Flight Path Angle Reversal of an Air-to-Air Missile in Minimum Time Using Pseudo-spectral Method

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Abstract: The pseudo-spectral (PS) method has been applied in this paper to design an optimal guidance strategy for an air-to-air missile to completely reverse its flight path angle in minimum time, while simultaneously assuring a designated state constraint (Mach number) at the end of the turning. The missile has to reverse its direction as quickly as possible so as to intercept the target in the rear hemisphere. The PS method is found to give good results while steering the vehicle in the reverse direction and almost exactly satisfying the final Mach number constraint. The key idea of PS method is to discretize the optimal control problem and solve for the resulting low-dimensional nonlinear programming problem in real time. PS discretization demands a relatively lower number of grid points which thereby reduces the computational time. Both Chebyshev and Legendre polynomial approximation have been successfully used to solve the problem and the results from both are consistent.

Keywords: Pseudospectral discretization; minimum time optimal control; Chebyshev and Legendre approximation.

1. INTRODUCTION

Pseudo-spectral methods are coming up in the guidance and control literature due to its wide applicability in solving partial differential equations. Moreover, now PS methods are also being extensively used to solve trajectory optimization problems in real time and hence using them for flight vehicle guidance applications. The key idea of pseudo-spectral trajectory optimization is to discretize the optimal control problem and solve for the resulting finite dimensional nonlinear programming problem (which fall under the the class of direct transcription methods).

Recently, it has also been shown that pseudo-spectral methods have better convergence properties. One can see literature for a detailed analysis of Chebyshev PS method for trajectory optimization(Fahroo and Ross [2002]). (Kang et al. [2007]) discusses the existence of feasible solution for discretized constraints around the continuous-time problem and also the convergence study of pseudo-spectral method. The developments in the algorithms and the convergence of the PS discretization have led to its wide applicability in solving highly complex nonlinear optimal control problems in real time.

This paper also discusses the application of the PS method for reversal of an air-to-air missile in minimum time, while assuring a desirable state constraint (Mach number) at the end of turning. A possible application would be to launch a missile from an aircraft in the forward direction, but then quickly turning it (in minimum time) to intercept a target in the rear hemisphere. This problem is inspired from Han and Balakrishnan [2002], where the authors have first converted the difficult free final time problem to a fixed final-variable problem (considering flight path angle as free variable) and then solving it using adaptive critic neural networks to propose an envelope of solutions. However, for successful synthesis (training) of adaptive critic networks, one should know the domain of states at every point in time, which can be difficult to guess prior to the control design. Moreover, for final time problems, a set of networks are needed at every grid point (Han and Balakrishnan [2002]) and hence there are issues like continuity and smoothness of control solution. On the other hand, the PS method is free from any such bottlenecks as the trajectory optimization problem is directly solved online.

The results obtained are quite promising. Both Chebyshev and Legendre polynomial approximation have been successfully used to solve the problem and the results from both are consistent.

2. PSEUDO-SPECTRAL METHOD: AN OVERVIEW

A generic overview of the pseudo-spectral method is given in this section. Consider a state and control constraint nonlinear optimization problem.

$$J = E(x(t_0), x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt \qquad (1)$$

subject to the following system dynamics

$$\dot{x}(t) = f(x(t), u(t), t) \qquad (2)$$

and that meets the end point conditions

$$e(x(t_0), x(t_f)) = 0 \qquad (3)$$

as well as the path constraints

$$h(x(t), u(t)) \leq 0 \tag{4}$$

where $x \in \Re^n$, $u \in \Re^m$, and $L : (\Re^n \times \Re^m \times \Re) \rightarrow \Re, f : (\Re^n \times \Re^m \times \Re) \rightarrow \Re^n, e$ and h are Lipschitz continuous over the domain with respect to their arguments(Fahroo and Ross [2002]).

The basic idea of solving the nonlinear optimal control problem using PS method is to discretize the states (and/or the control) using pseudo-spectral polynomial approximation and convert the problem to a finite dimensional nonlinear programming (NLP) problem. For this approximate $x(t)$ by an N^{th} order polynomial function as $x^N(t)$. This can be thought of as a function x^N, which satisfies the boundary condition and makes the residual, R given by, $\|R\| = \|\dot{x}^N - f\|$ small.

An approximation method for solving differential equations is the Weighted Residual Method. For this we approximate x with x^N by expanding with a linear combination of certain trial functions. The Chebyshev polynomials, Legendre polynomials, Fourier series etc can be selected as *trial functions*. Now we need to select a set of *test functions* $(\chi_0, \chi_1, ..., \chi_N)$ and define the smallness of the residual R, using the Hilbert space scalar product given by, $< \chi_n, R >= 0$, $\forall n \in \{0, 1, 2, 3, ...\}$.

There are various approaches to make R small. For the pseudospectral method, we choose a grid of collocation points (grid/nodes points) $t_k, k = 0, ..., N$, which satisfies the function exactly at these points.

The grids can be selected based upon quadrature rules. We can either go for uniform grid, Gauss, Gauss-Radau or Gauss Lobotto grids. The advantage of the above grids over uniform grid is that they have the distribution property that they cluster around the endpoints of the interval. The Gauss-Lobotto points are found to have the least error. The Chebyshev Gauss Lobotto points used in this paper are the roots of the Chebyshev differential equation discussed below.

The derivative of these interpolating polynomial at these node points are given exactly by the differentiation matrix. For the problem to be discussed below, both the Chebyshev and Legendre polynomial approximations are considered for comparison. The grid point distribution are selected based on the Chebyshev-Guass-Lobotto (CGL) (Fahroo and Ross [2002]) and Legendre-Guass-Lobotto (LGL) (Kang et al. [2007]) points for the two methods. Approximation of the states using Chebyshev and Legendre polynomial approximation are discussed below.

Chebyshev approximation Chebyshev polynomials are important in approximation theory because the roots of the Chebyshev polynomials of the first kind, which are also called Chebyshev nodes, are used as nodes in polynomial interpolation.

For the above problem Chebyshev polynomials of the first kind are considered. The CGL interpolation points are given in closed form as,

$$t_n = -cos(\frac{n\pi}{N}), \quad n = 0, ..., N \tag{5}$$

The points lying in the interval [-1,1] are the extrema of the N^{th} order Chebyshev polynomial $T_N(t)$, defined as

$$T_k(t) = cos(kcos^{-1}t), \quad k = 0, ..., N \tag{6}$$

The $(k+1)^{th}$ Chebyshev polynomials can be obtained by the recursive relation,

$$T_{k+1}(t) = 2tT_k(t) - 2T_{k-1}, \quad k = 1, ..., N-1 \tag{7}$$

For approximating the continuous equations, we consider the polynomial approximations of the form,

$$x^N(t) = \sum_{k=0}^{N} a_k T_k(t), \quad k = 0, ..., N \tag{8}$$

Taking differential of the state equation (8) we have,

$$\dot{x}^N(t) = \sum_{k=0}^{N} a_k \dot{T}_k(t), \quad k = 0, ..., N \tag{9}$$

Equation (8) can be represented in matrix form as $X = TA$, where the elements of $X \in \Re^{N+1}$ are the interpolation points and $A \in \Re^{N+1}$ is the coefficient vector, and $T \in \Re^{(N+1)\times(N+1)}$ form the interpolation polynomial matrix at the N+1 CGL nodes.

Legendre approximation The Legendre method uses Lagrange polynomials for the approximations as given in (Kang et al. [2007]), and Legendre Gauss Lobatto (LGL) points for the orthogonal collocation. The LGL nodes, $t_n, \quad i = 0, ..., N$ are distributed in the interval [-1,1] and $t_n, \quad n = 1, ..., N-1$ are the zeroes of $\dot{P}_n(t)$, the derivative of the N^{th} order Legendre polynomial, $P_n(t)$ (Kang et al. [2007]).

The polynomial approximations for the state at the LGL nodes are:

$$x^N(t) = \sum_{k=0}^{N} x(t_k)\phi_k(t), \quad k = 0, ..., N \tag{10}$$

where, $\phi_k(t)$ are the Lagrange polynomials of order N defined as (Fahroo and Ross [2001]),

$$\phi(t) = \frac{1}{N(N+1)P_n(t_i)} \frac{(t^2-1)\dot{P}_n(t)}{t - t_i} \tag{11}$$

Now, the derivative of the states are given as,

$$\dot{x}^N(t_k) = \sum_{l=0}^{N} x(t_l)\dot{\phi}_l(t_k) = \sum_{l=0}^{N} D_{kl}x(t_l) \tag{12}$$

where $D_{kl} = \frac{d\phi_l}{dt}(t_k)$ are the elements of the $(N+1) \times (N+1)$ first order differentiation matrix D (Kang et al. [2007]),

$$D = [D_{kl}] = \begin{cases} \dfrac{P_n(t_k)}{P_n(t_l)} \dfrac{1}{t_k - t_l} & k \neq l \\ \dfrac{N(N+1)}{4} & k = l = 0 \\ \dfrac{N(N+1)}{4} & k = l = N \\ 0 & \text{otherwise} \end{cases}$$

2.1 *Approximating the integral equation*

The integral term in the performance measure of (1) can be approximated using quadrature rules (Delves and Mohamed [1985]). A quadrature rule is an approximation of the definite integral of a function, which is the weighted sum of the function values at specified points within the domain of integration. Approximating the integral equation by usual discretization requires,$\Delta\ t$ small enough for a good approximation. This will demand a higher number of grids which will be computationally inefficient. Thus,

$$J^N = E(\hat{x}(t_f), u(t_f)) + \frac{t_f - t_0}{2} \sum_{k=0}^{N} w_k L(\hat{x}(t_k), u(t_k)) \quad (13)$$

where $w_k, k = 0, ..., N$ are the corresponding weighting functions.

2.2 *The nonlinear programming problem*

Thus the optimal control problem given by (1)-(4) is now reduced to a finite dimensional nonlinear programming problem given by,

Minimize,

$$J^N = E(\hat{x}(t_f), u(t_f)) + \frac{t_f - t_0}{2} \sum_{k=0}^{N} w_k L(\hat{x}(t_k), u(t_k))$$

Subject to

$$\sum_{l=0}^{N} D_{kl} x(t_l) = f(\hat{x}(t_k), \hat{u}(t_k), t_k)$$

with endpoint conditions,

$$e(\hat{x}(t_0), \hat{x}(t_f)) = 0$$

and path constraints,

$$h(\hat{x}(t_n), \hat{u}(t_n)) \leq 0 \quad 0 \leq n \leq N \quad (14)$$

There are many techniques available in literature for solving the NLP problem. For example the Kuhn Tucker conditions can be applied for the above problem for a solution. For the solution search of the problem discussed here, the fmincon function of the optimization toolbox of Matlab was used to solve the nonlinear programming problem stated above.

3. OPTIMAL TURNING OF AN AIR-TO-AIR MISSILE

3.1 *System Dynamics*

The problem has been inspired from (Han and Balakrishnan [2002]), which has been modified by including the shear angle (angle between thrust and velocity vector) as a guidance parameter in addition to the angle of attack (which is the angle between the body x-axis and velocity vector). This is done with the assumption of having thrust vector control as well to minimize the load on the angle of attack (and hence to restrict it within some meaningful bound). Figure 1 shows the schematic diagram of the missile in the vertical plane. The non-dimensional point mass equations of motion for the air-to-air missile in a vertical plane is given by

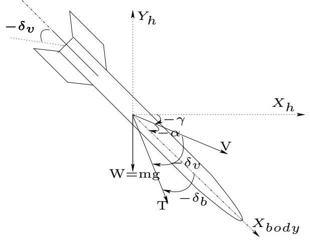

Fig. 1. Schematic diagram

$$M'(\tau) = -S_w M^2 C_D - sin\gamma + T_w cos(\alpha + \delta_b) \quad (15)$$

$$\gamma'(\tau) = \frac{1}{M}(S_w M^2 C_L + T_w sin(\alpha + \delta_b) - cos\gamma) \quad (16)$$

$$h'(\tau) = \frac{a^2 M}{g} sin\gamma; \quad x'(\tau) = \frac{a^2 M}{g} cos\gamma \quad (17)$$

where prime denotes differentiation with respect to the non dimensional time $\tau = \frac{g}{at}$ with initial Mach and flight path angle $M(\tau_0) = M_0$, $\gamma(\tau_0) = \gamma_0$. The non dimensional parameters are defined as follows:

$$\tau = \frac{g}{at}; \quad T_w = \frac{T}{mg}; \quad S_w = \frac{\rho a^2 S}{2mg}; M = \frac{V}{a} \quad (18)$$

Here M is the mach number,γ is the flight path angle, h is the vertical height, x is the horizontal range, a is the speed of sound, α is the angle of attack (AOA), g is the acceleration due to gravity, δ_b is the thrust vector angle which is the angle measured from the body X-axis direction to the thrust vector direction. The shear angle δ_v, as measured from the velocity vector direction to the thrust vector direction is given by, $\delta_v = \alpha + \delta_b$. C_L and C_D are usually functions of a and M obtained from the tabulated data.

3.2 *Cost Function and Boundary Condition*

We need to steer the missile completely in the reverse direction in minimum time with a desired final Mach number. Thus the cost function to minimize is

$$J = \int_0^{t_f} dt \quad (19)$$

subjected to the flight path constraints,$\gamma(0) = 0$ and $\gamma(t_f) = -180^0$. $M(t_0)$ is given. The desired final Mach number $M(t_f)$ is taken as 0.8. These form the hard constraint of the problem.

3.3 *Problem Formulation*

The above minimum time problem is difficult to solve and therefore getting inspired from the idea in Han and Balakrishnan [2002], it is converted into a tractable problem by taking γ as the independent variable instead. The independent variable, γ is chosen because it is expected to be a monotonic continuous function with respect to time. With this change of variable, the equations of motion are reformulated as follows:

$$\frac{dM}{d\gamma} = \frac{M'}{\gamma'} = \frac{M(-S_w M^2 C_D - sin\gamma + T_w cos(\alpha+\delta_b))}{(S_w M^2 C_L + T_w sin(\alpha+\delta_b) - cos\gamma)}$$
$$\frac{dt}{d\gamma} = \frac{1}{\gamma'} = \frac{aM}{g(S_w M^2 C_L + T_w sin(\alpha+\delta_b) - cos\gamma)}$$
$$\frac{dh}{d\gamma} = \frac{h'}{\gamma'} = \frac{a^2 M^2 sin\gamma}{g(S_w M^2 C_L + T_w sin(\alpha+\delta_b) - cos\gamma)}$$
$$\frac{dx}{d\gamma} = \frac{x'}{\gamma'} = \frac{a^2 M^2 cos\gamma}{g(S_w M^2 C_L + T_w sin(\alpha+\delta_b) - cos\gamma)} \tag{20}$$

Next, the cost function is equivalently transformed as

$$J = \int_{\gamma_0}^{\gamma_f} \frac{dt}{d\gamma} d\gamma = \int_0^{-\pi} \frac{aM}{g(S_w M^2 C_L + T_w sin(\alpha+\delta_b) - cos\gamma)} d\gamma$$

with limits on the flight path angle, γ. Thus a difficult minimum time problem has been converted to a relatively easier fixed final end point problem with the hard constraint, $M(\gamma_f) = 0.8$.

3.4 Guidance formulation using Chebyshev and Legendre polynomial approximation

The Chebyshev and Legendre polynomials discussed above are defined for the interval [-1,1]. For the problem specified in the interval $[\gamma_0, \gamma_f]$, the polynomials are shifted to the corresponding interval (Ross et al. [2009]). Thus we have the corresponding shifted polynomials and the shifted node points gives as,

$$\gamma_i = \frac{(\gamma_f - \gamma_0)\gamma + (\gamma_0 + \gamma_f)}{2} \tag{21}$$

where γ are the nodes corresponding to the interval [-1,1]. Here $\gamma_0 = 0$ and $\gamma_f = -180^0$

The dynamic equations given by equation (20) are discretized by imposing the condition that the derivatives of the state approximation satisfy the differential equation exactly at the node points. The Mach number, M and the control, α and δ are approximated using Chebyshev and Legendre approximations (see equations (8) and (10). The modified cost function J is discretized using Gauss quadrature rules as given by equation (13).

3.5 The nonlinear programming problem statement

Now we can formulate the discretized problem statement as follows

Minimize,

$$J^N = \sum_{k=0}^{N} w_k \frac{aM(\gamma_k)}{g(S_w M(\gamma_k)^2 C_L + T_w sin(\delta_v(\gamma_k)) - cos(\gamma_k))}$$

subject to the state constraints

For Chebyshev approximation

$$\sum_{k=0}^{N} a_k \dot{T}_k(\gamma_k) = \frac{(-S_w M(\gamma_k)^2 C_D - sin(\gamma_k) + T_w cos(\delta_v(\gamma_k))M(\gamma_k)}{S_w M(\gamma_k)^2 C_L + T_w sin(\delta_{vk}) - cos(\gamma_k)}$$

For Legendre approximation

$$\sum_{l=0}^{N} D_{kl} M(\gamma_l) = \frac{(-S_w M(\gamma_k)^2 C_D - sin(\gamma_k) + T_w cos(\delta_{vk}))M(\gamma_k)}{S_w M(\gamma_k)^2 C_L + T_w sin(\delta_{vk}) - cos(\gamma_k)}$$

and the endpoints conditions

$$e_0(x_0) = M(\gamma_0) = M_0, \text{where} \quad \mathrm{M}_0 \in [0.3, 0.8]$$
$$e_f(x_f) = M(\gamma_f) - 0.8 = 0 \tag{22}$$

4. RESULTS AND DISCUSSIONS

Initially the problem was studied without considering the shear angle control as in (Han and Balakrishnan [2002]). Similar results are obtained as in the paper, which has a high value of angle of attack (AOA). Figure 3 shows that the AOA is not within the admissible region as the values of the coefficient of lift, C_L are available only in the range $\alpha \in [-20^0, 20^0]$ and the the results obtained are not feasible. Figures 2 and 3 were plotted using extrapolated C_L values and without shear angle control.

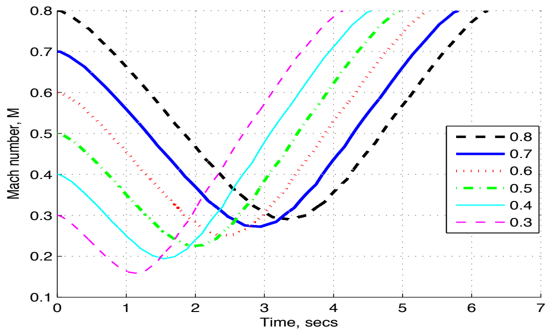

Fig. 2. Variation of Mach number with γ for $M_0 = 0.3$ to 0.8 without shear angle control

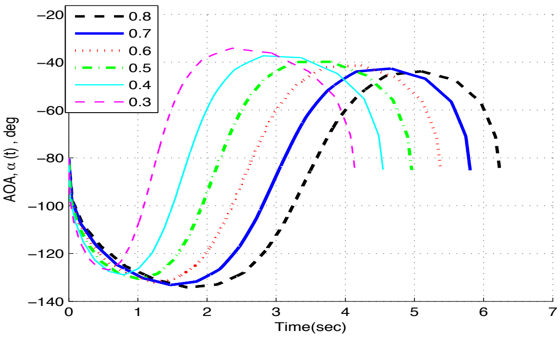

Fig. 3. Variation of AOA with γ for $M_0 = 0.3$ to 0.8 without shear angle control

The rest of the figures are plotted by considering the thrust vectoring for reducing the AOA to the feasible range.

Figures 4 to 8 gives the variation of various parameters $M, \alpha, \delta_b, \delta_v$ and γ for different initial values of Mach number varying from 0.3 to 0.8. Figure 4 shows that the hard constraint of $M(\gamma_f) = 0.8$ is exactly maintained. Figure 5 shows the variation of A0A and δ_b with time. The constraint on the AOA and δ_b for the analysis was taken as $-20 \leq \alpha \leq 20$ and $-72 \leq \alpha \leq 72$. It can be seen from the figure that for an initial value of $M_0 = 0.8$, the AOA remains almost constant at the expense of a higher value of δ_b. Figure 6 shows the variation of shear angle with time. Since the AOA is constrained to lie within $|\alpha| \leq 20$, the shear angle is higher for an initial Mach number of 0.8 which can be reduced by a higher value of AOA.

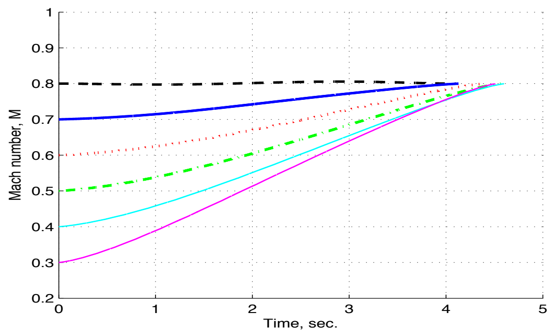

Fig. 4. Variation of Mach number with time ($M_0 = 0.3$ to 0.8)

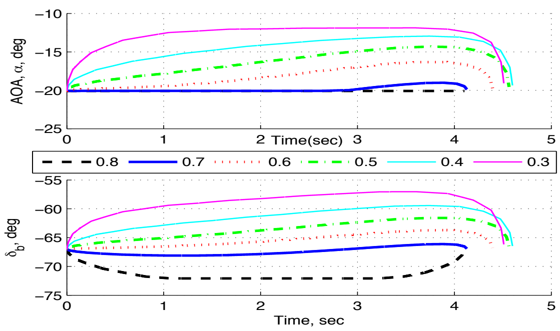

Fig. 5. AOA and δ_b for various initial Mach number($M_0 =$ 0.3 to 0.8)

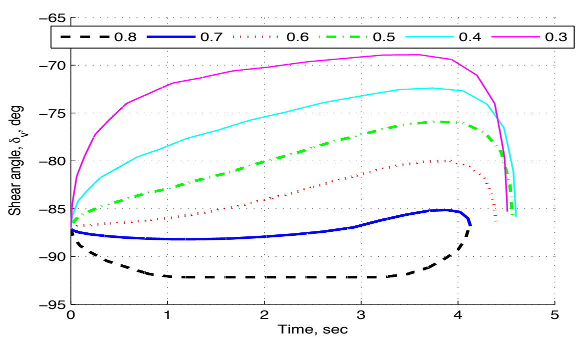

Fig. 6. Shear angle, δ_v with time for various initial Mach number,($M_0 = 0.3$ to 0.8)

Figure 7 shows that, the flight path angle, γ is a monotonic function with respect to time.

Figure 8 gives the envelope of the vehicle for which, the initial height was selected as 5 km. A lower Mach number shows a smaller turn radius, but takes longer time to reverse its direction.

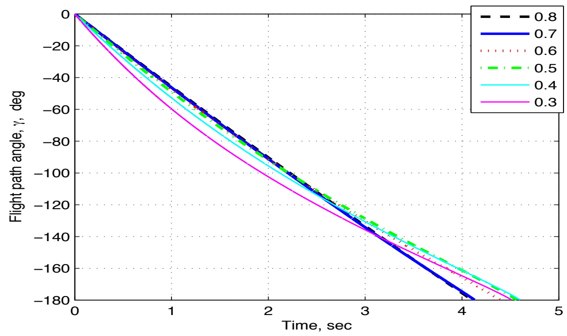

Fig. 7. Flight path angle,γ with time for various initial Mach number, $M_0 = 0.3$ to 0.8

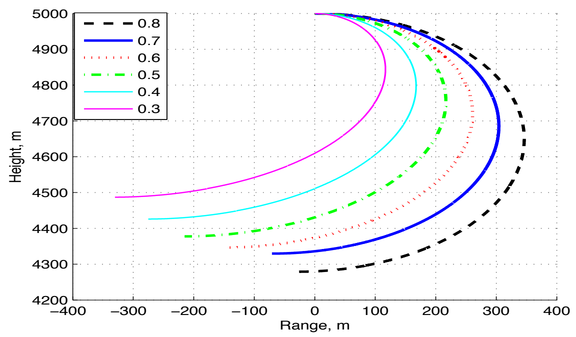

Fig. 8. Envelope of flight for various initial Mach number,$M_0 = 0.3$ to 0.8

Figure 9 gives a comparison of the AOA history for different grids points for the Chebyshev PS method. 20 grid points were selected for the above analysis as it is comparable with the higher grid points.

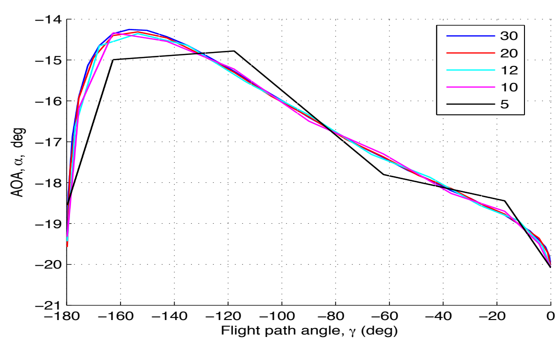

Fig. 9. Comparison of Angle of attack for various grids, $M(0) = 0.5$

In all the figures from 4 to 8, the variation was considered with the AOA constrained at -20. If the AOA is increased, it takes less time to reverse its direction as shown in figures 10 and 11. The C_L values were extrapolated for the AOA in the range $\alpha \in [-20, -30]$. Figure 11 shows that as the AOA increases, the time taken for reversing the flight path angle decreases. This shows that depending upon the mission requirements, the time required can be minimized by penalizing the AOA.

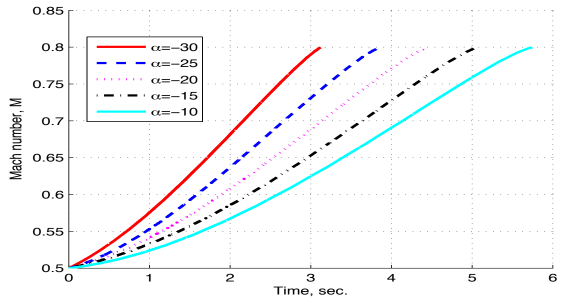

Fig. 10. Comparison of Mach number and time for varying angle of attack

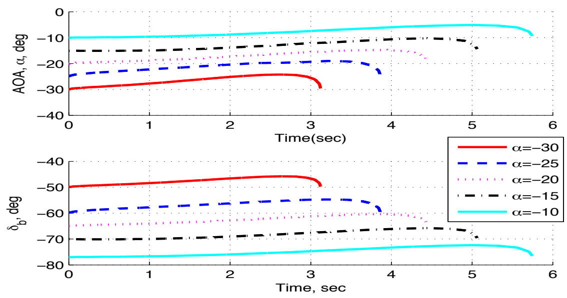

Fig. 11. AOA and δ_b for different initial angle of attack

From Figures 12 and 13 it is observed that both the Chebyshev and Legendre pseudospectral are attaining the hard contraint exactly. The time taken for the turn by the two methods are nearly the same. The difference in the cost function is very low (in the range of the second decimal place). Even the convergence rates of the two methods have been observed to be nearly the same. It is also observed that the relative energy error of the two control histories is in the range of 10^{-6}.

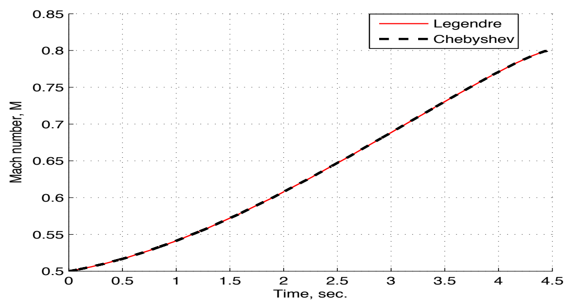

Fig. 12. Comparison of Legendre and Chebyshev approximation for Mach number, $M(0) = 0.5$

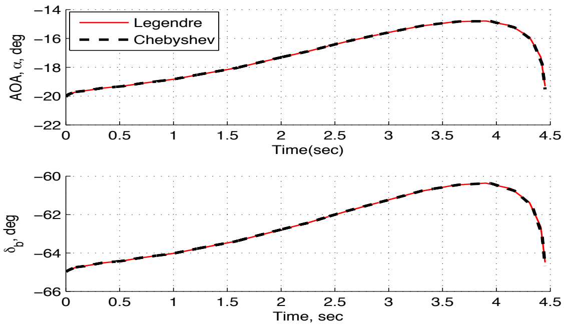

Fig. 13. Comparison of Legendre and Chebyshev approximation for control history, $M(0) = 0.5$

5. CONCLUSION

The pseudo-spectral discretization method for trajectory optimization has been discussed. The method is applied for reversing the trajectory of an air-to-air missile in minimum time. Both Chebyshev and Legendre polynomial approximation have been successfully used to solve the problem and the results from both are consistent. Convergence analysis studies need to be done to analyse the rate of convergence of the two methods for further comparison of the two methods of approximation.

REFERENCES

Q. Gong, F. Fahroo and I. M. Ross. A spectral algorithm for pseudospectral methods in optimal control *AIAA Journal of Guidance, Control and Dynamics*, Vol. 31, No. 3, pp.460-471, 2008.

Dongchen Han, S.N. Balakrishnan. State-constrained agile missile control with adaptive-critic-based neural networks. *IEEE Trans. on Control Systems Technology*, Vol. 10, No. 4, pp. 481-489, 2002.

Dongchen Han, S.N. Balakrishnan. Adaptive critic based neural network for control constrained agile missile control. *Proceedings of the American Control Conference*, San Diego, California, June 1999.

W. Kang, Q. Gong, I. M. Ross, F. Fahroo. On the convergence of nonlinear optimal control using pseudospectral methods for feedback linearizable systems. *Int. Journal of Robust and Nonlinear Control*, Vol 17, Issue 14, pp. 1251-1277,2007.

F. Fahroo and I. M. Ross. Costate estimation by a Legendre pseudospectral method. *AIAA Journal of Guidance, Control and Dynamics*, Vol. 24, No. 2, pp. 270-277, 2001.

L.M. Delves, J.L.Mohamed. Computational methods for integral equations. 1985, *Cambridge University Press.*

F. Fahroo and I. M. Ross. Direct trajectory optimization by a chebyshev pseudospectral method. *AIAA Journal of Guidance, Control and Dynamics*, Vol. 25,No. 1, 2002, pp.160 - 166.

Q. Gong, I. M. Ross, F. Fahroo. A Chebyshev pseudospectral method for nonlinear contrained optimal control problems. 48^{th} *Joint IEEE Conf. on Decision and Control and* 28^{th} *Chinese Control Conference*, Shanghai, P. R. China, Dec 16-18, 2009, pp. 5057-5062.