Mean-variance-optimal hedge

Sparsh Sah

Background

We often want to run a mean-variance-optimal investment strategy "beta-neutrally" to some basket B, such as a global stock-market index or a passive commodities-sector index.

Clearly, an easy attempt at running such a strategy (in the absence of trading costs and/or other constraints), is to calculate the MVO portfolio in the absence of beta-neutrality constraint, calculate that portfolio's beta to the basket, then simply short out that much of the basket (e.g. by shorting a futures contract or an ETF). Because beta is additive, the resulting portfolio (which is the sum of the unconstrained MVO portfolio plus the hedge portfolio) has zero beta.

We will show that this easy approach is actually mean-variance-optimal.

Foreground

Our investable universe is N passive assets A_n, with $N \times N$ covariance matrix Σ and $N \times 1$ ER column vector μ.

We have a standard mean-variance utility function rewarding expected portfolio profit and penalizing ex-ante portfolio risk,

$$U(w) = w'\mu - \frac{\gamma}{2} w'\Sigma w,$$

where γ is some investor-specific risk-aversion parameter.

Finally, we have some basket a linear combination of the assets,

$$B := \ell_1 A_1 + \cdots + \ell_N A_N,$$

where each ℓ_n is the basket B's loading on the nth asset.

In fact, let us reframe the investable universe: Rolling the basket up into a single meta-asset will be convenient for us in the proof, but we can't just add that meta-asset to the investable universe because we will get perfect multicollinearity and therefore a non-invertible covariance matrix. So, arbitrarily choose some asset A_n for which $\ell_n \neq 0$, and drop it from the universe, replacing it with B. For instance, if we had

$$B := 0.40A_2 + 0.30A_4 - 0.50A_7,$$

we could arbitrarily choose to drop A_4 from the investable universe, replacing it with B. You can of course easily recover the dropped asset: In this case, you can get A_4 by going long 3.33 units of B, short 1.33 units of A_2, and long 1.67 units of A_7. Netting the positions, we get

$$\begin{aligned}
&3.33B - 1.33A_2 + 1.67A_7 \\
&=: 3.33(0.40A_2 + 0.30A_4 - 0.50A_7) - 1.33A_2 + 1.67A_7 \\
&= 1.33A_2 + A_4 - 1.67A_7 - 1.33A_2 + 1.67A_7 \\
&= A_4
\end{aligned}$$

as desired. So we haven't changed anything, we've just written it differently.

Out of all this pops an $N \times 1$ column vector β of each passive asset's beta to the basket, where as usual we define

$$\beta_n := \frac{\text{Cov}(A_n, B)}{\text{Var}(B)}.$$

It's not important for the proof, but just notice that for the asset we replaced (in our example above, it was $m = 4$), we will have

$$\beta_m = \frac{\text{Cov}(B, B)}{\text{Var}(B)} = \frac{\text{Var}(B)}{\text{Var}(B)} = 1.$$

PROOF

Harry Markowitz showed us that the unconstrained MVO portfolio is $u := (\gamma\Sigma)^{-1}\mu$.

So let's set up and solve for the beta-neutral MVO portfolio. We want

$$\max_w \left(w'\mu - \frac{\gamma}{2} w'\Sigma w \right) \qquad \text{s.t.} \qquad w'\beta = 0.$$

We reframe this as an unconstrained maximization problem with a Langrangian term, and solve the first-order condition (gradient is zero):

$$\begin{aligned} \mathcal{L} &:= \left(w'\mu - \frac{\gamma}{2} w'\Sigma w \right) - \lambda w'\beta \\ \nabla_w \mathcal{L} &= \mu - \gamma\Sigma w - \lambda\beta \\ \implies \qquad 0 &= \mu - \gamma\Sigma w - \lambda\beta \\ w &= (\gamma\Sigma)^{-1}(\mu - \lambda\beta) \\ &= (\gamma\Sigma)^{-1}\mu - \lambda(\gamma\Sigma)^{-1}\beta \\ &=: u - \boxed{\lambda(\gamma\Sigma)^{-1}\beta}. \end{aligned}$$

So the optimal portfolio is the uncon MVO portfolio u, minus a hedge portfolio which we'll call H.

Now, remembering the formula at the end of the foreground, we write the vector β as

$$\beta := \frac{1}{\text{Var}(B)} \Sigma_{:,m},$$

where $\Sigma_{:,m}$ is suggestive notation for column m of Σ. So, we have

$$H = \lambda(\gamma \text{Var}(B)\Sigma)^{-1}\Sigma_{:,m},$$

and getting rid of scalars,

$$H \propto \Sigma^{-1}\Sigma_{:,m}.$$

Observant students of matrix multiplication will recognize the RHS as the mth column of $\Sigma^{-1}\Sigma$, which, Σ being a symmetric real matrix, is just the identity I. And the mth column of I is a vector that's zero everywhere except index m, where it is one. Hence, we know that H must also be zero everywhere except at index m. So, the hedge portfolio contains only the basket asset.

Once we know this, there's no reason to solve the Lagrangian any further. The hedge portfolio contains only the basket asset, whose beta is one, and the final portfolio has beta zero, which means that the hedge portfolio must have beta that exactly offsets the beta of the rest of the portfolio, which is the uncon MVO portfolio[1]. Which is all to say, indeed, you can construct the constrained MVO portfolio by calculating the beta of the uncon MVO portfolio and shorting that much of the basket asset, QED.

[1] Rather, it is the uncon MVO portfolio minus the weight on B used to reconstruct A when we dropped it earlier. So you'll have to keep in mind that the final position in B is the sum of the position used to reconstruct our uncon MVO position in A, plus whatever position is needed to beta-hedge that uncon MVO portfolio.