Seq no:

Name: **SR No.:** **Dept.:**

Maximum Points: 15 **E2-243: Quiz 3** **Duration**: 40 minutes

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- Tick the correct answer (**1+1+1+1+1 points. -0.5 point for each negative answer**)

1. Which of the following is a valid pdf

a)

$$f_X(x) = \begin{cases} 0, \text{ if } x < 0 \\ -\frac{x}{8} + \frac{1}{2}, \text{ if } 0 \le x < 4 \\ 24 \text{ if } x = 4 \\ 0, \text{ if } x > 4 \end{cases}$$

b)

$$f_X(x) = \begin{cases} 0, \text{ if } x \le -1 \\ 1, \text{ if } -1 < x < 0 \\ -\frac{x}{8} + \frac{1}{2}, \text{ if } 0 \le x \le 4 \\ 0, \text{ if } x > 4 \end{cases}$$

c)

$$f_X(x) = \begin{cases} 0, \text{ if } x \le -1 \\ 1, \text{ if } -1 < x < 0 \\ -\frac{x}{16} + \frac{1}{2}, \text{ if } 0 \le x \le 4 \\ 0, \text{ if } x > 4 \end{cases}$$

d)

$$f_X(x) = \begin{cases} 0, \text{ if } x < 0 \\ -\frac{x}{16} + \frac{1}{2}, \text{ if } 0 \le x < 8 \\ 24 \text{ if } x = 8 \\ 0, \text{ if } x > 8 \end{cases}$$

Answer (a)

Explanation For a function to be a valid pdf

$$\int_{-\infty}^{\infty} f_X(x) = 1$$

2. Which of the following is a valid CDF $F_X(x)$ for a non negative random variable X?

a)

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0 \\ e^{-x} \text{ if } x \ge 0 \end{cases}$$

b)

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0 \\ 1 - e^{-x}, \text{ if } x \ge 0 \end{cases}$$

c)

$$F_X(x) = \begin{cases} 0, \text{ if } x \le 0 \\ e^{-x}, \text{ if } 0 < x < \frac{1}{e} \\ 1 - e^{-x}, \text{ if } x \ge \frac{1}{e} \end{cases}$$

d)

$$F_X(x) = \begin{cases} 0, \text{ if } x \le 0 \\ 1 - e^{-x}, \text{ if } 0 < x < \frac{1}{e} \\ e^{-x}, \text{ if } x \ge \frac{1}{e} \end{cases}$$

Answer (b)

Explanation For a function to be a valid CDF, the following should hold

- $F_X(x)$ should be non decreasing
- $\lim_{x\to\infty} F_X(x) = 1$
- $\lim_{x\to-\infty} F_X(x) = 0$

3. X is defined as a continuous random variable which is uniformly distributed in $[0, 100]$. Then the probability that X will take any integer value is

a) $\frac{\pi}{6}$ b) 0 c) $\frac{\pi^2}{36}$ d) Not defined

Answer (b)

Explanation For a continuous random variable X, the probability mass at any single point is 0. Hence probability that X will take any integer value is 0.

4. Let $\Omega = \{1, 2, 3, 4\}$. Let $S_1 = \{1, 2, 4\}$. What is $\Sigma(S_1)$?

a) $\Sigma(S_1) = \{\{1\}, \{2\}, \{4\}, \Omega, \phi\}$

b) $\Sigma(S_1) = \{\{1, 2, 4\}, \{3\}, \Omega, \phi\}$

c) $\Sigma(S_1) = \{\{1\}, \{2\}, \{3\}, \{4\}, \Omega, \phi\}$

d) $\Sigma(S_1) = \{\{1, 2, 4\}, \Omega, \phi\}$

Answer (b)

5. Consider an experiment where two persons toss a coin. If outcomes of each of them differ then the game ends else tossing continues. What is the probability that the game ends in the second round of toss, if the coins are biased with probability of head being $\frac{3}{4}$.

a) 1 b) $\frac{60}{256}$ c) $\frac{16}{64}$ d) $\frac{5}{64}$

Answer (b)

Explanation Probability that the game ends in a round $q = P(\{H, T\}) + P(\{T, H\})$.

$$q = \frac{3}{4}\frac{1}{4} + \frac{1}{4}\frac{3}{4} = \frac{6}{16}$$

Probability that game ends in second round $= (1 - q)q = \frac{60}{256}$

- State true or false **(1+1+1+1 points. -0.5 point for each negative answer)**

1. If S_1 and S_2 are sigma algebra on Ω then $S_1 \backslash S_2$ is also a sigma algebra on Ω.

Answer False

Explanation As both S_1 and S_2 are sigma algebra on Ω, $S_1 \backslash S_2$ does not contain Ω and ϕ. Therefore, $S_1 \backslash S_2$ is not a sigma algebra.

2. Consider 2 events A and B such that $P(A \cup B) = 1$. Then, $A \cap B$ is independent of $A \cup B$

   Answer True

   Explanation As $P((A \cup B) \cap (A \cap B)) = P(A \cap B)$. If $A \cup B$ is independent of $A \cap B$ then the following should be true

$$P((A \cup B) \cap (A \cap B)) = P(A \cup B)P(A \cap B)$$

   The above expression holds true as $P(A \cup B) = 1$.

3. Consider the probability space $(\Omega, \mathbb{B}, \mathbb{P})$. Let $\{A_n\} \in \mathbb{B}, \forall n \in \{1, 2, .., N\}$ and $A_n \subsetneq A_{n-1}, \forall \;\; N \geq n > 1$. Then $P(A_N) = P(\bigcup_{i=1}^{N} A_i)$

   Answer False

   Explanation $P(A_N) = P(\bigcap_{i=1}^{N} A_i)$, as A_ns are decreasing.

4. Consider the probability space $(\Omega, \mathbb{B}, \mathbb{P})$. Let $\{A_n\}_{n \in \mathbb{N}} \in \mathbb{B}$. Then $\sum_{n=1}^{\infty} P(A_n) < \infty$ implies $P(\limsup_{n \to \infty} A_n) = 0$ if and only if $A_{n-1} \subset A_n, \forall \;\; n > 1$.

   Answer False

   Explanation From Borel Cantelli Lemma for any $\{A_n\}_{n \in \mathbb{N}} \in \mathbb{B}$, $\sum_{n=1}^{\infty} P(A_n) < \infty$ implies $P(\limsup_{n \to \infty} A_n) = 0$.

- Answer the following questions **(2+2+2 points)**

1. Consider the probability space $(\Omega, \mathbb{B}, \mathbb{P})$. If A, B, C are events in $\mathbb{B}$. Given that

   a) A and B are independent and

   b) A and C are independent

   Prove or disprove the following statement **(2 points)**
   B and C are independent.

   Explanation The above statement is false.

   Example 1: Consider $\Omega = \{1, 2, 3, 4\}$ such that $P(w_i) = \frac{1}{4}, \forall w_i \in \Omega$. Let us consider $A = \{1, 2\}, B = \{2, 3\}, C = \{1, 4\}$. Clearly, A and B are independent, A and C are independent, but B and C are not independent.

   Example:2 Consider the example discussed in class. Consider a unit square. Probability of an event is given by the area covered by the event. From the above figure

$$P(A) = \frac{1}{2}, P(B) = \frac{1}{2}, P(C) = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{4} = P(A)P(B)$$

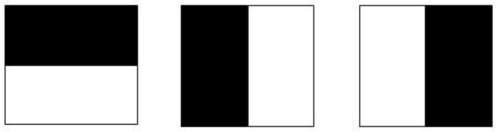

Figure 1: Portion in the black colour denotes the events A, B, C respectively.

Therefore, A and B are independent events.

$$P(A \cap C) = \frac{1}{4} = P(A)P(C)$$

Therefore, A and C are independent events. But,

$$P(B \cap C) = 0 \neq P(B)P(C)$$

Thus, B and C are not independent events.

2. Given that open interval (x, y) is a Borel set in $\mathbb{R}$, show that closed interval $[x, y]$ is a Borel set in $\mathbb{R}$, $(\forall x, y \in \mathbb{R})$. **(2 points)**

Explanation

$$(x - \frac{1}{n}, y + \frac{1}{n}) \in \mathbb{B}(\mathbb{R}), \quad \forall x, y \in \mathbb{R} \text{ and } \forall n \in \mathbb{N}$$

By definition of sigma algebra

$$\bigcap_{n=1}^{\infty} (x - \frac{1}{n}, y + \frac{1}{n}) \in \mathbb{B}(\mathbb{R})$$

Therefore,

$$[x, y] \in \mathbb{B}(\mathbb{R}), \quad \forall x, y \in \mathbb{R}$$

3. Consider the random experiment of rolling a fair die. Consider $\mathbb{B} = \{\Omega, \phi\}$

$$X(w) = \left\lceil \frac{w}{2} \right\rceil, \forall w \in \Omega$$

Is X a random variable on $(\Omega, \mathbb{B}, \mathbb{P})$? Explain. **(2 points)**

Explanation No X is not a random variable

$$R_X = \{1, 2, 3\}$$

$$X^{-1}(-\infty, 2) = \{1, 2\}$$

But $\{1, 2\} \notin \mathbb{B}$. Therefore, X is not a random variable.