74.5

E2:243 TEST II

(November 15, 2019) (2PM -4PM)

Name: SPARSH
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Sequence Number: 51

Answer All Questions (Maximum Marks:100)

15.5

I) In the following, in each question only one alternative is correct. Tick (√) the correct alternative:
(Correct Answer 2 Mark/Wrong Answer −0.5 Mark/Not Attempted 0 Mark)
(Total: 20 Marks)

1. Let $\{E_n\}$ be any sequence of events in a probability space $(\Omega, \mathcal{B}, P)$ such that $P(E_n) = \frac{1}{n^2}$. Then according to Borel-Cantelli Lemma,

$\sum P(E_n) < \infty$

(a) $P(\limsup_{n\to\infty} E_n) = 1$

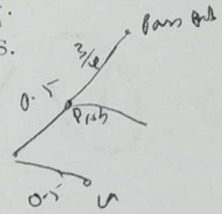
d

(b) $P(\liminf_{n\to\infty} E_n) = 1$

(c) $P(\liminf_{n\to\infty} E_n) = 0$

(d) $P(\limsup_{n\to\infty} E_n) = 0$

2. A student has to decide whether to register for the course on probability theory or Linear Algebra. If she takes Linear Algebra, she will pass with probability $\frac{2}{3}$; if she takes probability, she will pass with probability $\frac{3}{4}$. She make her decision to choose the course based on a fair coin toss. The probability that she passed the probability course is



b

a) $\frac{3}{4}$ b) $\frac{3}{8}$ c) $\frac{1}{2}$ d) $\frac{1}{3}$

3. A company sells CDs in packs of 10. The Cds are defective with probability 0.01. Each CD is defective or not independently of other CDs. A customer gets his/her money back only if more than one CD in a

P (at least 2 are defective)

Subtotal : 68.5

pack is defective. The probability that the customer gets his/her money back is

b

(a) $(0.99)^{10} + \{10 \times (0.01) \times (0.99)^9\}$
(b) $1 - [(0.99)^{10} + \{10 \times (0.01) \times (0.99)^9\}]$
(c) $0.99 \times (0.01) + 0.99 \times (0.01)^2$
(d) $1 - \{0.99 \times (0.01) + 0.99 \times (0.01)^2\}$

4. Let $\{A, B, C\}$ be an independent collection of events in a probability space $(\Omega, \mathcal{B}, P)$ such that $P(A) = P(B) = P(C) = 0.3$. Then $P(A \cup B \cup C) =$

c .657

a) $(0.3)^3$ b) $(0.7)^3$
c) $1 - (0.7)^3$ d) $1 - (0.3)^3$

$P(A)+P(B)+P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

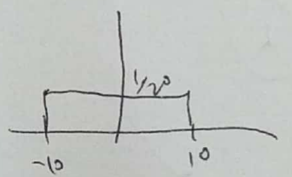
$0.3\times3 - 3\times0.3^2 + 0.3^3$

$0.9 - 3\times0.09 + .027$

$.657$

5. If a real valued random variable X is $\sim Uni[-10, 10]$ its cdf is given by

(a) $F_x(x) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x+10}{20} & \text{for } 0 \le x \le 10 \\ 1 & \text{for } x > 10 \end{cases}$



b

(b) $F_x(x) = \begin{cases} 0 & \text{for } x < -10 \\ \dfrac{x+10}{20} & \text{for } -10 \le x \le 10 \\ 1 & \text{for } x > 10 \end{cases}$

$\frac{x}{20} + c =$

$-\frac{10}{20} + c = \frac{1}{20}$

$c = \frac{11}{20}$

(c) $F_x(x) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x-10}{20} & \text{for } 0 \le x \le 10 \\ 1 & \text{for } x > 10 \end{cases}$

$0.3(3 - 0.9 + .09)$

$0.3(2.19)$

$.657$

$1 - .343 = .657$

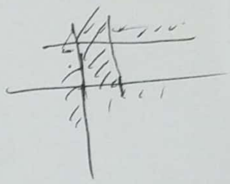
(d) $F_X(x) = \begin{cases} 0 & \text{for} \quad x < -10 \\ \dfrac{x-10}{20} & \text{for} \quad -10 \leq x \leq 10 \\ 1 & \text{for} \quad x > 10 \end{cases}$

$Var(X) + Var(Var(X)) + cov(X, Var X)$

6. If X is a real valued random variable with finite variance, then $Var(X+Var(X)) =$

b

a) $(Var(X))^2$ b) $Var(X) + Var(Var(X))$
c) $(Var(X))^2 + Var(X)$ d) $Var(X) \times Var(Var(X))$



7. Let X and Y be two real valued random variables on a probability space $(\Omega, \mathcal{B}, P)$ with joint cdf given by

$$F_{XY}(x,y) = \begin{cases} 0 & \text{if } x \text{ or/and } y < 0 \\ x(1-e^{-4y}) & \text{if } 0 \leq x \leq 1 \text{ and } y \geq 0 \\ 1 - e^{-4y} & \text{if } x > 1 \text{ and } y > 0 \end{cases}$$

$\int_{y=4}^{\infty}\int_{x=0}^{1} x(1-e^{-4y})\,dx\,dy$

$\left|\frac{x^2}{2} - xe^{-4y}\right|_0^1$

$\int_4^{\infty} \frac{1}{2} - e^{-4y}\,dy$

$\left|\frac{y}{2} - \frac{e^{-4y}}{-4}\right|_4^{\infty}$

The Probablity that $Y > 4$ is given by

a) $1-e^{-16}$ b) e^{-16} c) $(1-e^{-4})$ d) none of these

8. If X and Y are continuous real valued random variables with cdf $F_X(x)$ and $F_Y(y)$ respectively and joint cdf $F_{XY}(x,y)$ then $P(X < x | Y > y) =$

(a) $\dfrac{F_X(x)}{1 - F_Y(y)}$

(b) $\dfrac{F_Y(y)}{1 - F_X(x)}$

$P\left(\frac{X<x}{Y>y}\right) = \frac{P(X<x \cap Y>y)}{P(Y>y)}$

c (c) $\dfrac{F_X(x) - F_{XY}(x,y)}{1 - F_Y(y)}$

(d) $\dfrac{F_Y(y) - F_{XY}(x,y)}{F_Y(y)}$

9. Consider the following two statements:

I) $cov(X+Y, Z) = cov(X, Z) + cov(Y, Z)$
for any three real valued random variables X, Y, Z on a probability space $(\Omega, \mathcal{B}, P)$

II) $cov(\alpha X, \beta Y) = \alpha\beta cov(X, Y)$
for any two real valued random variables on a probability space $(\Omega, \mathcal{B}, P)$ and for any two real numbers α and β

Then

(a) Both (I) and (II) are TRUE

(b) (I) is FALSE and (II) is TRUE

(c) (I) is TRUE and (II) is FALSE

(d) Both (I) and (II) are FALSE

10. Consider the two discrete real valued random variables whose joint pmf is as given below:

	$Y=-1$	$Y=0$	$Y=1$	
$X=0$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$
$X=1$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

Consider the following two statements:

(I) X and Y are independent F

(II) $cov(X, Y) = 0$ T

(a) Both (I) and (II) are TRUE

(b) Both (I) and (II) are FALSE

$XY = \begin{cases} 0 & \frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{3}{3}\ (1) \\ -1 & 0 \\ 1 & 0 \end{cases}$

$E(XY) = 0$

$E(X) = \frac{1}{3}$

(d)

(c) (I) is TRUE and (II) is FALSE

(d) (I) is FALSE and (II) is TRUE

11.5

II) In the following, state TRUE or FALSE:
(Correct Answer 1.5 Mark/Wrong Answer −0.5 Mark/Not Attempted 0 Mark)
(Total :15 Marks)

1. If A, B are two events in a probability space $(\Omega, \mathcal{B}, P)$,
$$P(A \cup B) = P(A) + P(B) \iff A \cap B = \phi$$
T

2. If E, F, G are pairwise independent events in a probability space $(\Omega, \mathcal{B}, P)$ then the events $E \cap F$ and G must be independent F

3. Any two disjoint events in a probability space must be independent F

$P(A\cap B) = P(A)P(B)$

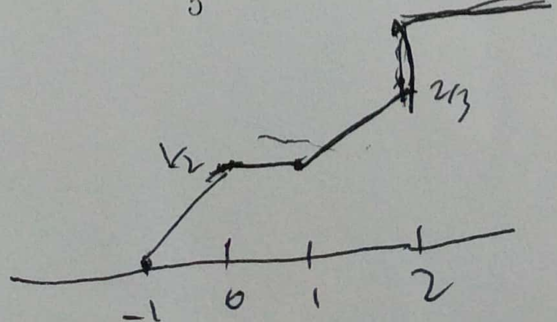
4. If E, F are two events in a probability space $(\Omega, \mathcal{B}, P)$ (with $P(E \cap F) > 0$) then
$$\frac{P(E|F)}{P(F|E)} = \frac{P(F)}{P(E)}$$
F

5. Let X, Y be continuous real valued random variables whose joint cdf is given by
$$F_{XY} = \begin{cases} 1 - e^{-3x} - e^{-4y^2} + e^{-3x-4y^2} & \text{for } x, y \text{ both} \geq 0 \\ 0 & \text{for other values of } x \text{ and } y \end{cases}$$
Then X and Y are independent random variables

$F_x F_y$

6. The following function cannot be the cdf of any real valued random variable:
$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+1}{2} & \text{for } -1 \leq x < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \\ \frac{x+2}{6} & \text{for } 1 \leq x \leq 2 \\ 1 & \text{for } x > 2 \end{cases}$$
T

4/6



7. For any real valued random variable X with finite variance, the following holds: T
$$Var(3X + 2) = Var(-3X + 4)$$

8. If a sequence $\{X_n\}$ of continuous real valued random variables converges in distribution to the real valued random variable X then X must also be a continuous real valued random variable F

9. Let X, Y, Z be three identically distributed real valued random variables and let X_n be the sequence defined as

$$X_n = \begin{cases} X & \text{for } n = 3k, (k = 1, 2, 3, \cdots) \\ Y & \text{for } n = 3k + 1, (k = 1, 2, 3, \cdots) \\ Z & \text{for } n = 3k + 2, (k = 1, 2, 3, \cdots) \end{cases}$$ T

Then $X_n \xrightarrow{d} X$, $X_n \xrightarrow{d} Y$ and $X_n \xrightarrow{d} Z$

10. If $X_n \sim Exp(\lambda)$ random variables then F

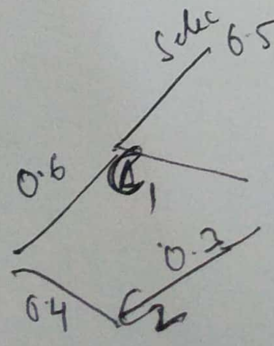
$$\frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow{p} \lambda$$

III) In the following **FILL IN THE BLANKS WITH APPROPRIATE ANSWERS:**
(Correct Answer 3 Marks/Wrong Answer or Not attempted 0 Mark)
(Total: 15 Marks)

1. DESE appoints two committees C_1 and C_2 to interview candidates who have applied for PhD admission and a candidate can opt to be interviewed by only one of the committees. The probability that an applicant A chooses Committee C_2 is 0.4. The probability that the candidate A gets selected if he appears before committee C_1 is 0.5 and if he appears before committee C_2 is 0.3. Then the

Probability that A gets selected = 0.42

12



$+ 0.4 \times 0.3$

2. Suppose that 5 good fuses and 2 defective ones have been mixed up. To find the defective fuses, we test them one by one. Then

$\frac{{}^2C_2}{{}^7C_2} = \frac{1}{21}$

$$\left\{ \begin{array}{l} \text{Probability that we find both the} \\ \text{defective ones in the first two tests} \end{array} \right\} = \frac{1}{21}$$

3. Let A and B be independent events with $P(A) = 0.25$ and suppose $P(A \cup B) = 2P(B) - P(A)$. Then

$= P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 2P(A) + P(B) - P(A)P(B)$

$2 \times 0.25 - 0.25\,P(A) = P(B)$

$$P(B) = 2/5 = 0.4$$

$$P(B'|A) = 0.6$$

$\frac{P(B' \cap A)}{P(A)} = \frac{0.6 \times 1/4}{1/4}$

$\mu = 9$

4. If $X \sim Exp(3)$ random variable and $Y = 2X + 3$ then

$\mu = 3$

$v = 1/3$

$X = 3e^{-3x}$

$x\,3e^{-3x} + 3\,dx$

$$F_Y(y) =$$

$1 - e^{-3x}$

$E(X)$

5. Let X_1, X_2, X_3, X_4 be independent identically distributed real valued random variables, with mean 10 and Variance 25, and let

$$Y = \frac{X_1 + X_2 + X_3 + X_4 - 40}{10}$$

Then

$E(Y) = E\left(\frac{X_1 + \cdots}{10}\right)$

$= \frac{E(X_1 \ldots) - 40}{10}$

$= \frac{4 \times 10 - 40}{10}$

$V(Y) = \frac{25 \times 4}{100}$

$$E(Y) = 0$$

$$Var(Y) = 1$$

IV) In the following give reasons for your answers and show the details of your working:
(Write the answers in the space provided below each question)

1. Let X and Y be independent discrete real valued random variables defined on a probability space $(\Omega, \mathcal{B}, P)$ with joint pmf matrix given partially as below:

	$Y=-1$	$Y=0$	$Y=1$	Marginal
$X=-2$	0.09	0.15	.06	0.3
$X=-1$	.075	0.125	.05	.25
$X=0$	.075	.125	.05	.25
$X=1$	.06	.10	0.04	.2
Marginal	0.3	0.5	.2	1

10

Answer the following:

(a) Fill in the other entries in the above joint pmf matrix
(b) Verify that $E(XY) = E(X)E(Y)$
(c) Find $E(X|Y=1)$

(7+3+2=12 Marks)

2. When the office of a CEO of a company transmits a message to the CEO, the probability that the message will be received by him is p. When the CEO receives the message, he transmits an acknowledgment signal to the office which reaches surely. If the office does not receive the acknowledgment signal, it sends the message again. Answer the following:

   (a) What is the pmf of N, the number of times the office sends the same message?

   (b) The company wants to limit the number of times the office has to send the same message. It has a goal of $P(N \leq 3) \geq 0.95$. What is the minimum value of p necessary to achieve the goal?

8

(5+5=10 Marks)

3. Consider independent trials consisting of rolling a pair of fair dice over and over. Answer the following:

   (a) Find the probability that event F that the sum of the numbers showing up on the two dice in the first roll is 5

   (b) Find the probability of the event G that the sum of the numbers showing up on the two dice in the first roll is 7

   (c) Find the probability of the event H that the sum of the numbers showing up on the two dice in the first roll is neither 5 nor 7

   (d) Let E be the event that a sum of 5 shows up before a sum of 7 does. Is $P(E|H) = P(E)$?

   (e) Find $P(E)$

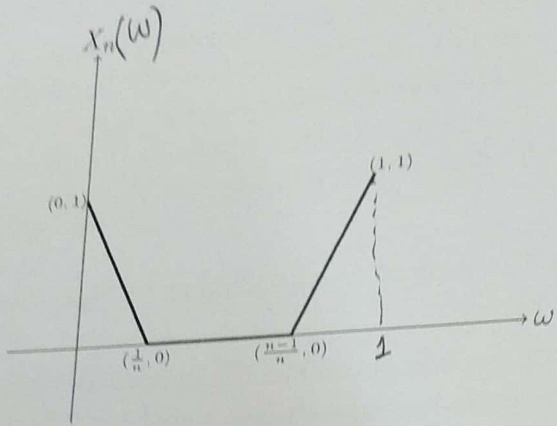
6

(2+2+2+2+2=10 Marks)

4. The average time of getting connected to a telephone line is 15 seconds and the standard deviation is 3 seconds. Find the estimate (using Chebychev inequality) that the connecting time is between 12 and 20. **(6 Marks)**

2½

5. Let X_n be the sequence of real valued random variables defined on the sample space $\Omega = [0, 1]$, (The event space is the Borel sets and the probability of an interval is its length), defined as shown in the figure below:



Using the respective definitions answer the following:

(a) Does the sequence converge in distribution to 0?

(b) Does the sequence converge in probability to 0?

(c) Does the sequence converge almost surely to 0?

(6+3+3=12 Marks)

As $n \to \infty$ $\frac{1}{n} \to 0$ $\frac{n-1}{n} \to 1$

Q2

(b)

XY	P(XY)
2	0.09
1	.115
0	.625
-1	.11
-2	.06

$E(X) = -2\times0.3 - 1\times.25 + 1\times.2$
$= -0.6 - .25 + .2$
$= -0.85 + 0.2$
$= -0.65$

$E(Y) = -1\times0.3 + 1\times.2$
$= -0.1$

$E(X)E(Y) = .065$

$E(XY) = 2\times.09 + 1\times.115 - 1\times.11 - 2\times.06$
$= .18 + .115 - .11 - .12$
$= .065$

RW
.09
.115
.625
.11
.06
1.000

.11 .18
.12 .115
.23 .295
-.23

$E(XY) = E(X)E(Y)$

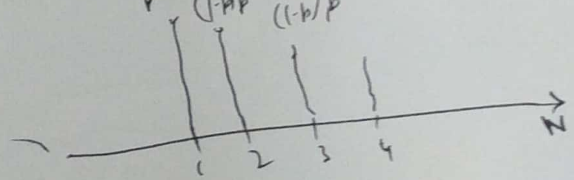
(c) $E(X/Y=1) = -2\times.06 - 1\times.05 + 1\times.04$
$= -.12 - 0.05 + .04$
$= -.17 + .04$
$= -.13$

.12
.05
.17
.04
.13

Q2

N	Probability (N=n)
2	p
2	$(1-p)p$
3	$(1-p)^2 p$
⋮	

PMF



(b) Goal

$P(N \leq 3) \geq 0.95$

$p + (1-p)p + (1-p)^2 p \geq 0.95$

$p + (p-p^2) + (1+p^2-2p)p \geq 0.95$

$2p - p^2 + p + p^3 - 2p^2 \geq 0.95$

$p^3 - 3p^2 + 3p - 0.95 \geq 0$

CDF of ~~Bernoulli~~ Binomial RV

~~PDF~~

~~$P(X \leq k) = {}^{n}C_k \, p^k (1-p)^{n-k}$~~

$(p-1)^3 + 0.25 \geq 0$

$(p-1)^3 \geq -0.25$

$(p-1) \geq (-0.25)^{1/3}$

$p \geq 1 + (-0.25)^{1/3}$

$(p-1)^3$

$p^3 - 1$

$-3p(p-1)$

$p^3 - 1$

$-3p^2 + 3p$

$(a^2-b)(a-b)$

(a^2+b^2-2ab)

$(a-b)$

$a^3 + ab^2$

$-2ab$

Q3

1,4
2,3
3,2
4,1

a) $P(F) = \dfrac{4}{36}$

1,6
2,5
3,4
4,3
5,2
6,1

b) $P(G) = \dfrac{6}{36}$

c) $P(H) = 1 - P(F) - P(G)$

$= 1 - \dfrac{4}{36} - \dfrac{6}{36}$

$= 1 - \dfrac{10}{36}$

$= \dfrac{26}{36}$

X. d) P(5 shows up before 7)

? $= \dfrac{4}{36} + \dfrac{26}{36} \times \dfrac{4}{36} + \left(\dfrac{26}{36}\right)^2 \times \dfrac{4}{36} + \left(\dfrac{26}{36}\right)^3 \times \dfrac{4}{36} + \cdots$

X d) $P(E/H) = \dfrac{P(E \cap H)}{P(H)} = \dfrac{22/36 + P(E)}{P(H)}$

$= \dfrac{22}{36} \times \dfrac{36}{26} + \dfrac{P(E)}{P(H)}$

$= \dfrac{22}{26} + \dfrac{P(E)}{26} \times 36$

$\neq P(E)$

Hence False

$P(E \cap H) = \dfrac{26}{36} + \dfrac{26}{36} \times \dfrac{4}{36} + \left(\dfrac{26}{36}\right)^2 \dfrac{4}{36} + \cdots$

$= \dfrac{26}{36} - \dfrac{4}{36} + \underbrace{\dfrac{4}{36} + \cdots}_{P(E)}$

$P(E \cap H) = \dfrac{22}{36} + P(E)$

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Q4 Let T a RV denotes time of connecting to telephone line

$E(T) = 15$

$Var(T) = 3$

$E((T-15)^2) = 3$

$P(|x-\mu|^2 \leq k^2) \leq \frac{Var\, X}{k^2}$

$P(12 \leq T \leq 20)$

$\equiv P(|T-16| < 4)$

$= P(|T-16|^2 < 4^2)$

$\frac{12}{2}$ $\frac{32}{2}$ 16

2/2

$-4 < T-16 < 4$

12

$\leq \frac{E(|T-16|^2)}{4^2}$

$\leq \frac{E(T^2+256-32T)}{4^2}$

$= \frac{E(T^2)+256-32E(T)}{16}$

Put here $E(T^2)$

$E(T^2+225-30T) = 3$

$E(T^2)+225-30E(T) = 3$

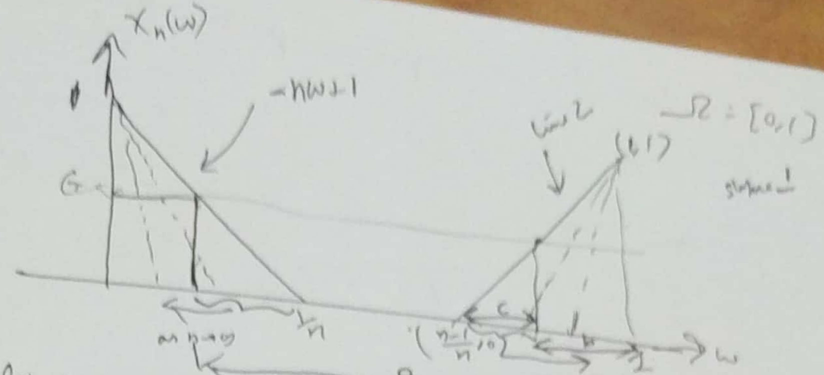
$E(T^2) = 3+30\times15-225$

$= 3+450-225$

$= 228$

Calculate $E(T^2)$

Q5



a)

Convergence in distribution

$$X_n(\omega) = \begin{cases} \cancel{-n\omega+1} & 0 \le \omega \le 1/n \\ 0 & \frac{1}{n} < \omega < \frac{n-1}{n} \\ ? & \omega > \frac{n-1}{n} \end{cases}$$

$-n\omega + 1 = x$

$n\omega = \frac{1-x}{n}$

$P(X_n \le x)$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{2x+n-2}{n} & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

5

As $n \to \infty$ $F_X(x) \xrightarrow{d} \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ & x > 1 \end{cases}$

Hence Yes it converges in distribution

Line 2

$$\frac{y-0}{\omega - \frac{n-1}{n}} = \frac{1}{1-\frac{n-1}{n}} = \frac{n}{1}$$

$y = \left(\omega - \frac{n-1}{n}\right) \times n$

$y = \left(\omega - \frac{n-1}{n}\right) \times n$

Put $y = x$

$x = n\omega - n + 1$

$x + n - 1 = n\omega$

$\omega = \frac{x+n-1}{n}$

Distance a

$\frac{x+n-1}{n} - \frac{1-x}{n}$

$\frac{x+n-1-1+x}{n} = \frac{2x+n}{n}$

Distance b

$= 1 - \frac{x+n-1}{n}$

$= \frac{n-x-n+1}{n}$

$= \frac{1-x}{n}$

Distance c

$\frac{x+n-1}{n} - \frac{n-1}{n}$

$n\left(\frac{2x}{n}+1\right)$

b Yes it converges in probability ; it follows from c.

Also we can find $n = N_\epsilon$ for which $|X_n - X| < \epsilon$

ie $P(w \in \Omega \mid |X_n - X| \geq \epsilon) = 0$

Use basic def. as given in question

1

c Yes sequence converges almost surely beyond which as for every w we can find N ~~for~~ which

$X_n(w) = X(w)$

$$X(w) = \begin{cases} 1 & w = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

3

$$X_n(w) \xrightarrow{a.s} X(w)$$