Name: **SR No.:** **Dept.:**

Maximum Points: 20 **E2-243: Quiz 4** **Duration**: 45 minutes

---

1. State whether the following are **TRUE** or **FALSE**. (**5** points)
   (**1** Point for correct answer, **-0.5** for wrong answer and **0** for no attempt).

   a) Any matrix $A \in \mathbb{C}^{n \times n}$ is diagonalizable over $\mathbb{C}$ if and only if the geometric multiplicity, g_j is equal to the algebraic multiplicity a_j for every eigen value λ_j of A. ______
   Answer: TRUE

   b) If $v_1, \ldots, v_r$ are eigen vectors that correspond to distinct eigen values $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set$\{v_1, \ldots, v_r\}$ is linearly independent. ______
   Answer: TRUE

   c) Any vector $x \in \mathbb{C}^n$ can be uniquely decomposed as the sum of a vector in W and a vector orthogonal to W, where W is a subspace of $\mathbb{C}^n$. ______
   Answer: TRUE

   d) $A^\star A$ is Hermition Matrix $\implies$ A is Hermition Matrix. ______
   Answer: FALSE

   e) Pseudo Inverse, $A^\dagger$, of a matrix A always exist. ______
   Answer: TRUE

2. If x is an eigen vector for A corresponding to the eigen value λ , what is A^3x? (**4** points)

   Answer: we know that
   $Ax = \lambda x$
   Left multiplying by A both sides
   $A(Ax) = A(\lambda x) \implies (A \cdot A)x = A(\lambda x) \implies A^2x = \lambda(Ax) \implies A^2x = \lambda(\lambda x) \implies A^2x = \lambda^2 x$
   Again multiplying by A both sides
   $A(A^2x) = A(\lambda^2 x) \implies (A \cdot A^2)x = A(\lambda^2 x) \implies A^3x = \lambda^2(Ax) \implies A^3x = \lambda^2(\lambda x)$
   $\implies A^3x = \lambda^3 x$

3. For $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$ with $\lambda_1 = 10$ and $\lambda_2 = 3$, find a basis for the eigen space corresponding to listed eigen values. (**5** points)

Answer:

$A_{\lambda_j} = (\lambda_j I - A)$

$\mathcal{N}_{A_{\lambda_j}} = \{x \in \mathbb{F}^n; A_{\lambda_j} x = \theta_n\}$

$A_{\lambda_1} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

For Null Space $\mathcal{N}_{A_{\lambda_1}}$

$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$6x_1 + 2x_2 = 0$ and $3x_1 + x_2 = 0$.

$\implies x_2 = -3x_1$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ -3\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\mathcal{N}_{A_{\lambda_1}} = \mathcal{L}\Big[\begin{pmatrix} 1 \\ -3 \end{pmatrix}\Big]$

Hence, $\Big\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\Big\}$ is a basis for $\mathcal{N}_{A_{\lambda_1}}$.

$A_{\lambda_2} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

For Null Space $\mathcal{N}_{A_{\lambda_2}}$

$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + 2x_2 = 0$ and $3x_1 - 6x_2 = 0$.

$\implies x_1 = 2x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\mathcal{N}_{A_{\lambda_2}} = \mathcal{L}\Big[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\Big]$

Hence, $\Big\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\Big\}$ is a basis for $\mathcal{N}_{A_{\lambda_2}}$

4. Find the singular value decomposition (product form) of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$ (**6** points)

Answer:

First, compute $A^TA = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$. The eigen values of A^TA are 18 and 0, with corresponding unit eigen vectors

$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

These unit vectors form the columns of V:

$V = [v_1 v_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

The singular values are $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = 0$. Since there is only one nonzero singular value, the "matrix" D may be written as a single number. That is $D = 3\sqrt{2}$. The matrix Σ is the same size as A , with D in its upper-left corner:

$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

To construct U, first construct Av_1 and Av_2 :

$Av_1 = \begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix}$, $Av_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

As a check on the calculations, verify that $||Av_1|| = \sigma_1 = 3\sqrt{2}$. Of course, $Av_2 = 0$ because $||Av_2|| = \sigma_2 = 0$. The only column found for U so far is

$u_1 = (1/3\sqrt{2}) \cdot Av_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$

The other columns of U are found by extending the set $\{u_1\}$ to an orthonormal basis for $\mathbb{R}^3$. In this case, we need two orthogonal unit vectors u_2 and u_3 that are orthogonal to u_1. Each vector must satisfy $u_1^Tx = 0$, which is equivalent to the equation $-x_1 - 2x_2 + 2x_3 = 0$.A basis for the solution set of this equation is

$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$,$w_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Apply Gram-Schmidt process to $\{w_1, w_2\}$, and obtain

$u_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$

Finally , set $U = [u_1 u_2 u_3]$, take Σ and V^T from above, and write

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3/\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$