PPHA 44100 Problem Set 4

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4.4

Consider the following three utility functions (in each case, $\alpha_1, \alpha_2 > 0$):
1. $U(x_1, x_2) = \alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2}$
2. $U(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$
3. $U(x_1, x_2) = \min(\alpha_1 x_1, \alpha_2 x_2)$

For each, answer the following:
(a) Are the preferences monotone? Strictly monotone?
(b) Are the preferences convex? Strictly convex?
(c) For each, calculate the demands for strictly positive prices $(p_1, p_2) >> 0$ and income I. (For utility functions (1) and (2), use the Kuhn-Tucker conditions. For part (3), explain why the Kuhn-Tucker Theorem does not apply, and then find the demands anyway.)

1.a)

Suppose there exist two bundles, $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

$U(x_1, x_2) = \alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2}$ is monotonic if $x \geq y$ implies $x \succsim y$.
Suppose $x \geq y$. Then, $x_1 \geq y_1$ and $x_2 \geq y_2$.

$$U(x) = \alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2}$$
$$U(y) = \alpha_1\sqrt{y_1} + \alpha_2\sqrt{y_2}$$

Given that $x_1 \geq y_1$, we can also say that $\sqrt{x_1} \geq \sqrt{y_1}$.
Then, given that $\alpha_1, \alpha_2 > 0$, we can also say that $\alpha_1\sqrt{x_1} \geq \alpha_1\sqrt{y_1}$.
Similarly, we can say that $\alpha_2\sqrt{x_2} \geq \alpha_2\sqrt{y_2}$. Therefore, $U(x) \geq U(y)$ for any $x \geq y$, and we say this function **is monotonic**.

Further, Suppose $x > y$. Then, $x_1 \geq y_1$ and $x_2 \geq y_2$, with at least one having strict inequality, $>$.

$$U(x) = \alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2}$$
$$U(y) = \alpha_1\sqrt{y_1} + \alpha_2\sqrt{y_2}$$

Given that $x_1 \geq y_1$, we can also say that $\sqrt{x_1} \geq \sqrt{y_1}$. Then, given that $\alpha_1, \alpha_2 > 0$, we can also say that $\alpha_1\sqrt{x_1} \geq \alpha_1\sqrt{y_1}$.
Similarly, we can say that $\alpha_1\sqrt{x_2} \geq \alpha_1\sqrt{y_2}$. Again, with one section having strict inequality, $>$.
Therefore, $U(x) \geq U(y)$ for any $x \geq y$ and we say this function **is strictly monotonic**.

1.b)

Suppose there exists two bundles, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. $U(x_1, x_2) = \alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2}$ satisfies convexity if $U(x) \geq U(y)$ implies $U(\lambda x + (1-\lambda)y) \geq U(y)$, $\forall \lambda \in [0, 1]$.
Suppose $x \geq y$. Then, $x_1 \geq y_1$ and $x_2 \geq y_2$.

$$\begin{aligned}
U(x) &= \alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2} \\
U(y) &= \alpha_1\sqrt{y_1} + \alpha_2\sqrt{y_2} \\
U(\lambda x + (1-\lambda)y) &= \lambda(\alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2}) + (1-\lambda)(\alpha_1\sqrt{y_1} + \alpha_2\sqrt{y_2}) \\
U(\lambda x + (1-\lambda)y) &= \lambda\alpha_1\sqrt{x_1} + \lambda\alpha_2\sqrt{x_2}) + (1-\lambda)\alpha_1\sqrt{y_1} + (1-\lambda)\alpha_2\sqrt{y_2}) \\
U(\lambda x + (1-\lambda)y) &= \lambda\alpha_1\sqrt{x_1} + \lambda\alpha_2\sqrt{x_2}) + \alpha_1\sqrt{y_1} - \lambda\alpha_1\sqrt{y_1} + \alpha_2\sqrt{y_2} - \lambda\alpha_2\sqrt{y_2}) \\
U(\lambda x + (1-\lambda)y) &= \lambda\alpha_1\sqrt{x_1} + \lambda\alpha_2\sqrt{x_2} - \lambda\alpha_2\sqrt{y_2} - \lambda\alpha_1\sqrt{y_1} + U(y) \\
U(\lambda x + (1-\lambda)y) &= \lambda(\alpha_1(\sqrt{x_1} - \sqrt{y_1}) + \alpha_2(\sqrt{x_2} - \sqrt{y_2}) + U(y)
\end{aligned}$$

Therefore, since $\lambda \in [0, 1]$ and $x_1 \geq y_1, x_2 \geq y_2$, we can say that $\lambda(\alpha_1(\sqrt{x_1} - \sqrt{y_1}) + \alpha_2(\sqrt{x_2} - \sqrt{y_2}) \geq 0$.
Therefore, $U(\lambda x + (1-\lambda)y = \lambda(\alpha_1(\sqrt{x_1} - \sqrt{y_1}) + \alpha_2(\sqrt{x_2} - \sqrt{y_2}) + U(y) \geq U(y)$ and we say that **convexity is satisfied**.

Now, suppose we have a third bundle, $z = z_1, z_2$, $z \succsim y$, $z \neq x$.

$$U(z) = \alpha_1\sqrt{z_1} + \alpha_2\sqrt{z_2}$$

Strict convexity is satisfied if $U(\lambda x + (1-\lambda)z) > U(y)$. Or, in other words, $U(\lambda x + (1-\lambda)z) - U(y) > 0$.
$U(\lambda x + (1-\lambda)z) - U(y) = \lambda\alpha_1(\sqrt{x_1} - \sqrt{z_1}) + \lambda\alpha_2(\sqrt{x_2} - \sqrt{z_2}) + \alpha_1(\sqrt{z_1} - \sqrt{y_1}) + \alpha_2(\sqrt{z_2} - \sqrt{y_2})$
Therefore, strict convexity is satisfied if

$$\lambda\alpha_1(\sqrt{x_1} - \sqrt{z_1}) + \lambda\alpha_2(\sqrt{x_2} - \sqrt{z_2}) + \alpha_1(\sqrt{z_1} - \sqrt{y_1}) + \alpha_2(\sqrt{z_2} - \sqrt{y_2}) > 0$$

However, since we cannot say anything about the relationship of x and z, this cannot be proven to be true.

1.c)

To find demand, we maximize the utility function with respect to the budget constraint.
$\max_{x_1,x_2\in X} U(x_1,x_2) = \alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2}$ subject to $p_1x_1 + p_2x_2 \le I$ and $x_1, x_2 \ge 0$.

$$\mathscr{L} = \max_{x_1,x_2\in X} \alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2} + \lambda(I - p_1x_1 - p_2x_2),\ x_1, x_2 \ge 0.$$

$$\frac{\partial\mathscr{L}}{\partial x_1} : \frac{1}{2}\alpha_1 x_1^{-(\frac{1}{2})} \le p_1\lambda, x_1 \ge 0$$

$$x_1\frac{\partial\mathscr{L}}{\partial x_1} = 0$$

$$\Rightarrow \frac{1}{2}\alpha_1\sqrt{x_1} = p_1x_1\lambda$$

$$\frac{\partial\mathscr{L}}{\partial x_2} : \frac{1}{2}\alpha_2 x_2^{-(\frac{1}{2})} \le p_2\lambda, x_2 \ge 0$$

$$x_2\frac{\partial\mathscr{L}}{\partial x_2} = 0$$

$$\Rightarrow \frac{1}{2}\alpha_2\sqrt{x_2} = p_2x_2\lambda$$

$$\frac{\partial\mathscr{L}}{\partial\lambda} : I \le p_1x_1 + p_2x_2, \lambda \ge 0$$

$$\lambda(I - p_1x_1 - p_2x_2) = 0$$

We consider the cases $\lambda = 0, \lambda > 0$. If $\lambda = 0$,

$$\frac{1}{2}\alpha_1 x_1^{-(\frac{1}{2})} \le 0$$

$$\Rightarrow x_1^{-(\frac{1}{2})} \le 0$$

$$\Rightarrow x_1 \le 0$$

Similarly, $\Rightarrow x_2 \le 0$. Therefore, $x_1^* = 0, x_2^* = 0$. If $\lambda > 0$,

$$\frac{\frac{1}{2}\alpha_1\sqrt{x_1}}{p_1x_1} = \lambda = \frac{\frac{1}{2}\alpha_2\sqrt{x_2}}{p_2x_2}.$$

$$\Rightarrow \frac{\alpha_1\sqrt{x_1}}{p_1x_1} = \frac{\alpha_2\sqrt{x_2}}{p_2x_2}$$

$$\Rightarrow \frac{\alpha_1\sqrt{x_1}x_2}{\alpha_2\sqrt{x_2}x_1} = \frac{p_1}{p_2}$$

$$\Rightarrow \frac{p_1}{p_2} = \frac{\alpha_1\sqrt{x_2}}{\alpha_2\sqrt{x_1}}$$

$$\Rightarrow x_1(\frac{\alpha_2 p_1}{\alpha_1 p_2})^{-\frac{1}{2}} = x_2$$

Also, given that $\lambda > 0$, we have that $I = p_1x_1 + p_2x_2$.

$$\Rightarrow I = \frac{\alpha_1 p_2 \sqrt{x_1}\sqrt{x_2}}{\alpha_2} + p_2x_2$$
$$\Rightarrow I = \frac{\alpha_2 p_1 \sqrt{x_1}\sqrt{x_2}}{\alpha_1} + p_1x_1$$
$$\Rightarrow I = \frac{\alpha_2 p_1 \sqrt{x_1}\sqrt{x_1(\frac{\alpha_2 p_1}{\alpha_1 p_2})^{-\frac{1}{2}}}}{\alpha_1} + p_1x_1$$
$$\Rightarrow I = (1 + (\frac{\alpha_2}{\alpha_1})^2\frac{p_1}{p_2})p_1x_1$$
$$\Rightarrow \frac{I}{(1 + (\frac{\alpha_2}{\alpha_1})^2\frac{p_1}{p_2})p_1} = x_1^*(I, p_1, p_2)$$

Similarly, we find x_2^*.

$$\frac{p_1\alpha_2}{p_2\alpha_1} = \frac{\sqrt{x_2}}{\sqrt{x_1}}$$
$$\Rightarrow \sqrt{x_1} = \frac{p_2\alpha_1}{p_1\alpha_2}\sqrt{x_2}$$
$$\Rightarrow x_1 = (\frac{p_2\alpha_1}{p_1\alpha_2})^{\frac{1}{2}}x_2$$
$$I = \frac{\alpha_2 p_1 \sqrt{x_1}\sqrt{x_2}}{\alpha_1} + p_1x_1$$
$$\Rightarrow I = p_2x_2 + \frac{\sqrt{p_1}\sqrt{p_2}\alpha_1}{\alpha_2}x_2$$
$$\Rightarrow I = x_2(p_2 + (\frac{p_1p_2\alpha_1}{\alpha_2})^{\frac{1}{2}})$$
$$\Rightarrow \frac{I}{p_2 + (\frac{p_1p_2\alpha_1}{\alpha_2})^{\frac{1}{2}}} = x_2^*(I, p_1, p_2)$$

2.a)

Suppose there exist two bundles, $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

$U(x_1, x_2) = \alpha_1x_1 + \alpha_2x_2$ is monotonic if $x \geq y$ implies $x \succsim y$.
Suppose $x \geq y$. Then, $x_1 \geq y_1$ and $x_2 \geq y_2$.

$$U(x) = \alpha_1x_1 + \alpha_2x_2$$
$$U(y) = \alpha_1y_1 + \alpha_2y_2$$

Given that $x_1 \geq y_1$, and $\alpha_1, \alpha_2 > 0$, we can say that $\alpha_1x_1 \geq \alpha_1y_1$.
Similarly, we can say that $\alpha_2x_2 \geq \alpha_2y_2$. Therefore, $U(x) \geq U(y)$ for any $x \geq y$, and we say this function **is monotonic**.
Further, Suppose $x > y$. Then, $x_1 \geq y_1$ and $x_2 \geq y_2$, with at least one having strict inequality, $>$.

$$U(x) = \alpha_1x_1 + \alpha_2x_2$$
$$U(y) = \alpha_1y_1 + \alpha_2y_2$$

Given that $x_1 \geq y_1$, and $\alpha_1, \alpha_2 > 0$, we can also say that $\alpha_1x_1 \geq \alpha_1y_1$. Again, with one section having strict inequality, $>$.
Therefore, $U(x) > U(y)$ and we say this function **is strictly monotonic**. #### 2.b) Suppose there exists two bundles, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. $U(x_1, x_2) = \alpha_1x_1 + \alpha_2x_2$ satisfies convexity if $U(x) \geq U(y)$ implies $U(\lambda x + (1-\lambda)y) \geq U(y)$, $\forall\lambda \in [0, 1]$.
Suppose $x \geq y$. Then, $x_1 \geq y_1$ and $x_2 \geq y_2$.

$$
\begin{aligned}
U(x) &= \alpha_1 x_1 + \alpha_2 x_2 \\
U(y) &= \alpha_1 y_1 + \alpha_2 y_2 \\
U(\lambda x + (1-\lambda)y) &= \lambda(\alpha_1 x_1 + \alpha_2 x_2) + (1-\lambda)(\alpha_1 y_1 + \alpha_2 y_2) \\
\Rightarrow U(\lambda x + (1-\lambda)y) &= \lambda\alpha_1 x_1 + \lambda\alpha_2 x_2 + (1-\lambda)\alpha_1 y_1 + (1-\lambda)\alpha_2 y_2 \\
\Rightarrow U(\lambda x + (1-\lambda)y) &= \lambda\alpha_1 x_1 + \lambda\alpha_2 x_2 + \alpha_1 y_1 + \alpha_2 y_2 - \lambda\alpha_1 y_1 - \lambda\alpha_2 y_2 \\
U(\lambda x + (1-\lambda)y) &\geq^? U(y) \\
\lambda\alpha_1 x_1 + \lambda\alpha_2 x_2 + \alpha_1 y_1 + \alpha_2 y_2 - \lambda\alpha_1 y_1 - \lambda\alpha_2 y_2 &\geq^? \alpha_1 y_1 + \alpha_2 y_2 \\
\lambda\alpha_1 x_1 + \lambda\alpha_2 x_2 - \lambda\alpha_1 y_1 - \lambda\alpha_2 y_2 &\geq^? 0 \\
\lambda(\alpha_1(x_1 - y_1) + \alpha_2(x_2 - y_2)) &\geq^? 0
\end{aligned}
$$

Therefore, since $\lambda \in [0,1]$ and $x_1 \geq y_1, x_2 \geq y_2$, we can say that $\lambda(\alpha_1(x_1 - y_1) + \alpha_2(x_2 - y_2)) \geq 0$.
Therefore, $U(\lambda x + (1-\lambda)y) \geq U(y)$ and we say that **convexity is satisfied**.

Now, suppose we have a third bundle, $z = z_1, z_2$, $z \succsim y$, $z \neq x$.

$$U(z) = \alpha_1 z_1 + \alpha_2 z_2$$

Strict convexity is satisfied if $U(\lambda x + (1-\lambda)z) > U(y)$. Or, in other words, $U(\lambda x + (1-\lambda)z) - U(y) > 0$. $U(\lambda x + (1-\lambda)z) - U(y) = \lambda\alpha_1(x_1 - z_1) + \lambda\alpha_2(x_2 - z_2) + \alpha_1(z_1 - y_1) + \alpha_2(z_2 - y_2)$ Therefore, strict convexity is satisfied if
$\lambda\alpha_1(x_1 - z_1) + \lambda\alpha_2(x_2 - z_2) + \alpha_1(z_1 - y_1) + \alpha_2(z_2 - y_2) > 0$.
However, since we cannot say anything about the relationship of x and z, this cannot be proven to be true.

2.c)

To find demand, we maximize the utility function with respect to the budget constraint.[1]
$\max_{x_1, x_2 \in X} U(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$ subject to $p_1 x_1 + p_2 x_2 \leq I$ and $x_1, x_2 \geq 0$.

[1] Given $\lambda = 0$, all equations degenerate to 0.

$$\mathscr{L} = \max_{x_1,x_2 \in X} \alpha_1 x_1 + \alpha_2 x_2 + \lambda(I - p_1 x_1 - p_2 x_2),\ x_1, x_2 \geq 0.$$

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial x_1} &: \alpha_1 \leq p_1 \lambda, x_1 \geq 0 \\
x_1 \frac{\partial \mathscr{L}}{\partial x_1} &= 0 \\
\Rightarrow \alpha_1 x_1 &= p_1 x_1 \lambda \\
\frac{\partial \mathscr{L}}{\partial x_2} &: \alpha_2 x_2 \leq p_2 \lambda, x_2 \geq 0 \\
x_2 \frac{\partial \mathscr{L}}{\partial x_2} &= 0 \\
\Rightarrow \alpha_2 x_2 &= p_2 x_2 \lambda \\
\frac{\partial \mathscr{L}}{\partial \lambda} &: I \geq p_1 x_1 + p_2 x_2, \lambda \geq 0 \\
\lambda(I - p_1 x_1 - p_2 x_2) &= 0
\end{aligned}$$

$$\begin{aligned}
For\ \lambda &> 0 \\
\lambda &= \frac{\alpha_1}{p_1} \\
\lambda &= \frac{\alpha_2}{p_2} \\
\Rightarrow \frac{p_1}{p_2} &= \frac{\alpha_1}{\alpha_2} \\
\alpha_2 x_2 &= p_2 x_2 \lambda \\
\Rightarrow \alpha_2 x_2 &= p_2 x_2 \frac{\alpha_1}{p_1} \\
I &= p_1 x_1 + p_2 x_2 \\
\Rightarrow p_2 x_2 &= I - p_1 x_1 \\
\Rightarrow \alpha_2 x_2 &= (I - p_1 x_1) \frac{\alpha_1}{p_1} \\
\Rightarrow x_2 &= \frac{I - \alpha_1 x_1}{\alpha_2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow I &= p_1 x_1 + p_2 \frac{I - x_1 \alpha_1}{alpha_2} \\
\Rightarrow I &= p_1 x_1 + I \frac{p_2}{\alpha_2} - p_1 x_1 \\
\Rightarrow I &= I \frac{p_2}{\alpha_2} \\
\Rightarrow \alpha_2 &= p_2
\end{aligned}$$

$$\begin{aligned}
\alpha_2 x_2 &= \alpha_2 x_2 \lambda \\
\Rightarrow \lambda &= 1 \\
\Rightarrow p_1 &= \alpha_1
\end{aligned}$$

If it were the case that $\alpha_1 \leq \alpha_2$, the decision maker optimizes as such,

$$\begin{aligned}
I &= \alpha_1 x_1 \\
x_1^* &= \frac{I}{\alpha_1} \\
x_2^* &= 0
\end{aligned}$$

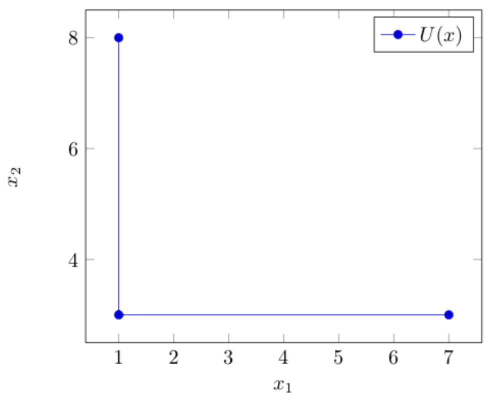
If it were the case that $\alpha_2 \leq \alpha_1$, the decision maker optimizes as such,

$$I = \alpha_2 x_2$$
$$x_2^* = \frac{I}{\alpha_2}$$
$$x_1^* = 0$$

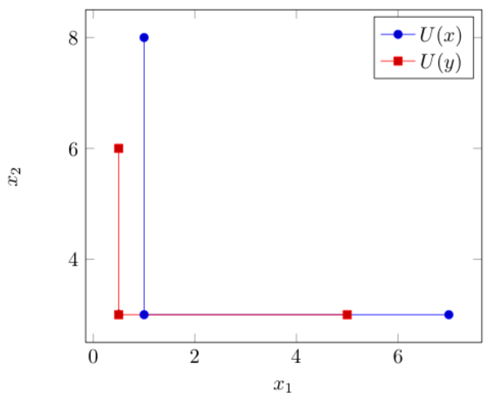
If $\alpha_1 = \alpha_2$, the decision maker is indifferent between the two goods and will randomize between them until the entire budget, I, is used.

3.a)

Plotting our utility function would give us the following general shape,



Then, if monotonicity was satisfied, adding another bundle y such that $x \geq y$ implies $x \succsim y$.



Clearly, at every point along the plot, $U(x) \geq U(y)$ for any arbitrarily chosen parameters. Therefore, $x \geq y$ implies $x \succsim y$ and we can say **monotonicity is satisfied**. Strict monotonicity would be satisfied if $x_1 \geq y_1$ and $x_2 \geq y_2$, with at least one having strict inequality, $>$, implied $x \succ y$. However, when looking at our previous graph that is clearly not the case. (At the kink, we see that x is strictly greater than y along the horizontal axis but, given that they are equal along the vertical axis, $U(x) = U(y)$ as long as the vertical axis input is the minimum.)
Therefore, this function is not strictly monotonic.

3.b)

Convexity exists if $U(x) \geq U(y)$ implies $U(\lambda x + (1-\lambda)y) \geq U(y)$, $\forall \lambda \in [0,1]$. Suppose that $U(x) \geq U(y)$,

$$\Rightarrow \min\{\alpha_1 x_1, \alpha_2 x_2\} \geq \min\{\alpha_1 y_1, \alpha_2 y_2\}$$

Then, $U(\lambda x + (1-\lambda)y) = \min\{\lambda\alpha_1 x_1 + (1-\lambda)\alpha_1 y_1, \lambda\alpha_2 x_2 + (1-\lambda)\alpha_2 y_2\}$

Case 1: Suppose $\min\{\lambda\alpha_1 x_1 + (1-\lambda)\alpha_1 y_1, \lambda\alpha_2 x_2 + (1-\lambda)\alpha_2 y_2\} = \lambda\alpha_1 x_1 + (1-\lambda)\alpha_1 y_1$

$$\Rightarrow U(\lambda x + (1-\lambda)y) = \lambda\alpha_1 x_1 - \lambda\alpha_1 y_1 + \alpha_1 y_1$$

If $U(y) = \alpha_1 y_1$, then

$$\lambda\alpha_1 x_1 - \lambda\alpha_1 y_1 + \alpha_1 y_1 \geq \alpha_1 y_1$$

because $\lambda\alpha_1 x_1 - \lambda\alpha_1 y_1 > 0$ due to the fact that we suppose $U(x) \geq U(y)$.

$$\Rightarrow U(\lambda x + (1-\lambda)y) \geq U(y)$$

If $U(y) = \alpha_2 y_2$, then

$$\lambda\alpha_1 x_1 - \lambda\alpha_1 y_1 + \alpha_1 y_1 \geq \alpha_2 y_2$$

We know this to be true due to the following:
1. $\lambda\alpha_1 x_1 \geq \lambda\alpha_1 y_1$ (from previous)
2. $U(y) = \alpha_2 y_2 \Rightarrow \min\{\alpha_1 y_1, \alpha_2 y_2\} = \alpha_2 y_2 \Rightarrow \alpha_1 y_1 \geq \alpha_2 y_2$

$$\Rightarrow U(\lambda x + (1-\lambda)y) \geq U(y)$$

Case 2: Suppose $\min\{\lambda\alpha_1 x_1 + (1-\lambda)\alpha_1 y_1, \lambda\alpha_2 x_2 + (1-\lambda)\alpha_2 y_2\} = \lambda\alpha_2 x_2 + (1-\lambda)\alpha_2 y_2\}$

$$\Rightarrow U(\lambda x + (1-\lambda)y) = \lambda\alpha_2 x_2 - \lambda\alpha_2 y_2 + \alpha_2 y_2$$

If $U(y) = \alpha_2 y_2$, then

$$\lambda\alpha_2 x_2 - \lambda\alpha_2 y_2 + \alpha_2 y_2 \geq \alpha_2 y_2$$

because $\lambda\alpha_2 x_2 - \lambda\alpha_2 y_2 > 0$ due to the fact that we suppose $U(x) \geq U(y)$.

$$\Rightarrow U(\lambda x + (1-\lambda)y) \geq U(y)$$

If $U(y) = \alpha_1 y_1$, then

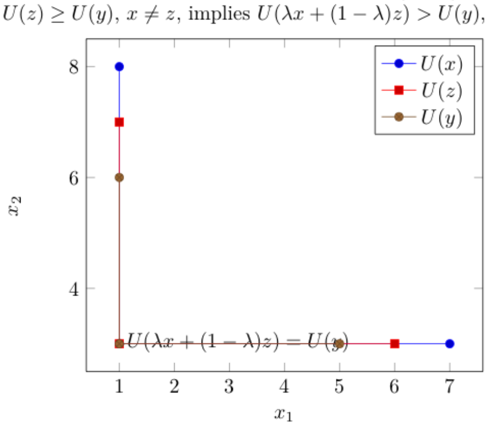
$$\lambda\alpha_2 x_2 - \lambda\alpha_2 y_2 + \alpha_2 y_2 \geq \alpha_1 y_1$$

We know this to be true due to the following: 1. $\lambda\alpha_2 x_2 \geq \lambda\alpha_2 y_2$ (from previous) 2. $U(y) = \alpha_1 y_1 \Rightarrow \min\{\alpha_1 y_1, \alpha_2 y_2\} = \alpha_1 y_1 \Rightarrow \alpha_2 y_2 \geq \alpha_1 y_1$

$$\Rightarrow U(\lambda x + (1-\lambda)y) \geq U(y)$$

Given that we find $U(\lambda x + (1-\lambda)y) \geq U(y)$ to be true in all cases, we say that **convexity is satisfied**.

Now, suppose an additional third bundle $z = (z_1, z_2)$ such that $U(z) \geq U(y), z \neq x$. Strict convexity is satisfied if $U(x) \geq U(y)$, $U(z) \geq U(y)$, $x \neq z$, implies $U(\lambda x + (1-\lambda)z) > U(y)$, $\forall \lambda \in (0, 1)$.



Obviously, strict convexity is not satisfied.

3.c)

Due to the shape of our utility function, we cannot use Kuhn Tucker conditions because it is not continuously differentiable (the graph has obvious kinks).

We solve for consumer demand as follows.

$$U = min\{\alpha_1 x_1, \alpha_2 x_2\}$$

such that

$$p_1 x_1 + p_2 x_2 \leq I, x_1 \geq 0, x_2 \geq 0$$

There are two cases, either $min\{\alpha_1 x_1, \alpha_2 x_2\} = \alpha_1 x_1$ or $min\{\alpha_1 x_1, \alpha_2 x_2\} = \alpha_2 x_2$.
Case 1: $min\{\alpha_1 x_1, \alpha_2 x_2\} = \alpha_1 x_1$

$$\mathscr{L} = \max_{x_1, x_2} \alpha_1 x_1 + \lambda_1[I - p_1 x_1 - p_2 x_2] + \lambda_2[\alpha_2 x_2 - \alpha_1 x_1]$$

$$\frac{\partial \mathscr{L}}{\partial x_1} : \alpha_1 \leq p_1 \lambda_1 + \alpha_1 \lambda_2,\ x_1 \geq 0,\ \alpha_1 = \frac{p_1 \lambda_1}{(1 - \lambda_2)}$$

$$\frac{\partial \mathscr{L}}{\partial x_2} :\ \alpha_2 \leq \partial_2,\ x_2 \geq 0, \alpha_2 = p_2$$

$$\frac{\mathscr{L}}{\partial \lambda_1} :\ I \geq \alpha_2 x_2 + \alpha_1 x_1, \lambda_1 \geq 0, I = p_1 x_1 + p_2 x_2$$

$$\frac{\mathscr{L}}{\lambda_2} :\ \alpha_2 x_2 \geq \alpha_1 x_1,\ \lambda_2 \geq 0, \alpha_2 x_2 = \alpha_1 x_1$$

From these first order conditions, we get the following. 1. $\frac{x_1}{x_2} = \frac{\alpha_2}{\alpha_1}$ 2. $I = p_1 x_1 + p_2 x_2$ 3. $\alpha_2 = p_2$ 4. $\alpha_1 = p_1 \frac{\lambda_1}{1 - \lambda_2}$

$$I = p_1 x_1 + \alpha_2 x_2$$
$$\Rightarrow \frac{I - \alpha_2 x_2}{p_1} = x_1$$
$$\frac{x_1}{x_2} = \frac{\alpha_2}{\alpha_1}$$
$$\Rightarrow x_1 = \frac{\alpha_2}{\alpha_1} x_2$$

$$\Rightarrow \frac{I - \alpha_2 x_2}{p_1} = \frac{\alpha_2}{\alpha_1} x_2$$
$$\Rightarrow x_2^* = \frac{\alpha_1 I}{(p_1 + \alpha_1)\alpha_2}$$

$$x_1 = \frac{\alpha_2}{\alpha_1} x_2^*$$
$$\Rightarrow x_1 = \frac{\alpha_2}{\alpha_1} \frac{\alpha_1 I}{(p_1 + \alpha_1)\alpha_2}$$
$$\Rightarrow x_1^* = \frac{I}{p_1 + \alpha_1}$$

Case 2: $min\{\alpha_1 x_1, \alpha_2 x_2\} = \alpha_2 x_2$

$$\mathscr{L} = \max_{x_1, x_2} \alpha_2 x_2 + \lambda_1[I - p_1 x_1 - p_2 x_2] + \lambda_2[\alpha_1 x_1 - \alpha_2 x_2]$$

$$\frac{\partial \mathscr{L}}{\partial x_2} : \alpha_2 \leq p_2 \lambda_1 + \alpha_2 \lambda_2,\ x_2 \geq 0,\ \alpha_2 = \frac{p_2 \lambda_1}{(1 - \lambda_2)}$$

$$\frac{\partial \mathscr{L}}{\partial x_1} :\ \alpha_1 \leq \partial_2,\ x_1 \geq 0, \alpha_1 = p_1$$

$$\frac{\mathscr{L}}{\partial \lambda_1} :\ I \geq \alpha_2 x_2 + \alpha_1 x_1, \lambda_1 \geq 0, I = p_1 x_1 + p_2 x_2$$

$$\frac{\mathscr{L}}{\lambda_2} :\ \alpha_2 x_2 \geq \alpha_1 x_1,\ \lambda_2 \geq 0, \alpha_2 x_2 = \alpha_1 x_1$$

From these first order conditions, we get the following. 1. $\frac{x_1}{x_2} = \frac{\alpha_2}{\alpha_1}$ 2. $I = p_1 x_2 + p_2 x_2$ 3. $\alpha_1 = p_1$ 4. $\alpha_2 = p_2 \frac{\lambda_1}{1-\lambda_2}$

$$I = \alpha_1 x_1 + p_2 x_2$$
$$\Rightarrow \frac{I - \alpha_1 x_1}{p_2} = x_2$$
$$\frac{x_1}{x_2} = \frac{\alpha_2}{\alpha_1}$$
$$\Rightarrow x_2 = \frac{\alpha_1}{\alpha_2} x_1$$

$$\Rightarrow \frac{I - \alpha_1 x_1}{p_2} = \frac{\alpha_1}{\alpha_2} x_1$$
$$\Rightarrow x_1^* = \frac{\alpha_2 I}{(p_2 + \alpha_2)\alpha_1}$$

$$x_2 = \frac{\alpha_1}{\alpha_2} x_1^*$$
$$\Rightarrow x_2 = \frac{\alpha_1}{\alpha_2} \frac{\alpha_2 I}{(p_2 + \alpha_2)\alpha_1}$$
$$\Rightarrow x_2^* = \frac{I}{p_2 + \alpha_2}$$

4.6

An infinitely lived consumer owns 1 unit of cake that she consumes over her lifetime. The cake is perfectly storable and she will receive no more than she has now. Consumption of cake in period t is denoted x_t, and her lifetime utility function is given by $U(x_0, x_1, ...) = \sum_{t=0}^{\infty} \delta^t \log x_t$, where $0 < \delta < 1$. Calculate her optimal level of cake consumption in each period. (Note: The statement of the Kuhn-Tucker theorem in class was for finite dimensional problems. But it also applies to this problem, even though there are infinitely many choice variables.)

Given the utility function $U(x_0, x_1, ...) = \sum_{t=0}^{\infty} \delta^t \log x_t$, where $0 < \delta < 1$, we divide by some constant to get $\sum_t \delta^t = 1$. Therefore, our function is strictly concave and the marginal utility of x_t is $\frac{\delta^t}{x_t}$. Therefore, given $\frac{\delta^t}{x_t} \to \infty$ as $x_t \to 0$, we know that any solution will be interior and we solve as follows.

$$U(x_0, x_1, ...) = \sum_{t=0}^{\infty} \delta^t \log x_t, \ 0 < \delta < 1, \ \sum_{t=0}^{\infty} x_t \leq 1$$
$$\mathscr{L} = \sum_{t=0}^{\infty} \max_{x_t = x_0, \ldots, x_\infty} \delta^t \log x_t + \lambda(1 - \sum_{t=0}^{\infty} x_t)$$
$$\mathscr{L} = \sum_{t=0}^{\infty} \max_{x_t = x_0, \ldots, x_\infty} \delta^t \log x_t + \lambda - \lambda x_0 - \lambda x_1 - \lambda \sum_{t=2}^{\infty} x_t)$$
$$\frac{\partial \mathscr{L}}{\partial x_0} : \frac{\delta^0}{x_0} = \lambda$$
$$\frac{\partial \mathscr{L}}{\partial x_1} : \frac{\delta^1}{x_1} = \lambda$$
$$.$$
$$.$$
$$.$$
$$\frac{\partial \mathscr{L}}{\partial x_1} : \frac{\delta^t}{x_t} = \lambda$$

And the constraint $1 = \sum_{t=0}^{\infty} x_t$.

The decision maker's optimal level of cake consumption is x_t^* such that the two constraints are met, $\frac{\delta^t}{\lambda} = x_t^*$ and $1 = \sum_{t=0}^{\infty} x_t$.

4.7

A consumer consumes 2 commodities, wheat and candy. His utility from consuming w units of wheat and c units of candy is $3 \log w + 2 \log c$. He faces 4 constraints:

- Consumption of each good must be nonnegative.
- The consumer has \$10 to spend, and the price of each good is \$1.
- The consumer is on a diet, and cannot consume more than 1550 calories. A unit of wheat has 150 calories, and a unit of candy has 200 calories.

Follow the following steps to solve this consumer's problem:

1.

Derive the Kuhn-Tucker optimality conditions for this consumer's problem. The Kuhn Tucker optimality conditions, taken from Proposition 4.1 in the notes, are
$\frac{\partial U}{\partial x_j}(x^*) \leq \lambda p_j$
$x_j^*(\frac{\partial U}{\partial x_j}(x^*) - \lambda p_j) = 0$
$px^* = I$
So, for this problem, the Kuhn Tucker optimality conditions are that w^*, c^* solve the consumer's problem given that there is $\lambda_1, \lambda_2 > 0$ such that,

$$\begin{aligned}
\frac{\partial U}{\partial w}(w^*, c^*) &\leq 1\lambda_1 + 150\lambda_2 \\
\frac{\partial U}{\partial c}(w^*, c^*) &\leq 1\lambda_1 + 200\lambda_2 \\
w^*(\frac{\partial U}{\partial w}(w^*, c^*) - 1\lambda_1 - 150\lambda_2) &= 0 \\
c^*(\frac{\partial U}{\partial w}(w^*, c^*) - 1\lambda_1 - 200\lambda_2) &= 0 \\
1w + 1c &= 10 \\
150w + 200c &= 1550
\end{aligned}$$

2.

Are the conditions derived above sufficient for this problem? Why or why not?
These conditions are sufficient to use the Kuhn-Tucker Theorem to solve this problem because the necessary assumptions are met (keeping in mind that the set X for this problem is $\mathbb{R}_+$, given that $w, c \geq 0$):

1. Preferences are continuous. $\checkmark U(w,c) = 3\log w + 2\log c$ is continuous for all $w, c \geq 0$
2. Preferences are locally non-satiated. $\checkmark$There exists $y \in X$ for any $x \in X, \epsilon > 0$, that $||x - y|| < \epsilon$ that $y \succ x$.
3. Preferences are strictly convex. $\checkmark U(x) \geq U(x')$ implies $U(\lambda x + (1-\lambda)x') > U(x')$
4. U is continuously differentiable. $\checkmark \frac{\partial U(x)}{\partial x}$ exists for all $x \in X$
5. For all $x \in \mathbb{R}_+^k$, we have $\mathbf{D}U(x) > 0$. $\checkmark$Meaning that we are experiencing demand for a normal good such that utility will increase with higher levels of consumption.

3.

Explain why the conditions from part (1) imply consumption of both commodities must be positive.
To start, we assume that $w, c \geq 0$, so we will show that $w, c \neq 0$.
From part (1), we have the following,

$$\begin{aligned}
\frac{3}{w^*} &= \lambda_1 + 150\lambda_2 \\
\frac{2}{c^*} &= \lambda_1 + 200\lambda_2 \\
w^*(\frac{3}{w^*} - \lambda_1 - 150\lambda_2) &= 0 \\
c^*(\frac{2}{c^*} - \lambda_1 - 200\lambda_2) &= 0 \\
w^* + c^* &= 10 \\
150w^* + 200c^* &= 1550
\end{aligned}$$

Suppose $w = 0$, then:

$$\begin{aligned}
c^* &= 10 \\
\frac{2}{10} &= \lambda_1 + 200\lambda_2 \\
10(\frac{2}{10} - \lambda_1 - 200\lambda_2) &= 0 \\
200(10) &= 1550 \\
\Rightarrow 2000 &= 1550 \leftarrow \textit{contradiction}
\end{aligned}$$

So, obviously, we find that $w^* \neq 0$.
Now, suppose $c^* = 0$, then:

$$\begin{aligned}
w^* &= 10 \\
\frac{3}{10} &= \lambda_1 + 150\lambda_2 \\
10(\frac{3}{10} - \lambda_1 - 150\lambda_2) &= 0 \\
150(10) &= 1550 \\
1500 &= 1550 \leftarrow \textit{contradiction}
\end{aligned}$$

4.

Explain why the conditions from part (1) imply that at least one of the budget and calorie constraint must bind.
If neither constraint binds, meaning if $w^* + c^* < 10$ and $150w^* + 200c^* < 1550$ there would be some $\epsilon > 0$ such that some input, w or c, could be increased and the new bundle $U(w, c) + \epsilon \succ U(w, c)$. Meaning, local nonsatiation implies that at least one of the constraints must bind.

5.

Look for a solution the the conditions from part (1) in which only the budget constraint binds.
A solution in which only the budget constraint binds implies

$$\begin{aligned}
\lambda_2 &= 0 \\
\frac{3}{w^*} &= \lambda_1 \\
\frac{2}{c^*} &= \lambda_1 \\
w^*(\frac{3}{w^*} - \lambda_1) &= 0 \\
c^*(\frac{2}{c^*} - \lambda_1) &= 0 \\
w^* + c^* &= 10 \\
\Rightarrow 10 &= \frac{3}{2}c^* + c^* \\
\Rightarrow 10 &= \frac{3c^* + 2c^*}{2} \\
\Rightarrow 20 &= 5c^* \\
c^* &= 4 w^* = 6
\end{aligned}$$

6.

Look for a solution the the conditions from part (1) in which only the calorie constraint binds.

$$\begin{aligned}
\lambda_1 &= 0 \\
\frac{3}{w^*} &= 150\lambda_2 \\
\frac{2}{c^*} &= 200\lambda_2 \\
w^*(\frac{3}{w^*} - 150\lambda_2) &= 0 \\
c^*(\frac{2}{c^*} - 200\lambda_2) &= 0 \\
150w^* + 200c^* &= 1550 \\
\Rightarrow \frac{3}{150w^*} = \lambda_2 &= \frac{2}{200c^*} \\
\Rightarrow 2c^* &= w^* \\
\Rightarrow 150w^* + 100w^* &= 1550 \\
\Rightarrow w^* &= 6.2 \\
\Rightarrow c^* &= 3.1
\end{aligned}$$

7.

Look for a solution the the conditions from part (1) in which both constraints bind.

$$\begin{aligned}
w^* &= 10 - c^* \\
w^* &= \frac{1550 - 200c^*}{150} \\
\Rightarrow 10 - c^* &= \frac{1550 - 200c^*}{150} \\
\Rightarrow c^* &= 1 \\
\Rightarrow w^* &= 9
\end{aligned}$$

4.9

In a three good world, a consumer has demands given by Table 4.3.3. Are these choices consistent with the maximization of a complete, transitive, and locally non-satiated preference relation?

bundle	p1	p2	p3	Y	x1	x2	x3
a	1	1	1	20	10	5	5
b	3	1	1	20	3	5	6
c	1	2	2	25	13	3	3
d	1	1	2	20	15	3	1

We can show that these demands are not consistent with the maximization of a complete, transitive, and locally non-satiated preference relation by comparing bundles c and d.

$$\begin{aligned}
p_c \times x_d = 23 &< Y_c \\
\Rightarrow x_c &\succ x_d \\
p_d \times x_c = 22 &< Y - d \\
\Rightarrow x_d &\succ x_c \\
\Rightarrow x_c \succ x_d &\succ x_c
\end{aligned}$$

This contradiction shows that these demands are inconsistent.