SOLUTIONS

E2:243 TEST 2

(OCTOBER 26, 2018) (2PM - 4PM)

Name: SR No.: Department:

Answer All Questions (Maximum Marks:70)

I) In the following, in each question, only one alternative is correct. Tick (√) the correct alternative: (Correct Answer 1 Mark/Wrong Answer −0.5 Mark/Not Attempted 0 Mark)

1. Let $\mathcal{S}$ be the subset of $\mathbb{R}^4$ defined as follows:

$$\mathcal{S} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 + x_2 = 0 \text{ and } x_3 + x_4 = a \text{ where } x_1, x_2, x_3, x_4, a \in \mathbb{R} \right\}$$

$\mathcal{S}$ is a subspace of $\mathbb{R}^4$,

$\theta_4 \in$ Subspace $\Rightarrow a = 0$.

   (a) for any $a \in \mathbb{R}$
   (b) if and only if $a \geq 0$
   (c) if and only if $a < 0$
   (d) ✓ if and only if $a = 0$

2. Let $\mathcal{W}$ be a subspace of $\mathbb{F}^n$. If $\mathcal{S}_1$ is a spanning set for $\mathcal{W}$ then $\mathcal{S}_1 \bigcup \mathcal{S}_2$ is also a spanning set for $\mathcal{W}$,

$\mathcal{L}[S_1] = W$

$\forall\ S_2 \in W$

$\mathcal{L}[S_1 \cup S_2] = W$.

   (a) ✓ for every subset $\mathcal{S}_2$ of $\mathcal{W}$
   (b) if and only if $\mathcal{S}_1 \subseteq \mathcal{S}_2$
   (c) if and only if $\mathcal{S}_2 \subseteq \mathcal{S}_1$
   (d) for every subset $\mathcal{S}_2$ of $\mathbb{F}^n$

3. Let $\mathcal{S}_1 = \{u_1, u_2, u_3, u_4, u_5\}$ be a basis for a subspace $\mathcal{W}$ of $\mathbb{F}^7$, and $\mathcal{S}_2 = \{v_1, v_2, v_3, v_4\}$ a linearly independent set in $\mathcal{W}$. Consider the following two statements:

   (A) Every vector in $\mathcal{S}_1$ is a linear combination of the vectors in $\mathcal{S}_2$
   (B) Every vector in $\mathcal{S}_2$ is a linear combination of the vectors in $\mathcal{S}_1$

   Then

   (a) $(\boldsymbol{A})$ is TRUE but $(\boldsymbol{B})$ is FALSE

   (b) $(\boldsymbol{A})$ is FALSE but $(\boldsymbol{B})$ is TRUE

   (c) Both $(\boldsymbol{A})$ and $(\boldsymbol{B})$ are FALSE

   (d) Both $(\boldsymbol{A})$ and $(\boldsymbol{B})$ are TRUE

4. If $\{u_1, u_2, u_3, u_4\}$ is a basis for a subspace $\mathcal{W}$ of $\mathbb{F}^7$ then the set $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ is linearly independent,

   (a) for any two vectors u_5, u_6 in $\mathcal{W}$

   (b) for any two vectors u_5, u_6 in $\mathbb{F}^7$

   (c) for any two vectors u_5, u_6 in $\mathbb{F}^7$ such that u_5, u_6 are not in $\mathcal{W}$

   (d) for any two linearly independent vectors u_5, u_6 in $\mathbb{F}^7$ such that u_5, u_6 are not in $\mathcal{W}$

5. Let $\mathcal{S} = \{u_1, u_2, u_3\}$ be a spanning set for a subspace $\mathcal{W}$ of $\mathbb{F}^n$. Suppose there exists a vector $x \in \mathcal{W}$ such that

$$x = u_1 + u_2 - 3u_3 \text{ and } x = u_1 - u_2 + 2u_3$$

   Then

   (a) $dimension\ of\ \mathcal{W} = 3$

   (b) $dimension\ of\ \mathcal{W} > 3$

   (c) $dimension\ of\ \mathcal{W} \leq 2$

   (d) $dimension\ of\ \mathcal{W} = 2$

6. If $A \in \mathbb{R}^{10\times 8}$ and $\rho_A = 5$ then $\nu_A - \nu_{A^T}$ is equal to

   $a)\ 2$ $\quad b)\ -5$ $\quad c)\ -2$ $\quad d) 5$

7. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of $\mathbb{F}^n$ such that $\mathcal{W}_1 \subsetneq \mathcal{W}_2$ then

   (a) $\mathcal{W}_1^{\perp} \subsetneq \mathcal{W}_2^{\perp}$
   (b) $\mathcal{W}_1^{\perp} \subseteq \mathcal{W}_2^{\perp}$
   (c) $\mathcal{W}_2^{\perp} \subsetneq \mathcal{W}_1^{\perp}$
   (d) $\mathcal{W}_2^{\perp} \subseteq \mathcal{W}_1^{\perp}$

8. Let $\{u_1, u_2\}$ be an orthonormal basis for a subspace $\mathcal{W}$ of $\mathbb{C}^n$. Let x be a unit vector in $\mathbb{C}^n$ such that $(x, u_1) = 0.5$ and $(x, u_2) = 0.6$. Then $\|x_{_{\mathcal{W}^\perp}}\|^2$, (where $x_{_{\mathcal{W}^\perp}}$ is the orthogonal projection of x onto $\mathcal{W}^\perp$), is given by

   a) 0.25   *b*) 0.36   *c*) 0.61   *d*) 0.39

9. Let $A \in \mathbb{C}^{m \times n}$ and b a fixed vector in $\mathbb{C}^m$. Then the set

$$\{x \in \mathbb{C}^n : Ax = b\}$$

   is a subspace of $\mathbb{C}^n$ if and only if

   a) $m = n$ *a*) $m < n$ *c*) $b \neq \theta_m$ *d*) $b = \theta_m$

10. If $A \in \mathbb{R}^{8\times 7}$ and $\rho_{_A} = 3$ then the number of vectors in any basis for $\mathcal{R}^{\perp}_{_{A^T}}$ must be

   a) 4   *a*) 5   *c*) 3   *d*) 1

II) In the following, state TRUE or FALSE: (Correct Answer 1 Mark/Wrong Answer −0.5 Mark/Not Attempted 0 Mark)

1. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of $\mathbb{F}^n$ then $\mathcal{W}$ defined below is also a subspace of $\mathbb{F}^n$:

$$\mathcal{W} \;=\; \{x \in \mathbb{F}^n : x = x_1 + x_2 : x_1 \in \mathcal{W}_1,\ x_2 \in \mathcal{W}_2\}$$

   TRUE

2. The subset of $\mathbb{R}^3$ defined as

$$\mathcal{S} = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ 4 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

   is a subspace of $\mathbb{R}^3$ FALSE

3. The set of vectors

$$\mathcal{S} = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for the following subspace of $\mathbb{R}^3$:

$$\mathcal{W} = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ \alpha - \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$ FALSE

4. Every nonempty subset of a linearly dependent set in $\mathbb{F}^n$ is linearly dependent FALSE

5. Consider the following two subsets of $\mathbb{F}^3$:

$$\mathcal{S}_1 = \left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{S}_2 = \left\{ v_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Then

$x \in \mathbb{F}^3$ can be expressed as a linear combination of the vectors in $\mathcal{S}_1$

$\Longleftrightarrow$

$x \in \mathbb{F}^3$ can be expressed as a linear combination of the vectors in $\mathcal{S}_2$ TRUE

6. If a subpace $\mathcal{W}$ of $\mathbb{F}^n$ has a spanning set containing 5 vectors, then no basis of $\mathcal{W}$ can have more than 5 vectors TRUE

7. If $A \in \mathbb{F}^{n \times n}$ then both A and A^T have the same rank and same nullity TRUE

8. Let $A \in \mathbb{C}^{3\times 3}$ and 2 and 3 be eigenvalues of A. If $\begin{pmatrix} 1 \\ 1 \\ -i \end{pmatrix}$ is an eigenvector corresponding to eigenvalue 2, the vector $\begin{pmatrix} 2 \\ -1 \\ -i \end{pmatrix}$ cannot be an eigenvector corresponding to eigenvalue 3

TRUE

9. For any matrix $A \in \mathbb{C}^{n \times n}$ the algebraic multiplicity of every eigenvalue is less than or equal to its geometric multiplicity FALSE

10. If $A \in \mathbb{C}^{n \times n}$ is such that $A^* = -A$ and $B = iA$ then all the eigenvalues of B are pure imaginary (where $i = \sqrt{-1}$) FALSE

III) In the following FILL IN THE BLANKS WITH APPROPRIATE AN SWERS: (Correct Answer 2 Marks/Wrong Answer or Not attempted 0 Mark)

1. If $\mathcal{W}$ is a subspace of $\mathbb{F}^{10}$ and dimension of $\mathcal{W}$ is 4,

$$\textit{dimension of } \mathcal{W}^{\perp} = 6$$

2. If $x, y \in \mathbb{R}^n$ are unit vectors then

$$||x + y||^2 + ||x - y||^2 = 4$$

* BONUS TO ALL

3. Let $\mathcal{W}$ be a subspace of $\mathbb{R}^4$. The orthogonal projection of the vector $x = \begin{pmatrix} 2 \\ 3 \\ -2 \\ -3 \end{pmatrix}$ onto $\mathcal{W}^{\perp}$ is the vector $\begin{pmatrix} -1 \\ -1 \\ 4 \\ -3 \end{pmatrix}$. Then the orthogonal projection of x onto the subspace $\mathcal{W}$ is given by

$$\begin{bmatrix} 3 \\ 4 \\ -6 \\ 0 \end{bmatrix}$$

4. If

$$A = \begin{pmatrix} 2 & 4 & 4 & 2 \\ 0 & 2 & 9 & -6 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

the eigenvalues of A and their algebraic multiplicities are given by

$\lambda_1 = 2 \quad , \quad a_1 = 3$

$\lambda_2 = -1 \quad , \quad a_2 = 1.$

5. A Hermitian matrix $A \in \mathbb{C}^{5\times 5}$ has characteristic polynomial

$$(\lambda - 1)^3(\lambda + 3)^2$$

dimension of Range of $(A + 3I)) = 3$

IV) In the following give reasons for your answers and show the details of your working:

1. Let $\mathcal{W}$ be a subspace of R^4 defined as,

$$\mathcal{W} = \left\{ \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \\ \alpha-\beta \end{pmatrix} : \alpha,\ \beta \in \mathbb{R} \right\}$$

Answer the following:

(a) Show that the set of vectors,

$$\mathcal{S} = \left\{ u = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix},\ v = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix},\ w = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix} \right\}$$

is a spanning set for $\mathcal{W}$ but not a basis for $\mathcal{W}$
(6 Marks)

$$W = \mathcal{L}\left[\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}\right\}\right]$$

$$S = \{u, v, w\}$$

$$2u + v = w$$

$$\Rightarrow \mathcal{L}[S] = \mathcal{L}[S \setminus \{w\}]$$

$$= \mathcal{L}[\{u, v\}]$$

$$\alpha u + \beta v = \alpha \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha+\beta \\ \alpha-\beta \\ 2\alpha \\ 2\beta \end{bmatrix} = \begin{bmatrix} \alpha+\beta \\ \alpha-\beta \\ (\alpha+\beta)+(\alpha-\beta) \\ (\alpha+\beta)-(\alpha-\beta) \end{bmatrix}$$

$$\Rightarrow \mathcal{L}[\{u, v\}] = W = \mathcal{L}[S]$$

since S is linearly dependent

$\Rightarrow$ S is not a basis

but $\mathcal{L}[S] = W$.

(b) Show that each of the following set of vectors is a basis for $\mathcal{W}$:

$$\begin{aligned} \mathcal{S}_1 &= \{u, v\} \\ \mathcal{S}_2 &= \{u, w\} \\ \mathcal{S}_3 &= \{v, w\} \end{aligned}$$

(6 Marks)

from previous, u & v are linearly independent

& $\mathcal{L}[\{u, v\}] = W$.

$\Rightarrow$ $\mathcal{L}[S_1] = W$ & S_1 is L.I.

$\Rightarrow$ S_1 is basis.

$S_2 = \{u, w\} = \{u, 2u+v\}$.

u & w are L.I.

& $\mathcal{L}[S_2] = W$.

$\Rightarrow$ Hence basis.

$S_3 = \{v, w\} = \{v, 2u+v\}$.

v & w are L.I.

$\mathcal{L}[S_3] = W$

$\Rightarrow$ Hence basis.

2. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ defined as

$$\mathcal{W} = \left\{ \begin{pmatrix} \alpha + 2\beta + 3\gamma \\ \alpha + \gamma \\ \alpha + 2\beta + \gamma \\ \alpha - \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

Answer the following:

(a) Show that the set of vectors

$$\mathcal{S} = \left\{ u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is a basis for $\mathcal{W}$
(3 Marks)

$$W = \mathcal{L}\left[\left\{ \overset{u}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}, \overset{v}{\begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}}, \overset{w}{\begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}} \right\} \right]$$

$\Rightarrow \quad W = \mathcal{L}[S]$

$\lambda_1 u + \lambda_2 v + \lambda_3 w = 0$

$$\lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$

$\lambda_1 + \lambda_3 = 0$

$\lambda_1 + 2\lambda_2 + \lambda_3 = 0$

$\lambda_1 - \lambda_3 = 0$

$\Rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = 0$

Hence basis.

(b) Apply Gram-Schmidt process to the above basis to get an orthonormal basis for $\mathcal{W}$
(5 Marks)

$$q_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$v_2 = u_2 - (u_2, q_1)\, q_1$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} - (1+1) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$v_3 = u_3 - (u_3, q_2)\, q_2 - (u_3, q_1)\, q_1$$

$$= \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} - 2 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad q_3 = \frac{v_3}{\|v_3\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

(c) Express the vector $x = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$ in $\mathcal{W}$ as a linear combination of the vectors in the orthonormal basis obtained above
(3 Marks)

$$\begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$2 = \frac{\lambda_1 + \lambda_2 + \lambda_3}{2} \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 4 \quad \text{— (I)}$$

$$4 = \frac{\lambda_1 - \lambda_2 + \lambda_3}{2} \Rightarrow \lambda_1 - \lambda_2 + \lambda_3 = 8 \quad \text{— (II)}$$

$$6 = \frac{\lambda_1 + \lambda_2 - \lambda_3}{2} \Rightarrow \lambda_1 + \lambda_2 - \lambda_3 = 12 \quad \text{— (III)}$$

$$8 = \frac{\lambda_1 - \lambda_2 - \lambda_3}{2} \Rightarrow \lambda_1 - \lambda_2 - \lambda_3 = 16 \quad \text{— (IV)}$$

add (I) & (IV) $\Rightarrow 2\lambda_1 = 20$, $\boxed{\lambda_1 = 10}$

(I) − (III) $\Rightarrow 2\lambda_3 = -8$, $\boxed{\lambda_3 = -4}$

(I) − (II) $\Rightarrow 2\lambda_2 = -4$, $\boxed{\lambda_2 = -2}$

$$\boxed{x = 10q_1 - 2q_2 - 4q_3}$$

3. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^5$ defined as

$$\mathcal{W} = \left\{ \begin{pmatrix} \alpha+\beta \\ 0 \\ \alpha-\beta \\ \alpha+\beta \\ \alpha-\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

Answer the following:

$$W = \mathcal{L}\left[\left\{ \overset{U}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \right\}, \left\{ \overset{V}{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}} \right\} \right]$$

(a) Is the vectors $\underbrace{\begin{pmatrix} 4 \\ 0 \\ 2 \\ 4 \\ 2 \end{pmatrix}}_{x}$ in $\mathcal{W}$

(4 Marks)

$$\boxed{x = 3u + v}$$

(b) For the vector $x = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$ in $\mathbb{R}^5$ find the orthogonal projection of x onto $\mathcal{W}^\perp$

(4 Marks)

u & v are L.I. & span W, Hence $\{u, v\}$ is a basis.

Also $(u, v) = 0 \Rightarrow$ orthogonal basis.

$\therefore q_1 = \dfrac{u}{\|u\|} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$, $q_2 = \dfrac{v}{\|v\|} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$$\boxed{\begin{aligned} x_{W^\perp} &= x - x_W. \\ x_{W^\perp} &= x - (6q_1 + 2q_2) \end{aligned}}$$

$\{q_1, q_2\}$ is o.n.b

$x \cdot q_1 = \frac{5}{2} + \frac{3}{2} + \frac{3}{2} + \frac{1}{2} = \frac{12}{2} = 6$

$x \cdot q_2 = \frac{5}{2} - \frac{3}{2} + \frac{3}{2} - \frac{1}{2} = \frac{4}{2} = 2$

$x_W = 6q_1 + 2q_2$

$x_W = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} - 6\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - 2\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$$\boxed{x_{W^\perp} = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 \end{bmatrix}^T}$$

4. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix prove the following:

(a) $(Ax, y) = (x, Ay)$ for all $x, y \in \mathbb{C}^n$
(2 Marks)

(b) (Ax, x) is real for every $x \in \mathbb{C}^n$
(2 Marks)

(c) All the eigenvalues of A are real
(2 Marks)

(d) Eigenvectors corresponding to distinct eigenvalues are orthogonal to each other
(3 Marks)

(a) $(Ax, y) = y^*(Ax)$

$= (y^* A) \cdot x$

$= (A^* y)^* \cdot x$

$= (x, A^* y)$

but $A = A^*$ (Hermitian)

$\Rightarrow (Ax, y) = (x, Ay)$

(b) From above let $y = x$

$(Ax, x) = (x, Ax)$

$(Ax, x) = \overline{(Ax, x)}$

$\Rightarrow (Ax, x)$ is real

(c) Let $\varphi \neq \theta_n$ such that $A\varphi = \lambda \varphi$

where λ is eigen value & φ is associated eigen vector

$(A\varphi, \varphi) = (\lambda\varphi, \varphi) = \lambda(\varphi, \varphi)$

$\Rightarrow \lambda = \dfrac{(A\varphi, \varphi)}{(\varphi, \varphi)} = \dfrac{\text{real}}{\text{real}} = \text{real}.$

(d) Let, λ & μ be 2 distinct eigen values such that

$$A\varphi = \lambda\varphi \quad \& \quad A\psi = \mu\psi \quad , \quad \varphi, \psi \neq \theta_n$$

$$\lambda(\varphi,\psi) = (\lambda\varphi,\psi) = (A\varphi,\psi) = (\varphi, A\psi) = (\varphi, \mu\psi) = \mu(\varphi,\psi)$$

$$\Rightarrow \lambda(\varphi,\psi) = \mu(\varphi,\psi)$$

$$(\lambda-\mu)(\varphi,\psi) = 0$$

since λ & μ are distinct

$$\Rightarrow \lambda-\mu \neq 0$$

$$\Rightarrow (\varphi,\psi) = 0$$

Hence osthogonal.