriental languages); notices of the more important critical reviews; notices of translations into English; notices of new journals and serials, or of changes in their style; and notices of atlases and maps. Each issue normally contains a leading article synthesizing authoritatively the literature on some particular subject and also items of world news. Annual subscription is 110 s. or \$16.50 U.S.A. or Canada, post-free.

Guide to the Use of Forestry Abstracts

Containing a directory of publishing sources, analysis of an abstract notice, a key to abbreviations and many other aids, now available in an enlarged 4-language edition (English, French, German, Spanish) for 15s. or \$2.25 post free, including five annual supplements.

Centralized Title Service

A quick-service postal auxiliary to the Abstracts bringing to subscribers four times a month exact copies of the index cards made from the world stream of forestry literature during the Bureau's day-to-day work, totalling about 7,000 annually. Full particulars and samples from: The Director, Commonwealth Forestry Bureau, Oxford.

Orders should be sent direct to:

**Central Sales Branch, Commonwealth Agricultural Bureaux,
Farnham Royal, Buckinghamshire, England**