EXERCISE 4

1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. Answer the following:

   (a) Determine if the collection
   $$\mathcal{E} = \{\phi, \Omega, \{1, 2, 3\}, \{4, 5, 6\}\}$$
   is a σ-algebra of subsets of Ω

   (b) Let $\mathcal{S}_1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$. Find $\Sigma(\mathcal{S}_1)$

   (c) Let $\mathcal{S}_2 = \{\{1, 2, 3\}\}$. Find $\Sigma(\mathcal{S}_2)$

2. Let Ω be any nonempty set. (If $\mathcal{S}$ is any nonempty collection of subsets of Ω we use the notation $\Sigma(\mathcal{S})$ to denote the σ-algebra generated by $\mathcal{S}$). Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be any two nonempty collections of subsets of Ω. Answer the following:

   (a) Prove that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Longrightarrow \Sigma(\mathcal{S}_1) \subseteq \Sigma(\mathcal{S}_2)$

   (b) Can we have collections $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_1 \subsetneqq \mathcal{S}_2$ such that $\Sigma(\mathcal{S}_1) = \Sigma(\mathcal{S}_2)$. (If so give such an example)

   (c) Let $\mathcal{S} = \Sigma(\mathcal{S}_1) \bigcup \Sigma(\mathcal{S}_2)$, and $\mathcal{X} = S_1 \bigcup \mathcal{S}_2$. Prove that
   $$\Sigma(\mathcal{S}) = \Sigma(\mathcal{X})$$

3. (a) Let x be any real number. Is the set $S = \{x\}$. consisting of the single point x a Borel set in $\mathbb{R}$?

   (b) Let $x_1, x_2, \cdots, x_N$ be N real numbers. Is the set
   $$S = \{x_1, x_2, \cdots, x_N\}$$
   a Borel set in $\mathbb{R}$?

   (c) Let $x_1, x_2, \cdots, x_n, \cdots$ be an infinite sequence of real numbers. Is the set
   $$S = \{x_1, x_2, \cdots, x_n, \cdots\}$$
   a Borel set in $\mathbb{R}$?

4. When a biased die is rolled, the probability of getting the numbers 1 to 6 are as given below:

$$\mathcal{P}(1) = \frac{1}{2}, \mathcal{P}(2) = \frac{1}{4}, \mathcal{P}(3) = \frac{1}{8}, \mathcal{P}(4) = \frac{1}{16}, \mathcal{P}(5) = \mathcal{P}(6) = \frac{1}{32},$$

   Find the probability of the following events:

   (a) E=The event that a prime number shows up

   (b) F=The event that an odd number shows up

   (c) G=The event that a multiple of three shows up

5. A fair die is rolled twice and the outcome is recorded in the order in which it occurs. Find the probability of the following events:

   (a) E= The event that the number 5 appears in the first roll

   (b) F= The event that the sum of the numbers showing up in the two rolls is > 8

6. Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5\}$. Consider the experiment of picking randomly a function from A to B. Answer the following:

   (a) How many elements are there in the sample space Ω?

   (b) When a function is picked randomly what is the probability that,

      i. the function takes exactly four distinct values

      ii. the function takes at most four distinct values

7. Each of three persons have to randomly choose a 4 digit ATM PIN using the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. Answer the following:

   (a) What is the probability all three have chosen the same PIN?

   (b) What is the probability that exactly two of them have chosen the same PIN?

   (c) What is the probability that all three have different PIN?

   (d) What is the probability that at least two of them have chosen the same first digit?

8. Let $(\Omega, \mathcal{B}, P)$ be a Probability space. For the following statements determine whether they are TRUE or FALSE?

(a) $A, B \in \mathcal{B}$ and $A \subseteq B \Longrightarrow P(A) \leq P(B)$

(b) $A \in \mathcal{B}$ and $P(A) > 0.5 \Longrightarrow P(A') > 0.5$

(c) If $\{A_n\}_{n=1}^{\infty}$ is any sequence of events then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

(d) If $\{A_n\}_{n=1}^{\infty}$ is any sequence of events then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

(e) If $\{A_n\}_{n=1}^{\infty}$ is any nonincreasing sequence of events then

$$P\left(\bigcup_{n=1}^{\infty} A'_n\right) = 1 - \lim_{n\to\infty} P(A_n)$$

9. Let $(\Omega, \mathcal{B}, P)$ be a probability space. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events, that is, $A_n \in \mathcal{B}$. Answer the following:

   (a) If A_n is a nondecreasing sequence, that is for $n = 1, 2, 3, \cdots$, we have $A_n \subseteq A_{n+1}$ and if $P(A_n) = \frac{1}{2} - e^{-n}$ for all n, find $P\left(\bigcup_{n=1}^{\infty} A_n\right)$

   (b) If A_n are all disjoint and $P(A_n) = \frac{1}{2}e^{-n}$ find $P\left(\bigcup_{n=1}^{\infty} A_n\right)$

10. Let $(\Omega, \mathcal{B}, P)$ be a Probability Space. Answer the following:

    (a) If $A, B \in \mathcal{B}$ show that

    $$P(A \cap B) \geq P(A) + P(B) - 1$$

    (b) Let A and B be mutually exclusive events (that is $A \cap B = \phi$). Suppose the experiment is repeated until either A or B occurs. (We consider it a "'success" if either A or B occurs). How does the sample space of this new experiment look like?

(c) In the experiment of (b) above show that the probability of the event A occuring before the event B is given by $\dfrac{P(A)}{P(A)+P(B)}$ (Hint: What is the probability that there is a success and the event A occurs on the nth trial?)

11. Suppose each of three persons tosses a coin. If the outcome of one of the three tosses differs from the other outcomes the game ends. If not the persons start over and retoss their coins. What is the probability that the game ends with the first round of tosses if

    (a) the coins are fair coins

    (b) the coins are biased with probability of a Head being 0.25 (for each of the coins)

12. A two bit code with at most one 1 in it is chosen at random. Answer the following:

    (a) Write down the sample space for this experiment

    (b) Let $A_1 = \{00, 10\}$ and $A_2 = \{01\}$. Find the σ-algebra $\mathcal{B}$ generated by the collection $\mathcal{C} = \{A_1, A_2\}$

    (c) Write down the probabilty of each event in $\mathcal{B}$

    (d) Define $X : \Omega \longrightarrow \mathbb{R}$ as

    $$X(\omega) = a_1 2^1 + a_2 \text{ for } \omega = a_1 a_2$$

    For every $x \in \mathbb{R}$ find $X^{-1}(I_x)$ where $I_x = (-\infty, x]$

    (e) Is X a random variable on $(\Omega, \mathcal{B}, P)$?

13. Consider the random experiment of rolling a fair die. Consider the probability space with $\mathcal{B}$ to be the collection of all subsets of Ω. Let X and Y be the random variables defined as

$$\begin{aligned} X(\omega) &= \left\lfloor \frac{\omega}{2} \right\rfloor \\ Y(\omega) &= \left\lceil \frac{\omega}{2} \right\rceil \end{aligned}$$

Answer the following:

(a) Find $\mathcal{R}_X$ and $\mathcal{R}_Y$ the set of values taken by X and Y

(b) Find the pmfs p_X and p_Y

14. A random variable X is such that $\mathcal{R}_X = \{1, 2, 3, 4\}$ with $P(X = k) = p_k$ for $k = 1, 2, 3, 4$. Answer the following:

    (a) Find the pmf p_X if $p_k = \dfrac{p_1}{k}$ for $k = 2, 3, 4$

    (b) Find the pmf p_X if $p_{k+1} = \dfrac{p_k}{2}$ for $k = 1, 2, 3$

15. When a biased die is rolled, the probability of getting the numbers 1 to 6 are given below:

$$P(1) = \frac{1}{2},\ P(2) = \frac{1}{4}, ; P(3) = \frac{1}{8},\ P(4) = \frac{1}{16},\ P(5) = P(6) - \frac{1}{32},$$

Let X be the random variable defined as

$$X(\omega) = \frac{1 + \omega}{2} \text{ for every outcome } \omega$$

Answer the following:

(a) Find the Range $\mathcal{R}_X$ of the random variable X

(b) Find the pmf of X