EXERCISE 1

1. State TRUE or FALSE:

   (a) A convergent sequence of real numbers must be bounded

   (b) A bounded sequence of real numbers must be convergent

   (c) Every convergent sequence of real numbers is a Cauchy sequence

   (d) Every Cauchy sequence of real numbers is convergent

   (e) Every nondecreasing sequence of real numbers is bounded below

   (f) Every nondecreasing sequence of real numbers is bounded above

   (g) Every nondecreasing sequence of real numbers bounded below converges to its *glb*

   (h) Every nondecreasing sequence of real numbers bounded above converges to its *lub*

   (i) Every nonincreasing sequence of real numbers is bounded below

   (j) Every nonincreasing sequence of real numbers is bounded above

   (k) Every nonincreasing sequence of real numbers bounded below converges to its *glb*

   (l) Every nonincreasing sequence of real numbers bounded above converges to its *lub*

   (m) If U_0 is the *lub* of a sequence of real numbers then for every $\varepsilon > 0$ there is at least one term of the sequence which is greater than $U_0 - \varepsilon$

   (n) If U_0 is the *lub* of a nondecreasing sequence of real numbers then for every $\varepsilon > 0$ all but a finite number of terms of the sequence are greater than $U_0 - \varepsilon$

   (o) If u_0 is the *glb* of a sequence of real numbers then for every $\varepsilon > 0$ there is at least one term of the sequence which is less than $U_0 + \varepsilon$

   (p) If u_0 is the *lub* of a nonincreasing sequence of real numbers then for every $\varepsilon > 0$ all but a finite number of terms of the sequence are less than $U_0 + \varepsilon$

   (q) A bounded sequence of real numbers is convergent if and only if

$$\liminf_{n\to\infty} f_n < \limsup_{n\to\infty} f_n$$

(r) A bounded sequence of real numbers is convergent if and only if

$$\liminf_{n\to\infty} f_n \leq \limsup_{n\to\infty} f_n$$

(s) A bounded sequence of real numbers is convergent if and only if

$$\liminf_{n\to\infty} f_n > \limsup_{n\to\infty} f_n$$

(t) A bounded sequence of real numbers is convergent if and only if

$$\liminf_{n\to\infty} f_n \geq \limsup_{n\to\infty} f_n$$

(u) A bounded sequence of real numbers is convergent if and only if

$$\liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$$

2. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of real numbers defined as

$$f_n = \frac{n^2+2}{2n^2+3}$$

Answer the following:

(a) Show that the sequence is bounded above by 1 and below by $\dfrac{1}{2}$

(b) Show that the sequence is a Cauchy sequence

(c) Is the sequence is nonincreasing

(d) Does the sequence converge and if so to what value f does the sequence converge?

(e) Use the definition of convergence to show that the sequence converges to the f you obtainned in (d) above

3. Use the definition to show that the following infinite sequences of real

numbers converge to the given limit :

Sl.No.	*Sequence* f_n	*Limit* f
1	$\frac{1}{n}$	0
2	$\frac{2n+1}{n}$	2
3	$\frac{3n-1}{2n}$	$\frac{3}{2}$
4	$\frac{3n+10}{2n}$	$\frac{3}{2}$
5	$\frac{sin(n)}{n}$	0
6	$\frac{3n^2+sin(n)}{4n^2}$	$\frac{3}{4}$
7	$\frac{3n^2-2n+sin(n)}{5n^2}$	$\frac{3}{5}$

4. For the following sequences find

$$1)\ \limsup_{n\to\infty} f_n \text{ and } 2)\ \liminf_{n\to\infty} f_n$$

Determine whether $\lim_{n\to\infty} f_n$ exists and if so find $\lim_{n\to\infty} f_n$:

(a) $f_n = \dfrac{(-1)^n}{n}$

(b) $f_n = \dfrac{(-1)^n}{n+1}$

(c) $f_n = n(-1)^n$

(d) $f_n = (-1)^n + (-1)^{n+1}$

(e) $f_n = (-1)^n + (-1)^{n+2}$

(f) $f_n = sin(n\pi) + cos(n\pi)$

(g) $f_n = 2(-1)^n + \dfrac{n}{n+1}$

(h) f_n is defined recursively as follows:

$$\begin{aligned} f_1 &= -2 \\ f_n &= 3f_{n-1} \text{ for } n \geq 2 \end{aligned}$$

(i) $f_]$ is defined recursively as follows:

$$\begin{aligned} f_1 &= -2 \\ f_n &= -f_{n-1} \text{ for } n \geq 2 \end{aligned}$$

5. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of real numbers defined as

$$f_n = \frac{n}{n+1}$$

Answer the following:

(a) Show that the sequence is an increasing sequence

(b) Show that the sequence is bounded

(c) Find the *glb* and *lub* of the sequence

(d) Does the sequence converge and if so to what value does it converge

6. Consider the following sequence of real numbers:

$$\frac{1}{2}, \left(1+\frac{1}{2}\right), \frac{1}{3}, \left(1+\frac{1}{3}\right), \cdots, \frac{1}{n}, \left(1+\frac{1}{n}\right), \cdots$$

that is $\{f_n\}_{n\in\mathbb{N}}$ is defined as

$$\left.\begin{aligned} f_{(2n-1)} &= \frac{1}{n}, \\ f_{2n} &= 1+\frac{1}{n} \end{aligned}\right\} \text{ for } n = 1, 2, 3, \cdots$$

Answer the following:

(a) Find the *lub* and *glb* of the sequence

(b) Find $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$

(c) Does the sequence converge?

7. For the sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ defined as

$$f_n = 1 + \frac{(-1)^n}{n}$$

answer the following:

(a) Find $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$

(b) Does the sequence converge and if so what does it converge to?

8. Cosider the sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ defined as

$$f_n = \sqrt{n+1} - \sqrt{n}$$

(a) Show that the sequence is nonincreasing

(b) Show that the sequence converges to 0