$$\dot{\mathbf{u}}_i + \mathbf{u}_i \cdot \nabla_j \mathbf{u}_k - \nabla_i \cdot \nu_{ij} \left(\nabla_i \mathbf{u}_j + \nabla_j \mathbf{u}_i \right) = \mathbf{f}_i - \nabla_i p \tag{1}$$

For constant viscosity

$$\begin{aligned} \nabla_i \cdot \nu_{ij} \left(\nabla_i \mathbf{u}_j + \nabla_j \mathbf{u}_i \right) &= \nu_{ij} \left(\nabla_i \cdot \left(\nabla_i \right) \mathbf{u}_j + \nabla_i \cdot \left(\nabla_j \right) \mathbf{u}_i \right) \\ &= \nu_{ij} \nabla_i^2 \mathbf{u}_j \end{aligned} \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3}$$

1 Weak form:

$$\int_\Omega \mathbf{v} \cdot \dot{\mathbf{u}} \, d\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla \cdot \nabla \mathbf{u} \, d\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega, \quad \forall \mathbf{v} \tag{4}$$

in 1D:

$$\int_a^b v \nabla \tau \; dx = \left. v\tau \right|_a^b - \int_a^b \nabla v \tau \; dx \tag{5}$$

in multiple dimensions:

$$\int_\Omega \mathbf{v} \cdot \nabla \cdot \nabla \mathbf{u} \, d\Omega = \int_\Gamma \mathbf{v} \cdot \nabla \mathbf{u} \cdot n \, d\Gamma - \int \nabla \mathbf{v} \cdot \nabla \mathbf{u} \tag{6}$$

$$\int_\Omega \mathbf{v} \cdot \dot{\mathbf{u}} \, d\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega + \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{u} \, d\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega + \int_\Gamma \mathbf{v} \cdot \mathbf{g} \, d\Gamma, \quad \forall \mathbf{v} \tag{7}$$

$$g = \nabla \mathbf{u} \cdot n \tag{8}$$

$$\int_\Omega q \nabla \cdot \mathbf{u} \, d\Omega = 0, \quad \forall q \tag{9}$$

2 Time discretisation of momentum equation:

$$\dot{\mathbf{u}} = \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \tag{10}$$

$$\begin{aligned} &\int_\Omega \mathbf{v} \cdot \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \, d\Omega + \int_\Omega \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \, d\Omega + \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{u}_{n-\alpha} \, d\Omega \\ &= \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega + \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega + \int_\Gamma \mathbf{v} \cdot \mathbf{g} \, d\Gamma, \quad \forall \mathbf{v} \end{aligned} \tag{11}$$

$$\nabla \mathbf{u}_{n-\alpha} = \nu \left(\nabla \mathbf{u}_{n-\alpha} + \left(\nabla \mathbf{u}_{n-\alpha} \right)^\top \right) \tag{12}$$

$$\mathbf{u}_{n-\alpha} = \alpha \mathbf{u}_{n-1} + (1-\alpha)\,\mathbf{u}_n \tag{13}$$

$$\mathbf{u}_{n-\alpha_{nl}} = \alpha_{nl} \mathbf{u}_{n-1} + (1-\alpha_{nl})\,\mathbf{u}_{n*} \tag{14}$$

Explicit Adams Bashforth

$$\bar{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,, \tag{15}$$

$$\tilde{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,. \tag{16}$$

Forward Euler

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{17}$$

$$\tilde{\mathbf{u}} = \mathbf{u}_{n-1}\,. \tag{18}$$

Implicit Adams Bashforth

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{19}$$

$$\tilde{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,. \tag{20}$$

Impicit - as Fluidity

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{21}$$

$$\tilde{\mathbf{u}} = \mathbf{u}_{n-\alpha_{nl}}\,. \tag{22}$$

3 Pressure/Conservation:

3.1 Fully coupled:

$$\begin{aligned} &\int_\Omega \mathbf{v}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\ d\Omega + \Delta t\int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\ d\Omega \\ &- \Delta t\int_\Omega \mathbf{v}\cdot\mathbf{f}\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\nabla p\ d\Omega - \Delta t\int_\Gamma \mathbf{v}\cdot\mathbf{g}\ d\Gamma = 0, \quad \forall\mathbf{v} \end{aligned} \tag{23}$$

$$\int_\Omega q\nabla\cdot\mathbf{u}_{n-\alpha}\ d\Omega = 0, \quad \forall q \tag{24}$$

Therefore:

$$\begin{aligned} &\int_\Omega \mathbf{v}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\ d\Omega + \Delta t\int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\ d\Omega \\ &- \Delta t\int_\Omega \mathbf{v}\cdot\mathbf{f}\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\nabla p\ d\Omega + \int_\Omega q\nabla\cdot\mathbf{u}_{n-\alpha}\ d\Omega - \Delta t\int_\Gamma \mathbf{v}\cdot\mathbf{g}\ d\Gamma = 0, \quad \forall q\ \forall\mathbf{v} \end{aligned} \tag{25}$$

$$\int_\Omega \mathbf{v}\cdot\nabla p\ d\Omega = \int_\Gamma \mathbf{v}\cdot p\cdot n\ d\Gamma - \int (\nabla\cdot\mathbf{v})\,p\ d\Omega \tag{26}$$

$$
\begin{aligned}
&\int_\Omega \mathbf{v}\cdot(\mathbf{u}_n-\mathbf{u}_{n-1})\ d\Omega+\Delta t\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\,d\Omega+\Delta t\int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\,d\Omega\\
&-\Delta t\int_\Omega \mathbf{v}\cdot\mathbf{f}\,d\Omega-\Delta t\int(\nabla\cdot\mathbf{v})\,p\,d\Omega+\int_\Omega q\nabla\cdot\mathbf{u}_{n-\alpha}\,d\Omega-\Delta t\int_\Gamma \mathbf{v}\\
&\cdot\mathbf{g}\,d\Gamma+\int_{\Gamma_{/N}}\mathbf{v}\cdot p\cdot n\,d\Gamma_{/N}+\int_{\Gamma_N}\mathbf{v}\cdot p_N\cdot n\,d\Gamma_N=0,\quad\forall q\,\forall\mathbf{v}
\end{aligned}
\tag{27}
$$

$$
\int_\Omega \mathbf{v}\cdot\mathbf{u}\cdot\nabla\mathbf{u}\,d\Omega=\int_\Gamma(\mathbf{v}\cdot\mathbf{u})\,(\mathbf{u}\cdot n)\ d\Gamma-\int_\Omega(\nabla\mathbf{v}\cdot\mathbf{u})\,\mathbf{u}\,d\Omega-\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{u})\,\mathbf{u}\,d\Omega \tag{28}
$$

3.2 Incremental pressure correction (IPCS):

Tentative velocity step:

$$
\begin{aligned}
&\int_\Omega \mathbf{v}\frac{\mathbf{u}_*-\mathbf{u}_{n-1}}{\Delta t}\,d\Omega+\int_\Omega \mathbf{v}\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\,d\Omega+\int_\Omega\nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\,d\Omega-\int_\Omega\mathbf{v}\mathbf{f}\,d\Omega+\int_\Omega\mathbf{v}\nabla p_{n-\frac{1}{2}}\,d\Omega\\
&-\int_\Gamma \mathbf{v}g\,d\Gamma=0,\quad\forall\mathbf{v}
\end{aligned}
\tag{29}
$$

Corrected velocity:

$$
\begin{aligned}
&\int_\Omega \mathbf{v}\cdot\frac{\mathbf{u}_n-\mathbf{u}_{n-1}}{\Delta t}\,d\Omega+\int_\Omega\mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\,d\Omega+\int_\Omega\nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\,d\Omega\\
&-\int_\Omega\mathbf{v}\mathbf{f}\,d\Omega+\int_\Omega\mathbf{v}\cdot\nabla p_{n+\frac{1}{2}}\,d\Omega-\int_\Gamma\mathbf{v}\cdot\mathbf{g}\,d\Gamma=0,\quad\forall\mathbf{v}
\end{aligned}
\tag{30}
$$

(26) - (25):

$$
\int_\Omega \mathbf{v}\cdot\frac{\mathbf{u}_n-\mathbf{u}_*}{\Delta t}\,d\Omega+\int_\Omega\mathbf{v}\cdot\nabla\left(p_{n+\frac{1}{2}}-p_{n-\frac{1}{2}}\right)\,d\Omega=0,\quad\forall\mathbf{v} \tag{31}
$$

left multiply by the mass matrix and the transpose of the pressure gradient operator:

$$
\int_\Omega \nabla q\cdot\frac{\mathbf{u}_n-\mathbf{u}_*}{\Delta t}\,d\Omega+\int_\Omega\nabla q\cdot\nabla\left(p_{n+\frac{1}{2}}-p_{n-\frac{1}{2}}\right)\,d\Omega=0,\quad\forall q \tag{32}
$$

$$
\int_\Omega\nabla q\cdot\mathbf{u}_n\,d\Omega-\int_\Omega\nabla q\cdot\mathbf{u}_*\,d\Omega+\Delta t\int_\Omega\nabla q\cdot\nabla\left(p_{n+\frac{1}{2}}-p_{n-\frac{1}{2}}\right)\,d\Omega=0,\quad\forall q \tag{33}
$$

$$
\begin{aligned}
\int_\Omega q\nabla\cdot\mathbf{u}_n\,d\Omega&=\int_\Gamma q\,(\mathbf{u}_n\cdot n)\ d\Gamma-\int_\Omega\nabla q\cdot\mathbf{u}_n\\
&=0,\quad\forall q
\end{aligned}
\tag{34}
$$

$$
\int_\Omega\nabla q\cdot\mathbf{u}_n=\int_\Gamma q\,(\mathbf{u}_n\cdot n)\ d\Gamma,\quad\forall q \tag{35}
$$

$$
\int_\Omega\nabla q\cdot\mathbf{u}_*\,d\Omega-\int_\Gamma q\,(\mathbf{u}_n\cdot n)\ d\Gamma=\Delta t\int_\Omega\nabla q\cdot\nabla\left(p_{n+\frac{1}{2}}-p_{n-\frac{1}{2}}\right)\,d\Omega,\quad\forall q \tag{36}
$$

4 Analytical solution:

$$\dot{\mathbf{u}} = 0 \tag{37}$$

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{38}$$

$$u = \sin(x)\cos(y) \tag{39}$$

$$v = -\cos(x)\sin(y) \tag{40}$$

$$\nu = 1.0 \tag{41}$$

$$p = \cos(x)\cos(y) \tag{42}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} \cdot \nabla \begin{bmatrix} u \\ v \end{bmatrix} - \nu \nabla^2 \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{f} - \nabla p \tag{43}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} - \nu \begin{bmatrix} u_{xx} + u_{yy} \\ v_{xx} + v_{yy} \end{bmatrix} = \mathbf{f} - \nabla p \tag{44}$$

$$\begin{aligned} &\begin{bmatrix} uu_x + vu_y \\ uv_x + yv_y \end{bmatrix} - \nu \begin{bmatrix} u_{xx} + u_{yy} \\ v_{xx} + v_{yy} \end{bmatrix} + \begin{bmatrix} p_x \\ p_y \end{bmatrix} \\ &= \mathbf{f} \\ &= \begin{bmatrix} \sin(x)\sin(y)^2\cos(x) + \sin(x)\cos(x)\cos(y)^2 + 2\sin(x)\cos(y) \\ \sin(x)^2\sin(y)\cos(y) + \sin(y)\cos(x)^2\cos(y) - 2\sin(x)\cos(y) - 2\sin(y)\cos(x) \end{bmatrix} \end{aligned} \tag{45}$$

5 DG Advection:

$$\nabla \cdot (\mathbf{u}c) - \nabla \cdot \left(\bar{\bar{k}} \cdot \nabla c \right) = s \tag{46a}$$

Assume diffusivity is isentropic. Multiply by test function, integrate over each element, sum over the domain and integrate both left-hand side terms by parts

$$\sum \int_{\Omega_E} \nabla \phi \left(k \cdot \nabla c - \mathbf{u}c \right) \, d\Omega_E = \sum \int_{\Omega_E} \phi s \, d\Omega_E + \sum \int_{\Gamma_E} \phi \, \mathbf{n} \cdot \left(k \cdot \nabla c - \mathbf{u}c \right) \, d\Gamma_E \tag{46b}$$

Splitting the internal and domain surfaces

$$\begin{aligned} \sum \int_{\Omega_E} \nabla \phi \left(k \cdot \nabla c - \mathbf{u}c \right) \, d\Omega_E = &\sum \int_{\Omega_E} \phi s \, d\Omega_E + \int_{\Gamma_n} \phi \, \mathbf{n} \cdot \left(k \cdot \nabla c - \mathbf{u}c \right) \, d\Gamma_n \\ &+ \sum \int_{\Gamma_{int}} \phi \, \mathbf{n} \cdot \left(k \cdot \nabla c - \mathbf{u}c \right) \, d\Gamma_{int} \end{aligned} \tag{46c}$$

But the internal surfaces are not consistent - the simplest solution is to take an average at the internal faces

$$\begin{aligned} \sum \int_{\Omega_E} \nabla \phi \left(k \cdot \nabla c - \mathbf{u}c \right) \, d\Omega_E = &\sum \int_{\Omega_E} \phi s \, d\Omega_E + \int_{\Gamma_n} \phi \, \mathbf{n} \cdot \left(k \cdot \nabla c - \mathbf{u}c \right) \, d\Gamma_n \\ &+ \sum \int_{\Gamma_{int}} |\phi \, \mathbf{n}| \cdot \{\{ k \cdot \nabla c - \mathbf{u}c \}\} \, d\Gamma_{int} \end{aligned} \tag{46d}$$

6 k-ϵ:

6.1 momentum equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial \tau^R_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + s_i \tag{47}$$

$$\tau_{ij} = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{48}$$

$$\tau^R_{ij} = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \tag{49}$$

6.1.1 weak form

$$\begin{aligned} &\int_\Omega \phi \frac{\partial u_i}{\partial t}\, dx + \int_\Omega \phi u_j \frac{\partial u_i}{\partial x_j}\, dx - \int_\Omega \phi \frac{\partial \tau_{ij}}{\partial x_j}\, dx - \int_\Omega \phi \frac{\partial \tau^R_{ij}}{\partial x_j}\, dx = \\ &- \int_\Omega \phi \frac{\partial p}{\partial x_i}\, dx + \int_\Omega \phi \rho_* g_i\, dx + \int_\Omega \phi s_i\, dx \end{aligned} \tag{50}$$

$$\begin{aligned} &\int_\Omega \phi \frac{\partial u_i}{\partial t}\, dx + \int_\Omega \phi u_j \frac{\partial u_i}{\partial x_j}\, dx + \int_\Omega \frac{\partial \phi}{\partial x_j} \tau_{ij}\, dx - \int_\Omega \phi \frac{\partial \tau^R_{ij}}{\partial x_j}\, dx = \\ &- \int_\Omega \phi \frac{\partial p}{\partial x_i}\, dx + \int_\Omega \phi \rho_* g_i\, dx + \int_\Omega \phi s_i\, dx + \int_\Gamma \phi \tau_{ij}\, dx \end{aligned} \tag{51}$$

Always dirichlet or zero neumann boundary conditions:

$$\begin{aligned} &\int_\Omega \phi \frac{\partial u_i}{\partial t}\, dx + \int_\Omega \phi u_j \frac{\partial u_i}{\partial x_j}\, dx + \int_\Omega \frac{\partial \phi}{\partial x_j} \tau_{ij}\, dx - \int_\Omega \phi \frac{\partial \tau^R_{ij}}{\partial x_j}\, dx = \\ &- \int_\Omega \phi \frac{\partial p}{\partial x_i}\, dx + \int_\Omega \phi \rho_* g_i\, dx + \int_\Omega \phi s_i\, dx \end{aligned} \tag{52}$$

6.2 turbulent kinetic energy

$$\frac{\partial k_i}{t} + \frac{\partial (\mathbf{u}k)_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\nu_T \frac{\partial k}{x_j} \right) = s + \frac{\partial u_i}{\partial x_j} \tau^R_{ij} - \epsilon \tag{53}$$

$$\frac{\partial k_i}{t} + \frac{\partial (\mathbf{u}k)_i}{\partial x_j} - \nu_T \frac{\partial}{x_j} \left(\frac{\partial k}{x_j} \right) - \frac{\partial \nu_T}{\partial x_j} \frac{\partial k}{\partial x_j} = s + \frac{\partial u_i}{\partial x_j} \tau^R_{ij} - \epsilon \tag{54}$$

$$\int_\Omega \eta \frac{\partial k_i}{t}\, dx + \int_\Omega \eta \frac{\partial (\mathbf{u}k)_i}{\partial x_j}\, dx - \int_\Omega \eta \frac{\partial}{\partial x_j} \left(\nu_T \frac{\partial k}{x_j} \right)\, dx = \int_\Omega \eta s\, dx + \int_\Omega \eta \frac{\partial u_i}{\partial x_j} \tau^R_{ij}\, dx - \int_\Omega \eta \epsilon\, dx \tag{55}$$