

# $\rho$ -correlated signal yields $\approx \rho$ -Sharpe pnl

Sparsh Sah

We have passive-asset return

$$r = s + \varepsilon,$$

the sum of an ex-ante observable signal plus unpredictable, independent white noise.  $s \sim \mathcal{N}(0, \sigma := \rho)$ , and  $\varepsilon \sim \mathcal{N}(0, \sigma := \sqrt{1 - \rho^2})$ . Notice that  $\sigma^2(r | s) = 1 - \rho^2$ , whereas  $\sigma^2(r) = \rho^2 + 1 - \rho^2 = 1$ , so that the fraction of  $r$ 's variance explained by  $s$  (AKA " $R^2$ ") is  $\rho^2$ , hence indeed  $\text{Corr}(r, s) = \rho$ .

Suppose we take active leverage (proportional to)  $s$ . Our active pnl

$$\pi := sr = s(s + \varepsilon) = s^2 + s\varepsilon.$$

Note  $s$  is equivalent to  $\rho Z$  where  $Z$  is a standard Normal random variable, so that  $s^2$  is equivalent to  $\rho^2 Z^2$ . Hence,  $s^2 \sim \rho^2 \chi_1^2$ . Abusing notation a bit,  $\mathbb{E}[\chi_k^2] = k$ , so that our (unconditional) ex-ante expected active pnl is

$$\boxed{\mathbb{E}[\pi] = \rho^2}.$$

Keep in mind (i) that  $s^2 \sim \rho^2 \chi_1^2$ , (ii) that  $\sigma^2(cx) = c^2 \sigma^2(x)$ , (iii) that  $\varepsilon \perp s$ , and (iv) that the variance of the product of independent zero-mean random variables is simply the product of their variances. Then, the (unconditional) ex-ante variance of our active pnl is—letting  $\sigma^2(x)$  denote variance and  $\sigma^2(x, y)$  denote covariance—

$$\begin{aligned} \sigma^2(\pi) &= \sigma^2(s^2 + s\varepsilon) = \sigma^2(s^2) + \sigma^2(s\varepsilon) + \sigma^2(s^2, s\varepsilon) \\ &= (\rho^2)^2(2) + \sigma^2(s)\sigma^2(\varepsilon) + \mathbb{E}[s^2 s\varepsilon] - \mathbb{E}[s^2]\mathbb{E}[s\varepsilon] \\ &= 2\rho^4 + \rho^2(1 - \rho^2) + \mathbb{E}[s^3 \varepsilon] - \mathbb{E}[s^2]\mathbb{E}[s]\mathbb{E}[\varepsilon] \\ &= 2\rho^2 \rho^2 + \rho^2 - \rho^2 \rho^2 + \mathbb{E}[s^3] \cdot 0 - \mathbb{E}[s^2]\mathbb{E}[s] \cdot 0 \\ &= 2\rho^2 \rho^2 + \rho^2 - \rho^2 \rho^2 = \rho^2 + \rho^2 \rho^2 = \rho^2(1 + \rho^2), \end{aligned}$$

so that

$$\boxed{\sigma(\pi) = \rho \sqrt{1 + \rho^2}}.$$

Thus, our unconditional ex-ante active Sharpe is

$$\text{Sharpe}(\pi) = \frac{\rho^2}{\rho \sqrt{1 + \rho^2}} = \frac{\rho}{\sqrt{1 + \rho^2}}, \quad \text{which for } \rho \text{ small} \quad \approx \frac{\rho}{\sqrt{1 + 0}} = \boxed{\rho}.$$

## COMMENTARY

Notice that for  $\rho = 1$ , we get unconditional ex-ante active Sharpe exactly  $\frac{1}{\sqrt{2}}$ . Why, when in this case we can predict  $r$  perfectly? Well, we will take some unconditional active volatility simply because, unconditionally,  $s$  is itself a random variable. This leads to a counterintuitive result: If  $\rho = 1$ , you ought to take active leverage (proportional to) not  $s$ , but rather the *reciprocal* of  $s$ . If you do so, your active pnl will be  $\pi = 1$  with certainty, and your active Sharpe will be infinite.

In other words, suppose we observe at time  $t = 0$  a timeseries of all future signals  $S := (s_1, \dots, s_T)$ . The setting is the same, except that this has become a repeated experiment with independent timesteps i.e.  $r_j \perp r_i | S$ . We want to maximize our active Sharpe as measured ex-post at time  $t = T + 1$ . This  $T$ -period single-asset problem is isomorphic to a single-period  $T$ -asset problem. Markowitz tells us we should allocate active risk (proportional to) conditional ex-ante passive Sharpe given  $S$ , which in this case (because every  $r_t$  has the same conditional volatility given  $S$ ) means taking active leverage (proportional to)  $S$ .

BUT: If  $\rho = 1$  i.e. each  $r_t = s_t + 0$ , you should take active leverage (proportional to)  $1/S$ ! Then, your active pnl will be  $\pi_t = 1$  every day, and your ex-post active Sharpe is guaranteed to be infinite. So: If your signal is imperfectly predictive of  $r$  (i.e.  $0 < \rho < 1$ ), you ought to listen to it; But if your signal is perfectly predictive of  $r$  (i.e.  $\rho = 1$ ), you ought to listen to the *inverse* of it! This "paradox" is resolved if you notice that when  $\rho = 1$ , conditional ex-ante passive Sharpe of every  $r_t$  given  $S$  becomes infinite: Each  $r_t$  has ER  $s_t$ , but no volatility. So, the logic of allocating "proportional to conditional ex-ante passive Sharpe" breaks down. We get a similar result when  $\rho = 0$ : Instead of allocating active risk (proportional to)  $S$ , you should simply take no active risk at all, i.e. active leverage zero every day. Then, your ex-post active Sharpe is guaranteed to be  $\frac{0}{0}$ , which is an indeterminate form but arguably better than the determinate form  $\frac{0}{\text{something positive}} = 0$  which is what you can anticipate if you *do* take nonzero active risk.