

Linear algebra.

Strategy in handling a matrix.

Reduce or decompose into simple forms (matrices).

LU decomposition.

Cholesky.

Triangularization \rightarrow Schur
 \rightarrow Jordan.

Diagonalization - Similarity / Unitary / Orthogonal.
all symmetric

Tridiagonalization.

QR decomposition.

Hessenberg.

Has advantages and limitations.

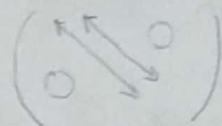
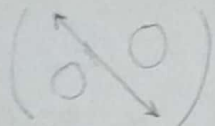
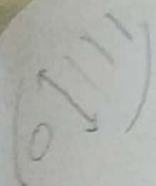
- 1) Applicable only for square matrices.
- 2) Not possible for all square matrices.
Cholesky - only for symmetric matrices.
Schur, Jordan - All matrices.

Universal decomposition for all square/rectangular matrices.

SVD - Singular value decomposition.

Diagonalize - Not for all
- possible for all symmetric

Can we triangularize Yes for all - Schur.



Jordan.

Generalization of all.

Summary:

Schur

Diagonalization

SVD.

All these revolve around eigen values and eigenvectors.

Diagonalization:

change of variables - Complex to Simple.

$$A: \mathbb{C}^{n \times n}$$

$$\begin{matrix} A & x & = & b. \\ n \times n & & & n \times 1 \end{matrix}$$

A is given, b given in \mathbb{C}^n

Find $x \in \mathbb{C}^n$

one equation and one unknown I know.

Can we reduce to n equations in n unknown each involving only one unknown?

means with one unknown.

one eqns with n unknown x.

Let $y = Cx$. which means C is $n \times n$ invertible matrix.

$y = C(x)$ (change of variable for unknown vector)

$z = Cb$ (change of variable for known vector)

Same change C is used in both cases.

$$A(C^{-1}y) = C^{-1}z$$

$$(CAC^{-1})y = z.$$

(let $p = C^{-1}$).

$$\Rightarrow (P^{-1}AP)y = z.$$

$$Ky = z.$$

Suppose K is diagonal matrix, by our choice on P .

Then $Ky = z$, becomes

$$\lambda_1 y_1 = z_1$$

\vdots

$$\lambda_n y_n = z_n$$

Suppose;

$$P^{-1}AP = \text{Diag } D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Hence, given an A in $\mathbb{C}^{n \times n}$ can we find, does there exist invertible $P \in \mathbb{C}^{n \times n}$ such that $P^{-1}AP = D \in \mathbb{C}^{n \times n}$

where D is diagonal matrix.

Not possible.

There exist matrices in $\mathbb{C}^{n \times n}$ such a P cannot exist.

Matrix $A \in \mathbb{C}^{n \times n}$ is said to be diagonalizable if $\exists P \in \mathbb{C}^{n \times n}$ invertible such that $P^{-1}AP = D$ diagonal in $\mathbb{C}^{n \times n}$.

In such a case how do we find such a P .

Example:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Show that $\nexists P \in \mathbb{C}^{n \times n} \Rightarrow P^{-1}AP$ is diagonal

Jordan found this - Any matrix split to diagonalizable part and such shown above.

$$A = \underbrace{D}_{\substack{\uparrow \\ \text{diagonal part}}} + \underbrace{N}_{N^k = 0}$$

diagonal part

After k generation of N , matrix dies.

Ex 2:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = 0.$$

Why not diagonalization?

Suppose $\exists P \in \mathbb{C}^{2 \times 2}$, s.t., invertible $P^{-1}AP = \begin{pmatrix} p & 0 \\ 0 & a \end{pmatrix}$

diagonal.

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$AP = P \begin{pmatrix} p & 0 \\ 0 & a \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & a \end{pmatrix}$$

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ap & ba \\ cp & da \end{pmatrix}$$

$$cp=0 \quad da=0$$

$$c=0 \text{ or } p=0 \quad d=0 \text{ or } a=0.$$

c and d are 0, c and a are 0, p and d are 0, p and a are 0,

Take each combination and argue.

Each one leads to Contradiction.

When is the matrix $A \in \mathbb{C}^{n \times n}$ diagonalizable?

Suppose, $A \in \mathbb{C}^{n \times n}$ is diagonalizable matrix,

$$\exists P \text{ s.t. } P^{-1}AP = \text{Diagonal}$$

$$AP = PD$$

$$P = [u_1 \ u_2 \ \dots \ u_n] \quad n \text{ columns}$$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$A * [u_1 \ u_2 \ \dots \ u_n] = [u_1 \ u_2 \ \dots \ u_n] \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$[Au_1 \quad Au_2 \quad \dots \quad Au_n] = [\lambda_1 u_1 \quad \lambda_2 u_2 \quad \dots \quad \lambda_n u_n]$$

$$Au_1 = \lambda_1 u_1$$

$$Au_2 = \lambda_2 u_2$$

$$\vdots$$

$$Au_n = \lambda_n u_n$$

A is diagonalizable implies \exists n linearly independent

vectors $u_1, u_2, \dots, u_n \in \mathbb{C}^n$

n Scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$

$$\text{s.t. } Au_j = \lambda_j u_j \quad 1 \leq j \leq n.$$

Converse is also true.

We have to look for n vectors and n scalars for diagonalization.

If A has to be diagonalized we must find those vectors and scalars.

Diagonalizability:

$A \in \mathbb{C}^{n \times n}$ is said to be diagonalizable if $\exists P \in \mathbb{C}^{n \times n}$ (invertible) such that $P^{-1}AP = D$; $D \in \mathbb{C}^{n \times n}$ diagonal matrix.

$$A, B \in \mathbb{C}^{n \times n}$$

A is similar to B if $\exists P \in \mathbb{C}^{n \times n}$ (invertible) s.t.

$$P^{-1}AP = B \quad (A \sim B)$$

A is said to be diagonalizable if $A \sim \text{Diagonal matrix}$.

However inverse computation is tedious, (inverse of P)
computationally tricky.

(Engineer)

$$\text{Now, } A^{-1} = \frac{1}{|A|} A_{\text{adj}}$$

$$|A|_{n \times n}$$

$$(A_{\text{adj}})_{n \times n}$$

computation is messy.

Are there matrices P for which inverse can be easily computed?

Rotational matrices are easy to handle.

A real $n \times n$ matrix is said to be orthogonal matrix if $P^T = P^{-1}$ or $P^T P = P P^T = I$.

(The columns are orthonormal vectors.)

* A complex matrix is said to be unitary if

$$P^* = P^{-1}; \quad \text{i.e., } P^* P = P P^* = I$$

$$\text{where } P^* = \overline{P^T}$$

$P^* \rightarrow$ Hermitian Conjugate of P .

The nice ^{real} matrices are those

i) Diagonalizable

ii) Diagonalizable with orthogonal matrix.

\exists orthogonal P s.t. $P^{-1} A P = \text{diagonal matrix}$.

Real - Theory of symmetric matrices.

Complex - Theory of Hermitian matrices.

Recall:

$$A \in \mathbb{C}^{n \times n}$$

Diagonalizable iff \exists n scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$

n vectors $v_1, v_2, \dots, v_n \in \mathbb{C}^n$

$$A v_j = \lambda_j v_j.$$

Look for n scalars and n vectors.

Where do we look for them?

where λ is one such scalar and u is one such corresponding vector.

$$\therefore Au = \lambda u \quad (u \neq 0_n)$$

$$(\lambda I - A)u = 0_n$$

$$Mu = 0_n \quad ; u \neq 0_n$$

$\Rightarrow M$ is not invertible, \therefore if it was then $u = M^{-1}0_n = 0_n$ contradiction.

$$\Rightarrow |M| = 0.$$

$$\Rightarrow |\lambda I - A| = 0$$

$$= \begin{vmatrix} \lambda - a_{11} & & & \\ & \lambda - a_{22} & & \\ & & \ddots & \\ & & & \lambda - a_{nn} \end{vmatrix}$$

$$0 = \lambda^n + ()\lambda^{n-1} + \dots + ()\lambda + ()$$

λ is root of polynomial

$$x^n + ()x^{n-1} + \dots + () = 0 \text{ of } |xI - A| = 0.$$

The scalars we are looking for are the roots of the polynomial $|xI - A| = 0$.

-characteristic polynomial.

monic polynomial - leading coefficient is one.

The ch polynomial $c(x)$ is a polynomial of degree n leading coefficient 1.

-monic polynomial of degree n .

We will have n roots, these roots can be real or complex, repeating roots.

Let $\lambda_1, \dots, \lambda_k$ be the distinct roots with multiplicity a_1, a_2, \dots, a_k .

$$i) a_1 + a_2 + \dots + a_k = n.$$

$$ii) 1 \leq a_i \leq n$$

$$iii) 1 \leq k \leq n$$

$$C(\lambda) = (\lambda - \lambda_1)^{a_1} (\lambda - \lambda_2)^{a_2} \dots (\lambda - \lambda_n)^{a_k}$$

a_i - algebraic multiplicity

λ_i - eigen values of matrix.

λ_j an eigen value, $\therefore M = (\lambda_j I - A)$ is not invertible.

$\therefore Ax = \lambda_j x$ has non zero solutions.

$$W_j = \{x \in \mathbb{C}^{n \times 1} : Ax = \lambda_j x\}.$$

W_j has atleast one non zero solution.

W_j has a subspace of \mathbb{C}^n .

$\Rightarrow W_j$ is non empty

$$\left. \begin{array}{l} x, y \in W_j \Rightarrow Ax = \lambda_j x \\ Ay = \lambda_j y \end{array} \right\} \Rightarrow \begin{array}{l} A(x+y) = \lambda_j(x+y) \\ \Rightarrow x+y \in W_j \end{array}$$

$$x \in W_j ; \alpha \in \mathbb{C} \Rightarrow Ax = \lambda_j x, \alpha \in \mathbb{C}$$

$$\Rightarrow A(\alpha x) = \lambda_j(\alpha x)$$

$\dim W_j$ is geometric multiplicity of λ_j

W_j is eigen subspace corresponding to eigen value λ_j

$$g_j = W_j$$

Given A ,

$c(\lambda)$ is ch polynomial

$$(\lambda - \lambda_1)^{a_1} \dots (\lambda - \lambda_k)^{a_k}$$

$\lambda_1, \lambda_2, \dots, \lambda_k$ - distinct eigenvalue.

a_1, a_2, \dots, a_k - algebraic multiplicity.

g_1, g_2, \dots, g_k - geometric multiplicity.

In general,

$$1 \leq g_j \leq a_j \quad \text{for } 1 \leq j \leq k.$$

Theorem: A matrix $A \in \mathbb{C}^{n \times n}$ is diagonalizable iff $g_j = a_j$ for each eigenvalue λ_j .

In particular if there are n distinct eigenvalues their $k=n$, $a_j = g_j = 1 \quad \forall j$; A is diagonalizable.

W_j are nice subspaces.

A lot of properties are nice

Recall:

$$A \in \mathbb{C}^{n \times n}$$

$c(\lambda) = |\lambda I - A|$ --- monic polynomial of degree n .

$\lambda_1, \lambda_2, \dots, \lambda_n$ distinct roots.

a_1, a_2, \dots, a_k multiplicities

$$c(\lambda) = (\lambda - \lambda_1)^{a_1} (\lambda - \lambda_2)^{a_2} \dots (\lambda - \lambda_k)^{a_k}$$

$\lambda_1, \lambda_2, \dots, \lambda_k$ distinct eigenvalues.

a_1, a_2, \dots, a_k alg multiplicities

For each λ_j let,

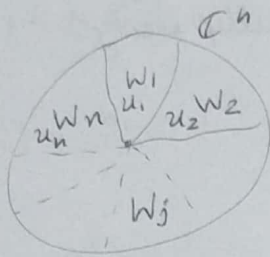
$$W_j = \{ x : Ax = \lambda_j x \}$$

This is a subspace called the eigenspace corresponding to λ_j

$$g_j = \dim(W_j) \geq 1.$$

Fact: $1 \leq g_j \leq a_j$ for any j

A diagonalizable means $g_j = a_j \forall j$



If u_1, u_2, \dots and $\dots u_k$ are non zero vectors in W_1, W_2, \dots, W_k respectively then u_1, u_2, \dots, u_k are linearly independent.

u_1 corresponding to λ_1 .

u_2 corresponding to λ_2 .

Lagrange polynomial:

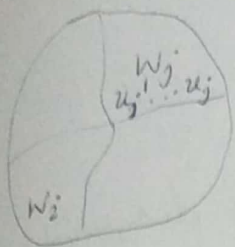
n points means, unique polynomial of degree $(n-1)$ which passes through n points.

W_j dimension is a_j

$$\dim W_j = a_j$$

Basis of W_j , $B_j = \{u_1^j, u_2^j, \dots, u_{a_j}^j\}$

Do it for $j = 1, 2, \dots, k$.



$$B_j = \{u_1^j, u_2^j, \dots, u_{a_j}^j\}$$

$$B_i = \{u_1^i, u_2^i, \dots, u_{a_i}^i\}$$

$B_1 \cup B_2 \cup \dots \cup B_K$ will form consists of $a_1 + a_2 + \dots + a_K$ vectors equal to n , and these are linearly independent and form the basis for \mathbb{C}^n .

$$P = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

first a_1 columns the B_1 vector

next a_2 columns the B_2 vector.

\vdots
 a_K columns the B_K vector.

$P_{n \times n}$ matrix columns are independent $\therefore P^{-1}$ exists

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ 0 & & & & \lambda_K \end{pmatrix}$$

$\underbrace{\hspace{1.5cm}}_{a_1} \quad \underbrace{\hspace{1.5cm}}_{a_2} \quad \underbrace{\hspace{1.5cm}}_{a_K}$

Given A ;

$\rightarrow C(\lambda)$.

\rightarrow eigenvalues are their multiplicities

\rightarrow capture eigenspaces. $W_j = \{x : Ax = \lambda_j x\}$

\rightarrow Basis for W_j , $g_j = \dim W_j$

\rightarrow Check if $a_j = g_j \forall j$

\rightarrow Construct $P = \begin{pmatrix} B_1 \text{ basis vector} & B_2 \text{ basis vector} & \dots \end{pmatrix}$

$\rightarrow P^{-1}AP = \text{Diagonalizable.}$

Now if $a_j \neq g_j$ for any j

No diagonalizability.

What to do?

→ Can we triangulize? (Compromise diagonalizability)

Yes!

The inverse computation is easy
works for square

→ Can we diagonalize else way? in other sense.

(Compromise the process of diagonalization).

idea: (works for both square and rectangular)

$$Ax = b$$

$$x = Py$$

$$b = Pz$$

change of variables gave this.

$$P^{-1}AP y = z$$

diagonal

Suppose, $x = Py$.

$$b = Qz.$$

$$APy = Qz$$

$$Q^{-1}APy = z$$

(Two Different Transformation
 Q and P)

Diagonal??

Can i find Q and P invertible such that $Q^{-1}AP$ is diagonal. Yes!

Q^{-1} Computation can be made easy.

*This is SVD.

Extends to rectangular matrices too.

Works for all matrices.

QR decomposition - Triangulize

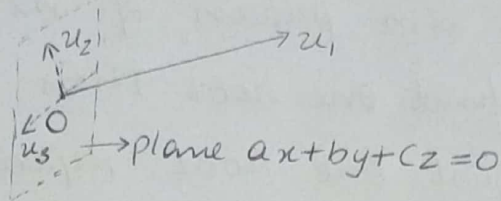
$$\begin{pmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{pmatrix}$$

$$A \in \mathbb{R}^{n \times n}$$

λ_1 is any eigen value of A

u_1 be the corresponding eigenvector

$$Au_1 = \lambda_1 u_1$$



$$u_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

get u_2, u_3, \dots, u_n such that all are orthonormal.

$u_1, u_2, u_3, \dots, u_n$ are all orthonormal.

$P = [u_1, u_2, \dots, u_n]$ Columns orthonormal, independent

Assume all u are real $P^T = P^{-1}$ (orthogonal).

$$AP = A[u_1, u_2, \dots, u_n]$$

$$= [Au_1, Au_2, \dots, Au_n]$$

$$= [\lambda_1 u_1, Au_2, \dots, Au_n]$$

$$P^{-1}AP = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} [\lambda_1 u_1, Au_2, \dots, Au_n]$$

$$= \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & & & \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix}$$

Continue the process.

Conclusion: We can make all entries below leading diagonal of the matrix by orthogonal/unitary transformation.

Observe all eigenvalues eventually end up diagonally. But to start with process of $n \times n$ matrix we just have to find one root. Then $(n-1) \times (n-1)$ matrix again find just one root. Repeat process.