Name: **SR No.:** **Dept.:**

Maximum Points: 20 **E2-243: Quiz 2** **Duration**: 30 minutes

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1. State whether the following are TRUE or FALSE. (**1+1+1+1+1** points)

   a) Let $\{f_n(x)\}_{n\geq 1}$ be a sequence of functions converging point-wise to $f(x)$. Then, all $f_n(x)$ are bounded $\Rightarrow$ $f(x)$ is bounded. **False**

   Explanation Refer Lecture notes part-2 for the solution.

   b) Let $\{f_n(x)\}_{n\geq 1}$ be a sequence of functions converging uniformly to $f(x)$. Then, all $f_n(x)$ are bounded $\Rightarrow$ $f(x)$ is not bounded. **False**

   Explanation Refer Lecture notes part-2 for the solution.

   c) Point-wise convergence preserves integrals. **False**

   Explanation Refer Lecture notes part-2 for the solution.

   d) Cauchy criteria $\Rightarrow$ Convergence in real numbers but not vice-a-verse.**False**

   Explanation From completeness axioms of real numbers, Cauchy criteria $\iff$ Convergence in real numbers

   e) Consider the sequence of functions defined over $|x| < 1$

   $$f_n(x) = 1 + x + x^2 + ... + x^n$$

   This sequence of functions does not converge point-wise. **False**

   Explanation This sequence converges point-wise to $\frac{1}{1-x}$, as $|x| < 1$.

2. Give an example to show that every uniformly convergent sequence of functions is also point-wise convergent and explain it in detail. (**5** points)

Solution Consider the sequence of functions defined on $[0, a]$, where $a > 0$ as shown below

$$f_n(x) = \frac{1}{n + x}$$

Point-wise convergence $\lim_{n\to\infty} f_n(x) = 0, \forall x \in [0, a]$

Uniform Convergence When $\lim_{n\to\infty} f_n(x) = f(x)$, if the sequence of functions are uniformly convergent, for a fixed ϵ there should exist unique N for all x.

$$|f_n(x) - f(x)| < \epsilon$$

$$|\frac{1}{n + x}| < \epsilon$$

As $x \in [0, a]$,

$$\frac{1}{n + x} \leq \frac{1}{n} < \epsilon$$

Hence $N > \frac{1}{\epsilon}$works for all x.Hence this is a uniformly convergent sequence.

(Or)

Using Weierstrass test

- $0 \leq \frac{1}{n+x} \leq M_n = \frac{1}{n}, \forall x \in [0, a]$.
- $\lim_{n\to\infty} M_n \to 0$

Thus, this is uniformly convergent.

3. Prove $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges. (**5** points)

Solution Consider a sequence $\{x_n\}_{n\geq 1}$ and let

$$S_n = x_1 + x_2 + x_3 + ... + x_n$$

and

$$S'_n = |x_1| + |x_2| + |x_3| + ... + |x_n|$$

When $m > n$, $S_m - S_n = x_{n+1} + x_{n+2} + ... + x_m$.

Similarly $S'_m - S'_n = |x_{n+1}| + |x_{n+2}| + ... + |x_m|$. Using,

$$x_i \leq |x_i| \ \forall i \in \{1, 2, 3, ..\}$$

We obtain

$$S_m - S_n \leq S'_m - S'_n \tag{1}$$

As it is given $\sum |x_n|$ converges, using Cauchy criterion for convergence of real number we can write

For every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m > n \geq N$ the following holds true

$$|S'_m - S'_n| < \epsilon$$

Using (1) we can write

$$|S_m - S_n| < \epsilon$$

Hence, $\sum x_n$ converges from cauchy criterion for convergence.

4. Consider a sequence of functions as defined on $(0, \infty)$ as shown below

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

Comment on point-wise convergence and uniform convergence of the sequence. (**2+3** points)

Solution

Point-wise convergence $\lim_{n\to\infty} f_n(x) = f(x) = \frac{x}{\frac{1}{n} + nx^2} = 0, \forall x \in (0, \infty)$ Hence this sequence converges point-wise.

Uniform convergence When $\lim_{n\to\infty} f_n(x) = f(x) = 0$, if the sequence of functions are uniformly convergent, for a fixed ϵ there should exist unique N for all x.

$$|f_n(x) - f(x)| < \epsilon$$

$$|\frac{nx}{1+n^2x^2}| < \epsilon$$

As $nx > 0$,

$$|\frac{nx}{1+n^2x^2}| < |\frac{nx}{n^2x^2}| < \frac{1}{nx} < \epsilon$$

Now, we should find $N \in \mathbb{N}$ such that when $n = N$, the following equation is satisfied $\forall x \in (0, \infty)$.

$$\frac{1}{nx} < \epsilon$$

But, there does not exist such N, hence this is not a uniformly convergent sequence.

(Or)

$f_n(x)$ has a maximum value of $\frac{1}{2}$ at $nx = 1$. Now fix $\epsilon = \frac{1}{4}$, the graph can't be trapped in ϵ-band. Hence this is not uniformly convergent.