$$\dot{\mathbf{u}}_i + \mathbf{u}_i \cdot \nabla_j \mathbf{u}_k - \nabla_i \cdot \nu_{ij} \left(\nabla_i \mathbf{u}_j + \nabla_j \mathbf{u}_i \right) = \mathbf{f}_i - \nabla_i p \tag{1}$$

For constant viscosity

$$\begin{aligned}\nabla_i \cdot \nu_{ij} \left(\nabla_i \mathbf{u}_j + \nabla_j \mathbf{u}_i \right) &= \nu_{ij} \left(\nabla_i \cdot \left(\nabla_i \right) \mathbf{u}_j + \nabla_i \cdot \left(\nabla_j \right) \mathbf{u}_i \right) \\ &= \nu_{ij} \nabla_i^2 \mathbf{u}_j\end{aligned} \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3}$$

1 Weak form:

$$\int_\Omega \mathbf{v} \cdot \dot{\mathbf{u}} \, d\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla \cdot \nabla \mathbf{u} \, d\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega, \quad \forall \mathbf{v} \tag{4}$$

in 1D:

$$\int_a^b v \nabla \tau \, dx = v\tau|_a^b - \int_a^b \nabla v \tau \, dx \tag{5}$$

in multiple dimensions:

$$\int_\Omega \mathbf{v} \cdot \nabla \cdot \nabla \mathbf{u} \, d\Omega = \int_\Gamma \mathbf{v} \cdot \nabla \mathbf{u} \cdot n \, d\Gamma - \int \nabla \mathbf{v} \cdot \nabla \mathbf{u} \tag{6}$$

$$\int_\Omega \mathbf{v} \cdot \dot{\mathbf{u}} \, d\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega + \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{u} \, d\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega - \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega + \int_\Gamma \mathbf{v} \cdot \mathbf{g} \, d\Gamma, \quad \forall \mathbf{v} \tag{7}$$

$$g = \nabla \mathbf{u} \cdot n \tag{8}$$

$$\int_\Omega q \nabla \cdot \mathbf{u} \, d\Omega = 0, \quad \forall q \tag{9}$$

2 Time discretisation of momentum equation:

$$\dot{\mathbf{u}} = \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \tag{10}$$

$$\begin{aligned}&\int_\Omega \mathbf{v} \cdot \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{\Delta t} \, d\Omega + \int_\Omega \mathbf{v} \cdot \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \, d\Omega + \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{u}_{n-\alpha} \, d\Omega \\ &= \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega + \int_\Omega \mathbf{v} \cdot \nabla p \, d\Omega + \int_\Gamma \mathbf{v} \cdot \mathbf{g} \, d\Gamma, \quad \forall \mathbf{v}\end{aligned} \tag{11}$$

$$\nabla \mathbf{u}_{n-\alpha} = \nu \left(\nabla \mathbf{u}_{n-\alpha} + \left(\nabla \mathbf{u}_{n-\alpha} \right)^\top \right) \tag{12}$$

$$\mathbf{u}_{n-\alpha} = \alpha \mathbf{u}_{n-1} + (1-\alpha)\, \mathbf{u}_n \tag{13}$$

$$\mathbf{u}_{n-\alpha_{nl}} = \alpha_{nl} \mathbf{u}_{n-1} + (1-\alpha_{nl})\, \mathbf{u}_{n*} \tag{14}$$

Explicit Adams Bashforth

$$\bar{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,, \tag{15}$$

$$\tilde{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,. \tag{16}$$

Forward Euler

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{17}$$

$$\tilde{\mathbf{u}} = \mathbf{u}_{n-1}\,. \tag{18}$$

Implicit Adams Bashforth

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{19}$$

$$\tilde{\mathbf{u}} = \frac{3}{2}\mathbf{u}_{n-1} - \frac{1}{2}\mathbf{u}_{n-2}\,. \tag{20}$$

Impicit - as Fluidity

$$\bar{\mathbf{u}} = \mathbf{u}_{n-\alpha}\,, \tag{21}$$

$$\tilde{\mathbf{u}} = \mathbf{u}_{n-\alpha_{nl}}\,. \tag{22}$$

3 Pressure/Conservation:

3.1 Fully coupled:

$$\begin{aligned} &\int_\Omega \mathbf{v}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\ d\Omega + \Delta t \int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\ d\Omega + \Delta t\int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\ d\Omega \\ &- \Delta t\int_\Omega \mathbf{v}\cdot\mathbf{f}\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\nabla p\ d\Omega - \Delta t\int_\Gamma \mathbf{v}\cdot\mathbf{g}\ d\Gamma = 0, \quad \forall\mathbf{v} \end{aligned} \tag{23}$$

$$\int_\Omega q\nabla\cdot\mathbf{u}_{n-\alpha}\ d\Omega = 0, \quad \forall q \tag{24}$$

Therefore:

$$\begin{aligned} &\int_\Omega \mathbf{v}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\ d\Omega + \Delta t \int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\ d\Omega + \Delta t\int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\ d\Omega \\ &- \Delta t\int_\Omega \mathbf{v}\cdot\mathbf{f}\ d\Omega + \Delta t\int_\Omega \mathbf{v}\cdot\nabla p\ d\Omega + \int_\Omega q\nabla\cdot\mathbf{u}_{n-\alpha}\ d\Omega - \Delta t\int_\Gamma \mathbf{v}\cdot\mathbf{g}\ d\Gamma = 0, \quad \forall q\ \forall\mathbf{v} \end{aligned} \tag{25}$$

$$\int_\Omega \mathbf{v}\cdot\nabla p\ d\Omega = \int_\Gamma \mathbf{v}\cdot p\cdot n\ d\Gamma - \int (\nabla\cdot\mathbf{v})\, p\ d\Omega \tag{26}$$

$$
\begin{aligned}
&\int_\Omega \mathbf{v}\cdot(\mathbf{u}_n-\mathbf{u}_{n-1})\ d\Omega+\Delta t\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\,d\Omega+\Delta t\int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\,d\Omega\\
&-\Delta t\int_\Omega \mathbf{v}\cdot\mathbf{f}\ d\Omega-\Delta t\int(\nabla\cdot\mathbf{v})\,p\ d\Omega+\int_\Omega q\nabla\cdot\mathbf{u}_{n-\alpha}\,d\Omega-\Delta t\int_\Gamma \mathbf{v}\\
&\cdot\mathbf{g}\,d\Gamma+\int_{\Gamma_{/N}}\mathbf{v}\cdot p\cdot n\ d\Gamma_{/N}+\int_{\Gamma_N}\mathbf{v}\cdot p_N\cdot n\ d\Gamma_N=0,\quad\forall q\,\forall\mathbf{v}
\end{aligned}
\tag{27}
$$

$$
\int_\Omega \mathbf{v}\cdot\mathbf{u}\cdot\nabla\mathbf{u}\,d\Omega=\int_\Gamma(\mathbf{v}\cdot\mathbf{u})(\mathbf{u}\cdot n)\ d\Gamma-\int_\Omega(\nabla\mathbf{v}\cdot\mathbf{u})\,\mathbf{u}\,d\Omega-\int_\Omega(\mathbf{v}\cdot\nabla\mathbf{u})\,\mathbf{u}\,d\Omega \tag{28}
$$

3.2 Incremental pressure correction (IPCS):

Tentative velocity step:

$$
\begin{aligned}
&\int_\Omega \mathbf{v}\frac{\mathbf{u}_*-\mathbf{u}_{n-1}}{\Delta t}\ d\Omega+\int_\Omega \mathbf{v}\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\ d\Omega+\int_\Omega\nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\ d\Omega-\int_\Omega \mathbf{v}\mathbf{f}\ d\Omega+\int_\Omega \mathbf{v}\nabla p_{n-\frac{1}{2}}\ d\Omega\\
&-\int_\Gamma \mathbf{v}g\ d\Gamma=0,\quad\forall\mathbf{v}
\end{aligned}
\tag{29}
$$

Corrected velocity:

$$
\begin{aligned}
&\int_\Omega \mathbf{v}\cdot\frac{\mathbf{u}_n-\mathbf{u}_{n-1}}{\Delta t}\ d\Omega+\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\ d\Omega+\int_\Omega\nabla\mathbf{v}\cdot\nabla\mathbf{u}_{n-\alpha}\ d\Omega\\
&-\int_\Omega \mathbf{v}\mathbf{f}\ d\Omega+\int_\Omega \mathbf{v}\cdot\nabla p_{n+\frac{1}{2}}\ d\Omega-\int_\Gamma \mathbf{v}\cdot\mathbf{g}\ d\Gamma=0,\quad\forall\mathbf{v}
\end{aligned}
\tag{30}
$$

(26) - (25):

$$
\int_\Omega \mathbf{v}\cdot\frac{\mathbf{u}_n-\mathbf{u}_*}{\Delta t}\ d\Omega+\int_\Omega \mathbf{v}\cdot\nabla\left(p_{n+\frac{1}{2}}-p_{n-\frac{1}{2}}\right)\ d\Omega=0,\quad\forall\mathbf{v} \tag{31}
$$

left multiply by the mass matrix and the transpose of the pressure gradient operator:

$$
\int_\Omega \nabla q\cdot\frac{\mathbf{u}_n-\mathbf{u}_*}{\Delta t}\ d\Omega+\int_\Omega\nabla q\cdot\nabla\left(p_{n+\frac{1}{2}}-p_{n-\frac{1}{2}}\right)\ d\Omega=0,\quad\forall q \tag{32}
$$

$$
\int_\Omega\nabla q\cdot\mathbf{u}_n\ d\Omega-\int_\Omega\nabla q\cdot\mathbf{u}_*\ d\Omega+\Delta t\int_\Omega\nabla q\cdot\nabla\left(p_{n+\frac{1}{2}}-p_{n-\frac{1}{2}}\right)\ d\Omega=0,\quad\forall q \tag{33}
$$

$$
\begin{aligned}
\int_\Omega q\nabla\cdot\mathbf{u}_n\ d\Omega&=\int_\Gamma q\,(\mathbf{u}_n\cdot n)\ d\Gamma-\int_\Omega\nabla q\cdot\mathbf{u}_n\\
&=0,\quad\forall q
\end{aligned}
\tag{34}
$$

$$
\int_\Omega\nabla q\cdot\mathbf{u}_n=\int_\Gamma q\,(\mathbf{u}_n\cdot n)\ d\Gamma,\quad\forall q \tag{35}
$$

$$
\int_\Omega\nabla q\cdot\mathbf{u}_*\ d\Omega-\int_\Gamma q\,(\mathbf{u}_n\cdot n)\ d\Gamma=\Delta t\int_\Omega\nabla q\cdot\nabla\left(p_{n+\frac{1}{2}}-p_{n-\frac{1}{2}}\right)\ d\Omega,\quad\forall q \tag{36}
$$

4 Analytical solution:

$$\dot{\mathbf{u}} = 0 \tag{37}$$

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{38}$$

$$u = \sin(x)\cos(y) \tag{39}$$

$$v = -\cos(x)\sin(y) \tag{40}$$

$$\nu = 1.0 \tag{41}$$

$$p = \cos(x)\cos(y) \tag{42}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} \cdot \nabla \begin{bmatrix} u \\ v \end{bmatrix} - \nu\nabla^2 \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{f} - \nabla p \tag{43}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} - \nu \begin{bmatrix} u_{xx} + u_{yy} \\ v_{xx} + v_{yy} \end{bmatrix} = \mathbf{f} - \nabla p \tag{44}$$

$$\begin{aligned} &\begin{bmatrix} uu_x + vu_y \\ uv_x + yv_y \end{bmatrix} - \nu \begin{bmatrix} u_{xx} + u_{yy} \\ v_{xx} + v_{yy} \end{bmatrix} + \begin{bmatrix} p_x \\ p_y \end{bmatrix} \\ &= \mathbf{f} \\ &= \begin{bmatrix} \sin(x)\sin(y)^2\cos(x) + \sin(x)\cos(x)\cos(y)^2 + 2\sin(x)\cos(y) \\ \sin(x)^2\sin(y)\cos(y) + \sin(y)\cos(x)^2\cos(y) - 2\sin(x)\cos(y) - 2\sin(y)\cos(x) \end{bmatrix} \end{aligned} \tag{45}$$

5 DG Advection:

$$\nabla \cdot (\mathbf{u}c) = s \tag{46a}$$

Assume diffusivity is isentropic. Multiply by test function, integrate over each element, sum over the domain and integrate both left-hand side terms by parts

$$\begin{aligned} -\int_\Omega \nabla\phi\,(\mathbf{u}c) &= -\sum_E \int_E \nabla\phi\,(\mathbf{u}c) \\ &= \sum_E \int_E \phi s - \sum_E \int_{\partial E} \phi\,\mathbf{n}\cdot\mathbf{u}c \end{aligned} \tag{46b}$$

$$-\sum_E \int_E \nabla\phi_h\,(\mathbf{u}_h c_h) = \sum_E \int_E \phi_h s_h - \sum_E \int_{\partial E} \phi_h\,\mathbf{n}\cdot\mathbf{u}_h^0 c_h^0 \tag{46c}$$

$$\sum_E \int_\Omega \phi_h \nabla\,(\mathbf{u}_h c_h) + \sum_E \int_{\partial E} \phi_h\,\mathbf{n}\cdot\left(\mathbf{u}_h^0 c_h^0 - \mathbf{u}_h c_h\right) - \sum_E \int_\Omega \phi_h s_h = 0 \tag{46d}$$

$$\begin{aligned} &\sum_E \int_\Omega \phi_h \nabla\,(\mathbf{u}_h c_h) + \sum_e \left[\int_{\partial e'} \phi_h\,\mathbf{n}\cdot\left(\mathbf{u}_h^0 c_h^0 - \mathbf{u}_h c_h\right) + \int_{\partial e_D} \phi_h\,\mathbf{n}\cdot(\mathbf{u}_h g_D - \mathbf{u}_h c_h)\right] \\ &\quad - \sum_E \int_\Omega \phi_h s_h = 0 \end{aligned} \tag{46e}$$

Using upwinding

$$\mathbf{u}_h^0 c_h^0 = \mathbf{u}_h^{up} c_h^{up} \tag{47a}$$

$$\int_{\partial e'} \phi_h \, \mathbf{n} \cdot \left(\mathbf{u}_h^0 c_h^0 - \mathbf{u}_h c_h\right) = \int_{\partial e'} \phi_h^+ \, \mathbf{n}^+ \cdot \left(\mathbf{u}_h^{up} c_h^{up} - \mathbf{u}_h^+ c_h^+\right) + \int_{\partial e'} \phi_h^- \, \mathbf{n}^- \cdot \left(\mathbf{u}_h^{up} c_h^{up} - \mathbf{u}_h^- c_h^-\right) \tag{47b}$$

$$\gamma = \frac{(\mathbf{n} \cdot \mathbf{u} + |\mathbf{n} \cdot \mathbf{u}|)}{2\,|\mathbf{n} \cdot \mathbf{u}|}, upwind\ 1.0, downwind\ 0.0 \tag{47c}$$

$$\begin{aligned} &\int_{\partial e'} \phi_h^+ \, \mathbf{n}^+ \cdot \left(\mathbf{u}_h^{up} c_h^{up} - \mathbf{u}_h^+ c_h^+\right) + \int_{\partial e'} \phi_h^- \, \mathbf{n}^- \cdot \left(\mathbf{u}_h^{up} c_h^{up} - \mathbf{u}_h^- c_h^-\right) = \\ &\int_{\partial e'} \left(\phi_h^+ \, \mathbf{n}^+ + \phi_h^- \, \mathbf{n}^-\right) \cdot \left(\gamma^+ \mathbf{u}_h^+ c_h^+ + \gamma^- \mathbf{u}_h^- c_h^-\right) - \int_{\partial e'} \left(\phi_h^+ \, \mathbf{n}^+ \cdot \mathbf{u}_h^+ c_h^+ + \phi_h^- \, \mathbf{n}^- \cdot \mathbf{u}_h^- c_h^-\right) \\ &= \\ &\int_{\partial e'} [[\phi_h \, \mathbf{n}]] \cdot \left(\gamma^+ \mathbf{u}_h^+ c_h^+ + \gamma^- \mathbf{u}_h^- c_h^-\right) - \int_{\partial e'} [[\phi_h \, \mathbf{n} \cdot \mathbf{u}_h c_h]] \end{aligned} \tag{47d}$$

$$\begin{aligned} &\sum_E \int_\Omega \phi_h \nabla \left(\mathbf{u}_h c_h\right) \\ &+ \sum_e \left[\int_{\partial e'} [[\phi_h \, \mathbf{n}]] \cdot \left(\gamma^+ \mathbf{u}_h^+ c_h^+ + \gamma^- \mathbf{u}_h^- c_h^-\right) - \int_{\partial e'} [[\phi_h \, \mathbf{n} \cdot \mathbf{u}_h c_h]] + \int_{\partial e_D} \phi_h \, \mathbf{n} \cdot \left(\mathbf{u}_h g_D - \mathbf{u}_h c_h\right)\right] \\ &- \sum_E \int_\Omega \phi_h s_h = 0 \end{aligned} \tag{48}$$

$$\begin{aligned} -\sum_e \int_\Omega \nabla \phi_h \left(\mathbf{u}_h c_h\right)\ d\Omega = &\sum_e \int_\Omega \phi_h s_h \ d\Omega - \sum_f \int_{\Gamma'} \phi_h \, \mathbf{n} \cdot \mathbf{u}_h^0 c_h^0 \ d\Gamma' \\ &- \sum_f \int_{\Gamma_D} \phi_h \, \mathbf{n} \cdot \mathbf{u}_h g_D \ d\Gamma_D - \sum_f \int_{\Gamma_N} \phi_h \, \mathbf{n} \cdot \mathbf{u}_h c_h \ d\Gamma_N \end{aligned} \tag{49a}$$

$$-\sum_e \int_\Omega \nabla \phi \left(\mathbf{u} c\right)\ d\Omega = \sum_e \int_\Omega \phi s \ d\Omega - \sum_f \int_\Gamma \phi \, \mathbf{n} \cdot \mathbf{u} c \ d\Gamma \tag{49b}$$

Using upwinding

$$\begin{aligned} \sum_f \int_\Gamma \phi \, \mathbf{n} \cdot \mathbf{u} c \ d\Gamma &= \sum_f \left[\int_\Gamma \phi_{down} \ (\mathbf{n} \cdot \mathbf{u} c)_{up} \ d\Gamma + \int_\Gamma \phi_{up} \ (\mathbf{n} \cdot \mathbf{u} c)_{up} \ d\Gamma\right] \\ &= \sum_f \left(\phi_{up} + \phi_{down}\right) (\mathbf{n} \cdot \mathbf{u} c)_{up} \ d\Gamma \end{aligned} \tag{49c}$$

In FEniCS +/- is abitrary and normal is oriented to each element. Jump is defined as:

$$[[\phi]] = \phi_+ - \phi_- \tag{49d}$$

Therefore:

$$\gamma = \frac{1}{2}\left(\mathbf{n}\cdot\mathbf{u} + |\mathbf{n}\cdot\mathbf{u}|\right) \tag{49e}$$

$$\sum_f \left(\phi_{up} + \phi_{down}\right)\left(\mathbf{n}\cdot\mathbf{u}c\right)_{up}\ d\Gamma = \sum_f \int_\Gamma [[\phi]]\left(\gamma_+ c_+ - \gamma_- c_-\right)\ d\Gamma \tag{49f}$$

$$-\sum_e \int_\Omega \nabla\phi\left(\mathbf{u}c\right)\ d\Omega + \sum_f \int_\Gamma [[\phi]]\left(\gamma_+ c_+ - \gamma_- c_-\right)\ d\Gamma = 0 \tag{50}$$

6 DG Diffusion:

$$-\nabla\cdot(\bar{\bar{\kappa}}\cdot\nabla c) = s \tag{51}$$

Assume diffusivity is isentropic. Second derivative of DG is unreal so we discretise the derivative first

$$-\nabla\cdot(\bar{\bar{\kappa}}\cdot\mathbf{q}) = s \tag{52}$$

$$-\sum_e \int_\Omega \phi\nabla\cdot(\kappa\mathbf{q})\ d\Omega = \int_\Omega \phi s\ d\Omega \tag{53}$$

$$\sum_e \int_\Omega \nabla\phi\cdot(\kappa\mathbf{q})\ d\Omega = \sum_e \int_\Omega \phi s\ d\Omega + \sum_f \int_\Gamma \phi\mathbf{n}\cdot\kappa\mathbf{q}\ d\Gamma \tag{54}$$

$$\sum_e \int_\Omega \nabla\phi\cdot(\kappa\mathbf{q})\ d\Omega = \sum_e \int_\Omega \phi s\ d\Omega + \sum_f \int_\Gamma \left(\phi_{up}\cdot\mathbf{n}_{up} + \phi_{down}\cdot\mathbf{n}_{down}\right)\left(\kappa\mathbf{q}\right)_{up}\ d\Gamma \tag{55}$$

$$\nabla c = \mathbf{q} \tag{56}$$

$$\int_\Omega \mathbf{w}\cdot\nabla c\ d\Omega = \int_\Omega \mathbf{w}\cdot\mathbf{q}\ d\Omega \tag{57}$$

$$-\sum_e \int_\Omega \nabla\cdot\mathbf{w}c\ d\Omega = \int_\Omega \mathbf{w}\cdot\mathbf{q}\ d\Omega - \sum_f \int_\Gamma \mathbf{w}\cdot\mathbf{n}\,c\ d\Gamma \tag{58}$$

$$\begin{aligned}\sum_f \int_\Gamma \mathbf{w}\cdot\mathbf{n}\,c\ d\Gamma &= \sum_f \left[\int_\Gamma (\mathbf{n}\cdot\mathbf{w})_{down}\ c_{up}\ d\Gamma + \int_\Gamma (\mathbf{n}\cdot\mathbf{w})_{up}\ c_{up}\ d\Gamma\right] \\ &= \sum_f \int_\Gamma \left(\mathbf{n}_{up}\cdot\mathbf{w}_{up} + \mathbf{n}_{down}\cdot\mathbf{w}_{down}\right) c_{up}\ d\Gamma\end{aligned} \tag{59}$$

$$\int_\Omega \mathbf{w}\cdot\mathbf{q}\ d\Omega = \sum_f \int_\Gamma \left(\mathbf{n}_{up}\cdot\mathbf{w}_{up} + \mathbf{n}_{down}\cdot\mathbf{w}_{down}\right) c_{up}\ d\Gamma - \sum_e \int_\Omega \nabla\cdot\mathbf{w}c\ d\Omega \tag{60}$$

7 k-ϵ:

7.1 momentum equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial \tau_{ij}^R}{\partial x_j} = -\frac{\partial p}{\partial x_i} + s_i \tag{61}$$

$$\tau_{ij} = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{62}$$

$$\tau_{ij}^R = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \tag{63}$$

7.1.1 weak form

$$\begin{gathered}\int_\Omega \phi \frac{\partial u_i}{\partial t}\, dx + \int_\Omega \phi u_j \frac{\partial u_i}{\partial x_j}\, dx - \int_\Omega \phi \frac{\partial \tau_{ij}}{\partial x_j}\, dx - \int_\Omega \phi \frac{\partial \tau_{ij}^R}{\partial x_j}\, dx = \\ -\int_\Omega \phi \frac{\partial p}{\partial x_i}\, dx + \int_\Omega \phi \rho_* g_i\, dx + \int_\Omega \phi s_i\, dx\end{gathered} \tag{64}$$

$$\begin{gathered}\int_\Omega \phi \frac{\partial u_i}{\partial t}\, dx + \int_\Omega \phi u_j \frac{\partial u_i}{\partial x_j}\, dx + \int_\Omega \frac{\partial \phi}{\partial x_j} \tau_{ij}\, dx - \int_\Omega \phi \frac{\partial \tau_{ij}^R}{\partial x_j}\, dx = \\ -\int_\Omega \phi \frac{\partial p}{\partial x_i}\, dx + \int_\Omega \phi \rho_* g_i\, dx + \int_\Omega \phi s_i\, dx + \int_\Gamma \phi \tau_{ij}\, dx\end{gathered} \tag{65}$$

Always dirichlet or zero neumann boundary conditions:

$$\begin{gathered}\int_\Omega \phi \frac{\partial u_i}{\partial t}\, dx + \int_\Omega \phi u_j \frac{\partial u_i}{\partial x_j}\, dx + \int_\Omega \frac{\partial \phi}{\partial x_j} \tau_{ij}\, dx - \int_\Omega \phi \frac{\partial \tau_{ij}^R}{\partial x_j}\, dx = \\ -\int_\Omega \phi \frac{\partial p}{\partial x_i}\, dx + \int_\Omega \phi \rho_* g_i\, dx + \int_\Omega \phi s_i\, dx\end{gathered} \tag{66}$$

7.2 turbulent kinetic energy

$$\frac{\partial k_i}{t} + \frac{\partial (\mathbf{u}k)_i}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\nu_T \frac{\partial k}{x_j} \right) = s + \frac{\partial u_i}{\partial x_j} \tau_{ij}^R - \epsilon \tag{67}$$

$$\frac{\partial k_i}{t} + \frac{\partial (\mathbf{u}k)_i}{\partial x_j} - \nu_T \frac{\partial}{x_j} \left(\frac{\partial k}{x_j} \right) - \frac{\partial \nu_T}{\partial x_j} \frac{\partial k}{\partial x_j} = s + \frac{\partial u_i}{\partial x_j} \tau_{ij}^R - \epsilon \tag{68}$$

$$\int_\Omega \eta \frac{\partial k_i}{t}\, dx + \int_\Omega \eta \frac{\partial (\mathbf{u}k)_i}{\partial x_j}\, dx - \int_\Omega \eta \frac{\partial}{\partial x_j} \left(\nu_T \frac{\partial k}{x_j} \right)\, dx = \int_\Omega \eta s\, dx + \int_\Omega \eta \frac{\partial u_i}{\partial x_j} \tau_{ij}^R\, dx - \int_\Omega \eta \epsilon\, dx \tag{69}$$