Probabilistic Stability Analysis of Planar Robots with Piecewise Constant Derivative Dynamics

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Abstract

In this paper, we study the probabilistic stability analysis of a subclass of stochastic hybrid systems, called the *Planar Probabilistic Piecewise Constant Derivative Systems (Planar PPCD)*, where the continuous dynamics is deterministic, constant rate and planar, the discrete switching between the modes is probabilistic and happens at boundary of the invariant regions, and the continuous states are not reset during switching. These aptly model piecewise linear behaviors of planar robots. Our main result is an exact algorithm for deciding *absolute* and *almost sure stability* of Planar PPCD under some mild assumptions on mutual reachability between the states and the presence of non-zero probability self-loops. Our main idea is to reduce the stability problems on planar PPCD into corresponding problems on Discrete Time Markov Chains with edge weights. Our experimental results on planar robots with faulty angle actuator demonstrate the practical feasibility of this approach.

1 Introduction

Stability of robotic trajectories is a desirable property, as it guarantees the correct tracking of the path plan even in the presence of external disturbances. In this paper, we investigate the stability of planar robots with constant rate dynamics where the sensor/actuator noise is captured using probabilistic mode switchings. More precisely, we study *Probabilistic Piecewise Constant Derivative Systems (PPCD)*, that consist of a finite number of discrete states representing different modes of operation each associated with a constant rate dynamics, and probabilistic mode switches enabled at certain polyhedral boundaries. These fall under the umbrella of Stochastic Hybrid Systems (SHS) [31].

Safety analysis of SHS has been extensively studied in the context of both non-stochastic as well as stochastic hybrid systems [29, 19, 9, 1, 20]; stability on the other hand is relatively less explored, especially, from a computational point of view. It is well-known that even for non-stochastic hybrid systems decidability (existence of exact algorithms) for safety is achievable only under restrictions on the dynamics and the dimension [16]. More recently, decidability of stability of hybrid systems has been explored in the non-stochastic

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setting [27]. The main contribution of this paper is the identification of a practically useful subclass of stochastic hybrid systems for which stability is decidable along with an exact stability analysis algorithm.

The classical stability analysis techniques build on the notion of Lyapunov functions that provide a certificate of stability. While the notion of Lyapunov functions have been extended to the hybrid system setting, computing them is a challenge. Typically, they require solving certain complex optimization problems, for instance, to deduce coefficients of polynomial templates, and more importantly, need the exploration of increasingly complex templates. In this paper, we take an alternate route where we present graph theory based reductions to show the decidability of stability analysis.

Our broad approach is to reduce a planar PPCD, that is a potentially infinite state probabilistic system, to that of a Finite State Discrete Time Markov Chain such that the stability of the planar PPCD can be deduced exactly by algorithmically checking certain properties of the reduced system. We study two notions of stability, namely, absolute convergence and almost sure convergence. In the former, we seek to ensure that every execution converges, while in the latter, we require that the probability of the set of system executions that converge be 1. Absolute convergence ignores the probabilities associated with the transitions, and hence, can be solved using previous results on stability analysis of Piecewise Constant Derivative systems [26], where one checks for certain diverging transitions and cycles. Checking almost sure convergence is much more challenging. We show that almost sure convergence can be characterized by certain constraints based on the stationary distribution of the reduced system. For this result to hold, we need mild conditions on the PPCD that ensure the existence of this stationary distribution. The proof relies on several insights, including the properties of planar dynamics, and convergence results on infinite sequences of random variables.

We have implemented the algorithm in a Python toolbox. Our experimental results on multiple PPCDs modeling planar mobile robots demonstrate the feasibility of our approach. We have omitted proofs of the results due to space constraints; details can be found in the extended draft: `https://gitlab.cs.ksu.edu/-/ide/project/spandan/public/edit/main/-/main_icra.pdf`.

1.1 Related Work

Stability is a well studied problem in classical control theory, where Lyapunov function based methods have been extensively developed. They have been extended to hybrid systems using multiple and common Lyapunov functions [5, 10, 21, 33]. However, constructing Lyapunov functions is computationally challenging, hence, alternate approximate methods have been explored. For example, in one approach the state space is divided into certain regions and shown that the system inevitably ends up in a certain region, thus ensuring stability [13, 14, 23, 24]. Another approach is based on abstraction, where a simplified model (known as the abstract model) is created based on the original model and stability analysis on the simplified model is mapped back to the original one [2, 6, 28, 11, 1, 9, 25, 26].

While stability has been extensively studied in non-probabilistic setting, investigations of stability for probabilistic systems are limited. Sufficient conditions for stability of Stochastic Hybrid Systems via Lyapunov functions is discussed in the survey [32]. Almost sure expo-

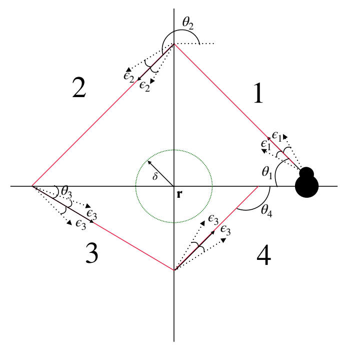

Figure 1: Motion of planar robot with faulty heading angle actuator

nential stability [7, 8, 12, 17] and asymptotic stability in distribution [35, 34] for Stochastic Hybrid Systems have also been studied. Most of these works on probabilistic stability analysis provide approximate mehtods for analysis. We provide a simple class of Stochastic Hybrid Systems that have practical application in modeling planar robots, and an exact decidable algorithm for probabilistic stability analysis.

2 Case Study: Planar robot with a faulty actuator

Consider a robot navigating in a 2D plane at some constant speed v as shown in Figure 1. The plane is divided into four regions R_1, R_2, R_3, R_4 corresponding to the four quadrants, and the robot has a unique direction θ_i (mode of operation) in which it moves while in the region R_i, and changes its mode of operation at the boundary of the regions. Due to faulty actuator, the robot heading angle may deviate from θ_i by an amount ϵ_i. We model this as probabilistically choosing one of the k_i uniformly distanced angles $\theta_i^1, \cdots, \theta_i^{k_i}$ in the interval $[\theta_i - \epsilon_i, \theta_i + \epsilon_i]$ with probabilities $p_i^1, \cdots, p_i^{k_i}$, respectively. The whole system can be modelled as a planar PPCD with $\sum_{i=1}^4 k_i$ modes, where for every i and $1 \leqslant j \leqslant k_i$, the mode q_i^j corresponds to the robot traversing with heading angle θ_i^j with speed v in the region R_i. The mode switching is possible between R_i and R_j if they are neighbors, that is, they share a common boundary. For instance, we can switch between quadrants 1 and 2 or 4 and 1 but not 1 and 3. We can move to any mode corresponding to a neighbor q_i^j with probability p_i^j.

The objective of the navigation is to reach a target point r on the 2D plane arbitrarily closely. More precisely, we want to check whether the robot reaches within a $\delta > 0$ ball around r for any arbitrarily small δ. We want to check if all executions of the robot have this property (absolute convergence) as well as if the probability of convergence is 1 (almost sure convergence).

3 Preliminaries

In this section, we will discuss important concepts related to Discrete Time Markov Chain (DTMC), Weighted Discrete Time Markov Chain (WDTMC) and convergence of WDTMC.

3.1 Discrete Time Markov Chain

Let $Dist(S)$ denote the set of all probability distributions on the set S. Let us define Discrete Time Markov Chain (DTMC) on the set of states S.

Definition 3.1 (Discrete Time Markov Chain)**.** *The Discrete Time Markov Chain (DTMC) is defined as the tuple $\mathcal{M} = (S, P)$ where*

- *S is a set of states.*
- *$P : S \mapsto Dist(S)$ is a function from the set of states S to the set of all probability distributions over S, $Dist(S)$.*

We use $P(s_1, s_2)$ to denote $P(s_1)(s_2)$ and $P^n(s_1, s_2)$ to denote the probability of going from s_1 to s_2 in n-steps.

A path of a DTMC $\mathcal{M}$ is a sequence of states $\sigma = s_1, s_2, \ldots$ such that for all $i < |\sigma|$, $P(s_i, s_{i+1}) > 0$, where $|\sigma|$ is the length of the sequence. A path of length 2 is called an edge. The i^{th} state of the path σ is denoted by σ_i and the last state of σ is denoted as σ_{end}. $\sigma[i : j]$ denotes the subsequence $(\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$. We say s_2 is reachable from s_1 (denoted $s_1 \rightsquigarrow s_2$) if there is a path σ on $\mathcal{M}$ such that $\sigma_1 = s_1$ and $\sigma_{end} = s_2$. The set of all finite paths of a DTMC $\mathcal{M}$ is denoted as $Paths_{fin}(\mathcal{M})$ and the set of all infinite paths is denoted as $Paths(\mathcal{M})$.

The probability of a path σ, denoted $P(\sigma)$, is the product of the probabilities of each of its edges, $P(\sigma) := \prod_{i<|\sigma|} P(\sigma_i, \sigma_{i+1})$. The probability of σ with respect to a distribution ρ, denoted $P_\rho(\sigma)$ is the product of $P(\sigma)$ and the probability of σ_1 under ρ, $P_\rho(\sigma) := \rho(\sigma_1) \cdot P(\sigma)$. We can associate a probability measure Pr to the set of infinite execution paths $Paths(\mathcal{M})$ of a DTMC $\mathcal{M}$ using probability of the cylinder sets of the finite paths as discussed in [4].

Next we define some subclasses of DTMC and show that it has some nice convergence properties.

Definition 3.2 (Irreducibility)**.** *A DTMC $\mathcal{M}$ is called irreducible if for any $s_1, s_2 \in S$, $s_1 \rightsquigarrow s_2$ and $s_2 \rightsquigarrow s_1$.*

Definition 3.3 (Periodicity)**.** *A state $s \in S$ in a DTMC $\mathcal{M}$ is called periodic if for any path σ starting and ending at s, $|\sigma|$ is a multiple of some natural number greater than* 1*. A DTMC $\mathcal{M}$ is called aperiodic if none of its states is periodic.*

We say a probability distribution is stationary for a DTMC $\mathcal{M}$ if the next step distribution remains unchanged.

Definition 3.4 (Stationary Distribution)**.** *A probability distribution $\rho^* \in Dist(S)$ is called the stationary distribution of DTMC $\mathcal{M}$ if,*

$$\rho^*(s) = \sum_{s' \in S} \rho^*(s') P(s', s), \quad \forall s \in S.$$

For finite, irreducible DTMC, the stationary distribution is unique. The following theorem (see [30]) guarantees existence of limiting distribution for finite, irreducible and aperiodic DTMC and associates it with the stationary distribution of the DTMC.

Theorem 3.5. *For a finite, irreducible and aperiodic DTMC* $\lim_{n\to\infty} P^n(s_1, s_2)$ *exists for all* $s_1, s_2 \in S$ *and* $\lim_{n\to\infty} P^n(s_1, s_2) = \rho^*(s_2)$ *where* $\rho^* \in Dist(S)$ *is the unique stationary distribution of* $\mathcal{M}$*.*

Note that $P^n(s_1, s_2)$ does not depend on s_1 as $n \to \infty$.

We will also be interested in probabilities of conditional events on a DTMC. On a DTMC, the probability of the current event only depends on the last observed event which helps in analysis of conditional events. This is called the *memoryless* property. For example, suppose an execution starts at state s and after k-steps the finite path σ' is observed. We denote the probability of this event by $P^k(\sigma' \mid s)$. Given that this event has occurred, we are interested in the probability that after n-steps ($n \geqslant k + |\sigma'| - 1$) from s, the finite path σ will be observed. We denote this probability by $P^n(\sigma \mid s \overset{k}{\rightsquigarrow} \sigma')$. Due to memoryless property this probability is same as $P^{(n-(k+|\sigma'|-1))}(\sigma \mid \sigma'_{end})$.

3.2 Weighted Discrete Time Markov Chain

Let us now define Weighted Discrete Time Markov Chain (WDTMC) that extend DTMC with weighted edges.

Definition 3.6 (Weighted DTMC)**.** *The weighted DTMC (WDTMC)* $\mathcal{M}_W = (S, P, W)$ *is a tuple such that* (S, P) *is a DTMC and* $W : \mathcal{E} \mapsto \mathbb{R}$ *is a weight function where* $\mathcal{E}$ *is the set of all possible edges of* $\mathcal{M}_W$*.*

With the weight function W defined, it is possible to associate weights to individual paths of $\mathcal{M}_W$.

Definition 3.7 (Weight of a path)**.** *The weight of a path* σ *of WDTMC* $\mathcal{M}_W$*, denoted* $W(\sigma)$*, is defined as,*

$$W(\sigma) := \sum_{i<|\sigma|} W(\sigma_i, \sigma_{i+1})$$

A simple path is a path without state repetition and a simple cycle is a path where only the starting and the ending states are same. We use the notation $\mathcal{SP}$ for the set of all simple paths and the notation $\mathcal{SC}$ for the set of all simple cycles of a WDTMC $\mathcal{M}_W$.

3.3 Convergence of Weighted Discrete Time Markov Chain

Let us define the notions of absolute and probabilistic convergence of WDTMC. A WDTMC is said to be absolutely convergent if the weight of every infinite path is upper bounded by a constant.

Definition 3.8 (Absolute Convergence of WDTMC)**.** *A WDTMC* $\mathcal{M}_W$ *is said to be absolutely convergent if* $\exists K \geqslant 0$ *such that for all infinite path* $\sigma \in Paths(\mathcal{M}_W)$*,* $W(\sigma) \leqslant K$*.*

Further, a WDTMC is said to be almost surely convergent if the weight of an infinite path is upper bounded by a constant with probability 1.

Definition 3.9 (Almost Sure Convergence of WDTMC)**.** *We say that a WDTMC* $\mathcal{M}_W$ *is almost surely convergent if* $\exists K \geqslant 0$ *such that for any path* σ *of* $\mathcal{M}_W$*,* $W(\sigma) \leqslant K$ *with probability* 1*. In other words,* $Pr\{\sigma \in Paths(\mathcal{M}_W) \mid W(\sigma) \leqslant K\} = 1$*.*

3.4 Polyhedral Sets

We denote the set of all polyhedral subsets of $\mathbb{R}^n$ by $Poly(n)$. The facets of a polyhedral subset A are the largest polyhedral subsets of the boundary of A. We denote the boundary of a polyhedral subset A by $\partial(A)$ and the set of all facets of A by $\mathbb{F}(A)$. We say a polyhedral subset P is positive scaling invariant if for all $x \in P$ and $\alpha > 0$, $\alpha x \in P$.

4 Analyzing Convergence of Weighted Discrete Time Markov Chains

In this section, we discuss necessary and sufficient conditions for absolute and almost sure convergence of WDTMC. For our analysis, we will assume all paths of the WDTMC start from a single state called the *initialization point* (denoted s_{init}) of the DTMC. In other words we restrict our attention to the set of paths $\Sigma' := \{\sigma \in Paths(\mathcal{M}_W) \mid \sigma_1 = s_{init}\}$.

4.1 Analyzing absolute convergence of Weighted DTMC

Here we provide a necessary and sufficient condition for analyzing absolute convergence of a WDTMC. We begin with the following theorem which states that for any finite path $\sigma \in Paths_{fin}(\mathcal{M}_W)$, we can get one simple path and a set of simple cycles such that their total weight equals the weight of σ.

Theorem 4.1. *For any finite path σ of $\mathcal{M}_W$ there exist a simple path $\sigma_s \in \mathcal{SP}$ and a set of simple cycles $\mathcal{SC}_\sigma \subseteq \mathcal{SC}$ such that $W(\sigma) = W(\sigma_s) + \sum_{\mathcal{C} \in \mathcal{SC}_\sigma} W(\mathcal{C})$.*

Proof. We traverse σ and whenever a cycle $\mathcal{C}$ is encountered, remove its edges from σ and add the cycle to the set $\mathcal{SC}_\sigma$. Thus when σ is traversed entirely, $\mathcal{SC}_\sigma$ contains only simple cycles and the remaining edges form a simple path $\sigma_s = \sigma - (\cup\{\mathcal{C} \mid \mathcal{C} \in \sigma_c\})$. Let $S^{\sigma_s}_{\rightarrow}$ denote the set of edges of σ_s and for each $\mathcal{C} \in \mathcal{SC}_\sigma$, $S^{\mathcal{C}}_{\rightarrow}$ denote the set of edges of $\mathcal{C}$. Clearly, $\{S^{\sigma_s}_{\rightarrow}\} \cup \{S^{\mathcal{C}}_{\rightarrow} \mid \mathcal{C} \in \mathcal{SC}_\sigma\}$ is a partition of the set of edges of σ. Thus $W(\sigma) = W(\sigma_s) + \sum_{\mathcal{C} \in \mathcal{SC}_\sigma} W(\mathcal{C})$. Hence, our claim is proved. ☐

We use Theorem 4.1 to prove the following main theorem which states that a WDTMC is absolutely convergent iff there is no edge of infinite weight and no cycle of weight greater that 1 reachable from the initial point.

Theorem 4.2 (Equivalent condition for absolute convergence)**.** *The WDTMC $\mathcal{M}_W$ is absolutely convergent iff,*

1. *There does not exist a edge (s_1, s_2) reachable from s_{init} such that $W(s_1, s_2) = \infty$.*
2. *For any simple cycle $\mathcal{C}$ reachable from s_{init}, $W(\mathcal{C}) \leqslant 0$.*

Proof. ($\Rightarrow$) The conditions 1 and 2 are necessary if for negation of either of them, absolute convergence of $\mathcal{M}_W$ is lost. If condition 1 is false then there is a edge (s_1, s_2) with $W(s_1, s_2) = \infty$ such that for some finite path σ starting from s_{init}, $\sigma_{|\sigma|-1} = s_1$ and $\sigma_{|\sigma|} = s_2$. But that implies $W(\sigma) = \sum_{i=1}^{|\sigma|-1} W(\sigma_i, \sigma_{i+1}) = \infty \nleq K$ for any $K \geqslant 0$. So for any infinite path σ' with prefix σ, $W(\sigma') = \infty$. Thus $\mathcal{M}_W$ is not absolutely convergent. On the other hand if we suppose condition 2 is false then there is a simple cycle $\mathcal{C} \in \mathcal{SC}$ with $W(\mathcal{C}) > 1$ such that for some finite path σ starting from s_{init}, there exists an index j such that $\mathcal{C} = \sigma[j : |\sigma|]$. Now we can easily construct the following infinite path $\sigma_\infty = \sigma \cdot \mathcal{C} \cdot \mathcal{C} \ldots$ by concatenating $\mathcal{C}$ infinite times to σ. Clearly, σ_∞ starts at s_{init} since σ starts at s_{init} and $W(\sigma_\infty) = W(\sigma) + \sum_{n \in \mathbb{N}} W(\mathcal{C}) = \infty \nleqslant K$ for any $K \geqslant 0$. Again, $\mathcal{M}_W$ is not absolutely convergent.

($\Leftarrow$) Conversely, suppose both conditions 1 and 2 hold. Now we know for any finite path σ of $\mathcal{M}_W$ there exist a simple path σ_s and a set of simple cycles $\mathcal{SC}_\sigma$ such that $W(\sigma) = W(\sigma_s) + \sum_{\mathcal{C} \in \mathcal{SC}_\sigma} W(\mathcal{C})$ (Theorem 4.1). Let σ start from s_{init}. Then σ_s and the set of simple cycles $\mathcal{SC}_\sigma$ that are reachable from s_{init}. Now $W(\sigma_s)$ is at most $\sum_{(s_1,s_2) \in S_\rightarrow} \max\{W(s_1, s_2) \mid (s_1, s_2) \in S_\rightarrow\} < \infty$. By condition 2 each simple cycle in $\mathcal{SC}_\sigma$ has weight less than 1. Thus $W(\sigma) = W(\sigma_s) + \sum_{\mathcal{C} \in \sigma_c} W(\mathcal{C}) < K$ where K is any positive real number greater than $\sum_{(s_1,s_2) \in S_\rightarrow} \max\{W(s_1, s_2) \mid (s_1, s_2) \in S_\rightarrow\}$. Now consider any infinite path π of $\mathcal{M}_W$ starting from s_{init}. For each $i \in \mathbb{N}$, the prefix $\pi[1 : i]$ of π is a finite path of $\mathcal{M}_W$ and by above observation $W(\pi[1 : i]) < K$. Thus $W(\pi) = \lim_{i \to \infty} W(\pi[1 : i]) < K$, i.e., $\mathcal{M}_W$ is absolutely convergent. □

4.2 Analyzing almost sure convergence of Weighted DTMC

In this subsection, we will provide a necessary and sufficient condition for almost sure convergence of a WDTMC. We assume a WDTMC $\mathcal{M}_W$ is finite, irreducible and aperiodic and thus has the limiting distribution equal to its stationary distribution ρ^* (Theorem 3.5). The main theorem basically states that a WDTMC is almost surely convergent iff the value obtained by multiplying the weight of each edge with its probability with respect to ρ^* and then summing it over all edges cannot cross 0 from below.

Theorem 4.3. *Let $\mathcal{M}_W$ be a WDTMC. $\mathcal{M}_W$ is almost surely convergent iff $\sum_{e \in \mathcal{E}} P_{\rho^*}(e) W(e) \leqslant 0$, where $\mathcal{E}$ is the set of all edges on $\mathcal{M}_W$.*

We say $\sum_{e \in \mathcal{E}} P_{\rho^*}(e) W(e)$ is the effective weight of the WDTMC $\mathcal{M}_W$ and denote it as $W_\mathcal{E}$. We define partial average weight upto n for an infinite path σ as

$$\frac{(S_\sigma)_n}{n} := \frac{\sum_{i=1}^{n} W(\sigma_i, \sigma_{i+1})}{n}$$

Next we state the main lemma of this subsection which essentially says the average weight of an infinite path is $W_\mathcal{E}$ almost surely.

Lemma 4.4. *$\frac{(S_\sigma)_n}{n} \to \sum_{i=1}^{N} P_{\rho^*}(e_i) W(e_i)$ as $n \to \infty$ almost surely, where $P_{\rho^*}(e_i)$ is the probability of the single edge e_i with respect to the stationary distribution ρ^*.*

Observe that $W(\sigma) = \lim_{n \to \infty} n \cdot ((S_\sigma)_n / n)$. Thus using Lemma 4.4 we can easily prove Theorem 4.3 (detailed proof in Appendix 7.2).

To prove Lemma 4.4 we enumerate edges of $\mathcal{M}_W$ as $e_1, \ldots, e_N$ and define random variables $\{X_j^i \mid i \in [N]; j \in \mathbb{N}\}$ so that X_j^i completely captures the information of which edge is appearing on which step of an infinite path σ.

$$X_j^i = \begin{cases} 1 \text{ if } (\sigma_j, \sigma_{j+1}) = e_i \\ 0 \text{ else.} \end{cases}$$

Observe that,

$$\frac{(S_\sigma)_n}{n} = \frac{\sum_{i=1}^{N}(\#\text{ times } e_i \text{ appears on } \sigma[1:n+1]) \cdot W(e_i)}{n}$$
$$\text{and } \frac{\sum_{j=1}^{n} X_j^i}{n} = \frac{(\#\text{ times } e_i \text{ appears in } \sigma[1:n+1])}{n}.$$
$$\text{Thus, } \frac{(S_\sigma)_n}{n} = \frac{\sum_{i=1}^{N}\left(\sum_{j=1}^{n} X_j^i\right) \cdot W(e_i)}{n}$$

Our approach is to show that $\frac{\sum_{j=1}^{n} X_j^i}{n} \to P_{\rho^*}(e_i)$ as $n \to \infty$ almost surely, which we do by using a variant of strong law of large numbers obtained as the corollary of a result discussed in [22].

Theorem 4.5. *Let $\{Y_i\}_{i\in\mathbb{N}}$ be non-iid L_2 random variables such that for all i $E[Y_i] = 0$, $E[Y_i^2] \leqslant 1$ for some $M > 0$ and for any $i, j \in \mathbb{N}$, there exist constants $C > 0$ and $\lambda \in (0,1)$ such that $|Cov(Y_i, Y_j)| \leqslant C\lambda^{|i-j|}$, then $\frac{\sum_{i=1}^{n} Y_i}{n} \to 0$ as $n \to \infty$ almost surely.*

However to use Theorem 4.5, we need to show that $|Cov(X_j^i, X_k^i)|$ decays exponentially which is one of our important contributions.

Lemma 4.6. *There exist constants $C > 0$ and $\lambda \in (0,1)$ such that $|Cov(X_j^i, X_k^i)| \leqslant C\lambda^{|j-k|}$.*

Proof. The proof of Lemma 4.6 depends on results on the rate of convergence of the WDTMC to its stationary distribution as number of steps $n \to \infty$. We mention these results with proof in Appendix 7.1.

Observe that, we can assume without loss of generality $j \geqslant k$, since Cov is a symmetric function (i.e., $Cov(X_j^i, X_k^i) = Cov(X_k^i, X_j^i)$). Hence it is enough to prove $|Cov(X_j^i, X_k^i)| \leqslant C\lambda^{(j-k)}$. Observe that the product random variable $X_j^i X_k^i$ is Bernoulii with

$$X_j^i X_k^i = \begin{cases} 1 \text{ if } X_j^i = 1 \text{ and } X_k^i = 1 \\ 0 \text{ else.} \end{cases}$$

We know if a random variable X is Bernoulii then $E[X] = Pr(X = 1)$. Using this fact we get,

$$
\begin{aligned}
&|Cov(X_j^i, X_k^i)| \\
=&|E[X_j^i X_k^i] - E[X_j^i]E[X_k^i]| \\
=&|Pr(X_j^i X_k^i = 1) - Pr(X_j^i = 1)Pr(X_k^i = 1)| \\
=&Pr(X_k^i = 1)|Pr(X_j^i = 1 \mid X_k^i = 1) - Pr(X_j^i = 1)| \\
&\left(\text{since } Pr(X_j^i = 1 \mid X_k^i = 1) = \frac{Pr(X_j^i X_k^i = 1)}{Pr(X_k^i = 1)}\right) \\
=&|Pr(X_j^i = 1 \mid X_k^i = 1) - P_{\rho*}(e_i) \\
&- Pr(X_j^i = 1) + P_{\rho*}(e_i)| \quad \big(\text{since } Pr(X_k^i = 1) \leqslant 1\big) \\
\leqslant&|Pr(X_j^i = 1 \mid X_k^i = 1) - P_{\rho*}(e_i)| \\
&+ |Pr(X_j^i = 1) - P_{\rho*}(e_i)| \quad (\text{using triangle inequality}) \\
\leqslant&C'\lambda^{(j-k)} + C'\lambda^j \text{ for some } C' > 0, \lambda \in (0,1) \\
&(\text{using Lemma 7.2 and Corollary 7.3}) \\
\leqslant&2C'\lambda^{(j-k)} \quad (\text{since } \lambda^{(j-k)} \geqslant \lambda^j) \\
\leqslant&C\lambda^{(j-k)} \quad (\text{where } C = 2C')
\end{aligned}
$$

Thus our claim is proved. □

Using Theorem 4.5 we prove that partial average of X_j^i upto n converges to partial average of $E[X_j^i]$ upto n (average is taken over j) almost surely.

Lemma 4.7. *As $n \to \infty$, $\frac{\sum_{j=1}^n X_j^i}{n} \to \frac{\sum_{j=1}^n E(X_j^i)}{n}$ almost surely.*

Proof. To satisfy conditions of Theorem 4.5, we define random variables $\{\hat{X}_j^i \mid j \in \mathbb{N}\}$ (for a fixed i), where $\hat{X}_j^i = X_j^i - E[X_j^i]$. Clearly then $E[\hat{X}_j^i] = 0$ and $E[(\hat{X}_j^i)^2] = E[\left(X_j^i - E[X_j^i]\right)^2] = Var(X_j^i) \leqslant 1$ (since X_j^i is Bernoulii). Now $Cov(\hat{X}_j^i, \hat{X}_k^i) = E[\hat{X}_j^i \hat{X}_k^i] = E[X_j^i X_k^i] - E[X_j^i]E[X_k^i] = Cov(X_j^i, X_k^i)$. We have already shown in Lemma 4.6 that there exist $C > 0$ and $\lambda \in (0,1)$ such that $|Cov(X_j^i, X_k^i)| \leqslant C\lambda^{|j-k|}$. Hence we can use Theorem 4.5 to conclude that,

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{\sum_{j=1}^n \hat{X}_j^i}{n} \to 0 \text{ almost surely} \\
\implies &\lim_{n\to\infty} \left(\frac{\sum_{j=1}^n X_j^i}{n} - \frac{\sum_{j=1}^n E[X_j^i]}{n}\right) = 0 \text{ almost surely} \\
\implies &\lim_{n\to\infty} \left(\frac{\sum_{j=1}^n X_j^i}{n}\right) = \lim_{n\to\infty} \left(\frac{\sum_{j=1}^n E[X_j^i]}{n}\right) \\
&\text{almost surely.}
\end{aligned}
$$

Hence our claim is proved. □

Finally, we prove partial average of $E[X_j^i]$ upto n (average is taken over j) converges to $P_{\rho^*}(\sigma)$ as $n \to \infty$, which gives us our desired result.

Lemma 4.8. $\frac{\sum_{j=1}^{n} E[X_j^i]}{n} \to P_{\rho^*}(e_i)$ *as* $n \to \infty$*.*

Proof. Since $E[X_j^i] = Pr(X_j^i = 1)$, from Corollary 7.3 we have,

$$\begin{aligned}
&\left| \frac{\sum_{j=1}^{n} E[X_j^i]}{n} - P_{\rho^*}(e_i) \right| \\
=&\frac{1}{n} \left| \sum_{j=1}^{n} (E[X_j^i] - P_{\rho^*}(e_i)) \right| \\
=&\frac{1}{n} \sum_{j=1}^{n} |E[X_j^i] - P_{\rho^*}(e_i)| \quad \text{(using triangle inequality)} \\
\leqslant&\frac{1}{n} \sum_{j=1}^{n} C\lambda^j
\end{aligned}$$

for some $C > 0$ and $\lambda \in (0, 1)$. Now,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} C\lambda^j \\
\leqslant &\lim_{n\to\infty} \frac{C\lambda}{n(1-\lambda)} = 0
\end{aligned}$$

Hence $\lim_{n\to\infty} \left| \frac{\sum_{j=1}^{n} E[X_j^i]}{n} - P_{\rho^*}(e_i) \right| \leqslant 0$. But $\left| \frac{\sum_{j=1}^{n} E[X_j^i]}{n} - P_{\rho^*}(e_i) \right| \geqslant 0$ for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} \frac{\sum_{j=1}^{n} E[X_j^i]}{n}$ must be equal to $P_{\rho^*}(e_i)$ and our claim is proved. $\square$

4.3 Computability

Based on Theorems 4.2 and 4.3 we present two algorithms here for checking absolute and almost sure convergence of a WDTMC. Assuming the WDTMC is finite, Algorithm 1 first checks for existence of an infinite weight edge by Breadth First Search (BFS) [18] and then for a strictly positive weight cycle using a variant of the Bellman-Ford algorithm [18]. If neither of them is found then the WDTMC is deemed absolutely convergent by Theorem 4.2. Since BFS takes time linear to the size of its input and Bellman-Ford takes time quadratic to the size of its input, the time complexity of Algorithm 1 is $O(|S|^2)$, where S is the set of states of $\mathcal{M}_W$.

Assuming the WDTMC is finite, irreducible and aperiodic, Algorithm 2 first checks for existence of an infinite weight edge by Breadth First Search (BFS). If such an edge exists then the WDTMC is deemed not almost surely convergent (by Theorem 4.3). Otherwise the stationary distribution ρ^* of the WDTMC is calculated solving a set of linear equations mentioned in Definition 3.4. The value $\sum_{e\in\mathcal{E}} P_{\rho^*}(e)W(e)$ is then calculated (where $\mathcal{E}$ is the set of transitions of the WDTMC) and compared to 0. The WDTMC is deemed almost

Data: A WDTMC $\mathcal{M}_W := (S, P, W)$
Result: Yes/No
Convert $\mathcal{M}_W$ to a weighted graph $G = (V, E, W')$ where $V = S$,
$E = \{(s_1, s_2) \in S \times S \mid P(s_1, s_2) > 0\}$, and $W' : E \to \mathbb{R}$ defined as
$W'(e) := -W(e)$;
Run BFS on G to check existence of edge with weight $-\infty$;
if *(edge with $-\infty$ weight exists)* **then**
| Return No;
end
Run Bellman-Ford algorithm on G for checking existence of a negative weight cycle;
if *(cycle with negative weight is found)* **then**
| Return No;
else
| Return Yes;
end

Algorithm 1: Checking absolute convergence of WDTMC

Data: A WDTMC $\mathcal{M}_W := (S, P, W)$
Result: Yes/No
Convert $\mathcal{M}_W$ to a weighted graph $G = (V, E, W')$ where $V = S$,
$E = \{(s_1, s_2) \in S \times S \mid P(s_1, s_2) > 0\}$, and $W' : E \to \mathbb{R}$ defined as $W'(e) := W(e)$;
Run BFS on G to check existence of edge with weight ∞;
if *(edge with ∞ weight exists)* **then**
| Return No;
end
Calculate stationary distribution ρ^* of $\mathcal{M}_W$ by solving the set of linear equations,

$$\rho^*(s) = \sum_{s' \in S} \rho^*(s') P(s', s), \quad \forall s \in S$$
$$\sum_{s \in S} \rho^*(s) = 1$$

;
$asWeight \leftarrow 0$;
for $e \in E$ **do**
| $asWeight = asWeight + P_{\rho^*}(e) W'(e)$;
end
if $asWeight \leqslant 0$ **then**
| Return Yes;
else
| Return No;
end

Algorithm 2: Checking almost sure convergence of WDTMC surely convergent only if $\sum_{e \in \mathcal{E}} P_{\rho^*}(e) W(e) \leqslant 0$. Since BFS takes time linear to its input size and solving a set of linear equations takes time at most cubic to the number of variables, the time complexity of Algorithm 2 is $O(|S|^3)$, where S is the set of states of $\mathcal{M}_W$.

5 Probabilistic Piecewise Constant Derivative Systems

In this section, we present the details of the Probabilistic Piecewise Constant Derivative Systems (PPCD) and provide a characterization of absolute and almost sure convergence by a reduction to that of DTMCs.

5.1 Formal Definition of PPCD

We model PPCDs as consisting of a discrete set of modes, each associated with an invariant and probabilistice transitions between modes that are enabled at the boundaries of the invariants.

Definition 5.1 (PPCD)**.** *The Probabilistic Piecewise Constant Derivative System (PPCD) is defined as the tuple $\mathcal{H} := (Q, \mathcal{X}, Inv, Flow, Edges)$ where*

- *Q is the set of discrete locations,*
- *$\mathcal{X} = \mathbb{R}^n$ is the continuous state space for some $n \in \mathbb{N}$,*
- *$Inv : Q \to Poly(n)$ is the invariant function which assigns a positively invariant polyhedral subset of the statespace to each location $q \in Q$,*
- *$Flow : Q \to \mathcal{X}$ is the Flow function which assigns a flow vector, say $Flow(q) \in \mathcal{X}$, to each location $q \in Q$,*
- *$Edges \subseteq Q \times (\cup_{q \in Q} \mathbb{F}(Inv(q))) \times Dist(Q)$ is the probabilistic edge relation such that $(q, f, \rho) \in Edges$ where for every (q, f), there is a at most one ρ such that $(q, f, \rho) \in Edges$ and $f \in \mathbb{F}(Inv(q))$. f is called a* Guard *of the location q.*

Next, we discuss the semantics of the PPCD. An execution starts from a location $q_0 \in Q$ and some continuous state $x_0 \in \mathcal{X}$ and evolves continuously for some time T according to the dynamics of q_0 until it reaches a facet f_0 of the invariant of q_0. Then a probabilistic discrete transition is taken if there is an edge (q_0, f_0, ρ_0) and the state q_0 is probabilistically changed to q_1 with probability $\rho_0(q_1)$. The execution (tree) continues with alternating continuous and discrete transitions.

Formally, for any two continuous states $x_1, x_2 \in \mathcal{X}$ and $q \in Q$, we say that there is a *continuous transition* from x_1 to x_2 with respect to q if $x_1, x_2 \in Inv(q)$, there exists $T \geqslant 0$ such that $x_2 = x_1 + Flow(q) \cdot T$, $x_1 + Flow(q) \cdot t \notin \partial(Inv(q_0))$ for any $0 \leqslant t < T$ and $x_2 \in \partial(Inv(q_0))$. We note that there is a unique continuous transition from any state (q, x) since it requires the state to evolve until it reaches the boundary for the first time, which corresponds to a unique time evolution T. Further, if for all $t \geqslant 0$, $x_1 + Flow(q) \cdot t \in Inv(q)$ then we say x_1 has an infinite edge with respect to q. For two locations $q_1, q_2 \in Q$, we say there is a *discrete transition* from q_1 to q_2 with probability p via $\rho \in Dist(Q)$ and $f \in \mathbb{F}(q_1)$ if $f \subseteq Inv(q_2)$, $(q_1, f, \rho) \in Edges$ and $p = \rho(q_2)$.

We capture the semantics of a PPCD using a WDTMC, wherein we combine a continuous transition and a discrete transition to represent a probabilistic transition of the DTMC. In addition, to reason about convergence, we also need to capture the relative distance of the states from the equilibrium point, which is captured using edge weights. Let us fix 0 as the equilibrium point for the rest of the section. The weight on a transition from (q_1, x_1) to (q_2, x_2) captures the logarithm of the relative distance of x_1 and x_2 from 0, that is, it is $(||x_2||_\infty/||x_1||_\infty)$, where $||x||_\infty$ captures the distance of state x from 0.

Definition 5.2 (Semantics of PPCD)**.** *Given a PPCD $\mathcal{H}$, we can construct the WDTMC $\mathcal{M}_\mathcal{H} := (S_\mathcal{H}, P_\mathcal{H}, W_\mathcal{H})$ where,*

- $S_\mathcal{H} = Q \times \mathcal{X}$
- *$P_\mathcal{H}$ and $W_\mathcal{H}$ are defined as follows for any (q_1, x_1) and (q_2, x_2):*
  - *If there is a continuous transition from x_1 to x_2 with respect to q_1 and there is a discrete transition from q_1 to q_2 with probability p via some $\rho \in Dist(Q)$ and $f \in \mathbb{F}(q_1)$, and $x_2 \in f$, then $P_\mathcal{H}((q_1, x_1), (q_2, x_2)) = p$ and $W_\mathcal{H}((q_1, x_1), (q_2, x_2)) = \log(||x_2||_\infty/||x_1||_\infty)$*
  - *If x_1 has an infinite edge with respect to q_1, then $P_\mathcal{H}((q_1, x_1), (q_2, x_2)) = 1$ if $(q_1, x_1) = (q_2, x_2)$ and 0, otherwise, and $W_\mathcal{H}((q_1, x_1), (q_1, x_1)) = \infty$.*
  - *Otherwise, $P_\mathcal{H}((q_1, x_1), (q_2, x_2)) = W_\mathcal{H}((q_1, x_1), (q_2, x_2)) = 0$.*

We use the semantics to define the stability of PPCD. More precisely, the absolute and almost sure stability of a PPCD $\mathcal{H}$ is defined as the absolute and almost sure convergence of its semantics $\mathcal{M}_\mathcal{H}$. Note that the convergence of the $\mathcal{M}_\mathcal{H}$ implies that executions of PPCD that start close to 0 remain close to 0 which is the standard definition of stability. Consider an infinite path σ of $\mathcal{M}_\mathcal{H}$. For all $n \in \mathbb{N}$, $W(\sigma[1:n]) = \log(d(\sigma_n, 0)/d(\sigma_0, 0))$, where $d((q, x), 0) = ||x||_\infty$. This implies that if $\mathcal{M}_\mathcal{H}$ is absolutely convergent then there exists $K \geqslant 0$ such that for any path σ, $d(\sigma_n, 0) \leqslant e^K \cdot d(\sigma_0, 0)$ for all $n \in \mathbb{N}$. Thus for all $\epsilon > 0$, there exists $0 < \delta < \epsilon/e^K$ such that $d(\sigma_0, 0) \leqslant \delta$ implies $d(\sigma_n, 0) \leqslant \epsilon$ for all $n \in \mathbb{N}$, as in the definition of Lyapunov stability.

5.2 Characterization of Stability of 2D PPCD

In this subsection, we focus on 2D PPCD and provide characterizations for stability. Our broad approach is to reduce the analysis of stability to the analysis of convergence of a finite WDTMC. More precisely, we construct a quotient WDTMC $\mathcal{H}^{red}$ for a 2-dimensional PPCD $\mathcal{H}$ such that $\mathcal{H}^{red}$ has the exact behavior of the semantics $\mathcal{M}_\mathcal{H}$.

Our main observation for the reduction is that if there is a transition between two points x_1 and x_2 on facets f_1 and f_2 using a flow rate r, the scaling $\log(||x_2||_\infty/||x_1||_\infty)$ is independent of x_1 and x_2 and only depends on f_1, f_2 and r due to the fact that f_1 and f_2 belong to the boundaries of a positively scaled polyhedral set.

Lemma 5.3. *Let $e = ((q_1, x_1), (q_2, x_2)), e' = ((q_1, x_1'), (q_2, x_2'))$ such that $P_\mathcal{H}(e), P_\mathcal{H}(e') > 0$, $f_1, f_2 \in \bigcup_{q \in Q} \mathbb{F}(Inv(q))$, such that $x_1, x_1' \in f_1$ and $x_2, x_2' \in f_2$. Then $P_\mathcal{H}(e) = P_\mathcal{H}(e')$ and $W_\mathcal{H}(e) = W_\mathcal{H}(e')$.*

Proof. It is easy to observe that $P_{\mathcal{H}}(e) = P_{\mathcal{H}}(e')$ since by definition both $P_{\mathcal{H}}(e)$ and $P_{\mathcal{H}}(e')$ depend only on q_1 and f_2. Let us now prove that $W_{\mathcal{H}}(e) = W_{\mathcal{H}}(e')$ as well. Since $\mathcal{X} = \mathbb{R}^2$ and $s_0 = (q_0, 0)$, any face can be depicted by a formula of the form $y = kx$, where $k \in \mathbb{R}$. Let $x_1 = (x_1(1), x_1(2))$ and $x_2 = (x_2(1), x_2(2))$. By property of PPCD, $x_2 = x_1 + \mathit{Flow}(q_1) \cdot T$ for some $T \geqslant 0$. Thus,

$$\begin{aligned}&(x_2(1), x_2(2))\\ =&(x_1(1), x_1(2)) + (\mathit{Flow}(q_1)(1), \mathit{Flow}(q_1)(2))T\end{aligned} \tag{1}$$

Let $f_1 : y = k_1 x$ and $f_2 : y = k_2 x$. So,

$$x_2(2) = k_2 \cdot x_2(1) \tag{2}$$

$$x_1(2) = k_1 \cdot x_1(1) \tag{3}$$

Using equations 1,2,3 we can write $x_2(1) = c \cdot x_1(1)$ where c depends on k_1, k_2, $\mathit{Flow}(q_1)(1)$ and $\mathit{Flow}(q_1)(2)$. Thus $\frac{||x_2||_\infty}{||x_1||_\infty}$ can also be written in terms of k_1, k_2, $\mathit{Flow}(q_1)(1)$ and $\mathit{Flow}(q_1)(2)$ since $\frac{||x_2||_\infty}{||x_1||_\infty}$ is equal to either $|x_2(2)|/|x_1(2)|$ or $|x_2(2)|/|x_1(1)|$ or $|x_2(1)|/|x_1(2)|$ or $|x_2(1)|/|x_1(1)|$ and x_1 and x_2 dependent terms on numerator and denomenator always cancel off each other. $\square$

Lemma 5.3 shows that to calculate weight of a transition, it is enough to consider the facets and the locations it corresponds to. Next, we define a reduced system, the quotient WDTMC $\mathcal{H}^{red}$ for $\mathcal{H}$, which is a finite WDTMC with states $Q \times \bigcup_{q \in Q} \mathbb{F}(q)$.

Definition 5.4 (Quotient of PPCD). *Let $\mathcal{H}$ be a 2-dimensional PPCD and $\mathcal{M}_{\mathcal{H}}$ be its semantics. We define the WDTMC $\mathcal{H}^{red} = (S^{red}, P^{red}, W^{red})$ as follows,*

- $S^{red} = Q \times \bigcup_{q \in Q} \mathbb{F}(q)$
- $P^{red}((q_1, f_1), (q_2, f_2)) = P_{\mathcal{H}}((q_1, x_1), (q_2, x_2))$ *for some $x_1 \in f_1$ and $x_2 \in f_2$ such that $P_{\mathcal{H}}((q_1, x_1), (q_2, x_2)) > 0$, and 0 otherwise.*
- $W^{red}((q_1, f_1), (q_2, f_2)) = W_{\mathcal{H}}((q_1, x_1), (q_2, x_2))$ *for some $x_1 \in f_1$ and $x_2 \in f_2$ such that $P_{\mathcal{H}}((q_1, x_1), (q_2, x_2)) > 0$, and 0 otherwise.*

The above definition is well-defined, that is, the choice of x_1 and x_2 do not matter due to Lemma 5.3.

We assume that we start at some state $x_0 \in f$ for some $f \in \bigcup_{q \in Q} \mathbb{F}(q)$. We also assume $\mathcal{H}^{red}$ is aperiodic and irreducible. Irreducibility can be achieved by ensuring that any facet is reachable from any other facet, and aperiodicity can be achieved by adding a self-loop with non-zero probability to each discrete state. This is a reasonable assumption since at any point there is positive probability that the robot fails to move.

Next, we prove some important connections between the PPCD and the reduced system. First, we claim that for every infinite path in $\mathcal{M}_{\mathcal{H}}$, there is a path in $\mathcal{H}^{red}$ that has the same weight.

Theorem 5.5 (Conservation of weight). *For every infinite path σ of $\mathcal{M}_{\mathcal{H}}$, there is a path π in $\mathcal{H}^{red}$ such that $W(\sigma) = W(\pi)$.*

Proof. Let $\sigma = (\sigma_1, \sigma_2, \dots)$ be an infinite path of $\mathcal{M}_\mathcal{H}$. By assumption, $\sigma_i \in f_i$ where $f_i \in \bigcup_{q \in Q} \mathbb{F}(q)$ is a facet, for each $i \in \mathbb{N}$. Suppose for each i, $\sigma_i = (q_i, x_i)$. Since for each i, there is an edge between (q_i, x_i) and (q_{i+1}, x_{i+1}) in $\mathcal{M}_\mathcal{H}$, there should be an edge between (q_i, f_i) and (q_{i+1}, f_{i+1}) in $\mathcal{H}^{red}$. Using Lemma 5.3 we can conclude that for all i, $W((q_i, x_i), (q_{i+1}, x_{i+1})) = W((q_i, f_i), (q_{i+1}, f_{i+1}))$. Thus we can construct the infinite path $\pi = ((q_1, f_1), (q_2, f_2), \dots)$ such that $W(\pi) = W(\sigma)$. □

In fact, the converse of Theorem 5.5 is true as well. The proof relies on the observation in Lemma 5.3.

Theorem 5.6 (Converse of Theorem 5.5)**.** *For every infinite path π of $\mathcal{H}^{red}$, there is a path σ of $\mathcal{M}_\mathcal{H}$, such that $W(\pi) = W(\sigma)$.*

Proof. We prove by induction that for any $n \in \mathbb{N}$ if there is a path π of length n in $\mathcal{H}^{red}$ then there is a path σ of length n in $\mathcal{M}_\mathcal{H}$ with same weight as π.
Base case: Suppose $((q_1, f_1), (q_2, f_2))$ is an edge of $\mathcal{H}^{red}$. Then there exist $x_1 \in f_1$ and $x_2 \in f_2$ such that $x_2 = x_1 + Flow(q_1) \cdot t$, for some $t \geqslant 0$, i.e., there is an edge between (q_1, x_1) and (q_2, x_2) in $\mathcal{M}_\mathcal{H}$. Also by Lemma 5.3, $W((q_1, x_1), (q_2, x_2)) = W((q_1, f_1), (q_2, f_2))$. Hence base case is proved.

Now suppose $((q_1, f_1), \dots, (q_n, f_n), (q_{n+1}, f_{n+1}))$ is a path of $\mathcal{H}^{red}$ and by induction hypothesis we have a path $((q_1, x_1), \dots, (q_n, x_n))$ in $\mathcal{M}_\mathcal{H}$ such that $W((q_1, f_1), \dots, (q_n, f_n)) = W((q_1, x_1), \dots, (q_n, x_n))$. Since there is an edge between (q_n, f_n) and (q_{n+1}, f_{n+1}), there exist $x'_n \in f_n$ and $x'_{n+1} \in f_{n+1}$ such that

$$x'_{n+1} = x'_n + Flow(q_n) \cdot t \tag{4}$$

for some $t \geqslant 0$. Since $\mathcal{X} = \mathbb{R}^2$, f_n and f_{n+1} are rays. By Equation 4, there is a straight line of slope $Flow(q_n)$ that intersects both of them. But then any straight line with slope $Flow(q_n)$ intersecting f_n will also intersect f_{n+1}, in fact, if we take the straight line with slope $Flow(q_n)$ passing through x_n, it will intersect f_{n+1}. That means there exists $t \geqslant 0$ and $x_{n+1} \in f_{n+1}$ such that $x_{n+1} = x_n + Flow(q_n) \cdot t$. This is because for t to be negative, f_n and f_{n+1} must intersect and x_n and x'_n must lie on opposite sides of this intersection point on f_n. But this is impossible since f_n and f_{n+1} intersect only at 0 and both of them get terminated at 0. Thus there exist $x_{n+1} \in f_{n+1}$ such that $((q_n, x_n), (q_{n+1}, x_{n+1}))$ is an edge of $\mathcal{M}_\mathcal{H}$. By Lemma 5.3, $W((q_n, x_n), (q_{n+1}, x_{n+1})) = W((q_n, f_n), (q_{n+1}, f_{n+1}))$. Hence our claim is proved for all $n \in \mathbb{N}$, i.e., it holds for infinite paths of $\mathcal{H}^{red}$ as well. □

Thus, any result that is true for $\mathcal{H}^{red}$ remains true for $\mathcal{M}_\mathcal{H}$ and vice versa. Now, we state our main result for verifying stability of PPCDs that comes directly from the above observation,

Theorem 5.7 (Characterization for Stability)**.** *A PPCD $\mathcal{H}$ is absolutely stable with respect to 0 iff the quotient WDTMC $\mathcal{H}^{red}$ is absolutely convergent. A PPCD $\mathcal{H}$ is almost surely stable with respect to 0 iff the quotient WDTMC $\mathcal{H}^{red}$ is almost surely convergent.*

		Observed		Time (sec)		
N	Locs	AS	ASS	T_{conv}	T_{abs}	T_{as}
1	96	Yes	Yes	24.135	0.066	0.091
2	96	No	Yes	24.626	0.071	0.089
3	96	No	No	24.477	0.009	0.009

Figure 2: Table 1

6 Experiments and Observations

In this section, we provide the details of the implementation and the observations from our experiments. We represent the PPCD and the quotient WDTMC as annotated graphs and use *networkx* [15] module of Python to store and manipulate these entities. To calculate the edge weights, we need to solve linear optimization problem (as discussed in [26]) and that is done by *pplpy*, a Python module for Parma Polyhedra Library (PPL) [3]. We use *networkx* [15] functions to find the existence of infinite weighted edges and simple cycles with weight greater than 0 to check for absolute stability of the PPCD. To compute almost sure stability, we need to calculate the stationary distribution of the quotient WDTMC, which requires solving a set of linear equations mentioned in Definition 3.4. We use pplpy [3] for solving the feasibility of linear equations. Our experiments have been performed on macOS Big Sur with Quad-Core Intel Core i7 2.8GHz$\times$ 1 Processor and 16GB RAM.

We have analyzed three examples of PPCD with 96 locations each. To construct these examples, we partitioned the XY plane into 8 regions, corresponding to splitting each of the quadrants diagonally into two, and assigned 12 locations to each region. The guard for all locations corresponding to a region is same, and is set to the facet of that region that makes a larger angle with positive X-axis. The Flow for each location is of the form $a\dot{x} + b\dot{y} = 0$, where a and b depend on the corresponding region and experiment. For example in the first experiment, for regions 1 and 2, $a = 1$ and b is randomly chosen between $\{1, \dots, 5\}$, for regions 3 and 4, $b = 1$ and $-a$ is randomly chosen between $\{1, \dots, 5\}$, for regions 5 and 6, $a = -1$ and $-b$ is randomly chosen between $\{1, \dots, 5\}$, and finally, for regions 7 and 8, $b = -1$ and a is randomly chosen between $\{1, \dots, 5\}$. In the second experiment, same is done except for the first location where we have chosen $a = 50$ and $b = 1$ so that the PPCD will not be absolutely stable. For the third experiment also, we have only changed flow for the first location with $a = 1$ and $b = -5$, so that the PPCD becomes unstable. The probabilities for change of locations are chosen randomly. Note that all linear equations have integer coefficients and all probabilities are rational since PPL cannot handle linear equations with real coefficients.

We present our observations in Table 1. Here N represents the experiment number, *Locs* represents the number of locations in the PPCD, *AS* denotes absolute convergence, *ASS* denotes almost sure convergence, T_{conv} is the processor time to construct quotient WDTMC, T_{abs} is the processor time required to check absolute stability on the reduced systems and T_{as} is the processor time required to check almost sure stability. In all the three cases, the experimental result agreed with the expected result. Note that computing the quotient WDTMC is the most expensive part of the analysis, since it requires solving linear optimization problems proportional to $|Q|^2$, where Q is the set of locations of the PPCD. Analyses of both absolute and almost sure stability are fast compared to the quotient construction time, however, checking almost sure stability takes relatively longer since it requires the computation of the stationary distribution.

7 Conclusion

In this paper, we showed the decidability of absolute and almost sure convergence of Planar Probabilistic Piecewise Constant Derivative Systems (PPCD), that are a practically useful subclass of stochastic hybrid systems and can model motion of planar robots with faulty actuators. We give a computable characterization of absolute and almost sure convergence through a reduction to a finite state DTMC. In the future, we plan to extend these ideas to analyze higher dimensions PPCD and SHS with more complex dynamics. In particular, the idea of reduction can be applied to higher dimensional PPCD but we will need to extend our analysis to a Markov Decision Process that will appear as the reduced system.

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Appendix

7.1 Rate of Convergence of DTMC to the Stationary Distribution

The following theorem (see [30]) shows that for finite, irreducible and aperiodic DTMC, $P^n(s_1) \to \rho^*$ exponentially as $n \to \infty$.

Theorem 7.1. *Given an irreducible, aperiodic DTMC $\mathcal{M}$ with stationary distribution ρ^*, there exist constants $C > 0$ and $\lambda \in (0,1)$ such that $\forall n \geqslant 0$,*

$$|P^n(x,y) - \rho^*(y)| \leqslant C\lambda^n \quad \forall x, y \in S$$

Based on Theorem 7.1, we can show that on a finite, irreducible, aperiodic DTMC, the conditional probability that the same edge is observed on j^{th} step provided that it was observed on k^{th} step before, converges to the probability of that edge with respect to ρ^*, as the difference between j and k grows large.

Lemma 7.2. *For any $j, k \in \mathbb{N}$ with $j \geqslant k$, there exist $C > 0$ and $\lambda \in (0,1)$ such that*

$$|Pr(X^i_j = 1 \mid X^i_k = 1) - P_{\rho^*}(e_i)| \leqslant C\lambda^{j-k}.$$

Proof. Let us observe that the event $(X^i_j = 1 \mid X^i_k = 1)$ denotes the event e_i is observed after j-steps given e_i is also observed after k-steps. Thus,

$$\begin{aligned} Pr(X^i_j = 1 \mid X^i_k = 1) &= P^j(e_i \mid s_{init} \overset{\text{k}}{\rightsquigarrow} e_i) \\ &= P^{(j-k-1)}(e_i \mid (e_i)_{end}), \end{aligned}$$

by memoryless property of DTMC. Now we can write

$$\begin{aligned} &P^{(j-k-1)}(e_i \mid (e_i)_{end}) \\ =& \left(P^{(j-k-1)}((e_i)_{end}, (e_i)_1)\right) \cdot \left(P((e_i)_1, (e_i)_{end})\right). \end{aligned}$$

Similarly by expanding $P_{\rho^*}(e_i)$ we can write

$$P_{\rho^*}(e_i) = \rho^*((e_i)_1) \cdot P((e_i)_1, (e_i)_{end}).$$

Now,

$$\begin{aligned} &|Pr(X^i_j = 1 \mid X^i_k = 1) - P_{\rho^*}(e_i)| \\ =&|P^{(j-k-1)}((e_i)_{end}, (e_i)_1) - \rho^*((e_i)_1)| \cdot P((e_i)_1, (e_i)_{end}) \\ \leqslant&|P^{(j-k-1)}((e_i)_{end}, (e_i)_1) - \rho^*((e_i)_1)| \\ &\text{(since } P((e_i)_1, (e_i)_{end}) \leqslant 1) \\ \leqslant&C'\lambda^{(j-k-1)} \text{ for some } C' > 0, \lambda \in (0,1) \\ &\text{(by Theorem 7.1)} \\ \leqslant&C\lambda^{(j-k)} \quad \text{(where } C = C'/\lambda) \end{aligned}$$

Hence our claim is proved. ∎

From Lemma 7.2 it is easy to observe the following corollary which states that the probability of observing an edge on the j^{th} step also converges to the probability of that edge with respect to ρ^* as $j \to \infty$.

Corollary 7.3. *For any $j \in \mathbb{N}$, there exist $C > 0$ and $\lambda \in (0, 1)$ such that*

$$|Pr(X_j^i = 1) - P_{\rho^*}(e_i)| \leqslant C\lambda^j.$$

7.2 Proof of Theorem 4.3

We have shown in Lemma 4.4 that for an infinite path σ,

$$\frac{(S_\sigma)_n}{n} := \frac{\sum_{i=1}^n W(\sigma_i, \sigma_{i+1})}{n}$$

almost surely. Now, $W(\sigma) = \lim_{n\to\infty} n\cdot((S_\sigma)_n/n)$. Thus, $W(\sigma) = \lim_{n\to\infty} n\cdot\big(\sum_{e\in\mathcal{E}} P_{\rho^*}(e)W(e)\big)$ almost surely. But $\lim_{n\to\infty} n\cdot\big(\sum_{e\in\mathcal{E}} P_{\rho^*}(e)W(e)\big) \leqslant K$ for some fixed $K \geqslant 0$ iff $\sum_{e\in\mathcal{E}} P_{\rho^*}(e)W(e) \leqslant 0$. Thus $W(\sigma) \leqslant K$ for some fixed $K \geqslant 0$ iff $\sum_{e\in\mathcal{E}} P_{\rho^*}(e)W(e) \leqslant 0$, i.e., $\mathcal{M}_W$ is almost surely convergent iff $\sum_{e\in\mathcal{E}} P_{\rho^*}(e)W(e) \leqslant 0$.