EXERCISE 2

Exercises on Functions

1. For the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(t) = t^2 - 1$ answer the following:
   (a) Find S_3 and S_{-4} (wherel $S_b = f^{(-1)}(\{b\}) = \{r \in \mathbb{R} : f(r) = b\}$)
   (b) Is f one-one?
   (c) Is onto?
   (d) If E is the interval $[0, 3]$ find $f^{(-1)}(E)$
2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined as
   $$f(x) = x^2 - 3x + 2$$
   Answer the folowing:
   (a) For the following subsets E of $\mathbb{R}$ find $f^{-1}(E)$:
      i. $E = (-\infty, 0)$
      ii. $E = (-\infty, -0.5)$
      iii. $E = (0, \infty)$
      iv. $E = \{0\}$
   (b) Is this function one-one?
   (c) Is this function onto?
3. If $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5\}$ and $\mathcal{B} = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ answer the following:
   (a) How many functions can be defined from $\mathcal{A}$ to $\mathcal{B}$?
   (b) How many one-one functions can be defined from $\mathcal{A}$ to $\mathcal{B}$?
   (c) How many onto functions can be defined from $\mathcal{A}$ to $\mathcal{B}$?
4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined as $f(t) = exp(-|t|)$. Answer the following:
   (a) Find the Range of f
   (b) Is f one-one?

(c) Is f onto?

5. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ Examine if the following are True or false:

   (a) $E \subseteq \mathcal{B}$ is a subset of $\mathcal{B} \Longrightarrow f^{(-1)}(E') = \left(f^{(-1)}(E)\right)'$

   (b) $E_1, E_2 \subseteq \mathcal{B} \Longrightarrow f^{(-1)}\left(E_1 \bigcup E_2\right) = f^{(-1)}(E_1) \bigcup f^{(-1)}(E_2)$

   (c) $E_1, E_2 \subseteq \mathcal{B} \Longrightarrow f^{(-1)}\left(E_1 \bigcap E_2\right) = f^{(-1)}(E_1) \bigcap f^{(-1)}(E_2)$

   (d) $E_1, E_2, \cdots, E_N$ are a finite number of subsets of $\mathcal{B} \Longrightarrow$
   $$f^{(-1)}\left(\bigcup_{n=1}^{N} E_n\right) = \bigcup_{n=1}^{N} f^{(-1)}(E_n)$$

   (e) $E_1, E_2, \cdots, E_N$ are a finite of subsets of $\mathcal{B} \Longrightarrow$
   $$f^{(-1)}\left(\bigcap_{n=1}^{N} E_n\right) = \bigcap_{n=1}^{N} f^{(-1)}(E_n)$$

   (f) $E_1, E_2, \cdots, E_n, \cdots$ is an infinite sequence of subsets of $\mathcal{B} \Longrightarrow$
   $$f^{(-1)}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} f^{(-1)}(E_n)$$

   (g) $E_1, E_2, \cdots, E_n, \cdots$ is an infinite sequence of subsets of $\mathcal{B} \Longrightarrow$
   $$f^{(-1)}\left(\bigcap_{n=1}^{\infty} E_n\right) = \bigcap_{n=1}^{\infty} f^{(-1)}(E_n)$$

Exercises on sequences

1. Use the definition of convergence of a sequence of real numbers to prove the following:

   (a) Every convergent sequence of real numbers $\{f_n\}_n$ must be bounded

   (b) For a convergent sequence of real numbers the limit must be unique

   (c) If $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are sequences of real numbers converging respectively to f and g then prove the following:

      i. The sequence $\{f_n + g_n\}_n$ converges to $f + g$

      ii. The sequence $\{f_n - g_n\}_n$ converges to $f - g$

      iii. The sequence $\{f_n g_n\}_n$ converges to fg

iv. If $g_n \neq 0$ for all n and $g \neq 0$ then the sequence $\left\{\frac{f_n}{g_n}\right\}_n$ converges to $\frac{f}{g}$

2. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that it satisfies the following properties:

   (a) $f_n > 0$ for all n and

   (b) $f_{n+1} \leq r f_n$ for all n where $0 < r < 1$

   Answer the following:

   (a) Show that
   $$f_n \leq r^{n-1} f_1 \text{ for all } n \geq 2$$

   (b) Does the sequence $\{f_n\}_{n\in\mathbb{N}}$ converge to zero - Give reasons

   (c) Is the above convergence result true if we had
   $$f_n \leq r^{n-1} f_1$$
   beyond a certain stage N, that is,
   $$f_{n+1} \leq r f_n \text{ for all } n \geq N \text{ instead of for all } n$$

3. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $f_n > 0$ for all n and
   $$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} \text{ exists and } = L < 1$$
   Show that the sequence converges to zero

4. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that
   $$\lim_{n\to\infty} \left|\frac{f_{n+1}}{f_n}\right| \text{ exists and } = L < 1$$
   Show that the sequence converges to zero

5. Determine whether the following sequences converge:

(a) $f_n = \dfrac{100n^2 - 5}{4n^2 + 100n - 4}$
(Hint: In the above sequence note that bothnumerator and denominator are polynomials in n)

(b) $f_n = \dfrac{3n^2 + 2n}{2^n}$
(Hint: Use Exercise 4)

(c) $f_n = \sqrt{n^2 + 3n} - n$
$\left(\text{Hint:} \sqrt{a} - b = \dfrac{a - b^2}{\sqrt{a} + b}\right)$

(d) $f_n = \dfrac{sin(n^2 + 1)}{n^2 + 1}$ (Hint: Sandwich Theorem)

(e) $f_n = \dfrac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}}$
(Hint: Similar Idea as in (d) above)

(f) $f_n = \dfrac{1}{n2^n}$

(g) $f_n = \dfrac{n}{2^n}$

(h) $f_n = \dfrac{e^n}{n!}$

(i) $f_n = \dfrac{n^p}{e^n}$ where p is a positive real number
(Hint:For the above exercises (f) to (i) use Exercise 4)