Seq no:

Name: **SR No.:** **Dept.:**

Maximum Points: 15 **E2-243: Quiz 2** **Duration**: 40 minutes

---

1. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of real valued functions defined as $f_n(x) = \frac{x^2}{n}$.

   a) Does the sequence converge point-wise to $f(x) = 0$ on the interval $I = [-1, 1]$? (2 points).

   Explanation Fix any $x_1 \in I$, let us now look at the sequence $f_n(x_1)$. $\forall\epsilon > 0, \exists$ a positive integer N_{ϵ,x_1} such that

   $\forall n \geq N_{\epsilon,x_1}$, $| f_n(x_1) - f(x_1) | < \epsilon$

   $| \frac{x_1^2}{n} - 0 | < \epsilon$

   As $-1 \leq x_1 \leq 1$

   $\frac{x_1^2}{n} < \frac{1}{n} < \epsilon$

   $n > \frac{1}{\epsilon} = N_{\epsilon,x_1}$

   This is true for all $x \in I$

   Hence f_n converges pointwise to f on I.

   b) Does it converge uniformly to $f(x) = 0$ on the interval $I = [-1, 1]$? (2 points)

   Explanation As $-1 \leq x \leq 1$

   $f_n(x) = \frac{x^2}{n} < \frac{1}{n}, \forall n \in \mathbb{N}, \forall x \in I$.

   Let us choose $M_n = \frac{1}{n}$

   $\lim_{n\to\infty} M_n = 0$

   Hence by M_n test f_n converges uniformly to zero function on the interval $I = [-1, 1]$.

   c) Does it converge uniformly to $f(x) = 0$ on the Interval $I = [0, \infty)$? (2 points)

   Explanation We shall show that this sequence does not converge to 0 uniformly. If we choose $x_n = n$ for every n then $f_n(x_n) = n \geq 1$. Since all f_ns cross the $\epsilon = \frac{1}{2}$ barrier, f_n does not converge uniformly to zero function on the interval $I = [0, \infty)$.

   d) For what values of p does f_n converge to f in $L^p[0, \infty)$?(1 Point)

   Explanation $\lim_{n\to\infty} \left(\int_0^\infty (\frac{x^2}{n})^p \, dx \right)^{\frac{1}{p}} = 0$

   $$\lim_{n\to\infty} \left(\frac{1}{(2p+1) * n^p} \left[x^{2p+1} \right]_0^\infty \right)^{\frac{1}{p}} = \infty$$

   Thus the sequence does not L^p converge to f for any $p \geq 1$ on the interval $I = [0, \infty)$.

   e) For what values of p does f_n converge to f in $L^p[0, 1]$? (2 points)

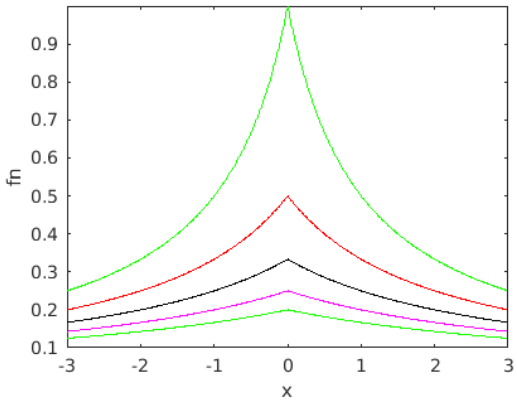
Explanation $\lim_{n\to\infty}\left(\int_0^\infty (\frac{x^2}{n})^p\,dx\right)^{\frac{1}{p}} = 0$

$\lim_{n\to\infty}\left(\left[\frac{x^{2p+1}}{(2p+1)*n^p}\right]_0^1\right)^{\frac{1}{p}} = \lim_{n\to\infty}\left(\frac{1}{(2p+1)*n^p}\right)^{\frac{1}{p}} = 0$

The sequence L^p converges to $f(x)$ for all $p \geq 1$ values on the interval $I = [0,1]$.

2. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of real valued functions defined on the interval $I = (-\infty, \infty)$ as $f_n(x) = \frac{1}{n+|x|}$.

   a) Sketch the graph of f_n. (2 points)



   b) Does the sequence converge point-wise? If so what is the limit function? (2 points)

Explanation The Limit function is f(x)= 0. Fix any $x_1 \in I = (-\infty, \infty)$, let us now look at the sequence $f_n(x_1)$.

$\forall \epsilon > 0, \exists$ a positive integer N_{ϵ,x_1} such that

$\forall n \geq N_{\epsilon,x_1}, \mid f_n(x_1) - f(x_1) \mid < \epsilon$

$\mid \frac{1}{n+|x_1|} - 0 \mid < \epsilon$

$\frac{1}{n+|x_1|} < \frac{1}{n} < \epsilon$ (as $x_1 \in (-\infty, \infty), \mid x \mid \in [0, \infty)$)

$N_{\epsilon,x_1} = \frac{1}{\epsilon}$

The above argument is true for all $x \in I$

Hence f_n converges pointwise to f.

   c) Does it converge uniformly to the limit function obtained in 2(b)? (2 points)

Explanation $f_n(x) = \frac{1}{n+|x|} < \frac{1}{n}, \forall n \in \mathbb{N}, \forall x \in (-\infty, \infty)$

Let us choose $M_n = \frac{1}{n}$

$\lim_{n\to\infty} M_n = 0$

Hence by M_n test f_n converges uniformly to zero function.