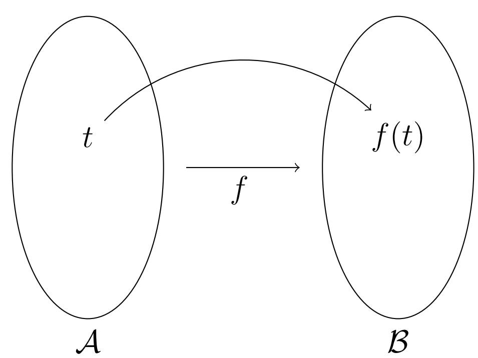
Chapter 1

Some Basic Ideas in Analysis

1.1 Functions

In general, a function f is a rule which associates with each t in a set $\mathcal{A}$ a value $f(t)$ from a set $\mathcal{B}$. We write

$$f : \mathcal{A} \longrightarrow \mathcal{B}$$



The set $\mathcal{A}$ is called the "**Domain**" of the function f and the set $\mathcal{B}$ is called the "**Codomain**" of the function f. For the domains set $\mathcal{A}$ and the codomain set $\mathcal{B}$ we shall be considering the following types of sets:

1. a finite set
$$\{1, 2, 3, \cdots, N\}$$

2. an infinite discrete set
$$\{1, 2, 3, \cdots\}$$

or

$$\{0, \pm 1, \pm 2, \pm 3, \cdots, \}$$

3. a finite interval on the real line

$$[0, T]$$

(The interval can also be taken as open interval or open on one side and closed on the other side)

4. a semi infinite interval on the real line, such as

$$[a, +\infty] \text{ or } \mathcal{A} = [-\infty, a] \text{ where } a \in \mathbb{R}$$

(Again the intervals above can also be open on the right or left respectively)

5. the infinite interval, that is the domain is the set $\mathbb{R}$ of real numbers

$$(-\infty, \infty)$$

In general the domain and codomain can be any arbitrary nonempty sets. For example, the domain $\mathcal{A}$ itself can be a collection of functions, and the codomain $\mathcal{B}$ a collection of sets, or $\mathcal{A}$ can be a collection of subsets of a fixed set Ω, and $\mathcal{B}$ the set $\mathbb{R}$ of real numbers.

Corresponding to any function $f : \mathcal{A} \longrightarrow \mathcal{B}$ we associate several types of sets of points, some of which are described below:

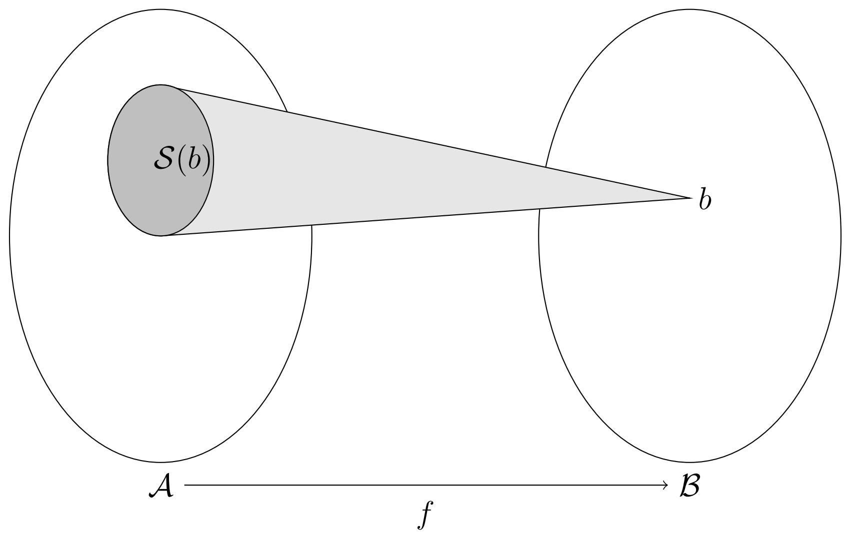
1. The domain set $\mathcal{A}$ and the codomain set $\mathcal{B}$

2. The set of all values taken by the function , called the "**Range of the function**" denoted by $\mathcal{R}_f$. We have

$$\begin{aligned} \mathcal{R}_f &= \{f(a) : a \in \mathcal{A}\} \\ &= \{b \in \mathcal{B} \ni \exists a \in \mathcal{A} \text{ such that } f(a) = b\} \end{aligned}$$

3. For any point b in $\mathcal{B}$ consider the set $\{b\}$, consisting of the single element b. We collect all those points in $\mathcal{A}$ whose image under f is b, that is, we define the set

$$\mathcal{S}(b) \stackrel{def}{=} \{a \in \mathcal{A} : f(a) = b\}$$



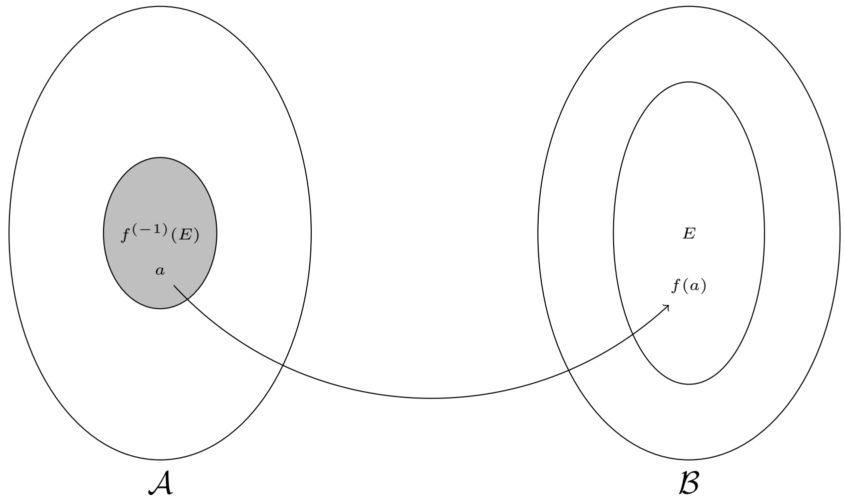
It may happen that $\mathcal{S}(b)$ is empty for some $b \in \mathcal{B}$ since there may not be any $a \in \mathcal{A}$ such that $f(a) = b$. Note that the set of all those $b \in \mathcal{B}$ for which $\mathcal{S}(b)$ is nonempty the Range of the function f

4. **Inverse image of a set under *f***

We next define the inverse image of a set under f. If $E \subseteq \mathcal{B}$ is any subset of $\mathcal{B}$ then we define $f^{(-1)}(E)$ as follows:

$$\begin{aligned} f^{(-1)}(E) &\stackrel{def}{=} \bigcup_{b \in E} \mathcal{S}(b) \\ &= \{a \in \mathcal{A} : f(a) \in E\} \end{aligned} \tag{1.1.1}$$

Thus $f^{(-1)}(E)$ is the set of all those points in $\mathcal{A}$ whose image under f fall in the set E. Note that for some sets E the set $f^{(-1)}(E)$ could be empty.



We classify the following special types of functions:

"one-one" function:

A function $f : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be a one-one function if for every $b \in \mathcal{B}$, $\mathcal{S}(b) = \phi$ or $\mathcal{S}(b)$ is a singleton. This means that

$$x, y \in \mathcal{A} \text{ and } f(x) = f(y) \Longrightarrow x = y$$

That is, any element $b \in \mathcal{R}_f$ has exactly one preimage $a$in $\mathcal{A}$.

"Onto" function;

A function $f : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be a onto function if $\mathcal{S}(b) \neq \phi$ for every $b \in \mathcal{B}$. This means that

$$b \in \mathcal{B} \Longrightarrow \exists a \in \mathcal{A} \ni f(a) = f(b)$$

That is, any element $b \in \mathcal{B}$ has at least one preimage $a \in \mathcal{A}$ under f.

1-1 correspondence:

A function $f : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be a 1-1 correspondence between $\mathcal{A}$ and $\mathcal{B}$ if it is both one-one and onto, that is $\mathcal{S}(b)$ is a singleton for every $b \in \mathcal{B}$. This means that distinct elements in the domain $\mathcal{A}$ have distinct images in the codomain $\mathcal{B}$ under f, and every element b in the codomain $\mathcal{B}$ has exactly one preimage in the domain $\mathcal{A}$. We can then use this unique preimage to define a new function $f^{(-1)} : \mathcal{B} \longrightarrow \mathcal{A}$ as

$$f^{(-1)}(b) = \text{ the unique preimage in } \mathcal{A} \text{ of } b \text{ under } f$$

Remark 1.1.1 Note that the inverse image under any function f of a set E, namely $f^{-1}(E)$ is defined for any subset E of the codomain. On the other hand the inverse function $f^{-1} : \mathcal{B} \longrightarrow A$ is defined if and only if the function f is one-one and onto

1.2 Convergence of a Sequence of Real Numbers

An infinite sequence of real numbers is a function

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$

The value of this function at n, namely $f[n]$, is a real number which we call the nth term of the sequence. We also write this sequence as

$$f[1], f[2], \cdots, f[n], \cdots$$

or as

$$\{f[n]\}_n$$

or sometimes as

$$\{f_n\}$$

or as

$$f_1, f_2, \cdots, f_n, \cdots$$

We recall the notion of convergence of such a sequence. We have

Definition 1.2.1 An infinite sequence $\{f_n\}_n$, of real numbers, is said to converge to a real number f if
for every positive real number $\varepsilon > 0$ there exists a positive integer,
(which may depend on ε), N_ε such that
$n \geq N_\varepsilon \Longrightarrow |f_n - f| < \varepsilon,$ that is
$n \geq N_\varepsilon \Longrightarrow f - \varepsilon < f_n < f + \varepsilon$

Remark 1.2.1 This says that the terms of the sequence can be made to differ from f by less than any tolerance threshold ε if the terms are chosen sufficiently far off in the sequence.

Remark 1.2.2 The above definition is useful to verify convergence if we know the limit f *ápriori*. Otherwise we can use the above definition to show that the sequence does NOT converge to a real number as follows:
To show that f_n does not converge to a we show the following: There exists a real number $\varepsilon > 0$ such that for every positive integer N there exists a positive integer $n > N$ for which $|f_n - a| \geq \varepsilon$

We next introduce the notion of **Cauchy sequences**. This tries to look at convergence of a sequence without an *ápriori* knowledge of the limit. We have seen above that a sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ converges to a real number f if the terms of the sequence differ from f by as small an error as needed, if we look at sufficiently far off in the sequence. If all the far off terms differ from f by a small error, then the terms far off themselves must be differing from each other by small error . Formalizing this idea gives us the notion of Cauchy sequences of real numbers. We have

Definition 1.2.2 A sequence of real numbers $\{f_n\}_n$ is said to be a Cauchy sequence if
for every positive real number $\varepsilon > 0$ there exists a positive integer N_ε, (which, in general, will depend on ε), such that
$n, m \geq N_\varepsilon \Longrightarrow |f_n - f_m| < \varepsilon$

Remark 1.2.3 It is easy to see that every convergent sequence of real numbers must be a Cauchy sequence. The Completeness axiom of the real numbers guarantees that every Cauchy sequence of real numbers converges. (For a Cauchy sequence, for sufficiently large n, the terms of the sequence give a good approximation of the limit of the sequence)

We next look at the notion of "**bounded sequences**".

Definition 1.2.3 A sequence of real numbers $\{f_n\}_n$ of real numbers is said to be bounded

$$\left.\begin{array}{l}\text{above}\\ \text{below}\end{array}\right\} \text{ if there exists a real number } \left.\begin{array}{c}M\\ m\end{array}\right\} \text{ such that } \left.\begin{array}{c}f_n \leq M\\ m \leq f_n\end{array}\right\} \text{ for every } n$$

A sequence is said to be bounded if it is both bounded below and bounded above.

Remark 1.2.4 It is not difficult to verify that every convergent sequence is bounded. However a bounded sequence need not be convergent as demostrated by the sequence

$$f_n = (-1)^{n+1}$$

Remark 1.2.5 From the above Remark it follows that a necessary condition for a sequence to converge is that it is bounded. Hence we shall first analyse bounded sequences.

We now define two important notions, namely the **supremum** and **infimum** of a bounded sequence

Definition 1.2.4 Let $\{f_n\}_n$ be a bounded infinite sequence of real numbers. A real number M_0 is said to be the supremum (or the least upper bound) of the sequence if

1. M_0 is an upper bound, that is, $f_n \leq M_0$ for every n, and

2. M_0 is the smallest among all upper bounds, that is, if M is any upper bound for the sequence then $M_0 \leq M$

Analogously,
A real number m_0 is said to be the infimum (or the greatest lower bound) of the sequence if

1. m_0 is a lower bound, that is, $m_0 \leq f_n$ for every n, and

2. m_0 is the largest among all lower bounds, that is, if m is any lower bound for the sequence then $m \leq m_0$

Remark 1.2.6 The Completeness Axiom guarantees that every sequence bounded above has a (unique) supremum and that every sequence bounded below has a (unique) infimum

Remark 1.2.7 We have the following characterization of supremum and infimum of a sequence of real numbers.

1. $\boldsymbol{M_0 = \sup_n f_n}$
   $\Longleftrightarrow$
   1) $\boldsymbol{M_0}$ **is an upper bound for the sequence, that is,** $\boldsymbol{f[n] \leq M_0}$ **for every** $\boldsymbol{n}$ **and**
   2) **for every** $\boldsymbol{\varepsilon > 0}$ **there exists a positive integer** $\boldsymbol{n_\varepsilon}$ **such that** $\boldsymbol{M_0 - \varepsilon < f_{n_\varepsilon} \leq M_0}$

2. $\boldsymbol{m_0 = \inf_n f_n}$

$\Longleftrightarrow$

1) $\boldsymbol{m_0}$ **is a lower bound for the sequence , that is,**
$\boldsymbol{m_0 \leq f_n}$ **for every** $\boldsymbol{n}$ **and**

2) **for every** $\boldsymbol{\varepsilon > 0}$ **there exists a positive integer** $\boldsymbol{n_\varepsilon}$ **such that**
$\boldsymbol{m_0 \leq f_{n_\varepsilon]} < m_0 + \varepsilon}$

Remark 1.2.8 We also denote the $\sup_n f_n$ as $lub(f_n)$ and $\inf_n f_n$ as $glb(f_n)$

1.3 Monotone Sequences

We saw above that not every bounded infinite sequence of real numbers converges. We shall now look at a class of bounded sequences which necessarily converge. These are called monotone sequences.

Definition 1.3.1 An infinite sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ of real numbers is said to be

$$\left.\begin{array}{l}\text{nondecreasing}\\ \text{nonincreasing}\end{array}\right\}\text{if}\left\{\begin{array}{l}f_n \leq f_{n+1}\\ f_{n+1} \leq f_n\end{array}\right.\quad\text{for all } n$$

Such sequences are called monotone sequences

Remark 1.3.1 A nondecreasing sequence is said to be increasing if $f_n < f_{n+1}$ for all n and a nonincreasing sequence is said to be decreasing if $f_{n+1} < f_n$ for all n

Remark 1.3.2 A nondecreasing sequence is always bounded below since f_1 is a lower bound for the sequence. Hence in order that a nondecreasing sequence is bounded it is enough if it is bounded above. Similarly,
A nonincreasing sequence is always bounded above since f_1 is an upper bound for the sequence. Hence in order that a nonincreasing sequence is bounded it is enough if it is bounded below.

Remark 1.3.3 If a nondecreasing sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded above then its least upper bound is denoted by $\sup_n f_n$

If a nondecreasing sequence $\{f_n\}_{n\in\mathbb{N}}$ is not bounded above then we define its least upper bound as $\sup_n f_n = +\infty$

Similarly,
If a nonincreasing sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded below then its greatest lower bound is denoted by $\inf_n f_n$
If a nonincreasing sequence $\{f_n\}_{n\in\mathbb{N}}$ is not bounded below then we define its greatest lower bound as $\inf_n a_n = -\infty$

We shall now look at the convergence of such monotone bounded sequences. We shall look at a nondecreasing sequence bounded above and then we can make an analogous statement for a nonincreasing sequence.
Let $\{f_n\}_n$ be a nondecreasing sequence bounded above. Since it is bounded above it has the least upper bound. Let

$$M_0 = \sup_n f_n$$

By the characterization of supremum of a sequence (See Remark 1.2.7) we have
For every $\varepsilon > 0$ there exists a positive integer N_ε such that

$$M_0 - \varepsilon < f_{N_\varepsilon} \leq M_0 < M_0 + \varepsilon \tag{1.3.1}$$

Since the sequence is nondecreasing we have

$$n \geq N_\varepsilon \Longrightarrow f_{N_\varepsilon} < f_n \leq M_0 < M_0 + \varepsilon \tag{1.3.2}$$

Combininig the above two equations we see that
For every $\varepsilon > 0$ there exists a positive integer N_ε such that

$$n \geq N_\varepsilon \Longrightarrow M_0 - \varepsilon < f_n \leq M_0 < M_0 + \varepsilon \tag{1.3.3}$$

Hence the sequence converges to M_0. Hence we get the following:

Theorem 1.3.1 Every nondecreasing sequence of real numbers bounded above converges to its supremum

Remark 1.3.4 For a nondecreasing sequence not bounded above we say that its limit is $+\infty$

Analogously we have

Theorem 1.3.2 Every nonincreasing sequence of real numbers bounded below converges to its infimum

Remark 1.3.5 For a nonincreasing sequence not bounded below we say that its limit is $-\infty$

We next see how the analysis of convergence of a bounded sequence (which may or may not be monotone) can be converted to that of the analysis of bounded monotone sequences.
Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. We define a new sequence $\{g_n\}_{n\in\mathbb{N}}$ as follows:

$$g_1 = \sup_k f_k = \sup\{f_1, f_2, \cdots, f_n, \cdots\} = \sup_{k\geq 1} f_k \quad (1.3.4)$$

$$g_2 = \sup\{f_2, \cdots, f_n, \cdots\} = \sup_{k\geq 2} f_k \quad (1.3.5)$$

and so on and in general

$$g_n = \sup\{f_n, f_{n+1}, \cdots\} = \sup_{k\geq n} f_k \quad (1.3.6)$$

This sequence $\{g_n\}_{n\in\mathbb{N}}$ is a nonincreasing sequence which is bounded below and hence by Theorem 1.3.2 it converges to its infimum. Hence the sequence $\{g_n\}_n$ converges to $\inf_n g_n$. We denote this by $\limsup_{n\to\infty} f_n$. Thus we have

$$\begin{aligned}\limsup_{n\to\infty} f_n &= \inf_n g_n \\ &= \inf_n \sup_{k\geq n} f_n\end{aligned}$$

Thus we have

Definition 1.3.2 For any bounded sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ we define

$$\limsup_{n\to\infty} f_n = \inf_n \sup_{k\geq n} f_n \quad (1.3.7)$$

Analogously we define

$$\liminf_{n\to\infty} f_n = \sup_n \inf_{k\geq n} f_n \quad (1.3.8)$$

We can show that in general

$$\liminf_{n\to\infty} f_n \leq \limsup_{n\to\infty} f_n \quad (1.3.9)$$

The convergent sequences are precisely those for which the equality holds. We shall state the following theorem without proof.

Theorem 1.3.3 A bounded sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ converges if and only if

$$\liminf_{n\to\infty} f_n \quad = \quad \limsup_{n\to\infty} f_n \tag{1.3.10}$$

1.4 Generalization to Sets

We shall now generalize the ideas of limsup and liminf to a sequence of subsets of a given set. Let Ω be a nonempty set and let $\mathcal{C}$ be a collection of subsets of Ω.

Upper and Lower Bounds

Definition 1.4.1 A subset K of Ω, (not necessarily a member of the collection $\mathcal{C}$), is said to be an upper bound for the collection $\mathcal{C}$ if

$$A \quad \subseteq \quad K \text{ for every } A \in \mathcal{C} \tag{1.4.1}$$

Note that clearly Ω is an upper bound for any collection $\mathcal{C}$ of subsets of Ω. Similarly,

Definition 1.4.2 A subset L of Ω, (not necessarily a member of the collection $\mathcal{C}$), is said to be a lower bound for the collection $\mathcal{C}$ if

$$L \quad \subseteq \quad A \text{ for every } A \in \mathcal{C} \tag{1.4.2}$$

Note that clearly the empty set ϕ is a lower bound for any collection $\mathcal{C}$ of subsets of Ω.

Supremum and Infimum

Definition 1.4.3 A subset K_0 of Ω, (not necessarily a member of the collection $\mathcal{C}$), is said to be supremum (or Least Upper Bound) of the collection $\mathcal{C}$ if

1. K_0 is an upper bound for $\mathcal{C}$, that is,

$$A \subseteq K_0 \text{ for every } A \in \mathcal{C}$$

   and

2. K_0 is the least among upper bounds, that is, if K is any upper bound for $\mathcal{C}$ then $K_0 \subseteq K$

Note that clearly the union of all the sets in $\mathcal{C}$ is the supremum for $\mathcal{C}$. Thus we have

$$sup\, \mathcal{C} \quad = \quad \bigcup_{A \in \mathcal{C}} A \tag{1.4.3}$$

Similarly we have,

Definition 1.4.4 A subset L_0 of Ω, (not necessarily a member of the collection $\mathcal{C}$), is said to be infimum (or Greatest Lower Bound) of the collection $\mathcal{C}$ if

1. L_0 is an lower bound for $\mathcal{C}$, that is,

   $$L_0 \subseteq A \text{ for every } A \in \mathcal{C}$$

   and

2. L_0 is the largest among lower bounds, that is, if L is any upper bound for $\mathcal{C}$ then $L \subseteq L_0$

Note that clearly the intersection of all the sets in $\mathcal{C}$ is the infimum for $\mathcal{C}$. Thus we have

$$inf\, \mathcal{C} \quad = \quad \bigcap_{A \in \mathcal{C}} A \tag{1.4.4}$$

Sequences of subsets of Ω

Analogously we define the following concepts for infinite sequences of subsets of a set Ω as follows: Let $A_1, A_2, \cdots, A_n, \cdots$ be an infinite sequence of subsets of Ω. Then clearly

$$\sup_n A_n = \bigcup_{n=1}^{\infty} A_n \text{ and } \inf_n A_n = \bigcap_{n=1}^{\infty} A_n$$

Let us now define a new sequence of subsets as

$$\begin{aligned} B_1 &= \sup_{k \geq 1} A_k = \bigcup_{k=1}^{\infty} A_k \\ B_2 &= \sup_{k \geq 2} A_k = \bigcup_{k=2}^{\infty} A_k \end{aligned}$$

and in general

$$B_n = \sup_{k\geq n} A_k = \bigcup_{k=n}^{\infty} A_k$$

It is easy to see that $B_{n+1} \subseteq B_n$ for any n. Hence B_n is a non increasing sequence and hence we define its limit as the infimum, that is,

$$\lim_{n\to\infty} B_n \stackrel{def}{=} \inf_n B_n = \bigcap_{n=1}^{\infty} B_n$$

Substituting for B_n we can write this as

$$\lim_{n\to\infty} \sup_{k\geq n} A_k = \lim_{n\to\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

This limit is denoted by $\limsup\limits_{n\to\infty} A_n$.

Analogously we define the sequence of subsets C_n as

$$C_n = \inf_{k\geq n} A_k = \bigcap_{k=n}^{\infty} A_k$$

Then the sequence C_n is nondecreasing and we define its limit as its sup, that is

$$\lim_{n\to\infty} C_n \stackrel{def}{=} \sup_n C_n = \bigcup_{n=1}^{\infty} C_n$$

Substituting for C_n we can write this as

$$\lim_{n\to\infty} \inf_{k\geq n} A_k = \lim_{n\to\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

This limit is denoted by $\liminf\limits_{n\to\infty} A_n$.

Thus we have

Definition 1.4.5 If $\{A_n\}_{n=1}^{\infty}$ is any infinite sequence of subsets of a set Ω then

$$\limsup_{n\to\infty} A_n \stackrel{def}{=} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \tag{1.4.5}$$

$$\liminf_{n\to\infty} A_n \stackrel{def}{=} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \tag{1.4.6}$$

Note that $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are both subsets of Ω

Remark 1.4.1 Let us now look at the structure of the set $\limsup_{n\to\infty} A_n$. We have

$$\begin{aligned}
x \in \limsup_{n\to\infty} A_n &\iff x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
&\iff x \in \bigcup_{k=n}^{\infty} A_k \text{ for every } n \\
&\iff \text{for every } n \text{ there exists a } k \geq n \text{ such that } x \in A_k \\
&\iff \text{however far we look in the sequence there exists a set} \\
&\qquad \text{in the sequence to which } x \text{ belongs} \\
&\iff x \text{ is an element of infinitely many of the } A_n \text{ sets}
\end{aligned}$$

Thus we have
$\limsup_{n\to\infty} A_n$ is the set consisting precisely of those elements that are elements in infinitely many of the A_n sets
Similarly we can see that
$\liminf_{n\to\infty} A_n$ is the set consisting precisely of those elements that are elements of all the A_n sets beyond a certain stage

1.5 Some Simple Tests For Convergence

We shall now look at simple methods by which we can test the convergence of a sequence of real numbers.
We first observe that if a sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ is such that the sequence of real numbers $\{|f_n|\}_n$ converges, it does not necessarily mean that the sequence of real numbers $\{f_n\}$ converges . (For example the sequence deefined as $f_n = (-1)^n$ does not converge whereas the sequence of absolure values $|f_n| = 1$ converges to 1). However, the sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ convergence to zero if and only if the sequence of absolute values $\{|f_n|\}_{n\in\mathbb{N}}$ also converges to zero. We have

Proposition 1.5.1 The sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ converges to zero
$\iff$
The sequence of real numbers $\{|f_n|\}_n$ converges to zero

We next observe that for a sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$ the following holds:

Proposition 1.5.2

$$\{f_n\}_{n\in\mathbb{N}} \text{ converges to } f \iff \text{the sequence } \{f_n - f\}_{n\in\mathbb{N}} \text{ converges to } 0$$

In view of the above Proposition 1.5.2 it follows that we need tests for convergence to zero. We shall now look at such simple tests.
We shall first state a useful theorem (without proof):

Theorem 1.5.1 **Sandwich Theorem**
Suppose $\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ are sequences of real numbers such that

1. $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ both converge to the same limit ℓ, and
2. $g_n \leq f_n \leq h_n$ for all n

Then
The sequence $\{f_n\}_{n\in\mathbb{N}}$ also converges to ℓ

Remark 1.5.1 Since convergence depends only on how the tail of the sequence behaves, the above theorem holds even if we replace condition (2) above by the following:
There exists a positive integer N_0 such that

$$g_n \leq f_n \leq h_n \text{ for all } n \geq N_0$$

Remark 1.5.2 In particular, if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers, (that is $f_n \geq 0$ for all n), and $\{h_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers converging to zero such that $f_n \leq h_n$ for all n then by taking $g_n = 0$ for all n and applying the Sandwich Theorem 1.5.1 we get that the sequence $\{f_n\}_n$ also converges to zero.

Remark 1.5.3 In the above Remark we can also replace $f_n \leq h_n$ by $f_n \leq K h_n$ where K is some positive constant.

We shall therefore look at some simple sequences that converge to zero that can be used to apply the Sandwich Theorem

1. **The sequence of real numbers defined as $\boldsymbol{a_n = \frac{1}{n^p}}$ converges to zero if and only if $\boldsymbol{p > 0}$**

   Example 1.5.1 We shall now use this to show that the sequence defined as
   $$f_n = \frac{n^2+1}{n^3+4}$$
   converges to zero.
   We have
   $$\begin{aligned} f_n &= \frac{n^2+1}{n^3+4} \\ \Longrightarrow & \\ f_n &\leq \frac{2n^2}{n^3+4} \text{ since } 1 \leq n^2 \text{ for all } n \\ &\leq \frac{2n^2}{n^3} = \frac{2}{n} \text{ since } \frac{1}{n^3+4} \leq \frac{1}{n^3} \text{ for all } n \end{aligned}$$

   Hence we get
   $$0 \leq f_n \leq \frac{2}{n}$$
   Now by choosing $h_n = \frac{1}{n}$ and using the Remark 1.5.3 we get the sequence $\{f_n\}_n$ also converges to zero.

2. **In general the sequence defined as**
   $$\boldsymbol{f_n = \frac{P(n)}{Q(n)}}$$
   where $\boldsymbol{P(n)}$ and $\boldsymbol{Q(n)}$ are polynomials in $\boldsymbol{n}$, converges to zero if $\boldsymbol{degree\ of\ P(n) < degree\ of\ Q(n)}$

   Example 1.5.2 We apply this to the sequence in Example 1.5.1 We have
   $$\begin{aligned} P(n) &= n^2+1 \\ Q(n) &= n^3+4 \\ degree\ of\ P(n) &= 2 \\ degree\ of\ Q(n) &= 3 \end{aligned}$$

Since $degree\ of\ P(n) < degree\ of\ Q(n)$ we have that the given sequence converges to zero

3. **More generally, if $\boldsymbol{f_n = \dfrac{F(n)}{G(n)}}$ where $\boldsymbol{F(n)}$ and $\boldsymbol{G(n)}$ are expressions each term of which is a power of $\boldsymbol{n}$, then the sequence converges to zero if**

   $\boldsymbol{highest\ exponent\ of\ n\ in\ F(n) < highest\ exponent\ of\ n\ in\ G(n)}$

   Example 1.5.3 Consider the sequence defined as

   $$f_n = \frac{n^2+1}{(\sqrt{n})^5+1}$$

   Then we have

   $$\begin{aligned} F(n) &= n^2+1 \\ G(n) &= (\sqrt{n})^5+1 \\ \text{highest exponent of } n \text{ in } F(n) &= 2 \\ \text{highest exponent of } n \text{ in } G(n) &= \frac{5}{2} \end{aligned}$$

   Since

   $$highest\ exponent\ of\ n\ in\ F(n) < highest\ exponent\ of\ n\ in\ G(n)$$

   the sequence converges to zero

4. **If $\boldsymbol{a_n = \dfrac{F(n)}{G(n)}}$ where $\boldsymbol{F(n)}$ and $\boldsymbol{G(n)}$ are expressions each term of which is a power of $\boldsymbol{n}$, and if**

   $\boldsymbol{highest\ exponent\ of\ n\ in\ F(n) = highest\ exponent\ of n\ in\ G(n)}$

   then the sequence converges to the ratio of the coefficients of the highest exponent terms in the numerator and denominator respectively

Example 1.5.4 The sequence defined as

$$f_n = \frac{3(\sqrt{n})^5 - n^2 + 1}{(2\sqrt{n})^5 - 3n - 1}$$

converges to $\dfrac{3}{2}$

5. A very useful sequence of real numbers that converges is the geometric sequence. We have
The sequence $\{r^{n-1}\}_{n\in\mathbb{N}}$ coverges to zero if $0 \leq r < 1$
We can easily see this as follows:
Let $\varepsilon > 0$ (Without loss of generality we assume $\varepsilon < 1$. Let N_ε be a positive integer such that

$$N_\varepsilon > \frac{\ell n(\varepsilon)}{\ell n(r)}$$

Then we have

$$\begin{aligned} n \geq N_\varepsilon &\implies n > \frac{\ell n(\varepsilon)}{\ell n(r)} \\ &\implies n\ell n(r) < \ell n(\varepsilon) \text{ (since } \ell n(r) < 0) \\ &\implies \ell n(r^n) < \ell n(\varepsilon) \\ &\implies r^n < \varepsilon \end{aligned}$$

Thus for any $\varepsilon > 0$ we can find a positive integer N_ε such that

$$n \geq \varepsilon \implies -\varepsilon < 0 \leq r^n < \varepsilon$$

Hence $f_n = r^{n-1}$ converges to zero

Remark 1.5.4 Using Proposition 1.5.2 we can now see that
the sequence of real numbers $\{r^{n-1}\}_n \in \mathbb{N}$ converges if $|r| < 1$

Remark 1.5.5 The sequence $f_n = r^{n-1}$ does not converge if $|r| > 1$. When $|r| = 1$ it converges if $r = 1$ and does not converge if $r = -1$

Example 1.5.5 The sequences $\left\{\left(\dfrac{1}{2}\right)^n\right\}_n$ and $\left\{\left(-\dfrac{1}{2}\right)^n\right\}_n$ both converge.
The sequence $\{(-1)^n\}_n$ does not converge whereas the sequence $\{(1)^n\}_n$ converges.
The sequence $\{2^n\}_n$ dose not converge.

1.6 Example

Let us consider the following sequence of real numbers:

$$f_n = \frac{3n^2+2}{4n^2+3}$$

We first observe that

$$\begin{aligned} f_n &= \frac{3n^2+2}{4n^2+3} \\ &= \frac{n^2(3+2n^{-2})}{n^2(4+3n^{-2})} \\ &= \frac{3+2n^{-2}}{4+3n^{-2}} \end{aligned}$$

We look at some of the terms of the sequence

n	f_n
10^2	$\frac{3.0002}{4.0003}$
10^3	$\frac{3.000002}{4.000003}$
10^4	$\frac{3.00000002}{4.00000003}$

1. It appears from the above table of values of f_n that the terms f_n are getting closer to $\frac{3}{4}$ as n increases and hence intuitively we see the limit as $\frac{3}{4}$. We shall now prove, using the definition of convergence, that the sequence indeed converges to $\frac{3}{4}$.
   In order to do this we need to show the following:
   We must show that for any given $\varepsilon > 0$ we can find a positive integer N_ε such that
   $$n \geq N_\varepsilon \Longrightarrow \left| f_n - \frac{3}{4} \right| < \varepsilon$$

Let us first estimate the error $\left|f_n - \frac{3}{4}\right|$. We have

$$\begin{aligned} f_n - \frac{3}{4} &= \frac{3n^2+2}{4n^2+3} - \frac{3}{4} \\ &= \frac{(12n^2+8) - (12n^2+9)}{4(4n^2+3)} \\ &= \frac{-1}{4(4n^2+3)} \end{aligned}$$

Thus

$$f_n - \frac{3}{4} = \frac{-1}{4(4n^2+3)} \tag{1.6.1}$$

Hence we get

$$\begin{aligned} |f_n - \frac{3}{4}| &= \frac{1}{4(4n^2+3)} \\ &\leq \frac{1}{(4n^2+3)} \\ &\leq \frac{1}{4n^2} \\ &\leq \frac{1}{n^2} \end{aligned}$$

Hence $\left|f_n - \dfrac{3}{4}\right|$ will be $< \varepsilon$ if
$\dfrac{1}{n^2} < \varepsilon$, that is if,
$n > \dfrac{1}{\sqrt{\varepsilon}}$

Hence given any $\varepsilon > 0$ we choose a positive integer $N_\varepsilon > \dfrac{1}{\sqrt{\varepsilon}}$. Then

$$n \geq N_\varepsilon \Longrightarrow \left|f_n - \frac{3}{4}\right| < \frac{1}{n^2} \leq \frac{1}{N_\varepsilon^2} < \varepsilon$$

Hence the sequence converges to $f = \dfrac{3}{4}$

2. Next we shall verify that the sequence is a Cauchy sequence.
   In order to do this we need to show the following:
   We must show that for any given $\varepsilon > 0$ we can find a positive integer N_ε such that
   $$m, n \geq N_\varepsilon \Longrightarrow |f_n - f_m| < \varepsilon$$
   Let us first estimate $|f_n - f_m|$. Without loss of generality let us assume $m > n$. We have
   $$\begin{aligned} f_n - f_m &= \frac{3n^2+2}{4n^2+3} - \frac{3m^2+2}{4m^2+3} \\ &= \frac{(12n^2m^2 + 9n^2 + 8m^2 + 6) - (12n^2m^2 + 8n^2 + 9m^2 + 6)}{(4n^2+3)(4m^2+3)} \\ &= \frac{n^2 - m^2}{(4n^2+3)(4m^2+3)} \end{aligned}$$
   Hence we get
   $$\begin{aligned} |f_n - f_m| &= \frac{m^2 - n^2}{(4n^2+3)(4m^2+3)} \\ &\leq \frac{m^2}{(4n^2+3)(4m^2+3)} \\ &\leq \frac{1}{4n^2+3} \\ &\leq \frac{1}{n^2} \end{aligned}$$
   Hence as above given any $\varepsilon > 0$ we choose a positive integer $N_\varepsilon > \dfrac{1}{\sqrt{\varepsilon}}$. Then
   $$n \geq N_\varepsilon \Longrightarrow |f_n - f_m| < \frac{1}{n^2} \leq \frac{1}{N_\varepsilon^2} < \varepsilon$$
   Thus the given sequence is a Cauchy sequence

3. Next we shall verify whether the sequence is a monotone sequence. The sequence will be nondecreasing if
   $$f_n \leq f_{n+1}$$

This will be true if

$$\frac{3n^2+2}{4n^2+3} \leq \frac{3(n+1)^2+2}{4(n+1)^2+3}$$

if

$$12n^2(n+1)^2+9n^2+8(n+1)^2+6 \leq 12n^2(n+1)^2+8n^2+9(n+1)^2+6$$

if

$$n^2 \leq (n+1)^2$$

which is true. Hence the sequence is nondecreasing.

4. Since the sequence is nondecreasing and bounded it must converge to its *lub*. But we have already proved in (1) above that the sequence converges to $\dfrac{3}{4}$. Hence $\dfrac{3}{4}$ must be its *lub*. We shall now directly verify that $\dfrac{3}{4}$ is indeed its *lub*.
   In order to show that $\frac{3}{4}$ is the *lub* we must show the following:

   (a) it is an upper bound, that is,

   $$f_n \leq \frac{3}{4} \text{ for all } n$$

   and

   (b) it is the least among all upper bounds, that is,

   for every $\varepsilon > 0$ there exists a term f_N of the sequence such that $f_N > \frac{3}{4} - \varepsilon$

   We prove (a) as follows:

   $$f_n \leq \frac{3}{4} \text{ if } \frac{3n^2+2}{4n^2+3} \leq \frac{3}{4}$$

   This is true if

   $$12n^2+8 \leq 12n^2+9$$

   which is obviously true for all n. Hence $\dfrac{3}{4}$ is an upper bound.
   We prove (b) as follows:
   We have observed in equation 1.6.1 above that

   $$f_n - \frac{3}{4} = \frac{-1}{4(4n^2+3)}$$

We want to find n such that

$$f_n \geq \frac{3}{4} - \varepsilon$$

that is we want to find n such that

$$f_n - \frac{3}{4} \geq -\varepsilon$$

Substituting for $f_n - \frac{3}{4}$ from equation 1.6.1 we get that we want to find n such that

$$\frac{-1}{4(4n^2+3)} \geq -\varepsilon$$

that is

$$\frac{1}{4(4n^2+3)} \leq \varepsilon$$

Now as in (1) we can choose N to be any positive integer $> \dfrac{1}{\sqrt{\varepsilon}}$ and we will get

$$f_N > \frac{3}{4} - \varepsilon$$

This proves (b).

Hence we see that $\dfrac{3}{4}$ is the *lub* of the sequence.

1.7 Sequences of Functions

We shall now investigate the idea of convergence of a sequence of functions. Let

$$f_n : \mathcal{D} \longrightarrow \mathbb{R},\ n = 1, 2, 3, \cdots$$

be a sequence of real valued functions defined on a set $\mathcal{D}$. Without loss of generality, for illustration, let us assume that the set $\mathcal{D}$ is an interval $\mathcal{I}$ on the real line. Hence we have

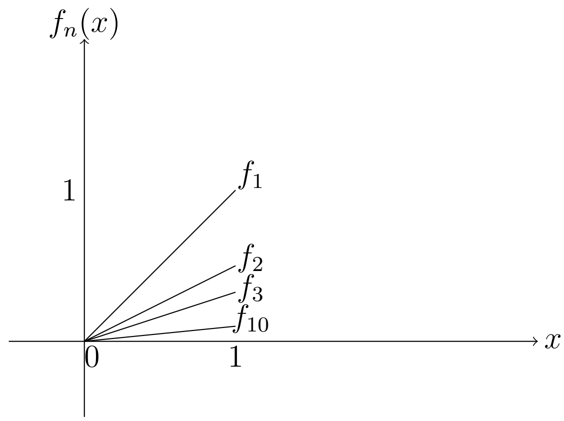
$$f_n : \mathcal{I} \longrightarrow \mathbb{R}$$

If we now look at the graphs of these functions, each function gives rise to a path in the plane and we would like to see if these paths approach some fixed path as $n \to \infty$.

Example 1.7.1 Let $\mathcal{I} = [0,1]$, the closed interval in $\mathbb{R}$. Let the sequence of functions be defined as

$$f_n(x) = \frac{x}{n}$$

If we now plot these functions we get a family of straight lines.



From the figure above it appears that these curves (straight lines) representing the functions $f_n(x)$ are approaching the zero function

$$f(x) = 0 \text{ for all } x$$

We want to now give a mathematical meaning to this type of convergence of a sequence of functions. We first introduce the notion of pointwise convergence which is the first simple way of giving meaning to this convergence.
Consider a sequence of real valued functions

$$f_n : \mathcal{D} \longrightarrow \mathbb{R} \qquad n = 1, 2, 3, \cdots$$

and a real valued function

$$f : \mathcal{D} \longrightarrow \mathbb{R}$$

For each fixed $x_0 \in \mathcal{D}$, the sequence of values $\{f_n(x_0)\}_{n\in\mathbb{N}}$ of the values of the functions at the point x_0 form a sequence of real numbers. If this sequence of real numbers converges to the real number $f(x_0)$, (namely the value of the function at the point x_0) we say that the given sequence of real valued functions converges to f at the point x_0. Thus we say
A sequence of real valued functions $f_n : \mathcal{D} \longrightarrow \mathbb{R}$ converges to the real valued function $f : \mathcal{D} \longrightarrow \mathbb{R}$ at a point $x_0 \in \mathcal{D}$, if

for every $\varepsilon > 0$ there exists a positive integer $N_0(\varepsilon)$, (which will, in general, depend not only on ε but also on the point x_0), such that

$$n \geq N_0(\varepsilon) \Longrightarrow |f_n(x_0) - f_(x_0)| < \varepsilon$$

If the sequence of functions converges at every point $x \in \mathcal{D}$ to $f(x)$ we then say that the sequence of real valued functions f_n converges pointwise to the function f. Hence we have the following definition:

Definition 1.7.1 **A sequence of real valued functions $f_n : \mathcal{D} \longrightarrow \mathbb{R}$ is said to converge to a real valued function $f : \mathcal{D} \longrightarrow \mathbb{R}$ if for every fixed every $x \in \mathcal{D}$, for any $\varepsilon > 0$, there exists a positive integer $N_x(\varepsilon)$, (which will, in general, depend on the point x), such that**

$$n \geq N_x(\varepsilon) \Longrightarrow |f_n(x) - f(x)| < \varepsilon$$

We then write

$$f_n \overset{pw\ (\mathcal{D})}{\longrightarrow} f$$

Example 1.7.2 Consider the sequence, $f_n(x) = \dfrac{x}{n}$ on the interval $[0, 1]$, of Example 1.7.1
For $x = 0$ we have $f_n(0) = 0$ for all n and hence $f_n(x)$ converges to 0 at $x = 0$.
For any fixed $x \neq 0$ and $0 < x \leq 1$ we have

$$\begin{aligned} 0 \leq f_n(x) &= \frac{x}{n} \\ &\leq \frac{1}{n} \\ &\longrightarrow 0 \end{aligned}$$

Hence this sequence of real valued functions converges pointwise to the zero function. We have

$$f_n \overset{pw\ [0,1]}{\longrightarrow} 0$$

We shall see this by using the ε definition of point wise convergence.
For $x = 0$ we already have $f_n(0) = 0$ for all n. Hence for any $\varepsilon > 0$ we can take $N_\varepsilon = 1$ and we have

$$n \geq N_\varepsilon \Longrightarrow |f_n(0)| = 0 < \varepsilon$$

Hence we have convergence to zero at the point $x = 0$
For $x \neq 0$ we have $0 < x \leq 1$ and

$$\begin{aligned} |f_n(x)| &= \frac{x}{n} \\ &\leq \frac{1}{n} \text{ since } 0 < x \leq 1 \end{aligned}$$

For any $\varepsilon > 0$ choose N_ε to be a positive integer $> \frac{1}{\varepsilon}$. Then we have from above

$$\begin{aligned} |f_n(x)| &\leq \frac{1}{n} \\ &\leq \frac{1}{N_\varepsilon} \text{ for all } n \geq N_\varepsilon \\ &< \varepsilon \text{ by the choice of } N_\varepsilon \end{aligned}$$

Hence $f_n(x)$ converges to zero for all $x \in [0, 1]$. Thus we have

$$f_n \overset{pw\ [0,1]}{\longrightarrow} 0$$

Example 1.7.3 By a similar argument as in Example 1.7.2 we get

$$f_n \overset{pw\ [0,\infty)}{\longrightarrow} 0$$

Example 1.7.4 Consider the sequence of real valued functions

$$f_n : [0, 1] \longrightarrow \mathbb{R}$$

defined as

$$f_n(x) = \frac{1}{n + x}$$

We have for any $x \in [0, 1]$,

$$\begin{aligned} 0 \leq |f_n(x)| &= \frac{1}{n + x} \\ &\leq \frac{1}{n} \end{aligned}$$

We have

$$f_n \overset{pw\ [0,1]}{\longrightarrow} 0$$

By a similar argument we also get

$$f_n \overset{pw\ [0,A]}{\longrightarrow} 0 \text{ for any positive real number } A$$

and

$$f_n \overset{pw\ [0,\infty)}{\longrightarrow} 0$$

Let $\mathcal{I}$ be an interval in $\mathbb{R}$. Suppose a sequence of real valued functions $f_n : \mathcal{I} \longrightarrow \mathbb{R}$ converge pointwise to a function $f : \mathcal{I} \longrightarrow \mathbb{R}$ and the functions $f_n(x)$ of the sequence all have some nice property, (if not all f_n at least all f_n beyond a certain stage). Then will the limit function $f(x)$ also possess this nice property - in other words does this pointwise limiting property preserves nice properties of the functions?
Which nice properties are of interest?
For example we would be interested in the answers to the following questions?
Suppose

$$f_n \overset{pw\ \mathcal{I}}{\longrightarrow} f$$

We list below some fundamental questions:

1. If all f_n, (if not all f_n at least all f_n beyond a certain stage), are bounded functions then will f be bounded - that is, does pointwise convergence preserve boundedness?

2. If all f_n, (if not all f_n at least all f_n beyond a certain stage), are continuous functions in $\mathcal{I}$ then will f be continuous in $\mathcal{I}$ - that is, does pointwise convergence preserve continuity?

3. If all f_n, (if not all f_n at least all f_n beyond a certain stage), are differentiable functions in $\mathcal{I}$ then will f be differentiable in $\mathcal{I}$, and if so will the derivative f' of the limit function be the limit of the sequence of derivatives, $\{f'_n\}_{n\in\mathbb{N}}$ - that is, does pointwise convergence preserve differentiability?

4. If all f_n, (if not all f_n at least all f_n beyond a certain stage), are integrable over $\mathcal{I}$ then will f be integrable over $\mathcal{I}$ and is

$$\lim_{n\to\infty} \int_{\mathcal{I}} f_n(x)dx = \int_{\mathcal{I}} \lim_{n\to\infty} f_n(x)dx$$

- that is, will the following hold:

$$\lim_{n\to\infty}\int_{\mathcal{I}} f_n(x)dx = \int_{\mathcal{I}} f(x)dx$$

that is, does pointwise convergence preserve integrals?

The answer to all these questions is, in general, these properties are not preserved by pointwise convergence. We shall illustrate this aspect by a few examples.

Example 1.7.5 Consider the sequence of real valued functions $f_n : (0,1) \longrightarrow \mathbb{R}$ defined as

$$f_n(x) = \frac{n}{1+nx}$$

We have, for any $x \in (0,1)$,

$$\begin{aligned} f_n(x) &= \frac{n}{1+nx} \\ &= \frac{1}{\frac{1}{n}+x} \\ &\longrightarrow \frac{1}{x} \end{aligned}$$

Thus we have

$$f_n \overset{pw\ (0,1)}{\longrightarrow} f(x)$$

where $f : (0,1) \longrightarrow \mathbb{R}$ is the function defined as

$$f(x) = \frac{1}{x}$$

We have for any $x \in (0,1)$, for every n,

$$|f_n(x)| = \left|\frac{n}{1+nx}\right| \leq n \text{ since } 1+nx \geq 1$$

Hence, for every n

$$|f_n(x)| \leq n \text{ for every } x \in (0,1)$$

Hence for every n, f_n is a bounded function on $(0,1)$. However the limit function $f(x)$ is not a bounded function on $(0,1)$. Thus we see that pointwise convergence does not, in general, preserved boundedness.

Example 1.7.6 Consider the sequence of real valued functions $f_n : [0,1] \longrightarrow \mathbb{R}$ defined as

$$f_n(x) = x^n$$

Clearly, we have

$$f_n \overset{pw\ [0,1]}{\longrightarrow} f(x)$$

where $f : [0,1] \longrightarrow \mathbb{R}$ is the function defined as

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

We see that for every n, f_n is a continuous function on $[0,1]$ but the limit function $f(x)$ is not continuous at $x = 1$. Thus we see that pointwise convergence does not, in general, preserve continuity.

Example 1.7.7 In the above example, we see that for every n, f_n is a differenttiable function on $[0,1]$ but the limit function $f(x)$ is not differentiable at $x = 1$ - it is not even continuous at $x = 1$. Thus we see that pointwise convergence does not, in general, preserve differentiability.

Example 1.7.8 In the above example the limit function f was not even differentiable whereas all the f_n were. Suppose the limit function is differentiable, will the derivetaive of the limit be the limit of the derivatives? We shall now show, in this example, that even this may not happen in general. Consider the sequence of real valued functions $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$f_n(x) = \frac{x}{1 + nx^2}$$

For $x = 0$ we have $f_n(0) = 0$ for every n and hence f_n converges to 0 at $x = 0$.
For $x \neq 0$ we have

$$\begin{aligned} 0 \leq |f_n(x)| &= \left| \frac{x}{1 + nx^2} \right| \\ &= \frac{|x|}{n\left(\dfrac{1}{n} + x^2\right)} \\ &= \frac{1}{n} \frac{|x|}{1 + x^2} \\ &\longrightarrow 0 \text{ for every } x \neq 0 \end{aligned}$$

Hence we have

$$f_n \overset{pw\,(\mathbb{R})}{\longrightarrow} 0$$

Further we see that for every n, f_n is a differentiable function, and

$$\begin{aligned} f_n'(x) &= \frac{(1+nx^2)(1) - x(2nx)}{(1+nx^2)^2} \\ &= \frac{1-nx^2}{(1+nx^2)^2} \end{aligned}$$

For $x = 0$ we have

$$\begin{aligned} f_n'(0) &= \lim_{x\to 0} \frac{f_n(x) - f_n(0)}{x} \\ &= \lim_{x\to 0} \frac{f_n(x)}{x} \text{ (since } f_n(0) = 0) \\ &= \lim_{x\to 0} \frac{1}{1+nx^2} \\ &= 1 \end{aligned}$$

Thus

$$\begin{aligned} f_n'(0) &= 1 \text{ for all } n \\ &\longrightarrow 1 \end{aligned}$$

For $x \neq 0$ we have

$$\begin{aligned} f_n'(x) &= \frac{1-nx^2}{(1+nx^2)^2} \\ &= \frac{n\left(\frac{1}{n} - \frac{x^2}{n}\right)}{n^2(\frac{1}{n}+x^2)^2} \\ &= \frac{1}{n}\,\frac{\frac{1}{n} - \frac{x^2}{n}}{\left(\frac{1}{n}+x^2\right)^2} \\ &\longrightarrow 0 \end{aligned}$$

Thus the derivative sequence $f_n'(x)$ is such that

$$f_n' \overset{pw\,\mathbb{R}}{\longrightarrow} g$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is the function defined as

$$g(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases}$$

On the other hand since the limit function f of the sequence f_n is the zero function we have

$$f'(x) \quad = \quad 0 \text{ for every } x$$

Hence $f'(x) \neq g(x)$ since $f'(0) \neq g(0)$

Thus we see that even if the derivative sequence converges pointwise to a function it need not be the derivative of the limit function.

Example 1.7.9 Consider the sequence of real valued functions $f_n : [0, \infty) \longrightarrow \mathbb{R}$ be the functions whose graph is as shown below:

We have

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if} \quad 0 \leq x \leq n \\ 0 & \text{if} \quad x > n \end{cases}$$

It is easy to see that

$$f_n \overset{pw\ [0,\infty)}{\longrightarrow} 0$$

For every n the function f_n is integrable over $[0, \infty)$ and

$$\int_0^\infty f_n(x)dx \quad = \quad 1$$

On the other hand the limit function f being the zero function we have

$$\int_0^\infty f(x)dx \quad = \quad 0$$

Thus

$$\lim_{n \longrightarrow \infty} \int_0^\infty f_n(x)dx \neq \int_0^\infty \lim_{n \longrightarrow} f_n(x)dx$$

Thus we see that pointwise convergence does not, ingeneral, preserve integrals

We shall next introduce different notions of convergence of sequence of functions where some of these defects are taken care of.

1.8 More Notions of Convergence of Sequences of Real Valued Functions

We shall next introduce other notions of convergence of a sequence of functions. In all these notions of convergence, we get the respective notions based on how we try to control the error between the functions f_n in the sequence and the limit function f. We shall first introduce the notion of uniform convergence. We shall begin with the two simple examples.

Example 1.8.1 Let N be a fixed positive integer and let $\mathcal{D}$ be the finite set,

$$\mathcal{D} = \{1, 2, 3, \cdots, N\}$$

Let f_n and f be the functions defined as follows;

$$\begin{aligned} f_n(x) &= \frac{x}{n} \text{ for all } x \in \mathcal{D} \\ f(x) &= 0 \text{ for all } x \in \mathcal{D} \end{aligned}$$

We have tabulated these functions below :

	$x = 1$	$x = 2$	$= x3$	$\cdots$	$x = N$
$f_1(x)(x)$	1	$\frac{2}{1}$	$\frac{3}{1}$	$\cdots$	$\frac{N}{1}$
$f_2(x)$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\cdots$	$\frac{N}{2}$
$f_3(x)$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\cdots$	$\frac{N}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f_n(x)$	$\frac{1}{n}$	$\frac{2}{n}$	$\frac{3}{n}$	$\cdots$	$\frac{N}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In order to look at the convergence of f_n at a point $x \in \mathcal{D}$ we have to look at the x-th column, (in the table above), which gives us the sequence of values $f_1(x), f_2(x), f_3(x), \cdots, f_n(x), \cdots$. From the table it follows that

$$\lim_{n \longrightarrow \infty} f_n(x) \quad = \quad 0 \text{ for every } x \in \mathcal{D}$$

Hence the sequence converges to the zero function pointwise, that is

$$f_n \stackrel{pw\ (\mathcal{D})}{\longrightarrow} 0$$

Consider any $\varepsilon > 0$. For the sake of illustration let us take $\varepsilon = 10^{-4}$ and we choose N_ε to be a positive integer $> \frac{x}{\varepsilon}$. Then

$$|f_n(x) - f(x)| = \frac{x}{n} < \varepsilon = 10^{-4} \text{ if } n \geq N_\varepsilon > x(10)^4$$

Thus we have a single stage N_ε beyond which

$$|f_n(x) - f(x)| < 10^{-4} \text{ for all } x \in \mathcal{D}$$

Thus, in this example we see that there exists a single positive integer N_ε which depends only on ε and not on the point $x \in \mathcal{D}$ such that the above inequality holds beyond this N_ε.

Example 1.8.2 Consider now the set $\mathcal{D} = \mathbb{N} = \{1, 2, 3, \cdots\}$, the infinite set of all positive integers, and as in the above example let the sequence $f_n : \mathcal{D} \longrightarrow \mathbb{R}$ be defined as

$$f_n(x) \quad = \quad \frac{x}{n} \text{ for all } x \in \mathcal{D}$$

These functions are tabulated below:

	$x = 1$	$x = 2$	$x = 3$	$\cdots$	$x = N$	$\cdots$	$\cdots$
$f_1(x)(x)$	1	$\frac{2}{1}$	$\frac{3}{1}$	$\cdots$	$\frac{N}{n}$	$\cdots$	$\cdots$
$f_2(x)$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\cdots$	$\frac{N}{2}$	$\cdots$	$\cdots$
$f_3(x)$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\cdots$	$\frac{N}{3}$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f_n(x)$	$\frac{1}{n}$	$\frac{2}{n}$	$\frac{3}{n}$	$\cdots$	$\frac{N}{n}$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

From the table it follows that

$$\lim_{n \longrightarrow \infty} f_n(x) \quad = \quad 0 \ (= f(x)) \text{ for all } x \in \mathcal{D}$$

Hence we get

$$f_n \overset{pw\ (\mathcal{D})}{\longrightarrow} 0$$

We observe that, in this example, for any $\varepsilon > 0$, we need different N_ε at different points in $\mathcal{D}$ to make $|f_n(x) - f(x)| < \varepsilon$ - and no single positive integer N_ε works for all $x \in \mathcal{D}$ - whereas in the previous example we could get one N_ε that worked for all points.

We next distinguish these two cases by introducing the notion of uniform convergence.

Definition 1.8.1 . A sequence of real valued functions, $f_n : \mathcal{D} \longrightarrow \mathbb{R}$, is said to converge "uniformly" to the function, $f : \mathcal{D} \longrightarrow \mathbb{R}$ on $\mathcal{D}$, if
For every $\varepsilon > 0$ there exists a positive integer N_ε (depending only on ε) such that

$$n \geq N_\varepsilon \Longrightarrow |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in \mathcal{D}$$

Thus the sequence in Example 1.8.1 converges uniformly to the zero function while the sequence in the Example 1.8.2 does not converge to the zero function uniformly, even though it converges pointwise to the zero function. If a sequence of functions f_n converges uniformly to a function f we then write,

$$f_n \stackrel{uc\,(\mathcal{D})}{\longrightarrow} f$$

Remark 1.8.1 It is easy to see that

$$f_n \stackrel{uc\,(\mathcal{D})}{\longrightarrow} f \implies f_n \stackrel{pw\,(\mathcal{D})}{\longrightarrow} f$$

On the other hand, from Example 1.8.2, we see that

$$f_n \stackrel{pw\,(\mathcal{D})}{\longrightarrow} f \not\Rightarrow f_n \stackrel{uc\,(\mathcal{D})}{\longrightarrow} f$$

Remark 1.8.2 It is easy to see that $f_n \stackrel{uc\,(\mathcal{D})}{\longrightarrow} f$ is equivalent to $f_n - f \stackrel{uc\,(\mathcal{D})}{\longrightarrow} 0$. It is, therefore, enough if we know how to test for uniform convergence to the zero function.

We shall therefore look at a simple test for uniform convergence of a sequence of functions to the zero function. This is similar to the Sandwich Theorem (Theorem 1.5.1).

Weierstrass' M_n Test:

Let $f_n : \mathcal{D} \longrightarrow \mathbb{R}$ be a sequence of real valued functions such that there exists a sequence $\{M_n\}$ of nonnegative real numbers such that,

1. **For every n, $|f_n(x)| \leq M_n$ for all $x \in \mathcal{D}$ and**

2. **$M_n \longrightarrow 0$**

Then $f_n \stackrel{uc\,(\mathcal{D})}{\longrightarrow} 0$

Example 1.8.3 Consider $\mathcal{D} = [-1, 1]$ and the sequence $f_n : \mathcal{D} \longrightarrow \mathbb{R}$ defined as

$$f_n(x) = \frac{x^2 + x + 1}{n^2 + 1}$$

We shall use Weierstrass' Test to show that this sequence converges to 0 uniformly on $\mathcal{D}$. We have

$$\begin{aligned} |x^2+x+1| &\leq |x^2|+|x|+1 \\ &\leq 1+1+1 \text{ for all } x\in\mathcal{D}, \text{ and} \\ n^2+1 &\geq n^2 \text{ and hence} \\ \frac{1}{n^2+1} &\leq \frac{1}{n^2} \end{aligned}$$

Thus we have

$$\begin{aligned} |f_n(x)| &= \frac{|x^2+x+1|}{n^2+1} \\ &\leq \frac{3}{n^2} \end{aligned}$$

Now if we choose

$$M_n = \frac{3}{n^2}$$

and we have $M_n \longrightarrow 0$. Hence all the conditions of Weierstrass' Test are satisfied and we get therefore $f_n \overset{uc\,(\mathcal{D})}{\longrightarrow} 0$

Example 1.8.4 Consider the sequence $f_n : [0,1] \longrightarrow \mathbb{R}$ defined as

$$f_n(x) = \frac{x^2+n^2x+x}{n^2+1}$$

We shall show that this sequence converges uniformly to the function

$$f : [0,1] \longrightarrow \mathbb{R}$$

defined as $f(x) = x$. This means that we have to show that the sequence $f_n(x) - f(x)$ converges uniformly to the zero function. We have

$$\begin{aligned} |f_n(x)-f(x)| &= \left|\frac{x^2+n^2x+x}{n^2+1} - x\right| \\ &= \left|\frac{x^2+n^2x+x-n^2x-x}{n^2+1}\right| \\ &= \frac{x^2}{n^2+1} \\ &\leq \frac{1}{n^2+1} \text{ since } x\in[0,1] \end{aligned}$$

Choosing $M_n = \frac{1}{n^2+1}$ and applying Weierstrass' test we get

$$f_n - f \overset{uc\ (\mathcal{I})}{\longrightarrow} 0$$

and hence

$$f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f$$

We shall next see that uniform convergence "preserves" "nice" properties of the functions of the sequence.
Without loss of generality we shall consider that the functions are all defined in an interval $\mathcal{I}$ in $\mathbb{R}$

1. **Uniform convergence preserves boundedness**

   $f_n : \mathcal{I} \longrightarrow \mathbb{R}$ be all bounded functions and $f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f$
   $\Longrightarrow$
   f is bounded

   Proof:
   f_n bounded $\Longrightarrow$ For every n there exists a positive real number M_n such that

   $$|f_n(x)| \leq M_n \text{ for every } x \in \mathcal{I}$$

   and hence

   $$-M_n \leq f_n(x) \leq M_n \text{ for every } x \in \mathcal{I} \tag{1.8.1}$$

   $f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f \Longrightarrow$
   For every $\varepsilon > 0$ there exists a positive integer N_ε such that

   $$n \geq N_\varepsilon \implies |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in \mathcal{I} \tag{1.8.2}$$

   Choosing $\varepsilon = 1$ and $n = N_1$, (the positive integer corresponding to $\varepsilon = 1$) we get from equation 1.8.2

   $$\begin{aligned} |f_{N_1}(x) - f(x)| &< 1 \text{ for all } x \in \mathcal{I} \\ &\Longrightarrow f_{N_1}(x) - 1 < f(x) < f_{N_1} + 1 \text{ for all } x \in \mathcal{I} \\ &\Longrightarrow -M_{N_1} - 1 < f(x) < M_{N_1} + 1 \text{ (by equation 1.8.1)} \\ &\Longrightarrow f \text{ is bounded} \end{aligned}$$

2. <u>**Uniform convergence preserves continuity**</u>

   $f_n : \mathcal{I} \longrightarrow \mathbb{R}$ be all continuous functions on $\mathcal{I}$ and $f_n \overset{uc\,(\mathcal{I})}{\longrightarrow} f$ $\Longrightarrow$
   f is continuous on $\mathcal{I}$
   <u>Proof:</u>
   We have to show that $f(x)$ is continuous at every point $a \in \mathcal{I}$, that is, we have to show that
   for every $a \in \mathcal{I}$, the error $|f(x) - f(a)|$ can be made as small as we please by choosing x sufficiently close to the point a
   This means we have to show that, given $a \in \mathcal{I}$,

$$\left\{ \begin{array}{l} \text{for every } \varepsilon > 0 \text{ there exists a } \delta_\varepsilon(a) > 0 \text{ such that} \\ |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon \end{array} \right\} \tag{1.8.3}$$

   We show equation 1.8.3 as follows:

   $f_n \overset{uc\,(\mathcal{I})}{\longrightarrow} f \Longrightarrow$
   for every $\varepsilon > 0$ there exists a positive integer N_ε such that

$$n \geq N_\varepsilon \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon \text{ for every } x \in \mathcal{I} \tag{1.8.4}$$

   This gives

$$|f_{N_\varepsilon}(x) - f(x)| \quad < \quad \varepsilon \text{ for every } x \in \mathcal{I} \tag{1.8.5}$$

   Since all the $f_n(x)$ are given to be continuous on $\mathcal{I}$ we have $f_{N_\varepsilon}(x)$ is continuous on $\mathcal{I}$
   Hence $f_{N_\varepsilon}(x)$ is continuous at $x = a$
   Thus given $\varepsilon > 0$ choose $\varepsilon_1 = \dfrac{\varepsilon}{3}$
   By the continuity of f_{N_ε} at the point a, we get a $\delta_{\varepsilon_1}(a) > 0$ (and hence $\delta_\varepsilon(a)$) such that

$$|x - a| < \delta_\varepsilon(a) \quad \Longrightarrow \quad |f_{N_\varepsilon}(x) - f_{N_\varepsilon}(a)| < \varepsilon_1 \tag{1.8.6}$$

   Then we have

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f_{N_\varepsilon}(x) + f_{N_\varepsilon}(x) - f_{N_\varepsilon}(a) + f_{N_\varepsilon}(a) - f(a)| \\ &\leq |f(x) - f_{N_\varepsilon}(x)| + |f_{N_\varepsilon}(x) - f_{N_\varepsilon}(a)| + |f_{N_\varepsilon}(a) - f(a)| \end{aligned}$$

The first and third terms are $< \varepsilon_1$ by equation 1.8.5 and the middle term is $< \varepsilon_1$ by equation 1.8.6 if $|x-a| < \delta_\varepsilon$.
Thus we have for every $\varepsilon > 0$ there exists a $\delta_\varepsilon(a)$ such that

$$|x-a| < \delta_\varepsilon \Longrightarrow |f(x) - f(a)| < 3\,\varepsilon_1 = \varepsilon$$

Hence $f(x)$ is continuous at every point $a \in \mathcal{I}$ and hence f is continuous on $\mathcal{I}$

Example 1.8.5 Consider the sequence given in Example 1.8.3
We have

$$\begin{aligned} |f_n(x)| &= \frac{|x^2+x+1|}{n^2+1} \\ &\leq \frac{|x^2|+|x|+1|}{n^2+1} \\ &\leq \frac{3}{n^2+1} \text{ since } x \in [-1,1] \end{aligned}$$

Hence the given sequence is bounded. We have seen in Example 1.8.3 that

$$f_n \stackrel{uc\ (\mathcal{I})}{\longrightarrow} 0$$

The limit function,being the zero function, is bounded.
Thus we see that uniform convergence has preserved boundedness.
We also see that all the f_n are continuous on $\mathcal{I}$ and the limit function,being the zero function, is continuous on $\mathcal{I}$.
Thus we see that uniform convergence has preserved continuity.

Example 1.8.6 Consider the sequence in Example 1.8.4 We had

$$f_n \stackrel{uc\ (\mathcal{I})}{\longrightarrow} f \text{ where } f(x) = x$$

We have

$$\begin{aligned} |f_n(x)| &= \left|\frac{x^2+n^2x+x}{n^2+1}\right| \\ &\leq \frac{2+n^2}{n^2+1} \end{aligned}$$

Hence all the f_n are bounded. Further we have for the limit function,

$$|f(x) \quad = \quad |x| \leq 1$$

and hence the limit function is also bounded. Thus we see that uniform convergence has preserved boundedness.
Further all the f_n are continuous functions on $\mathcal{I}$ and the limit function $f(x) = x$ is also continuous on $\mathcal{I}$. Thus we see that uniform convergence has preserved continuity.

3. We shall next investigate whether uniform convergence preserves differentiability. We shall first look at an example.

Example 1.8.7 Consider the sequence $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ of Example 1.7.8. We have

$$f_n(x) = \frac{x}{1 + nx^2}$$

In Example 1.7.8 we had the following facts:

$$f_n \overset{pw\ (\mathbb{R})}{\longrightarrow} 0 \tag{1.8.7}$$

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2} \tag{1.8.8}$$

$$f_n'(x) \overset{pw\ (\mathbb{R})}{\longrightarrow} g(x) \tag{1.8.9}$$

where

$$g(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases} \tag{1.8.10}$$

Now we shall see that the sequence $\{f_n\}_n$ converges uniformly to the zero function $f(x) = 0$. We have from 1.8.8 that the critical points of $f_n(x)$ are at $x = \pm\frac{1}{\sqrt{n}}$ and using second derivative we can show that the maximum occurs at $x = \frac{1}{\sqrt{n}}$ and the minimum occurs at $x = -\frac{1}{\sqrt{n}}$. The maximum value of $f_n(x)$ is

$$f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2\sqrt{n}}$$

and the minimum value of $f_n(x)$ is

$$f_n\left(-\frac{1}{\sqrt{n}}\right) = -\frac{1}{2\sqrt{n}}$$

Thus we have

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}} \text{ for all } x \in \mathbb{R}$$

Now choosing $M_n = \dfrac{1}{2\sqrt{n}}$ and applying Weierstrass' M_n test we see that

$$f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f(x) \text{ where } f(x) = 0 \text{ for all } x$$

Now by 1.8.8 and 1.8.9, f_n are all differentiable and $f_n' \overset{pw\ (\mathbb{R})}{\longrightarrow} g(x)$. But $f'(x) \neq g(x)$ at $x = 0$. Thus we see that even though all the f_n are differentiable and the convergence is uniform and the sequence of derivatives converge pointwise, the derivative of the limit function $f(x)$ is not the same as the limit of the derivative functions. Thus preserving differentiability is not, in general true, under uniform convergence.

We shall state the following theorem (without proof) about preserving differentiation under uniform convergence:

Theorem 1.8.1 **Let $\boldsymbol{f_n : \mathcal{I} \longrightarrow} \mathbb{R}$, $\boldsymbol{f : \mathcal{I} \longrightarrow} \mathbb{R}$ and $\boldsymbol{g : \mathcal{I} \longrightarrow} \mathbb{R}$**

$$\left\{\begin{array}{ll} \boldsymbol{i)} & \boldsymbol{f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f} \\ \boldsymbol{ii)} & \boldsymbol{f_n} \textbf{ are all differentiable in } \boldsymbol{\mathcal{I}} \\ \boldsymbol{iii)} & \boldsymbol{f_n' \overset{uc\ (\mathcal{I})}{\longrightarrow} g} \end{array}\right\}$$

$\Longrightarrow$

$\boldsymbol{f}$ **is differentiable and** $\boldsymbol{f'(x) = g(x)}$

4. Finally we shall see whether uniform convergence preserves integrals. We shall first look at two examples

Example 1.8.8 Consider the sequence of Example 1.8.4. We had

$$f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f$$

where $f(x) = x$. We have

$$\begin{aligned}
\int_{\mathcal{I}} f_n(x)dx &= \int_0^1 \frac{x^2 + n^2x + x}{n^2+1}dx \\
&= \frac{1}{n^2+1}\left\{\frac{1}{3} + \frac{n^2}{2} + \frac{1}{2}\right\} \\
&= \frac{1}{n^2+1}\left\{\frac{5}{6} + \frac{n^2}{2} + \frac{1}{2}\right\} \\
&= \frac{\frac{5}{6}}{n^2+1} + \frac{n^2}{2n^2+2} \\
&\longrightarrow \frac{1}{2} \text{ as } n \to \infty
\end{aligned}$$

Thus we have

$$\lim_{n\to\infty} \int_{\mathcal{I}} f_n(x)dx = \frac{1}{2} \tag{1.8.11}$$

We also have

$$\int_{\mathcal{I}} f(x)dx = \int_0^1 xdx = \frac{1}{2} \tag{1.8.12}$$

Thus we have

$$\lim_{n\to\infty} \int_{\mathcal{I}} f_n(x)dx = \int_{\mathcal{I}} \lim_{n\to\infty} f_n(x)dx$$

Thus in this example, uniform convergence has preserved the integrals

Example 1.8.9 Consider the sequence of Example 1.7.9. We have seen that the sequence converges pointwise to the zero function. Further we have, for every n,

$$|f_n(x) \leq \frac{1}{n} \text{ for every } x \in \mathcal{I}$$

where $\mathcal{I} = [0, \infty)$.
Choosing $M_n = \dfrac{1}{n}$, it follows by Weierstrass' test that

$$f_n \xrightarrow{uc\ (\mathcal{I})} f$$

where f is the zero function. We have seen in Example 1.7.9 that

$$\int_{\mathcal{I}} f_n(x)dx = 1 \not\longrightarrow 0 = \int_{\mathcal{I}} f(x)dx$$

Thus we see that even though we have uniform convergence of f_n to f the integral of f_n over the interval $\mathcal{I}$ does not converge to that of f

The difference between the two examples above is that the interval $\mathcal{I}$ in Example 1.8.8 is an infinite interval whereas it is finite in Example 1.8.9. We can prove that in general,

Theorem 1.8.2

$$\left\{ \begin{array}{ll} i) & \mathcal{I} \text{ is a finite interval} \\ & \\ ii) & f_n \overset{uc\ (\mathcal{I})}{\longrightarrow} f \\ & \\ iii) & f_n \text{ is integrable over } \mathcal{I} \end{array} \right\}$$

$\Longrightarrow$

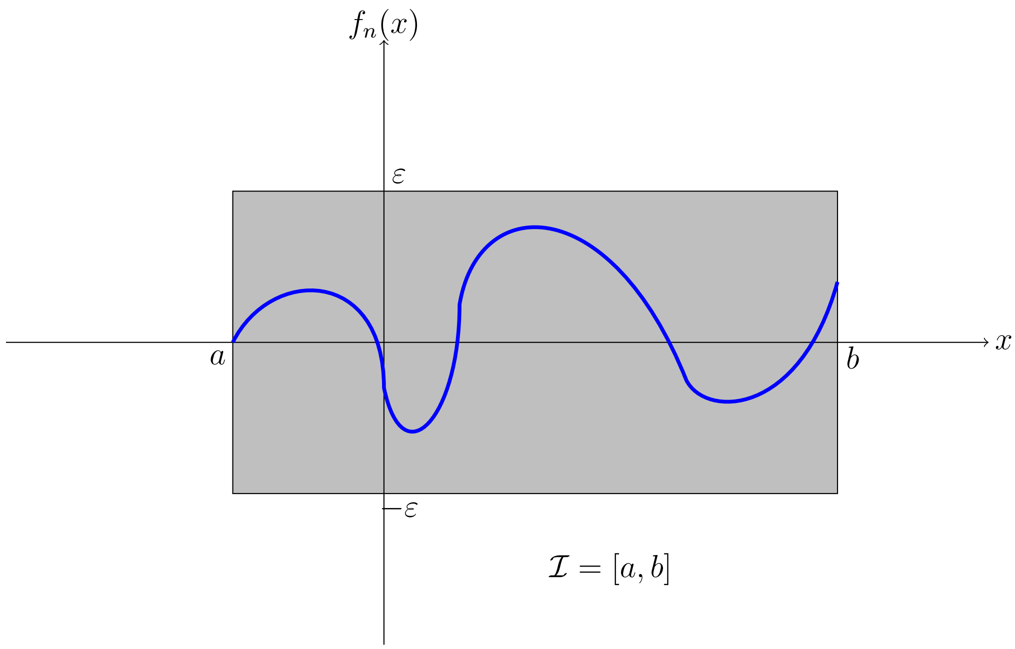
$$\left\{ \begin{array}{ll} i) & f \text{ is integrable over } \mathcal{I} \\ ii) & \lim_{n\to\infty} \int_{\mathcal{I}} f_n(x)dx = \int_{\mathcal{I}} \lim_{n\to\infty} f_n(x)dx = \int_{\mathcal{I}} f(x)dx \end{array} \right\}$$

Remark 1.8.3 However, when the interval is an infinite interval, uniform convergence may preserve integrals in some special cases.

We shall next look at what it means to say that a sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly to the function f. In view of Remark 1.8.2 it is enough to look at what it means to say that a sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly to the function the zero function. Without loss of generality let us assume $\mathcal{D}$ is an interval $\mathcal{I}$ on the real line.
A sequence of functions $f_n : \mathcal{I} \longrightarrow \mathbb{R}$ converges uniformly to the zero function if

$$\left\{ \begin{array}{l} \textbf{for every } \varepsilon > 0 \textbf{ there exists a positive integer } N_\varepsilon \textbf{ such that} \\ n \geq N_\varepsilon \Longrightarrow |f_n(x)| < \varepsilon \textbf{ for every } x \in \mathcal{I} \end{array} \right.$$

This means that if we sketch the graphs of the functions $f_n(x)$ then beyond the stage $n \geq N_\varepsilon$ the graphs of all the functions $f_n(x)$ will be trapped inside the band formed by the lines $y = \varepsilon$ and $y = -\varepsilon$



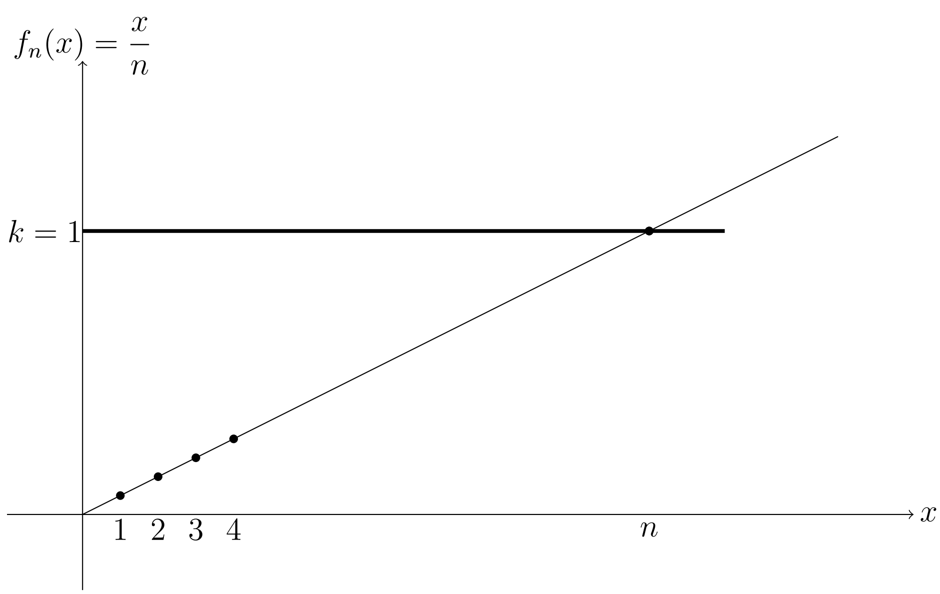
The graphs of all the functions $f_n(x)$ (for all n beyond a certain stage), must be trapped in the shaded region, and this must be possible for every $\varepsilon > 0$

Remark 1.8.4 The above observation tells us that the sequence f_n will fail to converge uniformly on $\mathcal{I}$, to the zero function, if there exists an $\varepsilon > 0$ for which we cannot trap the graph of all the functions $f_n(x)$ (beyond a certain stage) in the ε band around the zero function.
This means that the sequence f_n will fail to converge uniformly on $\mathcal{I}$ to the zero function if there exists an $\varepsilon > 0$ such that for every positive integer N we can find a positive integer $n > N$ and a point $x_n \in \mathcal{I}$ such that $|f_n(x_n)| \geq \varepsilon$.
This means that the sequence f_n will fail to converge uniformly on $\mathcal{I}$ to the zero function if there exists an $\varepsilon > 0$ such that the graph of $f_n(x)$ crosses the $+\varepsilon$ or the $-\varepsilon$ barrier infinitely often.
A simple situation where this happens is when the graph of all the functions, beyond a certain stage in the sequence, cross the ε barrier. This means that the sequence f_n will fail to converge uniformly on $\mathcal{I}$ to the zero function if there exists an $\varepsilon > 0$ and a positive integer N such that, for all $n \geq N$, the

graph of $f_n(x)$ crosses the $+\varepsilon$ or the $-\varepsilon$.
A simple situation where this happens is when there exist a positive real number $k > 0$ such that every $|f_n(x)|$ attains a value $\geq k$ at least at one point $x_n \in \mathcal{I}$. (In such a case we could choose the required ε to be $= \frac{k}{2}$ or any positive real number $< k$).

Example 1.8.10 Consider the sequence of 1.8.2, namely, $f_n : \mathbb{N} \longrightarrow \mathbb{R}$ defined as

$$f_n(x) = \frac{x}{n}$$



It is easy to see that the (discrete) graph of the function $f_n(x)$ consists of those points, on the straight line passing through the origin and with slope $\frac{1}{n}$, corresponding to positive integral values of x and therefore crosses the ε barrier
Now we see that if we take $k = 1$ then for every n we have $x_n = n$ such that $f_n(x_n) = 1$ and hence by the above Remark 1.8.4 since all f_n cross the $\varepsilon = \frac{1}{2}$ barrier, this sequence of real valued functions cannot converge uniformly on $\mathbb{N}$ to the zero function.

Example 1.8.11 Let $\mathcal{I} = [0, \infty)$. Consider the sequence of real valued functions $\{f_n\}_{n \in \mathbb{N}}$ defined on $\mathcal{I}$ as follows:

$$f_n(x) = \frac{x^2 + x + 1}{n^2 + 1}$$

It is easy to see that this sequence converges pointwise to the zero function on $\mathcal{I}$.
We shall show that this sequence does not converge to 0 uniformly. We shall use the Remark 1.8.4 for this purpose. We show the following:
Let $\varepsilon = 1$. We shall show that for every $n \in \mathbb{N}$ we can find an $x_n \in \mathcal{I}$ such that $f_n(x_n) \geq 1$. We have for any $x_n \in \mathcal{I}$,

$$\begin{aligned} |f_n(x_n| &= \frac{|x_n^2 + x_n + 1|}{n^2 + 1} \\ &\geq \frac{x_n}{n^2 + 1} \text{ (since } x_n \text{ is nonnegative in } \mathcal{I}) \end{aligned}$$

Hence if we choose $x_n = n^2 + 1$, then $x_n \in \mathcal{I}$ and $f_n(x_n) \geq 1$. Thus for every n at least at one point $x_n \in \mathcal{I}$ the function $f_n(x)$ crosses the value 1. Hence the sequence does not converge uniformly to zero on $\mathcal{I}$

We shall next briefly mention other types of convergences. In all these types of convergences the key is the manner in which the error $|f_n - f|$ is quantified and controlled. In pointwise convergence we controlled the error at each point separately. In uniform convergence we tried to control the error at all points simultaneously. Next we shall look at notions of converge that arise by trying to control the average error (the average being defined by various types of integrals).
We shall first introduce some terminologies:
A real valued function $F : \mathcal{I} \longrightarrow \mathbb{R}$ defined on an interval $\mathcal{I}$ is said to be in

$$L^1(\mathcal{I}) \quad \text{if} \quad \int_{\mathcal{I}} |F(x)| dx < \infty$$

$$L^2(\mathcal{I}) \quad \text{if} \quad \int_{\mathcal{I}} |F(x)|^2 dx < \infty$$

(and in general for $1 \leq p < \infty$)

$$L^p(\mathcal{I}) \quad \text{if} \quad \int_{\mathcal{I}} |F(x)|^p dx < \infty$$

Consider a sequence of real valued functions $\{f_n\}_{n\in\mathbb{N}}$ defined on an interval $\mathcal{I}$ in $\mathbb{R}$. Let f be a real valued function defined on $\mathcal{I}$. Then we have the following notions of convergence:

1. **L^1 convergence**

   If f_n and f are all in $L^1(\mathcal{I})$ and
   $$\lim_{n\to\infty} \int_{\mathcal{I}} |f_n(x) - f(x)| dx = 0$$
   we say that the sequence f_n converges to f in $L^1(\mathcal{I})$. We denote this convergence as
   $$f_n \xrightarrow{L^1(\mathcal{I})} f$$

2. **L^2 convergence**

   If f_n and f are all in $L^2(\mathcal{I})$ and
   $$\lim_{n\to\infty} \int_{\mathcal{I}} |f_n(x) - f(x)|^2 dx = 0$$
   we say that the sequence f_n converges to f in $L^2(\mathcal{I})$. We see that we can also write this as
   $$\lim_{n\to\infty} \sqrt{\int_{\mathcal{I}} |f_n(x) - f(x)|^2 dx} = 0$$
   We denote this convergence as
   $$f_n \xrightarrow{L^2(\mathcal{I})} f$$
   (This is also called RMS (Root Mean Square) Convergence)

3. **L^p convergence $(1 \leq p < \infty)$**

   If f_n and f are all in $L^p(\mathcal{I})$ and
   $$\lim_{n\to\infty} \int_{\mathcal{I}} |f_n(x) - f(x)|^p dx = 0$$
   we say that the sequence f_n converges to f in $L^p(\mathcal{I})$. We see that we can also write this as
   $$\lim_{n\to\infty} \left\{ \int_{\mathcal{I}} |f_n(x) - f(x)|^p dx \right\}^{\frac{1}{p}} = 0$$

We denote this convergence as

$$f_n \xrightarrow{L^p(\mathcal{I})} f$$

(This is also called p-mean Convergence)