Linear algebra.

Strategy in handling a matrix.

Reduce or decompose into simple forms (matrices

LU decomposition.

Choleske.

Triangularization → Schur
↳ Jordan.

Diagonalization – Similarity / $\underbrace{\text{Unitary / orthogonal}}_{\text{all symmetric}}$.

Tridiagonilization.

QR decomposition.

Hessenberg.

Has advantages and limitations.

1) Applicable only for square matricies.
2) Not possible for all square matricies

Choleske – only for symmetric matricies.

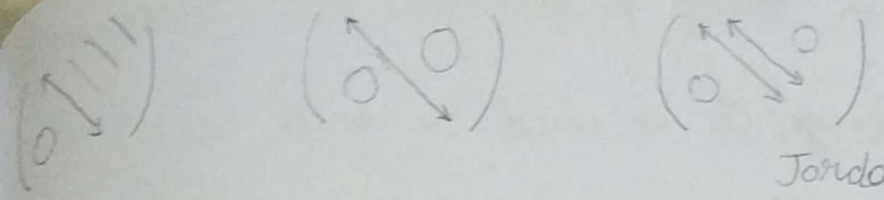
Schur, Jordan – All matricies.

universal decomposition for all square / rectangular matricies.

SVD – Singular value decomposition.

Diagonalize – Not for all
– possible for all symmetric

Can we triangularize Yes for all – Schur.



Jordan.

SVD - generalization of all.

Discussing:

Schur

Diagonalization

SVD.

All these revolve around eigen values and eigenvect or

Diagonalization:

change of variables - complex to simple.

$A : \mathbb{C}^{n \times n}$.

$\underset{n\times n}{A} x = \underset{n \times 1}{b}$. A is given, b given in $\mathbb{C}^n$

Find $x \in \mathbb{C}^n$

One equation and one unknown i know.

can we reduce to n equations in n unknown each involving only one unknown?

n eqns with one unknown.

One eqns with n unknown x.

Let $y = \mathcal{C} x$. which means $\mathcal{C}$ is $n \times n$ invertible matrix.

$y = \mathcal{C}(x)$ (change of variable for unknown vector)

$Z = \mathcal{C} b$ (change of variable for known vector)

Same change $\mathcal{C}$ is used in both cases.

$$A(\mathcal{C}^{-1} y) = \mathcal{C}^{-1} z$$

$$(\mathcal{C} A \mathcal{C}^{-1}) y = z.$$

(let $P = \mathcal{C}^{-1}$).

$$\Rightarrow (P^{-1} A P) y = z.$$

$$K y = z.$$

Suppose K is diagonal matrix, by our choice on P.

Then $Ky = z$, becomes

$$\lambda_1 y_1 = z_1$$

$$\vdots$$

$$\lambda_n y_n = z_n$$

Suppose;

$$P^{-1} A P = \text{Diag } D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Hence, Given an A in $\mathbb{C}^{n \times n}$ can we find does there exist invertible $P \in \mathbb{C}^{n \times n}$ such that $P^{-1} A P = D \in \mathbb{C}$

where D is diagonal matrix.

Not possible.

There exit matricies in $\mathbb{C}^{n \times n}$ such a P cannot exist.

Matrix $A \in \mathbb{C}^{n \times n}$ is said to be diagonalizable if $\exists\, P \in \mathbb{C}^{n \times n}$ invertible such that $P^{-1} A P = D$ diagonal in $\mathbb{C}^{n \times n}$.

In such a case how do we find such a P.

Example:

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Show that $\nexists P \in \mathbb{C}^{n\times n} \Rightarrow P^{-1}AP$ is diagonal

Jordan found this - Any matrix Split to diagonalizable part and such shown above.

$A = D + \underbrace{N}$

$\uparrow \quad N^k = 0.$

diagonal part

After k generation of N, matrix dies.

Ex 2:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = 0.$$

Why not diagonalization?

Suppose $\exists P \in \mathbb{C}^{2\times 2}$, s.t. invertible $P^{-1}AP = \begin{pmatrix} p & 0 \\ 0 & a \end{pmatrix}$ diagonal.

$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$

$AP = P \begin{pmatrix} p & 0 \\ 0 & a \end{pmatrix}.$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & a \end{pmatrix}$$

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ap & ba \\ cp & da \end{pmatrix}$$

$cp = 0 \qquad da = 0$

$c = 0$ or $p = 0 \qquad d = 0$ or $a = 0$.

c and d are 0, c and a are 0, p and d are 0, p and a are 0,

Take each combination and argue.

Each one leads to contradiction.

When is the matrix $A \in \mathbb{C}^{n\times n}$ diagonalizable?

Suppose, $A \in \mathbb{C}^{n\times n}$ is diagonalizable matrix,

$\exists P$ s.t. $P^{-1}AP = \text{Diagonal}$

$$AP = PD$$

$P = [u_1\ u_2 \ldots u_n]$ n columns

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$A * [u_1\ u_2 \ldots u_n] = [u_1\ u_2 \ldots u_n] \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_2 \end{pmatrix}$$

$$[Au_1\ \ Au_2 \ \ldots\ Au_n] = [\lambda_1 u_1\ \lambda_2 u_2 \ldots \lambda_n u_n]$$

$$Au_1 = \lambda_1 u_1$$
$$Au_2 = \lambda_2 u_2$$
$$\vdots \qquad \vdots$$
$$Au_n = \lambda_n u_n$$

A is diagonalizable implies $\exists$ n linearly independ[ent]

vectors $u_1, u_2 \ldots u_n \in \mathbb{C}^n$

n scalars $\lambda_1, \lambda_2 \ldots \lambda_n \in \mathbb{C}$

S.T $Au_j = \lambda_j u_j \quad 1 \le j \le n$.

Converse is also true.

We have to look for n vectors and n scalars for diagonilization.

If A has to be diagonilzed we must find these vectors and scalars.

Diagonilizability:

$A \in \mathbb{C}^{n\times n}$ is said to be diagonalizable if $\exists\, P \in \mathbb{C}^{n\times n}$ (invertible) such that $P^{-1}AP = D$; $D \in \mathbb{C}^{n\times n}$ diagonal matrix.

$A, B \in \mathbb{C}^{n\times n}$

A is similar to B if $\exists\, P \in \mathbb{C}^{n\times n}$ (invertible) s.t.

$P^{-1}AP = B$ $(A \sim B)$

A is said to be diagonalzable if $A \sim$ Diagonal matrix.

*However inverse computation is tideous, (inverse of P) computationally tricky.

(Engineer)

How, $A^{-1} = \frac{1}{|A|} A_{adj}$

$|A|_{n\times n}$

$(A_{adj})_{n\times n}$.

Computation is messy.

Are there matricies P for which inverse can be easily computed?

Rotational matricies are easy to handle.

A real $n\times n$ matrix is said to be orthogonal matrix if $P^T = P^{-1}$ or $P^TP = PP^T = I$.

(The columns are orthonormal vectors.)

* A complex matrix is said to be unitary if

$P^* = P^{-1}$; i.e., $P^*P = PP^* = I$

where $P^* = \overline{P^T}$

$P^* \to$ Hermitian conjugate of P.

The nice real matricies are those

i) Diagonalizable

ii) Diagonalizable with orthogonal matrix.

$\exists$ orthogonal P s.t. $P^{-1}AP =$ diagonal matrix.

Real - Theory of symmetric matricies.

Complex - Theory of Hermitian matricies.

Recall:

$A \in \mathbb{C}^{n\times n}$

Diagonalizable iff $\exists$ n Scalars $\lambda_1, \lambda_2 \ldots \lambda_n \in \mathbb{C}^n$

n vectors $u_1, u_2 \ldots u_n \in \mathbb{C}^n$

$Au_j = \lambda_j u_j$.

Look for n Scalars and n vectors.

Where do we look for them?

Suppose λ is one such scalar and u is one such corresponding vector.

$\therefore Au = \lambda u \quad (u \neq \theta_n)$

$\underbrace{(\lambda I - A)}_{M} u = \theta_n$

$Mu = \theta_n \quad ; u \neq \theta_n$.

$\Rightarrow$ M is not invertible, $\because$ if it was then $u = M^{-1}\theta_n = \theta_n$

contradiction.

$\Rightarrow |M| = 0$.

$\Rightarrow |\lambda I - A| = 0$

$$= \begin{vmatrix} \lambda - a_{11} & & & \\ & \lambda - a_{22} & & \\ & & \ddots & \\ & & & \lambda - a_{nn} \end{vmatrix}$$

$0 = \lambda^n + (\ )\lambda^{n-1} + \ldots\ldots + (\ )\lambda + (\ )$

λ is root of polynomial

$x^n + (\ )x^{n-1} + \ldots + (\ ) = 0$. of $|xI - A| = 0$.

$\therefore$ The scalars we are looking for are the roots of the polynomial $|xI - A| = 0$.

- characteristic polynomial.

monic polynomial - leading coefficient is one.

The ch polynomial $c(x)$ is a polynomial of degree n leading coefficient 1.

- monic polynomial of degree n.

We will have n roots, these roots can be real or complex repeating roots.

Let $\lambda_1 \dots \lambda_k$ be the distinct roots with multiplicity $a_1\ a_2 \dots a_k$.

i) $a_1 + a_2 + \dots + a_k = n.$

ii) $1 \leq a_i \leq n$

iii) $1 \leq k \leq n$

$$c(\lambda) = (\lambda-\lambda_1)^{a_1} (\lambda-\lambda_2)^{a_2} \dots (\lambda-\lambda_n)^{a_k}$$

a_i - algebraic multiplicity

a_i - eigen values of matrix.

λ_j an eigen value. $\therefore M = (\lambda_j I - A)$ is not invertible.

$\therefore Ax = \lambda_j x$ has non zero ~~too~~ solutions.

$W_j = \{x \in \mathbb{C}^{n \times n} : Ax = \lambda_j x\}.$

W_j has atleast one non zero solution.

W_j has a subspace of $\mathbb{C}^n$.

$\Rightarrow W_j$ is non empty

$$x, y \in W_j \Rightarrow \left.\begin{array}{l} Ax = \lambda_j x \\ Ay = \lambda_j y \end{array}\right\} \begin{array}{l} \Rightarrow A(x+y) = \lambda_j (x+y) \\ \Rightarrow x+y \in W_j \end{array}$$

$$x \in W_j \ ; \ \alpha \in \mathbb{C} \Rightarrow Ax = \lambda x, \ \alpha \in \mathbb{C}$$

$$\Rightarrow A(\alpha x) = \lambda_j (\alpha x)$$

Dim W_j is geometric multiplicity of λ_j

W_j is eigen subspace corresponding to eigen value λ_j

$g_j = W_j$

Given A,

$c(\lambda)$ is ch polynomial

$(\lambda-\lambda_1)^{a_1} \cdots (\lambda-\lambda_K)^{a_K}$

$\lambda_1, \lambda_2, \ldots \lambda_K$ — distinct eigenvalue.

$a_1, a_2, \ldots a_K$ — algebraic multiplicity.

$g_1, g_2, \ldots g_K$ — geometric multiplicity.

In general,

$1 \leq g_j \leq a_j$ for $1 \leq j \leq k$.

Theorem: A matrix $A \in \mathbb{C}^{n \times n}$ is diagonalizable iff $g_j = a_j$ for each eigenvalue λ_j

In particular if there are n distinct eigenvalue then $K=n$, $a_j = g_j = 1$ $\forall j$; A is diagonalizable.

W_j are nice subspaces.

A lot of properties are nice

Recall:

$A \in \mathbb{C}^{n \times n}$

$c(\lambda) = |\lambda I - A|$ .... monic polynomial of degree n.

$\lambda_1, \lambda_2 \ldots \lambda_n$ distinct roots.

$a_1, a_2 \ldots a_K$ multiplicities

$c(\lambda) = (\lambda-\lambda_1)^{a_1}(\lambda-\lambda_2)^{a_2} \ldots (\lambda-\lambda_K)^{a_K}$

$\lambda_1, \lambda_2 \ldots \lambda_K$ distinct eigenvalues.

$a_1, a_2 \ldots a_K$ alg multiplicities

For each λ_j let,

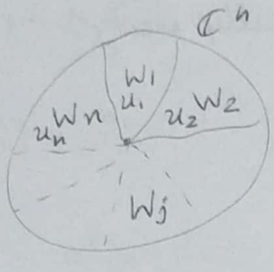
$W_j = \{x : Ax = \lambda_j x\}$

This is a subspace called the eigenspace corresponding to λ_j

$g_j = \dim(W_j) \geq 1$.

Fact: $1 \leq g_j \leq a_j$ for any j

A diagonalizable means $g_j = a_j \ \forall j$



If $u_1, u_2 \ldots$ and $\ldots u_K$ are non zero vectors in $W_1, W_2 \ldots\ldots W_k$ respectively then $u_1, u_2 \ldots u_k$ are linearly independent.

u_1 corresponding to λ_1.

u_2 corresponding to λ_2.

Lagrange polynomial:

n points means, unique polynomial of degree $(n-1)$ which passes through n points.

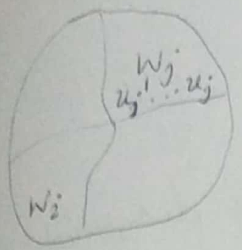
W_j dimension is a_j

$\dim W_j = a_j$

Basis of W_j, $B_j = \{u^j_1, u^j_2 \ldots u^j_{a_j}\}$

Do it for $j = 1, 2, \ldots, k$.

Scanned by CamScanner



$B_j = \{ u_1^j \, u_2^j \dots u_{a_j}^j \}$

$B_i = \{ u_1^i, u_2^i \dots u_{a_i}^i \}$.

$B_1 \cup B_2 \cup \dots \cup B_K$ will form consists of $a_1 + a_2 + \dots + a_K$ vectors equal to n, and these are linearly independent and form the basis for $\mathbb{C}^n$.

$P_1 = [\qquad]$

first a_1 columns the B_1 vector

next a_2 colums the B_2 vector.

⋮

a_K colums the B_K vector.

$P_{n \times n}$ matrix colums are independent $\therefore P^{-1}$ exists

$$P^{-1}AP = \begin{pmatrix} \overbrace{\lambda_1 \cdots \lambda_1}^{a_1} & & & 0 \\ & \underbrace{\lambda_2 \cdots \lambda_2}_{a_2} & & \\ 0 & & \ddots & \\ & & & \overbrace{\lambda_K \cdots}^{a_K} \end{pmatrix}$$

Given A;

→ $C(\lambda)$.

→ eigenvalues are their multiplicities

→ Capture eigenspaces. $W_j = \{ x : Ax = \lambda_j x \}$

→ Basis for W_j, $g_j = \dim W_j$

→ Check if $a_j = g_j \ \forall j$ ✓

→ Construct $P = (B_1 \text{ basis vector} \quad B_2 \text{ basis vector} \quad \dots)$

→ $P^{-1}AP$ = Diagonalizable.

Now if $a_j \neq g_j$ for any j

No diagonalizability.

What to do?

→ Can we triangulize? (compromise diagonalizability)

Yes!

The inverse computation is easy

works for square

→ Can we diagonalize else way? in other sense.

(compromise the process of diagonalization).

idea: (works for both square and rectangular)

$Ax = b$

$x = Py$

$b = Pz$

Change of variables gave this.

$\underbrace{P^{-1}AP}_{\text{diagonal}} y = z$

Suppose, $x = Py$.

$b = Qz$.

$APy = Qz$

$\underbrace{Q^{-1}AP}_{\text{Diagonal ??}} y = z$

(Two Different Transformation Q and P)

Can i find Q and P invertible such that $Q^{-1}AP$ is diagonal. Yes!

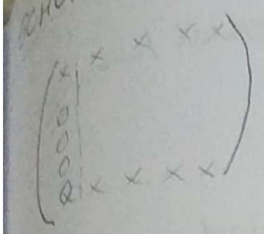
Q^{-1} Computation can be made easy.

This is SVD.

extends to rectangular matrices too.

Works for all matricies.

SCHUR decomposition – Triangulize

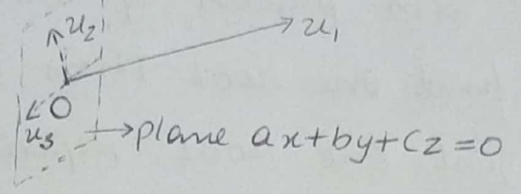


$A \in \mathbb{C}^{n \times n}$

λ_1 is any eigen value of A

u_1 be the corresponding eigenvector

$Au_1 = \lambda_1 u_2$.



$u_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

get $u_2\ u_3 \cdots u_n$ such that all are orthonormal.

$u_1\ u_2\ u_3 \cdots u_n$ are all orthonormal.

$P = [u_1\ u_2 \cdots u_n]$ Columns orthonormal, independent

Assume all u are real $P^T = P^{-1}$ (orthogonal).

$$AP = A[u_1\ u_2 \cdots u_n]$$
$$= [Au_1\ Au_2 \cdots Au_n]$$
$$= [\lambda_1 u_1\ Au_2 \cdots Au_n]$$

$$P^{-1}AP = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} [\lambda_1 u_1\ Au_2 \cdots Au_n]$$

$$= \left(\begin{array}{c|ccc} \lambda_1 & * & * & * \\ \hline 0 & & & \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{array}\right)$$

construct your own examples

Continue the process.

Conclusion: We can make all entries below leading diagonal of the matrix by orthogonal / unitary transformation.

Observe all eigenvalues eventually end up diagonally. But to start with process of $n \times n$ matrix we just have to find one root. Then $(n-1)\times(n-1)$ matrix again find just one root. Repeat process.