Mean-variance utility function does not obey Von-Neumann-Morgenstern rationality axioms

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Background

The Von Neumann-Morgenstern rationality axioms (link, archive) are a simple set of axioms that characterize a rational utility function.

A statement of Axiom 4 (the independence axiom)—which you will surely agree with—is that, for lotteries L, M, N such that $N \succ M$, we must have that

$$N' \quad := \quad \boxed{N} \cdot p \; + \; L \cdot (1-p) \qquad \succ \qquad \boxed{M} \cdot p \; + \; L \cdot (1-p) \quad =: \quad M',$$

with $p \in (0, 1]$ and where I've boxed the difference between the two sides.

That is, if you prefer N to M, then it doesn't matter with what probability you get them—as long as the probability of getting them is the same, and the payoff in the alternative case is also the same.

Mean-variance utility theory admits a utility function that ridiculously violates this axiom.

The gambles

I flip a coin.

- Gamble A: You get <u>$3</u> if the coin lands heads, else $2. Mean payoff is $2.50, while standard deviation of payoffs is $0.50.
- Gamble B: You get <u>$4</u> if the coin lands heads, else $2. Mean payoff is $3, while standard deviation of payoffs is $1.

Utility analysis

Suppose the utility to you of a random variable X is

$$U(X) := \mathbb{E}(X) - \text{Var}(X).$$

Trivially, getting $4 (as in gamble B) is preferable to getting $3 (as in gamble A), assuming for a moment that we can get each with certainty i.e. zero risk.

But of course in the actual gambles there is risk involved. Specifically, we have

$$U(A) = 2.50 - 0.50^2 = 2.50 - 0.25 = 2.25,$$

and

$$U(B) = 3 - 1^2 = 3 - 1 = 2.$$

So, A is the preferable gamble – despite the fact that it is (weakly) dominated by B in every state of the world. You prefer a gamble that is *guaranteed* to give you less money no matter what!

Salvaging mean-variance utility

MV utility theory (for a risk-averse agent) can be salvaged if we introduce free leverage. That is, if you have the option of a lottery represented by a random variable X, then you also have the option of a modified lottery represented by cX for arbitrary c. The decision rule becomes: Choose the gamble with the highest Sharpe ratio, then lever (or delever) it until it meets your ER requirement and/or respects your volatility limit. For instance, we could construct a modified gamble $A' := 2A$, meaning that we get $6 if the coin lands heads else $4. Now, the risk of A' is identical to B's, but the expected payoff is strictly greater, not just in expectation but in fact in every state of the world. Or, $A'' := 6A/5$, so that the expected payoff of A'' is identical to B's, but the risk is strictly less.