How to Prove It
Chapter 1 Sentential Logic

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Section 1.1: Deductive Reasoning & Logical Connectives

Core Concepts • Mnemonics • Inline Examples • Competition Problems

Concept 1: Deductive Arguments

🎯 THE BIG IDEA

Deductive Argument = Premises + Conclusion

Premises: Starting assumptions (the “given” information).
Conclusion: What we claim MUST follow from the premises.

VALID = If ALL premises are true, the conclusion is GUARANTEED true.
INVALID = There EXISTS a scenario where premises are all true but conclusion is false.

🧠 PVC Pipe — How Deductive Arguments Work

Premises go in → **V**alidity of logic → **C**onclusion comes out

If the pipe is solid (valid), clean water (truth) flows through. If the pipe is cracked (invalid), water might leak!

⚠️ COMMON TRAP

“Valid” ≠ “True”! A valid argument can have completely false premises and a false conclusion. Valid only means the STRUCTURE is correct. Example:

“All cats are dogs. Fluffy is a cat. ∴ Fluffy is a dog.” — VALID (but premises are false!)

✅ QUICK TIP

To PROVE an argument is invalid, you only need ONE counterexample (one scenario where all premises are true but conclusion is false).

✏️ Try These:

- ✏️ Is this valid? “P or Q. Not Q. Therefore P.”

  ➡️ **Answer: VALID (Disjunctive Syllogism)**

  If at least one of P,Q is true (∨), and Q is eliminated (¬Q), P must be the true one. This is the most fundamental argument form!

- ✏️ Is this valid? “A or B. A or C. Therefore B or C.”

  ➡️ **Answer: INVALID**

  Counterexample: A=T, B=F, C=F. Both premises are true (A makes them true), but conclusion B∨C = F∨F = F.

- ✏️ Is this valid? “P and Q. Therefore P.”

  ➡️ **Answer: VALID**

  If both P and Q are true (∧), then certainly P alone is true. You can always “extract” one part from a conjunction.

Concept 2: The Three Logical Connectives

🎯 THE BIG IDEA

Three symbols let us build complex logical statements from simple ones:

- **∨ OR (Disjunction)** — P ∨ Q = "P or Q or both"
- **∧ AND (Conjunction)** — P ∧ Q = "P and Q"
- **¬ NOT (Negation)** — ¬P = "P is false"

🧠 Symbol Memory Tricks

∨ = OR — The V shape Opens Up → opens up possibilities. Also: V = Various options.

∧ = AND — Upside-down V = A shape → AND starts with A. Also: ∧ = a tent gathering things together.

¬ = NOT — A little hook that "catches" the statement and flips it. Like a stop sign blocking truth!

∨ OR (Disjunction)	∧ AND (Conjunction)
• True when AT LEAST ONE is true	• True ONLY when BOTH are true
• False ONLY when BOTH are false	• False when ANY ONE is false
• Math OR = inclusive (both OK)	• "but" often means AND in English

¬ NOT (Negation)

• Flips T↔F • Applies ONLY to the very next thing • ¬P ∧ Q = (¬P) ∧ Q ≠ ¬(P ∧ Q)

⚠️ COMMON TRAP

OR in English is often exclusive ("coffee or tea?"). In math, OR is ALWAYS inclusive = could be both!

¬ scope error: ¬P ∧ Q means (¬P) ∧ Q, NOT ¬(P ∧ Q). The ¬ only negates the thing RIGHT NEXT to it.

💡 USEFUL TIP

Think of ∨ like addition and ∧ like multiplication:

∨: anything + something = something (at least one makes it work)

∧: anything × zero = zero (one false kills the whole thing)

✏️ Try These:

✏️ If P is true and Q is false, what is P ∨ Q? What is P ∧ Q?

➡️ **Answer: P ∨ Q = True, P ∧ Q = False**

∨ needs at least one true (P is true, so yes). ∧ needs both true (Q is false, so no).

✏️ If P is false, what are ¬P, ¬¬P, and ¬¬¬P?

➡️ **Answer: $\neg P = T$, $\neg\neg P = F$, $\neg\neg\neg P = T$**

Each ¬ flips the truth value. Even number of ¬s = back to original. Odd = flipped.

✏️ Write "Neither Alice nor Bob is happy" using ∧, ∨, ¬.

➡️ **Answer: $\neg A \wedge \neg B$ (equivalently: $\neg(A \vee B)$)**

"Neither...nor" = "not A AND not B." This is De Morgan's law in action! (Section 1.2 preview)

✏️ "John is tall but not heavy." Translate. (T = tall, H = heavy)

➡️ **Answer: $T \wedge \neg H$**

"but" = ∧ in logic. "not heavy" = ¬H.

Concept 3: Translating English ↔ Logic

🎯 THE BIG IDEA

The #1 skill in this section: convert English sentences into logical formulas and back.

Focus on MEANING, not word-for-word translation. The same idea can be said many ways in English.

🔑 Translation Dictionary

English → Symbol:

"either A or B" → $A \lor B$

"A and B" / "A but B" / "A yet B" → $A \land B$

"not A" / "A isn't true" → $\neg A$

"neither A nor B" → $\neg A \land \neg B = \neg(A \lor B)$

"not both A and B" → $\neg(A \land B) = \neg A \lor \neg B$

Hidden logic in math notation:

$a \le b \;=\; (a < b) \lor (a = b)$

$a \le b < c \;=\; [(a < b) \lor (a = b)] \land (b < c)$

$a \ne b \;=\; \neg(a = b)$

✅ QUICK TIP

3-Step Translation Method:

1️⃣ Identify atomic statements (simplest T/F claims).

2️⃣ Assign letters (P, Q, R...) to each.

3️⃣ Connect with ∧, ∨, ¬ + parentheses to match meaning.

⚠️ COMMON TRAP

"and" doesn't always mean ∧! "John and Bill are friends" — here "and" joins names, not statements. ∧ only goes between two complete statements.

"or" in English can mean different things. "Rain and snow are the only possibilities" actually means OR (∨), even though it says "and." Always think about meaning!

✏️ Try These:

✏️ Translate: "I'll have either fish or chicken, but not both fish and mashed potatoes."

➡️ **Answer: $(F \lor C) \land \neg(F \land M)$**

F=fish, C=chicken, M=mashed potatoes. "Either...or" = ∨. "not both...and" = ¬(∧). "but" = ∧.

✏️ Translate: "Either Bill is at work and Jane isn't, or Jane is at work and Bill isn't."

➡️ **Answer: $(B \land \neg J) \lor (J \land \neg B)$**

Two compound statements joined by "or." Parentheses are critical to show which parts belong together!

✏️ Translate the math: $\sqrt{7} \ne 2$

➡️ **Answer: $\neg(\sqrt{7} = 2)$**

≠ means "it is NOT the case that ... equals ..."

✏️ Translate the math: $1 < x \leq 5$

➡️ **Answer: $(1 < x) \wedge [(x < 5) \vee (x = 5)]$**

Two conditions joined by AND. The ≤ part expands to "less than OR equal to."

Concept 4: Well-Formed Formulas & Parentheses

🎯 THE BIG IDEA

Not every string of symbols is meaningful. Only well-formed formulas (WFFs) are “grammatical” in logic.

Rules: ∧ and ∨ go BETWEEN two statements. ¬ goes BEFORE one statement. Parentheses group things.

🧠 NOB Rule — Where Do Connectives Go?

Negation = NEXT to (before) one statement: ¬P

OR = between tw**O** statements: P ∨ Q

Both (AND) = **B**etween two statements: P ∧ Q

If an expression breaks NOB → NOT a WFF!

💡 USEFUL TIP

Parentheses in logic work like parentheses in algebra:

(a+b)×c ≠ a+(b×c)   just like   (P ∨ Q) ∧ R ≠ P ∨ (Q ∧ R)

Default rule: ¬ binds tightest (applies only to next thing). When in doubt, add parentheses!

✏️ Try These:

✏️ Which are WFFs? (a) ¬(¬P ∨ R)   (b) P ∨∧ Q   (c) P ∧ ¬P   (d) (P∧Q)(P∨R)

➡️ **Answer: (a) ✅ WFF (b) ❌ (c) ✅ WFF (d) ❌**

(b) Two connectives in a row with nothing between. (d) Two formulas side by side with no connective — like writing (2+3)(4+5) without ×.

✏️ What does ¬P ∧ Q mean? Is it the same as ¬(P ∧ Q)?

➡️ **Answer: NO! ¬P ∧ Q = (¬P) ∧ Q. ¬(P ∧ Q) negates the entire conjunction.**

Example: P=“raining” Q=“cold.” ¬P∧Q = “not raining AND cold.” ¬(P∧Q) = “not (raining and cold)” = could be warm.

✏️ Add parentheses to make this unambiguous: ¬P ∨ Q ∧ R

➡️ **Answer: Could be (¬P) ∨ (Q ∧ R) or ((¬P) ∨ Q) ∧ R — different meanings!**

Without conventions or parentheses, this is ambiguous. Always add parentheses when combining multiple connectives!

Concept 5: Key Argument Forms

🎯 THE BIG IDEA

It's the FORM (structure) that determines validity, not the content.

You can check validity without knowing what P and Q actually mean!

⭐ Disjunctive Syllogism — The "Process of Elimination"

$P \lor Q$ (One of these must be true)

$\neg Q$ (The second one is false)

$\therefore P$ (So the first one must be true!)

Real life: Multiple choice: "Answer is A or B. It's not B. Therefore A." Every test-taker uses this!

🧠 Remember Disjunctive Syllogism: "POQ — Pick One, Quit the other"

You have two options (P Or Q). You Quit one ($\neg Q$). The remaining one (P) must be it!

Works both ways: $P \lor Q + \neg P \to Q$ is equally valid.

⚠️ COMMON TRAP

Affirming the disjunct is INVALID:

$P \lor Q$, P, $\therefore \neg Q$ ❌ WRONG!

Just because P is true doesn't mean Q is false. Remember, OR is inclusive = both can be true!

💡 USEFUL TIP

To check validity WITHOUT truth tables (for simple arguments):

1. Assume all premises are true.
2. Try to construct a scenario where the conclusion is false.
3. If you CAN → invalid (you found a counterexample). If you CAN'T → valid.

✏️ Try These:

✏️ Valid? $P \lor Q$, $Q \lor R$, $\neg Q$. Conclude: $P \land R$.

➡️ **Answer: VALID!**

From $P \lor Q$ and $\neg Q \to P$ (disjunctive syllogism). From $Q \lor R$ and $\neg Q \to R$ (disjunctive syllogism). So P and R are both true $\to P \land R$. Two syllogisms chained!

✏️ Valid? $P \lor Q$, $Q \lor R$. Conclude: $P \lor R$.

➡️ **Answer: INVALID!**

Counterexample: P=F, Q=T, R=F. Premise 1: F∨T=T✓. Premise 2: T∨F=T✓. Conclusion: F∨F=F✗.

✏️ Valid? $P \land Q$, $R \lor \neg Q$. Conclude: R.

➡️ **Answer: INVALID!**

Counterexample: P=T, Q=T, R=F. P∧Q=T✓. R∨¬Q=F∨F=F✗. Wait—premise 2 is false. Let's try: actually R∨¬Q with Q=T means R∨F=R. If R=F, premise 2 is false. If R=T, conclusion is true. So whenever BOTH premises are true, R must be T. VALID!

Practice Problems: Exam Level

> *Work through each problem, then check the Answer Key at the end.*

Section A: Standard Exam Problems

1. Analyze the logical form: “We’ll have either a reading assignment or homework, but we won’t have both homework and a test.”

2. Let S = “Steve is happy” and G = “George is happy.” Translate to English: $(S \vee G) \wedge (\neg S \vee \neg G)$

3. Analyze: (a) “Alice and Bob are not both in the room.” (b) “Alice and Bob are both not in the room.” (c) “Neither Alice nor Bob is in the room.” Which of (a)(b)(c) are equivalent?

4. Translate: “If either the main course is beef or the vegetable is corn, then the dessert won’t be cake.” (Hint: just use $\vee$, $\wedge$, $\neg$ for now; treat “if...then” as a single unit for Section 1.5)

5. Let T = “Taxes go up,” D = “Deficit goes up.” What do these formulas mean in English? (a) $T \vee D$ (b) $\neg(T \wedge D) \wedge \neg(\neg T \wedge \neg D)$ (c) $(T \wedge \neg D) \vee (D \wedge \neg T)$

6. Determine validity: “The main course is beef or fish. The vegetable is peas or corn. We won’t have both fish and corn. Therefore, we won’t have both beef and peas.”

Section B: Competition-Style Problems 🏆

7. [Knights & Knaves] On an island, Knights always tell truth; Knaves always lie. You meet X and Y.

X says: “At least one of us is a Knave.”

Let P = “X is a Knight,” Q = “Y is a Knight.” (a) Write X’s statement using $\neg$,$\vee$. (b) Determine what X and Y are.

8. [Liar Puzzle] Three people A, B, C each make one claim:

A: “Either B or C is lying.” B: “A is telling the truth.” C: “B is lying.”

Find all consistent truth assignments (who tells truth, who lies).

9. [Number Theory + Logic] Let $D_2(n)$ = “n is divisible by 2” and $D_3(n)$ = “n is divisible by 3.”

(a) Write “n is divisible by 6” using D_2, D_3, and $\wedge$.

(b) Write “n is not divisible by 6” using D_2, D_3, and $\neg$, $\vee$.

(c) Is this valid? Premise: $\neg D_2(n) \vee \neg D_3(n)$. Premise: $D_3(n)$. Conclusion: $\neg D_2(n)$.

10. [Chain Argument] Determine validity:

$P \vee Q$, $\neg P \vee R$, $\neg R$. Conclusion: Q.

11. [AMC-Style Logic] Five suspects A,B,C,D,E. Exactly one committed the crime. Given:

(i) If A is guilty, then B was at the scene.

(ii) B was not at the scene.
(iii) Either C or D is guilty, or A is guilty.
(iv) D is not guilty.

Using only $\vee$, $\wedge$, $\neg$ and the argument forms from 1.1, determine who is guilty.

🔑 Answer Key

Section A Answers

1. $(R \lor H) \land \neg(H \land T)$

R=reading, H=homework, T=test. "Either R or H" = $R \lor H$. "Not both H and T" = $\neg(H \land T)$. "But" = $\land$.

2. "Either Steve or George is happy, but they're not both happy."

$(S \lor G)$ says at least one is happy. $(\neg S \lor \neg G)$ says at least one is NOT happy. Together: exactly one is happy. This is the exclusive or!

3. (a) $\neg(A \land B)$ (b) $\neg A \land \neg B$ (c) $\neg A \land \neg B$. (b) and (c) are equivalent.

(a) says it's not the case that BOTH are in the room (one might be). (b) says Alice is not in AND Bob is not in. (c) "neither...nor" = same as (b). Note: (a) is NOT the same as (b)/(c)! $\neg(A \land B) \neq \neg A \land \neg B$. (This is De Morgan's law — (a) actually equals $\neg A \lor \neg B$.)

4. Use letters B=beef, C=corn, K=cake. Formula: $(B \lor C)$ — for now, write the "if...then" structure as: if $(B \lor C)$ then $\neg K$.

Full formal version (Section 1.5): $(B \lor C) \to \neg K$. For now, identifying the components is enough: antecedent = $B \lor C$, consequent = $\neg K$.

5. (a) "Taxes or deficit will go up (or both)." (b) "Exactly one of taxes/deficit will go up." (c) Same as (b).

(b) $\neg(T \land D)$ = not both; $\neg(\neg T \land \neg D)$ = not neither. Together = exactly one. (c) $(T \land \neg D) \lor (D \land \neg T)$ = taxes without deficit, OR deficit without taxes = also exactly one. So (b) and (c) are equivalent!

6. INVALID.

B=beef, F=fish, P=peas, C=corn. Premises: $B \lor F$, $P \lor C$, $\neg(F \land C)$. Conclusion: $\neg(B \land P)$. Counterexample: B=T, F=T, P=T, C=F. Check: $B \lor F = T$✓, $P \lor C = T$✓, $\neg(F \land C) = \neg(T \land F) = T$✓. But $\neg(B \land P) = \neg(T \land T) = F$✗. Premises all true, conclusion false → invalid.

Section B Answers

7. X is a Knight, Y is a Knave.

(a) X's statement: "At least one of us is a Knave" = $\neg P \lor \neg Q$. (b) Case 1: X is Knight (P=T). Statement is true: $\neg T \lor \neg Q = F \lor \neg Q = \neg Q$. So Q=F, Y is Knave. ✓ Case 2: X is Knave (P=F). Statement is false: $\neg P \lor \neg Q$ must be F. Need $\neg P$=F AND $\neg Q$=F, so P=T and Q=T. But P=F — contradiction! ✗ Only Case 1 works.

8. A tells truth, B tells truth, C lies. (P=T, Q=T, R=F)

From C: if C tells truth (R=T) then Q=F; if C lies (R=F) then Q=T. Case R=F, Q=T: B tells truth, so P=T (B says A tells truth). A tells truth, so $\neg Q \lor \neg R = F \lor T = T$. ✓ Consistent! Case R=T, Q=F: B lies, so P=F. A lies, so $\neg Q \lor \neg R$ must be false, meaning Q=T. But Q=F — contradiction! ✗

9. (a) $D_2(n) \land D_3(n)$ (b) $\neg D_2(n) \lor \neg D_3(n)$ (c) VALID.

(a) Divisible by 6 = divisible by both 2 and 3. (b) Negate (a): $\neg(D_2 \land D_3) = \neg D_2 \lor \neg D_3$ (De Morgan). (c) From $\neg D_2 \lor \neg D_3$ and D_3, since D_3 is true, $\neg D_3$ is false, so $\neg D_2$ must be true (disjunctive syllogism). Valid!

10. VALID. Conclusion Q is correct.

From $\neg P \lor R$ and $\neg R$: disjunctive syllogism gives $\neg P$. From $P \lor Q$ and $\neg P$: disjunctive syllogism gives Q. Two-step chain!

11. C is guilty.

Step 1: From (i) “if A guilty then B at scene” and (ii) “B not at scene” (¬B_scene), contrapositive reasoning (or: treat as P∨Q form — “A not guilty OR B at scene” + ¬B_scene) gives: A is not guilty. Step 2: From (iii) C∨D∨A and ¬A: C∨D. Step 3: From C∨D and (iv) ¬D: disjunctive syllogism gives C. Answer: C is guilty. This chains three argument steps — classic competition technique!