E2:243 TEST 1

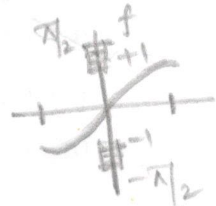
(September 14, 2018) **(2PM -3:30PM)**

Name: **SR No.:** **Department:**

Answer All Questions **(Maximum Marks:35)**

I) In the following, in each question only one alternative is correct. Tick ($\surd$) the correct alternative: (Correct Answer 1 Mark/Wrong Answer -0.5 Mark/Not Attempted 0 Mark)

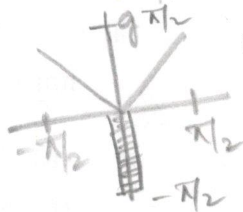
1. Let $\mathcal{I}$ be the closed interval $[-a, a]$ where $a = \frac{\pi}{2}$. Consider the functions f and g defined as



Not onto

$$f : \mathcal{I} \longrightarrow \mathcal{I} \text{ defined as } f(x) = sin(x)$$

$$g : \mathcal{I} \longrightarrow \mathcal{I} \text{ defined as } g(x) = |x|$$



Not one-on
not onto

   (a) Both f and g are one-one and onto

   (b) Only f is one-one and onto

   (c) Only g is one-one and onto

   ✓(d) Neither of them is one-one and onto

2. Let $\{A_n\}_n$ be a sequence of subsets of a set Ω. Consider the following two sets:

Refer notes

$$\begin{aligned} A &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\ L &= \{x \in \Omega : x \in A_n \text{ for all } n \text{ beyond a certain stage}\} \end{aligned}$$

   (a) Both A and L are equal to $\limsup_{n\to\infty} A_n$

   ✓(b) Both A and L are equal to $\liminf_{n\to\infty} A_n$

   (c) Only A is equal to $\limsup_{n\to\infty} A_n$

   (d) Only A is equal to $\liminf_{n\to\infty} A_n$

3. Consider the following two statements:

(A) The sequence of real numbers $\{a_n\}_n$ converges to a real number $a \Longrightarrow$
The sequence of real numbers $\{|a_n|\}_n$ converges to a real number $|a|$

(B) The infinite series of real numbers $\sum_{n=1}^{\infty} a_n$ converges and its sum is $a \Longrightarrow$

The infinite series of real numbers $\sum_{n=1}^{\infty} |a_n|$ converges and its sum is $|a|$

Then

Refer notes

(a) Both (A) and (B) are TRUE
(b) (A) is TRUE and (B) is FALSE
(c) (A) is FALSE and (B) is TRUE
(d) Both (A) and (B) are FALSE

4. Consider the sequence of real numbers $\{a_n\}_n$ where $a_n = \dfrac{n^2-1}{n^2+1}$. Then $\lim_{n\to\infty} a_n$

$$\lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} = 1$$

(a) does not exist
(b) exists and equal to -1
(c) exists and equal to 1
(d) exists and equal to 0

5. The infinite series of real numbers

$$\sum_{n=1}^{\infty} \underbrace{\frac{kn^2+3n+2}{4n^4+2}}_{S_n}$$

(where k is a real constant),

(a) converges absolutely if $|k| < 4$ and diverges for $|k| \geq 4$
(b) converges absolutely if $|k| \leq 4$ and diverges for $|k| > 4$
(c) converges absolutely for all values of $k > 0$
(d) converges absolutely for all real values of k

$$\frac{\frac{k+\frac{3}{n}+\frac{2}{n^2}}{4n^2+\frac{2}{n^2}}}{\frac{k}{4n^2}}$$

both $\Sigma S_n, \Sigma S_n'$ will converge or diverge together, as $\frac{1}{n^2}$ is absolutely convergent ΣS_n also converge

II) In the following, state TRUE or FALSE: (Correct Answer 1 Mark/Wrong Answer −0.5 Mark/Not Attempted 0 Mark)

1. Every nondecreasing sequence of real numbers bounded below converges False

2. A sequence of real numbers $\{a_n\}_n$ is convergent to a real number a
$\iff$
$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$ True By def.

3. By the Ratio Test it can be concluded that the infinite series of real numbers $\sum_{n=1}^{\infty}(-1)^n \frac{e^n}{n!}$ converges absolutely True
Use ratio test

4. Let $\mathcal{I}$ be the closed interval $[0, 1]$ in $\mathbb{R}$.
The sequence of real valued continuous functions $f_n : \mathcal{I} \longrightarrow \mathbb{R}$ converges pointwise to the function $f : \mathcal{I} \longrightarrow \mathbb{R}$
$\Longrightarrow$
$\int_0^1 f_n(x)dx$ converges to $\int_0^1 f(x)dx$ False.
Refer lecture notes

5. Let $\mathcal{I}$ be the closed interval $[0, 1]$ in $\mathbb{R}$. Let $f_n : \mathcal{I} \longrightarrow \mathbb{R}$ and $f : \mathcal{I} \longrightarrow \mathbb{R}$. Then

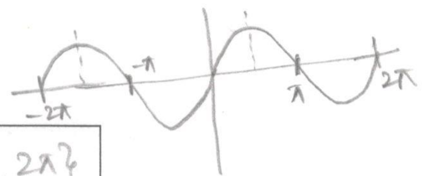
$$f_n \overset{u\ (\mathcal{I})}{\longrightarrow} f \iff f_n \overset{pw\ (\mathcal{I})}{\longrightarrow} f$$

False $f_n \overset{u(I)}{\longrightarrow} f \Rightarrow f_n \overset{pw(I)}{\longrightarrow} f$

III) In the following FILL IN THE BLANKS WITH APPROPRIATE AN SWERS: (Correct Answer 1 Marks/Wrong Answer or Not attempted 0 Mark)

1. Let $\mathcal{I}$ be the closed interval $[-2\pi, 2\pi]$ and $f : \mathcal{I} \longrightarrow \mathbb{R}$ be the function defined as $f(x) = sin(x)$. Let A, B be the subsets of $\mathbb{R}$ defined as

$$A = \{0, 1\} \text{ and } B = \{x \in \mathbb{R} : x^2 > 4\}$$



Then

$f^{-1}(A) = \{\frac{\pi}{2}, -3\pi/2, -2\pi, -\pi, \pi, 0, 2\pi\}$

$f^{-1}(B) = \emptyset$

2. Consider the sequence $\{A_n\}_n$ of subsets of $\mathbb{R}$ defined as

$$A_n \text{ is the interval } [n, 2n)$$

$$\limsup_{n\to\infty} A_n = \phi$$

$$\liminf_{n\to\infty} A_n = \phi$$

3. Consider the sequence of real numbers defined as

$$a_n = \begin{cases} \dfrac{1}{n} & \text{if } n \text{ is a multiple of } 3 \\ (-1)^n \dfrac{n}{n+1} & \text{if } n \text{ is not a multiple of } 3 \end{cases}$$

Then

$$\limsup_{n\to\infty} a_n = 1$$

$$\liminf_{n\to\infty} a_n = -1$$

4. The sequence of real valued functions $f_n : (0,1) \longrightarrow \mathbb{R}$ defined as

$$f_n(x) = \frac{n}{1+nx}$$

converges pointwise to the function $f : (0,1) \longrightarrow \mathbb{R}$ where

$$f(x) = \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{n}{1+nx} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}+x} = \frac{1}{x}$$

5. For what real values of k does the infinite series of real numbers of real numbers, $\sum_{n=1}^{\infty} \frac{n^2+3n+2}{2n^k+5}$ converge?

If and only if $k > 3$

as $\sum \frac{1}{n^p}$ converges if $p>1$

IV) In the following give reasons for your answers and show the details of your working:

1. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite sets both having the same number of elements, and $f : \mathcal{A} \longrightarrow \mathcal{B}$ a function from $\mathcal{A}$ to $\mathcal{B}$. Show that

$$f \text{ is one-one} \iff f \text{ is onto}$$

(4 Marks)

2 Marks f is one-one $\Rightarrow$ f is onto

2 Marks f is onto $\Rightarrow$ f is one-one.

$\Rightarrow$ One-one: if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

For f to be onto $\forall\, b \in B$ $\exists\, a \in A$ s.t $f(a) = b$

and also $|B|$ should be atmost $|A|$.

Given $|A| = |B|$, hence f is onto.

$\Leftarrow$ f is onto: For every $b \in B$, $\exists\, a \in A$ s.t $f(a) = b$.

As $|A| = |B|$ and f is a single valued function

every $a \in A$ has a unique mapping in B.

Hence f is one-one.

2. Show that the sequence of real numbers $\left\{ \dfrac{sin(4n^2)}{n^2+1} \right\}_n$ converges to zero (4 Marks)

Use sandwich theorem

(or)

Basic definition

3. Determine whether the following infinite series of real numbers converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2^n}{n\ \{(2n)!\}}$$

(3 Marks)

Use ratio test

Converges

4. Consider the sequence of real valued functions $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$f_n(x) = \frac{x^2}{1 + nx^4}$$

Answer the following:

(a) Does $f_n \overset{pw\,(\mathbb{R})}{\longrightarrow} 0$?

(b) Show that the sequence of derivatives $f_n'(x)$ converges pointwise and find the limit function $g(x)$

(c) Is $f'(x) = g(x)$?

(2 + 2 + 1 Marks)

(a) $\lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \dfrac{\frac{x^2}{n}}{\frac{1}{n} + x^4} = \dfrac{0}{0 + x^4} = 0.$

Yes $f_n \overset{PW(R)}{\longrightarrow} 0$ $f_n \to f$ $\therefore f(x) = 0 \; \forall x$

(b) $f_n'(x) = \dfrac{2x}{1+nx^4} - \dfrac{x^2 \times n \times 4 \times x^3}{(1+nx^4)^2}$

$= \dfrac{2x + 2nx^5 - 4nx^5}{(1+nx^4)^2}$

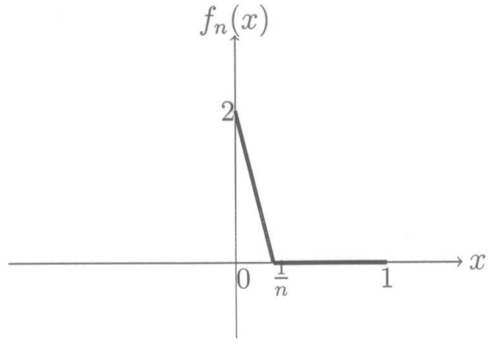
$= \dfrac{2x - 2nx^5}{(1+nx^4)^2}$

$g(x) = \lim\limits_{n \to \infty} f_n'(x) = \dfrac{\frac{2x}{n} - 2x^5}{\frac{1}{n} + 2x^4 + nx^8} = 0 \; \forall x$

(C) Yes

$f'(x) = g(x)$

5. Consider the sequence of functions $f_n : (0,1) \longrightarrow \mathbb{R}$ whose graphs are as shown below:



Answer the following:

(a) For each n, find a point $x_n \in (0,1)$ such that $f_n(x_n) = \dfrac{1}{2}$

(b) Does the sequence converge uniformly on $(0,1)$ to zero function?

(2 + 2 Marks)

(a) $f_n(x_n) = -2nx_n + 2$

$$\frac{1}{2} = -2nx_n + 2$$

$$-\frac{3}{2} = -2nx_n$$

$$x_n = \frac{3}{4n}$$

(b) Using (a), we see that $f_n(x_n) \geq \frac{1}{2}$ $\forall\ x_n \in (0, \frac{3}{4n}]$

Now, choose $\epsilon = \frac{1}{4}$

Then $|f_n(x_n) - 0| > \frac{1}{2}$ $\forall\ x_n \in (0, \frac{3}{4n}]$

As we cannot trap $f_n(x_n)$ however large n we choose, it is not uniformly convergent.