Sharpe-maximizing time-varying risk-taking

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Setting

Over T timesteps, we have a (random, real) T-vector of single-asset (nothing prevents this from being a strategy asset) returns,

$$R := (R_1,\, \dots,\, R_T)'.$$

We also have a (random, real) T-vector of signals,

$$S := (S_1,\, \dots,\, S_T)'.$$

Each

$$R_t = \mu + \rho S_t + \lambda \varepsilon_t \qquad \implies \qquad R_t \mid S_t \sim \mathcal{N}\left(\mu + \rho S_t,\, \lambda\right)$$

for given $\mu \geq 0$ and $0 \leq \rho < 1$ (with $\lambda := \sqrt{1-\rho^2}$ of standard Normal white noise ε_t), and the R_t's are independent. The S_t's are also i.i.d. standard Normal white noise. Notice that $\rho = \mathrm{Corr}(R_t,\, S_t)$.

Mean-variance-optimal allocation (given signals)

Any ex-ante MVO allocation to the R_t's (assuming a fixed policy, that is, assuming you must set your weights at $t = 0$, so that this T-period single-asset problem becomes isomorphic to a single-period T-asset problem [0]) must be proportional to their ex-ante Sharpes, so that the ex-ante MVO allocation given S (since each R_t has the same standard deviation) can be taken as

$$w_t^* \mid S = \mu + \rho S_t.$$

The aggregate ex-ante ER of this allocation (paying close attention to which variables are constants given S) is

$$\mathbb{E}\left[\sum_t w_t^* R_t \mid S\right] = \sum_t \mathbb{E}[w_t^* R_t \mid S]$$

$$= \sum_t \mathbb{E}\left[(\mu + \rho S_t)(\mu + \rho S_t + \lambda\varepsilon_t) \mid S\right] = \sum_t \mathbb{E}\left[(\mu + \rho S_t)(\mu + \rho S_t) + (\mu + \rho S_t)\lambda\varepsilon_t \mid S\right]$$

$$= \sum_t \left((\mu + \rho S_t)(\mu + \rho S_t) + \mathbb{E}\left[(\mu + \rho S_t)\lambda\varepsilon_t \mid S\right]\right) = \sum_t \left((\mu + \rho S_t)(\mu + \rho S_t) + (\mu + \rho S_t)\lambda\mathbb{E}[\varepsilon_t \mid S]\right)$$

$$= \sum_t \left((\mu + \rho S_t)(\mu + \rho S_t) + (\mu + \rho S_t)\lambda(0)\right) = \sum_t (\mu + \rho S_t)(\mu + \rho S_t) = \sum_t (\mu + \rho S_t)^2.$$

The aggregate ex-ante variance ($\implies$ aggregate ex-ante volatility) of this allocation (noting bilinearity of covariance) is

$$\sum_t {w_t^*}^2 \lambda^2 = \sum_t (\mu + \rho S_t)^2 \lambda^2 \qquad \implies \qquad \lambda\sqrt{\sum_t (\mu + \rho S_t)^2}.$$

The aggregate ex-ante Sharpe of the MVO allocation, then, is

$$\frac{1}{\lambda}\sqrt{\sum_t (\mu + \rho S_t)^2},$$

which we will for convenience write as

$$\frac{\sum_t (\mu + \rho S_t)(\mu + \rho S_t)}{\lambda\sqrt{\sum_t (\mu + \rho S_t)^2}} = \frac{\sum_t \left(\mu^2 + \mu\rho S_t + \mu\rho S_t + \rho^2 S_t^2\right)}{\lambda\sqrt{\sum_t \left(\mu^2 + \mu\rho S_t + \mu\rho S_t + \rho^2 S_t^2\right)}} = \frac{\sum_t \left(\mu^2 + \mu\rho S_t + \mu\rho S_t + \rho^2 S_t^2\right)}{\lambda\sqrt{\sum_t \left(\mu^2 + \mu\rho S_t + \mu\rho S_t + \rho^2 S_t^2\right)}}$$

$$= \frac{\sum_t \mu^2 + \sum_t \mu\rho S_t + \sum_t \mu\rho S_t + \sum_t \rho^2 S_t^2}{\lambda\sqrt{\sum_t \mu^2 + \sum_t \mu\rho S_t + \sum_t \mu\rho S_t + \sum_t \rho^2 S_t^2}} = \boxed{\frac{T\mu^2 + \mu\rho\sum_t S_t + \mu\rho\sum_t S_t + \rho\rho\sum_t S_t^2}{\lambda\sqrt{T\mu^2 + \mu\rho\sum_t S_t + \mu\rho\sum_t S_t + \rho^2\sum_t S_t^2}}}.$$

[0] I haven't studied whether the "unconstrained" multi-period problem has a different solution, e.g. whether you should lever up your future bets in an attempt to pull yourself out of the hole if you are allowed to observe past P&L and observe net losses.

Alternative allocation (again, given signals)

Consider now the alternative allocation

$$w_t \mid S := \mu + cS_t,$$

with $c > \rho$. (Notice that by applying

$$0 < x := 1 - \frac{\rho}{c} \leq 1$$

shrinkage toward μ—that is, putting x weight on μ and $1-x$ weight on w—we could recover w^*.)

The alternative allocation's aggregate ex-ante ER is

$$\sum_t (\mu + cS_t)(\mu + \rho S_t),$$

while its aggregate ex-ante variance ($\implies$ ex-ante volatility) is

$$\sum_t {w_t}^2 \lambda^2 = \sum_t (\mu + cS_t)^2 \lambda^2 \qquad \implies \qquad \lambda \sqrt{\sum_t (\mu + cS_t)^2}.$$

Its aggregate ex-ante Sharpe, then, is

$$\frac{\sum_t (\mu + cS_t)(\mu + \rho S_t)}{\lambda \sqrt{\sum_t (\mu + cS_t)^2}} = \frac{\sum_t \left(\mu^2 + \mu\rho S_t + \mu c S_t + \rho c S_t^2\right)}{\lambda \sqrt{\sum_t \left(\mu^2 + \mu c S_t + \mu c S_t + c^2 S_t^2\right)}} = \boxed{\frac{T\mu^2 + \mu\rho \sum_t S_t + \mu c \sum_t S_t + \rho c \sum_t S_t^2}{\lambda \sqrt{T\mu^2 + \mu c \sum_t S_t + \mu c \sum_t S_t + c^2 \sum_t S_t^2}}}.$$

Ex-ante Sharpe comparison (still given signals)

We can drop the $\frac{1}{\lambda}$ constant factor when comparing the two ex-ante Sharpes. Furthermore, although we still condition on S, we will let $T \to \infty$ so we can appeal to the Law of Large Numbers to assert that $\sum_t S_t := 0$ and $\sum_t S_t^2 := T$. So, we are comparing the MVO ex-ante Sharpe

$$\frac{T\mu^2 + \rho\rho T}{\sqrt{T\mu^2 + \rho^2 T}} \qquad \propto_{\sqrt{T}} \qquad \frac{\mu^2 + \rho\rho}{\sqrt{\mu^2 + \rho^2}}$$

to the alternative ex-ante Sharpe which is similarly $\propto_{\sqrt{T}}$

$$\frac{\mu^2 + \rho c}{\sqrt{\mu^2 + c^2}}.$$

Now: $\rho \geq 0$ and $c > \rho$, so both the numerator and denominator "look" bigger for the alternative allocation. Which one wins out? Well, let's consider for $h \in [\rho, \infty)$

$$\operatorname{sign}\left(\frac{\partial}{\partial h} \frac{\mu^2 + \rho h}{\sqrt{\mu^2 + h^2}}\right) \quad =^{1} \quad \operatorname{sign}\left(-\frac{\mu^2 (h - \rho)}{(h^2 + \mu^2)^{\frac{3}{2}}}\right) = -\operatorname{sign}\left(\frac{\mu^2 (h - \rho)}{(h^2 + \mu^2)^{\frac{3}{2}}}\right) \cong -\operatorname{sign}\left(\frac{[+][+]}{[+]^{\frac{3}{2}}}\right) = -[+] = [-].$$

The value is strictly decreasing in h: Thus, we know that the alternative is worse than the MVO.

[1] Query Uncle WolframAlpha for `partial derivative wrt h of (m^2 + r * h) / sqrt(m^2 + h^2)` .