Name: **SR No.:** **Dept.:**

Maximum Points: 20 **E2-243: Quiz 5** **Duration**: 45 minutes

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1. State whether the following are TRUE or FALSE. (**1+1+1** points)

   a) If X and Y are random variables, then $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$, where $\text{var}(X)$ is the variance of X. **False**

   Explanation $\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)+2\text{Cov}(X,Y)$. So the expression in the question is true only when X and Y are independent.

   b) Any three events A, B, C are independent $\implies$ they are pairwise independent.**True**

   Explanation Independence of random variable implies pairwise independence, but it is not true in other way.

   c) Consider any random variable X. Then $P(a < X < b) = F_X(b) - F_X(a)$.**False**

   Explanation $P(a < X < b) = F_X(b^-) - F_X(a^-)$ this seems wrong!! F_x(a)

2. Fill in the blanks. (**1+1+1** points)

   a) Let $(\Omega, \mathbb{B}, P)$ be a probability space. What is the smallest sigma algebra over all possible subsets of Ω? ______

   Explanation $\{\phi, \Omega\}$

   b) Let $(\Omega, \mathbb{B}, P)$ be a probability space. What is the largest sigma algebra over all possible subsets of Ω? ______

   Explanation Power set of Ω

   c) Let X be a uniform random variable over $[0, 1]$. What is $P(X = 0.57) =$ ______

   Explanation 0. As X is a continuous random variable

3. If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. Assume that an aircraft is present with probability 0.05. Find

   a) the probability of false alarm (a false indication of aircraft presence) **(2 marks)**

   b) the probability of missed detection (nothing registers, even though an aircraft is present) **(2 marks)**

Solution Let A and B be the events:
$A=$ {an aircraft is present}, and $B =$ {the radar registers an aircraft presence}.
and consider also their complements
$A^c =$ {an aircraft is not present} and $B^c =$ {the radar does not register an aircraft presence}.

$$P(\text{false alarm}) = P(A^c \cap B) = P(A^c)P(B|A^c) = 0.95 \times 0.10 = 0.095,$$
$$P(\text{missed detection}) = P(A \cap B^c) = P(A)P(B^c|A) = 0.05 \times 0.01 = 0.0005.$$

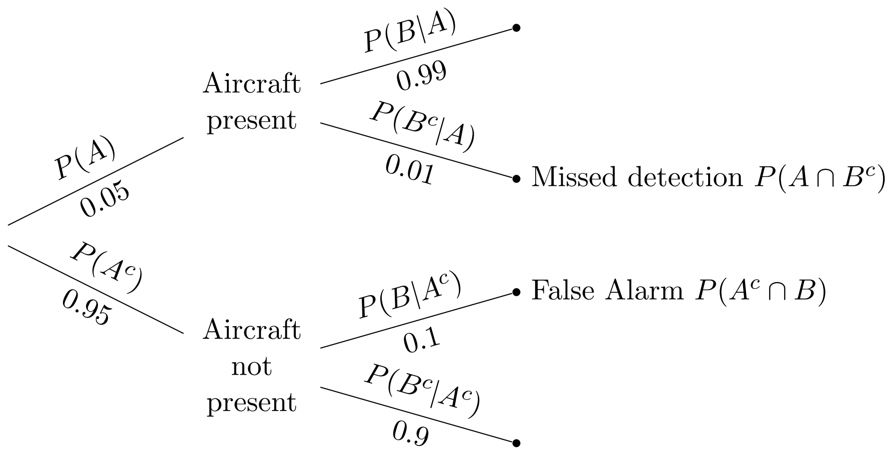

Figure 1: Sequential description of the sample space for the radar detection problem

4. If X is a random variable that is distributed exponentially (i.e. X has PDF $f_X(x) = \lambda e^{-\lambda x}$),
   a) Write an expression for the CDF of X (i.e. $P(X \leq t)$) **(2 marks)**

      Solution $\Pr(X \leq t) = \int_0^t \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda t}$

      You are expected to know that (i) λ should be greater than 0, and (ii) the exponential PDF takes nonzero values only in $[0, \infty)$.
   b) Compute the value of $\Pr(X > t)$ and use this to write an expression for the value of $\Pr(X > s + t \mid X > t)$ **(3 marks)**

Solution Note that s can't be negative since that would imply that X will take negative values in the case $t = 0$, which is not possible since X is exponentially distributed.

$\Pr(X > t) = 1 - \Pr(X \leq t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$

Now,

$$\Pr(X > s + t \mid X > t) = \frac{\Pr(\,(X > s + t)\ \cap\ (X > t)\,)}{\Pr(X > t)}$$

$$= \frac{\Pr(X > s + t)}{\Pr(X > t)}$$

($\because X > s + t \implies X > t$, and therefore $X > t$ is redundant in the numerator)

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s}$$

Note that $e^{-\lambda s} = \Pr(X > s)$.
Observe that the expression for $\Pr(X > s + t \mid X > t)$ is independent of t. This is known as the memorylessness property of the exponential distribution.

5. Let $(\Omega, \mathbb{B}, P)$ be a probability space. If $\{A_n\}_{n=1}^{\infty}$ be any sequence of events, prove

$$P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$$

(**5** points)

Solution Define

$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$.$$
$$.$$
$$B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i)$$

We observe the following from the above construction

a) $B_i \cap B_j = 0, \forall i \neq j$

b) $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

c) $B_i \subseteq A_i$

Using (B) we can write

$$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n)$$

Using (A) the above expression is equivalent to

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(B_n)$$

Using (C), we have $P(B_i) \leq P(A_i), \forall i \geq 1$. Finally using this inequality the above expression can be written as

$$P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$$