ρ-correlated signal yields $\approx$ ρ-Sharpe pnl

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We have passive-asset return
$$r = s + \varepsilon,$$
the sum of an ex-ante observable signal plus unpredictable, independent white noise. $s \sim \mathcal{N}(0, \sigma := \rho)$, and $\varepsilon \sim \mathcal{N}(0, \sigma := \sqrt{1-\rho^2})$. Notice that $\sigma^2(r \mid s) = 1 - \rho^2$, whereas $\sigma^2(r) = \rho^2 + 1 - \rho^2 = 1$, so that the fraction of r's variance explained by s (AKA "R^2") is ρ^2, hence indeed $\mathrm{Corr}(r, s) = \rho$.

Suppose we take active leverage (proportional to) s. Our active pnl
$$\pi := sr = s(s + \varepsilon) = s^2 + s\varepsilon.$$

Note s is equivalent to ρZ where Z is a standard Normal random variable, so that s^2 is equivalent to $\rho^2 Z^2$. Hence, $s^2 \sim \rho^2 \chi_1^2$. Abusing notation a bit, $\mathbb{E}[\chi_k^2] = k$, so that our (unconditional) ex-ante expected active pnl is
$$\boxed{\mathbb{E}[\pi] = \rho^2}.$$

Keep in mind (i) that $s^2 \sim \rho^2 \chi_1^2$, (ii) that $\sigma^2(cx) = c^2\sigma^2(x)$, (iii) that $\varepsilon \perp s$, and (iv) that the variance of the product of independent zero-mean random variables is simply the product of their variances. Then, the (unconditional) ex-ante variance of our active pnl is—letting $\sigma^2(x)$ denote variance and $\sigma^2(x, y)$ denote covariance—
$$\begin{aligned}
\sigma^2(\pi) &= \sigma^2(s^2 + s\varepsilon) = \sigma^2(s^2) + \sigma^2(s\varepsilon) + \sigma^2(s^2, s\varepsilon) \\
&= (\rho^2)^2(2) + \sigma^2(s)\sigma^2(\varepsilon) + \mathbb{E}[s^2 s\varepsilon] - \mathbb{E}[s^2]\mathbb{E}[s\varepsilon] \\
&= 2\rho^4 + \rho^2(1-\rho^2) + \mathbb{E}[s^3\varepsilon] - \mathbb{E}[s^2]\mathbb{E}[s]\mathbb{E}[\varepsilon] \\
&= 2\rho^2\rho^2 + \rho^2 - \rho^2\rho^2 + \mathbb{E}[s^3]\cdot 0 - \mathbb{E}[s^2]\mathbb{E}[s]\cdot 0 \\
&= 2\rho^2\rho^2 + \rho^2 - \rho^2\rho^2 = \rho^2 + 2\rho^2\rho^2 - \rho^2\rho^2 = \rho^2 + \rho^2\rho^2 = \rho^2(1+\rho^2),
\end{aligned}$$
so that
$$\boxed{\sigma(\pi) = \rho\sqrt{1+\rho^2}}.$$

Thus, our unconditional ex-ante active Sharpe is
$$\mathrm{Sharpe}(\pi) = \frac{\rho^2}{\rho\sqrt{1+\rho^2}} = \frac{\rho}{\sqrt{1+\rho^2}}, \qquad \text{which for } \rho \text{ small} \qquad \approx \frac{\rho}{\sqrt{1+0}} = \boxed{\rho}.$$

Commentary

Notice that for $\rho = 1$, we get unconditional ex-ante active Sharpe exactly $\frac{1}{\sqrt{2}}$. Why, when in this case we can predict r perfectly? Well, we will take some unconditional active volatility simply because, unconditionally, s is itself a random variable. This leads to a counterintuitive result: If $\rho = 1$, you ought to take active leverage (proportional to) not s, but rather the *reciprocal* of s. If you do so, your active pnl will be $\pi = 1$ with certainty, and your active Sharpe will be infinite.

In other words, suppose we observe at time $t = 0$ a timeseries of all future signals $S := (s_1, \ldots, s_T)$. The setting is the same, except that this has become a repeated experiment with independent timesteps i.e. $r_j \perp r_i \mid S$. We want to maximize our active Sharpe as measured ex-post at time $t = T + 1$. This T-period single-asset problem is isomorphic to a single-period T-asset problem. Markowitz tells us we should allocate active risk (proportional to) conditional ex-ante passive Sharpe given S, which in this case (because every r_t has the same conditional volatility given S) means taking active leverage (proportional to) S.

BUT: If $\rho = 1$ i.e. each $r_t = s_t + 0$, you should take active leverage (proportional to) $1/S$. Then, your active pnl will be $\pi_t = 1$ every day, and your ex-post active Sharpe is guaranteed to be infinite. So: If your signal is imperfectly predictive of r (i.e. $0 < \rho < 1$), you ought to listen to it; But if your signal is perfectly predictive of r (i.e. $\rho = 1$), you ought to listen to the *inverse* of it! This "paradox" is resolved if you notice that when $\rho = 1$, conditional ex-ante passive Sharpe of every r_t given S becomes infinite: Each r_t has ER s_t, but no volatility. So, the logic of allocating "proportional to conditional ex-ante passive Sharpe" breaks down. We get a similar result when $\rho = 0$: Instead of allocating active risk (proportional to) S, you should simply take no active risk at all, i.e. active leverage zero every day. Then, your ex-post active Sharpe is guaranteed to be $\frac{0}{0}$, which is an indeterminate form but arguably better than the determinate form $\frac{0}{\text{something positive}} = 0$ which is what you can anticipate if you *do* take nonzero active risk.