40.5

E2:243 TEST 1

(September 20, 2019) (2PM -3:30PM)

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Sequence Number: 51

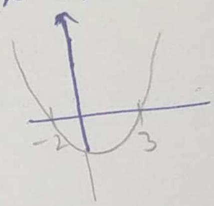
Answer All Questions (Maximum Marks:50)

I) In the following, in each question only one alternative is correct. Tick ($\surd$) the correct alternative:
(Correct Answer 1 Mark/Wrong Answer -0.5 Mark/Not Attempted 0 Mark)
(Total: 8 Marks)

8

1. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = x^2 - x - 6$ is

$(x-3)(x+2)$



   (a) one-one but not onto
   (b) onto but not one-one
   (c) both one-one and onto
   ✓ (d) neither one-one nor onto

2. If $f : \mathcal{A} \longrightarrow \mathcal{B}$ and E, F are subsets of $\mathcal{B}$, then $f^{-1}(E \cup F') =$

   ✓ (a) $f^{-1}(E) \cup (f^{-1}(F))'$
   (b) $f^{-1}(E) \cup (f^{-1}(F'))'$
   (c) $f^{-1}(E') \cup f^{-1}(F)$
   (d) $f^{-1}(E') \cup f^{-1}(F')$

3. The sequence of real numbers $\{a_n\}_{n \in \mathbb{N}}$ defined as $a_n = \dfrac{n+1}{n}$ is

$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$

   (a) nondecreasing and bounded above
   (b) nondecreasing but not bounded above
   ✓ (c) nonincreasing and bounded below
   (d) nonincreasing but not bounded below

$\lim_{n \to \infty} \frac{n(1+1/n)}{n} = 1$

4. The sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ defined as

$$a_n = \frac{n+2}{n} + \sqrt{n+1} - \sqrt{n}$$

$\frac{n(1+2/n)}{n} \to 1$ $\quad$ $\sqrt{n+1}-\sqrt{n} \to \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$

(a) converges to 0
(b) converges to 1
(c) converges to 2
(d) does not converge

5. Consider the sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ defined as follows:

$$a_{2n-1} = \frac{2n-1}{2n}, \quad n = 1, 2, 3\cdots$$
$$a_{2n} = \frac{2n+1}{2n+2}, \quad n = 1, 2, 3, \cdots$$

$\frac{n(2-1/n)}{2n}$

$\left(\frac{1}{2}\right), \left(\frac{3}{4}\right), \left(\frac{3}{4}\right), \left(\frac{5}{6}\right), \left(\frac{5}{6}\right), \left(\frac{7}{8}\right)\left(\frac{7}{6}\right)$

Then

(a) $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$
(b) $\limsup_{n\to\infty} a_n > \liminf_{n\to\infty} a_n$
(c) $\limsup_{n\to\infty} a_n < \liminf_{n\to\infty} a_n$
(d) $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ do not exist

6. Suppose the sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is such that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \text{ exists and } = 0.7$$

Then

(a) The sequence $\{a_n\}_{n\in\mathbb{N}}$ converges but the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ need not converge
(b) Both the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{|a_n|\}_{n\in\mathbb{N}}$ converge
(c) Both the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{|a_n|\}_{n\in\mathbb{N}}$ do not converge
(d) The sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges but the sequence $\{a_n\}_{n\in\mathbb{N}}$ need not converge

7. The sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ defined on the interval $\mathcal{I} = [0, \infty)$ as

$$f_n(x) = \frac{1}{1+nx}$$

$\left|\frac{1}{1+nx}\right| < \epsilon$

$\frac{1}{1+nx} < \epsilon$

$\frac{1}{nx} < \epsilon$

$n > \frac{1}{\epsilon x}$

(a) does not converge pointwise on $\mathcal{I}$ but converges uniformly on $\mathcal{I}$
(b) neither converges pointwise on $\mathcal{I}$ nor uniformly on $\mathcal{I}$
(c) converges uniformly on $\mathcal{I}$ and hence also converges pointwise on $\mathcal{I}$
(d) converges pointwise on $\mathcal{I}$ but not uniformly on $\mathcal{I}$

8. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous real valued functions on the interval $\mathcal{I} = [-1, 1]$ converging uniformly on $\mathcal{I}$ to the function f then

(a) $\int_0^1 f(x)dx$ may not exist
(b) $\int_0^1 f(x)dx$ must exist but $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ may not exist
(c) $\int_0^1 f(x)dx$ and $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ both exist but may not be equal
(d) $\int_0^1 f(x)dx$ and $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ both exist and must be equal

II) In the following, state TRUE or FALSE:
(Correct Answer 1 Mark/Wrong Answer −0.5 Mark/Not Attempted 0 Mark)
(Total :8 Marks)

6.5

1. If $f : [0, 1] \longrightarrow \mathbb{R}$ is a real valued function defined on $[0, 1]$ and E, F are subsets of $\mathbb{R}$ then

$$f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$$

TRUE

2. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers and $a \in \mathbb{R}$. Then

$$a_n \longrightarrow a \Longleftrightarrow |a_n| \longrightarrow |a|$$

FALSE

3. The sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ defined as

$$a_n = \frac{3n+4+(-1)^n sin(3n+2)}{(\sqrt{n})^3+\sqrt{n}+1}$$

converges to 3

FALSE ✓

4. The sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ defined on the interval $\mathcal{I}=[0,1)$ as $f_n(x)=n^2x^n$ does not converge pointwise on $\mathcal{I}$ to the zero function

FALSE ✓

$|n^2x^n - 0| < \epsilon$
$n^2x^n < \epsilon$
$n^2 < \epsilon$

5. Let $\mathcal{I}$ be any interval in $\mathbb{R}$. If a sequence of continuous real valed functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $\mathcal{I}$ to a function real valued function f then f must also be continuous on $\mathcal{I}$

TRUE ✓

6. If a sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ defined on $\mathcal{I}=[0,\infty)$ converges uniformly on $\mathcal{I}$ to a function f and $\int_0^\infty f_n(x)dx < \infty$ then $\int_0^\infty f(x)dx < \infty$ and

$$\int_0^\infty f_n(x)dx \text{ must converge to } \int_0^\infty f(x)dx$$

FALSE ✓

Doesn't preserve integrability over infinite interval

7. If $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are sequences of real valued functions converging uniformly on $\mathcal{I}$ to f and g respectively then the sequence $h_n = f_n \times g_n$ must converge uniformly on $\mathcal{I}$ to $f \times g$

TRUE ✗

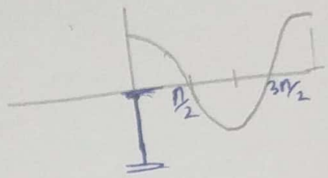
8. If the sequences of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ defined on the interval $\mathcal{I}$ converge uniformly on $\mathcal{I}$ to the functions f and g respectively then the sequence $\{h_n\}_{n\in\mathbb{N}}$ (where $h_n = f_n + g_n$) converges uniformly on $\mathcal{I}$ to the function $f+g$

TRUE ✓

III) In the following FILL IN THE BLANKS WITH APPROPRIATE ANSWERS:
(Correct Answer 1.5 Marks/Wrong Answer or Not attempted 0 Mark)
(Total: 9Marks)

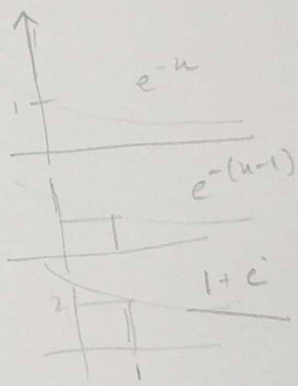
6

1. Let
$$f:[0,2\pi]\longrightarrow \mathbb{R}$$
be the function defined as $f(x)=cos(x)$. Let E be the subset of $\mathbb{R}$ defined as $E=\{x\in\mathbb{R}:x<0\}$. Then



$$\boxed{f^{-1}(E) = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)}$$ ✓

2. Let $\{E_n\}_{n\in\mathbb{N}}$ be the sequence of subsets of $\mathbb{R}^2$ defined as follows:
$$E_n=\left\{(x,y)\in\mathbb{R}^2:0\le x<1+e^{-(n-1)} \text{ and } 0\le y<1+e^{-(n-1)}\right\}$$
Then



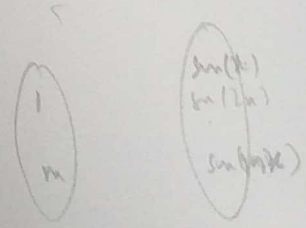
$$\boxed{\limsup_{n\to+\infty} E_n = \{(x,y)\in\mathbb{R}^2: 0\le x<1 \text{ and } 0\le y<1\}}$$ ✗

3. Let
$$A=\{1,2,3,\cdots,m\},\ B=\{sin(kx);1\le k\le n\}$$ $k\in\mathbb{Z}^+$, $x\in\mathbb{R}$

The total number of functions that can be defined from the set A to the set B is

$$\boxed{\infty}$$

✗



4. The sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ defined as

$$a_n = \frac{3n^2 + 4n + \sqrt{n}}{(5+2n)^2 + 3}$$

$$\frac{n^2\left(3 + \frac{4}{n} + \sqrt{\frac{1}{n^3}}\right)}{n^2\left(\left(\frac{5}{n}+2\right)^2 + \frac{3}{n^2}\right)}$$

$$\frac{3}{2^2} = \frac{3}{4}$$

converges to

$\boxed{3/4}$

$$\frac{4-9}{9+4}, \quad \frac{4-16}{16+4}$$

5. For the sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ defined as $\dfrac{4-n^2}{n^2+4}$

$$\frac{3}{5},\ 0,\ -\frac{5}{13},\ -\frac{12}{20},$$

$$\lim_{n\to\infty}\left\{\sup_{k\geq n} a_k - \inf_{k\geq n} a_k\right\}$$

is equal to

$\boxed{0}$

6. The sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ defined on the interval $\mathcal{I} = [0,\infty)$ as

$$f_n(x) = \frac{e^{nx}}{1+e^{nx}}$$

$$\frac{1}{\frac{1}{e^{nx}} + 1}$$

converges pointwise on $\mathcal{I}$ to the function f defined as

$$f(x) = \begin{cases} 1/2 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

IV) In the following give reasons for your answers and show the details of your working:
(Write the answers in the space provided below each question)

1. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of real numbers defined as

$$f_n = \frac{n^2+2}{2n^2+3}$$

9

Answer the following:

(a) Find the *lub* and *glb* of this sequence
(b) Show that the sequence is a Cauchy sequence
(c) Is the sequence nonincreasing?
(d) Does the sequence converge and if so to what value f does the sequence converge?
(e) Use the definition of convergence to show that the sequence converges to the f you obtainned in (d) above

(2 Marks for each part. Total 10 Marks)

a) $f_n = \frac{n^2+2}{2n^2+3}$ $n\in\mathbb{N}$

$\lim_{n\to\infty} \frac{n^2+2}{2n^2+3} = \lim_{n\to\infty} \frac{1+2/n^2}{2+3/n^2} = \frac{1}{2}$

$f_n = \left\{\frac{3}{5}, \frac{6}{11}, \frac{11}{21}, \frac{18}{35}, \right\}$ The sequence converges to $1/2$

Clearly it is a ~~d~~ non-increasing sequence, bounded below by $1/2$.

Hence

2

lub $= 3/5$
glb $= 1/2$

b) Cauchy Sequence: A sequence is called cauchy sequence if ~~that is~~ for every $\epsilon > 0$ $\exists$ a positive integer $N(\epsilon)$ s.t for $n,m \geq N(\epsilon) \Rightarrow |f_n - f_m| < \epsilon$ —(1)

Let $n > m$ $\left|\frac{n^2+2}{2n^2+3} - \frac{m^2+2}{2m^2+3}\right|$

$= \left|\frac{2m^2n^2 + 3n^2 + 2m^2 + 6 - [2m^2n^2 + 3m^2 + 4n^2 + 6]}{(2n^2+3)(2m^2+3)}\right|$

$= \left|\frac{-n^2+m^2}{(2n^2+3)(2m^2+3)}\right| < \epsilon$

1

$\frac{\cancel{n^2-m^2}\, m^2-n^2}{(2n^2+3)(2m^2+3)} < \epsilon$

$\frac{m^2\left(\frac{n^2}{m^2}+1\right)}{(2n^2+3)(2m^2+3)} < \epsilon$ $\because \frac{m^2}{2m^2+3} < 1$

$\frac{(n^2/m^2+1)}{(2n^2+3)} < \epsilon$

$\frac{n^2}{m^2(2n^2+3)} + \frac{1}{2n^2+3} < \epsilon$

$\frac{1}{m^2} + \frac{1}{2n^2+3} < \epsilon$

$\frac{1}{m^2} < \epsilon$

$\frac{1}{m} < \epsilon$

$m > \frac{1}{\epsilon}$

$\therefore$ For every $m > \frac{1}{\epsilon}$

i.e $N_\epsilon = \frac{1}{\epsilon}$.

$\Rightarrow |f_n - f_m| < \epsilon$ for $n,m \geq N_\epsilon$

c) From (A) clearly the sequence is non-increasing
But we will still prove it.

$$f_n = \frac{n^2+2}{2n^2+3}$$

$$f_{n+1} \leq f_n$$

$$\frac{(n+1)^2+2}{2(n+1)^2+3} \leq \frac{n^2+2}{2n^2+3}$$

$$\frac{n^2+1+2n+2}{2n^2+2+4n+3} \leq \frac{n^2+2}{2n^2+3}$$

$$\frac{n^2+2n+3}{2n^2+4n+5} \leq \frac{n^2+2}{2n^2+3}$$

$$\cancel{2n^4}+3n^2+\cancel{4n^3}+6n+6n^2+9 \leq \cancel{2n^4}+\cancel{4n^3}+5n^2+4n^2+8n+10$$

$$-1 \leq 2n$$

$$-\tfrac{1}{2} \leq n$$

2

$$n \in N$$

Hence our hypothesis $f_{n+1} \leq f_n$
and the sequence is non-increasing

d) Yes the sequence converges and it converges to $1/2$

2

e) A sequence $\{f_n\}$ now is said to converge to limit f is for every $\epsilon > 0$ there is a positive integer N s.t. $n \geq N(\epsilon)$

$$|f_n - f| < \epsilon \qquad \text{—— Eq 1}$$

$$\left| \frac{n^2+2}{2n^2+3} - \frac{1}{2} \right| < \epsilon$$

$$\left| \frac{\cancel{2n^2}+4-\cancel{2n^2}-3}{(2n^2+3)2} \right| < \epsilon$$

$$\left| \frac{1}{2(2n^2+3)} \right| < \epsilon$$

$$\frac{1}{2n^2+3} < \epsilon$$

2

$$\frac{1}{2n^2} < \epsilon$$

$$\frac{1}{n^2} < \epsilon$$

$$\frac{1}{n} < \epsilon$$

$$\boxed{n > \frac{1}{\epsilon}} = N_\epsilon$$

$\therefore$ For $N_\epsilon = \frac{1}{\epsilon}$ & $n > N_\epsilon$
$|f_n - f| < \epsilon$

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2. Use the definition of convergence of a sequence of real numbers to prove the following:

   (a) Every convergent sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ must be bounded

   (b) For a convergent sequence of real numbers the limit must be unique

(4 Marks for each part. Total 8 Marks)

4

a) From the definition of convergence, we have → Eq 1 Pg 7 Backside

$$|f_n - f| < \epsilon$$

$$-\epsilon < f_n - f < \epsilon$$

Adding f to both sides

$$f-\epsilon < f_n < f+\epsilon$$ ← $\forall\, n \geq N_\epsilon$ What about $n < N_\epsilon$

2

We can clearly see that the sequence f_n is bounded between $f-\epsilon$ and $f+\epsilon$

b) A sequence is convergen $\Rightarrow |f_n - f| < \epsilon$

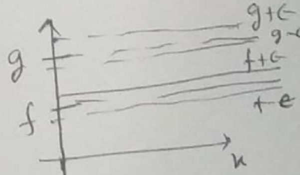
~~Here~~ the limit is f and we concluded in part (a) that the sequence will be bounded between $(f-\epsilon)$ and $(f+\epsilon)$

~~Let us assume~~

b

Hypothesis: Let us assume the sequence converges to another limit g. ($\neq f$)

That would imply $\{f_n\}_{n\in\mathbb{N}}$ is bounded b/w $(g-\epsilon)$ and $(g+\epsilon)$ as well



$$(g-\epsilon) < f_n < (g+\epsilon)$$

2

2

• But since $g \neq f$ we can have $g-\epsilon \neq f+\epsilon$, but for two limits the function should be above $(g-\epsilon)$ and below $(f+\epsilon)$

That is not possible. Hence our hypothesis is wrong, and the function can not have two limits

3. Consider the sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ defined on the interval $\mathcal{I} = (0, \infty)$ as

$$\mathcal{I} = [0,\infty) \qquad f_n(x) = \frac{n + e^x}{n + 3e^x}$$

Let f be the function defined on $\mathcal{I}$ as

$$f(x) = 1 \text{ for all } x \in \mathcal{I}$$

Answwer the following:

(a) Does $f_n \xrightarrow{pw\,(\mathcal{I})} f$?

(b) Show that for every n there exists a point $x_n \in \mathcal{I}$ such that $f_n(x_n) = \dfrac{1}{2}$

(c) Does $f_n \xrightarrow{uc\,(\mathcal{I})} f$?

(3+2+2=7 Marks)

7

$\frac{2+e^{\ln 2}}{2+3e^{\ln 2}}$ $\frac{4}{8}$

a)

3

~~$|f_n(x) - f(x)| < \epsilon$~~

~~$\left|\frac{n+e^x}{n+3e^x} - 1\right| < \epsilon$~~

~~$\left|\frac{n+e^x - n - 3e^x}{n+3e^x}\right| < \epsilon$~~ ~~$\frac{-2e^x}{n+3e^x}$~~

For $x = 0$ $\quad f_n(0) = \frac{n}{n} = 1$

For $x \neq 0$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n(1 + e^x/n)}{n(1+3e^x/n)}$$

$= 1$

$\therefore$ The function $f_n \xrightarrow{pw(\mathcal{I})} f$

Calculating the limit gives us $\lim_{n\to\infty} f_n(x) = 1$ for every x

$\therefore$ The sequence of functions $f_n(x)$ converges pointwise to $\underline{f}$

b)

$$f_n(x) = \frac{n + e^x}{n+3e^x}$$

Given $f_n(x) = 1/2$

$$\frac{1}{2} = \frac{n+e^x}{n+3e^x}$$

$$n + 3e^x = 2n + 2e^x$$

$$e^x = n \quad (Eq\,1)$$

$$\boxed{x = \ln n}$$

2

$\therefore$ From equation 2 we can ~~take different values~~ find the corresponding x for every n ie For example

For $n=1$ $\quad x = \ln 1$

$n=2$ $\quad x = \ln 2$

c) Since for every n there exists a point $x_n \in \mathcal{I}$ such that $f_n(x_n) = 1/2$ (From part b) Taking ϵ to be $1/4$ ie $\epsilon = 1/4$. All the functions crosses this bound and Hence, the sequence is NOT uniform convergent.

2