$$\dot{u} + u \cdot \nabla u - \nabla \cdot \sigma = f - \nabla p \tag{1}$$

$$\nabla u = 0 \tag{2}$$

$$\sigma = \nu \left(\nabla u + (\nabla u)^\top \right) \tag{3}$$

Several authors seem to neglect σ in this form and just have $\sigma = \nabla u$??

1 Weak form:

$$\int_\Omega v\dot{u}\ d\Omega + \int_\Omega vu \cdot \nabla u\ d\Omega - \int_\Omega v\nabla \cdot \sigma\ d\Omega = \int_\Omega vf\ d\Omega - \int_\Omega v\nabla p\ d\Omega \tag{4}$$

$$\int v\nabla \cdot \sigma = v\sigma - \int \nabla v \cdot \sigma \tag{5}$$

$$v\nabla \cdot \sigma = \nabla (v\sigma) - \nabla v \cdot \sigma \tag{6}$$

$$\begin{aligned} \int_\Omega \nabla (v\sigma)\ d\Omega &= \int_\Gamma (v\sigma) \cdot n\ d\Gamma \\ &= \int_\Gamma v (\sigma \cdot n)\ d\Gamma \end{aligned} \tag{7}$$

$$\int_\Omega v\dot{u}\ d\Omega + \int_\Omega vu \cdot \nabla u\ d\Omega + \int_\Omega \nabla v \cdot \sigma\ d\Omega = \int_\Omega vf\ d\Omega - \int_\Omega v\nabla p\ d\Omega + \int_\Gamma vg\ d\Gamma \tag{8}$$

$$g = \sigma \cdot n \tag{9}$$

$$\int_\Omega q\nabla \cdot u\ d\Omega = 0 \tag{10}$$

2 Time discretisation:

$$\dot{u} = \frac{u_n - u_{n-1}}{\Delta t} \tag{11}$$

$$\int_\Omega v\frac{u_n - u_{n-1}}{\Delta t}\ d\Omega + \int_\Omega v\tilde{u} \cdot \nabla \bar{u}\ d\Omega + \int_\Omega \nabla v \cdot \sigma_{n-\alpha}\ d\Omega = \int_\Omega vf\ d\Omega + \int_\Omega v\nabla p\ d\Omega + \int_\Gamma vg\ d\Gamma \tag{12}$$

$$\sigma_{n-\alpha} = \nu \left(\nabla u_{n-\alpha} + (\nabla u_{n-\alpha})^\top \right) \tag{13}$$

$$\int_\Omega q\nabla \cdot u_{n-\alpha}\, d\Omega = 0 \tag{14}$$

$$u_{n-\alpha} = \alpha u_{n-1} + (1-\alpha)\, u_n \tag{15}$$

$$u_{n-\alpha_{nl}} = \alpha_{nl} u_{n-1} + (1-\alpha_{nl})\, u_{n*} \tag{16}$$

Explicit Adams Bashforth

$$\bar{u} = \frac{3}{2}u_{n-1} - \frac{1}{2}u_{n-2}\,, \tag{17}$$

$$\tilde{u} = \frac{3}{2}u_{n-1} - \frac{1}{2}u_{n-2}\,. \tag{18}$$

Forward Euler

$$\bar{u} = u_{n-\alpha}\,, \tag{19}$$

$$\tilde{u} = u_{n-1}\,. \tag{20}$$

Implicit Adams Bashforth

$$\bar{u} = u_{n-\alpha}\,, \tag{21}$$

$$\tilde{u} = \frac{3}{2}u_{n-1} - \frac{1}{2}u_{n-2}\,. \tag{22}$$

Impicit - as Fluidity

$$\bar{u} = u_{n-\alpha}\,, \tag{23}$$

$$\tilde{u} = u_{n-\alpha_{nl}}\,. \tag{24}$$

Top three of these time discretisation schemes lead to a completely linear problem - no need for iteration if using fully coupled equation. ???

3 Pressure/Conservation:

3.1 Fully coupled:

$$\int_\Omega v\frac{u_n - u_{n-1}}{\Delta t}\, d\Omega + \int_\Omega v\tilde{u}\cdot\nabla\bar{u}\, d\Omega + \int_\Omega \nabla v\cdot\sigma_{n-\alpha}\, d\Omega - \int_\Omega vf\, d\Omega + \int_\Omega v\nabla p\, d\Omega - \int_\Gamma vg\, d\Gamma = 0 \tag{25}$$

$$\begin{split}\int_\Omega v\,(u_n - u_{n-1})\; d\Omega + \Delta t\int_\Omega v\tilde{u}\cdot\nabla\bar{u}\, d\Omega + \Delta t\int_\Omega \nabla v\cdot\sigma_{n-\alpha}\, d\Omega \\ - \Delta t\int_\Omega vf\, d\Omega + \Delta t\int_\Omega v\nabla p\, d\Omega - \Delta t\int_\Gamma vg\, d\Gamma = 0\end{split} \tag{26}$$

$$\int v\nabla p = vp - \int \nabla\cdot vp \tag{27}$$

$$v\nabla p = \nabla\cdot(vp) - (\nabla\cdot v)\, p \tag{28}$$

$$\int_{\Omega} \nabla \cdot (vp) \; d\Omega = \int_{\Gamma} (vp) \cdot n \; d\Gamma \tag{29}$$

$$\begin{aligned} \int_{\Omega} v \left(u_n - u_{n-1}\right) \; d\Omega + \Delta t \int_{\Omega} v\tilde{u} \cdot \nabla \bar{u} \; d\Omega + \Delta t \int_{\Omega} \nabla v \cdot \sigma_{n-\alpha} \; d\Omega \\ - \Delta t \int_{\Omega} vf \; d\Omega - \Delta t \int_{\Omega} (\nabla \cdot v) \, p \; d\Omega - \Delta t \int_{\Gamma} vh \; d\Gamma = 0 \end{aligned} \tag{30}$$

$$h = g + p \cdot n \tag{31}$$

$$\int_{\Omega} q \nabla \cdot u_{n-\alpha} \; d\Omega = 0 \tag{32}$$

Therefore:

$$\begin{aligned} \int_{\Omega} v \left(u_n - u_{n-1}\right) \; d\Omega + \Delta t \int_{\Omega} v\tilde{u} \cdot \nabla \bar{u} \; d\Omega + \Delta t \int_{\Omega} \nabla v \cdot \sigma_{n-\alpha} \; d\Omega \\ - \Delta t \int_{\Omega} vf \; d\Omega - \Delta t \int_{\Omega} (\nabla \cdot v) \, p \; d\Omega + \int_{\Omega} q \nabla \cdot u_{n-\alpha} \; d\Omega - \Delta t \int_{\Gamma} vh \; d\Gamma = 0 \end{aligned} \tag{33}$$

Is this right?? What do I do about the dirichlet pressure boundary condition?

3.2 Incremental pressure correction (IPCS):

Tentative velocity step:

$$\int_{\Omega} v \frac{u_* - u_{n-1}}{\Delta t} \; d\Omega + \int_{\Omega} v\tilde{u} \cdot \nabla \bar{u} \; d\Omega + \int_{\Omega} \nabla v \cdot \sigma_{n-\alpha} \; d\Omega - \int_{\Omega} vf \; d\Omega + \int_{\Omega} v \nabla p_{n-\frac{1}{2}} \; d\Omega - \int_{\Gamma} vg \; d\Gamma = 0 \tag{34}$$

Corrected velocity:

$$\int_{\Omega} v \frac{u_n - u_{n-1}}{\Delta t} \; d\Omega + \int_{\Omega} v\tilde{u} \cdot \nabla \bar{u} \; d\Omega + \int_{\Omega} \nabla v \cdot \sigma_{n-\alpha} \; d\Omega - \int_{\Omega} vf \; d\Omega + \int_{\Omega} v \nabla p_{n+\frac{1}{2}} \; d\Omega - \int_{\Gamma} vg \; d\Gamma = 0 \tag{35}$$

(26) - (25):

$$\int_{\Omega} v \frac{u_n - u_*}{\Delta t} \; d\Omega + \int_{\Omega} v \nabla \left(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}} \right) \; d\Omega = 0 \tag{36}$$

left multiply by $\nabla q \frac{1}{v}$:

$$\int_{\Omega} \nabla q \cdot \frac{u_n - u_*}{\Delta t} \; d\Omega + \int_{\Omega} \nabla q \cdot \nabla \left(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}} \right) \; d\Omega = 0 \tag{37}$$

$$\int_{\Omega} \nabla q \cdot \frac{u_n}{\Delta t} \; d\Omega - \int_{\Omega} \nabla q \cdot \frac{u_*}{\Delta t} \; d\Omega + \int_{\Omega} \nabla q \cdot \nabla \left(p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}} \right) \; d\Omega = 0 \tag{38}$$

Where do I go from here???