The two-child problem

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Setting

This is an extremely common brain-teaser, and it's been explained to death by much smarter people. My own attempt is a bit chaotic, but I try to explicitly relate how the calculations can be backed up by visual (via tables) and verbal intuition.

Here's the setting: You meet a mother on the street. She tells you she has two kids at home (yes, exactly two, and no, they are not twins). You want to know the probability—call it P—that they are both boys.

And here's the upshot:

- Before she says anything else, $\boxed{P = \frac{1}{4}}$.
- If she tells you that the first-born is a boy, then $\boxed{P = \frac{1}{2}}$.
- If instead she tells you merely that *at least one* is a boy, then $\boxed{P = \frac{1}{3}}$.
- But if she tells you that at least one is a boy born on a Tuesday... then $\frac{1}{3} \ll \boxed{P = \frac{13}{27}} < \frac{1}{2}$.

Now, what bearing has Tuesday possibly got on this?? Let's find out.

Definitions

Notation

We will indicate the gender of the first-born with an uppercase letter (B or G), and that of the second-born with a lowercase (b or g). We'll let B_0 (rsp G_0, b_0, g_0) represent the event that the first-born is a boy born in the first half of the week, similarly B_1 (rsp ...) for the second half. Similarly B_T for Tuesday specifically.

We will let $g(q)$ represent the probability of two independent and equally-probable events X and Y (where q is the probability of either event in isolation). That is, by inclusion-exclusion,

$$\Pr[X \cup Y] =: g(q) = q + q - q \cdot q = 2q - q^2.$$

Finally, in tabular calculations, we will use red to rule out cells, and green to remind us which cells represent "success".

What is probability?

This is not the only interpretation (and it might not be the most rigorous), but it will help keep our argument straight. We will say that an "event" is just a subset of the sample space. So, if we call the set of all possible states of the world as Ω, then B is the collection of every such state where the first-born child is a boy.

Similarly, b is the collection of every such state where the second-born is a boy, and $B \cap b$ is the intersection of those sets: That is, the collection of all states where both kids are boys.

On the other hand, $B \cup b$ is the collection of all states where *either* kid (i.e. at least one) is a boy. Notice that $B \cup b = \Omega - (G \cap g)$: The states where at least one kid is a boy, are all the states other than those where both are girls.

And naturally, $B \cap G = \varnothing$: There can be no states where the first-born is *both* a boy *and* a girl.

With all this in mind, the probability of an event is just the measure of the subset it represents.

Assumptions

We will assume that any given birth is independent of all others, and equally probable to occur on any day of the week. In the real world there are minor imbalances in the empirical distribution of births by day of the week, but it's not interesting for us.

CALCULATIONS

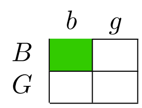
Scenario 0: No further information

Unconditionally, $P = \frac{1}{4}$. Pf:

First let's use (kinda) Bayes's rule:

$$P := \Pr[B \cap b] = \Pr[b \mid B] \cdot \Pr[B] = \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}.$$

Now, let's use a table (with the gender of the first-born in the rows, and the gender of the second-born in the columns) to confirm the same thing:



Very nice. There are four valid cells, and exactly one represents success. QED.

Scenario 1: First-born is a boy

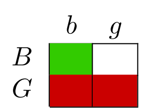
Suppose she tells you her first-born is a boy. Here, conditionally, $P = \frac{1}{2}$. Pf:

We know from above that

$$\Pr[B \cap b] = \Pr[b \mid B] \cdot \Pr[B] = \frac{1}{4}.$$

We can also modify this to account for "extra" conditioning by putting an exogenous "$[\cdots \mid \cdots, B]$" everywhere (below, the underlines just highlight the extra conditioning; they have no semantic significance):

$$P := \Pr[B \cap b \mid \underline{B}] = \Pr[b \mid B, \underline{B}] \cdot \Pr[B \mid \underline{B}] = \Pr[b \mid B] \cdot \Pr[B \mid B] = \frac{1}{2} \cdot 1 = \boxed{\frac{1}{2}}.$$



QED.

Scenario 2: *At least one* is a boy

$P = \frac{1}{3}$. Pf:

Let's write,

$$\Pr[(B \cap b) \cap (B \cup b)] = \Pr[B \cap b \mid B \cup b] \cdot \Pr[B \cup b].$$

Well, the first event is kind of redundant. So let's collapse it:

$$\Pr[\underline{B \cap b}] = \Pr[B \cap b \mid B \cup b] \cdot \Pr[B \cup b].$$

Now, we can substitute from above

$$\underline{\frac{1}{4}} = \Pr[B \cap b \mid B \cup b] \cdot \Pr[B \cup b].$$

Next, let's substitute

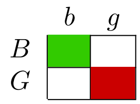
$$\frac{1}{4} = \Pr[B \cap b \mid B \cup b] \cdot \underline{g\left(\frac{1}{2}\right)} = \Pr[B \cap b \mid B \cup b] \cdot \left(2\frac{1}{2} - \left(\frac{1}{2}\right)^2\right)$$

$$= \Pr[B \cap b \mid B \cup b] \cdot \left(1 - \frac{1}{4}\right) = \Pr[B \cap b \mid B \cup b] \cdot \frac{3}{4},$$

finally rearranging to get

$$\frac{\frac{1}{4}}{\frac{3}{4}} = \Pr[B \cap b \mid B \cup b] =: P$$

$$= \boxed{\frac{1}{3}}.$$



QED.

Scenario 3: At least one is a boy born in the first half of his birthweek

This is sort of an intermediate scenario, just to help build up intuition for the canonical "Tuesday" scenario. We get $P = \frac{3}{7}$. Pf: For convenience, we're going to go about this one a bit circuitously. The information we pick up during the first pass will greatly ease our lives during the second pass.

We want to know the (conditional) probability that both her children are boys *given* that at least one child is a boy born in the first half of his birthweek. But let's first find the (unconditional, i.e. before the mother has revealed anything about her kids) probability that both children are boys *and* that at least one is a boy born in the first half of his birthweek. Now, if we just assume for a moment that both her kids *are* boys (bear with me), they're still equally likely to have been born on any permutation of halves of the week. So,

$$\Pr[(B \cap b) \cap (B_0 \cup b_0)] = \Pr[B_0 \cup b_0 \mid \underline{B \cap b}] \cdot \Pr[\underline{B \cap b}]$$

$$= g\left(\frac{1}{2}\right) \cdot \frac{1}{4} = \left(2 \cdot \frac{1}{2} - \left(\frac{1}{2}\right)^2\right) \cdot \frac{1}{4} = \left(\frac{2}{2} - \frac{1}{4}\right) \cdot \frac{1}{4} = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{4^2} = \frac{3}{2^4}.$$

Now our second pass, working backward to take advantage of Bayes's rule:

$$\frac{3}{2^4} = \Pr[(B \cap b) \cap (B_0 \cup b_0)] = \Pr[B \cap b \mid \underline{B_0 \cup b_0}] \cdot \Pr[\underline{B_0 \cup b_0}]$$

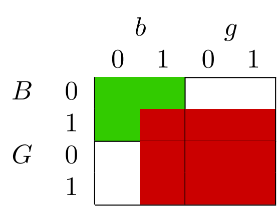
$$=: P \cdot \Pr[B_0 \cup b_0]$$

At which point which we must observe that, individually, the event that a specific child is a boy AND is born in the first half of his birthweek occurs independently and with probability $\frac{1}{2} \cdot \frac{1}{2}$. This lets us write

$$= P \cdot g\left(\frac{1}{2^2}\right) = P \cdot \left(2 \cdot \frac{1}{2^2} - \left(\frac{1}{2^2}\right)^2\right) = P \cdot \left(\frac{1}{2} - \frac{1}{2^4}\right) = P \cdot \frac{2^3 - 1}{2^4} = P \cdot \frac{7}{2^4}.$$

We collapse to get

$$\frac{3}{2^4} = P \cdot \frac{7}{2^4} \qquad \Longrightarrow \qquad P = \boxed{\frac{3}{7}}.$$



Wonderful. Seven valid cells, of which three represent success. QED *est demonstratum*, and in two different ways no less!

Scenario 4: At least one is a boy born on a Tuesday

$P = \frac{13}{27}$. Pf: We follow the template from Scenario 3, but adjust for the fact that we now consider <u>seven</u> equally-probable *days* of the week instead of <u>two</u> equally-probable *halves* of the week.

$$\Pr[(B \cap b) \cap (B_T \cup b_T)] = \Pr[B_T \cup b_T \mid B \cap b] \cdot \Pr[B \cap b]$$
$$= g\left(\frac{1}{7}\right) \cdot \frac{1}{4} = \left(2 \cdot \frac{1}{7} - \left(\frac{1}{7}\right)^2\right) \cdot \frac{1}{4} = \left(\frac{2}{7} - \frac{1}{49}\right) \cdot \frac{1}{4} = \left(\frac{14}{49} - \frac{1}{49}\right) \cdot \frac{1}{4} = \frac{13}{49} \cdot \frac{1}{4} = \frac{13}{2^2 \cdot 7^2}$$
$$= \Pr[(B \cap b) \cap (B_T \cup b_T)] = \Pr[B \cap b \mid B_T \cup b_T] \cdot \Pr[B_T \cup b_T]$$
$$=: P \cdot \Pr[B_T \cup b_T]$$
$$= P \cdot g\left(\frac{1}{2 \cdot 7}\right) = P \cdot \left(2 \cdot \frac{1}{2 \cdot 7} - \left(\frac{1}{2 \cdot 7}\right)^2\right) = P \cdot \left(\frac{1}{7} - \frac{1}{2^2 \cdot 7^2}\right) = P \cdot \frac{2^2 \cdot 7 - 1}{2^2 \cdot 7^2} = P \cdot \frac{27}{2^2 \cdot 7^2}.$$
$$\frac{13}{2^2 \cdot 7^2} = P \cdot \frac{27}{2^2 \cdot 7^2} \qquad \Longrightarrow \qquad P = \boxed{\frac{13}{27}}.$$

The table for this one is left as an exercise for the reader. QED :).

Commentary: What happened here?

It's natural that $P_1 = \frac{1}{2} > \frac{1}{4} = P_0$ (that is, P in Scenario 1 is bigger than P in Scenario 0), because in Scenario 1 we have done half the work by conditioning on the first-born being a boy. Now we just need to get the second-born to be a boy.

It's also natural that $P_2 = \frac{1}{3} > \frac{1}{4} = P_0$, as again we have done part of the work by ruling out the case where both children are girls.

But we also see that $P_2 = \frac{1}{3} < \frac{1}{2} = P_1$. Why? Well, think about it this way. If you know that the first-born is a boy, you've pinned down his gender, and now you just need to get the second-born to be a boy. But if you know only that *at least one of them* is a boy, you end up playing whack-a-mole: You want the first-born to be a boy, as before, but this time you can't just take this for granted, since it can slip out from under you and end up being that the first-born is a girl while the *second-born only* is a boy. So in some sense you're better off than in Scenario 0, since you can't have *both* children slip out from under you at once, but you're worse off than in Scenario 1, since one can still slip out from under you—just as in Scenario 1—but this time you don't know which one.

The table makes obvious that this is the classic story of distinguishable coin flips: Order matters, so even if you know that at least one is a boy, there's still only one way to get two boys but now two ways to get a mix.

Moving on, though, we begin conditioning on more identifying information than in Scenario 2, but still not pinning down exactly which child is the boy as in Scenario 1. And as we do so, P begins to climb upward from $P_2 = \frac{2}{3}$ and asymptotes at $P_1 = \frac{1}{2}$. Let's think through this by comparing the tables from Scenarios 2 and 3. We can imagine ourselves as detectives, ruling out possibilities. The possibilities get ruled out asymmetrically: As we get more identifying information, we start crossing out squares across the board, but we cross out *more* when one child is a girl than when both children are boys. In this way, statistics punishes both the two-boys and the boy-and-girl cases, but sneakily gives the boys the benefit of the doubt and punishes them less. We'll still never quite reach $\frac{1}{2}$—so the girls will always have the upper hand—but we can reduce their edge by using the same principle from above against them.

That is, as we went from Scenario 1 to Scenario 2, the girls gained an edge by giving themselves two ways to show up in the outcome while the boys got only one. But now the boys can reduce the girls' edge by giving themselves more ways to respect the conditions than the girls get. They must pigeonhole the girls while preserving their own wiggle room.

Said another way: As we place restrictions on events, they become less probable. So, for example, when we in Scenario 2 restrict the event $G \cap g$ entirely, its probability drops to zero. But that does *not* totally restrict the event $G \cup g$: There is still a nontrivial set of outcomes in the remainder, $(G \cup g) - (G \cap g)$. So, the boys need to somehow chip away at $(G \cup g) - (G \cap g)$. In fact, it's even okay if they end up chipping away at $B \cap b$ as they do so, as long as they chip away at $(G \cup g) - (G \cap g)$ faster.

The Great Game

And indeed, the boys *are* chipping away at $(G \cup g) - (G \cap g)$ faster than at $B \cap b$. You can see this in the table for Scenario 3—for instance, $G \cap b$ loses half its cells, and similarly for $B \cap g$, hence $(G \cup g) - (G \cap g)$ has lost half its probability mass, whereas $B \cap b$ loses only $\frac{1}{4}$—but let's try to get logical intuition too. Here's how I think about this: Once we know that at least one child is a boy, we can totally eliminate all four cells where both are girls. So this is already some asymmetric elimination. Now, you might say, "well, even then, there's a $\frac{2}{3}$ chance that at least one of the children is a girl – so how has this helped the boys?". But, that statement, while true, focuses on the wrong thing. For notice, that there's still a $\frac{2}{3}$ chance that exactly one of the children is a boy – but also a $\frac{1}{3}$ chance that *both* are boys, where the girls have no such opportunity at all.

So that's Scenario 2. But, once we know further that at least one child is a boy *born during the first half of his birthweek*, as in Scenario 3, something unique happens within each "block". To see this, consider the event that the first child is a girl. Then, the remaining child must not only be a boy, but also be born during the second half of his birth week. But, now consider what happens if the first child is a boy: The remaining child must be a boy—same as in the previous case—but now, he still has the freedom to have been born during the first half of his birthweek, because even if he isn't, *the first boy can pick up the slack*. So, if both children are boys, then the only cell eliminated is the cell where both are born during the second halves of their respective birthweeks. But if the first is a girl, then it doesn't matter when she was born: As soon as we find out that the second was born during the second half of his birthweek, that cell is dead.

This becomes even more extreme in Scenario 4: The moment you find out that the first child is a girl, you know the second must be a Tuesday boy. But if the first child is a boy, then the second still has a 1-out-of-7 chance—representing the probability that the first boy already took care of things by being born on a Tuesday—to be a non-Tuesday boy. It's a small absolute edge, but it's not nothing, and it becomes a progressively bigger *relative* edge—over the girls—as we make the identifying information more restrictive.

A bird's-eye view

In Scenario 2, we chip away asymmetrically at $G \cup g$ versus $B \cup b$: $G \cup g$ is eliminated entirely, whereas $B \cup b$ survives intact. And in Scenario 3, although we do end up chipping away at $B \cup b$, we lose only $\frac{5}{16}$ of its probability mass—and indeed only $\frac{1}{4}$ of $(B \cap b)$'s—whereas we lose fully $\frac{1}{2}$ of $(G \cup g)$'s already-smaller remaining mass. The coin-flip argument still applies: The girls have two "shots" at popping up—once as the first-born, once as the second-born—while the boys must nail both slots if they want to dominate. So, no matter how much identifying information we add, unless we are told unequivocally either that "the first-born is a boy for sure" or that "the second-born is a boy for sure", the probability that we turn up two boys will never reach $\frac{1}{2}$ as in Scenario 1. But, we *can* get it arbitrarily close.