Sharpe-maximizing time-varying risk-taking

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Setting

Over T timesteps, we have a (random, real) T-vector of single-asset (nothing prevents this from being a strategy asset) returns,

$$R \in \mathbb{R}^T \quad := (R_1,\, \dots,\, R_T)'.$$

We also have a (random, real) T-vector of signals,

$$S \in \mathbb{R}^T \quad := (S_1,\, \dots,\, S_T)'.$$

Each

$$R_t \in \mathbb{R} \quad = \mu + \rho S_t + \lambda \varepsilon_t \qquad \implies \qquad R_t \mid S_t \quad \sim \mathcal{N}\left(\mu + \rho S_t,\, \lambda\right)$$

for given $\mu \geq 0$ and $0 \leq \rho < 1$ (with $\lambda := \sqrt{1-\rho^2}$ of standard Normal white noise ε_t), and the R_t's are independent. The S_t's are also i.i.d. standard Normal white noise. Notice that, WLOG, unconditional $\text{Var}(R_t) = 1$ and $\text{Cov}(R_t,\, S_t) = \text{Corr}(R_t,\, S_t) = \rho$.

Mean-variance-optimal allocation (given signals)

Any ex-ante MVO allocation to the R_t's (assuming a fixed policy, that is, assuming you must set your weights at $t = 0$, so that this T-period single-asset problem becomes isomorphic to a single-period T-asset problem [0]) must be proportional to their ex-ante Sharpes, so that the ex-ante MVO allocation—conditional on S, and noting that each R_t has the same standard deviation so we can interpret the "$\propto$" below as an "="—can be taken as

$$w_t^* \mid S \quad = \text{SR}(R_t) \propto \mathbb{E}[R_t] = \mu + \rho S_t.$$

The aggregate ex-ante conditional ER of this allocation (paying close attention to which variables are constants given S) is

$$\mathbb{E}\left[\sum_t w_t^* R_t \mid S\right] = \sum_t \mathbb{E}[w_t^* R_t \mid S]$$

$$= \sum_t \mathbb{E}\left[(\mu + \rho S_t)(\mu + \rho S_t + \lambda\varepsilon_t) \mid S\right] = \sum_t \mathbb{E}\left[(\mu + \rho S_t)(\mu + \rho S_t) + (\mu + \rho S_t)\lambda\varepsilon_t \mid S\right]$$

$$= \sum_t \left((\mu + \rho S_t)(\mu + \rho S_t) + \mathbb{E}\left[(\mu + \rho S_t)\lambda\varepsilon_t \mid S\right]\right) = \sum_t \left((\mu + \rho S_t)(\mu + \rho S_t) + (\mu + \rho S_t)\lambda\mathbb{E}[\varepsilon_t \mid S]\right)$$

$$= \sum_t \left((\mu + \rho S_t)(\mu + \rho S_t) + (\mu + \rho S_t)\lambda(0)\right) = \sum_t (\mu + \rho S_t)(\mu + \rho S_t) = \sum_t (\mu + \rho S_t)^2.$$

The aggregate ex-ante conditional variance ($\implies$ aggregate ex-ante conditional volatility) of this allocation (noting bilinearity of covariance) is

$$\sum_t w_t^{*2}\lambda^2 = \sum_t (\mu + \rho S_t)^2\lambda^2 \qquad \implies \qquad \lambda\sqrt{\sum_t (\mu + \rho S_t)^2}.$$

The aggregate ex-ante conditional Sharpe of the MVO allocation, then, is

$$\frac{1}{\lambda}\sqrt{\sum_t (\mu + \rho S_t)^2},$$

which we will for convenience write as

$$\frac{\sum_t (\mu + \rho S_t)(\mu + \rho S_t)}{\lambda\sqrt{\sum_t (\mu + \rho S_t)^2}} = \frac{\sum_t \left(\mu^2 + \mu\rho S_t + \mu\rho S_t + \rho^2 S_t^2\right)}{\lambda\sqrt{\sum_t \left(\mu^2 + \mu\rho S_t + \mu\rho S_t + \rho^2 S_t^2\right)}} = \frac{\sum_t \left(\mu^2 + \mu\rho S_t + \mu\rho S_t + \rho^2 S_t^2\right)}{\lambda\sqrt{\sum_t \left(\mu^2 + \mu\rho S_t + \mu\rho S_t + \rho^2 S_t^2\right)}}$$

$$= \frac{\sum_t \mu^2 + \sum_t \mu\rho S_t + \sum_t \mu\rho S_t + \sum_t \rho^2 S_t^2}{\lambda\sqrt{\sum_t \mu^2 + \sum_t \mu\rho S_t + \sum_t \mu\rho S_t + \sum_t \rho^2 S_t^2}} = \boxed{\frac{T\mu^2 + \mu\rho\sum_t S_t + \mu\rho\sum_t S_t + \rho\rho\sum_t S_t^2}{\lambda\sqrt{T\mu^2 + \mu\rho\sum_t S_t + \mu\rho\sum_t S_t + \rho^2\sum_t S_t^2}}}.$$

[0] I haven't studied whether the "unconstrained" multi-period problem—wherein you are allowed to observe running cumulative P&L as time proceeds—has a different solution, e.g. whether you should lever up your future bets in an attempt to pull yourself out of the hole in response to net losses.

Alternative allocation (again, given signals)

Consider now the alternative allocation

$$w_t \mid S := \mu + cS_t,$$

with $c > \rho$. (Notice that by applying

$$0 < x := 1 - \frac{\rho}{c} \leq 1$$

shrinkage toward μ—that is, putting x weight on μ and $1 - x$ weight on w—we could recover w^*.)

The alternative allocation's aggregate ex-ante conditional ER is

$$\sum_t (\mu + cS_t)(\mu + \rho S_t),$$

while its aggregate ex-ante conditional variance ($\implies$ ex-ante conditional volatility) is

$$\sum_t {w_t}^2 \lambda^2 = \sum_t (\mu + cS_t)^2 \lambda^2 \qquad \implies \qquad \lambda \sqrt{\sum_t (\mu + cS_t)^2}.$$

Its aggregate ex-ante conditional Sharpe, then, is

$$\frac{\sum_t (\mu + cS_t)(\mu + \rho S_t)}{\lambda \sqrt{\sum_t (\mu + cS_t)^2}} = \frac{\sum_t \left(\mu^2 + \mu\rho S_t + \mu c S_t + \rho c S_t^2\right)}{\lambda \sqrt{\sum_t \left(\mu^2 + \mu c S_t + \mu c S_t + c^2 S_t^2\right)}} = \boxed{\frac{T\mu^2 + \mu\rho \sum_t S_t + \mu c \sum_t S_t + \rho c \sum_t S_t^2}{\lambda \sqrt{T\mu^2 + \mu c \sum_t S_t + \mu c \sum_t S_t + c^2 \sum_t S_t^2}}}.$$

Confirmation that the alternative is actually worse than the MVO

We can drop the $\frac{1}{\lambda}$ constant factor when comparing the two ex-ante Sharpes. Furthermore, although we still condition on S, we will let $T \to \infty$ so we can appeal to the Law of Large Numbers to assert that $\sum_t S_t := 0$ and $\sum_t S_t^2 := T$. So, we are comparing the MVO ex-ante Sharpe

$$\frac{T\mu^2 + \rho\rho T}{\sqrt{T\mu^2 + \rho^2 T}} \qquad \propto_{\sqrt{T}} \qquad \frac{\mu^2 + \rho\rho}{\sqrt{\mu^2 + \rho^2}}$$

to the alternative ex-ante Sharpe which is similarly $\propto_{\sqrt{T}}$

$$\frac{\mu^2 + \rho c}{\sqrt{\mu^2 + c^2}}.$$

Now: $\rho \geq 0$ and $c > \rho$, so both the numerator and denominator will be bigger for the alternative allocation as long as $\rho > 0$. (If $\rho = 0$, then we clearly see that the alternative allocation's Sharpe is worse than the MVO allocation's, so we don't need to worry about that case.) Which one wins out? Well, let's consider the sign of the partial derivative with respect to a dummy variable $h \in (\rho, \infty)$

$$\text{sign}\left(\frac{\partial}{\partial h} \frac{\mu^2 + \rho h}{\sqrt{\mu^2 + h^2}}\right) \quad =^1 \quad \text{sign}\left(-\frac{\mu^2(h - \rho)}{(\mu^2 + h^2)^{\frac{3}{2}}}\right) = -\text{sign}\left(\frac{\mu^2(h - \rho)}{(\mu^2 + h^2)^{\frac{3}{2}}}\right) \cong -\text{sign}\left(\frac{[+][+]}{[+]^{\frac{3}{2}}}\right) = -[+] = [-].$$

The derivative is strictly negative, meaning that the ratio is strictly decreasing in h: As h gets bigger than ρ, the final ratio gets smaller. And in our setup, c (which takes the place of h) is bigger than ρ; Hence we know that the alternative is worse than the MVO. [2]

My best intuition for this (at least, if we imagine c getting very big, and ignore the fact that $\frac{c}{c} = 1$) is as follows: If we squint, the numerator looks like ρc, whereas the denominator looks like $\sqrt{c^2} = 1c$. So, as c gets bigger, the numerator grows with a discount factor of $\rho < 1$, while the denominator grows with no discount. Thus, we have worsened (made smaller) their ratio.

[1] Query Uncle WolframAlpha for `partial derivative wrt h of (m^2 + r * h) / sqrt(m^2 + h^2)` .

[2] In fact, the expression for the derivative is valid even for $h \notin (\rho, \infty)$. So, when $h = \rho$ (so that $h - \rho = 0$), the derivative is zero; And when $h < 0$, the derivative is positive. We have thus shown that $h = \rho$ satisfies both the first- and second-order conditions to be a local maximizer (furthermore, the *only* interior maximizer, as the derivative has no other roots) of the ratio. That is, w_t^* is the best allocation of the form $\mu + hS_t$. But we already knew that, as we've already established that (from a mean-variance perspective) w^* is indeed the best-possible allocation of *any* form.