Name: **SR No.:** **Dept.:**

Maximum Points: 20 **E2-243: Quiz 1** **Duration**: 40 minutes

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1. State whether the following are TRUE or FALSE. (**1+1+1+2 points. -0.5 point for each negative answer**)

   a) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(\mathrm{x}) = |x|$. The function is one-to-one and onto.

      Answer False

      Explanation The function is neither one-to-one nor onto. As

      - f(x)=f(-x)
      - $\forall y < 0,\ f^{-1}(y)$ doesnt exist

   b) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = e^x$. The function is one-to-one but not onto.

      Answer True

      Explanation The range of $f(x)$ is (0,∞).The function is one-to-one as $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ on the range of $f(x)$.

      The function is not onto as $\forall y \in (-\infty, 0),\ f^{-1}(y)$ doesnt exist.

   c) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = 2x + 3$. The function is one-to-one and onto.

      Answer True

      Explanation The range of $f(x)$ is (-∞, ∞). The function is one-to-one as $\forall y \in (-\infty, \infty)$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

      The function is also onto as $\forall y \in (-\infty, \infty), f^{-1}(y)$ exists.

   d) Let Ω be a non empty set and let C be a collection of subsets of Ω. Then

      i. A subset L of Ω is said to be the lower bound for the collection C if $L \subseteq A$ for every $A \in C$. Hence $\underset{A\in C}{\cap} A$ is lower bound for the collection C.

         Answer True

         Explanation Clearly, $\underset{A\in C}{\cap} A \subseteq B$ for every $B \in C$. Hence by definition, it is the lower bound for the collection C.

      ii. A subset L_0 of Ω is said to be the infimum of the collection C if

         A. L_0 is a lower bound for C

         B. L_0 is the largest amongst lower bounds.

         $\underset{A\in C}{\cap} A$ is infimum for the collection C.

         Answer True

Explanation Refer lecture notes.

2. An increasing sequence $\{f_n\}_{n\geq 1}$ converges to 3. Specify whether the following statements are TRUE or FALSE. **(1 point each, -0.5 point for each negative answer)**

a) 3 must be the maximum element of the sequence $\{f_n\}$.

Answer False

Explanation Least upper bound need not always be an element of the sequence.

b) 3 must be the least upper bound of the sequence $\{f_n\}$.

Answer True

Explanation The increasing sequence would always converge to its least upper bound.

c) $\forall \epsilon > 0$, there are only finite number of points of the sequence greater than $3 - \epsilon$.

Answer False

Explanation Since the increasing sequence converges to 3, there are only finite points less than $3 - \epsilon$.

d) There exists an $\epsilon > 0$ such that $3 - \epsilon > f_n \ , \forall\ n$.

Answer False

Explanation Since 3 is the least upper bound for every $\epsilon > 0$, $\exists N$ such that $\forall\ n \geq N$, $3-\epsilon < f_n \leq 3$.

e) $\forall \epsilon > 0$ there exists N_ϵ such that $\forall\ m, n \geq N_\epsilon$, $|f_m - f_n| < \epsilon$.

Answer True

f) The first element of the sequence f_1 is the greatest lower bound for the sequence.

Answer True

Explanation Since the sequence is an increasing sequence.

g) 5 is an upper bound for the sequence.

Answer True

Explanation Since any value greater than least upper bound is an upper bound.

h) Since the sequence converges to 3, there exists an $\epsilon > 0$, such that the sequence also converges to $3+\epsilon$.

Answer False

Explanation The sequence always converges to a unique value.

i) Define a sequence $a_k = \sup_{n \geq k} f_n - \inf_{n \geq k} f_n$. The sequence a_k converges to 0 as $n \to \infty$.

Answer True

Explanation Since the sequence converges, lim sup $f_n =$ lim inf f_n.

j) Define a sequence $b_k = \sup_{n \geq k} f_n$. The sequence b_k is a non increasing sequence.

Answer True

Explanation Since the sequence f_n is an increasing sequence converging to 3, all elements of the sequence b_k are 3.

3. The sequence $\{f_n\}_{n \geq 1}$ is defined as $f_n = \frac{n}{e^n}$.

a) State whether the sequence is a non increasing sequence or a non decreasing sequence. Prove.(**2 points**)

Explanation The sequence is non increasing if $f_n - f_{n+1} \geq 0$

$$\frac{n}{e^n} - \frac{n+1}{e^{n+1}} \geq 0 \Rightarrow \frac{ne - n - 1}{e^{n+1}} \geq 0 \Rightarrow n(e-1) \geq 1.$$

Hence the sequence is non increasing.

b) Prove that the sequence converges to 0. (**2 points**)

Answer To prove that the sequence converges to 0, we need to prove $\forall \epsilon > 0$, $\exists N_\epsilon$ such that $|f_n - 0| < \epsilon$, $\forall n \geq N_\epsilon$

$$\frac{1}{\frac{e^n}{n}} < \epsilon,\ \forall\ n \geq\ N_\epsilon \Rightarrow \frac{1}{\frac{e^n}{n}} < \frac{1}{\frac{1+n+\frac{n^2}{2}+\frac{n^3}{3!}}{n}} < \frac{1}{\frac{1}{n}+1+n} < \frac{1}{n}$$

Hence $\forall\ n > 1/\epsilon$, the sequence converges to 0.

c) Specify the *lub* and *glb* of the sequence.(**1 point**)

Explanation Since the sequence is non increasing, the *lub* is the first term i.e. $\frac{1}{e}$. The *glb* is intuitively 0. This could be proven by proving that the sequence coverges to 0.