# Orthogonal Filter Banks

Ex: Haar wavelet

$$H_0(z) = G_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$$

$$H_1(z) = \frac{1}{\sqrt{2}}(1-z^{-1})$$

$$G_1(z) = \frac{1}{\sqrt{2}}(-1+z^{-1})$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$$= \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{H_p(z^2):\ \text{orthogonal matrix}} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

Obs: • $H_p(z^2)$ is independent of $z$

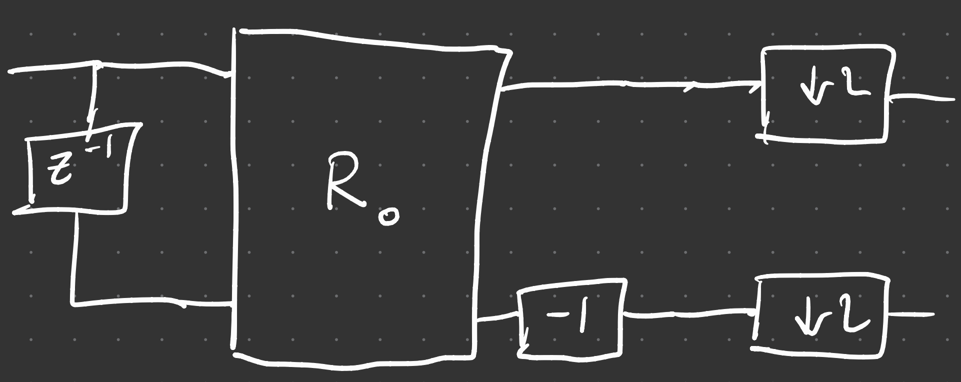
• $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is a rotation matrix

$$\begin{cases} G_0(z) = z^{-1} H_0(z^{-1}) \\ G_1(z) = z^{-1} H_1(z^{-1}) \end{cases}$$

If $H_0$ is min-phase, then $G_0$ is max-phase and vice-versa.

If $H_0(z)$ has all its zeros inside the unit circle, then $H_0(z^{-1})$ will flip them all to be outside, and vice-versa.

In general, for 1st-order systems, the analysis bank takes the form




$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

rotation matrix

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

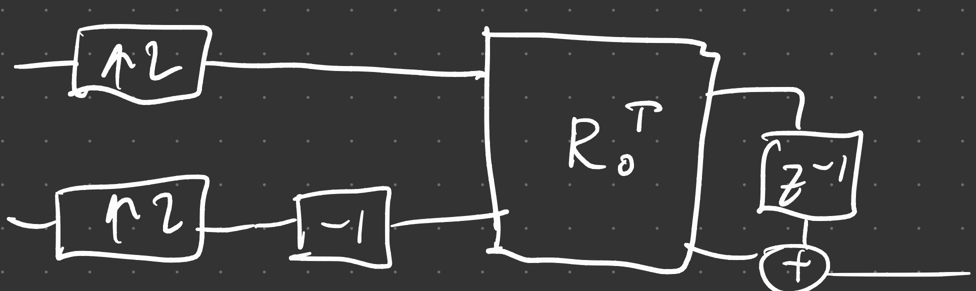
$$= \begin{bmatrix} c_0 & s_0 \\ s_0 & -c_0 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} c_0 + z^{-1} s_0 \\ s_0 - z^{-1} c_0 \end{bmatrix}$$

Obs: In the Haar case, $c_0 = \frac{1}{\sqrt{2}} \Rightarrow \theta_0 = \frac{\pi}{4}$.

Obs: $H_1(z) = -z^{-1} H_0(-z^{-1})$

In general, for 1st order systems, the synthesis bank takes the form



$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \begin{bmatrix} c_0 & -s_0 \\ s_0 & c_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \begin{bmatrix} c_0 & s_0 \\ s_0 & -c_0 \end{bmatrix}$$

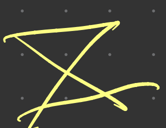
$$= \begin{bmatrix} s_0 + z^{-1} c_0 & -c_0 + z^{-1} s_0 \end{bmatrix}$$

<u>Obs:</u>
$$\begin{cases} G_0(z) = z^{-1} H_0(z^{-1}) \\ G_1(z) = z^{-1} H_1(z^{-1}) \end{cases}$$

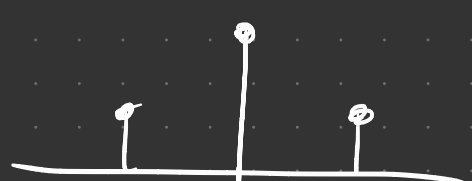
<u>Obs:</u> $P_0(z) = G_0(z) H_0(z)$ is a half-band filter.

$$= (s_0 + z^{-1} c_0)(c_0 + z^{-1} s_0)$$

lattice structure

$R_0$ 

$$= s_0 c_0 + z^{-1} + s_0 c_0 z^{-2}$$



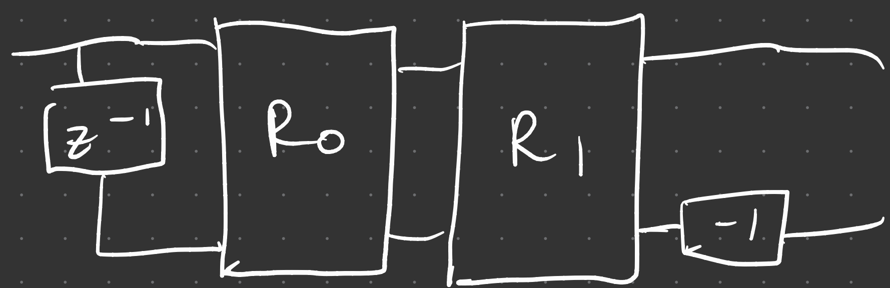
type I linear phase and half band

<u>Remark:</u> This is the general form of 1st order orthogonal filter bank.

<u>Obs:</u> From $H_0$ you know everything ($H_1, G_0, G_1$).

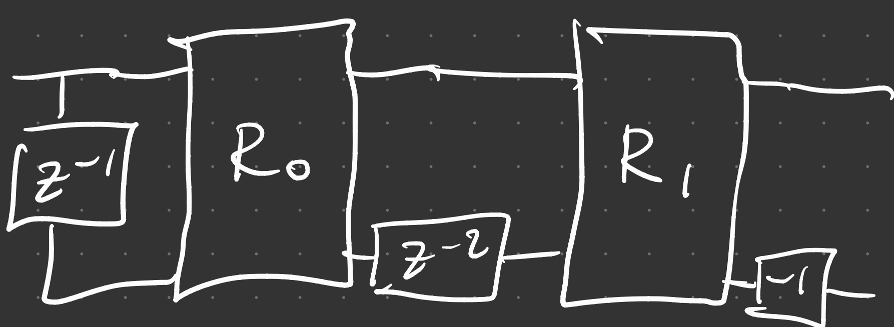
# Higher-order Systems

**Q:** Is this a higher-order filter bank?




**A:** No. A cascade of two rotations is still a rotation.

**Sol$^n$:** Add some delays.




$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-2} \end{bmatrix} \begin{bmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

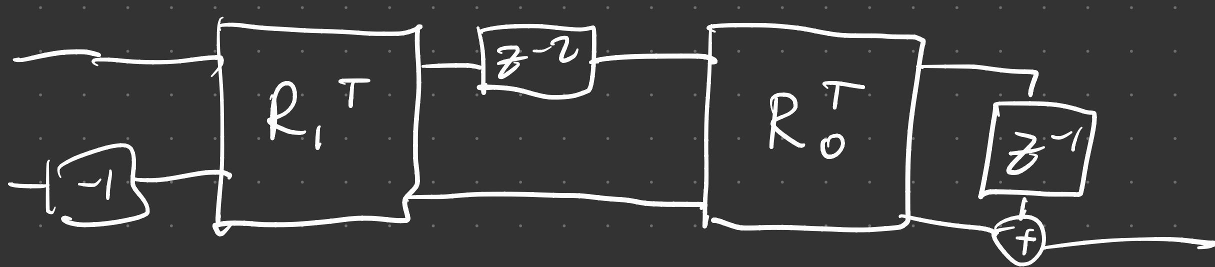
$$= \begin{bmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-2} \end{bmatrix} \begin{bmatrix} c_0 + z^{-1} s_0 \\ -s_0 + z^{-1} c_0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{bmatrix} \begin{bmatrix} c_0 + z^{-1} s_0 \\ -z^{-2} s_0 + z^{-3} c_0 \end{bmatrix}$$

$$= \begin{bmatrix} c_0 c_1 + z^{-1} s_0 c_1 - z^{-2} s_0 s_1 + z^{-3} c_0 s_1 \\ c_0 s_1 + z^{-1} s_0 s_1 + z^{-2} s_0 c_1 - z^{-3} c_0 c_1 \end{bmatrix}$$

Obs: $H_1(z) = -z^{-3} H_0(-z^{-1})$

For the synthesis bank, we have




$$[G_0(z)\ \ G_1(z)] = [z^{-1}\ \ 1] \begin{bmatrix} c_0 & -s_0 \\ s_0 & c_0 \end{bmatrix} \begin{bmatrix} z^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= [s_0 + c_0 z^{-1}\ \ \ c_0 - s_0 z^{-1}] \begin{bmatrix} z^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{bmatrix}$$

$$= [s_0 + c_0 z^{-1}\ \ \ c_0 - s_0 z^{-1}] \begin{bmatrix} z^{-2} c_1 & z^{-2} s_1 \\ s_1 & -c_1 \end{bmatrix}$$

$$= \begin{bmatrix} c_0 s_1 - z^{-1} s_0 s_1 + z^{-2} s_0 c_1 + z^{-3} c_0 c_1 \\ -c_0 c_1 + z^{-1} s_0 c_1 + z^{-2} s_0 s_1 + z^{-3} c_0 s_1 \end{bmatrix}$$

<u>Obs:</u> $\begin{cases} G_0(z) = z^{-3} H_0(z^{-1}) \\ G_1(z) = z^{-3} H_1(z^{-1}) \end{cases}$

<u>Remark</u>: For orthogonal filter banks, you only need to design $H_0$.

In general, you can get higher-order systems from lower-order systems by cascading more $R_K$ blocks with delays.

## Summary

$$\Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}$$

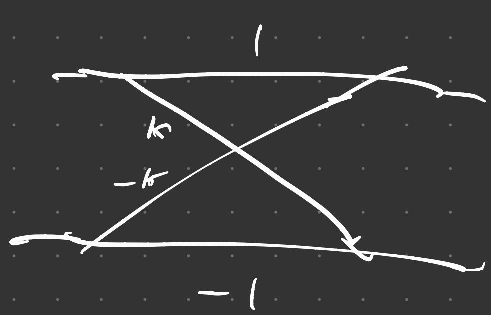
$$H_p(z) = \Lambda(-1) R_k \Lambda(z) R_{k-1} \Lambda(z) \cdots \Lambda(z) R_0,$$

where

$$R_\ell = \begin{bmatrix} \cos\theta_\ell & \sin\theta_\ell \\ -\sin\theta_\ell & \cos\theta_\ell \end{bmatrix} \quad \text{is a rot. matrix.}$$

$(K+1)$ parameters vs. $(4K+2)$ for a general factorization.

Lattice structure reduces the # of parameters.




$$R_\ell = \cos\theta \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}$$

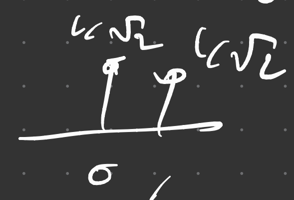
$$k = \frac{\sin\theta}{\cos\theta}$$

$\underline{Q:}$ Given $H_0$, how do we choose $(H_1, G_0, G_1)$ so that the system is orthogonal?

$\underline{A:}$
$$\left.\begin{cases} G_0(z) = z^{-L} H_0(z^{-1}) \\ H_1(z) = -z^{-L} H_0(-z^{-1}) \\ G_1(z) = z^{-L} H_1(z^{-1}) \end{cases}\right\}$$
$L$ is the order of $H_0$. $L$ is odd

__Remark!__ If these hold, you can show the lattice structure is sufficient.

__Exercise:__ Derive the Haar wavelet filters from $H_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$




__Sol$^n$:__ $L = 1$.

$$G_0(z) = z^{-1} H_0(z^{-1}) = \frac{z^{-1}}{\sqrt{2}}(1+z) = \frac{1}{\sqrt{2}}(1+z^{-1}) = H_0(z)$$

$$H_1(z) = -z^{-1} H_0(-z^{-1})$$
$$= -\frac{z^{-1}}{\sqrt{2}}(1-z^{-1}) = \frac{1}{\sqrt{2}}(1-z^{-1})$$

$$G_1(z) = z^{-1} H_1(z^{-1}) = \frac{z^{-1}}{\sqrt{2}}(1-z) = \frac{1}{\sqrt{2}}(-1+z^{-1})$$