EXERCISE 8

1. Let X_n be the Rayleigh random variables with

$$f_{X_n}(x) \quad = \quad \begin{cases} n^2 x \; exp\left(-\frac{n^2x^2}{2}\right) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

   Does $X_n \xrightarrow{p} 0$

2. Let $X_n \sim \cancel{Exp(-n)}$ random variables. Answer the following:

   (a) Show that $X_n \xrightarrow{p} 0$

   (b) Show that $X_n \xrightarrow{d} 0$

3. Let $X_n \sim N(0, \frac{1}{n})$, random variables, that is

$$f_{X_n}(x) \quad = \quad \frac{\sqrt{n}}{2\pi} exp\left(-\frac{nx^2}{2}\right)$$

   Let $\alpha_n > 0$ such that $\alpha_n \longrightarrow +\infty$. Answer the following:

   (a) Is $P(X_n \leq x) = P(\alpha_n X_n \leq \alpha_n x)$?

   (b) Use the above to find $\lim\limits_{n\to\infty} F_{X_n}(x)$ for $x > 0$

   (c) Use a similar idea to find $\lim\limits_{n\to\infty} F_{X_n}(x)$ for $x < 0$

   (d) Does $X_n \xrightarrow{d} 0$

4. Let X_n be the sequence of random variables defined on $\Omega = [0, 1]$, (with $P(any\ interval) = its\ length$), as follows:

$$X_n(\omega) \quad = \quad \begin{cases} n & \text{if} \quad 0 \leq \omega \leq \frac{1}{n} \\ 0 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$$

   Answer the following:

   (a) Does $X_n \xrightarrow{a.s} 0$

   (b) Does $X_n \xrightarrow{p} 0$?

   (c) Does $X_n \xrightarrow{d} 0$?

(d) For $r > 1$ does $X_n \xrightarrow{r.m} 0$

5. Let X_n be a sequence of real valued random variables such that

$$F_{X_n}(x) = \begin{cases} \dfrac{e^{n(x-1)}}{1+e^{n(x-1)}} & \text{if } \ x \geq \cancel{0} \\ \\ 0 & \text{if } \ x < \cancel{0} \end{cases}$$

Show that $X_n \xrightarrow{d} X$ where X is the constant random variable $X = 1$

6. Let X_n be a sequence of real valued random variables such that

$$F_{X_n}(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ \\ \dfrac{e^{nx} + xe^{n}}{e^{nx} + (\frac{n+1}{n})e^{n}} & \text{if } \ 0 \leq x \leq 1 \\ \\ \dfrac{e^{nx} + e^{n}}{e^{nx} + (\frac{n+1}{n})e^{n}} & \text{if } \ x > 1 \end{cases}$$

Show that $X_n \xrightarrow{d} Uni[0,1]$

7. Let X_n be the sequence of real valued random variables defined on a probability space $(\Omega, \mathcal{B}, P)$ such that

$$X_n = \begin{cases} n & \text{with probability } \ \dfrac{1}{n^2} \\ \\ 0 & \text{with probability } \ 1 - \dfrac{1}{n^2} \end{cases}$$

Answer the following:

(a) Show that $X_n \xrightarrow{p} 0$

(b) Show that $X_n \xrightarrow{r.m} 0$ for $r < 2$ and not for $r \geq 2$

(c) Show that $X_n \xrightarrow{a.s} 0$

8. Let X_n be a sequence of real valued random variables on a probability space $(\Omega, \mathcal{B}, P)$ such that the pdfs are given by

$$f_{X_n}(x) = \frac{n}{2}e^{-n|x|} \text{ for all } x \in \mathbb{R}$$

Show that $X_n \xrightarrow{p} 0$

9. Let X_n be a sequence of real valued random variables on a probability space $(\Omega, \mathcal{B}, P)$ such that the pdfs are given by

$$f_{X_n}(x) = \begin{cases} \dfrac{1}{nx^2} & \text{if } \; x > \frac{1}{n} \\ 0 & \text{if } \; x < \frac{1}{n} \end{cases}$$

Show that $X_n \xrightarrow{p} 0$

10. In the experiment of rolling a fair die consider the following random variables:

$$X_n(\omega) = \begin{cases} 1 & \text{if } \; \omega \text{ is even} \\ -1 & \text{if } \; \omega \text{ is odd} \end{cases}$$
$$Y_n(\omega) = \begin{cases} -1 & \text{if } \; \omega \text{ is even} \\ 1 & \text{if } \; \omega \text{ is odd} \end{cases}$$

~~Let X_n, Y_n be the constant sequences $X_n = X$ for all n and $Y_n = Y$ for all n.~~ Answer the following:

(a) Does $X_n \xrightarrow{d} X$?

(b) Does $Y_n \xrightarrow{d} X$?

(c) Does $X_n + Y_n \xrightarrow{d} X + X$?