

$$\dot{\mathbf{u}}_i + \mathbf{u}_i \cdot \nabla_j \mathbf{u}_k - \nabla_i \cdot \nu_{ij} (\nabla_i \mathbf{u}_j + \nabla_j \mathbf{u}_i) = \mathbf{f}_i - \nabla_i p \quad (1)$$

For constant viscosity

$$\nabla_i \cdot \nu_{ij} (\nabla_i \mathbf{u}_j + \nabla_j \mathbf{u}_i) = \nu_{ij} (\nabla_i \cdot (\nabla_i) \mathbf{u}_j + \nabla_i \cdot (\nabla_j) \mathbf{u}_i) \quad (2)$$

$$= \nu_{ij} \nabla_i^2 \mathbf{u}_j$$

$$\nabla \cdot \mathbf{u} = 0 \quad (3)$$

## 1 Weak form:

$$\int_{\Omega} \mathbf{v} \cdot \dot{\mathbf{u}} \, d\Omega + \int_{\Omega} \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega - \int_{\Omega} \mathbf{v} \cdot \nabla \cdot \nabla \mathbf{u} \, d\Omega = \int_{\Omega} \mathbf{v} \cdot \mathbf{f} \, d\Omega - \int_{\Omega} \mathbf{v} \cdot \nabla p \, d\Omega, \quad \forall \mathbf{v} \quad (4)$$

in 1D:

$$\int_a^b v \nabla \tau \, dx = v \tau|_a^b - \int_a^b \nabla v \tau \, dx \quad (5)$$

in multiple dimensions:

$$\int_{\Omega} \mathbf{v} \cdot \nabla \cdot \nabla \mathbf{u} \, d\Omega = \int_{\Gamma} \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{n} \, d\Gamma - \int \nabla \mathbf{v} \cdot \nabla \mathbf{u} \quad (6)$$

$$\int_{\Omega} \mathbf{v} \cdot \dot{\mathbf{u}} \, d\Omega + \int_{\Omega} \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega + \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{u} \, d\Omega = \int_{\Omega} \mathbf{v} \cdot \mathbf{f} \, d\Omega - \int_{\Omega} \mathbf{v} \cdot \nabla p \, d\Omega + \int_{\Gamma} \mathbf{v} \cdot \mathbf{g} \, d\Gamma, \quad \forall \mathbf{v} \quad (7)$$

$$g = \nabla \mathbf{u} \cdot \mathbf{n} \quad (8)$$

$$\int_{\Omega} q \nabla \cdot \mathbf{u} \, d\Omega = 0, \quad \forall q \quad (9)$$

## 2 Time discretisation of momentum equation:

$$\dot{\mathbf{u}} = \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \quad (10)$$

$$\begin{aligned} \int_{\Omega} \mathbf{v} \cdot \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \, d\Omega + \int_{\Omega} \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \, d\Omega + \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{u}_{n-\alpha} \, d\Omega \\ = \int_{\Omega} \mathbf{v} \cdot \mathbf{f} \, d\Omega + \int_{\Omega} \mathbf{v} \cdot \nabla p \, d\Omega + \int_{\Gamma} \mathbf{v} \cdot \mathbf{g} \, d\Gamma, \quad \forall \mathbf{v} \end{aligned} \quad (11)$$

$$\nabla \mathbf{u}_{n-\alpha} = \nu \left( \nabla \mathbf{u}_{n-\alpha} + (\nabla \mathbf{u}_{n-\alpha})^{\top} \right) \quad (12)$$

$$\mathbf{u}_{n-\alpha} = \alpha \mathbf{u}_{n-1} + (1 - \alpha) \mathbf{u}_n \quad (13)$$

$$\mathbf{u}_{n-\alpha_{nl}} = \alpha_{nl} \mathbf{u}_{n-1} + (1 - \alpha_{nl}) \mathbf{u}_{n*} \quad (14)$$

Explicit Adams Bashforth

$$\bar{\mathbf{u}} = \frac{3}{2} \mathbf{u}_{n-1} - \frac{1}{2} \mathbf{u}_{n-2}, \quad (15)$$

$$\tilde{\mathbf{u}} = \frac{3}{2} \mathbf{u}_{n-1} - \frac{1}{2} \mathbf{u}_{n-2}. \quad (16)$$

Forward Euler

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}, \quad (17)$$

$$\tilde{\mathbf{u}} = \mathbf{u}_{n-1}. \quad (18)$$

Implicit Adams Bashforth

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}, \quad (19)$$

$$\tilde{\mathbf{u}} = \frac{3}{2} \mathbf{u}_{n-1} - \frac{1}{2} \mathbf{u}_{n-2}. \quad (20)$$

Impicit - as Fluidity

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}, \quad (21)$$

$$\tilde{\mathbf{u}} = \mathbf{u}_{n-\alpha_{nl}}. \quad (22)$$

### 3 Pressure/Conservation:

#### 3.1 Fully coupled:

$$\begin{aligned} & \int_{\Omega} \mathbf{v} \cdot (\mathbf{u}_n - \mathbf{u}_{n-1}) d\Omega + \Delta t \int_{\Omega} \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} d\Omega + \Delta t \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{u}_{n-\alpha} d\Omega \\ & - \Delta t \int_{\Omega} \mathbf{v} \cdot \mathbf{f} d\Omega + \Delta t \int_{\Omega} \mathbf{v} \cdot \nabla p d\Omega - \Delta t \int_{\Gamma} \mathbf{v} \cdot \mathbf{g} d\Gamma = 0, \quad \forall \mathbf{v} \end{aligned} \quad (23)$$

$$\int_{\Omega} q \nabla \cdot \mathbf{u}_{n-\alpha} d\Omega = 0, \quad \forall q \quad (24)$$

Therefore:

$$\begin{aligned} & \int_{\Omega} \mathbf{v} \cdot (\mathbf{u}_n - \mathbf{u}_{n-1}) d\Omega + \Delta t \int_{\Omega} \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} d\Omega + \Delta t \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{u}_{n-\alpha} d\Omega \\ & - \Delta t \int_{\Omega} \mathbf{v} \cdot \mathbf{f} d\Omega + \Delta t \int_{\Omega} \mathbf{v} \cdot \nabla p d\Omega + \int_{\Omega} q \nabla \cdot \mathbf{u}_{n-\alpha} d\Omega - \Delta t \int_{\Gamma} \mathbf{v} \cdot \mathbf{g} d\Gamma = 0, \quad \forall q \forall \mathbf{v} \end{aligned} \quad (25)$$

$$\int_{\Omega} \mathbf{v} \cdot \nabla p d\Omega = \int_{\Gamma} \mathbf{v} \cdot p \cdot \mathbf{n} d\Gamma - \int_{\Omega} (\nabla \cdot \mathbf{v}) p d\Omega \quad (26)$$

$$\begin{aligned}
& \int_{\Omega} \mathbf{v} \cdot (\mathbf{u}_n - \mathbf{u}_{n-1}) \, d\Omega + \Delta t \int_{\Omega} \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \, d\Omega + \Delta t \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{u}_{n-\alpha} \, d\Omega \\
& - \Delta t \int_{\Omega} \mathbf{v} \cdot \mathbf{f} \, d\Omega - \Delta t \int_{\Omega} (\nabla \cdot \mathbf{v}) p \, d\Omega + \int_{\Omega} q \nabla \cdot \mathbf{u}_{n-\alpha} \, d\Omega - \Delta t \int_{\Gamma} \mathbf{v} \\
& \cdot \mathbf{g} \, d\Gamma + \int_{\Gamma_N} \mathbf{v} \cdot p \cdot n \, d\Gamma_N + \int_{\Gamma_N} \mathbf{v} \cdot p_N \cdot n \, d\Gamma_N = 0, \quad \forall q \, \forall \mathbf{v}
\end{aligned} \tag{27}$$

$$\int_{\Omega} \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega = \int_{\Gamma} (\mathbf{v} \cdot \mathbf{u}) (\mathbf{u} \cdot n) \, d\Gamma - \int_{\Omega} (\nabla \mathbf{v} \cdot \mathbf{u}) \mathbf{u} \, d\Omega - \int_{\Omega} (\mathbf{v} \cdot \nabla \mathbf{u}) \mathbf{u} \, d\Omega \tag{28}$$

### 3.2 Incremental pressure correction (IPCS):

Tentative velocity step:

$$\begin{aligned}
& \int_{\Omega} \mathbf{v} \frac{\mathbf{u}_* - \mathbf{u}_{n-1}}{\Delta t} \, d\Omega + \int_{\Omega} \mathbf{v} \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \, d\Omega + \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{u}_{n-\alpha} \, d\Omega - \int_{\Omega} \mathbf{v} \mathbf{f} \, d\Omega + \int_{\Omega} \mathbf{v} \nabla p_{n-\frac{1}{2}} \, d\Omega \\
& - \int_{\Gamma} \mathbf{v} \mathbf{g} \, d\Gamma = 0, \quad \forall \mathbf{v}
\end{aligned} \tag{29}$$

Corrected velocity:

$$\begin{aligned}
& \int_{\Omega} \mathbf{v} \cdot \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \, d\Omega + \int_{\Omega} \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \, d\Omega + \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{u}_{n-\alpha} \, d\Omega \\
& - \int_{\Omega} \mathbf{v} \mathbf{f} \, d\Omega + \int_{\Omega} \mathbf{v} \cdot \nabla p_{n+\frac{1}{2}} \, d\Omega - \int_{\Gamma} \mathbf{v} \cdot \mathbf{g} \, d\Gamma = 0, \quad \forall \mathbf{v}
\end{aligned} \tag{30}$$

(26) - (25):

$$\int_{\Omega} \mathbf{v} \cdot \frac{\mathbf{u}_n - \mathbf{u}_*}{\Delta t} \, d\Omega + \int_{\Omega} \mathbf{v} \cdot \nabla \left( p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}} \right) \, d\Omega = 0, \quad \forall \mathbf{v} \tag{31}$$

left multiply by the mass matrix and the transpose of the pressure gradient operator:

$$\int_{\Omega} \nabla q \cdot \frac{\mathbf{u}_n - \mathbf{u}_*}{\Delta t} \, d\Omega + \int_{\Omega} \nabla q \cdot \nabla \left( p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}} \right) \, d\Omega = 0, \quad \forall q \tag{32}$$

$$\int_{\Omega} \nabla q \cdot \mathbf{u}_n \, d\Omega - \int_{\Omega} \nabla q \cdot \mathbf{u}_* \, d\Omega + \Delta t \int_{\Omega} \nabla q \cdot \nabla \left( p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}} \right) \, d\Omega = 0, \quad \forall q \tag{33}$$

$$\begin{aligned}
\int_{\Omega} q \nabla \cdot \mathbf{u}_n \, d\Omega &= \int_{\Gamma} q (\mathbf{u}_n \cdot n) \, d\Gamma - \int_{\Omega} \nabla q \cdot \mathbf{u}_n \\
&= 0, \quad \forall q
\end{aligned} \tag{34}$$

$$\int_{\Omega} \nabla q \cdot \mathbf{u}_n \, d\Omega = \int_{\Gamma} q (\mathbf{u}_n \cdot n) \, d\Gamma, \quad \forall q \tag{35}$$

$$\int_{\Omega} \nabla q \cdot \mathbf{u}_* \, d\Omega - \int_{\Gamma} q (\mathbf{u}_n \cdot n) \, d\Gamma = \Delta t \int_{\Omega} \nabla q \cdot \nabla \left( p_{n+\frac{1}{2}} - p_{n-\frac{1}{2}} \right) \, d\Omega, \quad \forall q \tag{36}$$

## 4 Analytical solution:

$$\dot{\mathbf{u}} = 0 \quad (37)$$

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \quad (38)$$

$$u = \sin(x)\cos(y) \quad (39)$$

$$v = -\cos(x)\sin(y) \quad (40)$$

$$\nu = 1.0 \quad (41)$$

$$p = \cos(x)\cos(y) \quad (42)$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} \cdot \nabla \begin{bmatrix} u \\ v \end{bmatrix} - \nu \nabla^2 \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{f} - \nabla p \quad (43)$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} - \nu \begin{bmatrix} u_{xx} + u_{yy} \\ v_{xx} + v_{yy} \end{bmatrix} = \mathbf{f} - \nabla p \quad (44)$$

$$\begin{aligned} & \begin{bmatrix} uu_x + vv_y \\ uv_x + vu_y \end{bmatrix} - \nu \begin{bmatrix} u_{xx} + u_{yy} \\ v_{xx} + v_{yy} \end{bmatrix} + \begin{bmatrix} p_x \\ p_y \end{bmatrix} \\ &= \mathbf{f} \\ &= \begin{bmatrix} \sin(x)\sin(y)^2\cos(x) + \sin(x)\cos(x)\cos(y)^2 + 2\sin(x)\cos(y) \\ \sin(x)^2\sin(y)\cos(y) + \sin(y)\cos(x)^2\cos(y) - 2\sin(x)\cos(y) - 2\sin(y)\cos(x) \end{bmatrix} \end{aligned} \quad (45)$$

## 5 DG Advection:

$$\nabla \cdot (\mathbf{u}c) - \nabla \cdot (\bar{\bar{k}} \cdot \nabla c) = s \quad (46a)$$

Assume diffusivity is isotropic. Multiply by test function, integrate over each element, sum over the domain and integrate both left-hand side terms by parts

$$\sum \int_{\Omega_E} \nabla \phi (k \cdot \nabla c - \mathbf{u}c) \, d\Omega_E = \sum \int_{\Omega_E} \phi s \, d\Omega_E + \sum \int_{\Gamma_E} \phi \mathbf{n} \cdot (k \cdot \nabla c - \mathbf{u}c) \, d\Gamma_E \quad (46b)$$

Now surface integrals are multiply defined - we need to choose a solution

$$\begin{aligned} \sum \int_{\Omega_E} \nabla \phi (k \cdot \nabla c - \mathbf{u}c) \, d\Omega_E &= \sum \int_{\Omega_E} \phi s \, d\Omega_E + \int_{\Gamma_n} \phi \mathbf{n} \cdot (k \cdot \nabla c - \mathbf{u}c) \, d\Gamma_n \\ &+ \sum \int_{\Gamma_{int}} \phi \mathbf{n} \cdot (k \cdot \nabla c - \mathbf{u}c)_* \, d\Gamma_{int} \end{aligned} \quad (46c)$$

simplest is to take an average at the internal faces

$$\begin{aligned} \sum \int_{\Omega_E} \nabla \phi (k \cdot \nabla c - \mathbf{u}c) \, d\Omega_E &= \sum \int_{\Omega_E} \phi s \, d\Omega_E + \int_{\Gamma_n} \phi \mathbf{n} \cdot (k \cdot \nabla c - \mathbf{u}c) \, d\Gamma_n \\ &+ \sum \int_{\Gamma_{int}} |\phi \mathbf{n}| \cdot \{ \{ k \cdot \nabla c - \mathbf{u}c \} \} \, d\Gamma_{int} \end{aligned} \quad (46d)$$