Multivariable Calculus (Integration)

Consider a region R in the xy plane bounded by one (or) more curves, let $f(x,y)$ be a function defined at all points of R, let the region R be divided into small subregion each of area $\delta R_1, \delta R_2 \cdots \delta R_n$ which are pairwise non-overlapping. Let (x_i, y_i) be an arbitrary point within the subregion δR_i Consider the sum

$$f(x_1,y_1)\delta R_1 + f(x_2,y_2)\delta R_2 + \cdots + f(x_n,y_n)\delta R_n$$

If the sum tends to a finite limits as $n \to \infty$ such that $\max(\delta R_i) \to 0$ irrespective of the choice of (x_i, y_i), the limits is called the double integral of $f(x,y)$ over the region R and is denoted by the symbol

$$\iint_R f(x,y)\,dR \text{ (or) } \iint_R f(x,y)\,dx\,dy.$$

Properties of double Integrals:-

Properties of double Integrals are similar to those of definite integrals. ~~left~~ let f and g be functions of x and y, defined and continuous in a region R. then

(i) $\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$

ii) $\iint_R k f(x,y)\,dx\,dy = k\iint_R f(x,y)\,dx\,dy$ where k is constant

iii) $\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$

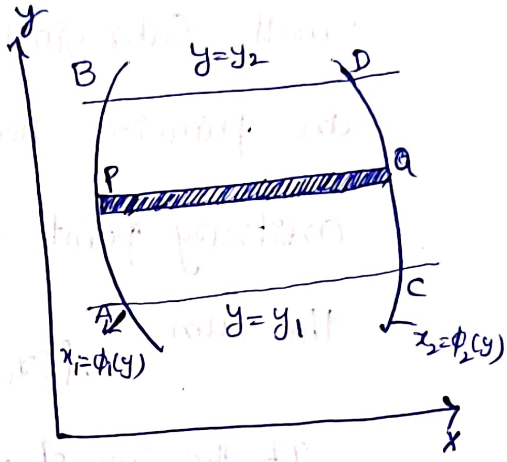
when R_1 and R_2 are two disjoint regions whose union is R.

Evalution of Double Integrals :-

The methods of evaluating the double integrals depend upon the nature of the curves bounding the region R. Let the region R be bounded by the curves $x=x_1, x=x_2$ and $y=y_1, y=y_2$

(i) when x_1, x_2 are functions of y and y_1, y_2 are constants:

let AB and CD be the curves $x_1=\phi_1(y)$ and $x_2=\phi_2(y)$.

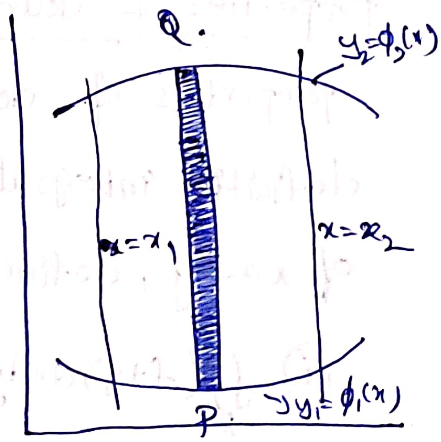


take a stripe which is parallel to x-axis because x-limits are variable limits. and we take the double Integral is evaluated first w.r.to x while treating y-as constant. The resulting expression which is a function y is integrated w.r.to y. between the limits $y=y_1$ and $y=y_2$ Thus,

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \boxed{\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\,dx}\, dy.$$

(ii) when y_1, y_2 are functions of x and x_1, x_2 are constants:-

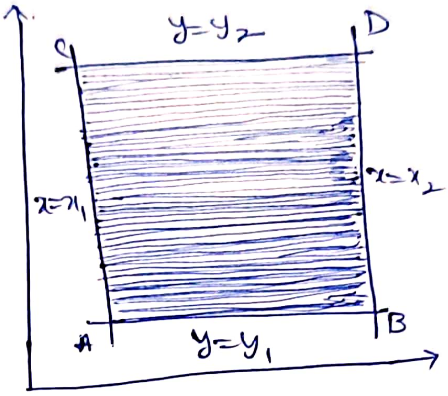
let AB and CD be the curves $y_1=\phi_1(x)$ $y_2=\phi_2(x)$. take stripe which parallel to y-axis because the limits are variable limits. take the double Integral is Evaluated first w.r.t y while treating x-as constant. The resulting expression which is a function x is integrated w.r.to x. between the limits $x=x_1$ and $x=x_2$ thus



$$\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2} \boxed{\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\,dy}\,dx.$$

(iii) When x_1, x_2, y_1, y_2 are Constants:-

Here the region of Integration R is the rectangle ABCD. If we Integrate first w.r.t x or we Integrate first w.r.to y. Thus the Order of Integration is immaterial, provided the limits of integration are changed accordingly.



$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy$$

(or)

$$\iint_R f(x,y)\,dy\,dx = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\,dx.$$

$$= \int_{-a}^{a} 2\left[x^2 y + \frac{y^3}{3}\right]^{\sqrt{a^2-x^2}} dx$$

(7)

(5) Evaluate $\iint (x+y)^2 \, dx dy$ over the area bounded by the ellipse $\frac{x^2}{}, \frac{y^2}{}$

(3)

(1) Evaluate $\int_0^2 \int_0^x y \, dy \, dx$

Sol:- The given Integral is

$$\int_{x=0}^{2} \int_{y=0}^{x} y \, dy \, dx = \int_{x=0}^{2} \left[\int_{y=0}^{x} y \, dy \right] dx$$

First we have to Evaluate the inner Integral keeping x as a constant

$$\therefore \text{Given Integral} = \int_{x=0}^{2} \left[\frac{y^2}{2}\right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{2} \frac{x^2}{2} \, dx = \left.\frac{x^3}{6}\right|_{x=0}^{2} = 8/6 = \frac{4}{3}$$

(2) Evaluate $\int_{x=0}^{a} \int_{y=0}^{b} (x^2+y^2) \, dy \, dx$

Sol:- The given Integral $\int_{x=0}^{a} \int_{y=0}^{b} (x^2+y^2) \, dy \, dx$

Sol:- $\int_{x=0}^{a} \left[\int_{y=0}^{b} (x^2+y^2) \right] dx$

$$\int_{x=0}^{a} \left[x^2 y + y^3/3\right]_{y=0}^{b} dx = \int_{x=0}^{a} \left[bx^2 + \frac{b^3}{3}\right] dx$$

$$= \left[b\frac{x^3}{3} + \frac{b^3 x}{3}\right]_{x=0}^{a} = \frac{ba^3}{3} + \frac{b^3 a}{3} = \frac{ab}{3}(a^2+b^2)$$

③ Evaluate $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$

Sol:- $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

$$\int_0^1\left[\int_x^{\sqrt{x}}(x^2+y^2)\,dy\right]dx = \int_0^1\left[x^2y+\frac{y^3}{3}\right]_{y=x}^{\sqrt{x}} dx$$

$$= \int_0^1\left[x^2\sqrt{x}+\frac{(\sqrt{x})^3}{3}-x^3-\frac{x^3}{3}\right]dx$$

$$= \int_0^1\left[x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\right]dx$$

$$= \frac{x^{7/2}}{7/2}+\frac{x^{5/2}}{3\times 5/2}-\frac{4x^4}{4.3}\Big|_0^1$$

$$= 2/7+2/15-\frac{1}{3} = \frac{30+14-35}{105} = \frac{9}{105} = 3/35$$

④ Evaluate $\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{a^2-x^2-y^2}\,dx\,dy$

Sol:- The given Integral $= \int_0^a\left[\int_0^{\sqrt{a^2-y^2}}\sqrt{(\sqrt{a^2-y^2})^2-x^2}\,dx\right]dy$

$$= \int_0^a\left[\frac{x}{2}\sqrt{a^2-y^2-x^2}+\frac{a^2-y^2}{2}\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a\frac{a^2-y^2}{2}\left[\sin^{-1}(1)-\sin^{-1}(0)\right]dy$$

$$= \int_0^a\frac{a^2-y^2}{2}\cdot\frac{\pi}{2}\,dy = \frac{\pi}{4}\left[a^2y-y^3/3\right]_0^a = \pi/6\,a^3$$

(4)

⑤ Evaluate $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dy\,dx}{1+x^2+y^2}$

Sol:- The given Integration $= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2} \right] dx$

$= \int_{x=0}^{1} \left[\int_{y=0}^{p} \frac{dy}{p^2+y^2} \right] dx$ where $p = \sqrt{1+x^2}$

$= \int_{x=0}^{1} \left[\frac{1}{p} \tan^{-1}(y/p) \right]_{x=0}^{p} dx = \int_{x=0}^{1} \frac{1}{p} \left[\tan^{-1}(1) - \tan^{-1} 0 \right] dx$

$\left(\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right)$

$= \int_{x=0}^{1} \frac{\pi}{4} \cdot \frac{dx}{\sqrt{1+x^2}}$ $(\because p = \sqrt{1+x^2})$

$\left(\log\left(x+\sqrt{x^2+a^2}\right) = \sinh^{-1}\left(\frac{x}{a}\right) \right)$

$= \frac{\pi}{4} \left[\log\left(x+\sqrt{(x^2+1)}\right) \right]_0^1 = \frac{\pi}{4} \log\left(1+\sqrt{2}\right)$

Note:- The above Integral is also Equal to

$= \pi/4 \left[\sinh^{-1}(x) \right]_0^1 = \frac{\pi}{4} \sinh^{-1}(1)$

⑥ Evaluate $\int_0^2 \int_0^x e^{x+y}\, dy\, dx$

Sol:- $\int_{x=0}^{2} \int_{y=0}^{x} e^{x+y}\, dy\, dx = \int_{x=0}^{2} \left[\int_{y=0}^{x} e^x \cdot e^y dy \right] dx$

$= \int_0^2 e^x \left[(e^y) \right]_0^x dx = \int_0^2 e^x (e^x - 1)\, dx$

$= \int_0^2 (e^{2x} - e^x)\, dx = \left(\frac{e^{2x}}{2} - e^x \right)_0^2 = \left(\frac{e^4}{2} - e^2 \right) - \left(\frac{1}{2} - 1 \right)$

$= \frac{e^4}{2} - e^2 + \frac{1}{2} = \frac{1}{2} (e^2 - 1)^2$

④ Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$

Sol:- $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_{y=0}^\infty e^{-y^2} \left[\int_0^\infty e^{-x^2} dx \right] dy$

(From Gamma function
$\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2]$

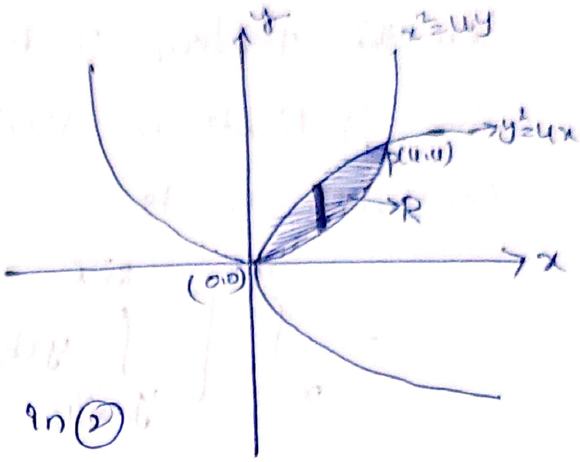
$= \sqrt{\frac{\pi}{2}} \left[\int_0^\infty e^{-y^2} dy \right]$ (again $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$ from Gamma function]

$= \sqrt{\pi}/2 \cdot \sqrt{\pi}/2 = \pi/4$

take x-as variable limits and y limits constants

Evaluation of double Integral, limits are not given:—

① Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2=4x$ and $x^2=4y$



Sol:— Given Parabolas

$y^2=4x$ — ① and

$x^2=4y$ — ②

To find the their points of intersection, solve ① and ②

from ① $y^2/4=x$ and subs in ②

$(y^2/4)^2=4y \Rightarrow \frac{y^4}{16}=4y \Rightarrow y^4=64y$

$y^4-64y \Rightarrow y(y^3-64)=0 \Rightarrow y=0,\ y^3=64$

$y=0,\ y=4.$

from ① If $y=0. \Rightarrow x=0$

If $y=4 \Rightarrow x^2=16 \Rightarrow x=4.$

. The Intersection points are $(x,y)=(0,0)$ and $(4,4)$

The shaded area between $y^2=4x$ and $x^2=4y$ is the region R of Intersection.

Here: $\iint_R y\,dx\,dy=\int_{x=0}^{4}\left(\int_{y=x^2/4}^{2\sqrt{x}} y\,dy\right)dx$

(Here we take one Variable limits as Constant and other Variable limits as interms Variables. i.e either x as constant limits, y should be Variable limits (or) y as constant limits we should take x as Variable limits, If we draw a stripe Parallel to x-axis we should take x-as Variable limits. and y limits constants

If we take a stripe parallel to y-axis we should take y limits as variable limits and x-limits are constants]

In this problem I drawn a stripe parallel to y-axis so y limits are variable limits and x-limits are constant limits.

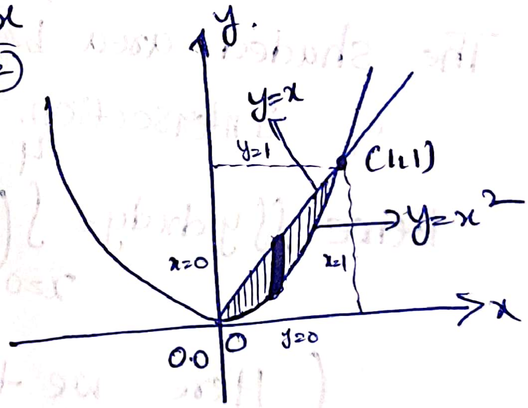
$$= \int_{x=0}^{4} \left[\int_{y=x^2/4}^{2\sqrt{x}} y\,dy \right] dx$$

$$= \int_{x=0}^{4} \left[\frac{y^2}{2} \right]_{x^2/4}^{2\sqrt{x}} dx = \frac{1}{2} \int_{0}^{4} \left[4x - \frac{x^4}{16} \right] dx$$

$$= \frac{1}{2} \left[2x^2 - \frac{x^5}{80} \right]_0^4 = \frac{1}{2} \left[32 - \frac{64}{5} \right] = \frac{48}{5}$$

② Evaluate $\iint_R xy(x+y)\,dx\,dy$ over the region R bounded by $y=x^2$ and $y=x$

Sol:- Given curves $y=x^2$ — ① and $y=x$ — ②



solve ① and ② for Intersection points.

Sub n ② in ①

$x = x^2 \Rightarrow x^2 - x = 0$

$x(x-1) = 0 \Rightarrow x = 0,\ x = 1$

If $x=0 \Rightarrow y=0$

If $x=1 \Rightarrow y=1$

$(x,y) = (0,0)$ and $(1,1)$

The shaded area between $y=x^2$ and $y=x$ is the region of Intersection.

Draw a stripe parallel to y-axis and take y-limits are Variable limits and x-limits are constants.

x varies from 0 to 1

y Varies from $y=x^2$ to $y=x$.

$$\iint_R xy(x+y)\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=x^2}^{x} xy(x+y)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{x} x^2 y\,dy + \int_{y=x^2}^{x} x y^2\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[x^2 \frac{y^2}{2}\Big|_{x^2}^{x} + x\frac{y^3}{3}\Big|_{x^2}^{x}\right]dx$$

$$= \int_{x=0}^{1}\left[\left(x^2.\frac{x^2}{2} - x^2.\frac{x^4}{2}\right) + \left(x.\frac{x^3}{3} - x.\frac{x^6}{3}\right)\right]dx$$

$$= \int_{x=0}^{1}\left[\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3}\right]dx$$

$$= \frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24}\Big|_0^1 = \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{9}{168} = 3/56$$

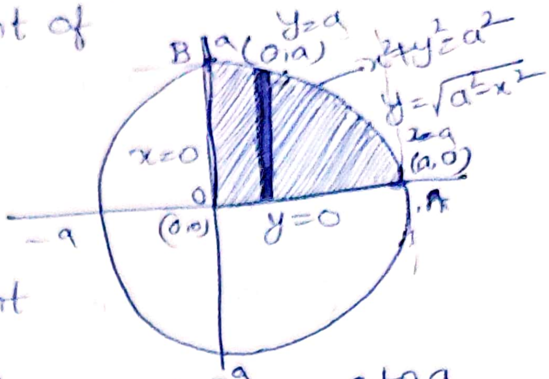
(3) Evaluate $\iint_R xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$

Sol:- Consider $\iint_R xy\,dx\,dy$

$x^2+y^2=a^2$ — (1) is circle with centre at $(0,0)$ and radius a units.

To find the Intersection points put $x=0$, in (1) we get $y=\pm a$, and put $y=0$ in (1) we get $x=\pm a$

The shaded area is the given region R of integration is bounded by in the positive quadrant of the circle $x^2+y^2=a^2$ i.e OABO



now draw a stripe parellel to y axis and we take y limits as Variable limits and x limits as constant

y-varies from $y=0$ to $y=\sqrt{a^2-x^2}$ and x-varies from 0 to a

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx = \int_{x=0}^{a}\left[\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\right]dx$$

$$= \int_{x=0}^{a}\left[x\frac{y^2}{2}\right]_{y=0}^{\sqrt{a^2-x^2}} dx = \int_{x=0}^{a}\left[\frac{x}{2}(a^2-x^2)-0\right]dx$$

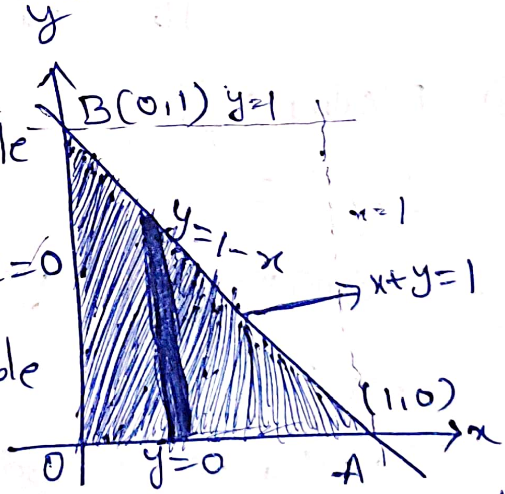
$$= \int_{x=0}^{a}\left(\frac{xa^2}{2}-\frac{x^3}{2}\right)dx$$

$$= \frac{a^2}{2}\frac{x^2}{2}-\frac{x^4}{8}\Big|_0^a = \frac{a^2}{4}a^2-\frac{a^4}{8}$$

$$= \frac{a^4}{4}-\frac{a^4}{8} = \frac{2a^4-a^4}{8} = \frac{a^4}{8}$$

1) Evaluate $\iint_R e^{2x+3y}\,dx\,dy$ over the triangle bounded by $x=0$, $y=0$ and $x+y=1$

Sol:- The region of integration is the triangle (Shaded area) OAB.



We draw a stripe Parallel to y-axis, and take y-limits as Variable limits and x-limits as constant.

Here x-Varies from 0 to 1 and y varies from x-axis upto

upto the line $x+y=1$ i.e $y=1-x$ (i.e y varies from 0 to $1-x$)

$\therefore$ The region R can be expressed as

$0 \le x \le 1,\ 0 \le y \le 1-x$

$$\therefore \iint_R e^{2x+3y}\,dx\,dy = \int_0^1\int_0^{1-x} e^{2x+3y}\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{3} e^{2x+3y}\right]_0^{1-x} dx$$

$$= \frac{1}{3}\int_0^1 \left[e^{2x+3(1-x)} - e^{2x}\right] dx$$

$$= \frac{1}{3}\int_0^1 \left[e^{3-x} - e^{2x}\right] dx$$

$$= \frac{1}{3}\left[\int_0^1 (e^{3-x} - e^{2x})\right] dx$$

$$= \frac{1}{3}\left[\int_0^1 e^3 \cdot e^{-x}\,dx - \int_0^1 e^{2x}\,dx\right]$$

$$= \frac{1}{3}\left[e^3\left[\frac{e^{-x}}{-1}\right]_0^1 - \left[\frac{e^{2x}}{2}\right]_0^1\right]$$

$$= \frac{1}{3}\left[e^3\left[\frac{e^{-1}}{-1}+1\right] - \frac{e^2}{2} + \frac{1}{2}\right]$$

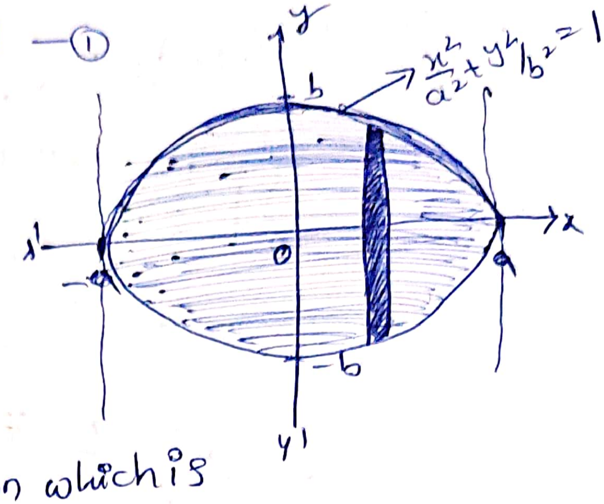
$$= \frac{1}{3}\left[-e^2 + e^3 - \frac{e^2}{2} + \frac{1}{2}\right] = \frac{1}{6}\left[-2e^2 + 2e^3 - e^2 + 1\right]$$

$$= \frac{1}{6}\left[-3e^2 + 2e^3 + 1\right]$$

$$= \frac{1}{6}\left[(e-1)^2(2e+1)\right]$$

(5) Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol:- Given curve $\frac{x^2}{a^2}+y^2/b^2=1$ — (1)



. put $x=0$ in (1) we get $\frac{y^2}{b^2}=1$

$y=\pm b$

put $y=0$ in (1) we get $\frac{x^2}{a^2}=1$

$x=\pm a$

take $a>b$

The shaded area is the region which is bounded by the ellipse $\frac{x^2}{a^2}+y^2/b^2=1$

we draw a stripe, which is parallel to the y-axis and take y-limits as variable limits and x-limits as constant limits.

y varies from $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

x varies from $-a$ to a

$$=\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}(x^2+y^2+2xy)\,dy\,dx$$

$$=\int_{-a}^{a}\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}}(x^2+y^2)\,dy\,dx+\int_{-a}^{a}\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}}2xy\,dy\,dx$$

$$=\int_{-a}^{a}\int_{0}^{b/a\sqrt{a^2-x^2}}2(x^2+y^2)\,dy\,dx+\int_{-a}^{a}[0]$$

[since x^2+y^2 is an even function of y and 2xy is an odd function of y]

$$= \int_{-a}^{a} 2\left[x^2 y + \frac{y^3}{3}\right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x^2 \frac{b}{a}\sqrt{a^2-x^2} + \frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right] dx$$

$$= 4\int_0^{a} \left[\frac{b}{a} x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

put $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

$$= 4\int_0^{\pi/2} \left[\frac{b}{a}\cdot a^2\sin^2\theta \cdot a\cos\theta + \frac{b^3}{3a^3}\cdot a^3\cos^3\theta\right] a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left[a^3 b\sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right]d\theta$$

$$= 4a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta + 4\,\frac{ab^3}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta.$$

$$\left[\because \int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta = \cancel{[(m-1)(m-3)(m-5)\cdots 5.3.1][(n-1)(n-3)(n-5)\cdots 5.3.1]}\right.$$

$$= \frac{[(m-1)(m-3)(m-5)\cdots 5.3.1][(n-1)(n-3)(n-5)\cdots 5.3.1]}{(m+n)(m+n-2)(m+n-4)\cdots 5.3.1}\left(\frac{\pi}{2}\right)$$

If both m and n are even.

$$\left. = \frac{[(m-1)(m-3)(m-5)\cdots(2\text{ or }1)][(n-1)(n-3)(n-5)\cdots(2\text{ or }1)]}{(m+n)(m+n-2)(m+n-4)\cdots(2\text{ or }1)}\right]$$

$$= 4a^3b\left[\frac{1.1}{4.2}\,\frac{\pi}{2}\right] + 4\,\frac{ab^3}{3}\left[\frac{3.1}{4.2}\right]\pi/2$$

$$= 4a^3b\,\frac{1}{8}\,\frac{\pi}{2} + 4\frac{ab^3}{3}\left[\frac{3}{8}\cdot\pi/2\right]$$

$$= \frac{4a^3b\pi}{\cancel{16}_{4}} + \frac{\cancel{12}ab^3\pi}{\cancel{48}_{4}} = ab\frac{\pi}{4}(a^2+b^2)$$

Evaluation of Double Integrals in polar Co-ordinates:- (1)

To Evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the lines $\theta=\theta_1, \theta=\theta_2$ and the curves $r=r_1$ and $r=r_2$ we first Integrate w.r.t r between limits $r=r_1$ and $r=r_2$ keeping θ fixed. The resulting Expression is integrated w.r.t θ from θ_1 to θ_2.

In this function Integral r_1, r_2 are functions of θ and θ_1 and θ_2 are Constants

1) Evaluate $\int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta$

$u=r^2$

$\sqrt{a^2-r^2}=u$

$a^2-r^2=u^2$

$-2r\,dr=2u\,du$

$\frac{1}{2}(a^2-r^2)^{-1/2}(-2r)$

$r\,dr=-(a^2-r^2)du$

$$I=\int_0^{\pi/2}\left[\int_0^{a^2\cos^2\theta} -\frac{1}{2}\sqrt{a^2-u}\,du\right]d\theta$$

$$=\int_0^{\pi/2}\left[-\frac{1}{2}\left(+\frac{1}{2}(a^2-u)^{-1/2}(-1)\right]_0^{a^2\cos^2\theta} d\theta$$

$$=+\frac{1}{4}\int_0^{\pi/2}\left[(a^2-u)^{-1/2}\right]_0^{a^2\cos^2\theta} d\theta.$$

$$=\frac{1}{4}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{-1/2}-(a^2)^{-1/2}\right]d\theta$$

$$=\frac{1}{4}\left[\int_0^{\pi/2} a^{-1}(1-\cos^2\theta)\,d\theta-\int_0^{\pi/2} a^{-1}\,d\theta\right]$$

$$=\frac{1}{4}\left[a^{-1}\left[\frac{\sin^3\theta}{2}\right]_0^{\pi/2}-a^{-1}[\theta]_0^{\pi/2}\right]$$

$$=\frac{1}{4a}\left[\frac{\sin^3\pi/2}{2}-0\right)-a^{-1}(\pi/2-0]=\frac{1}{4a}\left[\frac{1}{2}-\pi/2\right]$$

$\pi \quad a(1+\cos\theta)$

② Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Sol:- $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$

$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta = \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi}\frac{1-\cos2\theta}{2}d\theta = \frac{a^2}{4}\int_0^{\pi}(1-\cos2\theta)d\theta$

$= \frac{a^2}{4}\left(\theta - \frac{\sin2\theta}{2}\right)_0^{\pi} = \frac{a^2}{4}\left((\pi-0)-(0-0)\right) = \frac{a^2\pi}{4}$

③ Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr$

Sol:- $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr = \int_0^{\pi}$

put $r^2 = t \Rightarrow 2r\,dr = dt$

If $r=0 \Rightarrow t=0$

If $r=\infty \Rightarrow t=\infty$

$\int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty}\left(e^{-t}\frac{1}{2}dt\right)d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}dt\,d\theta$

$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}d\theta = \frac{-1}{2}\int_0^{\pi/2}(0-1)d\theta$

$= \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{1}{2}\left[\theta\right]_0^{\pi/2} = \frac{1}{2}(\pi/2) = \pi/4.$

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(4) Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$

Sol:- The given Integral is Equal to

$$\int_{\theta=0}^{\pi}\left(\int_{r=0}^{a(1+\cos\theta)} r\,dr\right)d\theta = \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2\,d\theta = \frac{a^2}{2}\int_{\theta=0}^{\pi} 4\cos^4\frac{\theta}{2}\,d\theta$$

Put $\theta/2 = t$ so that $d\theta = 2dt$

Also when $\theta=0$, $t=0$ and when $\theta=\pi$, $t=\pi/2$

$\therefore$ Given Integral $\frac{a^2}{\cancel{2}}\cdot 4\cdot\int_{t=0}^{\pi/2}\cos^4 t\,\cancel{2}dt$

$$= 4a^2\int_{t=0}^{\pi/2}\cos^4 t\,dt.$$

$\left[\because \int_0^{\pi/2}\cos^m\theta\,d\theta = \frac{(m-1)(m-3)(m-5)\cdots 5.3.1}{m(m-2)(m-4)\cdots 5.3.1}\,\pi/2\right.$

If m is even $\Big]$

$$= 4a^2\left[\frac{3.1}{4.2}\times\pi/2\right]$$

$$= 4a^2\left[\frac{3}{4}\cdot\frac{1}{2}\times\pi/2\right] = \cancel{4}a^2\left[\frac{3\pi}{\cancel{16}_4}\right] = \frac{3\pi a^2}{4}$$

(5) Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol:- $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = \int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\right\}d\theta$

$$= -\frac{1}{2}\int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{-2r}{\sqrt{a^2-r^2}}\,dr\right]d\theta \quad \left(\because \int\frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}\right)$$

$$= -\frac{1}{2}\int_0^{\pi/4} 2\left(\sqrt{a^2-r^2}\right)_0^{a\sin\theta} d\theta$$

$$= -\int_0^{\pi/4}\left(\sqrt{a^2-a^2\sin^2\theta}-\sqrt{a^2-0}\right) d\theta$$

$$= -\int_0^{\pi/4}\left(a\sqrt{1-\sin^2\theta} - a\right) d\theta = -\int_0^{\pi/4} a\left(\cos\theta - 1\right) d\theta$$

$$\Rightarrow -a\left[\sin\theta - \theta\right]_0^{\pi/4} = -a\left[\sin\pi/4 - \pi/4\right]$$

$$= -a\left[\frac{1}{\sqrt{2}} - \pi/4\right] = a\left(\pi/4 - \frac{1}{\sqrt{2}}\right)$$

① $$\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$$

$$I = \int_0^{\pi/2}\left[\int_0^{a\cos\theta} -\frac{1}{2}\sqrt{a^2-r^2}\,(-2r)dr\right] d\theta$$

put $a^2-r^2 = t \Rightarrow -2r\,dr = dt$

$$= \int_0^{\pi/2}\left[\int_{a^2}^{a^2\sin^2\theta} -\frac{1}{2}\, t^{1/2}\, dt\right] d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\frac{t^{3/2}}{3/2}\right]_{a^2}^{a^2\sin^2\theta} d\theta$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\int_0^{\pi/2}\left((a^2\sin^2\theta)^{3/2} - (a^2)^{3/2}\right) d\theta$$

$$= +\frac{1}{3}\left[\int_0^{\pi/2}\left(a^3\sin^3\theta - a^3\right) d\theta\right.$$

$$= -\frac{a^3}{3}\int_0^{\pi/2}\left(\sin^3\theta - 1\right) d\theta$$

$$\text{C.S } \sin^m\theta = \frac{m}{m-1}\cdot\frac{m-3}{m-2}\cdot\frac{m-5}{m-4}\cdots\frac{2}{3}\cdot 1$$

$$\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= -\frac{a^3}{3}\left[2/3 - \pi/2\right]$$

Change of Order of Integration :-

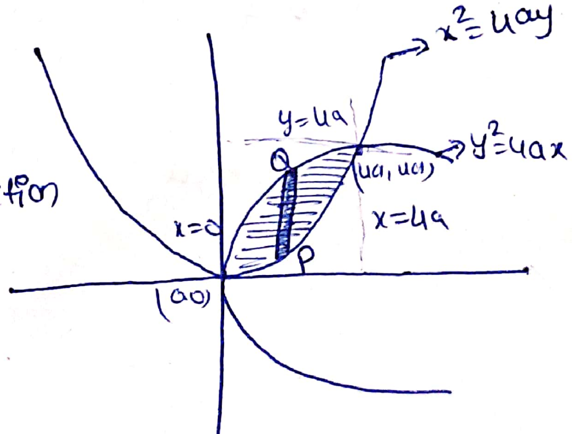
In a double Integral, if the limits of integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly. Thus,

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

But if the limits of integration are variable, a change in the order of integration necessitates change in the limits of integration. A rough sketch of the region of Integration helps in fixing the new limits of integration.

① Change the order of Integration in the following integral and evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol:- From the limits of Integration, it is clear that we have to integrate first w.r.t y which varies from $y = x^2/4a$ to $y = 2\sqrt{ax}$, and then w.r.t x which varies from $x = 0$ to $4a$. Thus integration is first performed along the vertical strip PQ which extends from a point P on the parabola $y = \frac{x^2}{4a}$ (i.e $x^2 = 4ay$) —① to the point Q on the parabola $y = 2\sqrt{ax}$ (i.e $y^2 = 4ax$). —②



Solve ① and ② for the intersection points $x^2 = 4ay$ —①

from $x = y^2/4a$

$$\frac{y^4}{16a^2} = 4ay \Rightarrow y^4 = 64a^3y$$

$y^4 - 64a^3 y = 0$

$y(y^3 - 64a^3) = 0$

$y = 0, \quad y^3 - 64a^3 = 0 \Rightarrow y^3 = 64a^3$

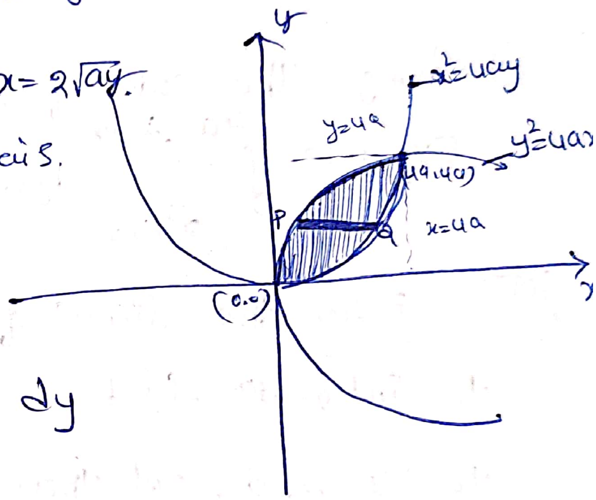
$y = 0, \quad y = 4a$

If $y = 0 \Rightarrow x = 0$

$y = 4a \Rightarrow x = 4a$

$(x, y) = (0,0)\ (4a, 4a)$

For changing the order of integration, we draw a strip into parallel to x-axis which extend from P to Q. i.e P on the parabola $y^2 = 4ax \Rightarrow x = y^2/4a$ to Q on the parabola $x^2 = 4ay \Rightarrow x = 2\sqrt{ay}$. Then this stripe is parallel to x-axis. and y-limits are constants.



$$\therefore \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} dx\, dy$$

$$= \int_0^{4a} \left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a} \left(2\sqrt{ay} - y^2/4a\right) dy$$

$$= \left[2\sqrt{a} \cdot \frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{4}{3}\sqrt{a} \cdot (4a)^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{4}{3}\sqrt{a} \cdot 8a^{3/2} - \frac{16a^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

② Change the order of Integration in the Integral

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy.$$

Sol:- From the given limits of Integration x varies from $x=0$ to $x=\sqrt{a^2-y^2}$ and $y=-a$ to a

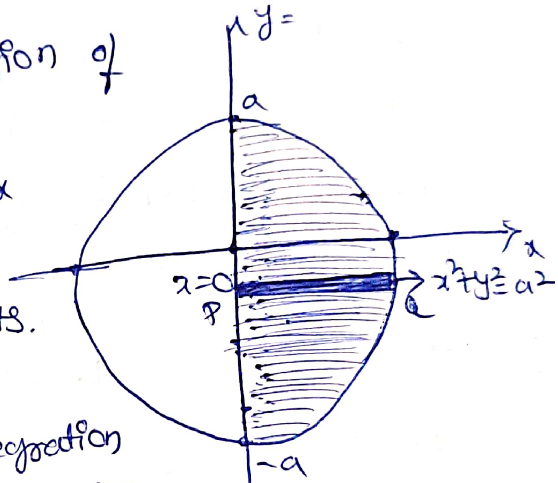
$x=\sqrt{a^2-y^2} \Rightarrow x^2=a^2-y^2 \Rightarrow x^2+y^2=a^2$ is circle.

with radius a and centre $(-g,-f)=(0,0)$

and the circle is bounded on y-axis from $-a$ to a

The shaded area is the region of Integration from

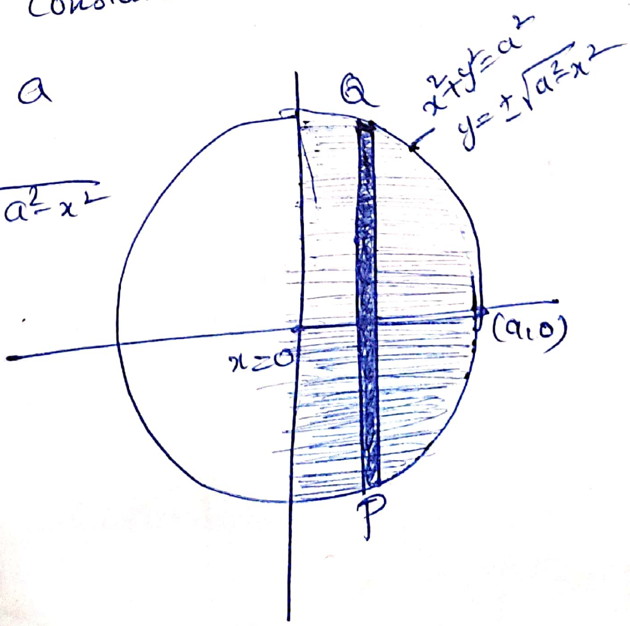
draw the strip parallel to x because x- limits are varies limits and y-limits are constants.



For chaning the order of Integration we draw a strip parallel to y-axis now y-limits will change into variable limits and x-limits will change into constant limits.

Now x-varies from 0 to a

and y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$



ie $$\int_{x=0}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$$

② Change the order of Integration and Evaluate

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$$

Sol:- From the given Integration limits x varies from 0 to a and y varies from $y=x/a$ to $y=\sqrt{x/a}$

$y=x/a$ —① (or) $ya=x$ $\qquad$ $y^2=x/a$ —② (or) $ay^2=x$

Solve eqn ① & ② for intersection points of bounded region which is bounded by $ay^2=x$ —③, $ya=x$ ④

Subn ④ in ③ we get

$ay^2=y\cancel{a} \Rightarrow y^2=y \Rightarrow y^2-y=0 \Rightarrow y(y-1)=0$

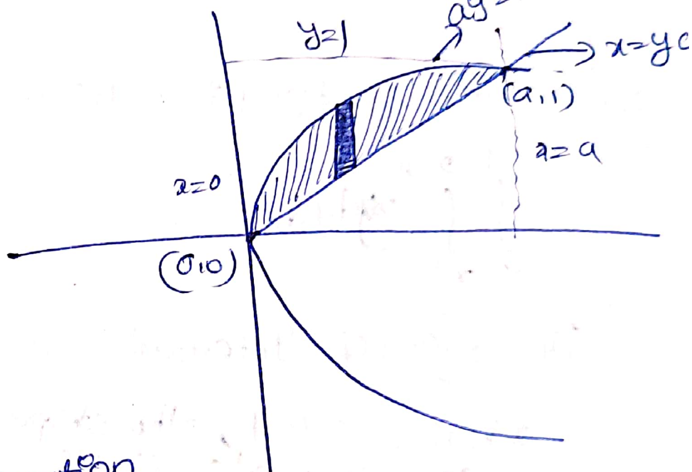
$y=0,\ y=1$

If $y=0 \Rightarrow x=0$,

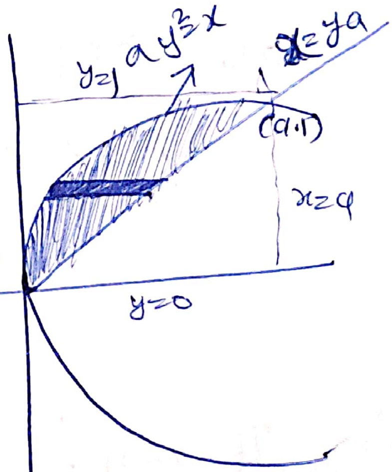
If $y=1 \Rightarrow x=a$

$\therefore (x,y)=(0,0)\ (a,1)$

are intersection points



For Change of order of Integration we draw graph with the strip is parallel to x-axis. now we take x-limits as variable limits and y-as constant.



$$\int_{x=0}^{a}\int_{y=x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx = \int_{y=0}^{1}\left[\int_{x=ay^2}^{x=ya} (x^2+y^2)\,dx\right]dy$$

$$= \int_{y=0}^{1}\left[\frac{x^3}{3}+y^2x\right]_{ay^2}^{ya} dy = \int_{y=0}^{1} \frac{y^3a^3}{3}+y^2ya-\left(\frac{a^3y^6}{3}+y^2\cdot ay^2\right]dy$$

Hence the point of ... of Integ...

$$\int_{x=0}^{1}\left(\frac{y^3a^3}{3}+ay^3-\frac{a^3y^6}{3}-y^4 a\right)dy$$

$$= \frac{a^3}{3}\frac{y^4}{4}+a\frac{y^4}{4}-\frac{a^3}{3}\frac{y^7}{7}-a\frac{y^5}{5}\Big|_0^1$$

$$= \frac{a^3}{12}+\frac{a}{4}-\frac{a^3}{21}-\frac{a}{5}$$

$$= \frac{a^3}{12}-\frac{a^3}{21}+\frac{a}{4}-a/5 = \frac{a^3}{28}+\frac{a}{20}.$$

(3) Change of order Integration $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence Evaluate the double Integral.

sol:- This Integral is to be written correctly in form

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx.$$

The region of Integration is given by $y: x^2 \to 2-x$

x varies 0 to 1, the stripe is parallel to y-axis because y-limits are variable limits.

i.e $y=x^2$ —(1) $y=2-x$ —(2) and $x=0$ and $x=1$

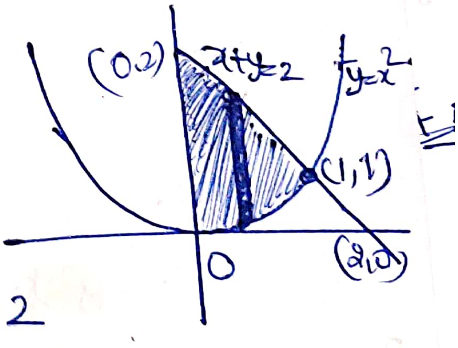
Solve (1) & (2) to get the Intersection points

$$x^2=2-x \Rightarrow x^2+x-2=0$$

$$x^2+2x-x-2=0$$

$$x(x+2)-1(x+2)=0$$

$$(x-1)(x+2)=0 \Rightarrow x=-1)\ x=-2$$



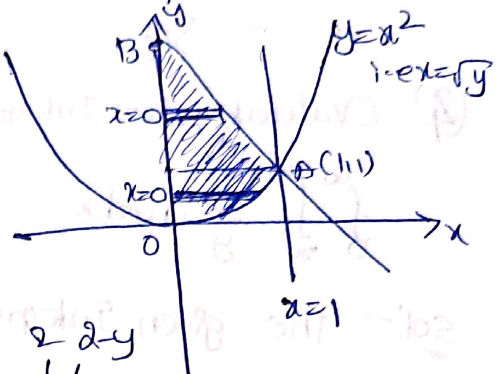
If $x=+1 \Rightarrow y=1$

$x=-2 \Rightarrow y=4$

Hence the point of intersection of curves are (-2,4) and (1,1)

The shaded area is the ~~area~~ region of Integration

we change the order of Integration. In the changed order we have to take two ~~hori~~ strips parallel to x-axis since during the sliding one edge of the strip remains on $x=0$ but the other edge of strip does not remain on single curve.



$$\therefore \int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$\int_{y=0}^{1}\left[\int_{x=0}^{\sqrt{y}} x\,dx\right] y\,dy + \int_{y=1}^{2}\left[\int_{x=0}^{2-y} x\,dx\right] y\,dy$$

$$= \int_{y=0}^{1}\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{y}} y\,dy + \int_{y=1}^{2}\left(\frac{x^2}{2}\right)_{x=0}^{2-y} y\,dy$$

$$= \int_{y=0}^{1} \frac{y}{2}\cdot y\,dy + \int_{y=1}^{2} \frac{(2-y)^2}{2} y\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 \cancel{+} \cancel{\frac{1}{2}\left[\frac{(2-y)^3}{3}\cdots\right]_1^2} \frac{1}{2}\left[\frac{1}{3}\right] + \frac{1}{6}\left[(2-y)^3\right]$$

$$= \cancel{\frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{6}\left[\right.}$$

$$\frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{6}\left[8 - y^3 + 3(2)y^2 \cancel{-3(4)y}\right]_1^2$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{6}\left[\cancel{8} - \cancel{8} + 24 - 12 - (8 - 1 + \cancel{6} - \cancel{6})\right.$$

$$= \frac{1}{6} + \frac{1}{6}(12 - 7) = \frac{1}{6} + \frac{1}{6}(5) = \frac{6+5}{6}$$

$= \frac{1}{2}\left[\int_{y=0}^{1} y^2\,dy\right] + \frac{1}{2}\int_{y=1}^{2}\left[4y - 4y^2 + y^3\right]dy$

$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[4\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4}\right]_1^2$

$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$

$= 3/8$

(3) Evaluate the Integral by changing order of Integration

$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

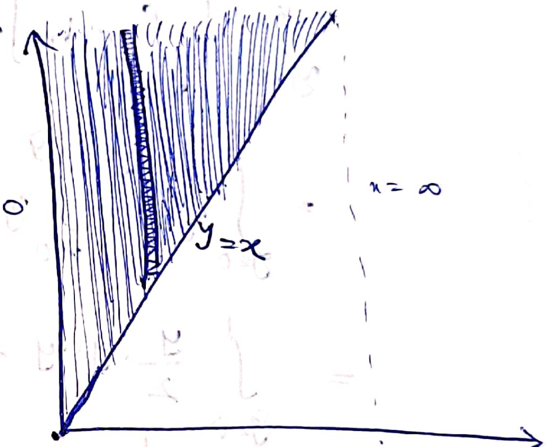
Sol:- The given Integral is $\int_{x=0}^{\infty}\left[\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\right]dx$

From the given Integral the given limits are

$x = 0$ to ∞ ; $y = x$ to ∞

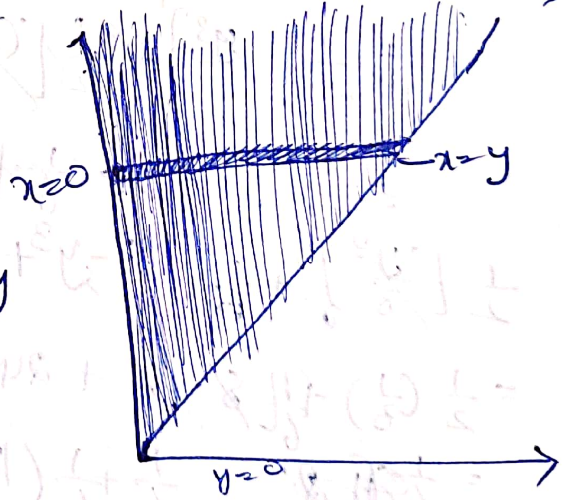
Draw a graph Curve for the bounded Region which is bounded by

$x = 0$ to ∞ ; $y = x$ to ∞



change the order of Integration we take x as Variable limits and y as constant limits

now draw a strip parallel to x-axis to take x as variable limits.



x — Varies from $x=0$ to $x=y$

y Varies from $y=0$ to ∞

$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx = \int_{y=0}^{\infty}\int_{x=0}^{y} e^{-y}/y\,dx\,dy$

$= \int_{y=0}^{\infty}\left[\frac{e^{-y}}{y}(x)_0^y\right]dy$

$= \int_{y=0}^{\infty}\left[\frac{e^{-y}}{y}(y)\right]dy = \int_{y=0}^{\infty} e^{-y}\,dy = \frac{e^{-y}}{-1}\Big|_0^\infty = -e^{-\infty} + e^{-0} = 1$

(5) By changing the order of Integration, Evaluate
$$\int_0^3 \int_1^{\sqrt{(4-y)}} (x+y)\,dx\,dy$$

Sol:- The given Integration $\int_0^3 \int_1^{\sqrt{(4-y)}} (x+y)\,dx\,dy$

x varies from 1 to $\sqrt{4-y}$

$x = 1$ —(1) and $x = \sqrt{4-y} \Rightarrow x^2 = 4-y$ —(2)

y varies from $y = 0$ to 3

from (1) & (2)

from (1) If $y = 0$, $x = 1$
$y = 3$ $x = 1$

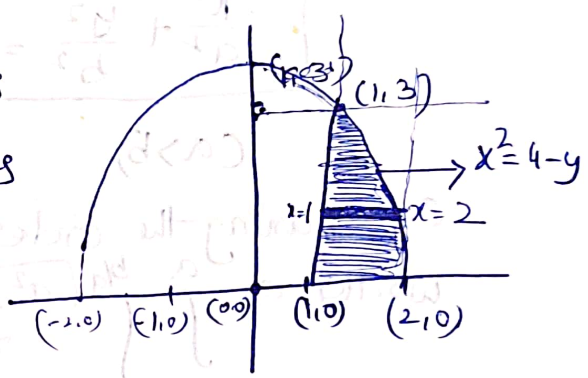
from (2) If $y = 0$ $x = \pm 2$
If $y = 3$ $x = \pm 1$

the Intersection points are $(x, y) = (1,0)\ (1,3)\ (2,0)$

change the order of Integration.

now the strip is parallel to y-axis

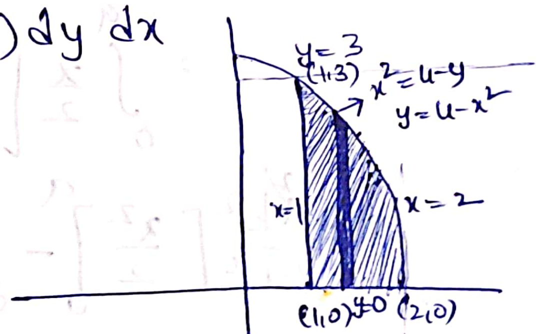
take y as variable limits and x as constant limits.



x varies from $x = 1$ to 2

y varies from 0 to $4-x^2$

i.e $$\int_{y=0}^{3} \int_{x=1}^{x=\sqrt{4-y}} (x+y)\,dx\,dy = \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\,dy\,dx$$



$$= \int_{x=1}^{2} \left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_{x=1}^{2} \left[x(4-x^2) + \frac{(4-x^2)^2}{2}\right] dx$$

$$= \int_{x=1}^{2} \left[4x - x^3 + \frac{16 + x^4 - 8x^2}{2}\right] dx =$$

$$= 4\frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{2}\left(16x + \frac{x^5}{5} - \frac{8x^3}{3}\right)\Big|_1^2$$

$$= \left[8 - 4 + \frac{1}{2}\left(32 + \frac{32}{5} - \frac{64}{3}\right)\right] - \left(2 - \frac{1}{4} + \frac{1}{2}\left(16 + \frac{1}{5} - \frac{8}{3}\right)\right)$$

$$= \frac{241}{60}$$

⑥ change the order of Integration and evaluate $\int_0^b \int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy\,dx\,dy$

<u>sol:</u>- The Given limits of Integration

y varies from 0 to b

x varies from 0 to $\frac{a\sqrt{b^2-y^2}}{b}$

$x = 0$, $x = \frac{a\sqrt{b^2-y^2}}{b} \Rightarrow bx = a\sqrt{b^2-y^2}$

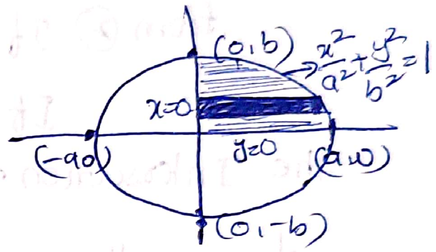
$$b^2x^2 = a^2(b^2-y^2)$$

$$b^2x^2 = a^2b^2 - a^2y^2$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ → is ellipse

$(a>b)$



On chaning the order of Integration, give Integral can be written as

$$\int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} xy\,dy\,dx = \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\frac{b\sqrt{a^2-x^2}}{a}} dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{b^2}{a^2}(a^2-x^2)\right]dx$$

$$= \frac{\cancel{a^2}b^2}{2\cancel{a^2}}\left[\frac{x^2}{2}\right]_0^a - \frac{b^2}{2a^2}\left[\frac{x^4}{4}\right]_0^a$$

$$\Rightarrow \frac{b^2}{2}\left[\frac{a^2}{2}\right] - \frac{b^2}{2a^2}\left[\frac{a^4}{4}\right] = \frac{a^2b^2}{4} - \frac{a^2b^2}{8} = \frac{a^2b^2}{8}$$

Triple Integrals :-

The triple integral is Evaluated as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$$

where the limits of z are z_1, z_2, which are either constants or functions of x and y. The y limits y_1, y_2 are either constants or functions of x, the x limits x_1, x_2 are constants.

The above multiple Integral is Evaluated as follows. First $f(x,y,z)$ is Integrated w.r.to z between the limits z_1 and z_2 keeping x and y are fixed. The resulting Expression is Integrated w.r.to y b/w the limits y_1 and y_2 keeping x constant. The result is finally integrated w.r.to x from x_1 to x_2

Thus the above multiple integral can be evaluated

Can be evaluated as $\int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx$

* when $x_1, x_2, y_1, y_2, z_1, z_2$ are all Constants, the triple Integral Can be evaluated in any order.

① Evaluated $\int_0^1 dx \int_0^2 dy \int_1^2 x^2 y z \, dz$

Sol:- $\int_0^1 dx \int_0^2 dy \left[\frac{x^2 y z^2}{2}\right]_1^2 = \int_0^1 dx \int_0^2 \left[\frac{4x^2y}{2} - \frac{x^2y}{2}\right] dy$

$= \frac{3}{2} \int_0^1 dx \int_0^2 x^2 y \, dy = \frac{3}{2} \int_0^1 dx \, x^2 \left[\frac{y^2}{2}\right]_0^2 = 3\int_0^1 x^2 dx = 1$

② Evaluate $\int_0^1 \int_1^2 \int_2^3 xyz \, dx\,dy\,dz$

Sol:- $\int_0^1 \int_1^2 yz \left[\frac{x^2}{2}\right]_2^3 dy\,dz$

$= \int_0^1 \int_1^2 yz \left[\frac{9}{2} - \frac{4}{2}\right] dy\,dz$

$\Rightarrow \int_0^1 \int_1^2 yz [5/2] \, dy\,dz = \frac{5}{2} \int_0^1 \int_1^2 yz \, dy\,dz$

$= \frac{5}{2} \left[\int_0^1 \left[\frac{y^2}{2}\right]_1^2 z \, dz \right.$

$\Rightarrow 5/2 \left[\int_0^1 \left(\frac{4}{2} - \frac{1}{2}\right) z \, dz = \frac{5}{2}\left[\int_0^1 3/2 \, z \, dz\right] \right.$

$= 5/2 \left[3/2 \, \frac{z^2}{2} \Big|_0^1 \right] = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}$

$= \frac{15}{8}$

(7) Evaluate $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$ (38)

sol. Given sphere is $x^2+y^2+z^2=a^2 \Rightarrow z=\sqrt{a^2-x^2-y^2}$

The projection of the sphere on the xy-plane is the circle $x^2+y^2=a^2$

So, this circle is covered as y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from 0 to a.

$$\therefore \iiint xyz\,dx\,dy\,dz = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} y\,dy \cdot \int_0^{\sqrt{a^2-x^2-y^2}} z\,dz$$

$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}}$$

$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy \left(\frac{a^2-x^2-y^2}{2}\right)$$

$$= \frac{1}{2}\int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} (ya^2 - x^2y - y^3)\,dy$$

$$= \frac{1}{2}\int_0^a x\,dx \left[\frac{a^2y^2}{2} - \frac{x^2y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x\,dx \left[\frac{a^2}{2}(a^2-x^2) - \frac{x^2(a^2-x^2)}{2} - \frac{(a^2-x^2)^2}{4}\right]$$

$$= \frac{1}{4}\int_0^a x\,dx \left[a^2(a^2-x^2) - x^2(a^2-x^2) - \frac{(a^2-x^2)^2}{2}\right]$$

$$= \frac{1}{4}\int_0^a x\,dx \left[(a^2-x^2)^2 - \frac{(a^2-x^2)^2}{2}\right] = \frac{1}{8}\left[\int_0^a x\,dx\,(a^2-x^2)^2\right]$$

$$= \frac{1}{8}\int_0^a (a^4-2a^2x^2+x^4)\,x\,dx = \frac{1}{8}\int_0^a (a^4x-2a^2x^3+x^5)$$

$$= \frac{1}{8}$$

$$= \frac{1}{8} \Big\{ \frac{a^4 x^2}{2} - \frac{2a^2 x^4}{4} + \frac{x^6}{6} \Big\}_0^a$$

$$= \frac{1}{8} \left[\frac{a^6}{2} - \frac{a^6}{2} + \frac{a^6}{2} \right] = \frac{1}{8} a^6 \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{a^6}{48}.$$

③ Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

Sol:— $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz = \int_{z=-1}^{1}\left[\int_{x=0}^{z}\left[\int_{y=x-z}^{x+z}(x+y+z)\,dy\right]dx\right]dz$

$$= \int_{z=-1}^{1}\left[\int_{x=0}^{z}\left[\left[xy+y^2/2+zy\right]_{x-z}^{x+z}dx\right]dz\right.$$

$$= \int_{z=-1}^{1}\left[\int_{x=0}^{z}\left(x(x+z)+\frac{(x+z)^2}{2}+z(x+z)-x(x-z)-\frac{(x-z)^2}{2}+z(x-z)\right]dx\right]dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[(x+z)\left[x+\frac{x+z}{2}+z\right]-(x-z)\left[x+\frac{x-z}{2}+z\right]\right]dx\Big]dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}(x+z)\left[\frac{2x+x+z+2z}{2}\right]-(x-z)\left[\frac{2x+x-z+2z}{2}\right]\Big]dx\Big]dz$$

$$= \int_{z=1}^{1}\int_{x=0}^{z}(x+z)\left[\frac{3x+3z}{2}\right]-(x-z)\left\{\frac{3x+2z}{2}\right]\Big)dx\Big]dz$$

$$\int_{x=-1}^{1}\int_{x=0}^{z}\left[\frac{3x^2+3xz+3xz+3z^2}{2}-\left[\frac{3x^2+2zx-3xz-2z^2}{2}\right]dx\right]dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\frac{(3x^2+3z^2+6xz)}{2}-\frac{(3x^2-2z^2-xz)}{2}\Big]dx\Big]dz$$

$$\int_{z=-1}^{1}\left[\frac{1}{2}\left[\frac{3x^3}{3}+3z^2x+6z\frac{x^2}{2}\right]_0^z-\frac{1}{2}\left(\frac{3x^3}{3}-2z^2x-z\frac{x^2}{2}\right]_0^z\right]dz$$

$$\int_{z=-1}^{1}\left[\frac{1}{2}\left[\frac{3z^3}{3}+3z^2+3zx^2-3z^2\right]-\frac{1}{2}\left[z^3-\frac{2}{3}z^2-\frac{z^3}{2}+2z^2\right]\right]dz$$

$$= \int_{z=-1}^{1}\left[\frac{1}{2}\left(z^3+3z^2x^2\right)-\frac{1}{2}\left(\frac{z^3}{2}\right)\right]dz$$

$$\frac{1}{2}\left[\frac{z^4}{4}+\frac{3z^2x^2}{2}\right]_{-1}^{1}-\frac{1}{2}\left[\frac{z^4}{8}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{1}{4}+\frac{3}{2}-\frac{1}{4}-\frac{3}{2}\right]$$

$$-\frac{1}{2}\left[\frac{1}{8}-1/8\right] = 0 //$$

(ii) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$

Sol:- Given $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$

$$= \int_1^e \left[\int_1^{\log y} \left[\int_1^{e^x} (\log z) dz \right] dx \right] dy$$

$$= \int_1^e \left[\int_1^{\log y} \left[z \log z - z \right]_1^{e^x} dx \right] dy$$

$$= \int_1^e \left[\int_1^{\log y} \left[e^x \log_e e^x - e^x (-1) \right] dx \right] dy$$

$$= \int_1^e \left[\int_1^{\log y} (x e^x - e^x + 1) \, dx \, dy \right.$$

$$= \int_1^e \left[x e^x - e^x - e^x \cancel{+x} \right]_0^{\log y} \cancel{dx} \, dy$$

$$= \int_1^e \left[\log y \cancel{e^t} \right. \quad \int_0^0 (x e^x - e^x)$$

$$= \int_1^e (y \log y + \log y - 2y + e - 1) \, dy$$

$$= \left[\left(\frac{y^2}{2} + y\right) \log y - \left(\frac{y^2}{4} + y \right) - y^2 + (e-1) y \right]_1^e$$

$$= \frac{e^2}{2} - 2e + \frac{13}{4} = \frac{1}{4} (e^2 - 8e + 13)$$

(5) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

Sol:- Integrating first w.r.to z keeping x and y constant we have

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{xy}{2}(1-x^2-y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} (xy-x^3y-xy^3)\,dy\,dx$$

Integrating now w.r.to 'y' keeping x constant, we have

$$= \frac{1}{2}\int_0^1\left[\frac{xy^2}{2}-\frac{x^3y^2}{2}-\frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{4}\left[\int_0^1 (x(1-x^2)-x^3(1-x^2)-\frac{x}{2}(1-x^2)^2\right]dx$$

$$= \frac{1}{4}\left[\int_0^1 (x-2x^3+x^5)\right]dx-\frac{1}{8}\int_0^1 x(1-x^2)^2\,dx$$

Finally Integrated w.r.t x we have

$$= \frac{1}{4}\left[\frac{x^2}{2}-\frac{2}{4}x^4+\frac{x^6}{6}\right]_0^1-\frac{1}{8}\int_1^0 t^2\left(-\frac{dt}{2}\right) \quad (\text{putting } 1-x^2=t)$$

$$= \frac{1}{4}\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right]-\frac{1}{16}\int_0^1 t^2\,dt = \frac{1}{24}-\frac{1}{16}\left(\frac{t^3}{3}\right)_0^1$$

$$= \frac{1}{24}-\frac{1}{48} = \frac{1}{48}$$

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$1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\frac{x^4}{17}+\dots$

⑥ Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$ where V is the region of space bounded by $x=0, x=1$ $y=0, y=2$ $z=0$ $z=3$,

sol:- $\iiint_V (xy+yz+zx)\,dx\,dy\,dz = \int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1}(xy+yz+zx)\,dx\,dy\,dz$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left[\frac{x^2}{2}y + xyz + \frac{zx^2}{2}\right]_0^1 dy\,dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left[y/2 + yz + \frac{z}{2}\right]dy\,dz$$

$$= \int_{z=0}^{3}\left[\frac{y^2}{4} + \frac{y^2}{2}z + \frac{zy}{2}\right]_0^2 dz$$

$$= \int_{z=0}^{3}\left[\frac{4}{4} + \frac{4}{2}z + \frac{2z}{2} - 0\right]dz$$

$$\Rightarrow \int_{z=0}^{3}[1+2z+z]\,dz = \left[z + \frac{2z^2}{2} + \frac{z^2}{2}\right]_0^3$$

$$= 3+9+\frac{9}{2}$$

$$= 12+\frac{9}{2} = \frac{24+9}{2} = \frac{33}{2}$$