Mean-variance utility function does not obey Von-Neumann-Morgenstern rationality axioms

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Background: The Von-Neumann-Morgenstern rationality axioms

The VNM rationality axioms (link, archive) are a simple set of axioms that characterize a rational utility function. A statement of Axiom 4 (the independence axiom)—which you will surely agree with—is that, for lotteries L, M, N such that $N \succ M$, we must have that

$$N' \quad := \quad \boxed{N} \cdot p \,+\, L \cdot (1-p) \qquad \succ \qquad \boxed{M} \cdot p \,+\, L \cdot (1-p) \quad =: \quad M',$$

with $p \in (0, 1]$ and where I've boxed the difference between the two sides.

That is, if you prefer N to M, then it doesn't matter with what probability you get them—as long as the probability of getting them is the same, and the payoff in the alternative case is also the same.

Expected value is a VNM-consistent utility function. However, we know that in the St Petersburg game (link), its recommendation feels silly. Mean-variance utility theory seems like an easy out, but I will show the uncomfortable fact that it admits a utility function that ridiculously violates the above axiom.

Setting: The gambles

I flip a coin.

- Gamble A: You get $\underline{\$3}$ if the coin lands heads, else \$2. Mean payoff is \$2.50, while standard deviation of payoffs is \$0.50.
- Gamble B: You get $\underline{\$4}$ if the coin lands heads, else \$2. Mean payoff is \$3, while standard deviation of payoffs is \$1.

Analysis: Mean-variance-optimal choice

Suppose the utility to you of a random variable X is described by the simplest mean-variance function:

$$U(X) := \mathbb{E}(X) - \mathrm{Var}(X).$$

Trivially, getting \$4 (as in gamble B) is preferable to getting \$3 (as in gamble A), assuming for a moment that we can get each with certainty i.e. zero risk. A constant, after all, is just a degenerate random variable, so we can plug it into U:

$$\boxed{U(B \mid \text{Heads})} = 4 - 0^2 = 4 - 0 = 4 \qquad > \qquad 3 = 3 - 0 = 3 - 0^2 = \boxed{U(A \mid \text{Heads})},$$

$$\boxed{U(B \mid \text{Tails})} = 2 - 0^2 = 2 - 0 = 2 \qquad = \qquad 2 = 2 - 0 = 2 - 0^2 = \boxed{U(A \mid \text{Tails})}.$$

But of course in the actual gambles there is risk involved. Specifically, we have

$$\boxed{U(B)} = 3 - 1^2 = 3 - 1 = 2 \qquad < \qquad 2.25 = 2.50 - 0.25 = 2.50 - 0.50^2 = \boxed{U(A)}.$$

So, A is the preferable gamble – despite the fact that it is (weakly) dominated by B in every state of the world. Each outcome in gamble B is (according to U) conditionally better than the corresponding outcome in gamble A, yet somehow the gamble B as a whole is (still according to U!) unconditionally worse. You prefer A, a gamble that is *guaranteed* to give you less money no matter what!

Commentary: Salvaging mean-variance utility

MV utility theory (for a risk-averse agent) can be salvaged if we introduce free leverage[1]. That is, if you have the option of a lottery represented by a random variable X, then you also have the option of a modified lottery represented by cX for arbitrary c. The decision rule becomes: Choose the gamble with the highest Sharpe ratio, then lever (or delever) it until it meets your ER requirement and/or respects your volatility limit. For instance, we could construct a modified gamble $A' := 2A$, meaning that we get \$6 if the coin lands heads else \$4. Now, the risk of A' is identical to B's, but its payoff is strictly greater, not just in expectation but in fact in every state of the world. Or, $A'' := 6A/5$, so that the expected payoff of A'' is identical to B's, but its risk is strictly less.

[1] Or, equivalently, quote all payoffs in excess-of-funding terms, which is the actual solution in practice