pm-stats-div

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1 PM STATISTICS OF A SIGNAL-DIVERSE STRATEGY

1.1 Definitions

- We have a single standardized asset with return stream $(A_t)_{t=0}^{t=T-1}$, with each A_t i.i.d. $\mathcal{N}(\mu := 0,\, \sigma := 1)$.
- We have two i.i.d. standardized signal streams $(X_t)_{t=0}^{t=T-1}$ and similarly for Y_t.
  - Each X_t is distributed as $\mathcal{N}(\mu := 0,\, \sigma := 1)$, and similarly for each Y_t.
- Across every t, $\mathrm{Corr}(X_{t+1},\, X_t)$ is some constant. Notice that $(X_{t+1},\, X_t)$ are unconditionally jointly distributed as MVN with zero mean and a nontrivial correlation matrix. So, any linear combination thereof is Normal. Let's call the difference $\Delta X_{t+1} := X_{t+1} - X_t$ distributed as $\mathcal{N}(\mu := 0,\, \sigma := \delta)$.
  - We have a similarly for Y that ΔY_{t+1} is distributed as $\mathcal{N}(\mu := 0,\, \sigma := \delta)$.
- The ex-ante Sharpe of each signal is $S > 0$, that is, for each t, calling $X_t A_t =: \alpha_t$ and $Y_t A_t =: \pi_t$, we have
$$\frac{\mathbb{E}(\alpha_t)}{\mathrm{Std}(\alpha_t)} =: S := \frac{\mathbb{E}(\pi_t)}{\mathrm{Std}(\pi_t)}.$$
  - See here for some results about S.

1.2 Leverage

Let's diversify our strategy, so that we have a new signal (note: because the signals have the same Sharpe, this happens to be the MVO weighting) $Z_t := \frac{1}{2}X_t + \frac{1}{2}Y_t$. Actually, for reasons that will become clear in a moment, let's make our new signal

$$Z_t := \frac{1}{\sqrt{2}}X_t + \frac{1}{\sqrt{2}}Y_t.$$

What is the distribution of Z_t? Well, its mean is clearly zero (it's the sum of two zero-mean RV's!). And an elementary calculation shows that its standard deviation is unity. Therefore,

$$X_t \cong Z_t \cong Y_t,$$

that is, Z_t is equivalent in distribution to each of its constituents. Hence it must have the same statistics as its constituents, and in particular takes the *same expected leverage*.

Some people find this unintuitive: The strategy is diversified, they reason, and so we know we must scale it up to hit the same risk, in particular multiplying the space-average signal by $\sqrt{2}$ (that's what we did when modifying the $\frac{1}{2}$ to $\frac{1}{\sqrt{2}}$). Shouldn't its time-average leverage therefore also scale up by $\sqrt{2}$? This misses the key point: The reason we must scale up the strategy, is that its volatility has shrunk. Equivalently, the time-average leverage the strategy takes in A has shrunk. After all, volatility is proportional to leverage, and vice versa! Just like the risk, the leverage has been partly “diversified away”. We scale up the strategy to hit the same time-average leverage, and thus the same risk, as before.

1.3 Turnover

Now consider ΔZ_{t+1}. It is

$$\begin{aligned}\Delta Z_{t+1} &= \frac{1}{\sqrt{2}}(X_{t+1} + Y_{t+1}) - \frac{1}{\sqrt{2}}(X_t + Y_t)\\ &= \frac{1}{\sqrt{2}}(X_{t+1} - X_t) + \frac{1}{\sqrt{2}}(Y_{t+1} - Y_t)\\ &=: \frac{1}{\sqrt{2}}\Delta X_{t+1} + \frac{1}{\sqrt{2}}\Delta Y_{t+1}.\end{aligned}$$

We know that each of ΔX_{t+1}, ΔY_{t+1} is a Normal random variable, since all of

$$(X_0,\, Y_0,\, \cdots,\, X_{t-1},\, Y_{t-1},\, X_t,\, Y_t,\, X_{t+1},\, Y_{t+1},\, \cdots\, X_{T-1},\, Y_{T-1})$$

are jointly a single multivariate Normal random variable of dimension $2T$, and any linear combination of elements of a MVN vector is Normal.

Furthermore, ΔX_{t+1} and ΔY_{t+1} are independent. Recall that ucorrelated Normal random variables are independent; We have

$$\begin{aligned}\mathrm{Cov}(\Delta Y_{t+1},\, \Delta X_{t+1}) &=: \mathrm{Cov}(Y_{t+1} - Y_t,\, X_{t+1} - X_t)\\ &= \mathrm{Cov}(Y_{t+1},\, X_{t+1} - X_t) + \mathrm{Cov}(-Y_t,\, X_{t+1} - X_t)\\ &= \mathrm{Cov}(Y_{t+1},\, X_{t+1}) + \mathrm{Cov}(Y_{t+1},\, -X_t) + \mathrm{Cov}(-Y_t,\, X_{t+1}) + \mathrm{Cov}(-Y_t,\, -X_t),\end{aligned}$$

with all four terms—by our own definition—zero.

Hence again an elementary calculation shows that ΔZ_{t+1} is Normal with mean zero and standard deviation δ. So, the new strategy also takes the *same expected turnover* as each of its constituents.

1.4 Note

We could generalize all this to spiritually similar results (just with averages instead of equalities) for non-i.i.d. X and Y – but these results do NOT hold if X and Y act on nontrivially different (meaning that there is at least one t such that $X_t \neq 0$ on some asset where Y is always zero, or vice versa) asset universes. For example, if X acts on asset A while Y acts on asset B, then the gross leverage of the combined strategy at t is not

$$\left|\frac{1}{\sqrt{2}}\left(X_t + Y_t\right)\right|,$$

where the thing between the $|$’s is identically distributed to X_t or Y_t hence must behave the same as either, but rather

$$\left|\frac{1}{\sqrt{2}}X_t\right| + \left|\frac{1}{\sqrt{2}}Y_t\right| = \frac{1}{\sqrt{2}}\left|X_t\right| + \frac{1}{\sqrt{2}}\left|Y_t\right|.$$

We can find the expected gross leverage in the latter case by linearity (and noting that X,Y are i.i.d.) as

$$\begin{aligned}\frac{1}{\sqrt{2}}\mathbb{E}\left(\left|X_t\right|\right) + \frac{1}{\sqrt{2}}\mathbb{E}\left(\left|Y_t\right|\right) &= \frac{1}{\sqrt{2}}\mathbb{E}\left(\left|X_t\right|\right) + \frac{1}{\sqrt{2}}\mathbb{E}\left(\left|X_t\right|\right)\\ &= \frac{2}{\sqrt{2}}\mathbb{E}\left(\left|X_t\right|\right) = \sqrt{2}\mathbb{E}\left(\left|X_t\right|\right).\end{aligned}$$

In this case, you DO get the result that the expected (gross) leverage of the diversified strategy is larger than that of each constituents by the "diversification factor".