

How to Prove It

Chapter 1 Sentential Logic

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Section 1.1: Deductive Reasoning & Logical Connectives

Core Concepts • Mnemonics • Inline Examples • Competition Problems

Concept 1: Deductive Arguments

THE BIG IDEA

Deductive Argument = Premises + Conclusion

Premises: Starting assumptions (the “given” information).

Conclusion: What we claim **MUST** follow from the premises.

VALID = If **ALL** premises are true, the conclusion is **GUARANTEED** true.

INVALID = There **EXISTS** a scenario where premises are all true but conclusion is false.

PVC Pipe — How Deductive Arguments Work

Premises go in → **V**alidity of logic → **C**onclusion comes out

If the pipe is solid (valid), clean water (truth) flows through. If the pipe is cracked (invalid), water might leak!

COMMON TRAP

“Valid” ≠ “True”! A valid argument can have completely false premises and a false conclusion.


Valid only means the **STRUCTURE** is correct. Example:

“All cats are dogs. Fluffy is a cat. ∴ Fluffy is a dog.” — VALID (but premises are false!)

QUICK TIP

To PROVE an argument is invalid, you only need ONE counterexample (one scenario where all premises are true but conclusion is false).

Try These:

 Is this valid? “P or Q. Not Q. Therefore P.”


 **Answer: VALID (Disjunctive Syllogism)**

If at least one of P,Q is true (\vee), and Q is eliminated ($\neg Q$), P must be the true one. This is the most fundamental argument form!

 Is this valid? “A or B. A or C. Therefore B or C.”

 **Answer: INVALID**

Counterexample: A=T, B=F, C=F. Both premises are true (A makes them true), but conclusion $B \vee C = F \vee F = F$.

 Is this valid? “P and Q. Therefore P.”

 **Answer: VALID**

If both P and Q are true (\wedge), then certainly P alone is true. You can always “extract” one part from a conjunction.

Concept 2: The Three Logical Connectives

THE BIG IDEA

Three symbols let us build complex logical statements from simple ones:

\vee OR (Disjunction) — $P \vee Q$ = “P or Q or both”

\wedge AND (Conjunction) — $P \wedge Q$ = “P and Q”

\neg NOT (Negation) — $\neg P$ = “P is false”

Symbol Memory Tricks

\vee = OR — The V shape Opens Up \rightarrow opens up possibilities. Also: V = Various options.

\wedge = AND — Upside-down V = A shape \rightarrow AND starts with A. Also: \wedge = a tent gathering things together.

\neg = NOT — A little hook that “catches” the statement and flips it. Like a stop sign blocking truth!

\vee OR (Disjunction)

- True when AT LEAST ONE is true
- False ONLY when BOTH are false
- Math OR = inclusive (both OK)

\wedge AND (Conjunction)

- True ONLY when BOTH are true
- False when ANY ONE is false
- “but” often means AND in English

\neg NOT (Negation)

- Flips T \leftrightarrow F
- Applies ONLY to the very next thing
- $\neg P \wedge Q = (\neg P) \wedge Q \neq \neg(P \wedge Q)$

COMMON TRAP

OR in English is often exclusive (“coffee or tea?”). In math, OR is ALWAYS inclusive = could be both!

\neg scope error: $\neg P \wedge Q$ means $(\neg P) \wedge Q$, NOT $\neg(P \wedge Q)$. The \neg only negates the thing RIGHT NEXT to it.


USEFUL TIP

Think of \vee like addition and \wedge like multiplication:

\vee : anything + something = something (at least one makes it work)

\wedge : anything \times zero = zero (one false kills the whole thing)

Try These:

 If P is true and Q is false, what is $P \vee Q$? What is $P \wedge Q$?

 Answer: $P \vee Q$ = True, $P \wedge Q$ = False

\vee needs at least one true (P is true, so yes). \wedge needs both true (Q is false, so no).

 If P is false, what are $\neg P$, $\neg\neg P$, and $\neg\neg\neg P$?

➡ **Answer:** $\neg P = T$, $\neg\neg P = F$, $\neg\neg\neg P = T$

Each \neg flips the truth value. Even number of \neg s = back to original. Odd = flipped.

✎ Write “Neither Alice nor Bob is happy” using \wedge , \vee , \neg .

➡ **Answer:** $\neg A \wedge \neg B$ (equivalently: $\neg(A \vee B)$)

“Neither...nor” = “not A AND not B.” This is De Morgan’s law in action! (Section 1.2 preview)

✎ “John is tall but not heavy.” Translate. (T = tall, H = heavy)

➡ **Answer:** $T \wedge \neg H$

“but” = \wedge in logic. “not heavy” = $\neg H$.

Concept 3: Translating English \leftrightarrow Logic

THE BIG IDEA

The #1 skill in this section: convert English sentences into logical formulas and back.

Focus on MEANING, not word-for-word translation. The same idea can be said many ways in English.

Translation Dictionary

English \rightarrow Symbol:

“either A or B” $\rightarrow A \vee B$

“A and B” / “A but B” / “A yet B” $\rightarrow A \wedge B$

“not A” / “A isn’t true” $\rightarrow \neg A$

“neither A nor B” $\rightarrow \neg A \wedge \neg B = \neg(A \vee B)$

“not both A and B” $\rightarrow \neg(A \wedge B) = \neg A \vee \neg B$

Hidden logic in math notation:

$a \leq b = (a < b) \vee (a = b)$

$a \leq b < c = [(a < b) \vee (a = b)] \wedge (b < c)$

$a \neq b = \neg(a = b)$

QUICK TIP

3-Step Translation Method:

- 1 Identify atomic statements (simplest T/F claims).
- 2 Assign letters (P, Q, R...) to each.
- 3 Connect with \wedge , \vee , \neg + parentheses to match meaning.

COMMON TRAP

“and” doesn’t always mean \wedge ! “John and Bill are friends” — here “and” joins names, not statements. \wedge only goes between two complete statements.

“or” in English can mean different things. “Rain and snow are the only possibilities” actually means OR (\vee), even though it says “and.” Always think about meaning!

Try These:

 Translate: “I’ll have either fish or chicken, but not both fish and mashed potatoes.”

 Answer: $(F \vee C) \wedge \neg(F \wedge M)$

F=fish, C=chicken, M=mashed potatoes. “Either...or” = \vee . “not both...and” = $\neg(\wedge)$. “but” = \wedge .

 Translate: “Either Bill is at work and Jane isn’t, or Jane is at work and Bill isn’t.”

 Answer: $(B \wedge \neg J) \vee (J \wedge \neg B)$

Two compound statements joined by “or.” Parentheses are critical to show which parts belong together!

 Translate the math: $\sqrt{7} \neq 2$

➡ **Answer:** $\neg(\sqrt{7} = 2)$

\neq means “it is NOT the case that ... equals ...”

✎ Translate the math: $1 < x \leq 5$

➡ **Answer:** $(1 < x) \wedge [(x < 5) \vee (x = 5)]$

Two conditions joined by AND. The \leq part expands to “less than OR equal to.”

Concept 4: Well-Formed Formulas & Parentheses

THE BIG IDEA

Not every string of symbols is meaningful. Only well-formed formulas (WFFs) are “grammatical” in logic.

Rules: \wedge and \vee go BETWEEN two statements. \neg goes BEFORE one statement. Parentheses group things.

NOB Rule — Where Do Connectives Go?

Negation = NEXT to (before) one statement: $\neg P$

OR = between two **O** statements: $P \vee Q$

Both (AND) = **B**etween two statements: $P \wedge Q$

If an expression breaks NOB \rightarrow NOT a WFF!


USEFUL TIP

Parentheses in logic work like parentheses in algebra:

$(a+b) \times c \neq a+(b \times c)$ just like $(P \vee Q) \wedge R \neq P \vee (Q \wedge R)$


Default rule: \neg binds tightest (applies only to next thing). When in doubt, add parentheses!

Try These:

 Which are WFFs? (a) $\neg(\neg P \vee R)$ (b) $P \vee \wedge Q$ (c) $P \wedge \neg P$ (d) $(P \wedge Q)(P \vee R)$


 Answer: (a)  WFF (b)  (c)  WFF (d) 

(b) Two connectives in a row with nothing between. (d) Two formulas side by side with no connective — like writing $(2+3)(4+5)$ without \times .

 What does $\neg P \wedge Q$ mean? Is it the same as $\neg(P \wedge Q)$?

 Answer: NO! $\neg P \wedge Q = (\neg P) \wedge Q$. $\neg(P \wedge Q)$ negates the entire conjunction.

Example: $P = \text{“raining”}$ $Q = \text{“cold.”}$ $\neg P \wedge Q = \text{“not raining AND cold.”}$ $\neg(P \wedge Q) = \text{“not (raining and cold)”} = \text{could be warm.}$

 Add parentheses to make this unambiguous: $\neg P \vee Q \wedge R$

 Answer: Could be $(\neg P) \vee (Q \wedge R)$ or $((\neg P) \vee Q) \wedge R$ — different meanings!

Without conventions or parentheses, this is ambiguous. Always add parentheses when combining multiple connectives!

Concept 5: Key Argument Forms

THE BIG IDEA

It's the FORM (structure) that determines validity, not the content.

You can check validity without knowing what P and Q actually mean!

★ Disjunctive Syllogism — The “Process of Elimination”

$P \vee Q$ (One of these must be true)
 $\neg Q$ (The second one is false)
 $\therefore P$ (So the first one must be true!)

Real life: Multiple choice: “Answer is A or B. It's not B. Therefore A.” Every test-taker uses this!

Remember Disjunctive Syllogism: “POQ — Pick One, Quit the other”

You have two options (P Or Q). You Quit one ($\neg Q$). The remaining one (P) must be it!

Works both ways: $P \vee Q + \neg P \rightarrow Q$ is equally valid.

⚠ COMMON TRAP

Affirming the disjunct is INVALID:

$P \vee Q, P, \therefore \neg Q$ ❌ **WRONG!**


Just because P is true doesn't mean Q is false. Remember, OR is inclusive = both can be true!

💡 USEFUL TIP

To check validity WITHOUT truth tables (for simple arguments):

1. Assume all premises are true.
2. Try to construct a scenario where the conclusion is false.
3. If you CAN \rightarrow invalid (you found a counterexample). If you CAN'T \rightarrow valid.

Try These:

 Valid? $P \vee Q, Q \vee R, \neg Q$. Conclude: $P \wedge R$.

 **Answer: VALID!**

From $P \vee Q$ and $\neg Q \rightarrow P$ (disjunctive syllogism). From $Q \vee R$ and $\neg Q \rightarrow R$ (disjunctive syllogism). So P and R are both true $\rightarrow P \wedge R$. Two syllogisms chained!

 Valid? $P \vee Q, Q \vee R$. Conclude: $P \vee R$.

 **Answer: INVALID!**

Counterexample: $P=F, Q=T, R=F$. Premise 1: $F \vee T=T \checkmark$. Premise 2: $T \vee F=T \checkmark$. Conclusion: $F \vee F=F \times$.

 Valid? $P \wedge Q, R \vee \neg Q$. Conclude: R.

 **Answer: INVALID!**

Counterexample: $P=T$, $Q=T$, $R=F$. $P \wedge Q = T \checkmark$. $R \vee \neg Q = F \vee F = F \text{ X}$. Wait—premise 2 is false. Let's try: actually $R \vee \neg Q$ with $Q=T$ means $R \vee F = R$. If $R=F$, premise 2 is false. If $R=T$, conclusion is true. So whenever BOTH premises are true, R must be T . VALID!

Practice Problems: Exam Level

Work through each problem, then check the Answer Key at the end.

Section A: Standard Exam Problems

1. Analyze the logical form: “We’ll have either a reading assignment or homework, but we won’t have both homework and a test.”
2. Let S = “Steve is happy” and G = “George is happy.” Translate to English: $(S \vee G) \wedge (\neg S \vee \neg G)$
3. Analyze: (a) “Alice and Bob are not both in the room.” (b) “Alice and Bob are both not in the room.” (c) “Neither Alice nor Bob is in the room.” Which of (a)(b)(c) are equivalent?
4. Translate: “If either the main course is beef or the vegetable is corn, then the dessert won’t be cake.” (Hint: just use \vee , \wedge , \neg for now; treat “if...then” as a single unit for Section 1.5)
5. Let T = “Taxes go up,” D = “Deficit goes up.” What do these formulas mean in English? (a) $T \vee D$ (b) $\neg(T \wedge D) \wedge \neg(\neg T \wedge \neg D)$ (c) $(T \wedge \neg D) \vee (D \wedge \neg T)$
6. Determine validity: “The main course is beef or fish. The vegetable is peas or corn. We won’t have both fish and corn. Therefore, we won’t have both beef and peas.”

Section B: Competition-Style Problems 🏆

7. [Knights & Knaves] On an island, Knights always tell truth; Knaves always lie. You meet X and Y .
 X says: “At least one of us is a Knave.”
 Let P = “ X is a Knight,” Q = “ Y is a Knight.” (a) Write X ’s statement using \neg, \vee . (b) Determine what X and Y are.
8. [Liar Puzzle] Three people A, B, C each make one claim:
 A : “Either B or C is lying.” B : “ A is telling the truth.” C : “ B is lying.”
 Find all consistent truth assignments (who tells truth, who lies).
9. [Number Theory + Logic] Let $D_2(n)$ = “ n is divisible by 2” and $D_3(n)$ = “ n is divisible by 3.”
 (a) Write “ n is divisible by 6” using D_2, D_3 , and \wedge .
 (b) Write “ n is not divisible by 6” using D_2, D_3 , and \neg, \vee .
 (c) Is this valid? Premise: $\neg D_2(n) \vee \neg D_3(n)$. Premise: $D_3(n)$. Conclusion: $\neg D_2(n)$.
10. [Chain Argument] Determine validity:
 $P \vee Q, \neg P \vee R, \neg R$. **Conclusion: Q .**
11. [AMC-Style Logic] Five suspects A, B, C, D, E . Exactly one committed the crime. Given:
 (i) If A is guilty, then B was at the scene.

- (ii) B was not at the scene.
- (iii) Either C or D is guilty, or A is guilty.
- (iv) D is not guilty.

Using only \vee , \wedge , \neg and the argument forms from 1.1, determine who is guilty.