EXERCISE 3

1. Let $\mathcal{I}$ be an interval in $\mathbb{R}$ and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued functions on $\mathcal{I}$ and f a real valued function on $\mathcal{I}$. Are the following statements TRUE or FALSE?

   (a) $f_n \overset{pw\ (\mathcal{I})}{\longrightarrow} f \Longrightarrow f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f$

   (b) $f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f \Longrightarrow f_n \overset{pw\ (\mathcal{I})}{\longrightarrow} f$

   (c) If f_n are all continuous on $\mathcal{I}$ and $f_n \overset{pw\ (\mathcal{I})}{\longrightarrow} f$ and f is not continuous on $\mathcal{I}$ then f_n does not converge uniformly to f on $\mathcal{I}$

   (d) If f_n are all continuous on $\mathcal{I}$ and $f_n \overset{pw\ (\mathcal{I})}{\longrightarrow} f$ and f is continuous on $\mathcal{I}$ then f_n must converge uniformly to f on $\mathcal{I}$

   (e) $f_n \overset{pw\ (\mathcal{I})}{\longrightarrow} f$ and f_n are all differentiable on $\mathcal{I}$
   $\Longrightarrow$
   f is differentiable and $f_n' \overset{pw\ (\mathcal{I})}{\longrightarrow} f'$

   (f) $f_n \overset{pw\ (\mathcal{I})}{\longrightarrow} f$ and f_n and f are all differentiable on $\mathcal{I}$
   $\Longrightarrow$
   $f_n' \overset{pw\ (\mathcal{I})}{\longrightarrow} f'$

   (g) $f_n \overset{pw\ (\mathcal{I})}{\longrightarrow} f$
   $\Longrightarrow$
   $\liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = f(x)$ for every $x \in \mathcal{I}$

   (h) $\mathcal{I} = [-10, 10]$ and $f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f$
   $\Longrightarrow$
   $$\int_{-10}^{10} f_n(x)dx \longrightarrow \int_{-10}^{10} f(x)dx$$

   (i) $\mathcal{I} = (-\infty, \infty)$ and $f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f$
   $\Longrightarrow$
   $$\int_{-\infty}^{\infty} f_n(x)dx \longrightarrow \int_{-\infty}^{\infty} f(x)dx$$

2. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of real valued functions defined on the interval $\mathcal{I} = [0, 1)$ as $f_n(x) = n^2x^n$. Does the sequence converge pointwise on $\mathcal{I}$ to the function $f(x) = 0$ for all $x \in \mathcal{I}$

3. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of real valued functions defined on the interval $\mathcal{I} = [0,1]$ as $f_n(x) = nx(1-x)^n$. Does the sequence converge pointwise on $\mathcal{I}$ to the function $f(x) = 0$ for all $x \in \mathcal{I}$

4. Show that the following sequences converge to the zero function uniformly:

   (a) $f_n(x) : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

   $$f_n(x) = \frac{sin(nx)}{n+1}$$

   (b) $f_n(x) : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

   $$f_n(x) = \frac{1 + cos(n^2 x)}{n^2 + 1}$$

5. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of real valued functions defined on the interval $\mathcal{I} = [-\frac{\pi}{2}, \frac{\pi}{2}]$ as $f_n(x) = cos^n(x)$ Answer the following:

   (a) Show that the sequence converges pointwise on $\mathcal{I}$ and find the pointwise limit function $f(x)$

   (b) Are the functions $f_n(x)$ continuous on $\mathcal{I}$?

   (c) Is the limit function $f(x)$ continuous on $\mathcal{I}$?

   (d) Does the sequence converge to f uniformly on $\mathcal{I}$

6. Let $f_n : [0,1] \longrightarrow \mathbb{R}$ and $f : [0,1] \longrightarrow \mathbb{R}$ be defined as follows:

   $$f_n(x) = \left(x - \frac{1}{n}\right)^2 \text{ and } f(x) = x^2$$

   Show that f_n converges uniformly to f

7. Let $f_n : [0,1] \longrightarrow \mathbb{R}$ and $f : [0,1] \longrightarrow \mathbb{R}$ be defined as follows:

   $$\begin{aligned} f_n(x) &= exp\left(-\frac{x^2}{n}\right) \\ f(x) &= 1 \text{ for all } x \in [0,1] \end{aligned}$$

   Answer the following:

(a) Show that f_n converges pointwise to f on $[0,1]$

(b) Show that for every positive integer N we can find an $x_{_N} \in [0,1]$ such that

$$|f_{_N}(x_{_N}) - f(x_{_N})| \geq \frac{1}{2}$$

(c) Does f_n converge to f uniformly on $[0,1]$?

8. Let $\mathcal{I} = (0, \infty)$ and the sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ on $\mathcal{I}$, be defined as

$$f_n(x) = \begin{cases} n^3 & \text{for} \quad 0 \leq x \leq \frac{1}{n} \\ 0 & \text{for} \quad x > \frac{1}{n} \end{cases}$$

Answer the following: Does the sequence converge pointwise on $\mathcal{I}$ to the function $f(x) = 0$ for all $x \in \mathcal{I}$

9. Let $f_n : [0, \infty) \longrightarrow \mathbb{R}$ be defined as follows:

$$f_n(x) = \frac{x^2}{1 + n^4 x^3}$$

Discuss the following convergences:

(a) Pointwise Converge of the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$

(b) Uniform Converge of the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$

(c) Pointwise Converge of the sequence of functions $\{f_n'\}_{n\in\mathbb{N}}$

(d) Uniform Converge of the sequence of functions $\{f_n'\}_{n\in\mathbb{N}}$

10. Let $f_n : [0,1] \longrightarrow \mathbb{R}$ and $f : [0,1] \longrightarrow \mathbb{R}$ be defined as follows:

$$\begin{aligned} f_n(x) &= \frac{1}{1 + nx} \\ f(x) &= \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad 0 < x \leq 1 \end{cases} \end{aligned}$$

Answer the following:

(a) Show that f_n converges pointwise to f on $[0,1]$

(b) Show that for every positive integer N we can find an $x_{_N} \in [0,1]$ such that
$$|f_{_N}(x_{_N}) - f(x_{_N})| \geq \frac{1}{2}$$

(c) Does f_n converge to f uniformly on $[0,1]$?

(d) Does f_n converge to f in $L^1[0,1]$?

(e) Does f_n converge to f in $L^2[0,1]$?

(f) If $p > 1$ does f_n converge to f in $L^p[0,1]$?

11. Let $\mathcal{I} = [0,1]$ and consider the sequence of real valued functions $\{f_n\}_{n \in \mathbb{N}}$, defined as

$$f_n(x) \quad = \quad x^n \tag{0.0.1}$$

Let $f(x)$ be the function defined as

$$f(x) \quad = \quad \begin{cases} 0 & \text{for} \quad 0 \leq x < 1 \\ 1 & \text{for} \quad x = 1 \end{cases}$$

Answer the following:

(a) Does $f_n \overset{pw(\mathcal{A})}{\longrightarrow} f$?

(b) Does $f_n \overset{u(\mathcal{A})}{\longrightarrow} f$?

(c) Does $f_n \overset{L^p(\mathcal{A})}{\longrightarrow} f$for $1 \leq p < \infty$

12. Let $\mathcal{I} = [0, \infty)$ and consider the sequence of real valued functions $\{f_n\}_{n \in \mathbb{N}}$, defined on $\mathcal{I}$ as

$$f_n(x) \quad = \quad \begin{cases} \frac{1}{n} & \text{for} \quad 0 \leq x < n \\ 0 & \text{for} \quad x \geq n \end{cases}$$

Answer the following:

(a) Sketch the graph of $f_n(x)$

(b) Discuss the pointwise and uniform convergence of f_n on $\mathcal{I}$

(c) Discuss the L^1, L^2 and L^p $(p > 1)$ convergence of f_n on $\mathcal{I}$

13. Let $\mathcal{I} = (-\infty, \infty)$ and consider the sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$, defined on $\mathcal{I}$ as

$$f_n(x) = \begin{cases} n(x+\frac{1}{n}) & \text{for} \quad -\frac{1}{n} \leq x < 0 \\ -n(x-\frac{1}{n}) & \text{for} \quad 0 \leq x \leq \frac{1}{n} \\ 0 & \text{if} \quad |x| > \frac{1}{n} \end{cases}$$

Answer the following:

(a) Sketch the graph of $f_n(x)$

(b) Discuss the pointwise, uniform, L^1, L^2 and L^p $(p > 1)$ convergences of f_n

14. Let $\mathcal{I} = [0, \infty)$ and consider the sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$, defined on $\mathcal{I}$ as

$$f_n(x) = \begin{cases} -\frac{1}{n^2}(x-n) & \text{for} \quad 0 \leq x \leq n \\ 0 & \text{for} \quad x \geq n \end{cases}$$

Answer the following:

(a) Sketch the graph of $f_n(x)$

(b) Discuss the pointwise, uniform, L^1, L^2 and L^p $(p > 1)$ convergences of f_n

15. Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of real valued functions defined on an interval $\mathcal{I}$ and converging uniformly on $\mathcal{I}$ respectively to the functions f and g. Define the sequence $\{h_n\}_{n\in\mathbb{N}}$ on $\mathcal{I}$ as $h_n(x) = f_n(x) + g_n(x)$. Show that h_n converges uniformly on $\mathcal{I}$ to the function $h(x) = f(x) + g(x)$

16. Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of real valued functions defined on an $\mathbb{R}$ as

$$\begin{aligned} f_n(x) &= x \text{ for all } x \in \mathbb{R} \text{ and for all } n \\ g_n(x) &= \frac{1}{n} \text{ for all } x \in \mathbb{R} \text{ and for all } n \end{aligned}$$

Let $f(x)$ and $g(x)$ be defined as

$$\begin{aligned} f(x) &= x \text{ for all } x \in \mathbb{R} \\ g(x) &= \frac{1}{n} \text{ for all } x \in \mathbb{R} \end{aligned}$$

Answer the following:

(a) Does f_n converge uniformly to f on $\mathbb{R}$?

(b) Does g_n converge uniformly to g on $\mathbb{R}$?

(c) Does $f_n + g_n$ converge uniformly to $f + g$ on $\mathbb{R}$?

(d) Does $f_n \times g_n$ converge uniformly to $f \times g$ on $\mathbb{R}$?