EXERCISE 7

1. The average score of a class in a test is 70, Find an upper bound for the probability that the score is at least 90

2. A biased coin has probability of landing on Heads as 0.2. Suppose the coin is flipped 20 times. Answer the following:

   (a) Find an bound for the probability that it lands on Heads at least 16 times using

      i. Markov Inequality

      ii. Chebychev Inequality

   (b) Compute the actual value of the probability that the coin turns up Heads at least 16 times

3. A biased coin has probability of landing on Heads as 0.1. It is flipped 200 times. Compare Markov, Chebychev and Chernoff upper bounds for the probability that it lands on its Heads at least 120 times

4. If X is a real valued random variable with $E(X) = \mu_X < \infty$ and $Var(X) = \sigma_X^2 < \infty$ then

   (a) $P(|X - c| \geq k) \leq \dfrac{\sigma_X^2 + (\mu_X - c)^2}{k^2}$
   for any positive real numbers k and c

   (b) $P(a < x < b) \geq 1 - \dfrac{\sigma_X^2 + (\mu_X - \frac{b+a}{2})^2}{(\frac{b-a}{2})^2}$
   for any two real numbers $a < b$

5. (One Sided Chebychev Inequalities - also known as Cantelli's ineqaulities) Let X be a real valued random variable with $E(X) = \mu < \infty$ and $Var(X) = \sigma_X^2 < \infty$. Let $Y = X - \mu_X$.Let $a > 0$. Answer the following:

   (a) Find $E(Y)$ and $Var(Y)$

   (b) For any $t > 0$ is the following true?

$$P(Y \geq a) = P(Y + t \geq a + t) \leq P((Y + t)^2 \geq (a + t)^2)$$

(c) Show that

$$P(P(Y \geq a) \leq \frac{\sigma_X^2 + t^2}{(a+t)^2}$$

(d) Show that the rhs above is minimum when $t = \dfrac{\sigma_X^2}{a}$

(e) Show that

$$P(X - \mu_X \geq a) \leq \frac{\sigma_X^2}{\sigma_X^2 + a^2}$$

(Note that we have assumed $a > 0$)

(f) Use the above result to show that when $a < 0$,

$$P(X - \mu_X < a) \leq \frac{\sigma_X^2}{\sigma_X^2 + a^2}$$

and hence

$$P(X - \mu_X \geq a) \geq \frac{a^2}{\sigma_X^2 + a^2}$$

6. Consider the experiment of rolling a fair die. Let $X(j) = j$ be the random variable. Answer the following:

   (a) Find $P(X \geq 6)$

   (b) Estimate $P(X \geq 6)$ using

      i. Markov inequality,

      ii. Chebychev inequality and

      iii. one sided Chebychev inequality

   (c) In the above estimates which one of the inequalities gives the best estimate to the actual value?

7. A fair coin is flipped 100 times. Find an upper bound for the probability that it lands on Heads is at least 60 or at most 40

8. Let X be a random variable. For any $\varepsilon > 0$ let $f(\varepsilon) = P(X \geq \varepsilon)$ and $g(\varepsilon)$ be the upper bound for $f(\varepsilon)$ obtained from Markov inequality. Sketch the functions $f(\varepsilon)$ and $g(\varepsilon)$ as functions of ε if

   (a) $X \sim Uni[0, A]$ (where A is a real positive constant)

(b) $X \sim Exp(\lambda)$ (where λ is a real positive constant)

(c) $f_X(x) = \begin{cases} \dfrac{x}{\alpha^2} exp\left(-\dfrac{x^2}{2\alpha^2}\right) & \text{if} \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases}$

(You can use the facts that $E(X) = \alpha\sqrt{\frac{\pi}{2}}$, $Var(X) = \left(2 - \frac{\pi}{2}\right)\alpha^2$)

(d) $f_X(x) = \begin{cases} \dfrac{\alpha A^\alpha}{x^{\alpha+1}} & \text{if} \quad x \geq A \\ 0 & \text{if} \quad x < A \end{cases}$ (where $\alpha > 1$)

(You can use the facts that $E(X) = \frac{\alpha A}{\alpha - 1}$, $Var(X) = \frac{\alpha A^2}{(\alpha-2)(\alpha-1)^2}$)

9. Let X be a random variable. For any $\varepsilon > 0$ let $f(\varepsilon) = P(X \geq \varepsilon)$ and $g(\varepsilon)$ be the upper bound for $f(\varepsilon)$ obtained from Chebychev inequality. Sketch the functions $f(\varepsilon)$ and $g(\varepsilon)$ as functions of ε if

   (a) $X \sim Uni[-A, A]$

   (b) $X \sim Lap(\lambda)$ (that is X is a Laplace random variable with $f_X(x) = \frac{\lambda}{2} exp(-\alpha|x|)$)

10. Let X be a real valued random variable on a probability space $(\Omega, \mathcal{B}, P)$. If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of real valued random variables on $(\Omega, \mathcal{B}, P)$ examine whether the following are true:

    (a) $\lim_{n\to\infty} E(|X_n - X|) = 0 \Longrightarrow X_n \xrightarrow{p} X$

    (b) $\lim_{n\to\infty} E(|X_n - X|^2) = 0 \Longrightarrow X_n \xrightarrow{p} X$

    (c) $\lim_{n\to\infty} E(|X_n - X|^k) = 0 \Longrightarrow X_n \xrightarrow{p} X$ (where k is any positive integer)