3) $N = 200$ here EXC - 7

Chernoff Bound :

$$P(X \geq K) \leq \min_{\alpha > 0} \{ e^{-\alpha k} E(e^{\alpha X}) \}$$

When,

$$X = \sum_{i=1}^{N} X_i$$

$X_i = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } (1-p) \end{cases}$

$Y_i = e^{\alpha X_i} = \begin{cases} e^{\alpha} & \text{with prob } p \\ 1 & \text{with prob } (1-p) \end{cases}$

X_i are iid.

$$E(e^{\alpha X}) = E(e^{\alpha \sum_{i=1}^{N} X_i}) = E(e^{\alpha X_1} e^{\alpha X_2} \dots e^{\alpha X_n})$$

$$\underset{iid}{=} E(e^{\alpha X_1}) E(e^{\alpha X_2}) \dots E(e^{\alpha X_n})$$

☆ $e^y \geq 1 + y \ \forall \ y \in \mathbb{R}$ ...

For any i : $E(e^{\alpha X_i}) = p e^{\alpha 1} + (1-p) e^{\alpha \cdot 0}$

$\forall \ i \in [N]$ $= p e^{\alpha} + (1-p)$.

$$E(e^{\alpha X_i}) = p(e^{\alpha} - 1) + 1 \quad \text{——} \ ①$$

define : $y = p(e^{\alpha} - 1)$

$$p(e^{\alpha} - 1) + 1 \leq e^{p(e^{\alpha} - 1)} \quad \text{(Bounding)}$$

From ①

$$E(e^{\alpha X_i}) \leq e^{p(e^{\alpha} - 1)}.$$

$$E(e^{\alpha X}) \leq e^{p(e^{\alpha} - 1)} e^{p(e^{\alpha} - 1)} \dots (e^{p(e^{\alpha} - 1)}) \quad N \text{ times}.$$

$$E(e^{\alpha X}) \leq e^{Np(e^{\alpha} - 1)}.$$

$$e^{-\alpha k} E(e^{\alpha X}) \leq e^{-\alpha K} e^{Np(e^{\alpha} - 1)} \quad \forall \ \alpha > 0.$$

$$\min \{ e^{-\alpha K} E(e^{\alpha X}) \} \leq \min_{\alpha} \{ e^{Np(e^{\alpha} - 1) - \alpha K} \}.$$

$$P(X \geq K) \leq \min_{\alpha > 0} \{ e^{Np(e^{\alpha} - 1) - \alpha K} \}$$

$$\iff \min_{\alpha > 0} Np(e^{\alpha} - 1) - \alpha K$$

Use calculus (Derivative)

$$\alpha = \log \frac{K}{np}$$

holds when $K \geq np$.

Substitute back and we will get bound.

4.) $E(X) = \mu_X < \infty$; $Var(X) = \sigma_X^2 < \infty$

a) chebechev Centred at some other value (Not at mean)

$a \geq b$ $\qquad$ $a, b \in \mathbb{R}^+$

$\Rightarrow a^2 \geq b$

$P(|X - c| \geq K)$

$K > 0$ and $|X - c| \geq 0$, Now we can square

$|X - c| \geq K \iff (X-c)^2 \geq K^2$.

(Dont use Markov for non negative random variables)

$$P[(X-c)^2 \geq K^2] \leq \frac{E[(X-c)^2]}{K^2}$$

$$(X-c)^2 = X^2 + c^2 - 2Xc + \mu^2 - 2X\mu - \mu^2 + 2X\mu .$$

b) $P(a < X < b) = P\left(\left|X - \left(\frac{a+b}{2}\right)\right| < \frac{b-a}{2}\right)$.

$$\left|X - \left(\frac{a+b}{2}\right)\right| < \frac{b-a}{2}$$

$X - \left(\frac{a+b}{2}\right) < \frac{b-a}{2}$ $\quad|\quad$ $X - \left(\frac{a+b}{2}\right) > -\left(\frac{b-a}{2}\right)$

$\Rightarrow X < b$ $\quad|\quad$ $\Rightarrow X > a$.

In a) we centred at c.

But;

$P(a<x<b) = P\left(\left|X - \left(\frac{a+b}{2}\right)\right| < \frac{b-a}{2}\right) = 1-P\left(\left|X - \frac{a+b}{2}\right| \geq \frac{b-a}{2}\right)$

5) $E(X) = \mu < \infty$

$Var(X) = \sigma_x^2 < \infty$

Let $Y = X - \mu_x$ and $a > 0$.

a) $E(Y) = 0$.

$Var(Y) = \sigma_x^2$.

b) $t > 0$; is this true?

$P(Y \geq a) = P(Y + t \geq a+t) \leq P((Y+t)^2 \geq (a+t)^2)$.

$Y+t \geq a+t \overset{?}{\Longrightarrow} (Y+t)^2 \geq (a+t)^2$

No true Y may not be non negative.

$|Y+t| \geq |a+t|$; $a>0, t>0$

$|Y+t| \geq a+t$. $a+t > 0$.

$|a+t| = a+t$

$(Y+t) \geq (a+t)$ | $(Y+t) \leq -(a+t)$

But Since $(a+t) > 0$ hence $(Y+t) > 0$.

$|Y+t| \geq (a+t) \iff (Y+t)^2 \geq (a+t)^2$

$P[(Y+t)^2 \geq (a+t)^2] = P[(Y+t \geq a+t) \cup (Y+t \leq -(a+t))]$

$\geq P(Y+t \geq a+t)$.

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$$P(Y \geq a) \leq \frac{\sigma_x^2 + t^2}{(a+t)^2}$$

$$P(Y \geq a) \underset{\uparrow \; b)}{\leq} P[(Y+t)^2 \geq (a+t)^2]$$

using chebychev from 4a)

$$\leq \frac{\sigma_Y^2 + (\mu_Y - t)^2}{(a+t)^2}.$$

From 5 ⓐ $\quad \sigma_Y^2 = \sigma_x^2 \qquad \mu_Y = 0$

$$= \frac{\sigma_x^2 + t^2}{(a+t)^2}.$$

d) $t = \frac{\sigma_x^2}{a}$

minimize

$$g''(t) = \frac{2a^2 + 8at - 6\sigma_x^2}{(a+t)^4}$$

$$t = \frac{\sigma_x^2}{a} \qquad \Rightarrow 8at = 8\sigma_x^2.$$

$$= \frac{2a^2 + 8\sigma_x^2 - 6\sigma_x^2}{(a+t)^4}.$$

e) Substituting.

f) $a < 0$ define $\quad b = -a$. since a is $-ve$; $b = +ve$

$$P(Y \geq a) = 1 - P(Y < a)$$

$$P(Y < a) = P(Y < -b)$$

$$= P(-Y > b)$$

b is positive.

$$\sigma_{-Y}^2 = \sigma_x^2$$

Using previous part.

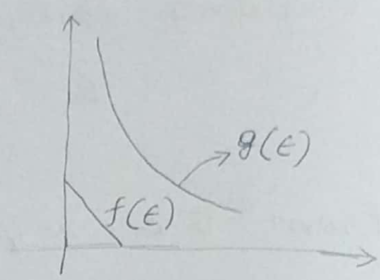
$$\leq \frac{\sigma_{-y}^2}{\sigma_{-y}^2 + b^2} = \frac{\sigma_x^2}{\sigma_x^2 + a^2}.$$

8) X is r.v; $\forall \epsilon > 0$

$f(\epsilon) = P(X \geq \epsilon)$

$f(\epsilon) = 1 - F_X(\epsilon)$.

$$g(\epsilon) = \frac{E(x)}{\epsilon}$$



$g(\epsilon) > f(\epsilon)$

How markov inequality behaves in various distribution.

c) Define $y = \frac{t^2}{2\alpha^2}$.

$y\,dy = \frac{t}{\alpha^2} dt$.

9)

b) $f_x(x) = \frac{\lambda}{2} \exp(-\alpha |x|)$

Split for $x < 0$ and $x > 0$.

$$\lim_{n\to\infty} P(\omega\in\Omega : |X_n(\omega) - X(\omega)| \geq \epsilon) = 0 \; ; \forall \epsilon > 0$$

$$\lim_{n\to\infty} P(|X_n(\omega) - X(\omega)| \geq \epsilon) = 0$$, converge in probability

2) $X_n \sim \exp(n)$

$E(X_n) = \frac{1}{n}$.

$P(|X_n(\omega) - X(\omega)| \geq \epsilon) = P(|X_n(\omega)| \geq \epsilon)$.

Apply Markov.

$$\leq \frac{E(X_n)}{\epsilon} = \frac{1}{n\epsilon}$$

~~$\forall \epsilon > 0$,~~ $\lim_{n\to\infty} \frac{1}{n\epsilon} \to 0 \; ; \forall \epsilon > 0$

3) $X_n \sim N(0, \frac{1}{n})$

$$f_{X_n} = \frac{\sqrt{n}}{2\pi} \exp\left(-\frac{nx^2}{2}\right)$$

Let $\alpha_n > 0$; s.t $\alpha_n \to +\infty$.

a) $P(X_n \leq x) = P(\alpha_n X_n \leq \alpha_n x)$?

True α_n is positive

b) $F_{X_n}(x) = P(X_n \leq x)$

$= P(\alpha_n X_n \leq \alpha_n x)$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} P(\alpha_n X_n \leq \alpha_n x)$$

$x > 0$, $\alpha_n x > 0$.

$\lim_{n\to\infty} \alpha_n x \to \infty$.

$= P(\lim_{n\to\infty} \alpha_n X_n \leq \underbrace{\lim_{n\to\infty} \alpha_n x}_{\infty})$

$= 1$

when $x<0$, $\alpha_n x < 0$

$\lim_{n\to\infty} \alpha_n x \longrightarrow -\infty$

$P(\_\_\_\_ \leq -\infty) = 0.$ (By definition).

$F_{X_n}(x) \to F_X(x)$ At 0 it doesnt matter.

$$F_X(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x \not> 0 \end{cases}$$

NOTE:

$X(\omega) = K \quad \forall \ \omega \in \Omega$

$P(X=K) = 1$

$P(X \leq x) = 0 \ \forall \ x < K.$

$\quad = 1 \ \forall \ x \geq K.$

$$F_X(x) = \begin{cases} 0 & \forall \ x < K. \\ 1 & \forall \ x \geq K. \end{cases}$$

5) $$F_{X_n}(x) = \begin{cases} \dfrac{e^{n(x-1)}}{1+e^{n(x-1)}} & \text{if } x \geq 1. \\ 0 & \text{if } x < 1. \end{cases}$$

$$\frac{1}{e^{-n(x-1)} + 1}.$$

as $\lim n\to\infty$

6) adjust things as shown above.

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1) Ma... ...ality

Exc 7.

10) $\lim_{n\to\infty} E(|X_n - X|^K) = 0 \Rightarrow X_n \xrightarrow{P} X$

$P(|X_n - X| > \epsilon) = P(|X_n - X|^K > \epsilon^K) \leq \dfrac{E(|X_n - X|^K)}{\epsilon^K} \to 0$

using Markov.

Exc 8.

9) $(\Omega\ \mathcal{B}\ P)$ ------ $\{X_n\}$

$$f_{X_n}(x) = \begin{cases} \dfrac{1}{nx^2} & \text{if } x > \frac{1}{n} \\ 0 & \text{if } x < \frac{1}{n} \end{cases}$$

Now, $X_n \xrightarrow{P} 0$

$= \int_{1/n}^{x} \frac{1}{nx^2}\,dx$

$$F_{X_n}(x) = \begin{cases} 1 - \dfrac{1}{nx} & \text{if } x > \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases}$$

$P(|X_n - 0| > \epsilon) = P(X_n > \epsilon) = 1 - F_X(\epsilon)$

$= 1 - \left(1 - \frac{1}{n\epsilon}\right)$

$= \dfrac{1}{n\epsilon}$.

10) $X_n(\omega) = Y_n(\omega)$. (ii/no)

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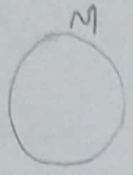
Linear algebra

Subspace : non empty set of vectors



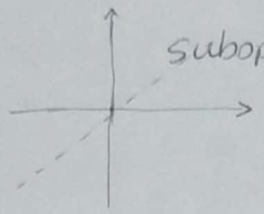
$x, y \in M$ (vectors)

$x + y \in M$

$\alpha \in R$ and $x \in M$, then $\alpha x \in M$

closed under addition Scalar multiplication.

$\mathbb{R}^2$,

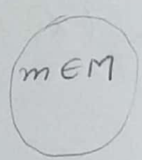


subspace in $\mathbb{R}^2$

Spanning set for a subspace:

$S = \{u_1\ u_2\ u_3 \ldots u_n\}$

iff i) $u_1, u_2, u_3, \ldots, u_n \in M$



ii) $m = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$

Linear independent set:

$S = \{u_1\ u_2\ u_3 \ldots u_n\}$ $\mathbb{R}^K$

$\sum_{j=1}^{n} \alpha_j u_j = \theta_K$; $\alpha_j = 0\ \forall\ j$.

Basis:

Spaning set which is linearly independent.

$S_1 = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$0u_1 + 0u_2 = \theta_2$

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orthonormal Basis:

$S = \{u_1, u_2, \ldots, u_n\} \in \mathbb{R}^n$.

conditions: i) $\langle u_i, u_j \rangle = 0 \quad \forall\, i \neq j$ --- orthogonal.

ii) $\langle u_i, u_j \rangle = 1 \quad \forall\, i = j$

$\|u_i\|^2 = 1$ (length 1).

Gram Schmidt's process:

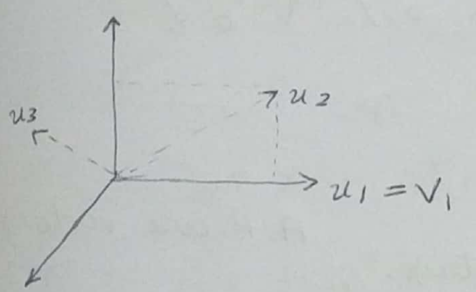
$(u_1, u_2, \ldots \ldots, u_n)$

$\Downarrow$ orthogonalization.

$(v_1 \;\; v_2 \;\ldots\ldots\; v_n)$

$\Downarrow$ Normalization

$(a_1 \;\; a_2 \ldots\ldots a_n)$



$v_1 = u_1$

$a_1 = \dfrac{v_1}{\|v_1\|}$.

$v_2 = u_2 - \dfrac{\langle u_2 v_1 \rangle}{\|v_1\|^2} v_1$.

$v_3 = u_3 - \dfrac{\langle u_3 v_1)}{\|v_1\|^2} v_1$.

$a_2 = \dfrac{v_2}{\|v_2\|}$.

$a_3 - \dfrac{v_3}{\|v_3\|}$.

Ex: $u_1 = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$

$S = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$a_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$a_2 = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Ex: $u_1 = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$ $u_2 = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix}$ $u_3 = \begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix}$

$$v_1 = u_1 = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}.$$

$$a_1 = \frac{1}{\sqrt{54}}\begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}.$$

$$a_3 = \frac{1}{\sqrt{74}}\begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix}.$$

$$v_2 = u_2 - \frac{\langle u_2 \, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix} - \frac{(-36)}{54}\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \qquad a_3 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

QR factorization:

$A = QR$ ↑upper Δ^{le} matrix. ↑ orthonormal matrix

A, R are vectors

$$A = [u_1 \; u_2 \; u_3].$$

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & -1 \\ 6 & -7 & 8 \end{pmatrix}$$

$$r_{ij} = \frac{\langle u_j \, v_i \rangle}{\|v_i\|} v_i$$

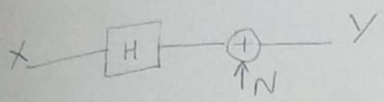
$r_{jj} = \|v_j\|$. $\quad i > j$

$r_{ij} = 0$.

$$R_3 = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

$A = QR.$

Communication system:



$Y = HX + N.$

$H = QR.$

$H^{-1}(Y-N)\cancel{H^{-1}} = \hat{x}$

H^{-1} is computationally intensive.

$H = QR.$

$\uparrow$ orthonormal.

$Q^* = Q^T$, R is upper Δ^{le}.

$Q^* Y = Q^* \underbrace{QR}_{H} X + Q^* N.$

$Q^* Y = RX + Q^* N.$

$\tilde{Y} = RX + \tilde{N}.$

$H = UDV^*$

$Y = HX + N$

$Y = (\underline{UDV^H})X + N$

$U^* YV = U^* UDV^H V X + N$

$U^* YV = DX + N$