Name: **SR No.:** **Dept.:**

Maximum Points: 20 **E2-243: Quiz 3** **Duration**: 30 minutes

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1. State whether the following are **TRUE** or **FALSE**. (**10** points)
(**1** Point for correct answer, **-0.5** for wrong answer and **0** for no attempt).

a) Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. Then the set $\{\theta_{\mathcal{V}}\}$, consisting of the zero vector, is a linearly independent set. ______

Answer: FALSE
If field is $\mathbb{R}$ then, $1 \cdot 0 = 0$, $2 \cdot 0 = 0$, $3 \cdot 0 = 0$, and so on. Hence $\alpha \cdot 0 = 0, \alpha \in \mathbb{R}$. Hence linearly dependent.

b) A subset S of $\mathbb{F}^n$ is linearly independent $\iff$ it is an orthonormal set. ______

Answer: FALSE
Take field $\mathbb{R}$ and $\mathcal{S} = \left\{ u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$

c) If $\mathcal{W}_1$ and $\mathcal{W}_2$ are two subspaces in the vector space $\mathcal{V}$ such that $\mathcal{W}_1 \subset \mathcal{W}_2$, then $\mathcal{W}_1 \cup \mathcal{W}_2$ is also a subspace. ______

Answer: TRUE
$\mathcal{W}_1 \subset \mathcal{W}_2 \implies \mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W}_2$. Hence subspace.

d) A finite set of vectors which spans a subspace is always Linearly Independent. ______

Answer: FALSE
The spanning set may or maynot be linearly independent.

e) If $\mathcal{W}$ is a subspace of a finite dimensional vector space $\mathcal{V}$, then any spanning set of $\mathcal{W}$ can be extended to form a basis for $\mathcal{V}$. ______

Answer: FALSE
Any basis (NOT spanning set) of $\mathcal{W}$ can be extended to form a basis for $\mathcal{V}$.

f) For $A \in \mathbb{R}^{m \times n}$
$Rank(A^T) + Nullity(A^T) =$ Number of Rows of A. ______

Answer: TRUE
Rank-Nullity Theorem

g) Let $u = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$ and $v = \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$ be two vectors in $\mathbb{C}^3$. Thus, Equality holds in Cauchy Schwartz Inequality for these two vectors. ______

Answer: TRUE
$v = i \cdot u, \implies u$ and v are linearly dependent.

h) Every non-empty subset of a linearly independent set is linearly independent. ______

Answer: TRUE
Refer class notes

i) For $A \in \mathbb{R}^{m \times n}$, $\mathcal{N}_A$ and $\mathcal{R}_{A^T}$ are orthogonal complement of each other. ______

Answer: TRUE
Refer class notes

j) The subset $S = \{x \in \mathbb{C}^4 : x_2 = 0\}$ is NOT a subspace of $\mathbb{C}^4$. ______

Answer: FALSE
All 3 properties are satisfied. Hence subspace.

2. For $A \in \mathbb{R}^{m\times n}$, Show that

a) Null Space of A is Subspace of $\mathbb{R}^n$. (**3** points)

Answer
Null Space of A is a subset of $\mathbb{R}^n$.
$A \cdot \theta_n = 0$
$\implies \theta_n \in \mathcal{N}_A$
If $u, v \in \mathcal{N}_A \implies A \cdot u = 0$ and $A \cdot v = 0$.
$\implies A \cdot u + A \cdot v = 0$
$\implies A \cdot (u + v) = 0. \implies u + v \in \mathcal{N}_A.$
For any scalar $c \in \mathbb{R}$
$A \cdot (cu) = c \cdot (Au) = c \cdot 0 = 0.$
$\implies cu \in \mathcal{N}_A$
Hence Null Space of A is a subspace.

b) Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and let $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if u belongs to the null space of A.

(**1** points)

Answer
$A \cdot u = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\implies u \in \mathcal{N}_A.$

3. Given $S = \left\{ u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Consider a set $S_1 = \left\{ v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$.

a) Show that S_1 is also a basis for $\mathbb{R}^3$. (**3** points)

Answer
Checking whether S_1 is linearly independent or not.

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0 \tag{1}$$

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$\alpha_1 + \alpha_3 = 0 \tag{3}$$

$$2\alpha_1 + 2\alpha_2 = 0 \tag{4}$$

$$3\alpha_2 + 3\alpha_3 = 0 \tag{5}$$

Solving equations (3),(4) and (5) simultaneously, we get

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \tag{6}$$

Hence S_1 is a linearly independent set.
Since dimension of $\mathbb{R}^3$ is 3, any linearly independent set of three vectors in $\mathbb{R}^3$ will form a basis for $\mathbb{R}^3$. Therefore $\mathcal{S}_1$ is a basis for $\mathbb{R}^3$.

b) Let a non-zero vector $x \in \mathbb{R}^3$ is represented as (α, β, γ) with S as basis, where $\alpha, \beta, \gamma \in \mathbb{R}$. How will the vector x be represented with S_1 as basis? (**3** points)

Answer

$$x = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

$$x = \alpha' \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta' \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + \gamma' \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \tag{8}$$

Equating (7) and (8), we get

$$\alpha' + \gamma' = \alpha \tag{9}$$

$$\alpha' + \beta' = \beta/2 \tag{10}$$

$$\beta' + \gamma' = \gamma/3 \tag{11}$$

Adding (9), (10) and (11)

$$\alpha' + \beta' + \gamma' = \alpha/2 + \beta/4 + \gamma/6 \tag{12}$$

(12)-(11) gives,

$$\alpha' = \alpha/2 + \beta/4 - \gamma/6 \tag{13}$$

(12)-(9) gives,

$$\beta' = -\alpha/2 + \beta/4 + \gamma/6 \tag{14}$$

(12)-(10) gives,

$$\gamma' = \alpha/2 - \beta/4 + \gamma/6 \tag{15}$$

Hence, x will be represented as $\begin{bmatrix} \alpha/2 + \beta/4 - \gamma/6 \\ -\alpha/2 + \beta/4 + \gamma/6 \\ \alpha/2 - \beta/4 + \gamma/6 \end{bmatrix}$ with S_1 as basis.