

# pctl-kurt

May 14, 2021

## 1 A Quick-and-Dirty Attempt to Maximize “Percentile Kurtosis”

```
[1]: from typing import Tuple, Iterable, Generator, Optional
import pandas as pd
import numpy as np
from functools import reduce
import matplotlib.pyplot as plt
import seaborn as sns; sns.set()
```

### 1.1 Motivation

Suppose someone tells you the event  $X = x$  is a “ $+4\sigma$  event” i.e. that its  $z$ -score  $z(X = x) = +4$ . That sounds statistically extreme! In other words,  $\Pr[X > x]$  is probably very low. Indeed, if  $X$  is Normal,  $X > x$  would be essentially impossible.

I want to try to construct a “pathological” example: A distribution with finite mean  $\mu$  and standard deviation  $\sigma$  where an event more extreme than  $X = x := \mu + 4\sigma$  (i.e. more than a  $+4\sigma$  event) is actually *very* likely. This is hard, but I’ll settle for making it *pretty* likely.

That is, I want to construct a distribution that is highly “percentile-kurtotic”: A distribution where  $\Pr[X > \mu + 4\sigma]$  is maximal.

### 1.2 Definitions

#### 1.2.1 $z$ -score

We define  $z$ -score

$$z(X = x) := \frac{x - \mu(X)}{\sigma(X)},$$

with

$$\sigma^2(X) := \int_{x=-\infty}^{x=\infty} (x - \mu(X))^2 \Pr[X = x] dx,$$

where  $X$  is a random variable and we abuse notation a bit to let  $\Pr[X = x]$  represent its probability density function. (We’re going to continue to be a bit hand-wavey with this, assuming some mild conditions e.g. PDF is continuous.)

### 1.2.2 percentile

We define percentile (AKA cumulative distribution function) the usual way, as

$$\Pr[X \leq x] := \int_{t=-\infty}^{t=x} \Pr[X = t] dt.$$

### 1.2.3 “percentile kurtosis”

Finally, as I alluded above, I’m going to define the “percentile kurtosis” (a made-up term)  $k$  of a random variable as

$$k(X) := 1 - \Pr[X \leq \mu(X) + 4\sigma(X)],$$

the complement-percentile of a  $+4\sigma$  event. Maximizing  $k$  is equivalent to minimizing the percentile.

Notice that  $k$  is location- and scale-invariant:

$$k(a + bX) = k(X) \quad \forall a, b \in \mathbb{R}^2.$$

## 1.3 Background

Using the definitions above, let’s answer a couple quick questions.

### 1.3.1 Can $k$ be literally 0, i.e. $\Pr[X \leq \mu + 4\sigma] = 1$ ?

Yes. Consider the Standard Uniform from 0 to 1.

$$\mu + 4\sigma = 0.5 + 4\sqrt{1/12} \approx 1.65 > 1,$$

so that a  $4\sigma$  event is not just “essentially” impossible, but “literally” impossible: It’s literally outside the support of the distribution.

This is not a very helpful example for our question, because we want to *maximize*  $k$ . But it’s just background.

### 1.3.2 Can $k$ be essentially but not literally 0?

Yes, we already mentioned the example of the Standard Normal.

### 1.3.3 Can $k$ be some small but nontrivial positive value?

Yes, the Standard  $\chi^2$  distribution has  $\mu = 1$  and  $\sigma = \sqrt{2}$ . In this case,  $k = 1 - \Pr[X \leq \mu + 4\sigma]$  turns out to be about 0.01.

### 1.3.4 Can $k$ be literally 1, i.e. $\Pr[X \leq \mu + 4\sigma] = 0$ ?

No. This sounds silly but it’s still worth convincing ourselves that it’s truly impossible.

Assume for simplicity that  $\mu = 0$  (this is WLOG, because if  $\mu \neq 0$ , we can just shift the entire distribution by  $-\mu$ ).

Suppose to the contrary that it is possible, i.e. there is some random variable  $X$  s.t.  $\Pr[X \leq +4\sigma] = 0$ . Then,

$$\Pr[X = x] = 0 \quad \forall x \in (-\infty, +4\sigma].$$

Hence, we can collapse the expression for variance to

$$\sigma^2(X) = \int_{x=+4\sigma}^{x=+\infty} x^2 \Pr[X = x] dx.$$

So we can think of the variance as a weighted average of squared values between  $+4\sigma$  and  $+\infty$ .

Standard deviation is of course nonnegative, so also  $+4\sigma \geq 0$ . So the smallest squared value in the weighted average is  $(+4\sigma)^2 = 16\sigma^2$ . Necessarily, then, the final weighted average must be at least  $16\sigma^2$ . (Actually, it must be just a smidge more than  $16\sigma^2$ , because we put zero density at exactly  $X = +4\sigma$ , hence the smallest-possible squared value is  $(+4\sigma + \varepsilon)^2$ , and if we accounted for this smidge, we'd run into a contradiction immediately. However, I find it more illuminating at this step to draw the weaker conclusion, which is still true, but lets us run further before hitting a contradiction.)

Hence we get

$$\sigma^2 \geq 16\sigma^2 \implies \sigma = 0.$$

So supposing that  $k$  can be 1 for some distribution, that distribution must have zero variance.

Well, according to a convention, the Dirac delta is the unique probability density function with zero variance (take this for granted, or if you insist on arguing, email me). It characterizes the density of a point mass.

So let's try it out. Supposing  $X$  is a Dirac-delta-distributed random variable (AKA a constant), we have

$$\Pr[X \leq +4\sigma] = \Pr[X \leq 0] = 1.$$

Sadly, this failed. We had assumed the exact opposite: that  $\Pr[X \leq 0] = 0$ . Hence by contradiction, we have Q.E.D.

### 1.3.5 What about the evil Cauchy (AKA Standard $t$ ) distribution?

The immediate concern with the Cauchy (that indeed disqualifies it from consideration) is that its variance (or even its mean) is not finite.

But it can be instructive to think about it. If we “define”  $\sigma(X) := \infty$  (acceptable) and  $\mu(X) := 0$  (please don't haunt my dreams Prof Blitzstein) for a Cauchy random variable, then  $\mu(X) + 4\sigma(X) = +\infty$ , which is the upper bound of support for the Cauchy, hence  $k(X) = 0$ , just like for the Standard Uniform.

The issue here is that the fat tails of the Cauchy are *so* fat, that they make its standard deviation infinite. At that point, it becomes impossible to push any probability mass at all beyond  $\mu + 4\sigma$ , because there's no more number line remaining after that.

## 1.4 Chebyshev-Cantelli upper bound

Obviously, the best would be to solve for the maximal  $k$  analytically in closed form. Letting WLOG  $\mu = 0$ , we want to solve

$$\max_{f \in F} \int_{x=\ell}^{x=+\infty} f(x) dx,$$

where  $F$  is the set of well-formed PDF's and

$$\ell := +4\sqrt{\int_{t=-\infty}^{t=+\infty} t^2 f(t) dt}.$$

This doesn't look impossible but it certainly looks very boring and I haven't got any good ideas about how to begin.

However, Chebyshev's Inequality gives us a weak (indeed, unattainable) upper bound  $k(X) \leq \frac{1}{4^2} = \frac{1}{16} = 0.0625$ .

Pf: By Adam's Law. WLOG assume  $\mu = 0$ . Take  $z_0 := 4$  as in our definition of  $k(X)$ . I suppress  $(X)$  in e.g.  $\sigma(X)$  for clarity.

$$\begin{aligned} \sigma^2 &= E[X^2] \\ &= E[X^2 \mid |X| > z_0\sigma] \cdot \Pr[|X| > z_0\sigma] + E[X^2 \mid |X| \leq z_0\sigma] \cdot \Pr[|X| \leq z_0\sigma] \\ &\geq E[X^2 \mid |X| > z_0\sigma] \cdot \Pr[|X| > z_0\sigma] \\ &> (z_0\sigma)^2 \cdot \Pr[|X| > z_0\sigma] \\ \frac{1}{z_0^2} &> \Pr[|X| > z_0\sigma] \\ \frac{1}{4^2} &> \Pr[|X| > 4\sigma] \\ \frac{1}{16} &> \Pr[|X| > 4\sigma], \end{aligned}$$

Q.E.D.

Further, the “one-sided” Chebyshev-Cantelli Inequality gives us the even tighter upper bound  $k \leq B := \frac{1}{1+z_0^2} = \frac{1}{1+4^2} = \frac{1}{17} \approx 0.0588$ , which I'm guessing we can similarly show is unattainable.

## 1.5 Solution (for general $z_0 > 0$ )

So fixing  $z_0 > 0$ , the best we can do is get arbitrarily close to the Chebyshev-Cantelli upper bound  $B$ . And in fact we can get there with the following zero-mean “Generalized Trinoulli” PMF, taking arbitrarily small  $\varepsilon > 0$ :

$$X = \begin{cases} -\frac{1}{z_0} & \text{w.p. } (1-B)(1-\varepsilon) \\ 0 & \text{w.p. } \varepsilon \\ z_0 & \text{w.p. } B(1-\varepsilon). \end{cases}$$

The mean  $\mu$  of this distribution is

$$\begin{aligned} &-\frac{1}{z_0} \left( \frac{z_0^2}{1+z_0^2} \right) (1-\varepsilon) + z_0 \left( \frac{1}{1+z_0^2} \right) (1-\varepsilon) \\ &= -\frac{z_0}{1+z_0^2} (1-\varepsilon) + \frac{z_0}{1+z_0^2} (1-\varepsilon) \\ &= \left( -\frac{z_0}{1+z_0^2} + \frac{z_0}{1+z_0^2} \right) (1-\varepsilon) \\ &= 0. \end{aligned}$$

Its variance  $\sigma^2$  is

$$\begin{aligned}
 & \frac{1}{z_0^2} \left( \frac{z_0^2}{1+z_0^2} \right) (1-\varepsilon) + z_0^2 \left( \frac{1}{1+z_0^2} \right) (1-\varepsilon) \\
 &= \frac{1}{1+z_0^2} (1-\varepsilon) + \frac{z_0^2}{1+z_0^2} (1-\varepsilon) \\
 &= \left( \frac{1}{1+z_0^2} + \frac{z_0^2}{1+z_0^2} \right) (1-\varepsilon) \\
 &= (1-\varepsilon).
 \end{aligned}$$

Hence its standard deviation  $\sigma < 1$ .

Therefore, its maximum value  $z_0$  is larger than  $\mu + z_0\sigma = 0 + z_0\sigma = z_0\sigma$ . It achieves its maximum with probability  $B - \varepsilon$ , hence by making  $\varepsilon$  arbitrarily small we can get arbitrarily close to  $B$ . For our original problem specification, simply set  $z_0 := +4$ .

```
[2]: PMF: type = pd.Series # numeric index, float data

EPSILON: float = 1e-8
ZSCORE: float = +4.00 # for pctl-kurt
CHEBYSHEV_CANTELLI_UBOUND: float = 1 / (1 + ZSCORE**2)

def check_pmf(pmf: PMF) -> bool:
    if not (pmf >= 0).all():
        raise ValueError(f"{pmf} not nonnegative!")
    if not np.isclose(pmf.sum(), 1):
        raise ValueError(f"{pmf} doesn't sum to unity!")
    return True

def calc_mean(pmf: PMF) -> float:
    # pmf-weighted average
    return sum(pmf.index * pmf)

def calc_std(pmf: PMF) -> float:
    squared_centered_values = (pmf.index - calc_mean(pmf=pmf))**2
    # again, just a pmf-weighted average
    return sum(squared_centered_values * pmf)**0.5

def calc_pctl(pmf: PMF, x: float=0) -> float:
    return pmf.loc[:x].sum()

def calc_pctl_kurt(pmf: PMF, zscore: float=ZSCORE) -> float:
    mu = calc_mean(pmf=pmf)
```

```

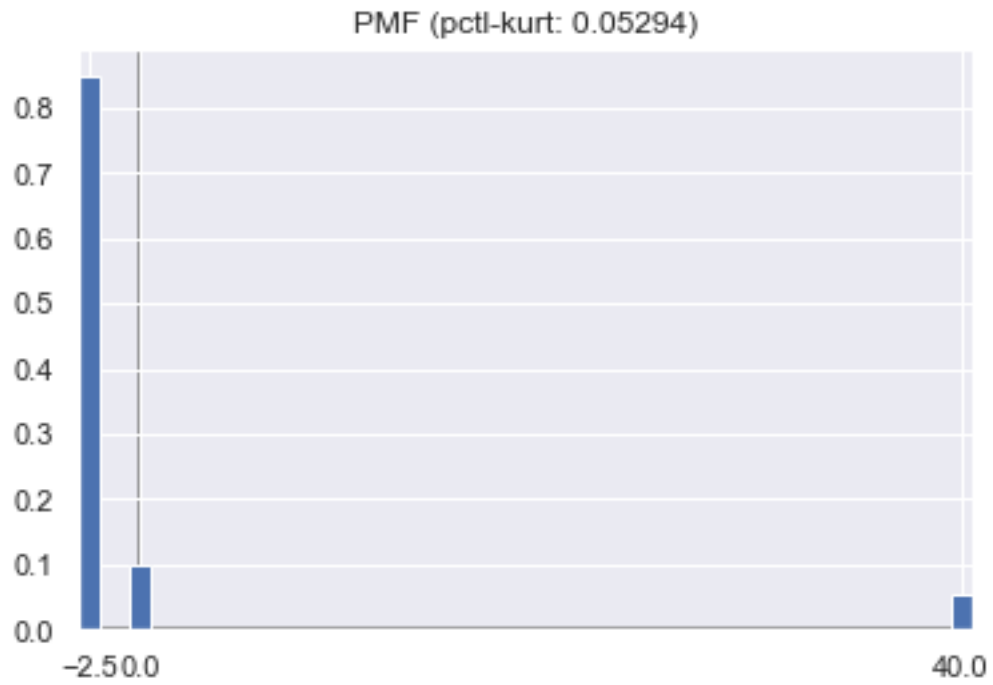
sigma = calc_std(pmf=pmf)
pctl = calc_pctl(pmf=pmf, x=mu + zscore*sigma)
return 1 - pctl

def plot_pmf(
    pmf: PMF, width: float=1.00,
    xlim_left: Optional[float]=None, xlim_right: Optional[float]=None
) -> float:
    check_pmf(pmf=pmf)
    pctl_kurt = calc_pctl_kurt(pmf=pmf)
    plt.bar(x=pmf.index, height=pmf.values, width=width)
    plt.axhline(y=0, zorder=0, color="gray")
    plt.axvline(x=0, zorder=0, color="gray")
    plt.xticks(ticks=pmf.index)
    plt.xlim(left=pmf.index.min()-0.5 if xlim_left is None else xlim_left,
              right=pmf.index.max()+0.5 if xlim_right is None else xlim_right),
    plt.ylim(bottom=0)
    plt.title(f"PMF (pctl-kurt: {pctl_kurt:.5f})")
    plt.show()
    return pctl_kurt

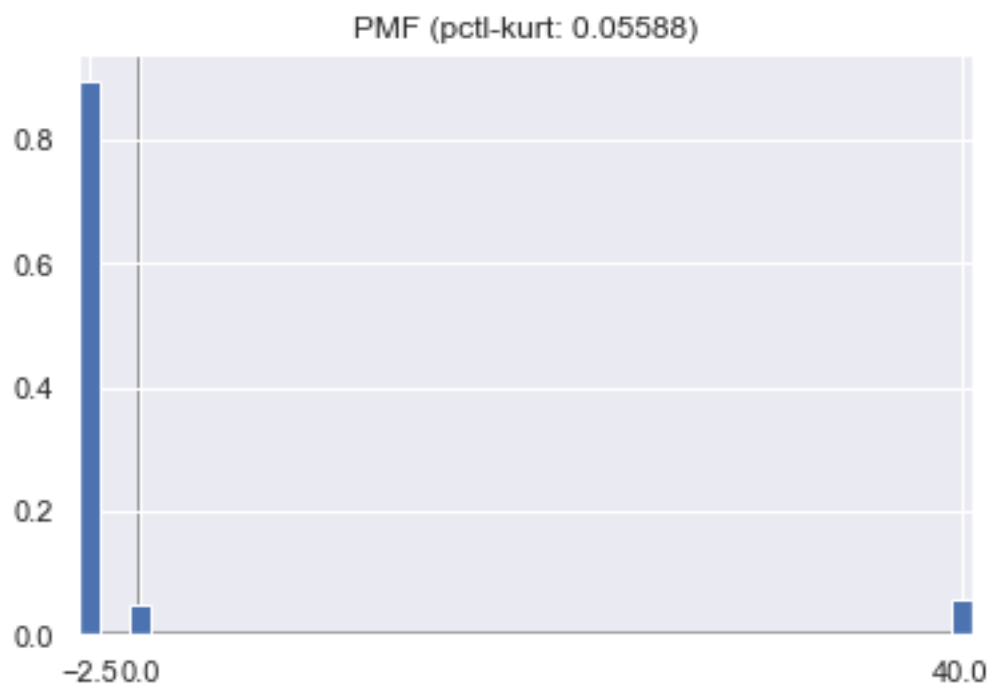
def solve(epsilon: float=EPSILON, legibility_scalar: float=10) -> float:
    """
    Get (arbitrarily close) to the Chebyshev-Cantelli upper bound,
    with a zero-mean "Generalized Trinoulli" PMF.
    """
    pctl_kurt = plot_pmf(pmf=pd.Series({
        -1 / ZSCORE * legibility_scalar:
            (1 - CHEBYSHEV_CANTELLI_UBOUND) * (1 - epsilon),
        0 * legibility_scalar:
            epsilon,
        ZSCORE * legibility_scalar:
            CHEBYSHEV_CANTELLI_UBOUND * (1 - epsilon),
    }))
    print(f"We are {pctl_kurt / CHEBYSHEV_CANTELLI_UBOUND - 1:.8f} away!\n")
    return pctl_kurt

if __name__ == "__main__":
    for epsilon in (1e-1, 5e-2, 1e-2, 1e-8):
        solve(epsilon=epsilon)

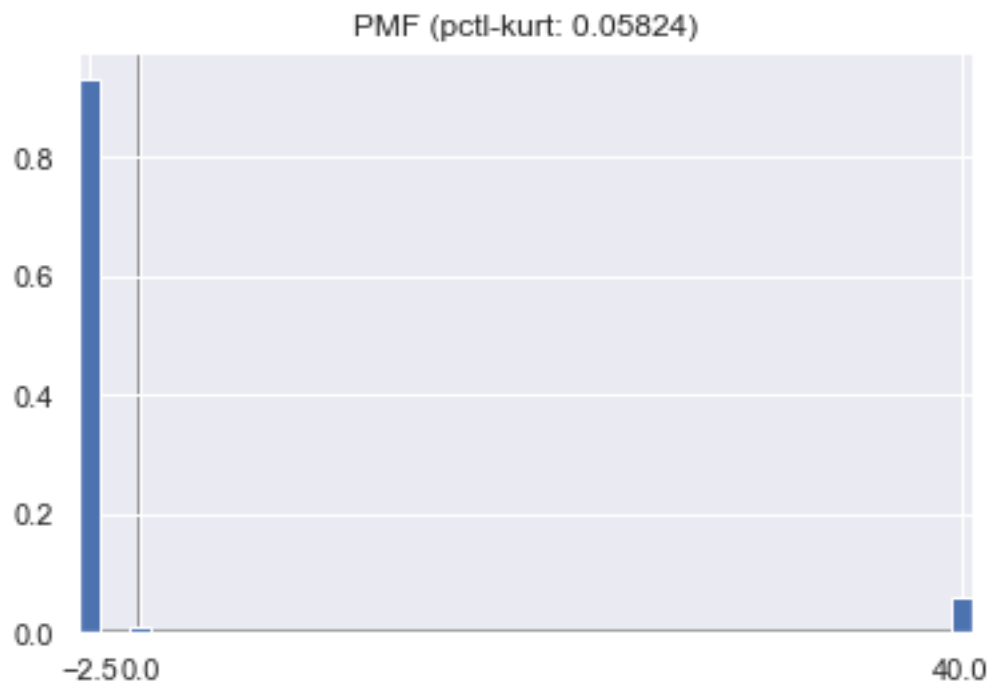
```



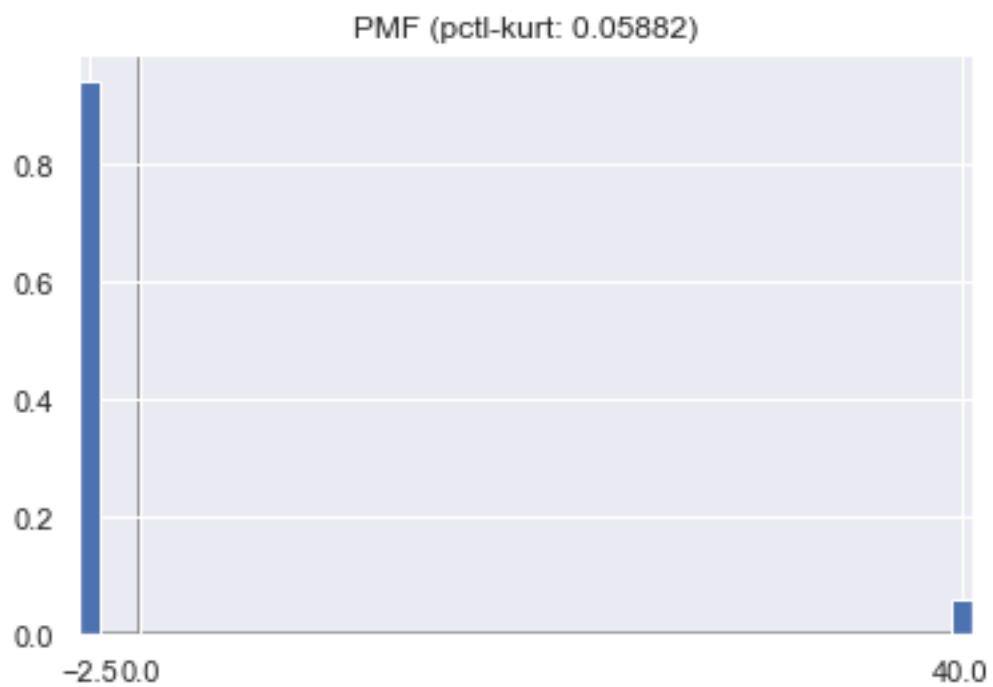
We are  $-0.10000000$  away!



We are  $-0.05000000$  away!



We are  $-0.01000000$  away!





We are -0.00000001 away!

## 1.6 Appendix: Some ways that will not work

### 1.6.1 A war on two fronts: Fighting Chebyshev-Cantelli *and* the CLT

One of my dumber friends suggested that I simply take a zero-mean base PMF like `pd.Series({-9: 0.10, +1: 0.90})` which gives a hefty 90% chance of being positive, and scaling its width by some constant  $c$ , the argument being that its maximum would increase with  $c$  whereas its  $\sigma$  would increase with  $\sqrt{c}$  and hence eventually its maximum value would “overtake” its  $+4\sigma$  threshold. At that point, there would necessarily be a 90% chance of a  $+4\sigma$  event. Obviously this is wrong because scaling a PMF by  $c$  also scales its  $\sigma$  by  $c$ :  $\sigma$  scales with  $\sqrt{c}$  when you *add together* i.i.d. variables, not just scale up a single one.

Not satisfied with one silly mistake for the night, before going to bed, he conceded that that wouldn't work, but suggested that I should simply then take instead the PMF of the sum of  $c$  i.i.d. random variables drawn from his base PMF. Then, the  $\sigma$  would indeed scale with  $\sqrt{c}$ . I am also dumb which is why we're friends, so I actually thought he'd solved it. I cracked a bottle of champagne and promised to buy him a beer when Covid ended.

Then I tried it. It does not work. Adding together those i.i.d. random variables (as opposed to simply scaling up any individual one) compresses their standard deviation precisely because probability mass flows inward *away from the tails* (toward zero, the mean) as the “sampling distribution” of their sample mean (which fixing  $c$  is just a deterministically scaled-down version of their sum) converges to a Normal according to the Central Limit Theorem.

```
[3]: def _get_pmf_of_indep_sum(pmf_a: PMF, pmf_b: PMF) -> PMF:
    """Get the PMF of the sum of two independent random variables."""
    pmf = pd.Series(dtype=float)
    for a, p_a in pmf_a.iteritems():
        for b, p_b in pmf_b.iteritems():
            pmf.loc[a + b] = pmf.get(key=a+b, default=0) + p_a*p_b
    return pmf

def get_pmf_of_indep_sum(pmfs: Iterable[PMF]) -> PMF:
    """Get the PMF of the sum of `len(pmfs)` independent random variables."""
    return reduce(_get_pmf_of_indep_sum, pmfs)

def try_sums() -> None:
    base_pmf = pd.Series({-9: 0.10, +1: 0.90})

    pmf = pd.Series({0: 1.00})
    for c in range(1, 8+1):
        print(f"Adding {c} i.i.d. r.v.'s from the base PMF:")
```

```

    pmf = _get_pmf_of_indep_sum(pmf_a=pmf, pmf_b=base_pmf)
    plot_pmf(pmf=pmf)

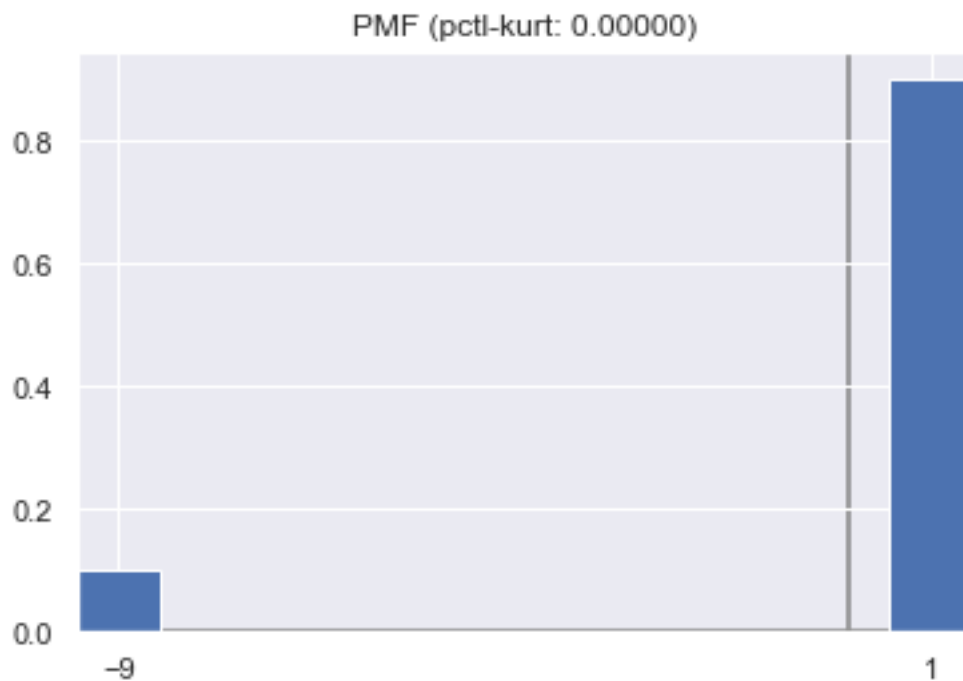
c = 32
print(f"Adding {c} i.i.d. r.v.'s from the base PMF:")
pmf = get_pmf_of_indep_sum(pmfs=[base_pmf]*c)
plot_pmf(pmf=pmf, width=2)

print("Zoom in to see that the PMF still has a long left tail,\n" \
      "but it's begun to converge to Normal as CLT kicks in:")
plot_pmf(pmf=pmf, width=2, xlim_left=-50, xlim_right=+50)

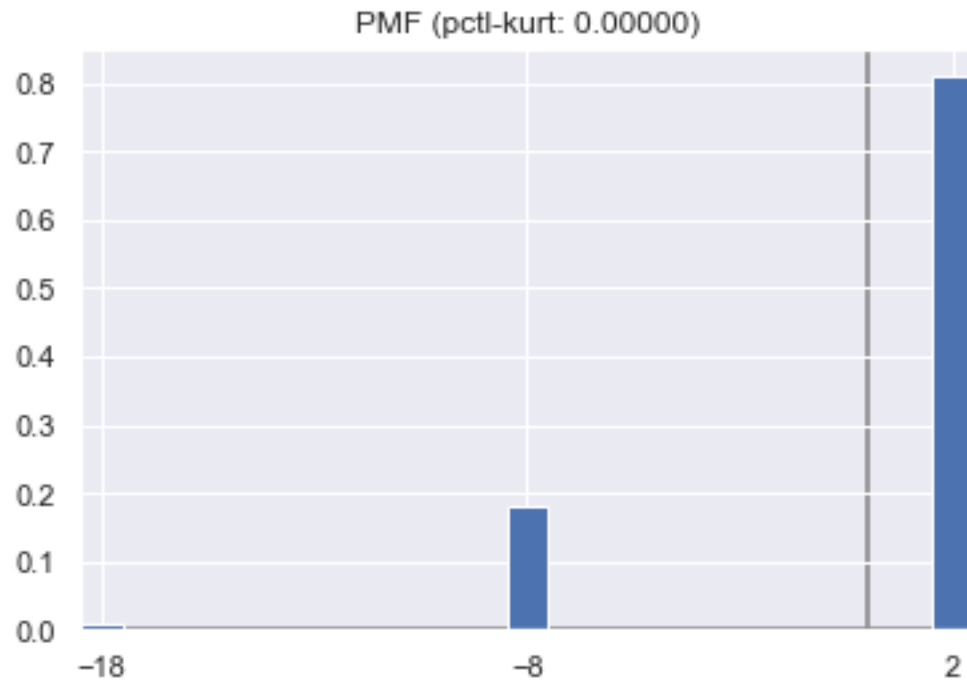
if __name__ == "__main__":
    try_sums()

```

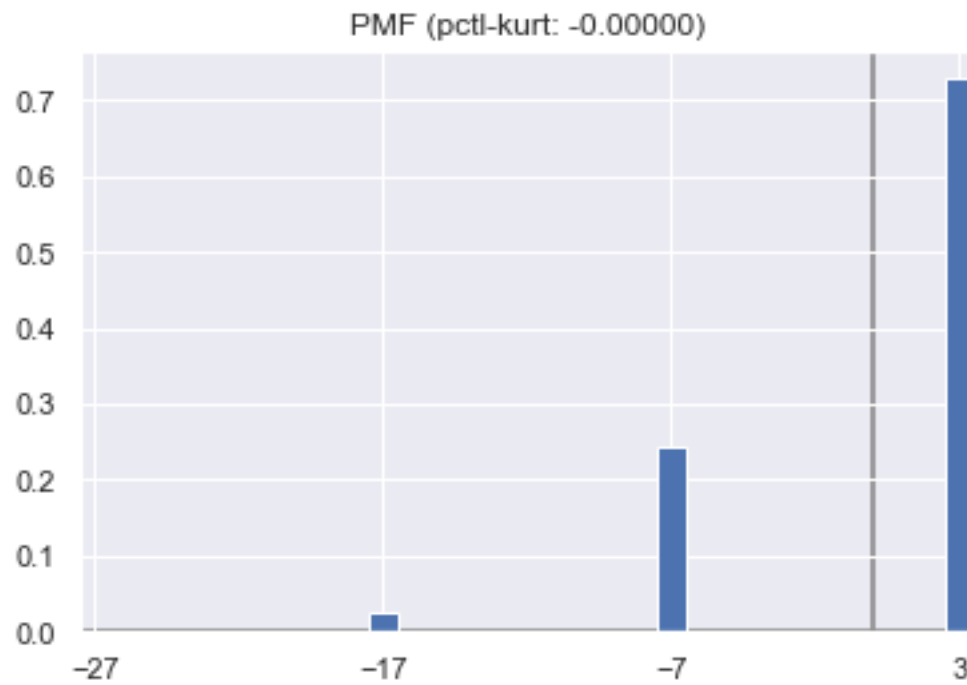
Adding 1 i.i.d. r.v.'s from the base PMF:



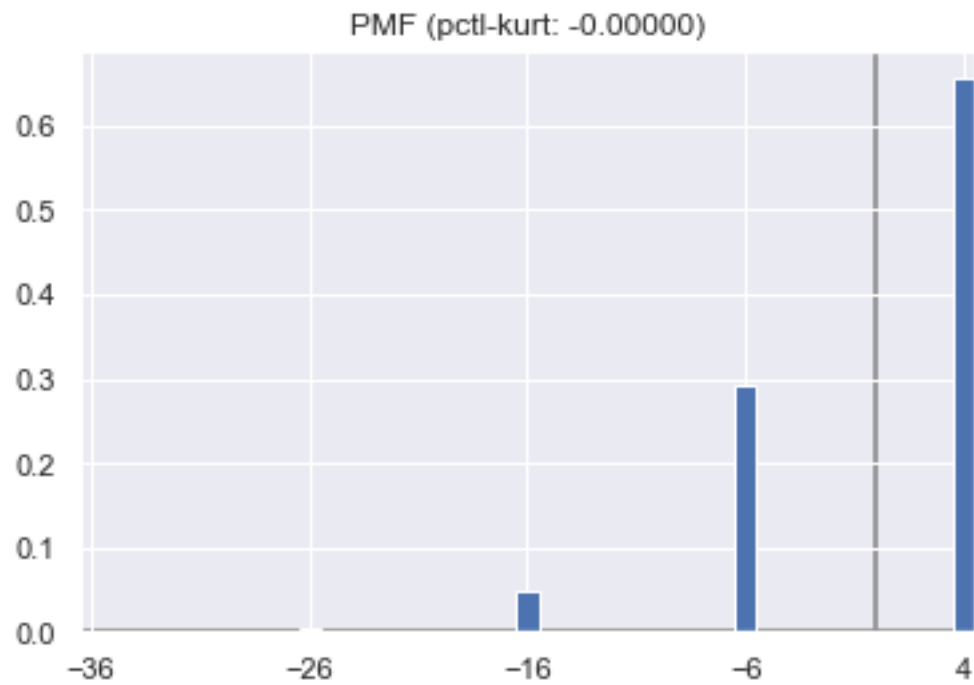
Adding 2 i.i.d. r.v.'s from the base PMF:



Adding 3 i.i.d. r.v.'s from the base PMF:



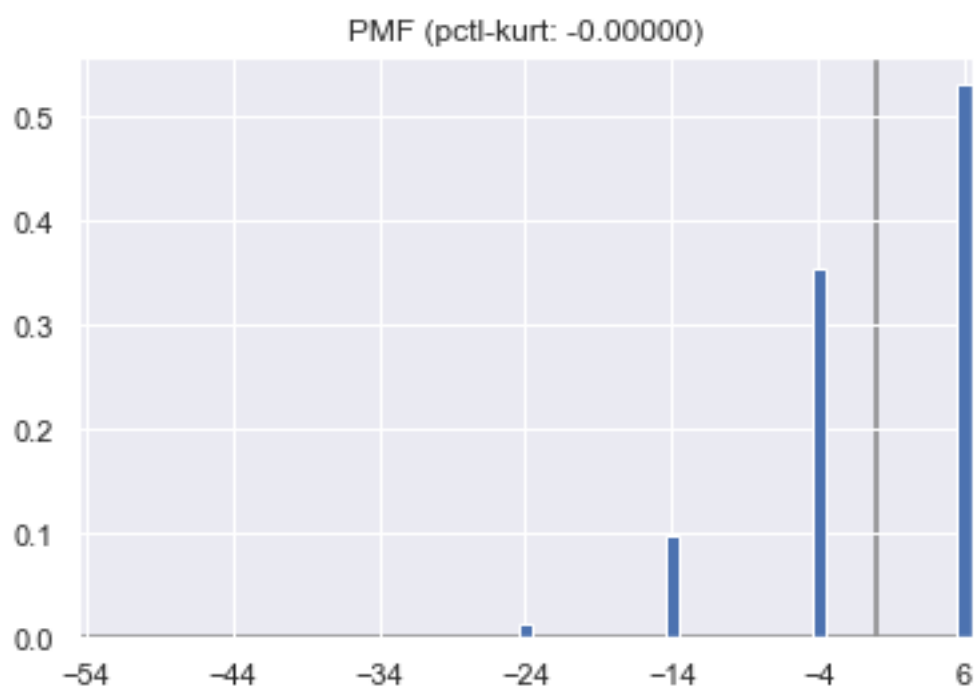
Adding 4 i.i.d. r.v.'s from the base PMF:



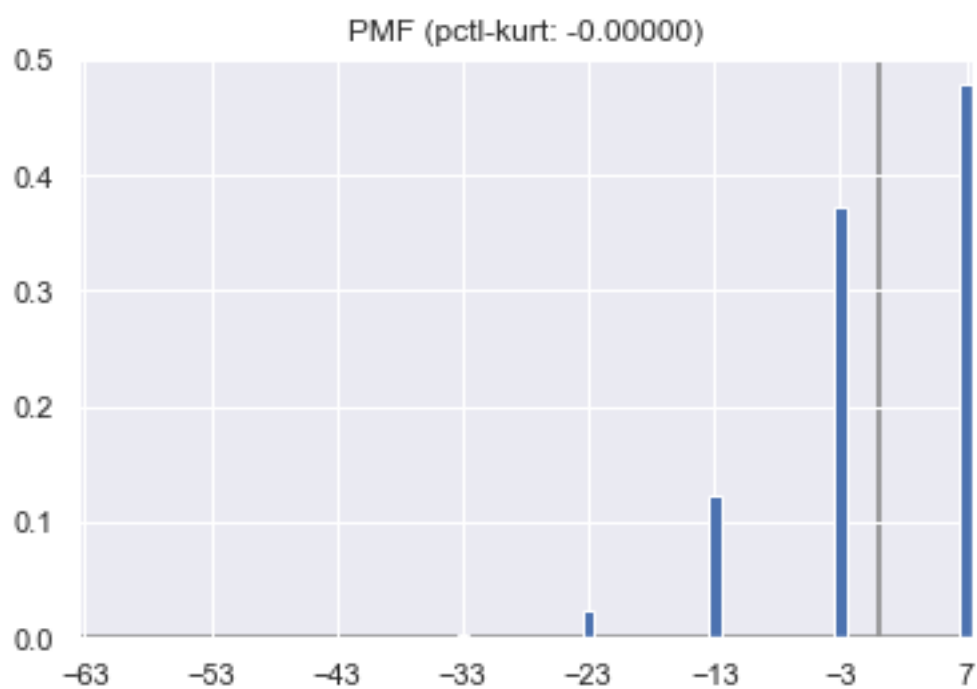
Adding 5 i.i.d. r.v.'s from the base PMF:



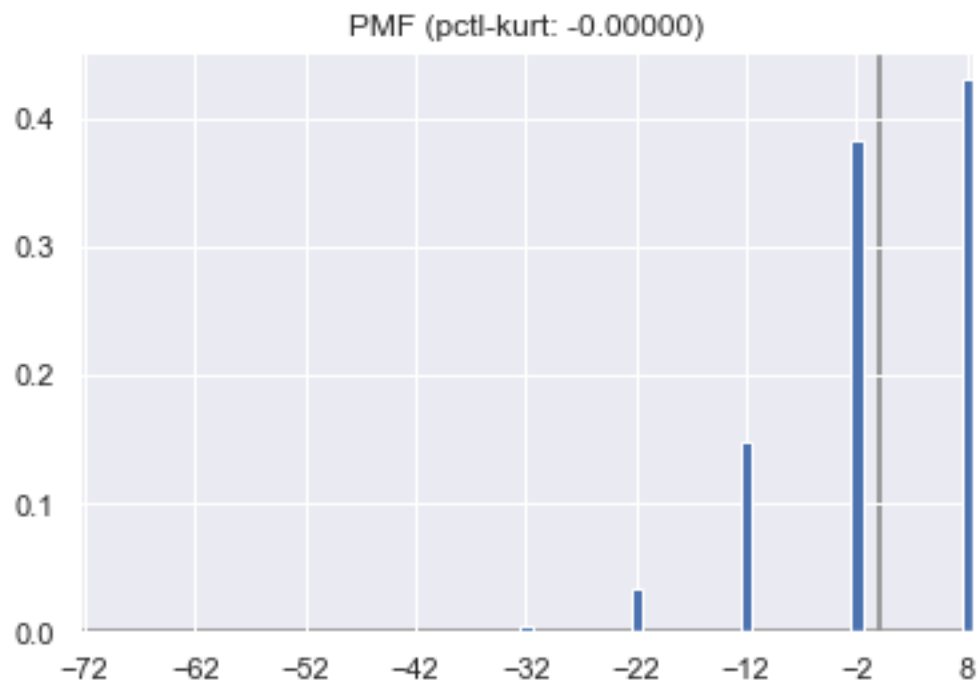
Adding 6 i.i.d. r.v.'s from the base PMF:



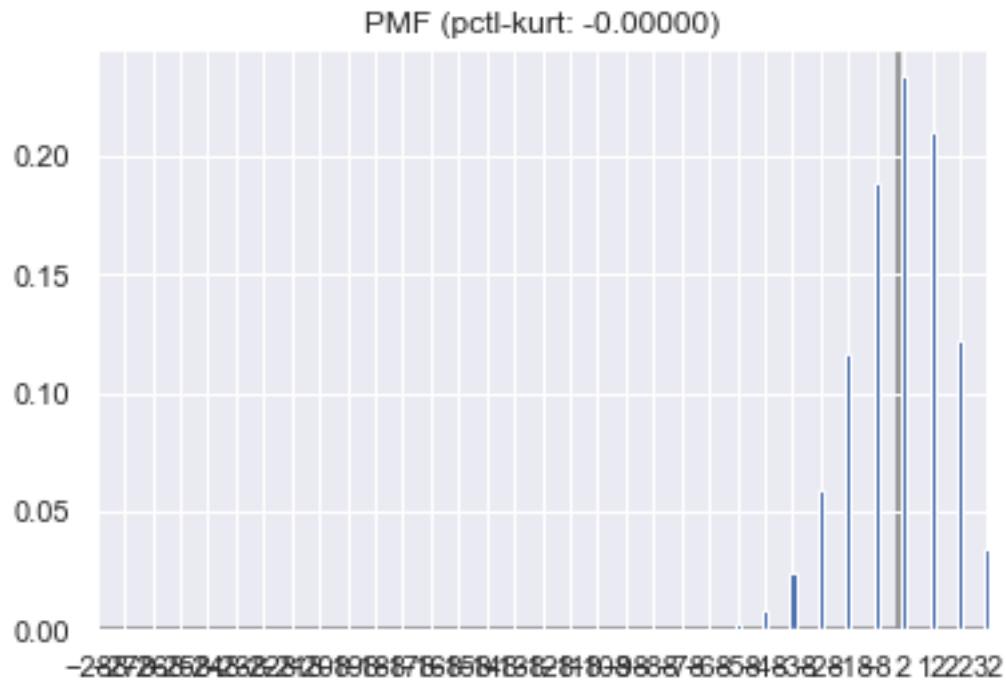
Adding 7 i.i.d. r.v.'s from the base PMF:



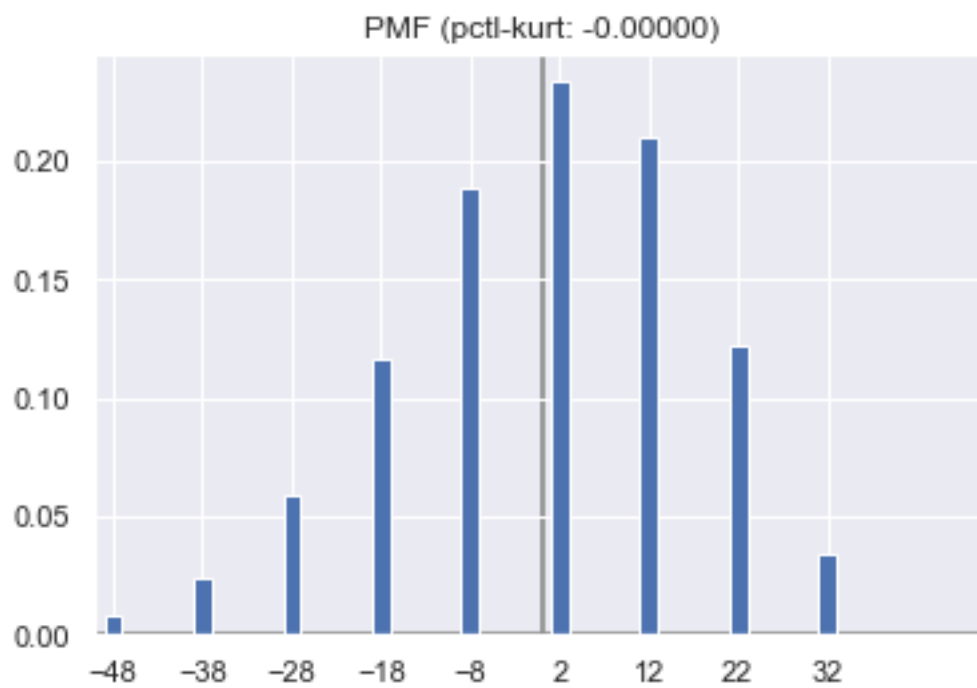
Adding 8 i.i.d. r.v.'s from the base PMF:



Adding 32 i.i.d. r.v.'s from the base PMF:



Zoom in to see that the PMF still has a long left tail,  
but it's begun to converge to Normal as CLT kicks in:



### 1.6.2 Brute force

**Stars and bars** My initial compromise was to select some high “resolution”  $N$  and simply systematically generate for analysis all the possible discrete PMF’s available at this resolution (since we normalize by  $N$  before analyzing, we can limit our search to “PMF”s that sum to  $N$ ).

In general, the number of permutations to search over is precisely the number of stars-and-bars solutions with  $N$  balls and  $N$  possibly-empty urns. This program’s time complexity is therefore  $O\left(\binom{2N-1}{N}\right)$ . Even  $N = 15$  took more than a day; at that rate  $N = 16$  would have taken almost a week.

**Spectral analysis** A good suggestion I had gotten here was a sort of “spectral analysis”, choosing a normalized basis of  $N_P$  parent PDF’s and then systematically generating child PDF’s as weighted averages of the parents, searching over grid of  $N_C$  coefficients between 0 and 1 for each parent.

The combinatorics here do not blow up nearly as fast: The number of permutations to search over is the number of stars-and-bars solutions with  $N_C$  and  $N_P$  possibly-empty urns. This program’s time complexity is therefore  $O\left(\binom{N_C+N_P-1}{N_C}\right) = O(N_C^{N_P})$ . Suppose we choose  $N_P = 8$  parent PDF’s and  $N_C = 21$  coefficients (I’m thinking of counting up from 0 to 1 in increments of 0.05): We could hit “Run” before going to bed and the results would be ready in time for breakfast.

However, I had already solved the problem by the time I got the suggestion, so I didn’t bother implementing it. It does however raise the interesting question of what the  $N_P = 8$  parent basis functions would have looked like.

```
[4]: N: int = 15 # resolution (higher is better) of our PMF's

def gen_perms(len_: int=N, sum_: int=N) -> Generator[Tuple[int], None, None]:
    """
    Generate permutations of Naturals  $\{0, 1, 2, \dots\}$ 
    with length `len_` and sum `sum_`.
    Each permutation is a tuple of the form
    `(n_0, n_1, n_2, ..., n_{len_-1})`.

    This is the "stars and bars" problem:
    How many ways can we sort
    `sum_` indistinguishable balls into
    `len_` labelled urns, possibly leaving some urns empty?

    This is pretty general code, but we're going to keep it simple by
    fixing some large `N`,
    and then analyzing `gen_perms(len_=N, sum_=N)`. In this usage,
    `N` represents our
    "resolution": Ultimately, we'll normalize (elementwise)
    each permutation by `N` itself and
    thereby yield a well-formed PMF. It's convenient to let
    `sum_ := len_` because
    then we can perfectly encode a Uniform PMF as an  $N$ -tuple of 1's.
```



You might notice that as  $N$  increases, so does the support of our PMF: That's OK. The key is that larger  $N$  gives us more flexibility to create finer and finer "shapes" for the PMF. It doesn't matter that in the process of creating these shapes, we stretch out the support, because you can always just imagine analyzing instead the PMF of the random variable  $X/N$  i.e. our random variable  $X$  divided by our fixed resolution  $N$ , thereby shrinking the support back down to the interval  $[0, 1]$ . Results will be equivalent for our purposes.

Recursive generator inspired by <https://stackoverflow.com/a/7748851>. We could have saved a bit of time by implementing a dynamic-programming solution, but that would require a ton of space to store all the permutations in memory. In this case, the space is the limiting resource, because there are many many permutations we want to iterate over, but generating any single permutation is actually reasonably fast relative to the analysis we subsequently run on it.

```

"""
if len_ < 1:
    raise ValueError(len_)
if sum_ < 0:
    raise ValueError(sum_)

# base case
if len_ == 1:
    # only choice is singleton tuple with `sum_` as its only element
    yield (sum_,)
# recursive case
else:
    # iterate over choices for head (first) element i.e.  $n_0$ 
    # `reversed` because i want mass to start at LHS and flow rightward,
    # e.g. first-choice PMF puts 100% weight on `0` not `len_-1`
    for head in range(sum_, -1, -1):
        """
        Now having fixed the head, recursively generate choices for
        the tail (remaining) elements i.e.  $n_1, \dots, n_{\{len_-1\}}$ .
        Tail must be  $len_- 1$  elements long, and sum to  $sum_- head$ .
        """
        for tail in gen_perms(len_=len_-1, sum_=sum_-head):
            # concatenate tuples
            yield (head,) + tail

def get_pmf_from_perm(perm: Tuple[int]) -> PMF:
    """
    Normalize a tuple of ints by its sum, creating a PMF.

```

```

input
-----
perm: Tuple[int], a permutation of Naturals
    e.g. `(3, 1, 0, 1)`.

output
-----
PMF, a well-formed PMF
    e.g. `pd.Series({0: 0.6, 1: 0.2, 2: 0.0, 3: 0.2})`.
"""
return pd.Series(perm) / sum(perm)

def get_pmf_and_pctl_kurt_from_perm(perm: Tuple[int]) -> Tuple[PMF, float]:
    # turn permutation into a probability mass function
    pmf = get_pmf_from_perm(perm=perm)
    # calculate its percentile kurtosis
    pctl_kurt = calc_pctl_kurt(pmf=pmf)
    assert not pd.isnull(pctl_kurt), \
        (pmf, pctl_kurt)
    return pmf, pctl_kurt

def try_perms() -> Tuple[float, PMF]:
    """
    Loop over permutations, selecting the winner based on
    highest percentile kurtosis.
    """
    # "do-while" structure
    perms = gen_perms()
    # PMF and value associated w/ maximal pctl-kurt seen so far ("do" part)
    argmax_pctl_kurt, max_pctl_kurt = \
        get_pmf_and_pctl_kurt_from_perm(perm=next(perms))
    # iterate through the remaining permutations ("while" part)
    for perm in gen_perms():
        pmf, pctl_kurt = get_pmf_and_pctl_kurt_from_perm(perm=perm)
        if pctl_kurt > max_pctl_kurt:
            max_pctl_kurt = pctl_kurt
            argmax_pctl_kurt = pmf
    plot_pmf(pmf=argmax_pctl_kurt)
    return argmax_pctl_kurt, max_pctl_kurt

if __name__ == "__main__":
    # try_perms() # takes forever
    pass

```

### 1.6.3 Playing the lottery

The worst would be Monte Carlo, which I've genuinely gotten as a suggestion here. Monte Carlo can be good for pinning down "averages" but is not necessarily always as good for pinning down "extremes". I'm not even going to dignify this suggestion with an implementation.