Welcome to ECE 251C

Filter Banks and Wavelets

Course will primarily focus on:

1. Multirate Signal Processing
2. Filter Banks
3. Wavelets

Lectures: Theory

(uncollected)

Quizzes & Midterm: Based on lectures & HW

(open ended)

Project: Theory and/or applications

Filters / LTI Systems (Review)

Two kinds of filters:

1. Finite Impulse Responce (FIR)
2. Infinite Impulse Response (IIR)

<u>Q:</u> What is the difference?

<u>Q:</u> What is an impulse response?

<u>Systems:</u>

Linear
Time / Translation
Invariant

Sampled signal
$n \in \mathbb{Z}$

<u>Linear Systems:</u>

<u>Q:</u> What does it mean for a system to be <u>linear</u>?

A: If: $x_1[n] \rightarrow \boxed{H} \rightarrow y_1[n]$

$x_2[n] \rightarrow \boxed{H} \rightarrow y_2[n]$

Then:

$\forall \alpha, \beta \in \mathbb{R}$

$$\alpha x_1[n] + \beta x_2[n] \rightarrow \boxed{H} \rightarrow \alpha y_1[n] + \beta y_2[n]$$

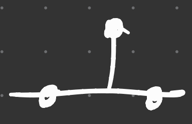
Linear Time/Translation Invariant Systems:

The system does not care when it is turned on.

$x[n] \rightarrow \boxed{H} \rightarrow y[n]$

delayed input $x[n-k] \rightarrow \boxed{H} \rightarrow y[n-k]$ delayed output

Impulse Response:



-1 0 1

$\delta[n] \rightarrow \boxed{H} \rightarrow h[n]$

kronecker delta

impulse response

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = (h * x)[n]$$

$$= \sum_{k \in \mathbb{Z}} h[k]\, x[n-k]$$

convolution

The sequence $h[n]$, $n \in \mathbb{Z}$, is called a <u>filter</u>. The <u>support</u> of $h[n]$ is

$$\operatorname{supp}(h) = \{ n \in \mathbb{Z} : h[n] \neq 0 \}$$

<u>Def^{n}</u>: A filter h is FIR if

$$|\operatorname{supp}(h)| < \infty.$$

<u>Def^{n}</u>: A filter h is IIR if

$$|\operatorname{supp}(h)| = \infty.$$

<u>Ex</u>: Consider the system

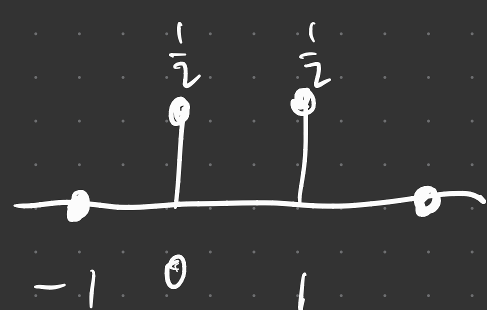
$$y[n] = \frac{1}{2}\left(x[n] + x[n-1] \right).$$

Q: What does this system do?

A:
- Moving average
- Smoothing
- Low-pass filter

Ex: What is the impulse response of this system?

$$h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right)$$



2-tap FIR filter

Q: How do we know this is low pass?

A: Look at its frequency response.

$$e^{j\omega n} \longrightarrow \boxed{H} \longrightarrow \sum_{k\in\mathbb{Z}} h[k]\, e^{j\omega(n-k)}$$

$$= e^{j\omega n}\left(\sum_{k\in\mathbb{Z}} h[k]\, e^{-j\omega k}\right)$$

$$= e^{j\omega n}\; H(e^{j\omega})$$

DTFT of $h[n]$

frequency response

<u>Obs:</u> Pure frequencies (complex exponentials) are <u>eigen functions</u> of LTI systems.

<u>Convolution Theorem:</u>

$$y[n] = (h * x)[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

<u>Obs:</u> Fourier transforms turn convolutions into multiplications.

Z-Transform

$$X(z) = \sum_{n \in \mathbb{Z}} x[n]\, z^{-n}, \quad z \in \mathbb{C}$$

$$X(e^{j\omega}) = X(z)\Big|_{z = e^{j\omega}}$$

Ex: $h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right)$

$$H(e^{j\omega}) = \frac{1}{2} + \frac{1}{2} e^{-j\omega}$$

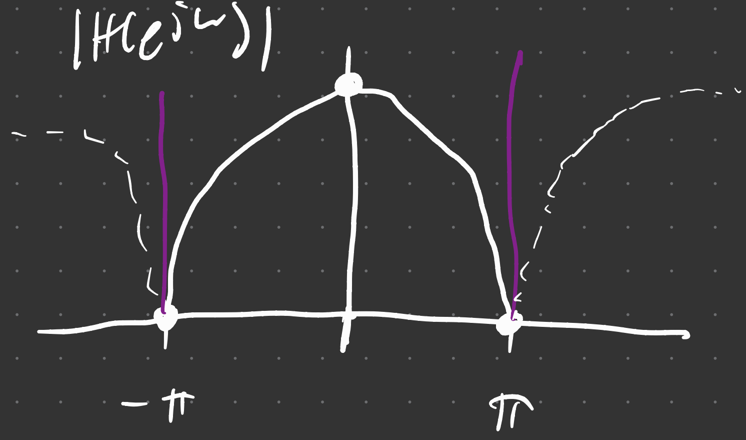
$$= e^{-j\frac{\omega}{2}} \left[\frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2} \right]$$

$$= \underbrace{e^{-j\frac{\omega}{2}}}_{\text{phase}} \; \underbrace{\cos\left(\frac{\omega}{2}\right)}_{\text{magnitude}}$$

$\hookrightarrow = e^{j\phi(\omega)} \Rightarrow \phi(\omega) = -\frac{\omega}{2}$

phase response

$|H(e^{j\omega})| = |\cos(\frac{\omega}{2})|$ is the magnitude responce.



Attenuates high frequencies $\Rightarrow$ low-pass

Linear-Phase Filters

Defn: $H(e^{j\omega}) = e^{j\Phi(\omega)} H_{ampl}(\omega)$

is called linear-phase if

$\Phi(\omega)$ is linear.

Ex: $\Phi(\omega) = -\frac{\omega}{2}$

slope $= -\frac{1}{2}$

$h[n]$

$\frac{1}{2}$ = center of symmetry

Symmetric / Antisymmetric filters provide a complete characterization of linear-phase filters.

Type I: odd length / symmetric

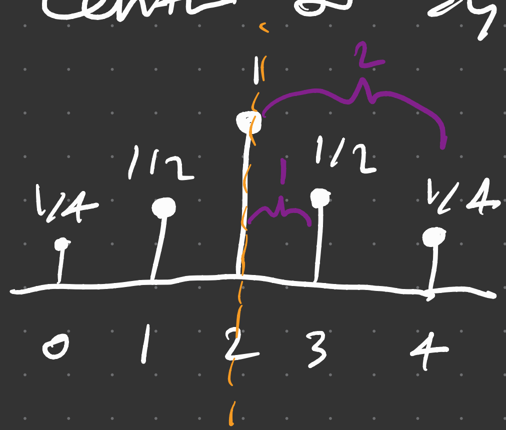
Type II: even length / symmetric

Type III: odd length / antisymmetric

Type IV: even length / antisymmetric

Type I: Center of Symmetry $L \in \mathbb{Z}$

Ex:



$L = 2 \in \mathbb{Z}$

$$H(e^{j\omega}) = e^{-j2\omega}\left[1 + 2\left(\tfrac{1}{2}\right)\cos(\omega) + 2\left(\tfrac{1}{4}\right)\cos(2\omega)\right]$$

General:

$$H(e^{j\omega}) = e^{-jL\omega} \sum_{n=0}^{L} a[n] \cos(n\omega)$$

Exercise: Express $a[n]$ in terms of $h[n]$.

Type II: Center of Symmetry $L - \frac{1}{2} \in \mathbb{Z}$

Ex:

$$L = \frac{3}{2} \Rightarrow L - \frac{1}{2} = 1 \in \mathbb{Z}$$

$$H(e^{j\omega}) = e^{-j\frac{3}{2}\omega}\left[2(1)\cos\left(\frac{\omega}{2}\right) + 2\left(\frac{1}{2}\right)\cos\left(\frac{3\omega}{2}\right)\right]$$

General:

$$H(e^{j\omega}) = e^{-jL\omega} \sum_{n=0}^{L-\frac{1}{2}} b[n] \cos\left(\left(n+\frac{1}{2}\right)\omega\right)$$

Exercise: Express $b[n]$ in terms of $h[n]$.

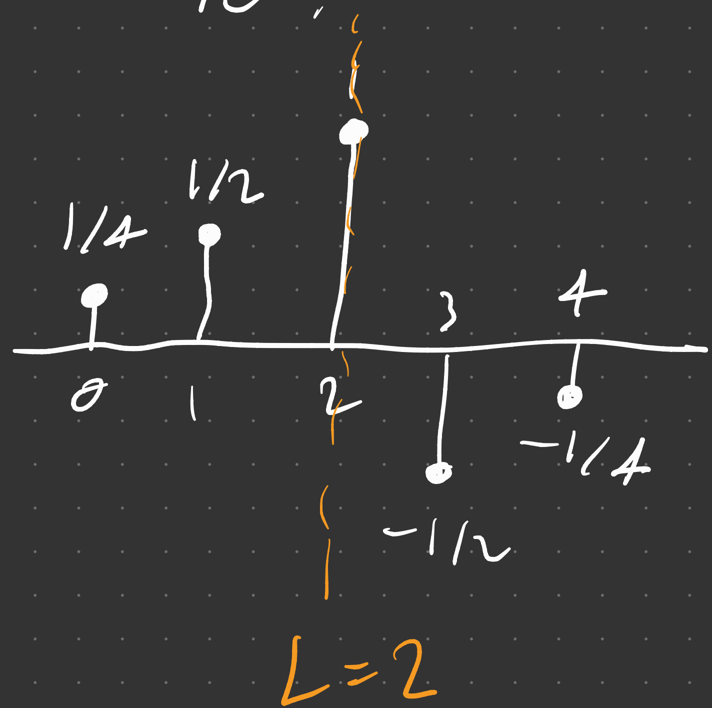
Q: What makes a filter $h[n]$ symmetric?

A: $h[n] = h[2L-n]$

Q: What makes a filter $h[n]$ antisymmetric?

A: $h[n] = -h[2L-n]$

Exercise: Is the following filter antisymmetric?



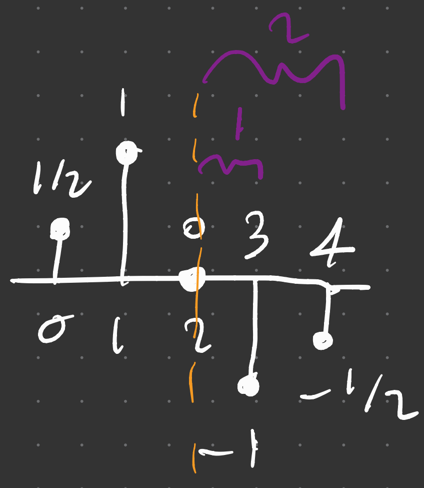
not antisymmetric!

Obs: $h[n] = -h[2L-n], \forall n \in \mathbb{Z}$

$\Rightarrow h[L] = -h[L] \Rightarrow \boxed{h[L] = 0}$

Type III: Center of anti-symmetry $L \in \mathbb{Z}$

Ex:



$L = 2 \in \mathbb{Z}$

$$H(e^{j\omega}) = e^{-j2\omega}\left[2(1)\,j\sin(\omega) + 2\left(\tfrac{1}{2}\right)\sin(2\omega)\right]$$

$$= e^{j\left(-2\omega + \frac{\pi}{2}\right)}\left[2\sin(\omega) + \sin(2\omega)\right]$$

$j = e^{j\frac{\pi}{2}}$

General:

$$H(e^{j\omega}) = e^{j\left(-L\omega + \frac{\pi}{2}\right)} \sum_{n=0}^{L} c[n]\sin(n\omega)$$

Exercise: Express $c[n]$ in terms of $h[n]$.

Type IV: Center of antisymmetry $L-\frac{1}{2} \in \mathbb{Z}$

Ex:

$L=\frac{3}{2} \Rightarrow L-\frac{1}{2} = 1 \in \mathbb{Z}$

$$H(e^{j\omega}) = e^{j\left(-\frac{3}{2}\omega+\frac{\pi}{2}\right)}\left[2(1)\sin\left(\frac{\omega}{2}\right)+2\left(\frac{1}{2}\right)\sin\left(\frac{3\omega}{2}\right)\right]$$

General:

$$H(e^{j\omega}) = e^{j\left(-L\omega+\frac{\pi}{2}\right)}\sum_{n=0}^{L-\frac{1}{2}} d[n]\sin\left(\left(n+\tfrac{1}{2}\right)\omega\right)$$

Exercise: Express $d[n]$ in terms of $h[n]$.

Summary of Linear-Phase Filters

$$H(e^{j\omega}) = e^{j(-L\omega + \beta)} H_{ampl}(\omega)$$

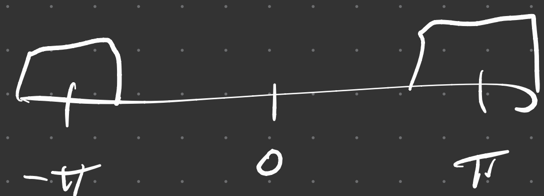
Phase response is <u>linear</u>: $\phi(\omega) = -L\omega + \beta$

$H_{ampl}(\omega)$	$L \in \mathbb{Z}$	$L - \frac{1}{2} \in \mathbb{Z}$
Symmetric $\beta = 0$ $h[n] = h[2L-n]$	I $\sum_{n=0}^{L} a[n] \cos(n\omega)$	II $\sum_{n=0}^{L-\frac{1}{2}} b[n] \cos\left((n+\frac{1}{2})\omega\right)$
Antisymm. $\beta = \frac{\pi}{2}$ $h[n] = -h[2L-n]$	III $\sum_{n=0}^{L} c[n] \sin(n\omega)$	IV $\sum_{n=0}^{L-\frac{1}{2}} d[n] \sin\left((n+\frac{1}{2})\omega\right)$

<u>Remark:</u> Many wavelet filters are linear-phase.

Exercise: Can a high-pass filter be Type II?

$|H(e^{j\omega})|$



No. **Proof:**

$$\cos\left(\left(n+\tfrac{1}{2}\right)\pi\right) = 0 \quad \forall n \in \mathbb{Z}.$$

Obs: Type II filters necessarily have a zero at π.