Conclusion: We can make all entries below leading diagonal of the matrix by orthogonal/unitary transformation.

Observe all eigenvalues eventually end up diagonally. But to start with process of $n \times n$ matrix we just have to find one root. Then $(n-1) \times (n-1)$ matrix again find just one root. Repeat process.

Ex: $A = \begin{pmatrix} 4 & 8 & -2 \\ -3 & 6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$

$$c(\lambda) = |\lambda I - A| = (\lambda+3)(\lambda+2)^2$$

eigen values are $\lambda_1 = -3$; $\lambda_2 = -2$,
$a_1 = 1$; $a_2 = 2$

Find eigen spaces and g.m

eigenspace corresponding to $\lambda_1 = -3$.

$A - \lambda_1 I = \theta_3$

$(A + 3I)x = \theta_3$.

$$\begin{pmatrix} 7 & 8 & -2 \\ -3 & -3 & 1 \\ 9 & 12 & -2 \end{pmatrix} x = \theta_3$$

Solutions are of the form $\alpha \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$

eigenspaces are $W_1 = \left\{ \alpha \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$

$u_1 = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ is the basis for W_1 (eigenvector corr to $\lambda_1 = -3$)

Dimension $W_1 = 1$
$g_1 = 1$

Eigenspace corresponding to $\lambda_2 = -2$.

$(A - \lambda_2 I)x = \theta_3$

$(A+2I)\, x = \theta_3$

$$\begin{pmatrix} 6 & 8 & -2 \\ -3 & -4 & 1 \\ 9 & 12 & -3 \end{pmatrix} x = \theta_3$$

solutions are of the form,

$$W_2 = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} : \alpha\beta \in \mathbb{C} \right\}$$

$$u_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

$\dim W_2 = 2 = g_2$

$\lambda_1 = -3$ $\quad \lambda_2 = -2$

$a_1 = 1$ $\quad a_2 = 2$

$g_1 = 1$ $\quad g_2 = 2$

The $a_j = g_j$ for every eigenvalue λ_j

$\therefore A$ is diagonalizable.

The diagonalizing matrix,

$P = (u_1 \ u_2 \ u_3)$

$$= \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ -3 & 3 & 4 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

verify

$AP = PD$.

However for $P^{-1}AP = P^{-1}PD$

inverse has to exist.

Ex 2: $A = \begin{pmatrix} 6 & 8 & -2 \\ -3 & -4 & 1 \\ 9 & 12 & -3 \end{pmatrix}$

$c(\lambda) = (\lambda+3)(\lambda+2)^2$

$\lambda_1 = -3, \ a_1 = 1$

$\lambda_2 = -2, \ a_2 = 2$

eigenspace corresponding to $\lambda_1 = -3$ is same as previous

eigenspace corresponding to $\lambda_2 = -2$ is

$$W_2 = \left\{ \alpha \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

$u_2 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$ is basis for W_2

Michael Athea-Talk
Index theory.

$\lambda_1 = -3;\ a_1 = g_1 = 1.$

$\lambda_2 = -2;\ a_2 = 2 > g_2 = 1$

Not diagonalizable.

$A = PDP^{-1}$ (decomposition) --- helps in defining functions of matrix.

$A^{-1} = PD^{-1}P^{-1}$

Schur decomposition: (weaker version)

The idea is the following

Given $A \in \mathbb{C}^{n\times n}$, we find a matrix $P \in \mathbb{C}^{n\times n}$ which is invertible such that,

$P^{-1}AP$ is a upper triangular matrix in $\mathbb{C}^{n\times n}$.

Can be done for all matrices A in $\mathbb{C}^{n\times n}$

1) $A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$

$c(\lambda) = (\lambda+3)(\lambda+2)^2$

Choose $\lambda_1 = -3$

Solve $(A+3I)x = \theta_3$

$u_1 = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$

Choose $u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $u_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

u_1, u_2, u_3 are linearly independent forms basis for $\mathbb{C}^3$

Steps: $A \in \mathbb{C}^{n\times n}$

1) Start with unit eigenvalue λ_1 of A.

2) Find an eigen vector u_1 correspon to eigenvalue λ_1.

3) choose any $(n-1)$ vectors $u_2\, u_3 \ldots u_n$ Such that $u_1\, u_2\, u_3 \ldots u_n$ forms the basis for $\mathbb{C}^n$

→ Apply Gram Schmidt to get orthonormal basis

Let $P = [u_1 \dots u_n]$

4) Find P

5) $P^{-1}AP = ?$

$$P = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ -3 & 0 & 0 \end{pmatrix}$$

$P^{-1}AP$ is of the form,

$$AP = A[u_1\ u_2\ u_3 \dots u_n]$$
$$= [Au_1\ Au_2 \dots Au_n]$$
$$= [\lambda_1 u_1\ Au_2 \dots Au_n]$$

Now $Au_2 \in \mathbb{C}^n$ and $u_1 \dots u_n$ is basis for $\mathbb{C}^n$

$\therefore$ Au_2 = linear combination of u_j

$$Au_2 = \alpha_{12} u_1 + \alpha_{22} u_2 + \alpha_{j2} u_j + \alpha_{n2} u_n.$$

Similarly

$$Au_j = \alpha_{1j} u_1 + \alpha_{2j} u_2 + \dots + \alpha_{jj} u_j + \dots + \alpha_{nj} u_n.$$

$$Au_1 = \lambda_1 u_1 + 0u_2 + \dots + 0u_n$$

$$T = \begin{pmatrix} \lambda_1 & \alpha_{12} & \alpha_{13} & \alpha_{1j} & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{2j} & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_{n2} & \alpha_{n3} & \alpha_{nj} & \alpha_{nn} \end{pmatrix}$$

Now $AP = [u_1 \dots u_n]\, T$

$$AP = PT$$

$$P^{-1}AP = T.$$

$$P^{-1}AP = \left(\begin{array}{c|c} \lambda_1 & K_{1\times(n-1)} \\ \hline 0 & \\ \vdots & A_{2(n-1\times n-1)} \\ 0 & \end{array}\right)$$

continued...

$$Au_2 = \begin{pmatrix} 0 \\ 5 \\ -6 \end{pmatrix}$$

$$Au_2 = 2u_1 + 4u_2 + 3u_3$$

$$Au_3 = \begin{pmatrix} -6 \\ -11 \\ 9 \end{pmatrix}$$

$$Au_3 = -3u_1 - 12u_2 - 8u_3$$

$$T = \left(\begin{array}{c|cc} -3 & 2 & -3 \\ \hline 0 & 4 & -12 \\ 0 & 3 & -8 \end{array}\right).$$

Repeat the process.

Symmetric matricies all always diagonalizable.

Recall:

$A \in \mathbb{C}^{n \times n}$

A is diagonalizable iff am = gm of every eigen value

Not all $A \in \mathbb{C}^{n \times n}$ are diagonalizable

Schur said all matrices $A \in \mathbb{C}^{n \times n}$ is triangularizable by a similarity transformation i.e. $\exists P \in \mathbb{C}^{n \times n}$ (invertible)

$P^{-1}AP$ = upper triangular matrix T.

Stronger version:

Every $A \in \mathbb{C}^{n \times n}$ is triangularizable by a unitary transformation i.e. $\exists P \in \mathbb{C}^{n \times n}$ s.t $P^* = P^{-1}$ and $P^* A P = T$, an upper triangular matrix.

There's a class of matricies in $\mathbb{C}^{n \times n}$ that are always diagonalizable by a unitary transformation.

$\forall A \in \mathcal{H}$ $\exists$ $P \in \mathbb{C}^{n \times n}$ s.t. $P^* = P^{-1}$ and $P^* AP$ is diagonal.

$\mathcal{H}$ - Hermitian matricies.

$P^* = P^{-1}$

Hermitian - Symmetric

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be Hermitian if $A^* = A$

$A^* = (\overline{A^T})$.

A real Hermitian matrix $A^T = A$ and these are called real symmetric matricies.

observations :

$A \in \mathbb{C}^{n \times n}$

Suppose $x, y \in \mathbb{C}^n$

$$(x,y) = y^* x = \sum_{j=1}^{n} x_j \overline{y_j} .$$

$$(x,x) \geq 0 = \|x\|^2$$

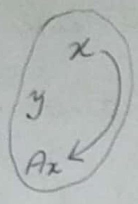
$$\|x\| = \sqrt{(x,x)}$$

$$(x,y) = \overline{(y,x)} \quad - \circledast$$

$$\overset{\parallel}{\sum_{j=1}^{n}} x_j \overline{y_j} = \overline{\sum_{j=1}^{n} y_j \overline{x_j}}$$

$A \in \mathbb{C}^{n \times n}$, then take any $x \in \mathbb{C}$

$Ax \in \mathbb{C}^n$



$\mathbb{C}^n$

Let $y \in \mathbb{C}^n$

Look at
$$\begin{aligned}(Ax, y) &= y^*(Ax). \\ &= y^*((A^*)^* x) \\ &= (y^*(A^*)^*) x \\ &= ((A^* y)^*) x \\ &= (x, A^* y)\end{aligned}$$

$\therefore A \in \mathbb{C}^{n \times n} \Rightarrow$

$$(Ax, y) = (x, A^* y) \quad \forall\, x, y \in \mathbb{C}^n.$$

In particular $A \in \mathbb{C}^{n \times n}$ is Hermitian.

$$(Ax, y) = (x, Ay) \quad \forall\, x, y \in \mathbb{C}^n. \qquad - ①$$

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$y^*(Ax) = (Ay)^* x \;\forall\; x, y \in \mathbb{C}^n$

This property gives two fundamental properties regarding eigenvalues and eigenvectors of a Hermitian matricies.

Let $A \in \mathcal{H}^{m\times n}$

1) Suppose λ is an eigenvalue of A

$\Rightarrow \exists\; u \neq \theta_n \ni Au = \lambda u$ (an eigenvector)

$\Rightarrow u^*(Au) = u^*(\lambda u)$

$\Rightarrow (Au, u) = (\lambda u, u)$

$= \lambda (u, u)$

$= \lambda \|u\|^2$ Since $\|u\| \geq 0$.

$\Rightarrow \lambda = \dfrac{(Au, u)}{\|u\|^2}$ ....

NOTE: $(Ax, x) = (x, Ax)$ if $A \in \mathcal{H}$

$\therefore (Ax, x)$ real $\forall x$. — ②

$\lambda = \dfrac{\overset{(Au,u)}{\text{real}}}{\underset{\|u\|^2}{\text{real}}} = \text{real}$.

Every eigenvalue of a Hermitian matrix must be real.

λ, μ are two distinct eigen values.

Let u be any eigenvector corresponding to λ

v be any - - - - - - - - - - to μ

$Au = \lambda u$ and $Av = \mu v$.

$\lambda (u v) = (\lambda u, v)$

$= (Au, v)$ Since Hermitian

$= (u, Av)$ $= A \in \mathcal{H}$

$= (u, \mu v)$

$= \mu(u,v)$

$\lambda(u,v) = u(u\,v)$

$(\lambda-\mu)(u,v) = 0 \Rightarrow (u,v) = 0 \quad (\because \lambda \neq 0)$

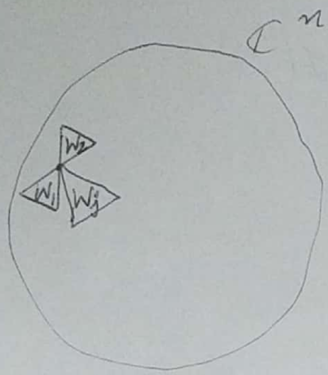
Eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

$\lambda_1, \ldots, \lambda_k$ are distinct eigenvalue of A. (real).

$a_1, \ldots, a_k$ are a.m.

$c(\lambda) = (\lambda-\lambda_1)^{a_1} \ldots (\lambda-\lambda_k)^{a_k}$

$W_j = \{x : Ax = \lambda_j x\}$ eigenspace corresponding to eigenvalue λ_j

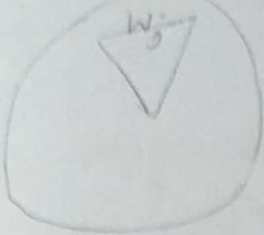


Every $x \in W_j$ is orthogonal to every W_i $(i \neq j)$

g_j = gm of λ_j = dim W_j

$A \in \mathbb{F}^{n\times n}$

$\Rightarrow g_j = a_j$ for every eigenvalue λ_j, $\therefore$ A is diagona-lizable.



Pick a basis $B_j = u_1^{(j)}, \dots \; u_{a_j}^{(j)}$ for W_j

$B_1 = u_1^{(1)}, u_2^{(1)}, \dots, u_{a_1}^{(1)}$

$B_2 = u_1^{(2)}, u_2^{(2)}, \dots, u_{a_2}^{(2)}$

$B_j = B_1^{(j)} \; B_2^{(j)} \dots . \; u_{a_j}^{(j)}$

$B_K = u_1^{(K)} \; u_2^{(K)} \dots \dots u_{a_K}^{(K)}$.

We can apply Gram Schmidt to each one of these and get o-n basis for W_j

w.l.g let these be the o.n basis.

Let P = Columns are these on vectors then $P^T A P = \text{diag}$

$P = [u_1^{(1)} \dots \; u_{a_1}^{(1)} \; u_1^{(2)} \dots \; u_{a_2}^{(2)} \dots \dots u_1^{(K)} \dots u_{a_K}^{(K)}]$

$P^* = P^{-1}$ and.

$$P^*AP = \begin{pmatrix} \underbrace{\lambda_1 \ddots}_{a_1 \text{ time}} & & \\ & \underbrace{\lambda_2 \ddots}_{a_2 \text{ time}} & \\ & & \underbrace{\lambda_K \ddots}_{a_K \text{ time}} \end{pmatrix} = \begin{pmatrix} \lambda_1 I_{a_1 \times a_1} & & \\ & \lambda_2 I_{a_2 \times a_2} & \\ & & \lambda_3 I_{a_K \times a_K} \end{pmatrix}.$$

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$\lambda_1, \lambda_2, \ldots \lambda_n$ eigenvalues

$u_1, u_2 \ldots u_n$ o.n eigenvectors

These form a basis for $\mathbb{C}^n$

$x \in \mathbb{C}^n \Rightarrow x$ can be expanded as a l.c of them basis vector.

$$x = (u_1^* x)u_1 + (u_2^* x)u_2 + \ldots + (u_n^* x)u_n.$$

$$\Rightarrow Ax = \lambda_1(u_1^* x)u_1 + \lambda_2(u_2^* x)u_2 + \ldots + \lambda_k(u_k^* x)u_k.$$

$$= \lambda_1 u_1(u_1^* x) + \lambda_2 u_2(u_2^* x) + \ldots + \lambda_k u_k u_k^* x.$$

$$= [\lambda_1 \underbrace{\underset{n\times 1}{u_1}\underset{1\times n}{u_1^*}}_{n\times n} + \lambda_2 u_2 u_2^* + \ldots + \lambda_k u_k u_k^*]x. \quad \forall x$$

$$\therefore \quad A = \lambda_1 u_1 u_1^* + \lambda_2 u_2 u_2^* + \ldots + \lambda_n u_n u_n^*.$$

eigendecomposition of A.

$\lambda_1 u_1 u_1^*$ – rank 1.

$\lambda_2 u_2 u_2^*$ – rank 2.

Recall:

$A \in \mathbb{C}^{n \times n}$ | $A \in \mathbb{R}^{n \times n}$

1) A Hermitian $A^* = A$ $\quad (A^* = \overline{A^T})$ | A is symmetric if $A^T = A$

2) $(Ax, y) = (x, Ay)\ \forall x, y \in \mathbb{C}^n$ | $(Ax, y) = (x, Ay)\ \forall x, y \in \mathbb{R}^n$

$y^*(Ax) = (Ay)^* x$ | $y^T(Ax) = (Ay)^T x$.

3) (Ax, x) is real $\forall x \in \mathbb{C}^n$ | obviously (Ax, x) real $\forall x \in \mathbb{R}^N$

$x^*(Ax)$ is real $\forall x \in \mathbb{C}^n$

4) All eigenvalues are real | All eigenvalues are real

5) eigenvectors corresponding to different eigenvalues are orthogonal

$\lambda \neq \mu$, u, v eigenvectors of λ and μ then $v^* u = 0$ | $v^T u = 0$.

6) $a \cdot m = g \cdot m$ for every eigenvalue | (same)

7) A is diagonalizable | (same)

* $A \in \mathbb{C}^{n \times n}$ Hermitian | $A \in \mathbb{R}^{n \times n}$ (Symmetric)

$\exists P \in \mathbb{C}^{n \times n}$ s.t. $P^* P = I$ (i.e. $P^{-1} = P^*$) and

$P^* A P = D$, diagonal

$\exists P \in \mathbb{R}^{n \times n}$ s.t. $P^T P = I$ i.e., $P^{-1} = P^T$ and

$P^T A P = D$, diagonal

Diagonal entries in D are eigenvalues

(Same)

8) Eigendecomposition

Σ (eigenvalue) times $(\text{eigenvector})_{n\times 1}$ $(\text{eigenvector})^*_{1\times n} = A$

Σ (eigenvalue)(eigenvector) $(\text{eigenvector})^T = A$

eigenvector chosen orthonormal.

Example :

1) $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}$ (real Symm matrix)

$c(\lambda) = (\lambda-6)(\lambda-3)(\lambda-2)$

$\lambda_1 = 6$ $\quad \lambda_2 = 3$ $\quad \lambda_3 = 2$ (all real)

$a_1 = 1$ $\quad a_2 = 1$ $\quad a_3 = 1$

$g_1 = 1$ $\quad g_2 = 1$ $\quad g_3 = 1$

$\therefore$ diagonalizable.

Eigenspaces :

$\rightarrow \lambda_1 = 6$

$(A - \lambda_1 I)x = 0_3$

Solved : $W_1 = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$

eigenvector $v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\rightarrow \lambda_2 = 3$

$W_2 = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$

$v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$u_2 = \frac{1}{\sqrt{3}} v_2$

$\lambda_3 = 2$

$$W_3 = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$$v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

$$u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{6}} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \end{pmatrix}$$

$$P^T P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

What does eigendecomposition say?

$$\overset{\lambda}{(6)} \overset{V}{\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}} \overset{V^*}{\frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}} + (3) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} +$$

$$2 \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}.$$

$$= \overset{\text{rank1}}{\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}} + \overset{\text{rank1}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} + \overset{\text{rank 1}}{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}.$$

$$A_{rank 3} = \sum (\text{Rank } 1)$$

Ex 2:

$$A = \begin{pmatrix} 5 & 3 & 1 & -1 \\ 3 & 5 & -1 & 1 \\ 1 & -1 & 5 & 3 \\ -1 & 1 & 3 & 5 \end{pmatrix}$$ real symm

$c(\lambda) = \lambda(\lambda-8)^2(\lambda-4)$

$\lambda_1 = 8, \quad \lambda_2 = 4, \quad \lambda_3 = 0$

$a.m = 2, \quad a_2 = 1 \quad a_3 = 1$

$g_1 = \quad g_2 = \quad g_3 =$

$\rightarrow W_1 \Rightarrow (A - 8I)x = \theta_4$

$$W_1 = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \beta \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$u_1^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad u_2^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$\rightarrow \lambda_2 = 4$

$$W_2 = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \\ -\alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$$v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$u_3^{(2)} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$\rightarrow \lambda_3 = 0$

$$W_3 = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ -\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$$v_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$u_1^{(3)} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$u_1^{(1)}\ u_2^{(1)}\ u_1^{(2)}\ u_1^{(3)}$ are Orthogonal.

$$P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/2 & 1/2 \\ 1/\sqrt{2} & 0 & -1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & -1/2 & 1/2 \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} 8 & & & \\ & 8 & & 0 \\ & & 4 & \\ 0 & & & 0 \end{pmatrix}$$

eigendecomposition,

$$(8)\, u_1^{(1)}(u_1^{(1)})^T + 8\, u_2^{(1)}(u_2^{(1)})^T + 4 u_1^{(2)}(u_1^{(2)})^T + 0\, u_1^{(3)}(u_1^{(3)})^T$$

$= A.$

A real Symmetric - Nice Theory.
Nice decomposition.

we use this to develop nice decomposition for general matrices.

$A \in \mathbb{R}^{m \times n}$

$N = A^* A$ $\qquad\qquad$ $M = AA^*$

$N = n \times n.$ $\qquad\qquad$ $M = m \times m.$ $\qquad\qquad$ choose $(n, m)_{min}$

Assuming $n < m$,

$N = A^T A \in \mathbb{R}^{n \times n}$ (Square)

$N^T = (A^T A)^T$

$= A^T A$

$N^T = N$ (Symmetric)

Now i can apply Symmetric matrix theory to N and get decomposition of N.

How to extract decomposition of A from decomposition of N.?

N is Square, Symmetric

3) $x \in \mathbb{R}^n \underset{\text{s.t}}{\Rightarrow} Ax = \theta_m$

$\Rightarrow \underset{n \times m}{A^T}\underset{m \times 1}{(Ax)} = A^T(\theta_m) = \theta_n$

Vector $x \in$ Null Space A

$\Rightarrow x \in$ Null Space N

$\Rightarrow Nx = \theta_n$

4) $Nx = \theta_n \Rightarrow A^T A x = \theta_n$

$\Rightarrow x^T(A^T A x) = x^T \theta_n = 0$

$\Rightarrow (Ax)^T (Ax) = 0$

$\|Ax\|^2 = 0$

$\Rightarrow Ax = \theta_n$

If $\underbrace{\{x \in \mathbb{R}^n \text{ s.t } Ax = \theta_n\}}_{\text{Null Space of A}} = \underbrace{\{x \in \mathbb{R}^n \; Nx = \theta_n\}}_{\text{Null Space of N}}$

$\dim(N_A) = \dim(N_N)$

Nullity A = Nullity N.

$\therefore$ Rank nullity theorem Says Rank(A) = Rank(N)

But, Rank(A) = Rank(A^T) = Rank(M).

- Rank (A) = Rank(N) = Rank(A^T) = Rank(M)
- Nullity (A) = Nullity N.