$$\dot{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \bar{\bar{\boldsymbol{\phi}}} = \mathbf{f} - \nabla p \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\bar{\bar{\boldsymbol{\phi}}} = \nu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \tag{3}$$

1 Weak form:

$$\int_\Omega \mathbf{v} \cdot \dot{\mathbf{u}} \, d\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla \cdot \bar{\bar{\boldsymbol{\phi}}} \, d\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega, \quad \forall \mathbf{v} \tag{4}$$

in 1D:

$$\int_a^b v \nabla \tau \, dx = v\tau|_a^b - \int_a^b \nabla v \tau \, dx \tag{5}$$

in multiple dimensions:

$$\int_\Omega \mathbf{v} \cdot \nabla \cdot \bar{\bar{\boldsymbol{\phi}}} \, d\Omega = \int_\Gamma \mathbf{v} \cdot \bar{\bar{\boldsymbol{\phi}}} \cdot n \, d\Gamma - \int \nabla \mathbf{v} \cdot \bar{\bar{\boldsymbol{\phi}}} \tag{6}$$

$$\int_\Omega \mathbf{v} \cdot \dot{\mathbf{u}} \, d\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega + \int_\Omega \nabla \mathbf{v} \cdot \bar{\bar{\boldsymbol{\phi}}} \, d\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega + \int_\Gamma \mathbf{v} \cdot \mathbf{g} \, d\Gamma, \quad \forall \mathbf{v} \tag{7}$$

$$g = \bar{\bar{\boldsymbol{\phi}}} \cdot n \tag{8}$$

$$\int_\Omega q \nabla \cdot \mathbf{u} \, d\Omega = 0, \quad \forall q \tag{9}$$

2 Time discretisation of momentum equation:

$$\dot{\mathbf{u}} = \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \tag{10}$$

$$\begin{aligned} &\int_\Omega \mathbf{v} \cdot \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \, d\Omega + \int_\Omega \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \, d\Omega + \int_\Omega \nabla \mathbf{v} \cdot \bar{\bar{\boldsymbol{\phi}}}_{n-\alpha} \, d\Omega \\ &= \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega + \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega + \int_\Gamma \mathbf{v} \cdot \mathbf{g} \, d\Gamma, \quad \forall \mathbf{v} \end{aligned} \tag{11}$$

$$\bar{\bar{\boldsymbol{\phi}}}_{n-\alpha} = \nu \left(\nabla \mathbf{u}_{n-\alpha} + (\nabla \mathbf{u}_{n-\alpha})^\top \right) \tag{12}$$

$$\mathbf{u}_{n-\alpha} = \alpha \mathbf{u}_{n-1} + (1 - \alpha) \, \mathbf{u}_n \tag{13}$$

$$\mathbf{u}_{n-\alpha_{nl}} = \alpha_{nl}\mathbf{u}_{n-1} + (1-\alpha_{nl})\,\mathbf{u}_{n*} \tag{14}$$

Explicit Adams Bashforth

$$\bar{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,, \tag{15}$$

$$\tilde{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,. \tag{16}$$

Forward Euler

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{17}$$

$$\tilde{\mathbf{u}} = \mathbf{u}_{n-1}\,. \tag{18}$$

Implicit Adams Bashforth

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{19}$$

$$\tilde{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,. \tag{20}$$

Impicit - as Fluidity

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{21}$$

$$\tilde{\mathbf{u}} = \mathbf{u}_{n-\alpha_{nl}}\,. \tag{22}$$

3 Pressure/Conservation:

3.1 Fully coupled:

$$\begin{aligned}\int_\Omega \mathbf{v}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\ d\Omega + \Delta t\int_\Omega \nabla\mathbf{v}\cdot\bar{\bar{\boldsymbol{\o}}}_{n-\alpha}\ d\Omega \\ -\Delta t\int_\Omega \mathbf{v}\cdot\mathbf{f}\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\nabla p\ d\Omega - \Delta t\int_\Gamma \mathbf{v}\cdot\mathbf{g}\ d\Gamma = 0, \quad \forall\mathbf{v}\end{aligned} \tag{23}$$

$$\int_\Omega q\nabla\cdot\mathbf{u}_{n-\alpha}\ d\Omega = 0, \quad \forall q \tag{24}$$

Therefore:

$$\begin{aligned}\int_\Omega \mathbf{v}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\ d\Omega + \Delta t\int_\Omega \nabla\mathbf{v}\cdot\bar{\bar{\boldsymbol{\o}}}_{n-\alpha}\ d\Omega \\ -\Delta t\int_\Omega \mathbf{v}\cdot\mathbf{f}\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\nabla p\ d\Omega + \int_\Omega q\nabla\cdot\mathbf{u}_{n-\alpha}\ d\Omega - \Delta t\int_\Gamma \mathbf{v}\cdot\mathbf{g}\ d\Gamma = 0, \quad \forall q\ \forall\mathbf{v}\end{aligned} \tag{25}$$

3.2 Incremental pressure correction (IPCS):

Tentative velocity step:

$$\int_\Omega \mathbf{v}\frac{\mathbf{u}_* - \mathbf{u}_{n-1}}{\Delta t}\, d\Omega + \int_\Omega \mathbf{v}\tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}\, d\Omega + \int_\Omega \nabla \mathbf{v} \cdot \bar{\bar{\bar{\boldsymbol{\varnothing}}}}_{n-\alpha}\, d\Omega - \int_\Omega \mathbf{v}\mathbf{f}\, d\Omega + \int_\Omega \mathbf{v}\nabla p_{n-\frac{1}{2}}\, d\Omega \\ - \int_\Gamma \mathbf{v}g\, d\Gamma = 0, \quad \forall \mathbf{v} \tag{26}$$

Corrected velocity:

$$\int_\Omega \mathbf{v} \cdot \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t}\, d\Omega + \int_\Omega \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}\, d\Omega + \int_\Omega \nabla \mathbf{v} \cdot \bar{\bar{\bar{\boldsymbol{\varnothing}}}}_{n-\alpha}\, d\Omega \\ - \int_\Omega \mathbf{v}\mathbf{f}\, d\Omega + \int_\Omega \mathbf{v} \cdot \nabla p_{n+\frac{1}{2}}\, d\Omega - \int_\Gamma \mathbf{v} \cdot \mathbf{g}\, d\Gamma = 0, \quad \forall \mathbf{v} \tag{27}$$

(26) - (25):

$$\int_\Omega \mathbf{v} \cdot \frac{\mathbf{u}_n - \mathbf{u}_*}{\Delta t}\, d\Omega + \int_\Omega \mathbf{v} \cdot \nabla \left(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}}\right)\, d\Omega = 0, \quad \forall \mathbf{v} \tag{28}$$

left multiply by the mass matrix and the transpose of the pressure gradient operator:

$$\int_\Omega \nabla q \cdot \frac{\mathbf{u}_n - \mathbf{u}_*}{\Delta t}\, d\Omega + \int_\Omega \nabla q \cdot \nabla \left(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}}\right)\, d\Omega = 0, \quad \forall q \tag{29}$$

$$\int_\Omega \nabla q \cdot \mathbf{u}_n\, d\Omega - \int_\Omega \nabla q \cdot \mathbf{u}_*\, d\Omega + \Delta t \int_\Omega \nabla q \cdot \nabla \left(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}}\right)\, d\Omega = 0, \quad \forall q \tag{30}$$

$$\int_\Omega q \nabla \cdot \mathbf{u}_n\, d\Omega = \int_\Gamma q\left(\mathbf{u}_n \cdot n\right)\, d\Gamma - \int_\Omega \nabla q \cdot \mathbf{u}_n \\ = 0, \quad \forall q \tag{31}$$

$$\int_\Omega \nabla q \cdot \mathbf{u}_n = \int_\Gamma q\left(\mathbf{u}_n \cdot n\right)\, d\Gamma, \quad \forall q \tag{32}$$

$$\int_\Omega \nabla q \cdot \mathbf{u}_*\, d\Omega - \int_\Gamma q\left(\mathbf{u}_n \cdot n\right)\, d\Gamma = \Delta t \int_\Omega \nabla q \cdot \nabla \left(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}}\right)\, d\Omega, \quad \forall q \tag{33}$$