EXERCISE 5

1. TRUE or FALSE? (Unless otherwise stated assume an underlying probability space $(\Omega, \mathcal{B}, P)$):

   (a) For any event $A \in \mathcal{B}$
   $$P(A|A) = 1$$

   (b) For any two events $A, B \in \mathcal{B}$
   $$P(A|B) = P(B|A)$$

   (c) For any two events $A, B \in \mathcal{B}$
   $$P(A) = P(A|B) + P(A|B')$$

   (d) For any two events $A, B \in \mathcal{B}$
   $$P(A) = P(A|B) + P(A'|B)$$

   (e) For any two events $A, B \in \mathcal{B}$
   $$P(A|B) = P(A|B)P(A)$$

   (f) For any two events $A, B \in \mathcal{B}$
   $$P(A|B) = \frac{P(B|A)P(B)}{P(A)}$$

   (g) For any two events $A, B \in \mathcal{B}$
   $$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

   (h) For any two events $A, B \in \mathcal{B}$
   $$P(A \cap B) = P(B|A)P(A)$$

   (i) For any two events $A, B \in \mathcal{B}$
   $$P(A \cap B) = P(A|B)P(B)$$

(j) For any two events $A, B \in \mathcal{B}$

$$P(A \cap B) = P(A|B)P(A)$$

(k) For any two events $A, B \in \mathcal{B}$

$$P(A \cap B) = P(A|B)P(B)$$

(l) If $A_1, A_2, \cdots, A_n, B$ are in $\mathcal{B}$ then $P\left(\left\{\bigcup_{j=1}^{n} A_j\right\}|B\right) = \sum_{j=1}^{n} P(A_j|B))$

(m) In an experiment with samplet space $\Omega = \{1, 2, 3, 4\}$, $P(\omega) = \frac{1}{4}$ for all $\omega \in \Omega$. The collection of events

$$\{A_1 = \{1, 3, 4\}\,,\ A_2 = \{2, 3, 4\}\,, A_3 = \phi\}$$

is an independent collection of events.

2. When a biased die is rolled, the probability of getting the numbers 1 to 6 are as given below:

$$\mathcal{P}(1) = \frac{1}{2}, \mathcal{P}(2) = \frac{1}{4}, \mathcal{P}(3) = \frac{1}{8}, \mathcal{P}(4) = \frac{1}{16}, \mathcal{P}(5) = \mathcal{P}(6) = \frac{1}{32},$$

Consider the following three events:

(a) E=The event that a prime number shows up
(b) F=The event that an odd number shows up
(c) G=The event that a multiple of three shows up

Answer the following:

(a) Determine which of the pairs of events are independent:

$$\{E, F\}\,,\ \{F, G\}\,,\ \{E, G\}$$

(b) Determine if the collection $\{E, F, G\}$ of these three events is independent

3. Let E and F be independent events in a probability space $(\Omega, \mathcal{B}, \mathcal{P})$. Prove the following:

(a) E and F' are independent

(b) E' and F are independent

(c) E' and F' are independent

4. Consider the random experiment of rolling a die, and consider the two cases:

   (a) Case 1: The die is fair

   (b) The die is biased with the probabilities given by

   $$\mathcal{P}(1) = \mathcal{P}(2) = \mathcal{P}(3) = \frac{1}{6} \text{ and } \mathcal{P}(4) = \frac{1}{4},\ \mathcal{P}(5) = \frac{1}{8},\ \mathcal{P}(6) = \frac{1}{8}$$

   For each of the two cases answer the following:
   For the three events,

   $$\begin{array}{lcllcl} E & = & \text{the event that an even number shows up} & = & \{2,4,6\} \\ F & = & \text{the event that a prime number shows up} & = & \{2,3,5\} \\ G & = & \text{the event that an odd prime number shows up} & = & \{3,5\} \end{array}$$

   answer the following:

   (a) Find the conditional probabilitites,

   $$\mathcal{P}(E|F),\ \mathcal{P}(F|E),\ \mathcal{P}(F|G),\ \mathcal{P}(G|F),\ \mathcal{P}(E|G),\ \mathcal{P}(G|E)$$

   (b) Which of the pairs of events, $\{E, F\}$, $\{F, G\}$, $\{E, G\}$ are independent?

5. Which of the following functions can be the CDF of a random variable (Give Reasons):

   (a) $F_X(x) = \begin{cases} 0 & \text{for } -\infty < x < -1 \\ 0.5 & \text{for } -1 \leq x < 0 \\ 0.75 & \text{for } 0 \leq x < 2 \\ 1 & \text{for } 2 \leq x < \infty \end{cases}$

   (b) $F_X(x) = \begin{cases} 0 & \text{for } -\infty < x \leq -1 \\ 0.5 & \text{for } -1 < x < 0 \\ 0.75 & \text{for } 0 \leq x < 2 \\ 1 & \text{for } 2 \leq x < \infty \end{cases}$

(c) $F_X(x) = \begin{cases} 0 & \text{for} \quad -\infty < x < -1 \\ 0.5 & \text{for} \quad -1 \le x < 0 \\ 0.75 & \text{for} \quad 0 \le x < 2 \\ x & \text{for} \quad 2 \le x < \infty \end{cases}$

(d) $F_X(x) = \begin{cases} \frac{1}{2|x|+4} & \text{for} \quad -\infty < x < -1 \\ 0.5 & \text{for} \quad -1 \le x < 0 \\ 0.75 & \text{for} \quad 0 \le x < 2 \\ 1 & \text{for} \quad 2 \le x < \infty \end{cases}$

(e) $F_X(x) = \begin{cases} \frac{1}{2|x|+4} & \text{for} \quad -\infty < x \le -1 \\ 0.5 & \text{for} \quad -1 < x < 0 \\ 0.75 & \text{for} \quad 0 \le x < 2 \\ 1 & \text{for} \quad 2 \le x < \infty \end{cases}$

(f) $F_X(x) = \begin{cases} \frac{1}{|x|+2} & \text{for} \quad -\infty < x < 0 \\ 1 - (0.5)e^{-x} & \text{for} \quad 0 \le x < \infty \end{cases}$

(g) $F_X(x) = \begin{cases} 0 & \text{for} \quad -\infty < x < 0 \\ \frac{e^x}{1+e^x} & \text{for} \quad 0 \le x < \infty \end{cases}$

6. In the above Exercise for those functions which represent CDF of a random variable X find the following probabilities:

   (a) $P(-1 \le X \le 2)$

   (b) $P(-1 \le X < 2)$

   (c) $P(-1 < X \le 2)$

   (d) $P(-1 < X < 2)$

   (e) $P(X = -1)$, $P(X = 0)$, $P(X = 1)$, $P(X = 2)$

7. What is the probability of three successes in five independent trials with success probability p.

8. To communicate one bit of information reliably, cellular phones transmit the same binary symbol five times. Thus the information 0 is transmitted as 00000 and the information one is transmitted as 11111. The receiver detects the correct information if three or more binary symbols are received correctly. By increasing the number of binary symbols per information bit from 1 to 5, show that the cellular phone reduces the probability of error from 0.1 to ~~0.0081~~.

0.00856

9. Let A, B, C, D be four events in $\mathcal{B}$ such that the following conditions are satisfied:

   (a) A and B are disjoint,

   (b) C and D are independent, and

   (c) $P(A) = \frac{1}{4}, P(B) = \frac{1}{8}, P(C) = \frac{5}{8}, P(D) = \frac{3}{8}$

   Answer the following:

   (a) Find $P(A \cap B)$

   (b) Find $P(A \cup B)$

   (c) Find $P(A \cap B')$

   (d) Find $P(A \cup B')$

   (e) Are A and B independent?

   (f) Find $P(C \cap D)$

   (g) Find $P(C \cup D)$

   (h) Find $P(C|D)$

   (i) Find $P(C \cap D')$

   (j) Find $P(C \cup D')$

   (k) Find $P(C' \cap D')$

   (l) Are C' and D' independent?

10. A particular operation has six components. Each component has a failure probability q independent of any other component. The operation is successful if both the following conditions hold:

   (a) At least one of the sets omponents $\{1, 2, 3\}$ and $\{4\}$ works and

   (b) Either component 5 or component 6 works

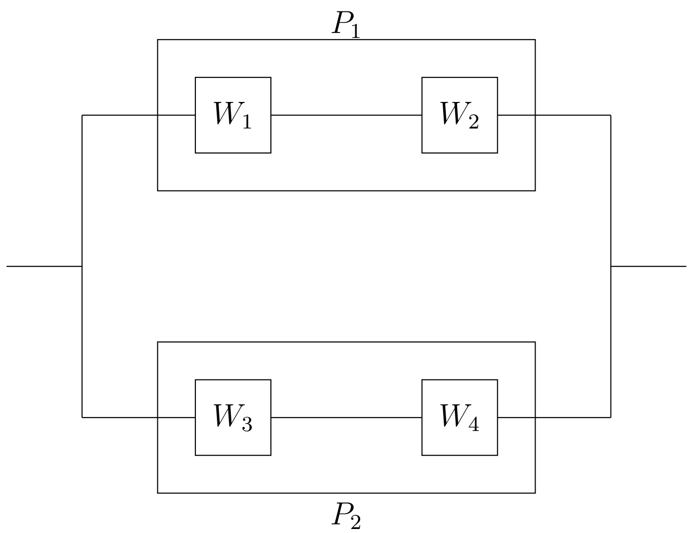
   What is the probability that the operation is successful?

11. Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. In each code word, a bit is a zero with probability 0.8, independent of any other bit. Answer the following:

(a) What is the probability of the code word 00111?

(b) What is the probability that a code word contains exactly three ones?

12. An operation consists of two parts P_1 and P_2. The first part has two components in series (W_1 and W_2) and the second part has two components in series (W_3 and W_4).
(See the following figure)



All components succeed with probability $p = 0.9$. Find the probability the operation succeeds

13. Monitor three consecutive phone calls going through a telephone switching office. Classify each one as a voice call v, if someone is speaking; or a data call d if the call is carrying a modem or fax signal. Your observation is a sequence of three letters (each one is either v or d). For example, three voice calls corresponds to vvv. The outcomes vvv and ddd each have probability 0.2 while the other outcomes each have probability 0.1. Let N_v be the number of voice calls in the three calls. Consider the four events $N_v = 0, N_v = 1, N_v = 2, N_v = 3$. Find the following probabilities:

(a) $P(vvd|N_v = 2)$

(b) $P(ddv|N_v = 2)$

(c) $P(N_v = 2|N_v \geq 1)$

(d) $P(N_v \geq 1|N_v = 2)$

14. A receiver R can receive messages from three different transmitters T_1, T_2, T_3. The probability that T_j transmits a message is p_j (for $j = 1, 2, 3$). The probability that a message transmitted by T_j gets corrupted in transmission is q_j (for $j = 1, 2, 3$). Answer the following:

    (a) For $j = 1, 2, 3$ find the probability that an uncorrupted message received by R is a transmission from T_j

    (b) Find the probability that an uncorrupted message received by R is a transmission from either T_1 or T_3

15. The cdf of a random variable X is given by

$$F_X(x) \quad = \quad \begin{cases} 0 & \text{for} \quad x < -1 \\ 0.5 & \text{for} \quad -1 \leq x \leq 0 \\ \dfrac{1+x}{2} & \text{for} \quad 0 \leq x \leq 1 \\ 1 & \text{for} \quad x > 1 \end{cases}$$

    Answer the following':

    (a) Plot the cdf

    (b) Is X a continuous rv?

    (c) Find the following probabilities:

$$P(X \leq -1), \; P(2 < X < 6), \; P(-0.5 < X < 0.5)$$

16. The cdf of a random variable is given by

$$F_X(x) \quad = \quad \begin{cases} 0 & \text{for} \quad x < 0 \\ 0.5 + \alpha sin^2\left(\frac{\pi x}{2}\right) & \text{for} \quad 0 \leq x \leq 1 \\ 1 & \text{for} \quad x > 1 \end{cases}$$

    Answer the following:

    (a) What are the possible values of α?

(b) Find the pdf of X

(c) Find $P(X > 0)$

17. The loose hand of a clock is spun hard and comes to rest making an angle θ (measured clockwise) with the vertical. Thus the sample space is $\Omega = [0, 2\pi]$. Let $\mathcal{B}$ be the Borel subsets of Ω and P be the probability measure defined as $P(any\ interval) = \frac{1}{2\pi}(length\ of\ the\ interval)$. Let X be the random variable defined as

$$X(\theta) = 2sin\left(\frac{\theta}{4}\right)$$

Find the cdf of X.

18. A biased die has the probability distribution as follows:

$$\begin{aligned} P(an\ odd\ number\ showing\ up) &= 0.4 \\ P(an\ even\ number\ showing\ up) &= 0.6 \end{aligned}$$

For the random experiment of rolling such a die (with the σ-algebra of events as the power set) let X and Y be the random variables defined as follows:

$$\begin{aligned} X(\omega) &= \begin{cases} 1 & \text{if} \quad \omega \text{ is even} \\ -1 & \text{if} \quad \omega \text{ is odd} \end{cases} \\ Y(\omega) &= \begin{cases} 2 & \text{if} \quad \omega > 2 \\ -2 & \text{if} \quad \leq 2 \end{cases} \end{aligned}$$

(a) Find the pmfs p_X and p_Y of X and Y

(b) Find the joint pmf p_{XY}

(c) Are X and Y independent random variables?

19. The waiting time X of a customer at a bus stop is zero if the customer finds a bus at the stop on arrival, and a uniformly distributed random length of time in the interval $[0, 1]$ (in hours) if no bus is found on arrival. The probability that the customer finds a bus at the stop on arrival is p. Find the cdf of X. Is X a continuous random variable?