Name: **SR No.:** **Dept.:**

Maximum Points: 20 **E2-243: Quiz 1** **Duration**: 30 minutes

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1. State whether the following are TRUE or FALSE. (**1+1+1+1+1** points)

a) If $\{x_n\}_{n\geq 1}$ is a non-negative, bounded sequence , so is $\{y_n\}_{n\geq 1}$, where $\forall n \geq 1, y_n = x_n^q$ for some finite $q > 0$.

Answer True.

Explanation As $\forall n \geq 1$, it is given that $\exists m, M \in \mathbb{R}$ such that $m \leq x_n \leq M$. Since $q > 0$, $\forall n \geq 1, m^q \leq y_n \leq M^q$. Hence $\{y_n\}_{n\geq 1}$ is also bounded.

b) Supremum of $[0, 99.999)$ is 100.

Answer False.

Explanation 100 is an upper bound. But 99.999 is the least upper bound hence it is the supremum.

c) $f[n]$ is defined recursively as follows:

$f[1] = -2$

$f[n] = -f[n-1]$ for $n \geq 2$

Then $\{f[n]\}_{n\geq 1}$ converges.

Answer False.

Explanation The sequence is $-2, 2, -2, 2, ... -2, 2, ...$ it doesn't converge.

d) If two sequence $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$convergence to A , then a sequence $\{a_n\}_{n\geq 1}$ such that $\alpha_n \leq a_n \leq \beta_n$ converges to $A' < A$.

Answer False.

Explanation It converges to A, follows from Sandwich Theorem.

e) For a sequence of sets $A_n = [0, \frac{n}{n+1})$, limit exists and is $[0, 1]$.

Answer False.

Explanation As the sequence is non-decreasing limit exists and us is $\cup_{n=1}^{\infty} A_n = [0, 1)$. As $\lim_{n\to\infty} A_n = [0, 1)$

2. Show that the sequence $\frac{1}{n^2+2}$ converges to 0, using the definition of convergence. (**5** points)

Solution To show convergence it has to be shown that for every $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$ the following holds

$$|\frac{1}{n^2+2} - 0| < \epsilon \tag{1}$$

As $n > 0$,

$$\frac{1}{n^2+2} < \frac{1}{n^2}$$

So, we can look at

$$\frac{1}{n^2} < \epsilon$$

Hence whenever $N > \frac{1}{\sqrt{\epsilon}}$, (1) holds true. Thus $\frac{1}{n^2}$ converges to 0.

3. Consider an infinite sequence $\{a_n\}_{n\geq 1}$ such that $a_n = n^{\sin(\frac{n\pi}{2})}$. Find $\limsup_{n\to\infty} a_n, \liminf_{n\to\infty} a_n$ and also comment on convergence of the sequence. (**5** points)

Solution On expanding the sequence we obtain

$$\{a_1, a_2, a_3, a_4, ...\} = 1, 1, \frac{1}{3}, 1, 5, 1, \frac{1}{7}, 1, 9, 1, \frac{1}{11}, ...$$

We could see three subsequences and their corresponding limits as shown below

- $\{a_1, a_5, a_9, a_13..\} = \{1, 5, 9, 13, ..\}$, limit does not exist(limit goes to ∞).
- $\{a_2, a_4, a_6, ...\} = \{1, 1, 1, ...\}$, limit is 1.
- $\{a_3, a_7, a_{11}, ...\} = \{\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, ...\}$, limit is 0.

The sequence is bounded below by 0. Hence $\limsup_{n\to\infty} a_n = \infty, \liminf_{n\to\infty} a_n = 0$. As

$$\limsup_{n\to\infty} a_n \neq \liminf_{n\to\infty} a_n,$$

limit of the sequence does exist and hence it does not converge.

4. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is (**1**+**1**+**2**+**1** points)

   a) neither one-to-one nor onto
   b) one-to-one but not onto
   c) onto but not one-to-one
   d) both one-to-one and onto

   You may express the function algebraically or sketch it.

Solution

(a) neither one-to-one nor onto: $y = x^2$, $y = e^{-|t|}$, $y = \sin(x)$, etc. You are encouraged to plot these curves, in Google (by entering the text 'plot x^2', for example, in Google search). Here, the important things to note is there there are at least 2 values of x for which y-values are the same, and there are some values of y that are not the image of any x. Another way to think about one-to-one and onto functions is that, in a one-to-one function every horizontal line (drawn from $-\infty$ to $+\infty$) will touch the function at one point at the most, and in an onto function, and every horizontal line will touch the function at one point at the least.

- $y = x^2$: $-a$ and $+a$ in the domain $\mathbb{R}$ have the same image a^2 in the co-domain $\mathbb{R}$, and no point in the entire negative y-axis is an image of any x in the domain.

- $y = e^{-|t|}$: There are some points (e.g., -1 and +1) in the domain that have the same image in the co-domain, namely $- e^{-t}$, and no point in the entire negative y-axis is an image of any x in the domain.
- $y = \sin(x)$: Since $\sin(x + 2\pi)) = \sin(x)$, there are multiple x-values that map to the same y-value, and no point in the positive y-axis beyond 1, and no point in the negative y-axis beyond -1 is an image of any x in the domain.

(b) one-to-one but not onto: $y = e^x, y = e^{-x}$. In general, 'e' can be replaced by any constant. Any horizontal line coincides graphs of these functions at point at the most, and no point in the entire negative y-axis is an image of any x in the domain.

(c) onto but not one-to-one: $y = x^3 + 4x^2 - x + 2$. There are horizontal lines that coincide graphs of these functions at more than one point, and every point in the y-axis is an image of some point x in the domain.

(d) both one-to-one and onto: $y = ax + b, (a, b$ constants and $a \neq 0)$. Any horizontal line intersects the graph of these functions at exactly one point.