

Last Time: MRA of $L^2(\mathbb{R})$ and Haar Wavelets

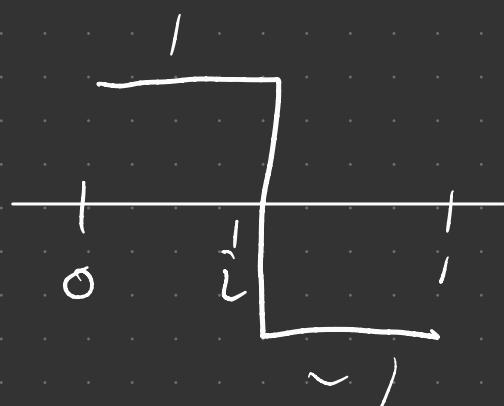
Haar scaling function:

$$\varphi(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{else} \end{cases}$$



Haar wavelet function:

$$\psi(t) = \begin{cases} 1, & 0 \leq t \leq \frac{1}{2} \\ -1, & \frac{1}{2} < t \leq 1 \\ 0, & \text{else} \end{cases}$$



Notation:

$$\varphi_{i,j,n}(t) = 2^{\frac{j}{2}} \varphi(2^i t - n)$$

translates and
dilates of the
scaling and wavelet
functions]

$$\psi_{i,j,n}(t) = 2^{\frac{j}{2}} \psi(2^i t - n)$$

Approximation and Wavelet Spaces:

$$V_i = \text{Span} \{ \varphi_{i,j,n} \}_{n \in \mathbb{Z}}$$

$$W_i = \text{Span} \{ \psi_{i,j,n} \}_{n \in \mathbb{Z}}$$

The collection $\{V_i\}_{i \in \mathbb{Z}}$ form a nested sequence of subspaces of $L^2(\mathbb{R})$

$$\{0\} \subset \dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots \subset L^2(\mathbb{R})$$

called a multiresolution analysis (MRA).

Recall: $V_0 \subset V_1$, $W_0 \subset V_1$

$$V_1 = V_0 \oplus W_0$$

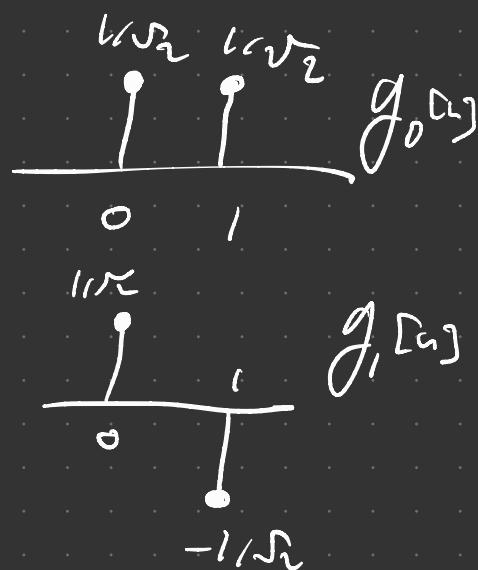
$\rightarrow W_0$ is the orthogonal complement of V_0 in V_1 .

$$\rightarrow W_0 = \{f \in V_1 : \langle f, g \rangle = 0, \forall g \in V_0\}$$

Two-Scale Equations:

$$\varphi(t) = \sum_{n \in \mathbb{Z}} g_0[n] \sqrt{2} \varphi(2t - n)$$

$$\psi(t) = \sum_{n \in \mathbb{Z}} g_1[n] \sqrt{2} \varphi(2t - n)$$



Obs: $g_0[n]$ & $g_1[n]$ are the Haar PB analysis filters.

Last Time:

$$V_1 = V_0 \oplus W_0$$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$

⋮
⋮

$$V_i = V_0 \oplus W_1 \oplus \cdots \oplus W_{i-1}$$

⋮
⋮

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{i=0}^{\infty} W_i \right)$$

Obs: $\{\varphi_{0,n}\}_{n \in \mathbb{Z}} \cup \bigcup_{i=0}^{\infty} \{\psi_{i,n}\}_{n \in \mathbb{Z}}$

V_0 W_i

is an ortho basis for $L^2(\mathbb{R})$.

Equivalently,

$$\{\varphi_{0,n}\}_{n \in \mathbb{Z}} \cup \{\varphi_{i,n}\}_{i \geq 0, n \in \mathbb{Z}}$$

is an ortho basis for $L^2(\mathbb{R})$.

Q: What does this mean for $f \in L^2(\mathbb{R})$?

A: f can be written as the sum of its projections onto $V_0, W_0, W_1, W_2, \dots$

$$f(t) = \text{Proj}_{V_0} f(t) + \sum_{i=0}^{\infty} \text{Proj}_{W_i} f(t).$$

$$\text{Proj}_{V_0} f(t) = \sum_{n \in \mathbb{Z}} \underbrace{\langle f, \varphi_{0,n} \rangle}_{a_0[n]} \varphi_{0,n}(t)$$

coarse approximation approx. coeffs.

$$\text{Proj}_{W_i} f(t) = \sum_{n \in \mathbb{Z}} \underbrace{\langle f, \psi_{i,n} \rangle}_{d_i[n]} \psi_{i,n}(t)$$

details detail coeffs.

$$f(t) = \sum_{n \in \mathbb{Z}} a_0[n] \varphi_{0,n}(t) + \sum_{i=1}^{\infty} \sum_{n \in \mathbb{Z}} d_i[n] \psi_{i,n}(t)$$

Synthesis Procedure

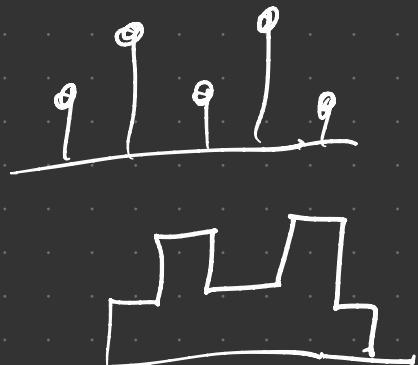
Analysis Procedure (Haar Wavelets)

Ex: finest resolution is $i=3$:

$$a_3[n] = \langle f, \varphi_{3,n} \rangle$$

$$f_3(t) = \sum_{n \in \mathbb{Z}} a_3[n] \varphi_{3,n}(t)$$

\checkmark_3



How do we get $a_2[n]$ & $d_2[n]$ from $a_3[n]$?

$$V_3 = V_2 \oplus W_2$$


↑ project onto these
subspaces

Project onto V_2

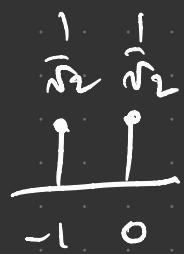
$$\text{Proj}_{V_2} f_3(t) = f_2(t) = \sum_{m \in \mathbb{Z}} \langle f_3, \varphi_{2,m} \rangle \varphi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \left\langle \sum_{n \in \mathbb{Z}} a_3[n] \varphi_{3,n}, \varphi_{2,m} \right\rangle \varphi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} a_3[n] \underbrace{\langle \varphi_{3,n}, \varphi_{2,m} \rangle}_{g_o[n-2m]} \varphi_{2,m}(t)$$

$g_o[n-2m]$ (last lecture)

$$f_2(t) = \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} a_3[n] g_o[n-2m] \right) \psi_{2,m}(t)$$



Define $h_o[k] = g_o[-k]$ time reversal

$$\Rightarrow a_2[m] = \sum_{n \in \mathbb{Z}} a_3[n] h_o[2m-n]$$

$$= (a_3 * h_o)[2m]$$

downsampling



Project onto W_2

$$\text{Proj}_{W_2} f_3(t) = g_2(t) = \sum_{m \in \mathbb{Z}} \langle f_3, \psi_{2,m} \rangle \psi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \left\langle \sum_{n \in \mathbb{Z}} a_3[n] \varphi_{3,n}, \psi_{2,m} \right\rangle \psi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \left[\sum_{n \in \mathbb{Z}} a_3[n] \langle \varphi_{3,n}, \psi_{2,m} \rangle \right] \psi_{2,m}(t)$$

$$\langle \varphi_{3,n}, \psi_{2,m} \rangle = \int_{-\infty}^{\infty} 2^{\frac{3}{2}} \varphi(2^3 t - n) 2 \psi(2^2 t - m) dt$$

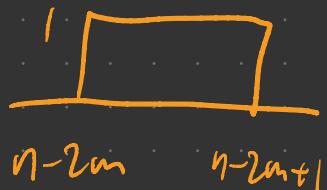
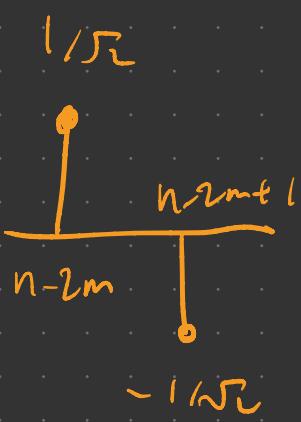
$d_2[m]$

$$\begin{cases} u = 2^3 t \\ du = 2^3 dt \end{cases}$$

$$= \frac{2^{\frac{5}{2}}}{2^3} \int_{-\infty}^{\infty} \varphi(u - n) \psi\left(\frac{u}{2} - m\right) dt$$

$\begin{cases} t = u - 2m \\ dt = du \end{cases}$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \varphi(t - (n - 2m)) \psi\left(\frac{t}{2}\right) dt$$



$$n-2m = 0 \Rightarrow S = 1$$

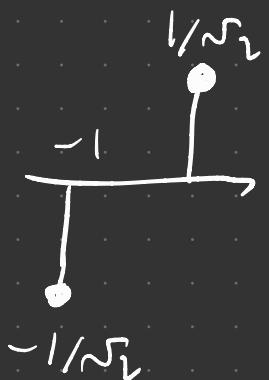
$$n-2m = 1 \Rightarrow S = -1$$

$$= g_1[n-2m]$$



$$g_2(t) = \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} a_3[n] g_1[n - 2m] \right) \psi_{2,m}(t)$$

$d_2[m]$



Define $h_1[n] = g_1[-n]$

$$\Rightarrow d_2[m] = \sum_{n \in \mathbb{Z}} a_3[n] h_1[2m - n]$$

downsampling

$$= (a_3 * h_1)[2m]$$



Obs: Given an approximation $a_i[n]$, we can get all other coefficients $a_k[n]$, $k \leq i$, and $d_l[n]$, $l \leq i$.

Extremely fast: Fast Wavelet Transform

Concrete implementation: DWT

General Wavelet Analysis Pipeline

$f(t)$: underlying analog signal

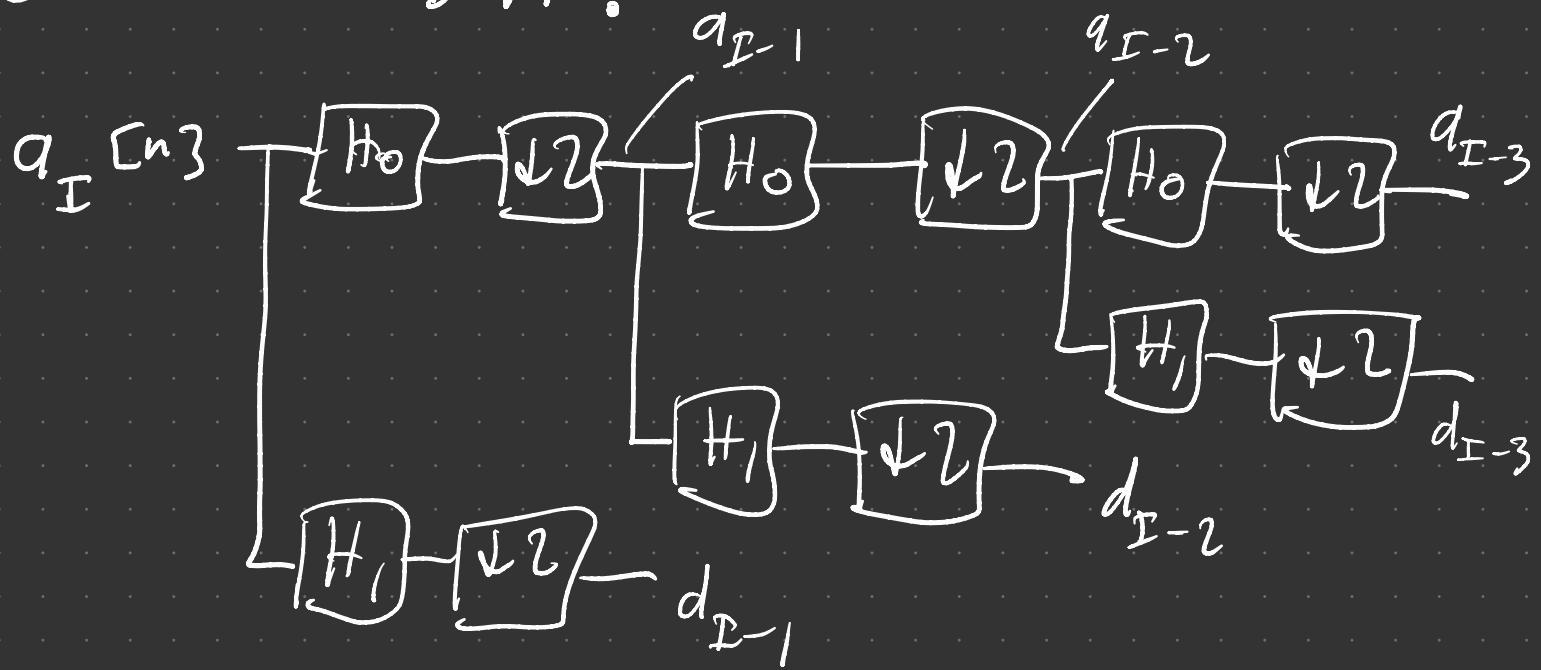
$a_I[n]$: discrete approximation at some high resolution I that you actually get to work with (e.g., image, audio, etc.)

$\varphi(t), \psi(t)$: scaling and wavelet function that gives rise to a MRA.

Fundamental assumption:

$$f(t) \approx \sum_{n \in \mathbb{Z}} a_I[n] \varphi_{I,n}(t)$$

① Perform a DWT:



- iterate as many times as you want
- filters come from scalings and wavelet functions

$$\bullet h_0[n] = g_0[-n], \quad g_0[n] = \langle e, e_{i,n} \rangle$$

$$\bullet h_1[n] = g_1[-n], \quad g_1[n] = \langle \psi, e_{i,n} \rangle$$

- ② Do some (nonlinear) processing on, e.g.,

$$d_{I-3}, d_{I-3}, d_{I-2}, d_{I-1}$$

→ For thresholding procedures people typically only threshold the detail coefficients

- ③ Perform an IDWT on the processed signals, resulting in $\hat{d}_I[n]$.

- ④ Processed analog signal:

$$\hat{f}(t) = \sum_{n \in \mathbb{Z}} \hat{d}_I[n] e_{I,n}(t)$$

Idea: "Think analog, act digital"

MRA

DWT

Obs: This whole pipeline is "blind" to the underlying MRA, ϵ , and ψ .

- We only need the filters h_0 & h_1 (or, equivalently g_0 & g_1).