E2-243
Programming Exercise - 3

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Instructions:

- **Attempt the following programming exercises using MATLAB.**
- **Do not submit your code, output files, etc.**
- **Please read all the questions carefully. All the questions are self explanatory.**
- **Error Handling: In your code, you should make sure that the input parameters to your function satisfy any assumptions that you make about them. For example, if we have asked you to write a function that takes a $m \times n$ matrix A, then you should check in your code that the matrix passed as input to your function is indeed of size $m \times n$. You should check for all possible pathological inputs.**

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1. Elementary Row Operations

   (a) Write a MATLAB function `EROType1(A,i,j)` that exchanges rows `i` and `j` in the given matrix `A`.

   (b) Write a MATLAB function `EROType2(A,alpha,i,j)` that adds `alpha` times row `j` (where `alpha` is a real number) to row `i`. (i.e., $R_i = R_i+$`alpha`$\times R_j$).

   (c) Write a MATLAB function `EROType3(A,alpha,i)` that replaces row `i` with `alpha` times row `i`. (i.e., $R_i =$`alpha`$\times R_i$).

   (d) Using the above elementary row operations, write a function `myRREF(A)` that produces the row-reduced echelon form of the given matrix `A`. You may assume that `A` is a square matrix and that its rows are all linearly independent (you still have to check in your code whether the input matrix `A` satisfies these assumptions). Verify your result using MATLAB's in-built function `rref()`.

2. **Gram-Schmidt orthonormalization and QR factorization**

   (a) Let $S = \{\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4, \ldots, \mathtt{x}_n\}$ be a set of `n` vectors in $\mathbb{R}^m$, and `x` be another vector in $\mathbb{R}^m$. Write a MATLAB function `IsLinearCombination(A,x)` where `A` is a $\mathtt{m} \times \mathtt{n}$ matrix that contains the vectors $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4, \ldots, \mathtt{x}_n$ as columns and outputs 1 or 0 depending on whether `x` is or is not (respectively) a linear combination of the vectors $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4, \ldots, \mathtt{x}_n$.

   (b) Let $S = \{\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4, \ldots, \mathtt{x}_n\}$ be a set of `n` vectors in $\mathbb{R}^m$. Write a MATLAB function `GetLinIndepVectors(A)` that takes a matrix A whose columns are the vectors in S as input, and returns a matrix B whose columns are such that (i) they are a subset of the columns in matrix A, (ii) they are linearly independent, and (iii) the number of columns in matrix B is the largest possible.

   (c) Write a MATLAB function `[V,Q] = GramSchmidt(B)` where B is the matrix returned by the function `GetLinIndepVectors()`, and matrices `V` and `Q` respectively are the matrices containing orthogonal and orthonormal vectors obtained from the columns of matrix B using the Gram-Schmidt procedure.

   (d) Write a MATLAB function `[Q,R] = QRFact(B)` that takes a square matrix B as input and returns two matrices Q, R such that Q is an orthogonal matrix and R is an upper-triangular matrix such that $B = QR$. You are required to use the `GramSchmidt()` function to generate matrix Q.

3. Plot a part of the plane in MATLAB (use the fill3() function) corresponding to the subspace $\mathscr{W} = \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \end{pmatrix} \in \mathbb{R}^3, \alpha, \beta \in \mathbb{R}$.

   Let u and v be two vectors in this subspace. Plot the vectors u and v and $\alpha u+\beta v$ (use the quiver3() function) for a few values of $\alpha, \beta \in \mathbb{R}$. Use the MATLAB rotation toolbar button to rotate the 3D plot and visually verify that all these vectors are in the same plane corresponding to subspace W .

4. Let $x, y, z, w \in \mathbb{R}^3$ be as defined below

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, z = \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}, w = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

   (a) Is the set $\{x, y\}$ linearly independent. Answer this first using hand calculation. Then form the matrix A with the vectors $x; y$ as columns and calculate its rank: A=[x y], r=rank(A).
   What values for r would show that the set x, y is linearly dependent?

   (b) Take an arbitrary linear combination v of x and z and calculate the rank of the augmented matrix:
   $v = rand(1) * x + rand(1) * z$, t=rank([$A$ v])
   Can you conclude from the value obtained for t that v is not in the subspace spanned by x, y? Why?

   (c) Form the matrix B with the vectors $x; y; z; w$ as columns and calculate its rank:
   B=[x y z w], r=rank(B)
   What can you conclude about the linear independence or dependence of the set of vectors $x; y; z; w$? Can you make this same conclusion for every set of four vectors in $\mathbb{R}^3$? Why?

5. Consider the matrix $H = \begin{bmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{bmatrix}$

   Find its eigenvalues and eigenvectors from the eigen equation $H\psi = \lambda\psi$, where ψ is an eigenvector of H and λ is the corresponding eigenvalue. Verify your results with the MATLAB function eig(). Examine whether the eigenvectors form a basis for $\mathbb{C}_3$.

6. Perform a singular value decomposition(SVD) on the matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ i.e.

   find the metrices U, S, V for $A = USV'$. You can use the function eig() for this problem.

7. Let $A = \begin{bmatrix} 7 & -9 & -4 & 5 & 3 & -3 & -7 \\ -4 & 6 & 7 & -2 & -6 & -5 & 5 \\ 5 & -7 & -6 & 5 & -6 & 2 & 8 \\ -3 & 5 & 8 & -1 & -7 & -4 & 8 \\ 6 & -8 & -5 & 4 & 4 & 9 & 3 \end{bmatrix}$

   Construct metrices C and N whose columns are bases for Range(A) and Nullspace(A), respectively, and construct a matrix R whose rows form a basis for Row space A.

8. Show that the columns of the matrix A are orthogonal by making an appropriate matrix calculation. State the calculations you use.

$$A = \begin{bmatrix} -6 & -3 & 6 & 1 \\ -1 & 2 & 1 & -6 \\ 3 & 6 & 3 & -2 \\ 6 & -3 & 6 & -1 \\ 2 & -1 & 2 & 3 \\ -3 & 6 & 3 & 2 \\ -2 & -1 & 2 & -3 \\ 1 & 2 & 1 & 6 \end{bmatrix}$$

9. In parts (a)-(d) let U be the matrix formed by normalizing each column of the matrix A in the previous question.

   (a) Compute $N = U^T U$ and $M = UU^T$. How do they differ?

   (b) Generate a random vector y in $\mathbb{R}^8$, and compute $\mathbf{p} = UU^T\mathbf{y}$ and $z = y - p$. Explain why p is in Range(A). Verify that z is orthogonal to p.

   (c) Verify that z is orthogonal to each column of U.

   (d) Notice that $y = p + z$, with p in Range(A). Explain why z is in $(Range(A))^\perp$? **CHECK**.