

ECE 172A: Introduction to Image Processing

Analog Images: Part I

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Outline

- Images as Functions
 - Vector-space formulation
 - Two-Dimensional **Systems**
- 2D Fourier Transform
 - Properties
 - Dirac Impulse, etc.
- Characterization of **LSI** Systems
 - Multidimensional **Convolution**
 - Modeling of Optical Systems
 - Examples of **Impulse Responses**

Images as Functions

- Analog = Continuously-Defined Image Representation
 - Images are functions of **two** real variables
- Vector-Space Formulation
 - All images are “points” in a vector space
- Vector Space of Finite-Energy Images
 - Mathematical framework for image representations
- Two-Dimensional Systems
- Linear, Shift-Invariant (LSI) Systems
 - Fundamental tool to “process” images

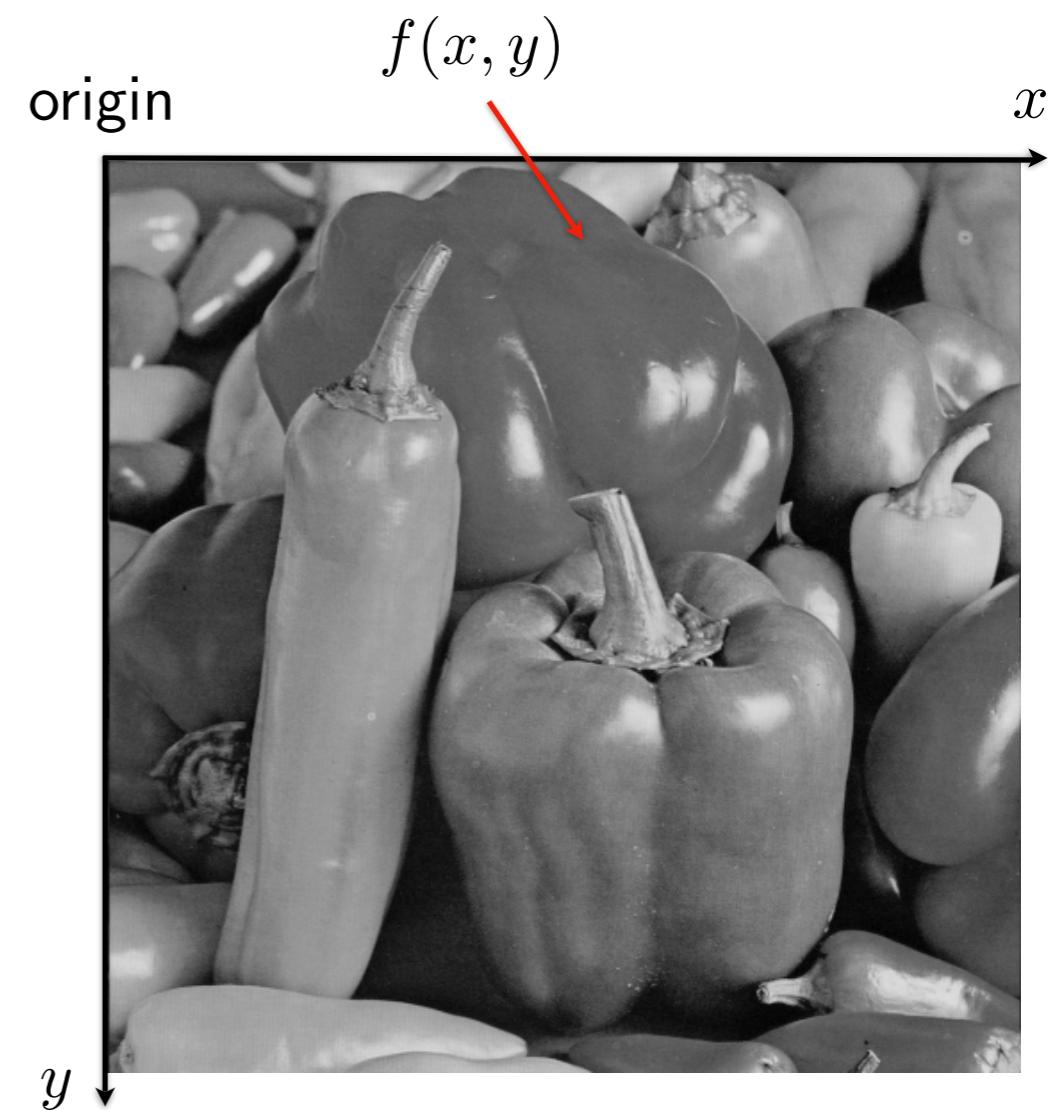
Analog Image Representation

Analog image

2D light intensity function: $f(x, y)$

- (x, y) are the spatial coordinates
- The output $f(x, y)$ is the **brightness** (or grayscale level) at (x, y)

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$



Vector-Space Formulation

What is a vector space?

Definition: A vector space is a set \mathcal{H} where, for every $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$, we have that

- **Associativity:** $f + (g + h) = (f + g) + h$
- **Commutativity:** $f + g = g + f$
- **Identity:** There exists $0 \in \mathcal{H}$ such that $f + 0 = f$
- **Inverse:** There exists $-f \in \mathcal{H}$ such that $f + (-f) = 0$
- **Compatibility With Scalar Multiplication:** $\alpha(\beta f) = (\alpha\beta)f$
- **Multiplication With Scalar Identity** $1f = f$ for $1 \in \mathbb{R}$
- **Distributivity I:** $\alpha(f + g) = \alpha f + \alpha g$
- **Distributivity II:** $(\alpha + \beta)f = \alpha f + \beta f$

Vector Space of Images

Do images (functions that map $\mathbb{R}^2 \rightarrow \mathbb{R}$) form a vector space?

- **Associativity:** $f + (g + h) = (f + g) + h$ ✓
- **Commutativity:** $f + g = g + f$ ✓ zero(x, y) = 0 for all $(x, y) \in \mathbb{R}^2$
- **Identity:** There exists $0 \in \mathcal{H}$ such that $f + 0 = f$ ✓
- **Inverse:** There exists $-f \in \mathcal{H}$ such that $f + (-f) = 0$ ✓
- **Compatibility With Scalar Multiplication:** $\alpha(\beta f) = (\alpha\beta)f$ ✓
- **Multiplication With Scalar Identity** $1f = f$ for $1 \in \mathbb{R}$ ✓
- **Distributivity I:** $\alpha(f + g) = \alpha f + \alpha g$ ✓
- **Distributivity II:** $(\alpha + \beta)f = \alpha f + \beta f$ ✓

Yes, images form a vector space

Vector Space of Finite-Energy Images

Definition: The **energy** of an image $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 dx dy$$

Definition: The **vector space** of finite-energy images is denoted $L^2(\mathbb{R}^2)$

$f \in L^2(\mathbb{R}^2)$ if and only if its energy is $< \infty$

$$\|f\|_{L^2}^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 dx dy$$

“squared L^2 -norm of f or energy of f ”

measures the “size” of f

Recall: Given a vector $\mathbf{x} \in \mathbb{R}^N$

$$\|\mathbf{x}\|_2^2 = \sum_{n=1}^N |x_n|^2$$

Inner Product of Finite-Energy Images

Recall: Given vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, their inner product is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = \sum_{n=1}^N x_n y_n$$

Definition: The **inner product** of $f, g \in L^2(\mathbb{R}^2)$ is

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) g^*(x, y) dx dy$$

conjugate if complex valued

Observation: The norm is **induced by** the inner product

$$\|f\|_{L^2}^2 = \langle f, f \rangle$$

$$L^2(\mathbb{R}^2) = \{f(x, y) : \|f\|_{L^2}^2 = \langle f, f \rangle < \infty\}$$

Examples of Finite-Energy Images

- 2D Gaussian

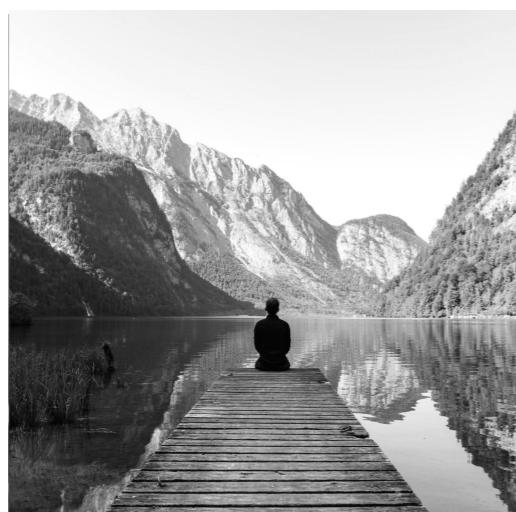
$$g(x, y) = \frac{1}{2\pi} \exp\left(-\frac{x^2 + y^2}{2}\right)$$

$$g \in L^2(\mathbb{R}^2)$$



- Finite support $\Omega \subset \mathbb{R}^2$ and bounded images

$$\begin{cases} f(x, y) = 0, & \text{for all } (x, y) \notin \Omega \\ |f(x, y)| < C, & \text{for all } (x, y) \in \mathbb{R}^2 \end{cases}$$



$$\Omega = [0, 1] \times [0, 1] = [0, 1]^2$$

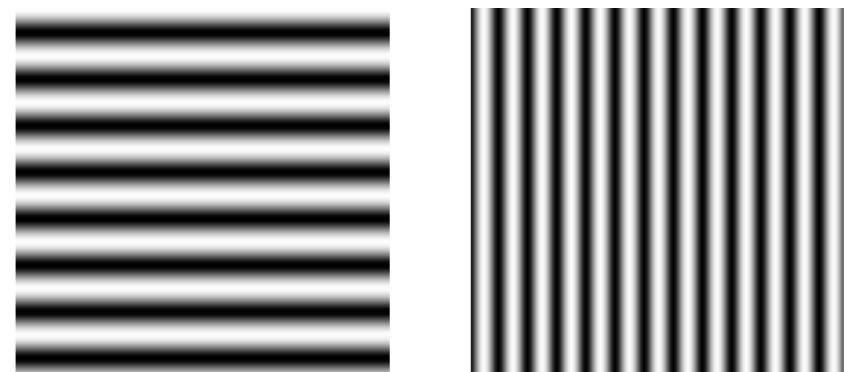
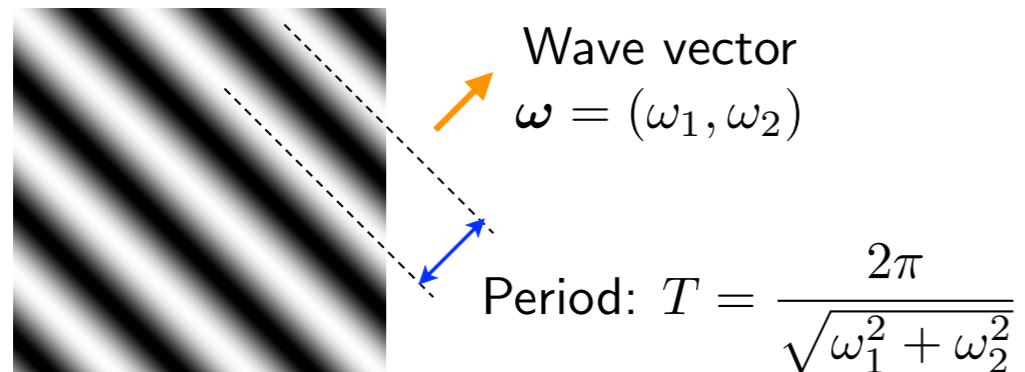
Exercise: Show that $f \in L^2(\mathbb{R}^2)$

$$\begin{aligned} \|f\|_{L^2}^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 dx dy \\ &= \iint_{\Omega} |f(x, y)|^2 dx dy \\ &< \iint_{\Omega} C^2 dx dy \\ &= C^2 \text{vol}(\Omega) < \infty \end{aligned}$$

Plane Waves

- Sinusoidal gratings

$$s(x, y) = A \cos(\omega_1 x + \omega_2 y + \phi)$$



Does s have finite energy?

No, $s \notin L^2(\mathbb{R}^2)$

However, $s(x, y) \cdot w(x, y) \in L^2(\mathbb{R}^2)$

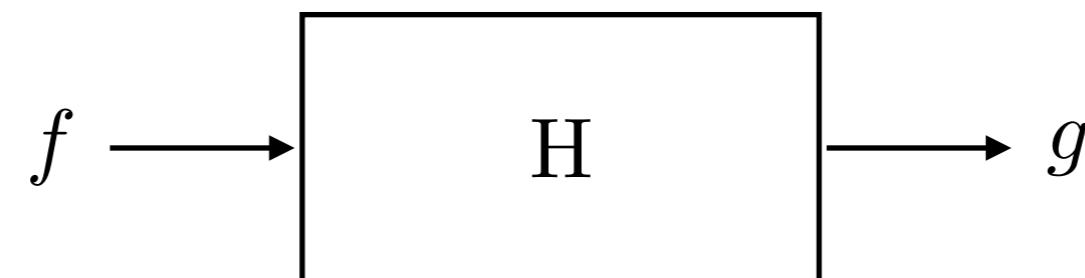
$w(x, y)$ is a finite-support and bounded **window function**

Example:

$$\begin{cases} w(x, y) = 1, & (x, y) \in [0, 1]^2 \\ w(x, y) = 0, & \text{else} \end{cases}$$

Two-Dimensional Systems

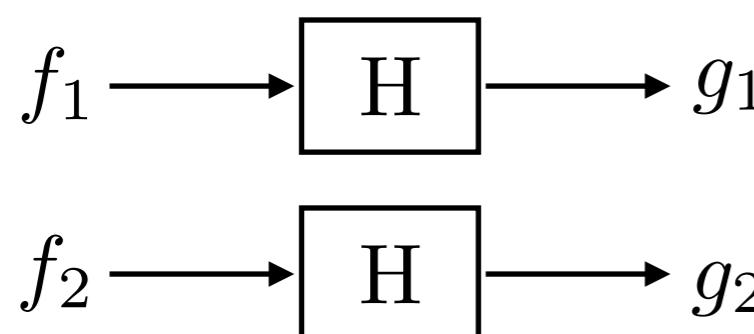
- Mapping from one image to another



$$H : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$$

$$g = H\{f\}$$

- The most important systems are **linear systems**



$$\forall \alpha, \beta \in \mathbb{R}$$
$$\alpha f_1 + \beta f_2 \rightarrow H \rightarrow \alpha g_1 + \beta g_2$$

$$H\{\alpha f_1 + \beta f_2\} = \alpha H\{f_1\} + \beta H\{f_2\}$$

Linearity Practice

- (Partial) derivative operators are linear or nonlinear? **Linear**

$$H_1\{f\} = \frac{\partial f(x, y)}{\partial x} \quad \text{and} \quad H_2\{f\} = \frac{\partial f(x, y)}{\partial y}$$

- The following operator is linear or nonlinear? **Linear**

$$H_3\{f\}(x, y) = f(x^2 + x + 1, y - \sqrt{y})$$

- Geometric operators are linear or nonlinear? **Linear**

$$H_4\{f\}(x, y) = f(G_1(x, y), G_2(x, y))$$

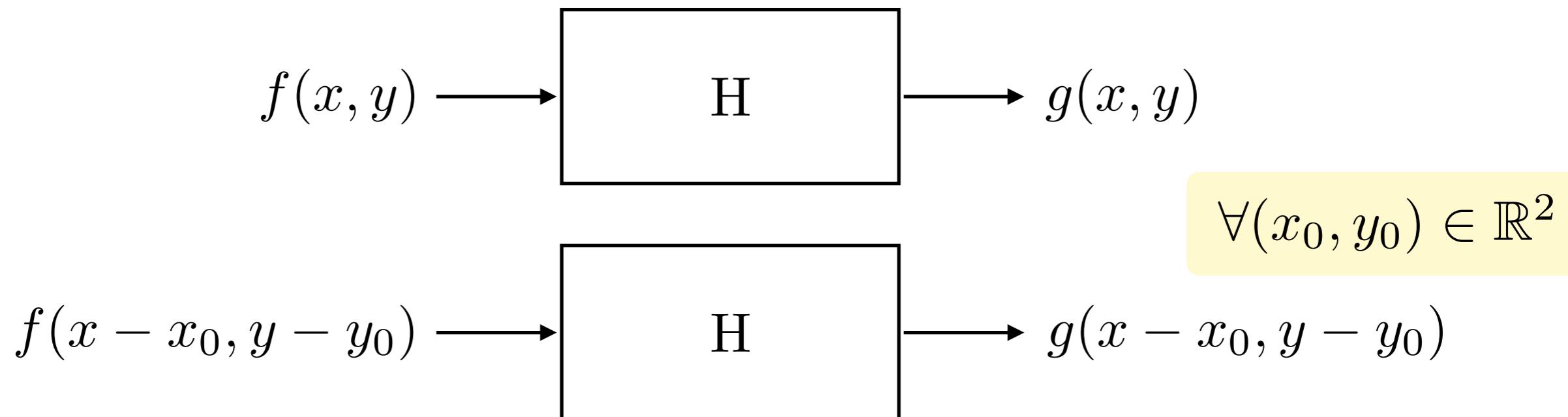
where $G_1(x, y)$ and $G_2(x, y)$ are arbitrary (nonlinear) transformations.

- The thresholding operator is linear or nonlinear? **Nonlinear**

$$H_5\{f\}(x, y) = \begin{cases} 1, & |f(x, y)| \geq T_0 \\ 0, & \text{else} \end{cases}$$

Linear, Shift-Invariant Systems (LSI)

Definition: A linear system H is **shift-invariant** if and only if shifted inputs correspond to shifted outputs.



$$H\{f(x - x_0, y - y_0)\} = H\{f\}(x - x_0, y - y_0)$$

- LSI systems model most physical imaging devices

LSI = realized by **convolution**: $H\{f\}(x, y) = (h * f)(x, y)$

“impulse response”



2D Fourier Transform

- Definition
- Separability
- Properties
- Dirac impulse
- Dirac related Fourier transforms
- Application: finding the orientation
- Importance of the phase

2D Fourier Transform: Definition

- 2D Fourier transform: $\hat{f}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j(\omega_1 x + \omega_2 y)} dx dy$
- Inverse Fourier transform: $f(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega_1, \omega_2) e^{j(\omega_1 x + \omega_2 y)} d\omega_1 d\omega_2$

Vector notation:

Spatial variables: $\mathbf{x} = (x, y) \in \mathbb{R}^2$

Frequency variables: $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$

$$\boldsymbol{\omega}^\top \mathbf{x} = \omega_1 x + \omega_2 y$$

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-j\boldsymbol{\omega}^\top \mathbf{x}} d\mathbf{x}$$

$$\uparrow \downarrow \mathcal{F}$$

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) e^{j\boldsymbol{\omega}^\top \mathbf{x}} d\boldsymbol{\omega}$$

Plancherel, Parseval, and Finite-Energy

- Fourier analysis on $L^2(\mathbb{R}^2)$ (Plancherel)

$$f \in L^2(\mathbb{R}^2) \text{ if and only if } \hat{f} \in L^2(\mathbb{R}^2)$$

- Parseval's formula for $f, g \in L^2(\mathbb{R}^2)$

$$\langle f, g \rangle = \frac{1}{(2\pi)^2} \langle \hat{f}, \hat{g} \rangle$$

- Plancherel's theorem for $f \in L^2(\mathbb{R}^2)$

$$\|f\|_{L^2}^2 = \frac{1}{(2\pi)^2} \|\hat{f}\|_{L^2}^2$$

What does this mean?

Fourier analysis is well-matched to finite-energy functions

Separability

- Separability of complex exponential: $e^{-j(\omega_1 x + \omega_2 y)} = e^{-j\omega_1 x} e^{-j\omega_2 y}$

2D Fourier transform = sequence of two 1D Fourier transforms

Fourier in x then y or Fourier in y then x

Exercise: Show that this is true.

1D Fourier transform in x : $\int_{-\infty}^{\infty} f(x, y) e^{-j\omega_1 x} dx = \hat{f}_y(\omega_1)$

1D Fourier transform in y : $\int_{-\infty}^{\infty} \hat{f}_y(\omega_1) e^{-j\omega_2 y} dy$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j\omega_1 x} dx e^{-j\omega_2 y} dy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j\omega_1 x} e^{-j\omega_2 y} dx dy = \hat{f}(\omega_1, \omega_2)$$

2D Fourier transform inherits most properties from 1D Fourier transform!

Separability (cont'd)

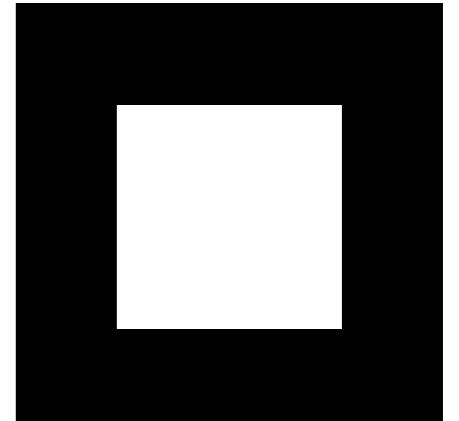
Definition: $f(x, y)$ is called separable if $f(x, y) = f_1(x)f_2(y)$ for some $f_1(x)$ and $f_2(y)$.

Exercise: For separable functions, show that $\hat{f}(\omega_1, \omega_2) = \hat{f}_1(\omega_1)\hat{f}_2(\omega_2)$.

What is an example of a separable function?

$$f(x, y) = \begin{cases} 1, & \text{if } (x, y) \in [0, 1]^2 \\ 0, & \text{else} \end{cases}$$

“box” or “rect”
function



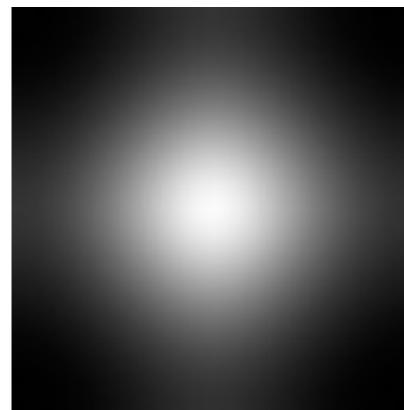
$$f_1(x) = \begin{cases} 1, & \text{if } x \in [0, 1] \\ 0, & \text{else} \end{cases}$$

$$f(x, y) = f_1(x)f_1(y)$$

Separability (cont'd)

- 2D Gaussian

$$g(x, y) = \exp\left(-\frac{x^2 + y^2}{2}\right)$$
$$= e^{-x^2/2}e^{-y^2/2}$$



$$\implies \hat{g}(\omega_1, \omega_2) = \hat{f}(\omega_1)\hat{f}(\omega_2) \quad \text{where } f(x) = e^{-x^2/2}$$



↑
↓ \mathcal{F}

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-j\omega x} dx = \sqrt{2\pi}e^{-\omega^2/2}$$

$$= \sqrt{2\pi}e^{-\omega_1^2/2} \cdot \sqrt{2\pi}e^{-\omega_2^2/2} = 2\pi \exp\left(-\frac{\omega_1^2 + \omega_2^2}{2}\right)$$

Fourier transform of a Gaussian is a Gaussian (just like 1D)

Fourier Properties

- Duality:

$$\hat{f}(\mathbf{x}) \xleftrightarrow{\mathcal{F}} (2\pi)^2 f(-\boldsymbol{\omega})$$

- Symmetry:

$$f(\mathbf{x}) \text{ real} \Leftrightarrow \hat{f}^*(\boldsymbol{\omega}) = \hat{f}(-\boldsymbol{\omega})$$

- Energy-Preservation:

$$\|f\|_{L_2}^2 = (2\pi)^{-2} \|\hat{f}\|_{L_2}^2$$

- Shift:

$$f(\mathbf{x} - \mathbf{x}_0) \xleftrightarrow{\mathcal{F}} e^{-j\boldsymbol{\omega}^\top \mathbf{x}_0} \hat{f}(\boldsymbol{\omega})$$

- Modulation:

$$e^{j\boldsymbol{\omega}_0^\top \mathbf{x}} f(\mathbf{x}) \xleftrightarrow{\mathcal{F}} \hat{f}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$$

- Scaling:

$$f(\mathbf{x}/\alpha) \xleftrightarrow{\mathcal{F}} |\alpha|^2 \hat{f}(\alpha \mathbf{x})$$

- Affine Transformation:

$$f(\mathbf{A}\mathbf{x}) \xleftrightarrow{\mathcal{F}} |\det \mathbf{A}|^{-1} \hat{f}((\mathbf{A}^{-1})^\top \boldsymbol{\omega})$$

- Differentiation:

$$\frac{\partial^n f(\mathbf{x})}{\partial x^n} \xleftrightarrow{\mathcal{F}} (j\omega_1)^n \hat{f}(\boldsymbol{\omega})$$

$$\frac{\partial^n f(\mathbf{x})}{\partial y^n} \xleftrightarrow{\mathcal{F}} (j\omega_2)^n \hat{f}(\boldsymbol{\omega})$$

- Moments: $\mu_f^{m,n} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^m y^n f(x, y) dx dy = j^{m+n} \left. \frac{\partial^{m+n} \hat{f}(\boldsymbol{\omega})}{\partial \omega_1^m \partial \omega_2^n} \right|_{\omega_1=0, \omega_2=0}$

In particular, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{x}) dx = \hat{f}(\mathbf{0}) = \hat{f}(0, 0)$

Dirac Impulse

- Recall the 1D Dirac impulse $\delta(x)$: $\langle f, \delta \rangle = \int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$
- Properties:

Normalized integral: $\int_{-\infty}^{\infty} \delta(x) dx = 1$

Fourier transform: $\delta(x) \xleftrightarrow{\mathcal{F}} 1$

Convolution: $(g * \delta)(x) = \int_{-\infty}^{\infty} \delta(u) g(x - u) du = g(x)$

Exercise: Prove these three properties using the definition.

Normalized integral: $f(x) = 1$

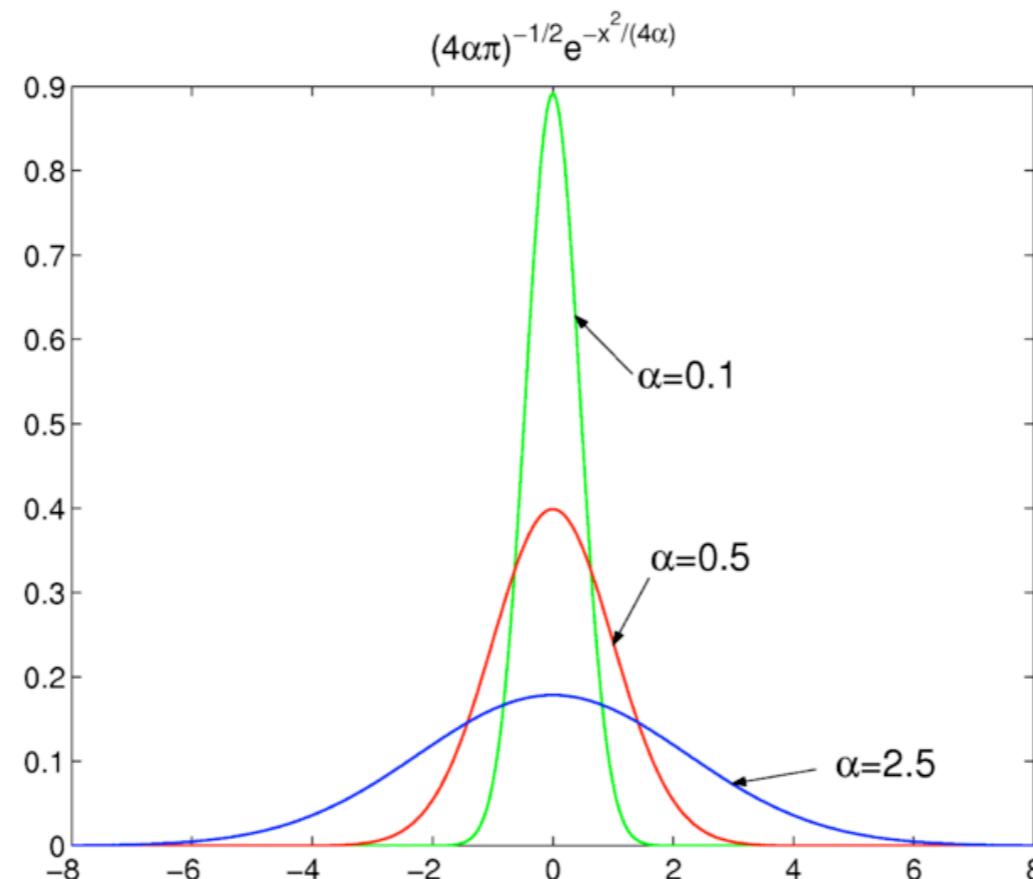
Fourier transform: $f(x) = e^{j\omega x}$

Convolution: $f(u) = g(x - u)$

Explicit Construction of the Dirac Impulse

- Consider any window function $\varphi(x)$ such that $\int_{-\infty}^{\infty} \varphi(x) dx = 1$
- Observe that $\int_{-\infty}^{\infty} \frac{1}{|\alpha|} \varphi\left(\frac{x}{\alpha}\right) dx = 1$ “integral-preserving dilation/contraction”
- $\delta(x) = \lim_{\alpha \rightarrow 0} \left(\frac{1}{|\alpha|} \varphi\left(\frac{x}{\alpha}\right) \right)$

e.g., φ is a Gaussian, rectangle, triangle, etc.



2D Dirac Impulse

- A reasonable definition: $\langle f, \delta \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x, y) dx = f(0, 0)$

What could give us this?

$$\delta(x, y) = \delta(x)\delta(y) \xleftrightarrow{\mathcal{F}} 1 \cdot 1 = 1$$

The Dirac impulse is **separable!**

Exercise: Prove that this is the 2D Dirac impulse.

- Properties:

- Normalized integral: $\int_{\mathbb{R}^2} \delta(\mathbf{x}) d\mathbf{x} = 1$
- Fourier transform: $\delta(\mathbf{x}) \xleftrightarrow{\mathcal{F}} 1$
- Multiplication: $f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}_0)\delta(\mathbf{x} - \mathbf{x}_0)$
- Sampling: $\langle f(\mathbf{x}), \delta(\mathbf{x} - \mathbf{x}_0) \rangle = \int_{\mathbb{R}^2} f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0) d\mathbf{x} = f(\mathbf{x}_0)$
- Convolution: $(f * \delta)(\mathbf{x}) = f(\mathbf{x})$
- Scaling: $\delta(\mathbf{x}/\alpha) = |\alpha|^2 \delta(\mathbf{x})$

These properties are deduced from the 1D Dirac properties.

Dirac-Related Fourier Transforms

- Constant

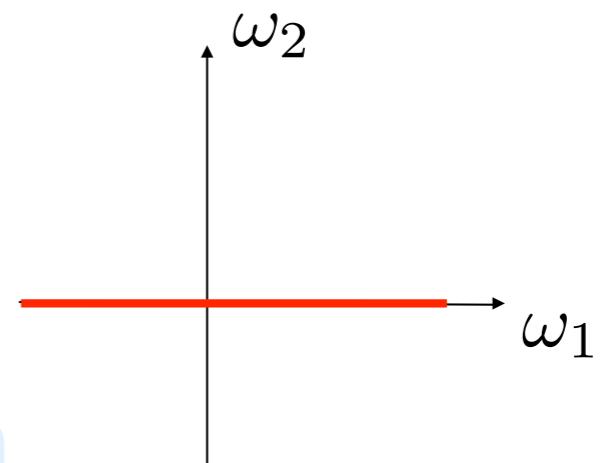
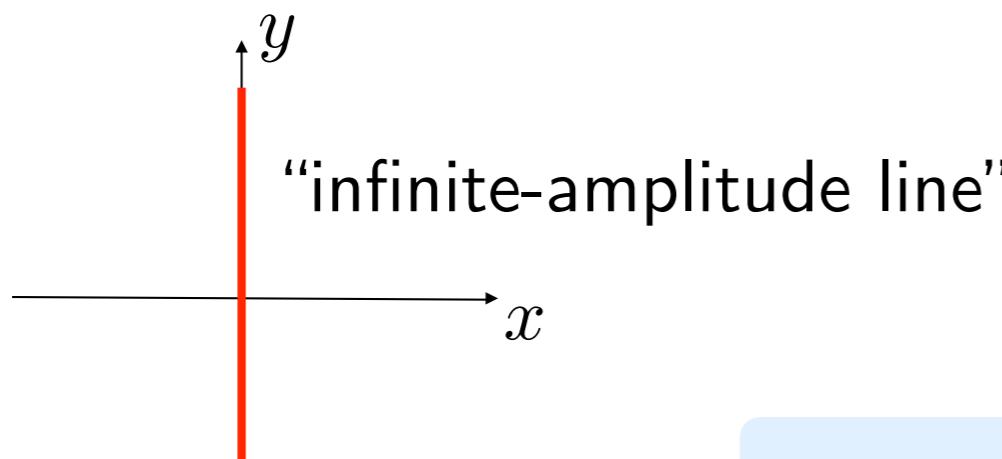
One-dimensional: $1 \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} e^{-j\omega x} dx = ???$

$$= \lim_{A \rightarrow \infty} \int_{-A}^A e^{-j\omega x} dx = 2\pi \delta(\omega) \quad (\text{or by duality})$$

Two-dimensional: $1 \xleftrightarrow{\mathcal{F}} (2\pi)^2 \delta(\omega) = (2\pi)^2 \delta(\omega_1, \omega_2)$

- Dirac line (or “ideal” line)

$$f(x, y) = \delta(x) \cdot 1 = f_1(x)f_2(y) \xleftrightarrow{\mathcal{F}} \hat{f}(\omega_1, \omega_2) = \hat{f}_1(\omega_1)\hat{f}_2(\omega_2) = 1 \cdot 2\pi\delta(\omega_2)$$



What does this mean?

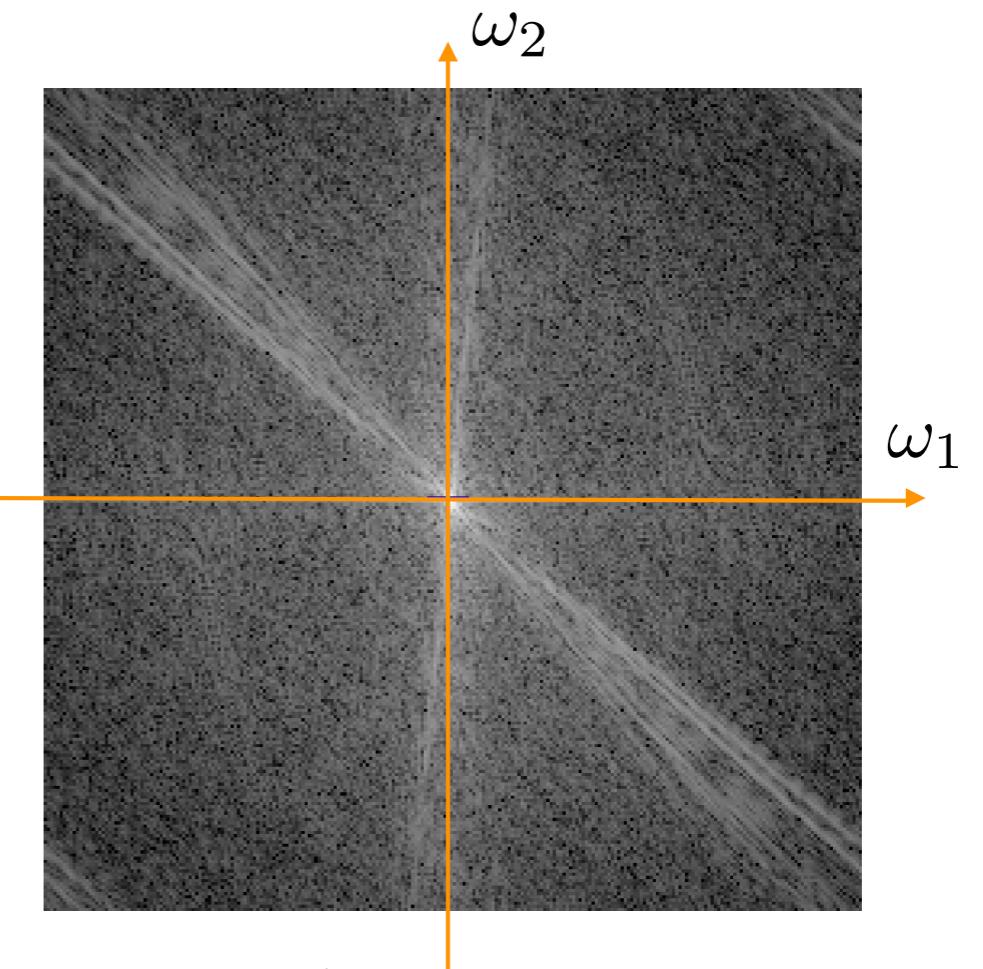
Example

Spatial Domain



$$f(x, y)$$

Fourier Domain



$$\hat{f}(\omega_1, \omega_2)$$

What are these two sets of lines?

More-Realistic Line Model

- Rectangular shape

$$f(x, y) = \text{rect}(x/a) \text{rect}(y/A) \longleftrightarrow |a|\text{sinc}\left(\frac{a\omega_1}{2\pi}\right) |A|\text{sinc}\left(\frac{A\omega_2}{2\pi}\right)$$



Reminder:

$$\text{rect}(x) = \begin{cases} 1, & \text{if } x \in [-1/2, 1/2] \\ 0, & \text{else} \end{cases} \longleftrightarrow \text{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2}$$