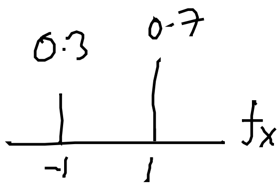
EXERCISE 6

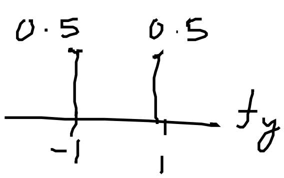
$$F_{X|B}(x) = \frac{P(X <= x, B)}{P(B)}$$

1. X and Y are two discrete random variables whose joint pmf is given as below:

$Y \rightarrow$ $X \downarrow$	$y_1 = -1$	$y_2 = 1$	*Row Sum*
$x_1 = -1$	0.2	0.1	
$x_2 = 1$	0.3	0.4	
Column Sum			

Answer the following:



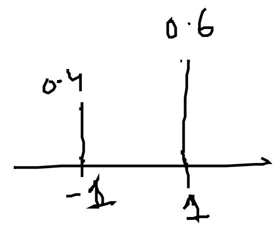


(a) Find the pmfs p_X and p_Y of X and Y respectively

(b) Are X and Y independent random variables? No

(c) Find $E(X), E(Y)$ $E(X) = 0.4$ $E(Y) = 0$

(d) Let $Z = XY$. Find the pmf of Z $E(Z) = 0.2$



(e) Find $E(Z)$

(f) Find $cov(X, Y)$ $= 0.2 - 0 = 0.2$

2. X and Y are two discrete random variables whose joint pmf is given in the table below. Answer the following:

(a) Find the pmfs p_X and p_Y of X and Y respectively

(b) Are X and Y independent random variables?

(c) Find $E(X), E(Y)$

(d) Let $Z = XY$. Find the pmf of Z

(e) Find $E(Z)$

(f) Find $cov(X, Y)$

$\mathrm{cov}(X,Y) =$

$$E[(X-u_x)(Y-u_y)] = E[XY - u_y X - u_x Y + u_x u_y]$$

$$= E[XY] - u_y E[X] - u_x E[Y] + u_x u_y$$

$$= E[XY] - u_x u_y$$

Q3 c

$$E(x) = \int_{-\infty}^{\infty} x f_x \, dx$$

$$= \int_{-\infty}^{\infty} x \, 2x \, dx = \frac{2}{3}|x^3|_0^1 = 2/3$$

$$\int u\,dv = uv - \int u'v$$

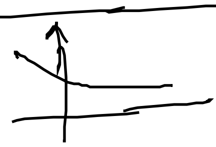
$$E(y) = \int_{-\infty}^{\infty} y f_y \, dy = \int_0^{\infty} y \, 3e^{-3y} \, dy = 3\int_0^{\infty} y e^{-3y} \, dy = 3\left[\left| \frac{y e^{-3y}}{-3} \right|_0^{\infty} + \int_0^{\infty} \frac{e^{-3y}}{3} \, dy \right]$$

$$= 3 \cdot 1/9$$

$$= 1/3$$

$$\lim_{y \to \infty} \frac{1}{-3e^{3y}} \cdot 3$$

$=$



$$\mathrm{Var}(x) = E((x-u_x)^2) = E[x^2] - (E(x))^2$$

$=$

b) Yes

c) $E(x) = -1\times 0.3 + 1\times 0.7 = 0.4$

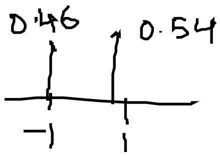
$E(y) = -1\times 0.4 + 1\times 0.6 = 0.2$

f) $cov(X,Y) = E[XY] - \mu_x\mu_y$

$= 0.08 - 0.4\times 0.2$

$= 0$ ✓

d) $Z = XY$



e) $E(Z)$

$= -1\times 0.46 + 1\times 0.54$

$= 0.08$

$Y \rightarrow$ $X \downarrow$	$y_1 = -1$	$y_2 = 1$	*Row Sum*
$x_1 = -1$	0.12	0.18	0.3
$x_2 = 1$	0.28	0.42	0.7
Column Sum	0.4	0.6	1

$f_x = \int_0^\infty 6xe^{-3y}\,dy = 6x\left.\frac{e^{-3y}}{-3}\right|_0^\infty$

$f_x = 2x$

$f_y(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$

3. Let X, Y be continuous random variables on a probability space $(\Omega, \mathcal{B}, P)$ whose joint pdf is given by

$$f_{XY}(x,y) = \begin{cases} 6xe^{-3y} & \text{for} \quad \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } y \ge 0\} \\ 0 & \text{for} \quad \text{other values of } (x,y) \end{cases}$$

$f_y = \int_0^1 6xe^{-3y}\,dx = 6\frac{e^{-3y}}{2} = 3e^{-3y}$

Answer the following:

$E(XY) = \int_0^1\int_0^\infty xy\,6xe^{-3y}\,dx\,dy$

(a) Find the pdfs $f_X(x)$ and $f_Y(y)$ of the random variables X and Y?

(b) Are X and Y independent random variables? Yes $f_{xy} = f_x f_y$

(c) Find $E(X), E(Y), Var(X), Var(Y)$

(d) Find $cov(X,Y)$ $E(XY) = 2/9$ $cov(X,Y) = \frac{2}{9} - \frac{1}{3}\times\frac{2}{3} = 0$

4. Let A_1, A_2, A_3, A_4 be disjoint events in a probability space $(\Omega, \mathcal{B}, P)$ such that $\Omega = A_1 \cup A_2 \cup A_3 \cup A_4$ and $P(A_1) = P(A_2)$ and $P(A_3) = P(A_4)$. Let X and Y be random variables on this probability space defined as follows:

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$$X(\omega) = \begin{cases} -1 & \text{if} \quad \omega \in A_1 \\ 1 & \text{if} \quad \omega \in A_2 \\ 0 & \text{if} \quad \omega \in A_3 \cup A_4 \end{cases}$$

$$Y(\omega) = \begin{cases} -1 & \text{if} \quad \omega \in A_3 \\ 1 & \text{if} \quad \omega \in A_4 \\ 0 & \text{if} \quad \omega \in A_1 \cup A_2 \end{cases}$$

$Z = XY$

$= \begin{cases} -1 & \times \quad \omega \in A_1 \;\&\; \omega \in A_4 \\ 1 & \times \\ 0 & \omega \in \Omega \end{cases}$

Answer the following:

$E[XY] = 0$

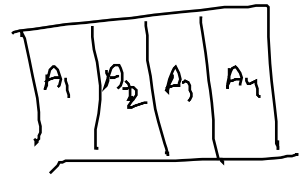
$cov(X,Y) = 0 - 0 = 0$

$X: \Omega \to \mathbb{R}$

$P(A_1) = P(A_2) = p/2$

$P(A_3) = P(A_4) = (1-p)/2$

$$X = \begin{cases} -1 & \omega \in A_1 \\ 1 & \omega \in A_2 \\ 0 & \omega \in A_3 \cup A_4 \end{cases}$$



$E(Y) = 0$

$E(X+Y) = 0$

$E(XY)$

$E(X) = \sum x_i p_i = -1 \times \frac{p}{2} + 1 \times \frac{p}{2} = 0$

(a) Find $E(X), E(Y), E(X+Y), E(XY), cov(X,Y)$

(b) Find the joint pmf p_{XY} of X and Y

(c) Are X and Y independent?

5. Let X be a random variable on a probability space $(\Omega, \mathcal{B}, P)$ with mean μ_X and variance σ_X^2. Let Y be the random variable defined as $K(X-L)$ (where $K \neq 0$ and L are real constants). Find the values of K and L for which the mean and variance of Y become 0 and 1 respectively

$L = \mu_x$

$K = \pm \frac{1}{\sigma_x}$

6. Let $X_1, X_2, \cdots, X_n$ be independent, identically distributed random variables with mean μ and variance σ^2. Find the values of K and L for which the random variable Y defined below has mean zero and variance one:

$Y = K(X_1 + X_2 + \cdots + X_n - L)$ (where $K \neq 0$ and L are real constants)

$L = \mu_1 + \dots \mu_n$
$= n\mu$

7. Let X be a random variable which takes only two values 4 and 0 with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Is the following True?

$$P(X \geq 4) = \frac{Exp(X)}{4}$$

Find $E(X|N)$ where N is the no. of turns till die lands in 6

8. A die has a probability of 0.3 of getting a 6. It is rolled until a 6 turns up. Let X be the number of 1s that turn up. Find the conditional expectation $E(X|\#rolls\ is\ 6)$
(Hint: The Expected Value of a Binomial $B(n,p)$ random variable is np)

9. Let X be a continuous random variable on a probability space $(\Omega, \mathcal{B}, P)$. For any event $B \in \mathcal{B}$ the conditional cdf $F_{X|B}(x)$ is defined as

$$F_{X|B}(x) = \frac{P(X \leq x | B)}{P(B)}$$

Answer the following:

(a) Find $F_{X|B}(x)$ if $B = \{2 \leq X \leq 4\}$.

(b) For the above B find the conditional pdf $f_{X|B}(x)$

(c) Answer the above two questions if $X \sim Uni[-4, 4]$ and find the conditional mean $E(X|B)$