univ-of-unif

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1 The Universality of the Uniform for Generating Random Samples

An Exponential r.v. X with rate parameter λ (hence mean $1/\lambda$) has CDF $\Pr[X \leq x] = 1 - \exp(-\lambda x)$. Here, $\Pr[X \leq x] \in [0, 1]$ represents the probability that X crystallizes below or at x. [1], [2]

By the marvelous Universality of the Uniform, this means we can simulate a random draw of X by plugging a Standard Uniform r.v. U into the inverse of the CDF, i.e. $-\ln(1-U)/\lambda$! (That's an exclamation mark, not a factorial sign. This is exciting stuff.) [3], [4]

[1]:
```
from typing import List
import numpy as np
import matplotlib.pyplot as plt
```

[1]By the way, every Exponential distribution is memoryless i.e. $E[X - t \mid X > t] = E[X]$. If you interpret X as a waiting time, this property says that the amount of *additional* time you can expect to wait before X hits, given that you've already waited for t minutes and X hasn't hit yet, is exactly the same as the amount of *total* time you had expected to wait when you originally started waiting. X doesn't "care" that you've already been waiting for t minutes: It's memoryless. Another way to write this is $E[X \mid X > t] = E[X] + t$: The amount of total time you can expect to wait before X hits, given that you've already waited for t minutes, is the same as the amount of total time you had expected to wait originally, plus t minutes. In fact, Exponential distributions are the *only* memoryless continuous distributions. (In discrete time, every Geometric distribution is memoryless, and in fact Geometric distributions are the *only* memoryless distributions.)

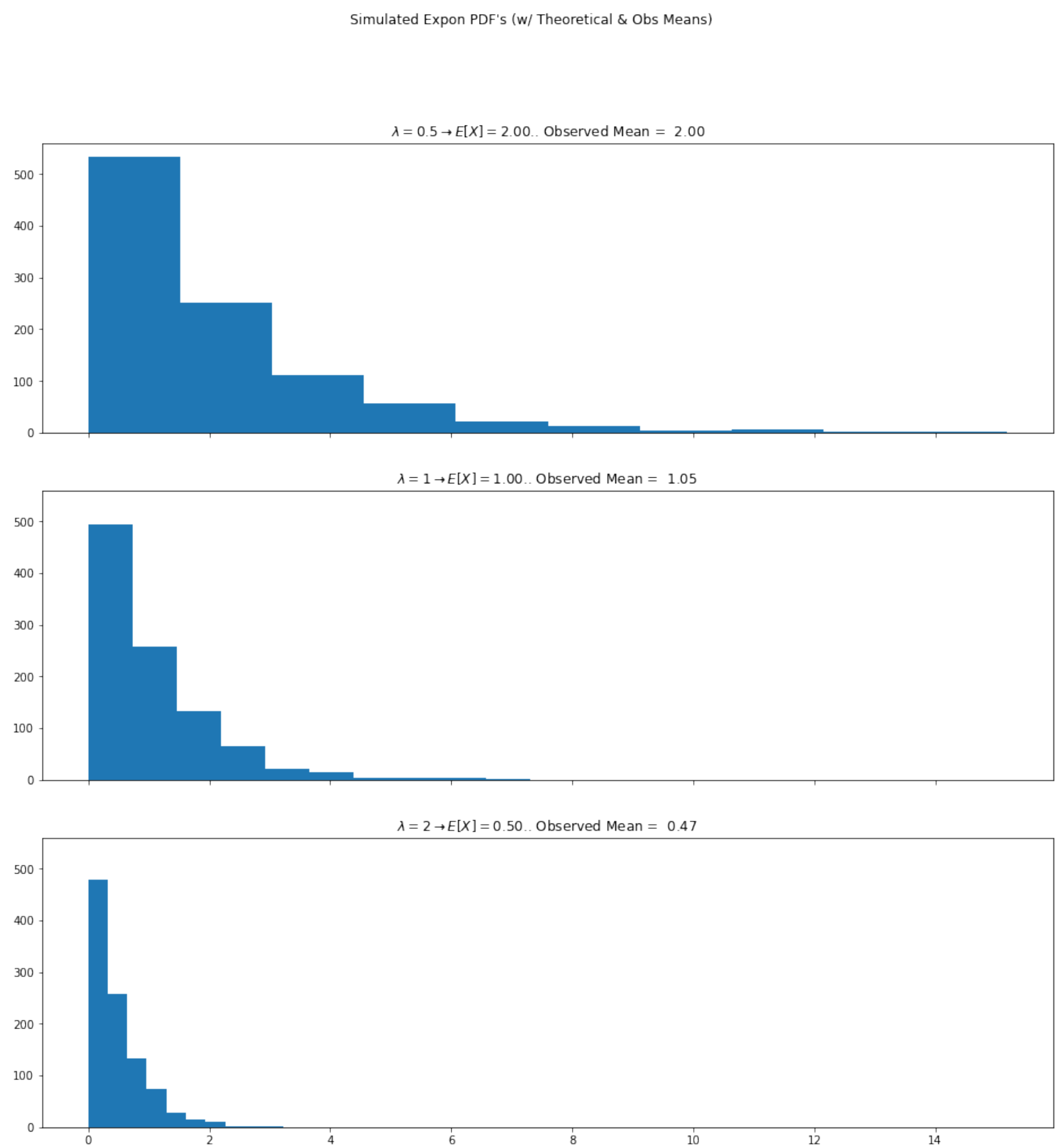
[2]Exponential distributions are also the only possible waiting-time distributions for Poisson counting processes. A Poisson counting process with rate parameter λ is a continuous-time stochastic process $N(t)$ over $t \geq 0$ characterized by the following four properties: 0. $N(t) = 0$; 1. Independent increments; 2. Stationary increments; 3. $N(t)$ follows a Poisson distribution with rate parameter λt. (In discrete time, Geometric distributions are the only possible waiting-time distributions for Binomial counting processes.)

[3]We could also use the identically-distributed but simpler form $-\ln(U)/\lambda$. However, this would mean that $U = 0$ maps to $X = +\infty$, whereas I prefer $U = 1$ mapping to $X = +\infty$, since according to the CDF, $x = +\infty$ maps to $Pr[X \leq x] = 1$.

[4]This is actually only one direction of the Universality of the Uniform, in particular the direction that says that for a random variable X with CDF F, we have $F^{-1}(U) \cong X$ where $\cong$ means "is identically distributed to". The other direction says that $F(X) \cong U$, an equally interesting but less useful (to us) result. You can use that other direction to prove that p-values are standard Uniform under the null hypothesis. To wit: Before you conduct your experiment, and assuming that the null hypothesis H_0 is true, you can view the final test statistic as a random variable S. The final p-value is defined as $p = \Pr[T \geq S \mid S, H_0]$, where T i.i.d. S. Hence, assuming you have a continuous test statistic, $p = 1 - Pr[T \leq S \mid S, H_0]$, whence $p = 1 - F_T(S)$ where F_T is the CDF of T under the null hypothesis. But since T i.i.d. S, we can write $p = 1 - F_S(S)$ which we now know is identically distributed to $1 - U$. Then we need only remember that for a standard Uniform U, $1 - U \cong U$, and we have QED.

[2]:
```python
def get_expon_rv(unif_rv: float, lambda_: float=1) -> float:
    return -np.log(1 - unif_rv) / lambda_


def sim_expon_rvs(lambda_: float=1, n: int=1_000) -> List[float]:
    return [get_expon_rv(unif_rv=u, lambda_=lambda_) for u in np.random.
 ↪random(size=n)]


def plot_sim(ax: object, lambda_: float=1) -> object:
    expected_mean = 1 / lambda_

    x = sim_expon_rvs(lambda_=lambda_)
    obs_mean = np.mean(x)

    ax.hist(x)
    ax.set_title(r"$\lambda = {lambda_} \rightarrow E[X] = {e: .2f}$.. Observed␣
 ↪Mean = {obs: .2f}".format(
        lambda_=lambda_, e=expected_mean, obs=obs_mean))
    return ax


def plot_sims(lambdas: List[float]):
    _, axs = plt.subplots(nrows=len(lambdas), ncols=1, sharex=True,␣
 ↪sharey=True, figsize=(16, 16))
    for i, lambda_ in enumerate(lambdas):
        plot_sim(ax=axs[i], lambda_=lambda_)
    plt.suptitle("Simulated Expon PDF's (w/ Theoretical & Obs Means)")
    plt.show()
```

[3]:
```python
np.random.seed(1337)
plot_sims(lambdas=[0.5, 1, 2])
```


Simulated Expon PDF's (w/ Theoretical & Obs Means)
$\lambda = 0.5 \rightarrow E[X] = 2.00$.. Observed Mean = 2.00
$\lambda = 1 \rightarrow E[X] = 1.00$.. Observed Mean = 1.05
$\lambda = 2 \rightarrow E[X] = 0.50$.. Observed Mean = 0.47
0
100
200
300
400
500
0
2
4
6
8
10
12
14