Introduction to Linear Programming

What is Linear Programming?

“**Linear programming** (**LP**; also called **linear optimization**) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. Linear programming is a special case of mathematical programming (mathematical optimization).” Wikipedia

- Linear Programming solves a type of optimization problem.
- Objectives of engineering, business and economic decisions frequently involve **maximizing** or **minimizing a function**.
- Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing.

Linear Programming (LP) has many applications

- Product mix
- Production Scheduling
- Marketing Research
- Portfolio selection
- Shipping & transportation
- Multiperiod scheduling
- Planning airline routes
- Telecommunications

Model Components

Decision variables - mathematical symbols representing levels of activity of a firm.

Objective function - a linear mathematical relationship describing an objective of the firm, in terms of decision variables - this function is to be maximized or minimized.

Constraints – requirements or restrictions placed on the firm by the operating environment, stated in linear relationships of the decision variables.

Parameters - numerical coefficients and constants used in the objective function and constraints.

LP problems standard (canonical) form:

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b} \\ \text{and} & \mathbf{x} \ge \mathbf{0} \end{array}$$

max $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$

…

…

$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \le b_n$

$x_i \ge 0$

For a particular application we begin with the problem scenario and data, then:

1) Define the decision variables
2) Formulate the LP model using the decision variables
   - Write the objective function equation
   - Write each of the constraint equations
3) Graph the feasible set.
4) Determine the vertices of the feasible
5) Evaluate the objective function at each vertex.
6) Determine the optimal point

LP Model Formulation
A Maximization Example

Product mix problem - Beaver Creek Pottery Company produces mugs and bowls. Each bowl requires one hour of labor and 4 lbs of clay. Each mug requires 2 hours of labor and 3 pounds of clay. The profit on each bowl is $40 and the profit for each mug is $50.

How many bowls and mugs should be produced to maximize profits given labor and materials constraints?

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (Lb./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50

LP Model Formulation
A Maximization Example

Resource Availability:	40 hrs of labor per day 120 lbs of clay
Decision Variables:	x_1 = number of bowls to produce per day x_2 = number of mugs to produce per day
Objective Function:	Maximize $Z = \$40x_1 + \$50x_2$ Where Z = profit per day
Resource Constraints:	$1x_1 + 2x_2 \leq 40$ hours of labor $4x_1 + 3x_2 \leq 120$ pounds of clay
Non-Negativity Constraints:	$x_1 \geq 0; x_2 \geq 0$

LP Model Formulation
A Maximization Example

Complete Linear Programming Model:

Maximize $\quad Z = \$40x_1 + \$50x_2$

subject to: $\quad 1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Feasible Solutions

A **feasible solution** does not violate **any** of the constraints:

Example: $x_1 = 5$ bowls

$x_2 = 10$ mugs

$Z = \$40x_1 + \$50x_2 = \$700$

Labor constraint check: $1(5) + 2(10) = 25 < 40$ hours

Clay constraint check: $4(5) + 3(10) = 50 < 120$ pounds

Infeasible Solutions

An **infeasible solution** violates **at least one** of the constraints:

Example: $x_1 = 10$ bowls
$x_2 = 20$ mugs
$Z = \$40x_1 + \$50x_2 = \$1400$

Labor constraint check: $1(10) + 2(20) = 50 > 40$ hours

Graphical Solution of LP Models

Graphical solution is limited to linear programming models containing ***only two decision variables*** (can be used with three variables but only with great difficulty).

Graphical methods provide ***visualization of how*** a solution for a linear programming problem is obtained.

Coordinate Axes
Graphical Solution of Maximization Model

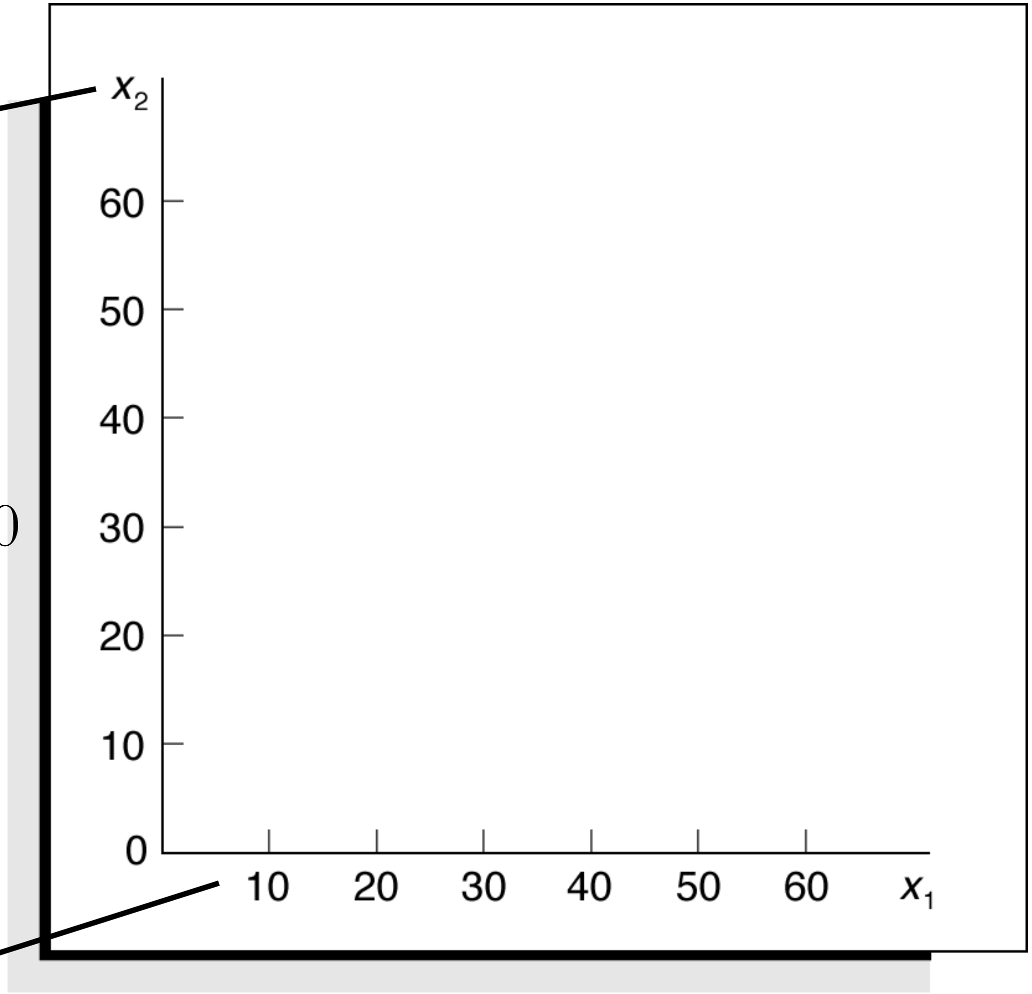
X_2 is mugs

Maximize $\$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



X_1 is bowls

Labor Constraint
Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

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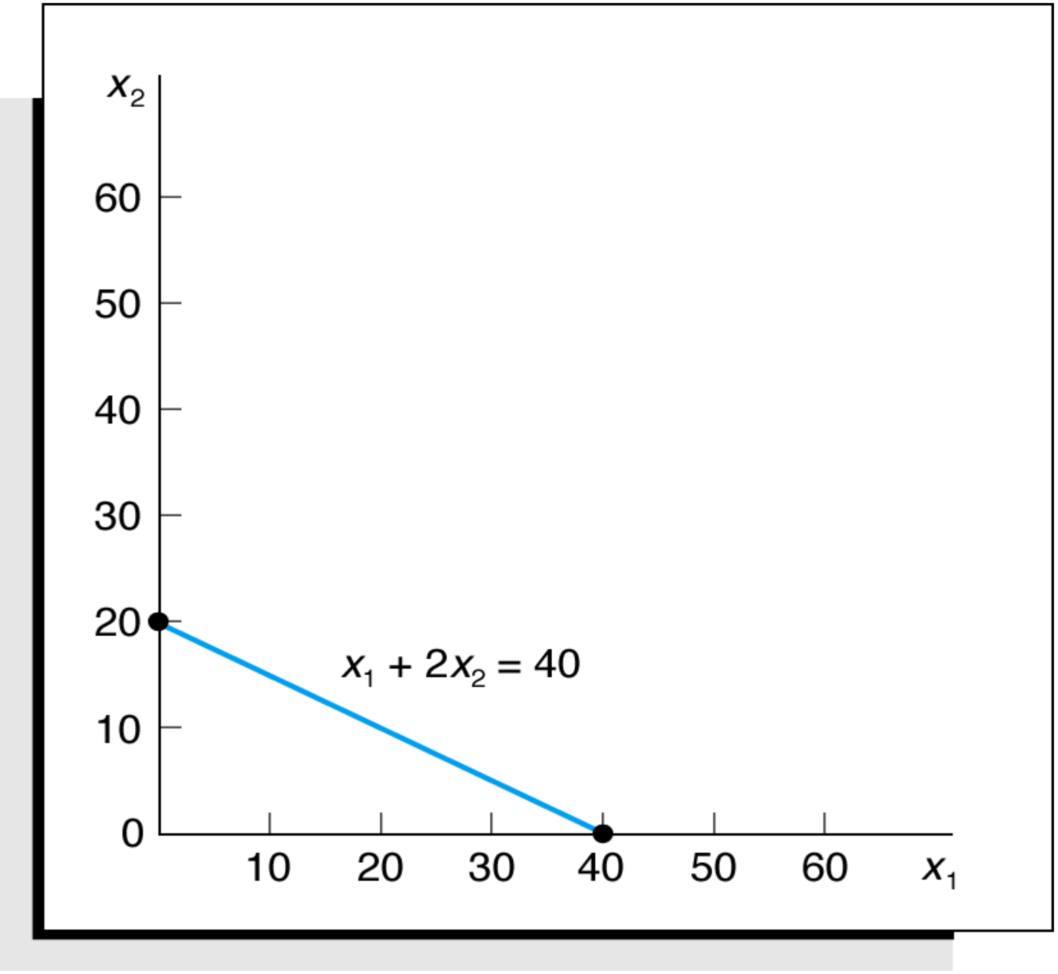
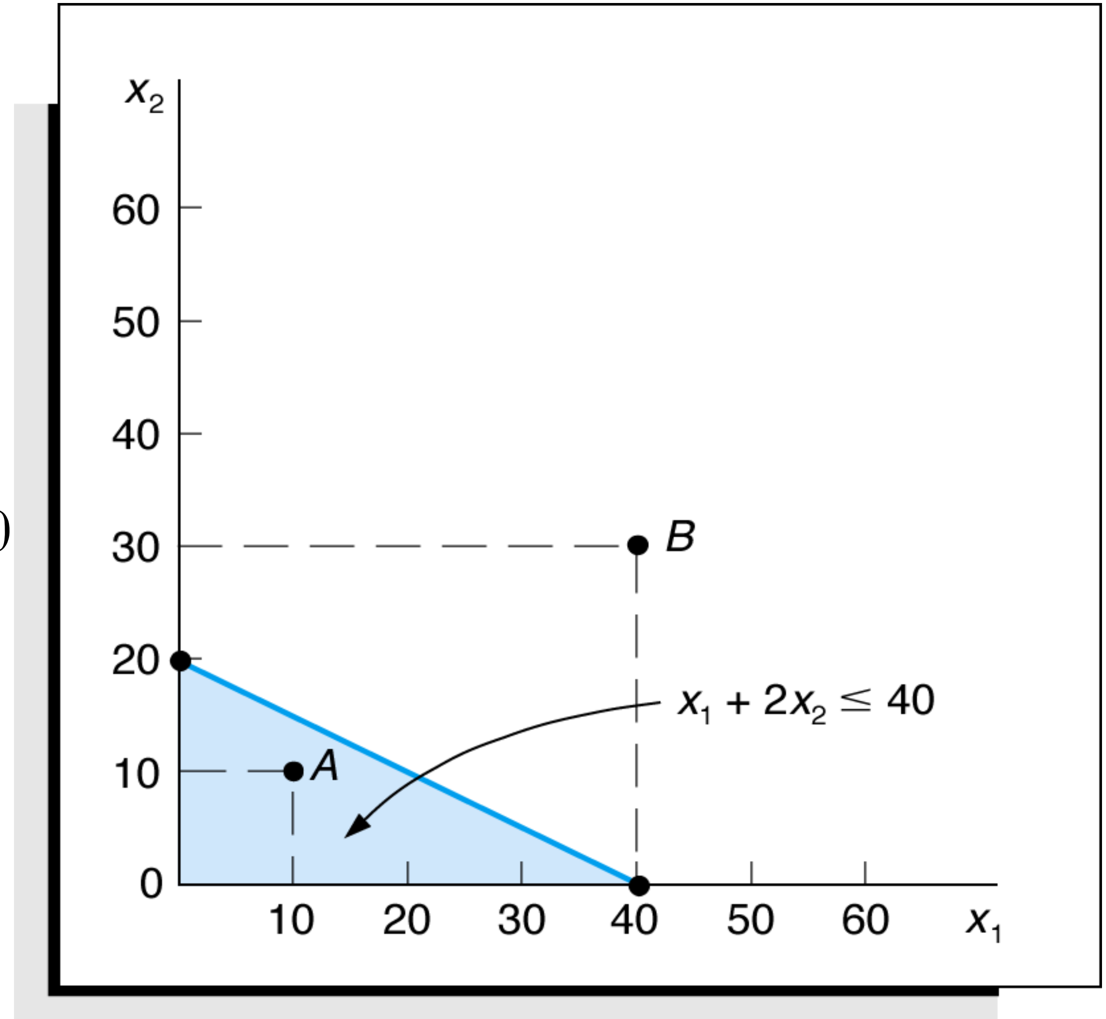


Labor Constraint Area
Graphical Solution of Maximization Model

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

$$\begin{aligned}\text{subject to:} \quad & 1x_1 + 2x_2 \leq 40 \\ & 4x_1 + 3x_2 \leq 120 \\ & x_1, x_2 \geq 0\end{aligned}$$



Labor Constraint Area

Clay Constraint Area
Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

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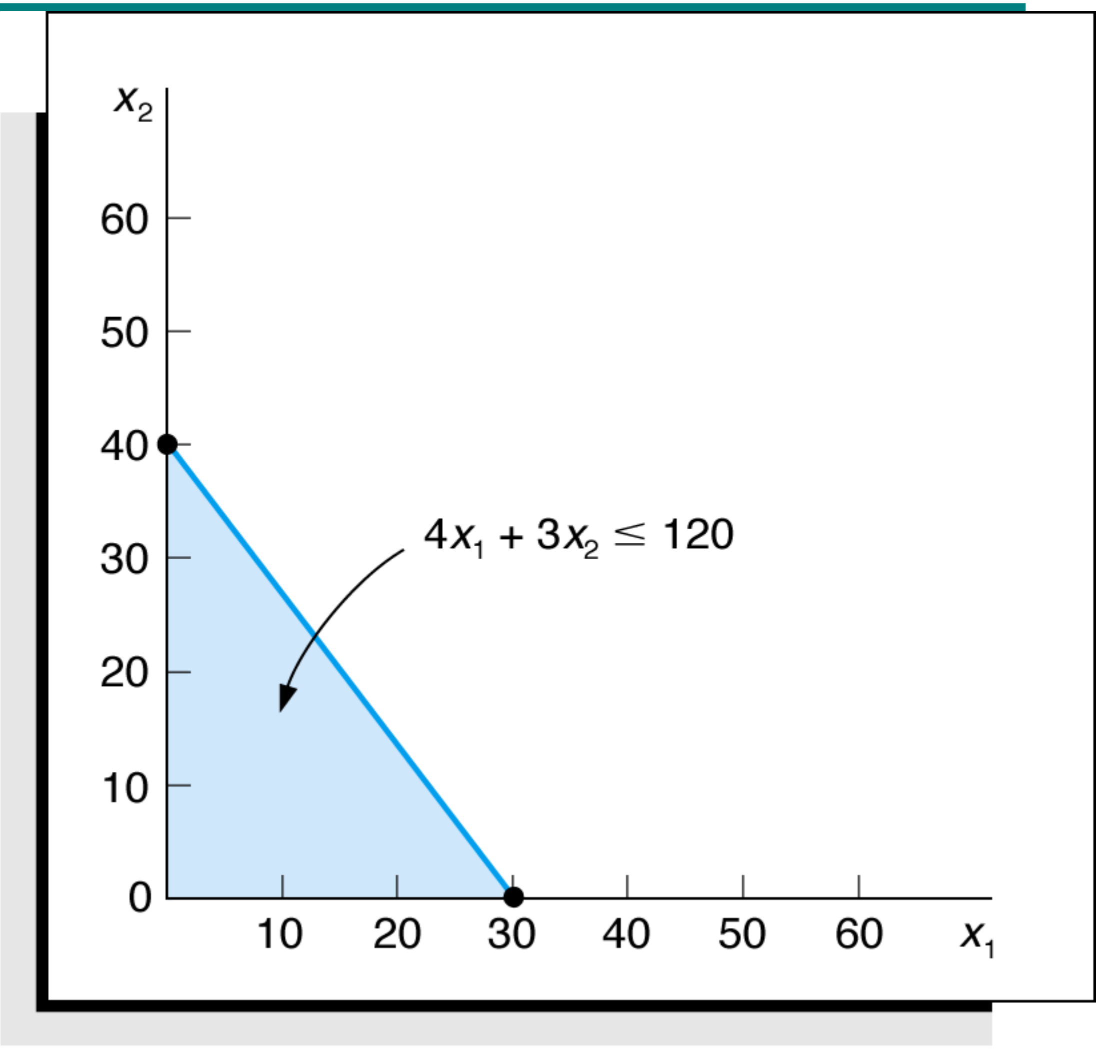

Clay Constraint Area

Both Constraints
Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$

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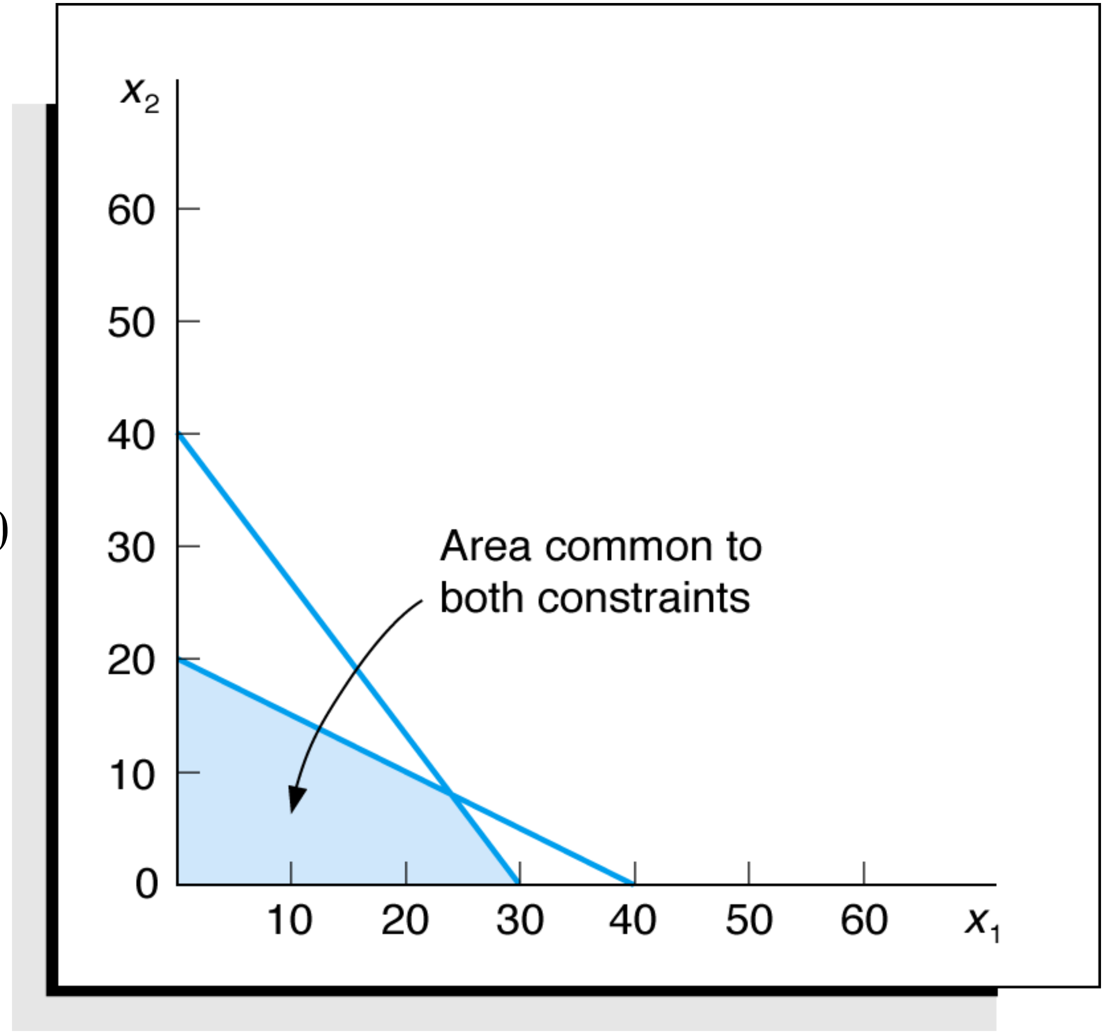

Graph of Both Model Constraints

Feasible Solution Area
Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

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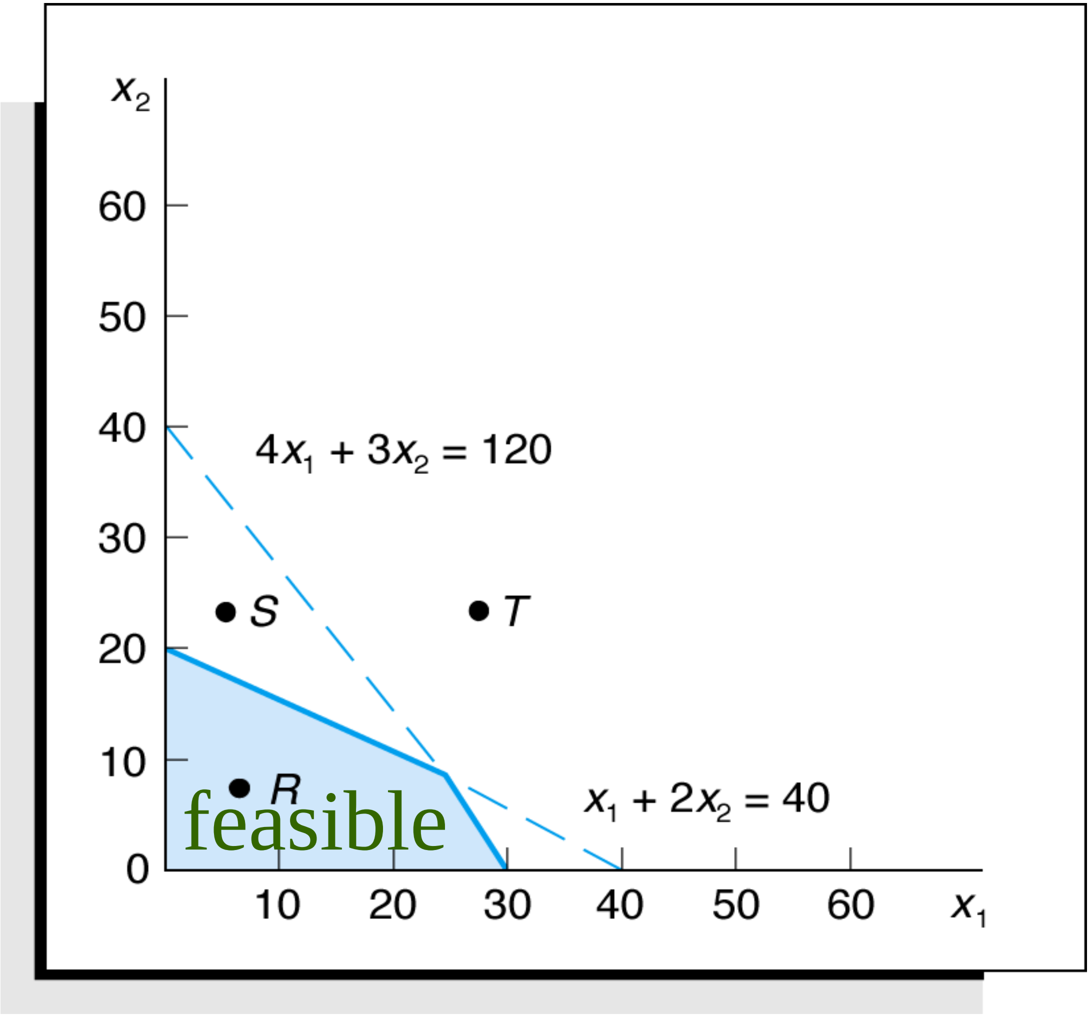
$x_1, x_2 \geq 0$



Feasible Solution Area

Objective Function Solution = \$800
Graphical Solution of Maximization

Maximize $Z = \$40x_1 + \$50x_2$

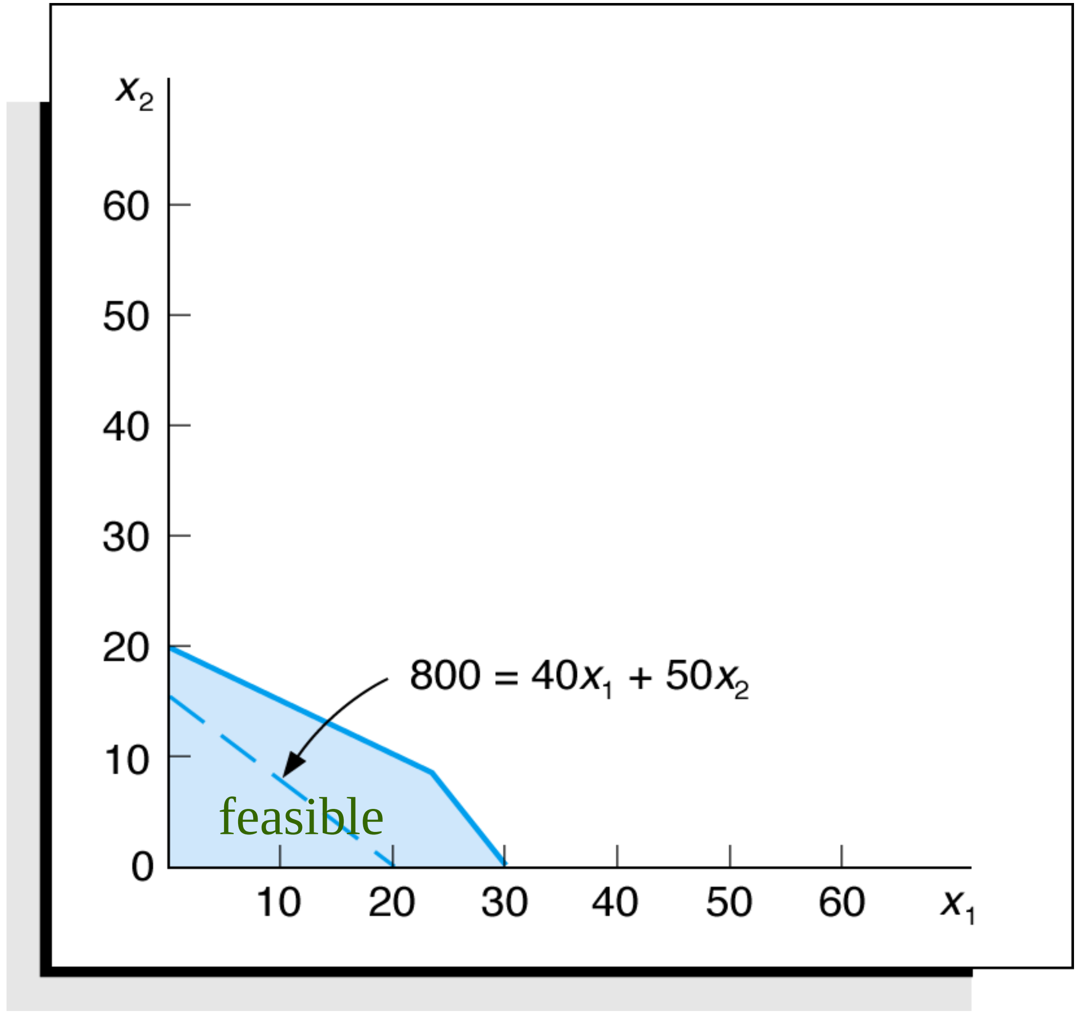
subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

One solution for Z = \$800 is $x_1 = 5$, $x_2 = 12$

However Z = \$800 is not optimal



Objection Function Line for Z = \$800

Alternative Objective Function Solution Lines
Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

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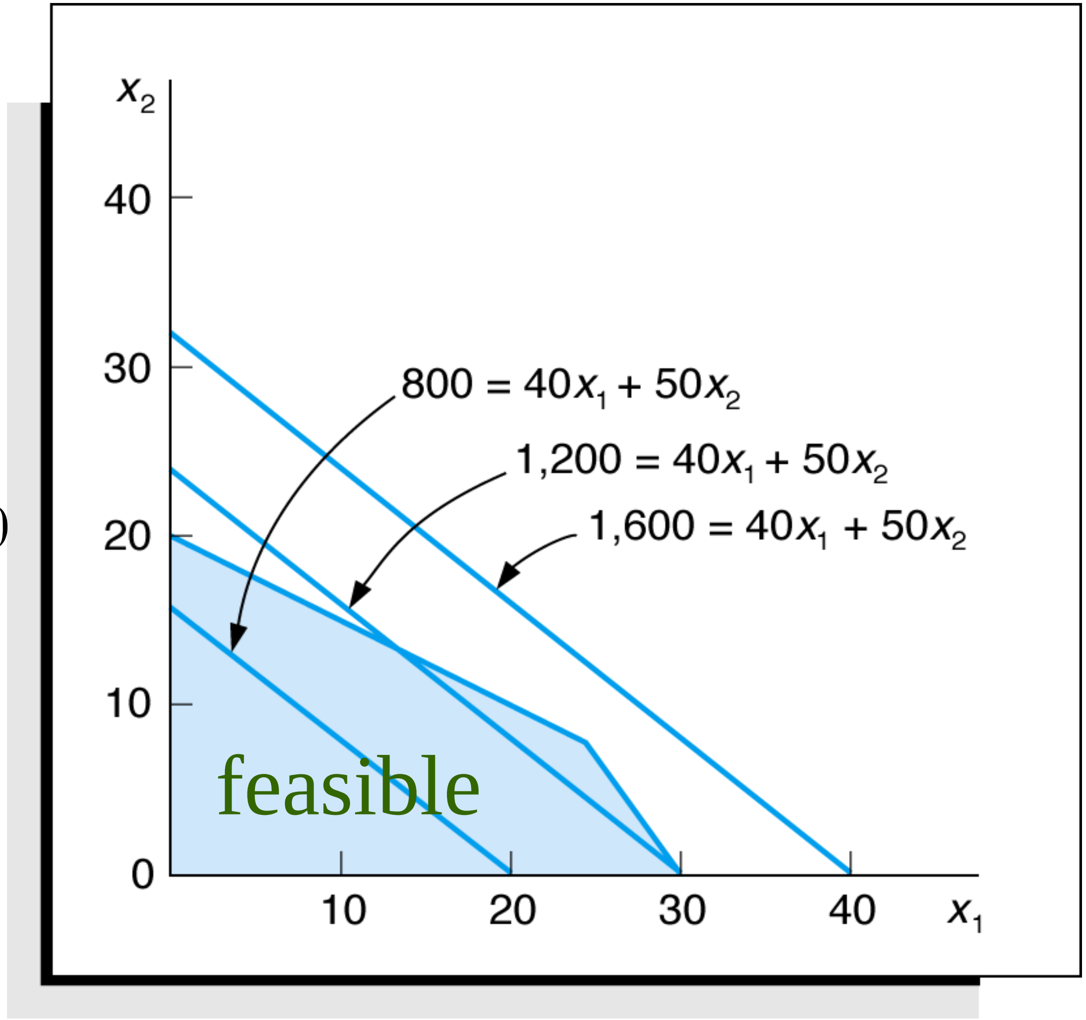
$x_1, x_2 \geq 0$



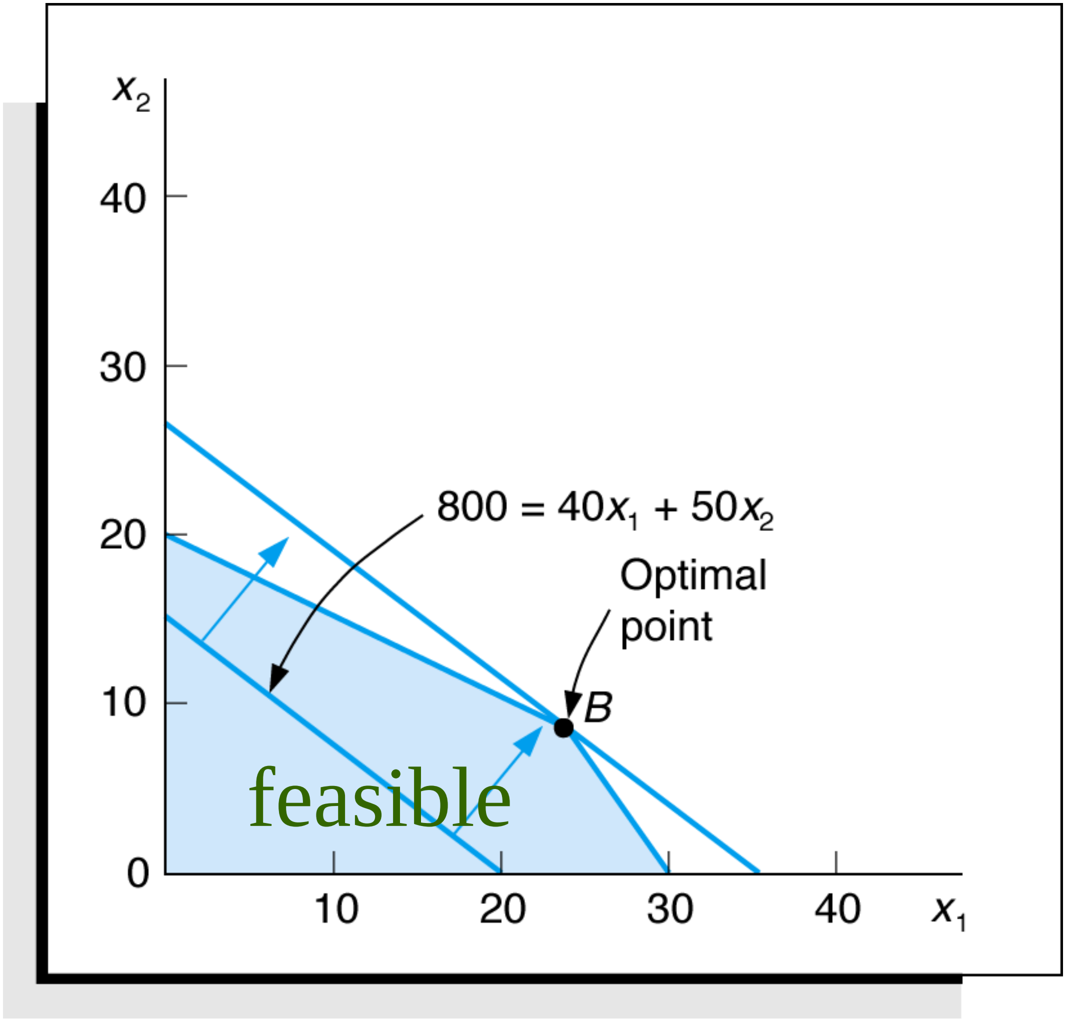
Alternative Objective Function Lines

Optimal Solution
Graphical Solution of Maximization Model

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

subject to:

$$1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$



Identification of Optimal Solution Point

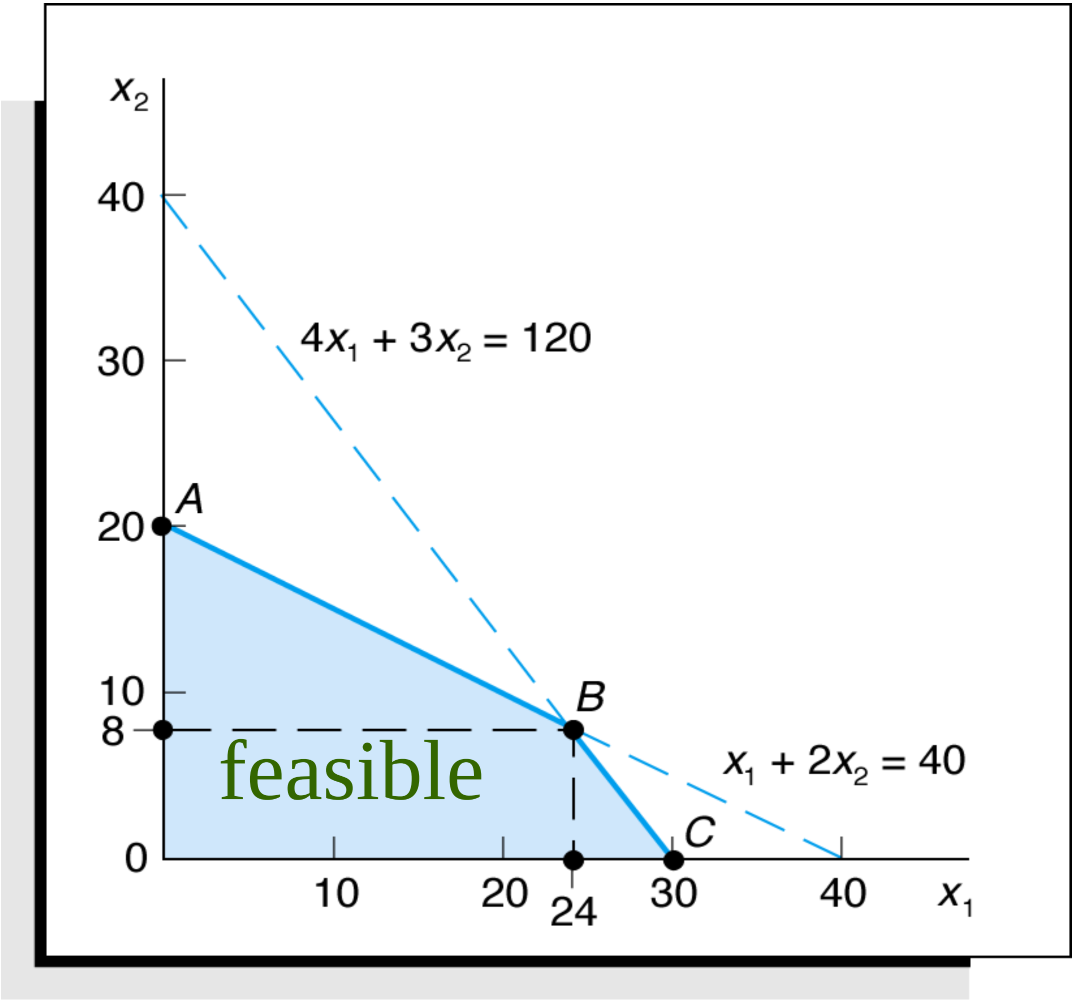
Optimal Solution Coordinates
Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Optimal Solution Coordinates

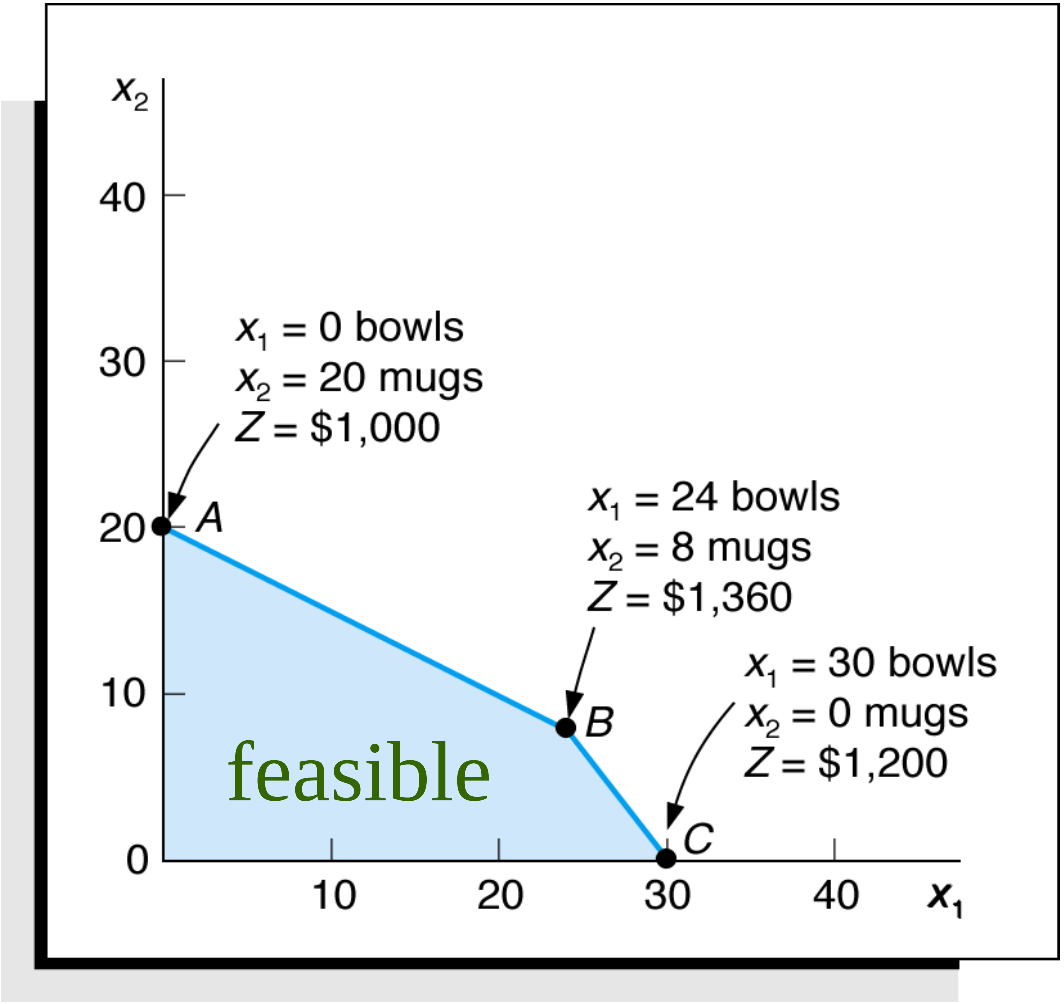
Extreme (Corner) Point Solutions
Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Solutions at All Corner Points

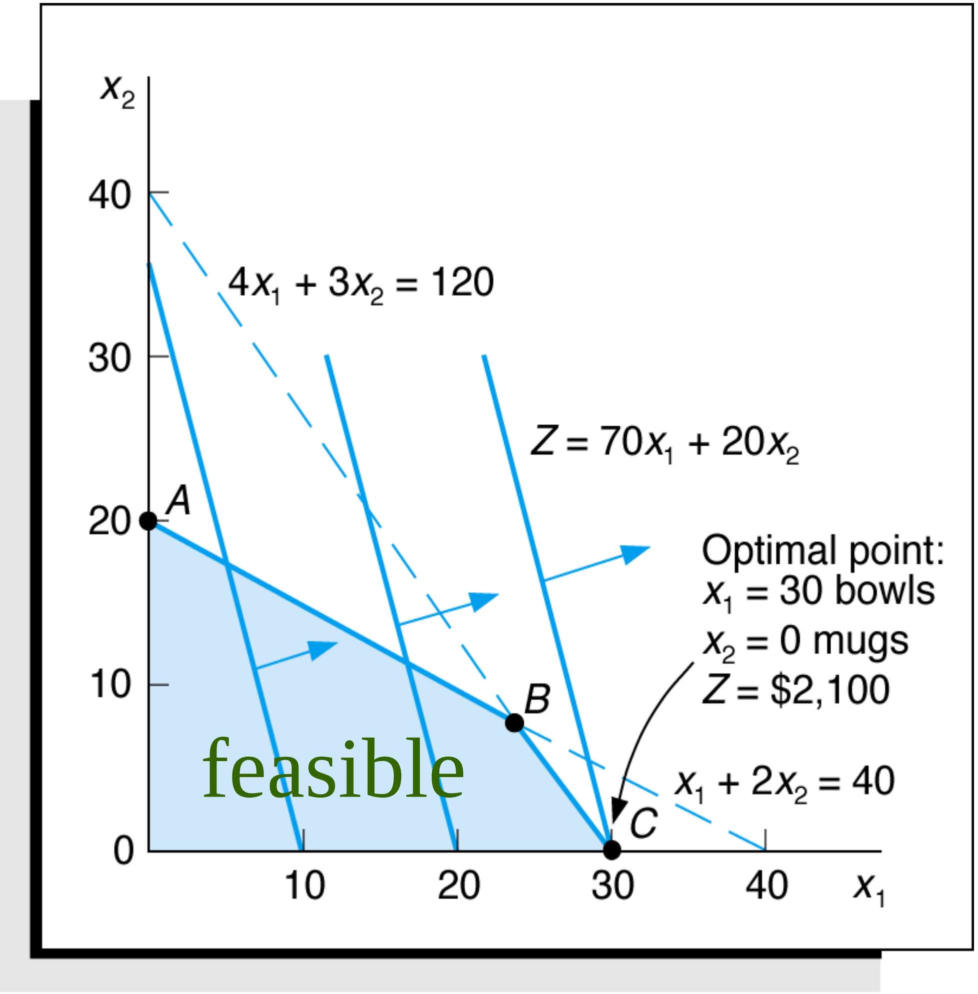
Optimal Solution for New Objective Function
Graphical Solution of Maximization Model

Maximize $Z = \$70x_1 + \$20x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Optimal Solution with $Z = 70x_1 + 20x_2$

Slack Variables

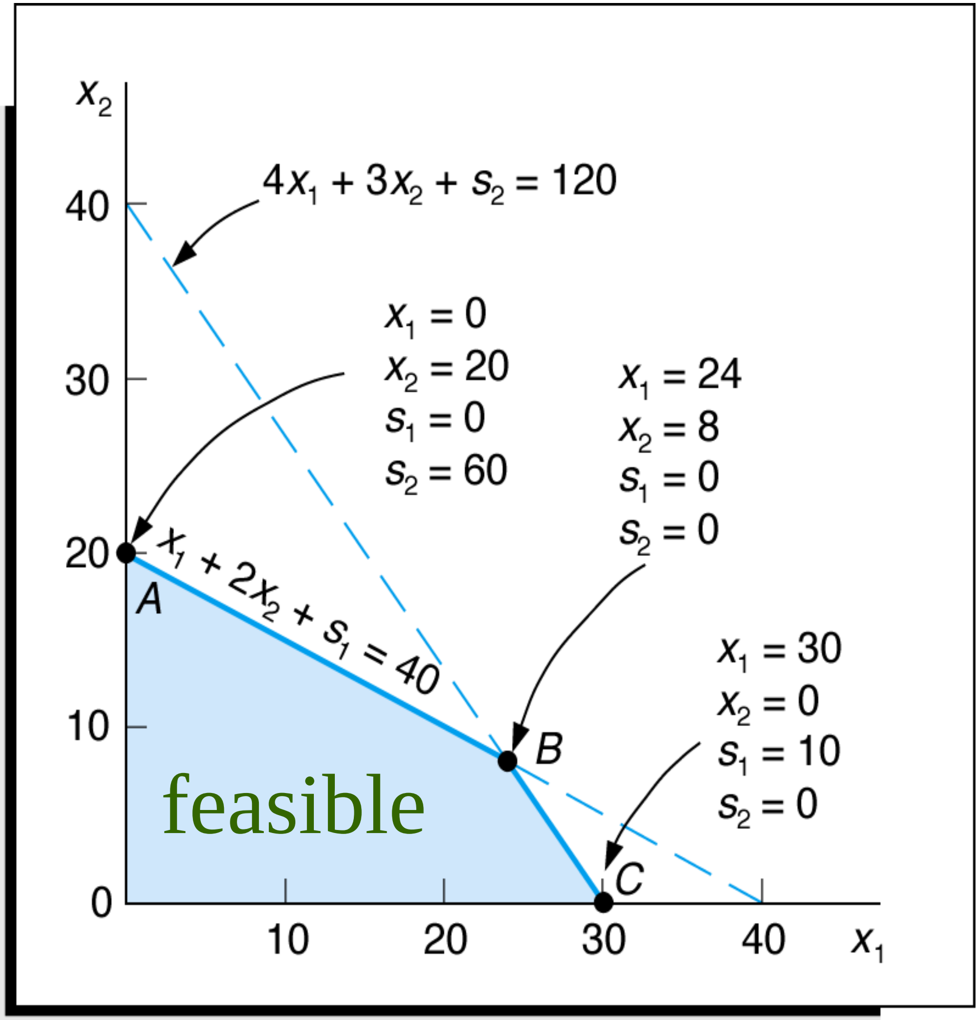
- Standard form requires that all constraints be in the form of equations (equalities).
- A slack variable is ***added to a $\leq$ constraint*** (weak inequality) to convert it to an equation (=).
- A slack variable typically represents an ***unused resource***.
- A slack variable ***contributes nothing*** to the objective function value.

Linear Programming Model: Standard Form

$$\text{Max } Z = 40x_1 + 50x_2 + 0s_1 + 0s_2$$
$$\text{subject to: } 1x_1 + 2x_2 + s_1 = 40$$
$$4x_1 + 3x_2 + s_2 = 120$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Where:

x_1 = number of bowls
x_2 = number of mugs
s_1, s_2 are slack variables



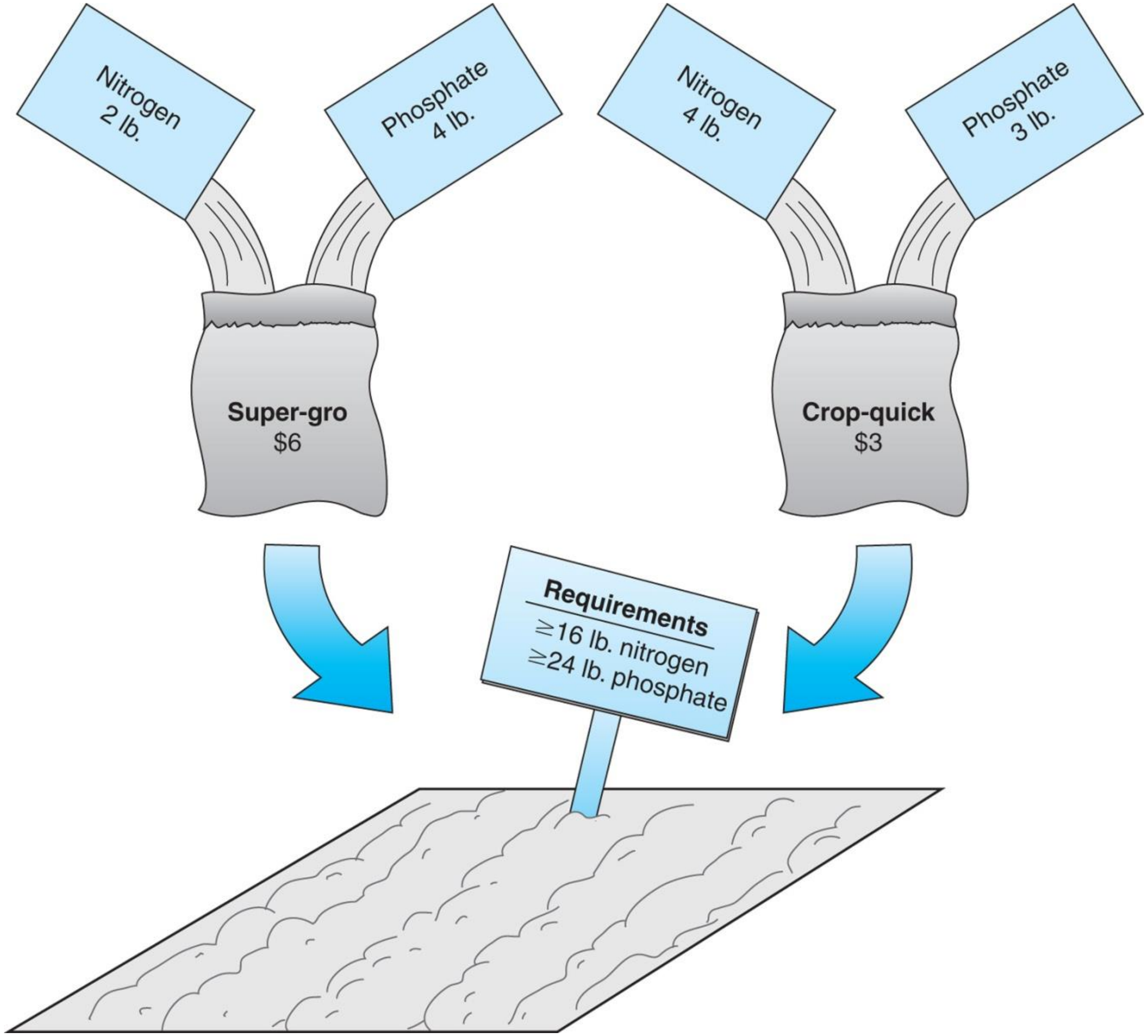
Solution Points A, B, and C with Slack

LP Model Formulation – Minimization

- Two brands of fertilizer available - Super-gro, Crop-quick.
- Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
- Super-gro costs $6 per bag, Crop-quick $3 per bag.
- Problem: How much of each brand to purchase to minimize total cost of fertilizer given following data ?

Brand	Chemical Contribution: Nitrogen (lb/bag)	Chemical Contribution: Phosphate (lb/bag)
Super-gro	2	4
Crop-quick	4	3

LP Model Formulation – Minimization



LP Model Formulation – Minimization

Decision Variables:

x_1 = bags of Super-gro
x_2 = bags of Crop-quick

The Objective Function:

Minimize $Z = \$6x_1 + 3x_2$
Where: $\$6x_1$ = cost of bags of Super-Gro
$\$3x_2$ = cost of bags of Crop-Quick

Model Constraints:

$2x_1 + 4x_2 \geq 16$ lb (nitrogen constraint)
$4x_1 + 3x_2 \geq 24$ lb (phosphate constraint)
$x_1, x_2 \geq 0$ (non-negativity constraint)

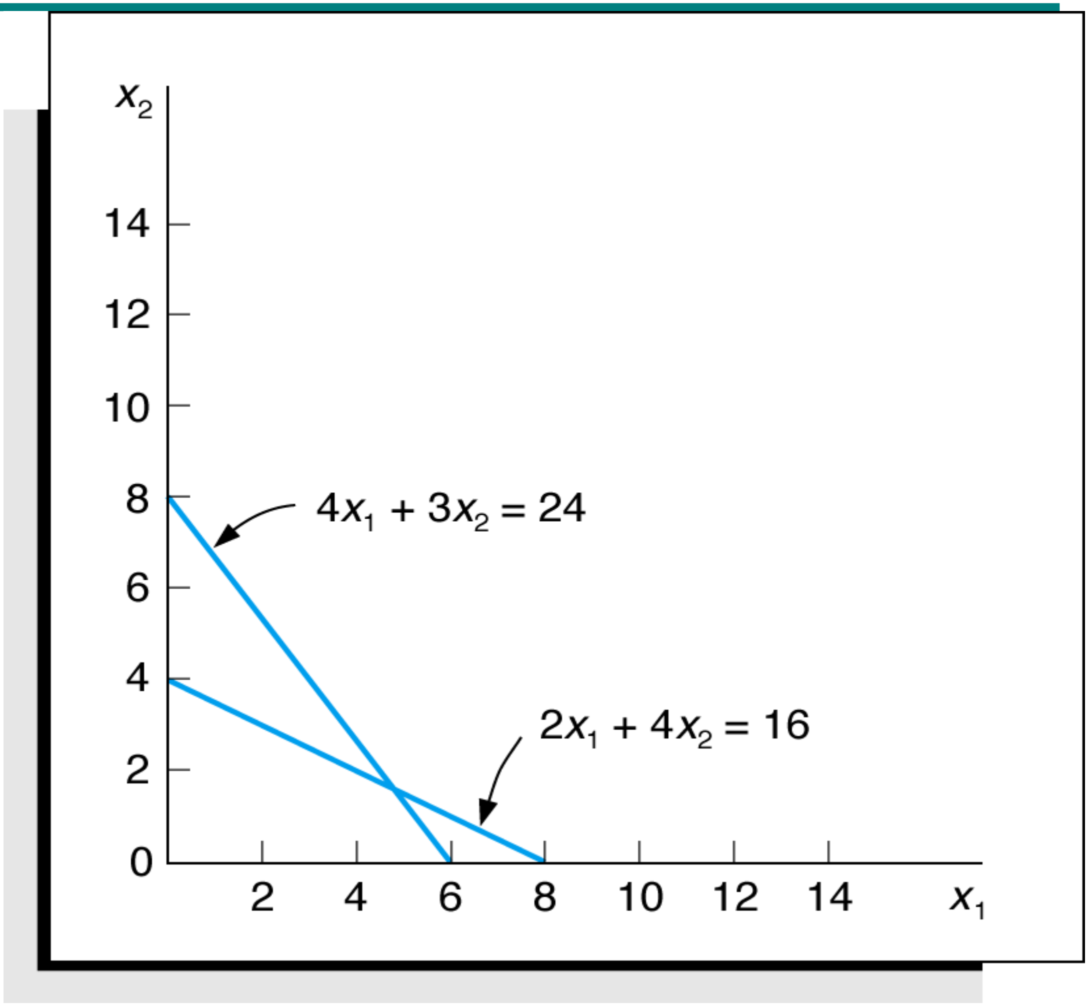
Constraint Graph – Minimization

Minimize $Z = \$6x_1 + \$3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$



Graph of Both Model Constraints

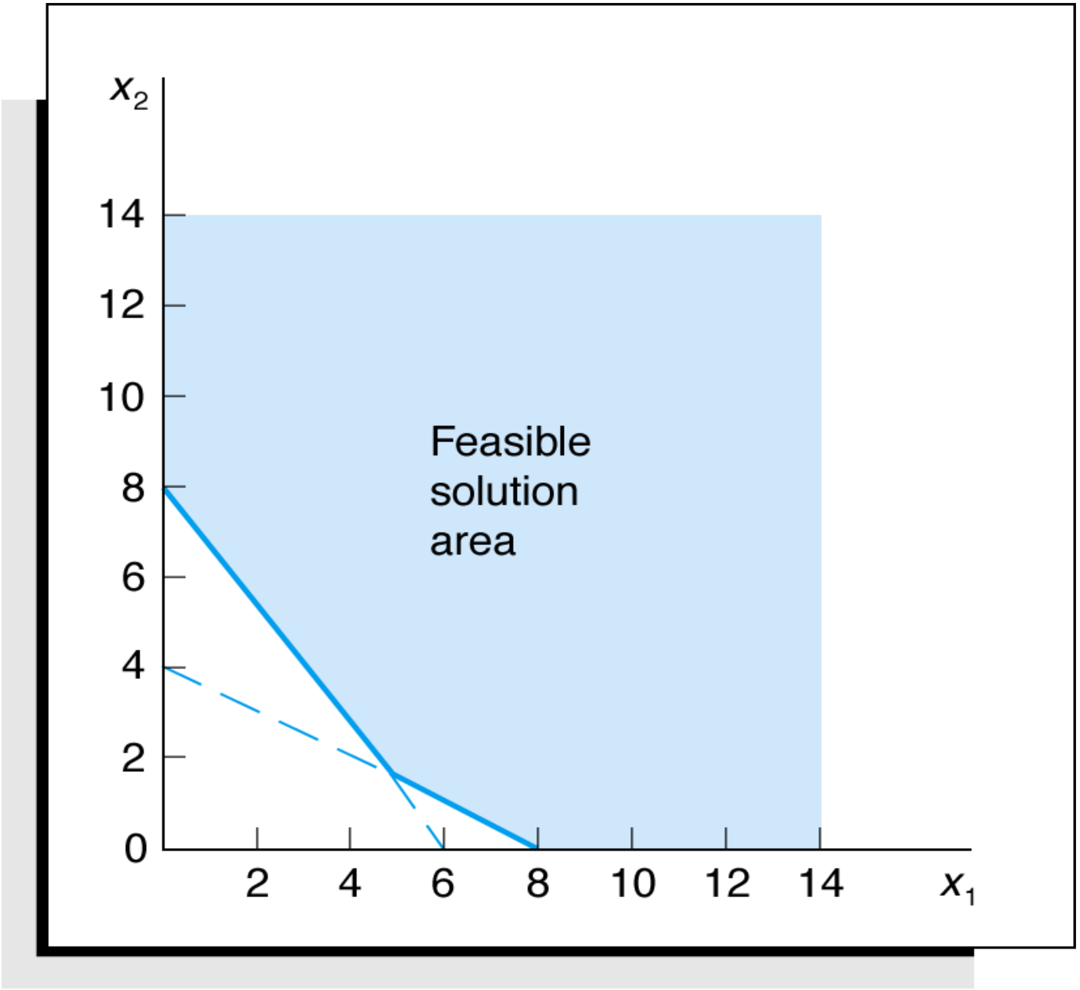
Feasible Region– Minimization

Minimize $Z = \$6x_1 + \$3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$



Feasible Solution Area

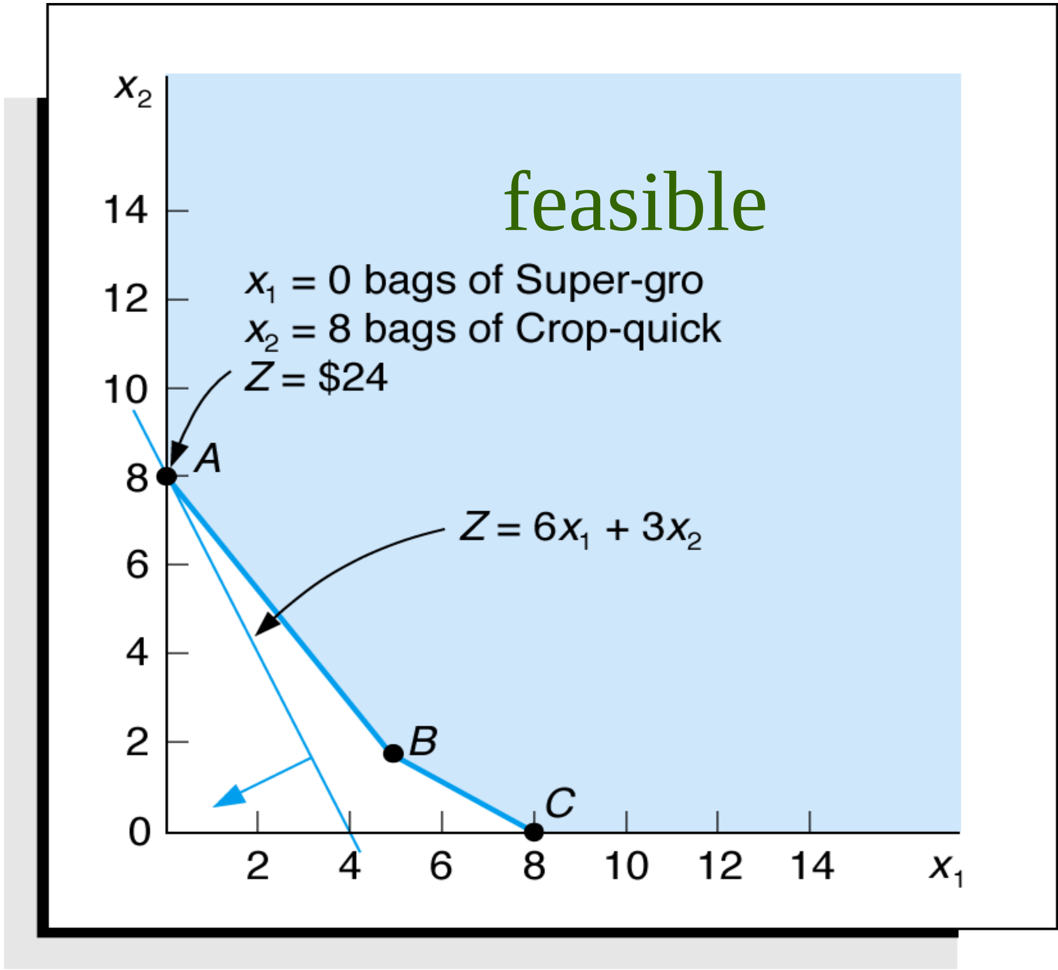
Optimal Solution Point – Minimization

Minimize $Z = \$6x_1 + \$3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$



Optimum Solution Point

Surplus Variables – Minimization

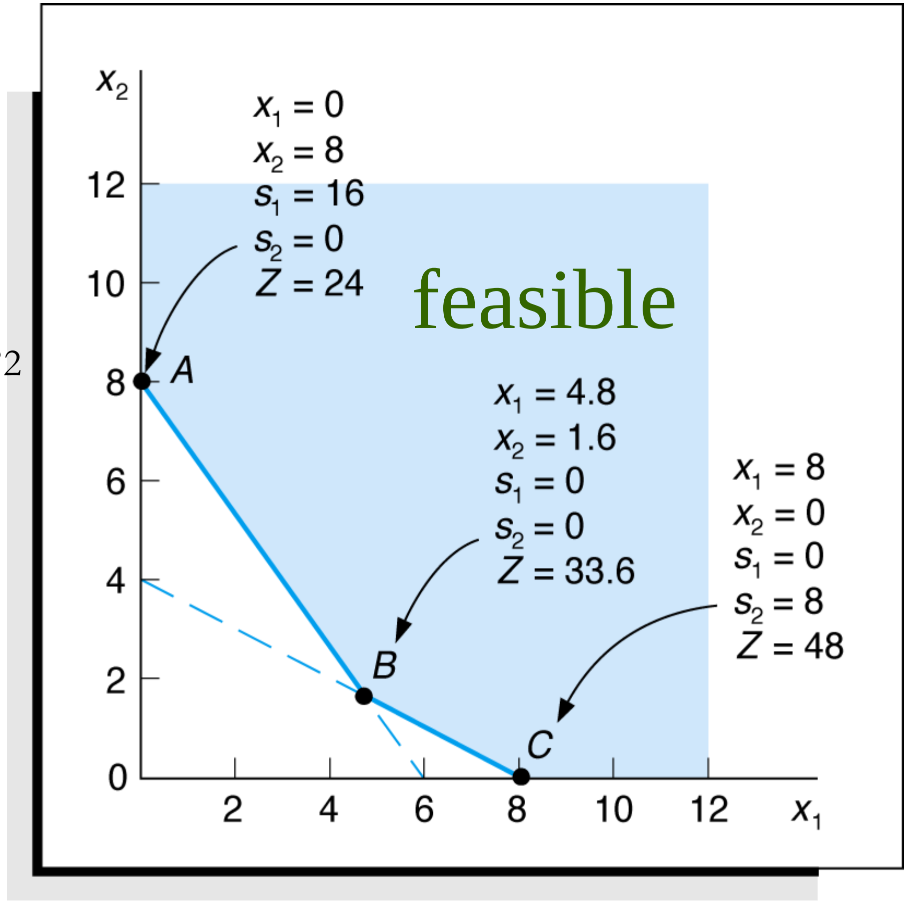
- A surplus variable is ***subtracted from a $\geq$ constraint*** to convert it to an equation (=).
- A surplus variable ***represents an excess*** above a constraint requirement level.
- A surplus variable ***contributes nothing*** to the calculated value of the objective function.
- Subtracting surplus variables in the farmer problem constraints:

$$2x_1 + 4x_2 - s_1 = 16 \text{ (nitrogen)}$$
$$4x_1 + 3x_2 - s_2 = 24 \text{ (phosphate)}$$

Graphical Solutions – Minimization

$$\text{Minimize } Z = \$6x_1 + \$3x_2 + 0s_1 + 0s_2$$

subject to:
$$2x_1 + 4x_2 - s_1 = 16$$
$$4x_1 + 3x_2 - s_2 = 24$$
$$x_1, x_2, s_1, s_2 \geq 0$$



Graph of Fertilizer Example

Irregular Types of Linear Programming Problems

For some linear programming models, the general rules do not apply.

Special types of problems include those with:

- Multiple optimal solutions
- Infeasible solutions
- Unbounded solutions

Multiple Optimal Solutions Beaver Creek Pottery

The objective function is **parallel** to a constraint line.

Maximize $Z = \$40x_1 + 30x_2$

subject to: $1x_1 + 2x_2 \leq 40$

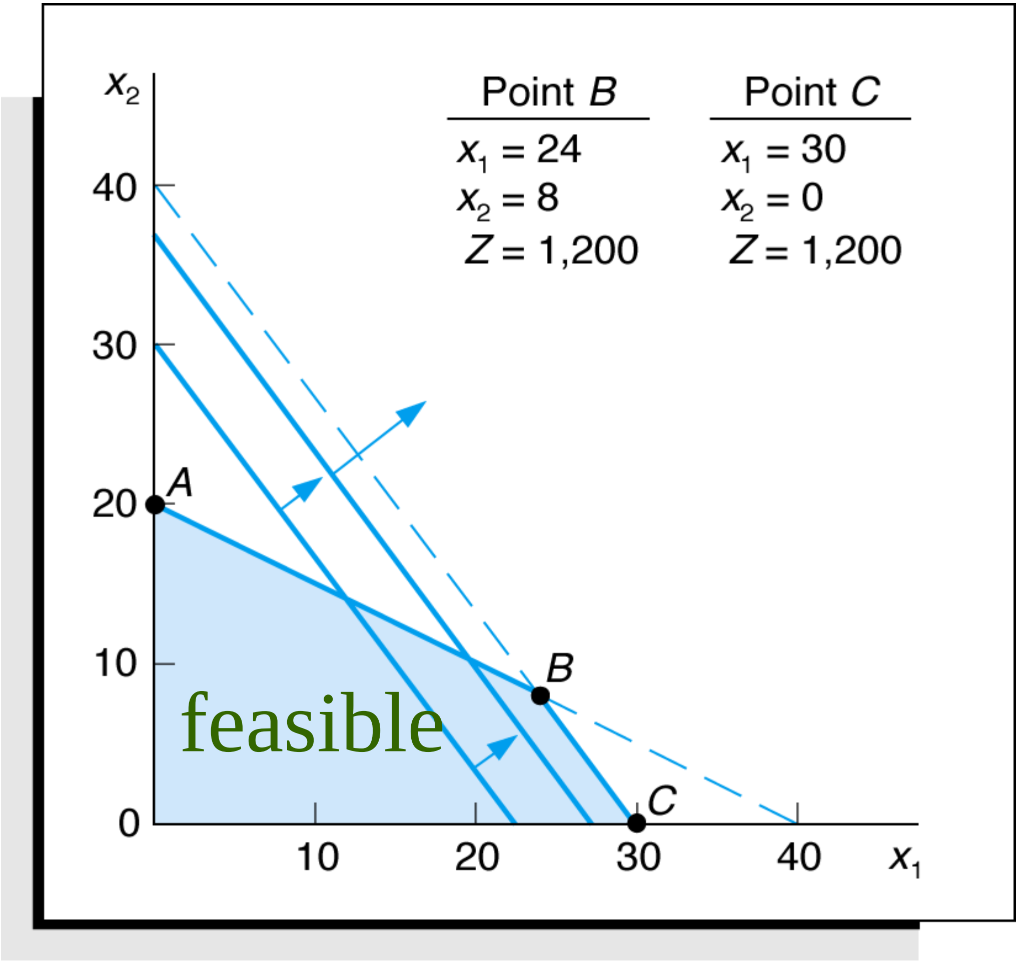
$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs



Example with Multiple Optimal Solutions

An Infeasible Problem

Every possible solution **violates** at least one constraint:

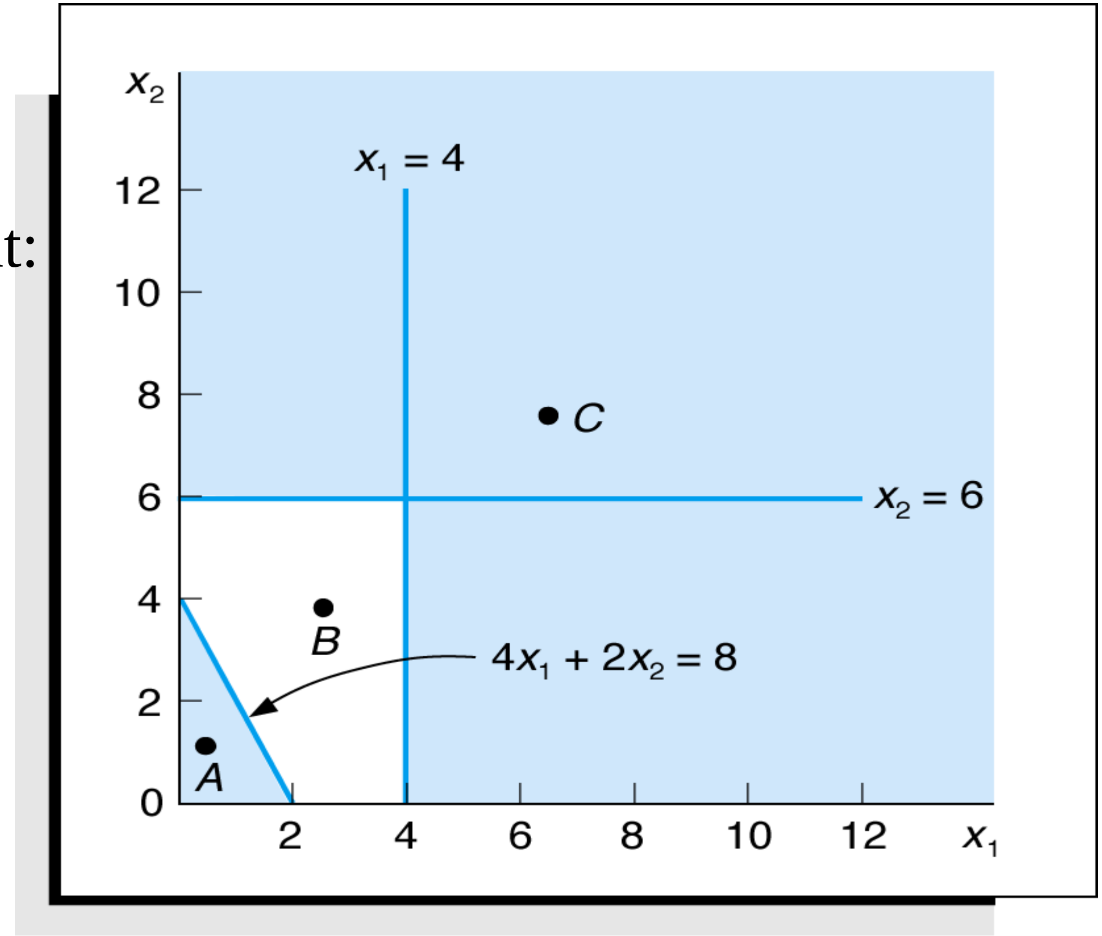
Maximize $Z = 5x_1 + 3x_2$

subject to: $4x_1 + 2x_2 \leq 8$

$x_1 \geq 4$

$x_2 \geq 6$

$x_1, x_2 \geq 0$



Graph of an Infeasible Problem

An Unbounded Problem

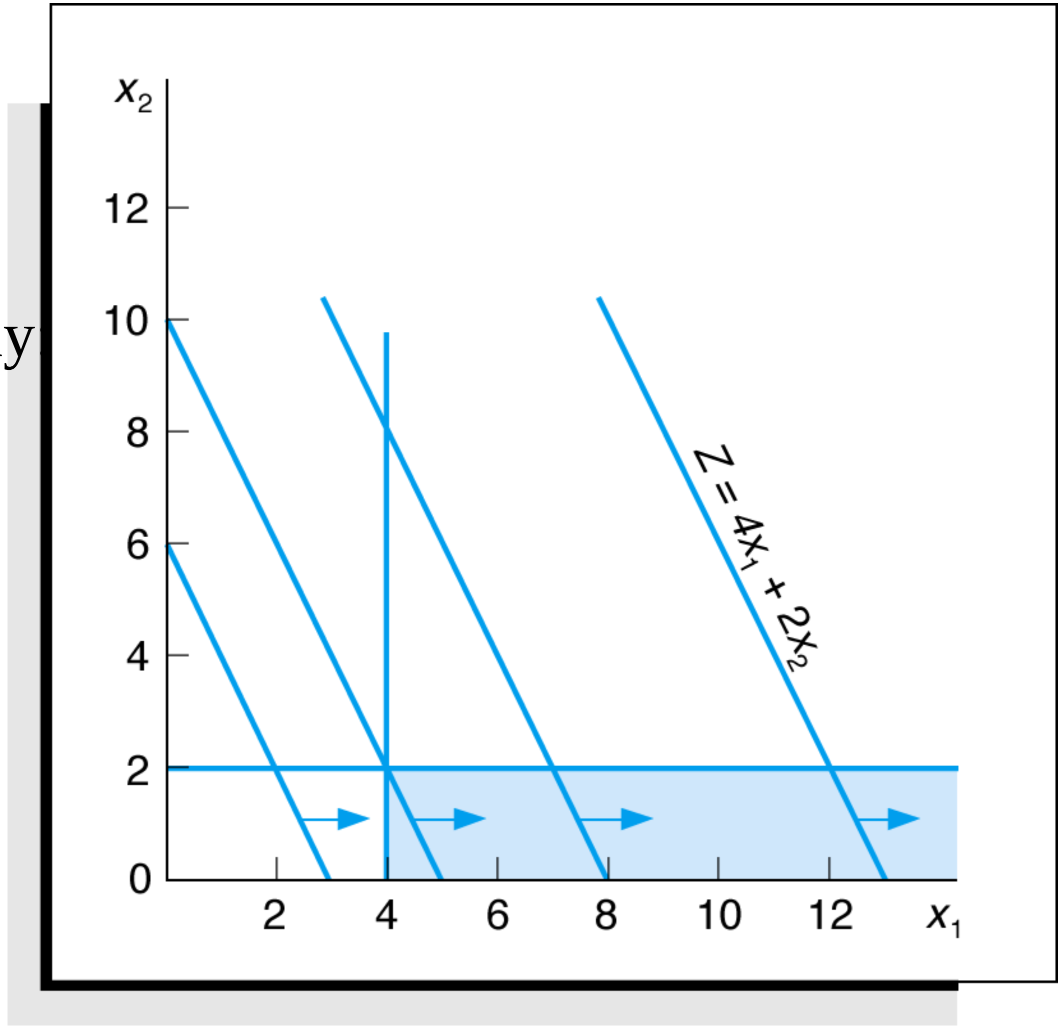
Value of the objective function increases indefinitely:

Maximize $Z = 4x_1 + 2x_2$

subject to: $x_1 \geq 4$

$x_2 \leq 2$

$x_1, x_2 \geq 0$



Graph of an Unbounded Problem

Characteristics of Linear Programming Problems

A decision amongst alternative courses of action is required.

The decision is represented in the model by **decision variables**.

The problem encompasses a goal, expressed as an **objective function**, that the decision maker wants to achieve.

Restrictions (represented by **constraints)** exist that limit the extent of achievement of the objective.

The objective and constraints must be definable by **linear** mathematical functional relationships.

Properties of Linear Programming Models

Proportionality - The rate of change (slope) of the objective function and constraint equations is constant.

Additivity - Terms in the objective function and constraint equations must be additive.

Divisibility -Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.

Certainty - Values of all the model parameters are assumed to be known with certainty (non-probabilistic).

Problem Statement
Example Problem No. 1

- Snack Trail mix in 1000-pound batches.
- Two ingredients, raisins (\$3/lb) and nuts (\$5/lb).
- Recipe requirements:

  at least 500 pounds of “raisins”

  at least 200 pounds of “nuts”
- Ratio of raisins to nuts must be at least 2 to 1.
- Determine optimal mixture of ingredients that will minimize costs.

Solution: Example Problem No. 1

Step 1:

Identify decision variables.

x_1 = lb of raisins in mixture

x_2 = lb of nuts in mixture

Step 2:

Formulate the objective function.

Minimize $Z = \$3x_1 + \$5x_2$

where Z = cost per 1,000-lb batch

$\$3x_1$ = cost of raisins

$\$5x_2$ = cost of nuts

Solution: Example Problem No. 1

Step 3:

Establish Model Constraints

$$x_1 + x_2 = 1{,}000 \text{ lb}$$
$$x_1 \geq 500 \text{ lb of raisins}$$
$$x_2 \geq 200 \text{ lb of nuts}$$
$$x_1/x_2 \geq 2/1 \text{ or } x_1 - 2x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

The Model: Minimize $Z = \$3x_1 + 5x_2$

subject to:

$$\begin{aligned} x_1 + x_2 &= 1{,}000 \text{ lb} \\ x_1 &\geq 500 \\ x_2 &\geq 200 \\ x_1 - 2x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Example Problem No. 2

Solve the following model graphically:

Maximize $Z = 4x_1 + 5x_2$

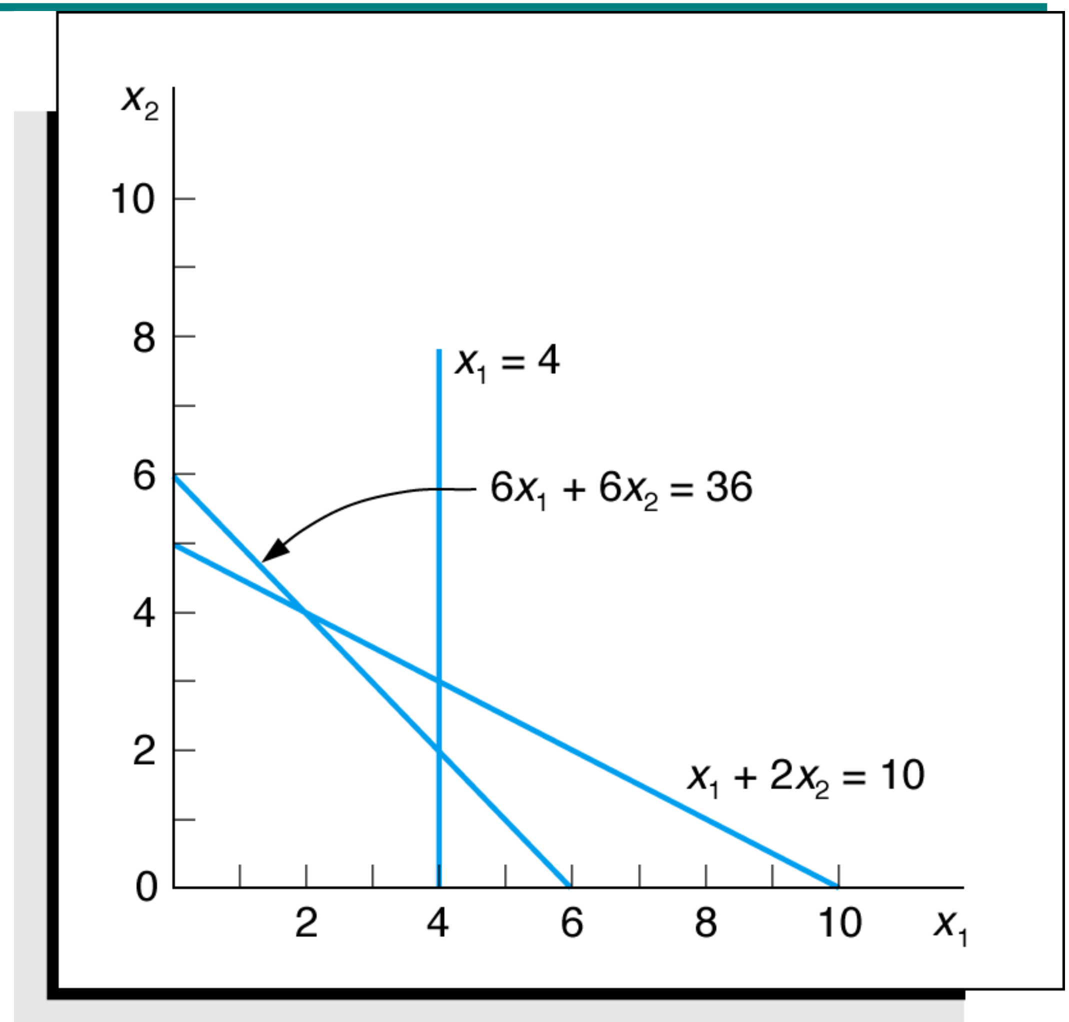
subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 1: Plot the constraints as equations



Constraint Equations

Example Problem No. 2

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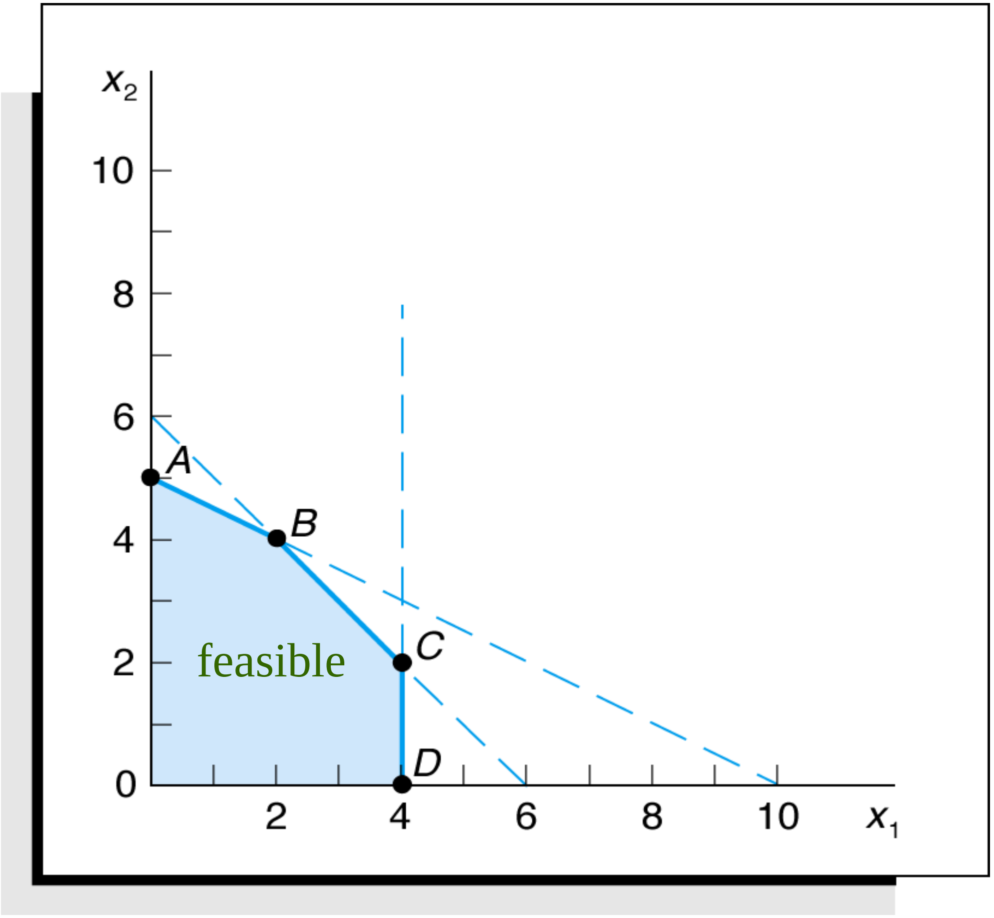
Step 2: Determine the feasible solution space



Feasible Solution Space and Extreme Points

Example Problem No. 2

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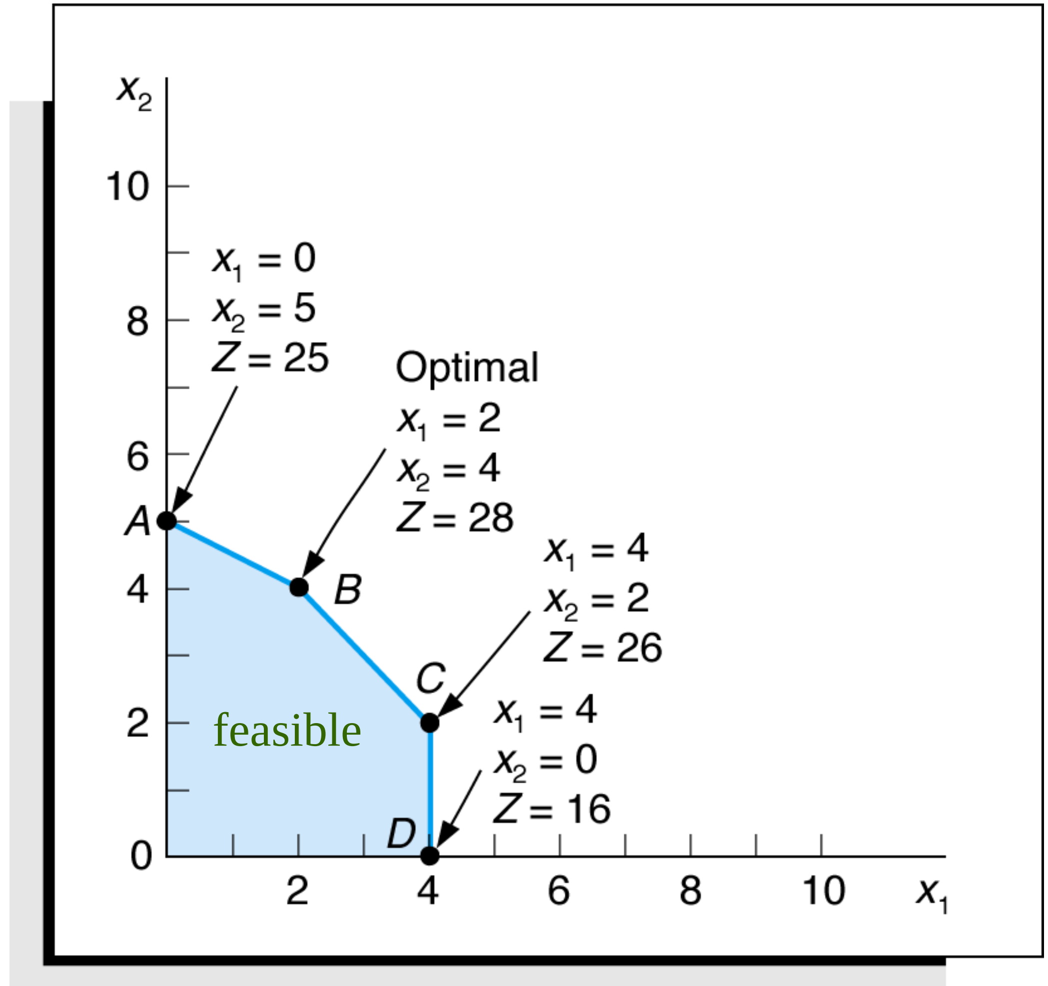
Step 3 and 4: Determine the solution points and optimal solution



Optimal Solution Point

Fundamental Theorem of Linear Programming

Fundamental Theorem of Linear Programming The maximum (or minimum) value of the objective function in a linear programming problem is achieved at one of the vertices of the feasible set.

The point that yields the maximum (or minimum) value of the objective function is called an optimal point.

Example Problem No. 2

Maximize $Z = 4x_1 + 5x_2$

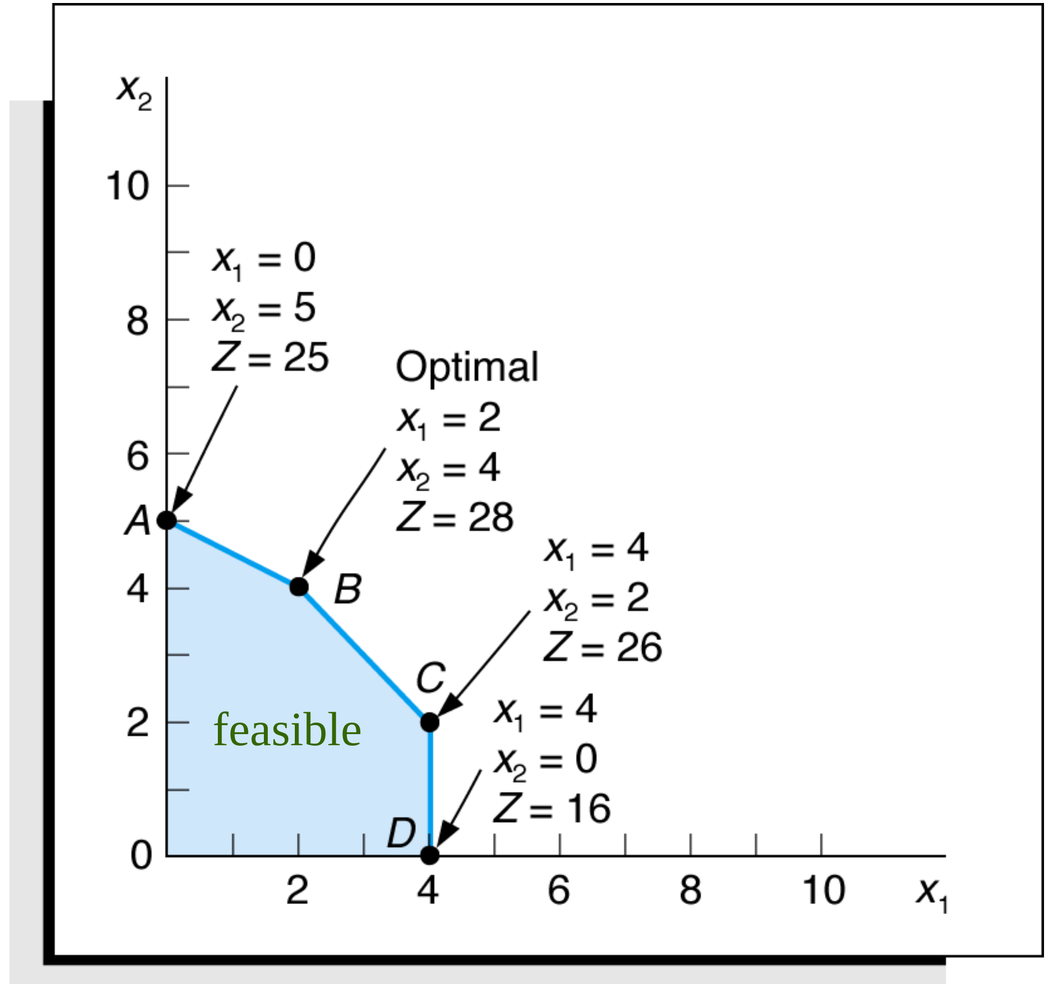
subject to: $x_1 + 2x_2 \le 10$

$6x_1 + 6x_2 \le 36$

$x_1 \le 4$

$x_1, x_2 \ge 0$

Step 3 and 4: Determine the solution points and optimal solution



Optimal Solution Point

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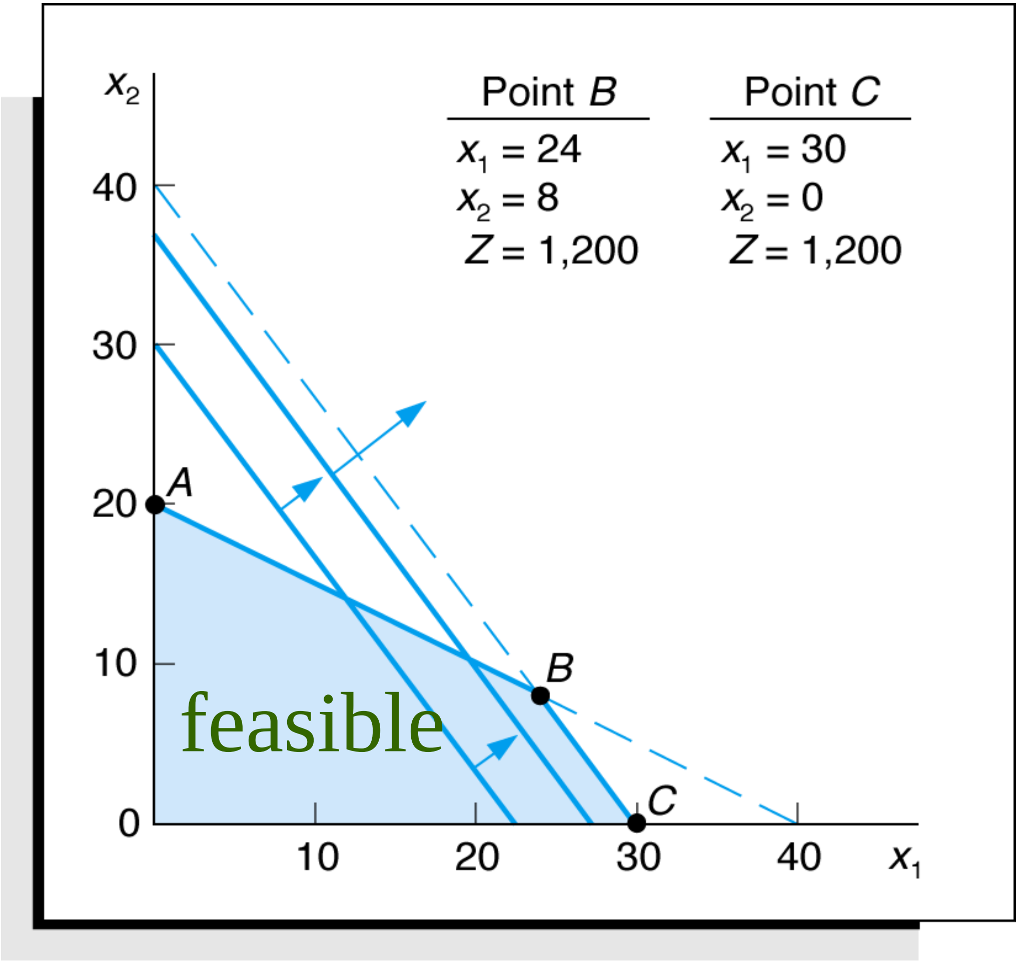

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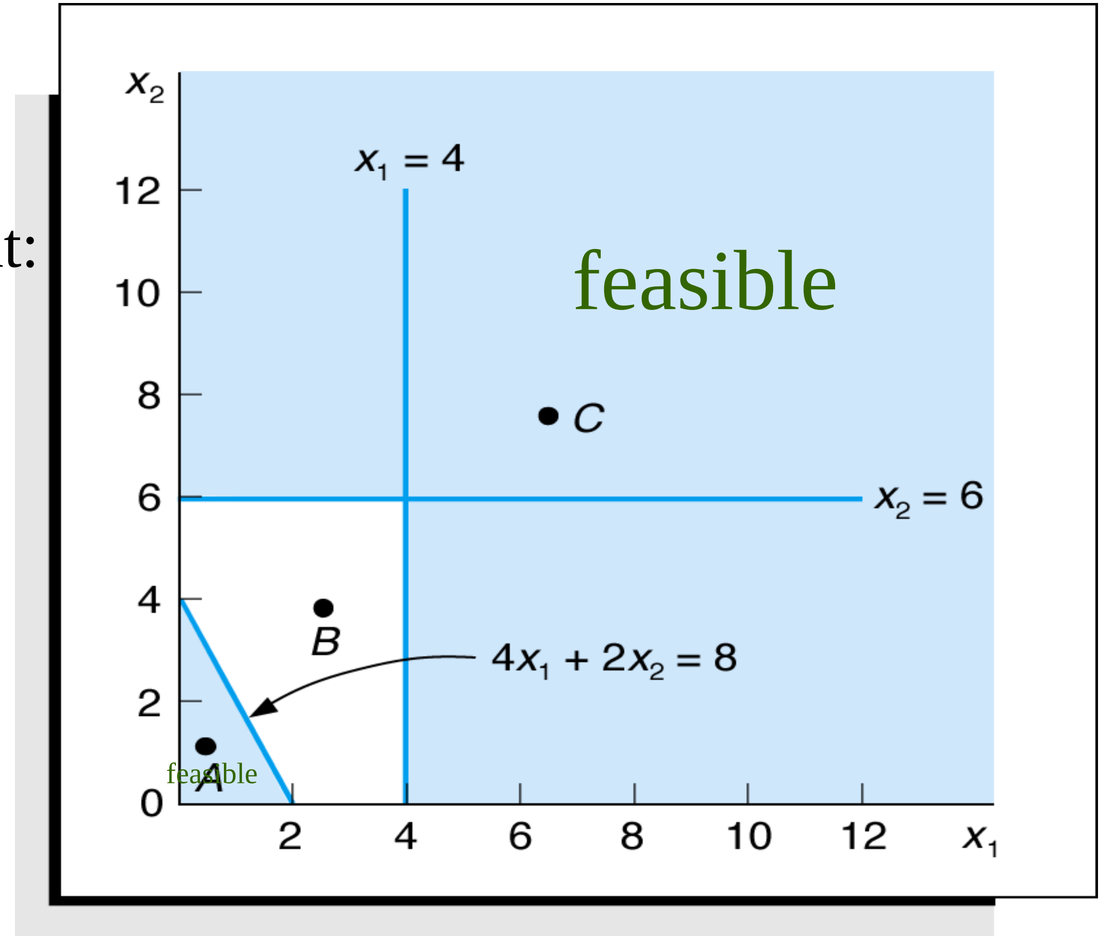


Graph of an Infeasible Problem

An Unbounded Problem

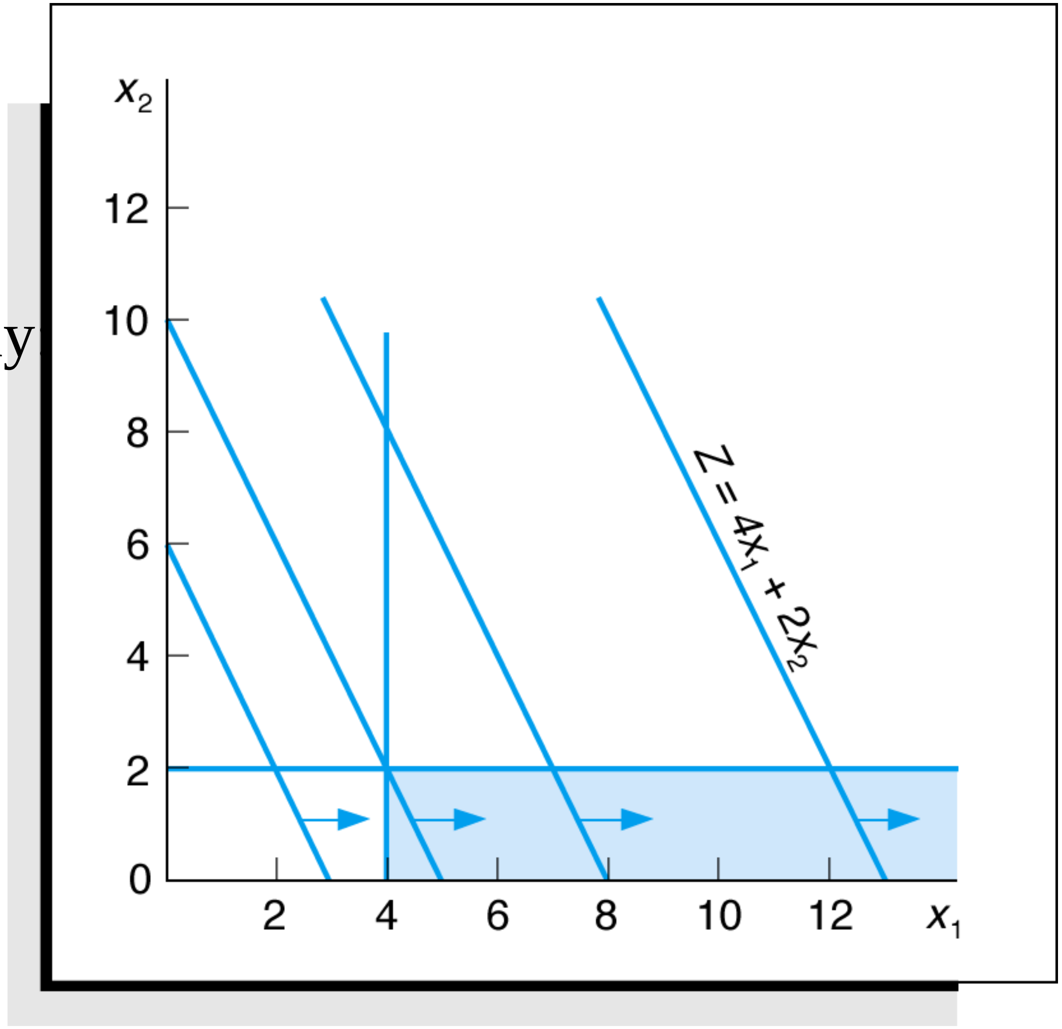
Value of the objective function increases indefinitely:

Maximize $Z = 4x_1 + 2x_2$

subject to: $x_1 \geq 4$

$x_2 \leq 2$

$x_1, x_2 \geq 0$



Graph of an Unbounded Problem

Question

A clothing company has 100 yards of cloth and produces shirts (x units) and vests (y units). Shirts require 10 units and have profit value of $5, while vests require 4 units and have profit value of $4.

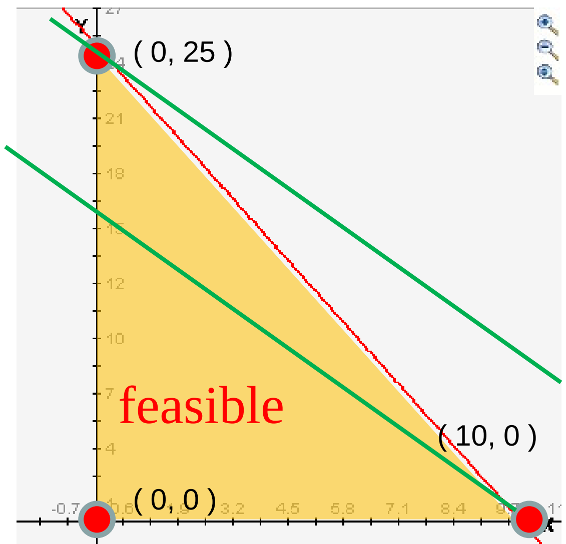
How many shirts and vests should be made to maximize profit? What is the maximum profit?

Solution

A clothing company has 100 yards of cloth and produces shirts (x units) and vests (y units). Shirts require 10 units and have profit value of \$5, while vests require 4 units and have profit value of \$4.

What is the optimal production solution?

Steps 3 & 4:
Feasible Region & Corner Points



Step 1 & 2:
Identify Components & Mixture Chart

1. Resources – Cloth (100)
2. Products – Shirts & Vests
3. Recipes – Shirts (10), Vests (4)
4. Profits – Shirts (\$5), Vests (\$4)
5. Objective – Maximize profit

Products	Resources Cloth (100)	Profit
Shirts (x units)	10	\$5.00
Vests (y units)	4	\$4.00

Point	Calculation of Profit Formula \$5.00x + \$4.00y = P
(0, 0)	\$5.00 (0) + \$4.00 (0) = \$0.00
(0, 25)	\$5.00 (0) + \$4.00 (25) = \$100.00
(10, 0)	\$5.00 (10) + \$4.00(0) = \$50.00

Question

A clothing company has 100 yards of cloth and produces shirts (x units) and vests (y units). Shirts require 10 units and have profit value of $5, while vests require 4 units and have profit value of $4. NOW they must produce at least 2 shirts and 10 vests.

How many shirts and vests must be made now to maximize profit? What is the new profit?

Solution

What if the company decides to also put a "non-zero constraint" on all production?
Must produce at least 2 shirts and 10 vests.

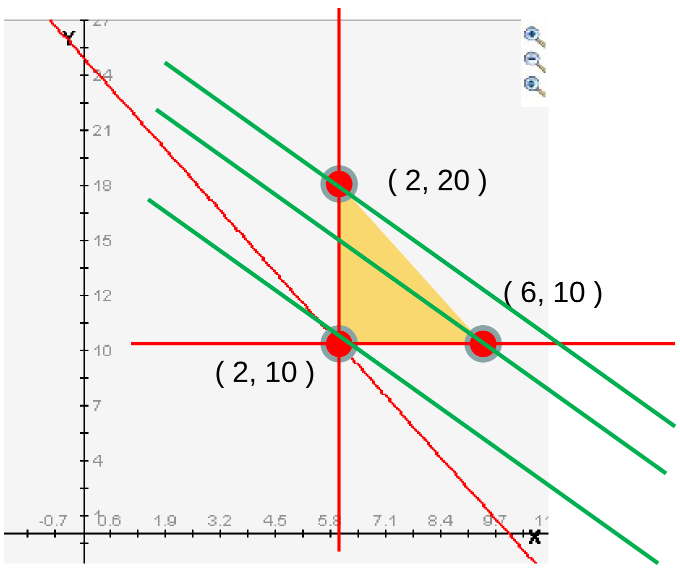
Constraints become:

$10x + 4y \leq 100$ …

$x \geq 2$

$y \geq 10$

Feasible Region becomes:



Corner Points:

Point	Calculation of Profit Formula $5.00x + $4.00y = P
(2, 10)	$5.00 (2) + $4.00 (10) = $50.00
(2, 20)	$5.00 (2) + $4.00 (20) = $90.00
(6, 10)	$5.00 (6) + $4.00(10) = $70.00

LP:Labor Planning

Addresses staffing needs over a specific time period.

Hong Kong Bank of Commerce:

- 12 Full time workers available, but may fire some.
- Use part time workers who has to work for 4 consequtive hours in a day.
- Luch time is one hour between 11a.m. and 1p.m. shared by full time workers.
- Total part time hours is less than 50% of the day's total requirement.
- Part-timers earn $4/hr (=$16/day) and full timers earn $50/day.

LP:Labor Planning

Time Period	Minimum labor required
9a.m.-10a.m.	10
10a.m.-11a.m.	12
11a.m.-noon	14
Noon-1p.m.	16
1p.m.-2p.m.	18
2p.m.-3p.m.	17
3p.m.-4p.m.	15
4p.m.-5p.m.	10

LP:Labor Planning

F : # Full time tellers per day

P_i : # Part time tellers who start work at time slot $i, i = 1,2...,5.$

$$Min\ Daily\ Personnel\ Cost = \$50F + \$16\sum P_i$$

$$\begin{aligned}
F + P_1 &\geq 10 \\
F + P_1 + P_2 &\geq 12 \\
0.5F + P_1 + P_2 + P_3 &\geq 14 \\
0.5F + P_1 + P_2 + P_3 + P_4 &\geq 16 \\
F \quad + P_2 + P_3 + P_4 + P_5 &\geq 18 \\
F \quad + P_3 + P_4 + P_5 &\geq 17 \\
F \quad + P_4 + P_5 &\geq 15 \\
F \quad + P_5 &\geq 10
\end{aligned}$$

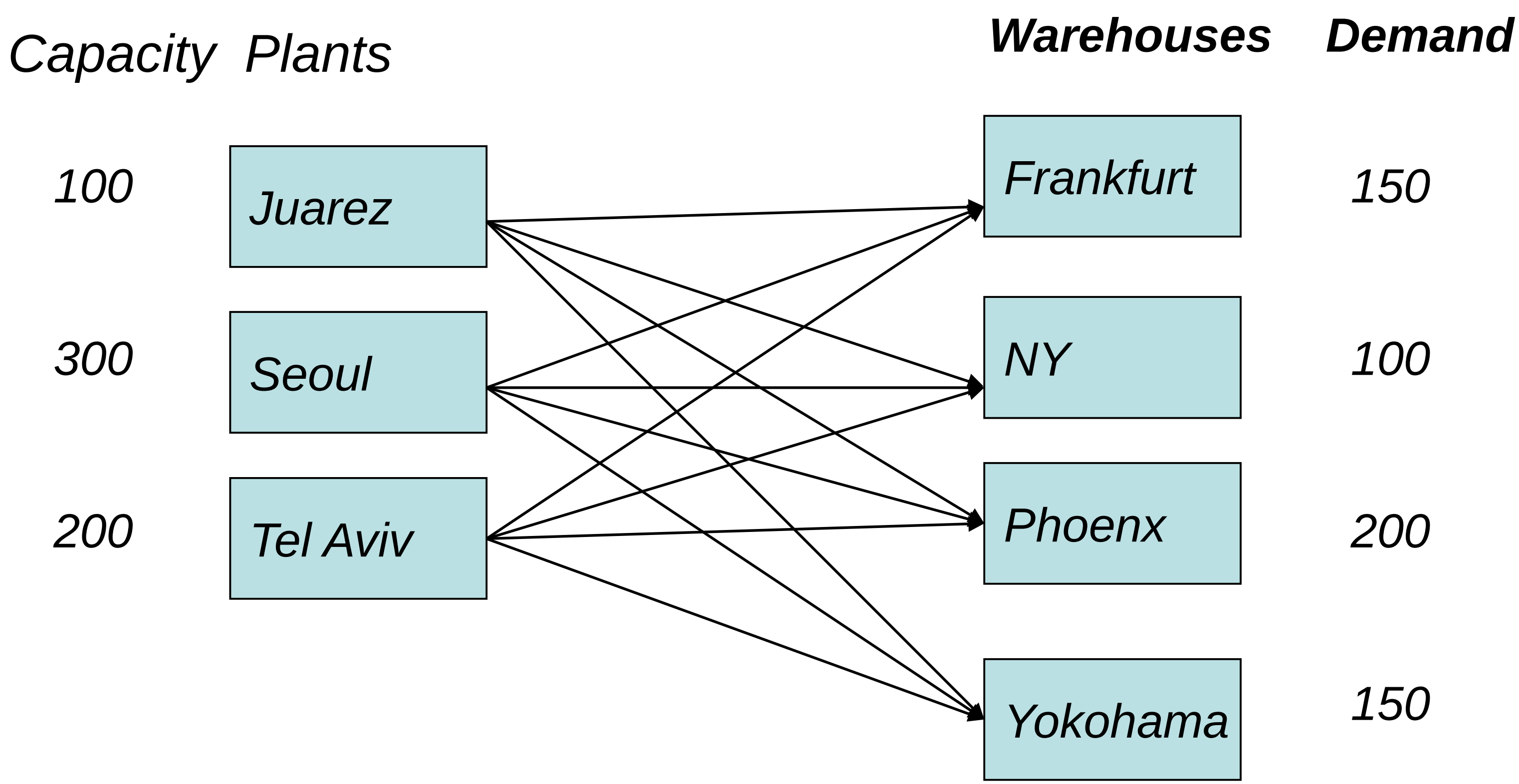
$$F \leq 12$$

$$4\sum P_i \leq 0.5(10 + 12 + 14 + ... + 10)$$

$$F, P_i \geq 0$$

Applications of LP:Transportation Models

Sporting goods company



LP:Transportation Models

What are the optimal shipping quantities from the plants to the warehouses, if the demand has to be met by limited capacities while the shipping cost is minimized?

Shipping Costs per pair of skis

	Destination			
From Plant	Frankfurt	NY	Phoenix	Yokohama
Juarez	$19	$7	$3	$21
Seoul	15	21	18	6
Tel Aviv	11	14	15	22

LP:Transportation Models

X_{ij}: Number of units shipped from plant i to warehouse j. i=1,2,3 and j=1,2,3,4.

Minimize shipping costs=$19X_{11}+7X_{12}+3X_{13}+21X_{14}$

$+15X_{21}+21X_{22}+18X_{23}+6X_{24}$

$+11X_{31}+14X_{32}+15X_{33}+22X_{34}$

From Plant	Destination				Capacity
	Frankfurt	NY	Phoenix	Yokohama	
Juarez	X11	X12	X13	X14	100
Seoul	X21	X22	X23	X24	300
Tel Aviv	X31	X32	X33	X34	200
Demand	150	100	200	150	600

LP:Transportation Models

subject to

#shipped from a plant can not exceed the capacity:

$$X_{11}+X_{12}+X_{13}+X_{14}\leq 100 \text{ (Juarez Plant)}$$

$$X_{21}+X_{22}+X_{23}+X_{24}\leq 300 \text{ (Seoul Plant)}$$

$$X_{31}+X_{32}+X_{33}+X_{34}\leq 200 \text{ (Tel Aviv Plant)}$$

#shipped to a warehouse can not be less than the demand:

$$X_{11}+X_{21}+X_{31}\geq 150 \text{ (Frankfurt)}$$

$$X_{12}+X_{22}+X_{32}\geq 100 \text{ (NY)}$$

$$X_{13}+X_{23}+X_{33}\geq 200 \text{ (Phoenix)}$$

$$X_{14}+X_{24}+X_{34}\geq 150 \text{ (Yokohama)}$$

Nonnegativity

$$X_{ij} \geq 0 \text{ for all } i,j.$$

LP:Transportation Models

Optimal Solution: Optimal cost=$6,250