

# Last Time: Two-Channel PR Filter Banks



$$Y(z) = z^{-L} X(z)$$

$$= \frac{1}{2} X(z) \left[ G_0(z) H_0(z) + G_1(z) H_1(z) \right]$$

*Distortion*

$$+ \frac{1}{2} X(-z) \left[ G_0(z) H_0(-z) + G_1(z) H_1(-z) \right]$$

*Aliasing*

## PR Conditions (Vetterli, 1986)

- $G_0(z) H_0(z) + G_1(z) H_1(z) = 2 z^{-L}$
- $G_0(z) H_0(-z) + G_1(z) H_1(-z) = 0$

Obs: The PR conditions are two equations with 4 unknowns ( $H_0, H_1, G_0, G_1$ ).

$$\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} 2z^{-L} \\ 0 \end{bmatrix}$$

Given the analysis filters, we can find the synthesis filters by inverting this matrix.

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2z^{-L}}{\Delta(z)} \begin{bmatrix} H_1(-z) & -H_1(z) \\ -H_0(-z) & H_0(z) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Synthesis

Filters

$$= \frac{2z^{-L}}{\Delta(z)} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$

Analy sis  
Filters

$$\Delta(z) = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$

Q: Suppose that  $H_0$  &  $H_1$  are FIR.

Are  $G_0$  &  $G_1$  FIR?

A: Not necessarily. The issue is  $\Delta(z)$ .

Might not even have causal or stable

IIR  $G_0$  &  $G_1$ ...

## FIR PR Filter Bank Design

Q: How do we force  $G_0$  &  $G_1$  to be FIR?

A: Force  $\Delta(z) = 2z^{-L} \Rightarrow$  FIR soln.

could put

any gain

and delay

Remark: This is one possible design choice.

$$2z^{-L} = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$

$$= \underbrace{H_0(z)G_0(z)}_{P_0(z)} - \underbrace{H_0(-z)G_0(-z)}_{P_0(-z)}$$

"product"

## Design Procedure :

1. Find  $P_o(z)$  that satisfies

$$P_o(z) - P_o(-z) = 2z^{-L}$$

How do

we do

this?

2. Factorize  $P_o(z)$  into  $H_o(z) \cdot G_o(z)$

3. Define  $H_1(z) = G_o(-z)$

$$G_1(z) = -H_o(-z)$$

} one possible choice.

→ Different filter banks / wavelets / properties

$$P_o(z) = P_{o,\text{even}}(z^2) + z^{-1} P_{o,\text{odd}}(z^2)$$

$$- P_o(-z) = P_{o,\text{even}}(z^2) - z^{-1} P_{o,\text{odd}}(z^2)$$

$$P_o(z) - P_o(-z) = 2z^{-1} P_{o,\text{odd}}(z^2) = 2z^{-L}$$

$\underbrace{\text{odd power}}$     $\underbrace{\text{only even powers}}$   $= 2z^{-(2k+1)}$

$\Rightarrow L$  must be odd

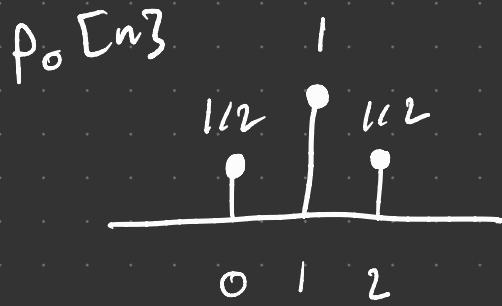
$$P_{o, \text{odd}}(z^2) = z^{-2K} \Rightarrow P_{o, \text{odd}}(z) = z^{-K}$$

pure delay

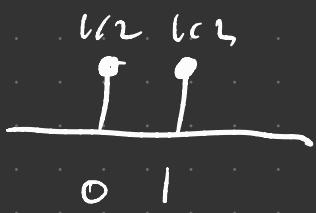
Obs:  $P_{o, \text{even}}(z)$  is the design choice even coeff.  
 $P_{o, \text{odd}}(z)$  must be a delay odd coeff.

Ex: Let

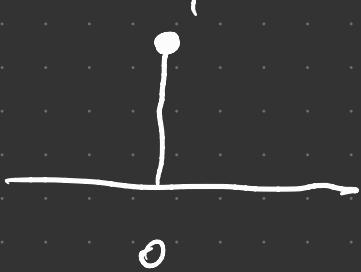
Is  $p_o[n]$  valid? Yes



$P_{o, \text{even}}[n]$



$P_{o, \text{odd}}[n] = \delta[n]$

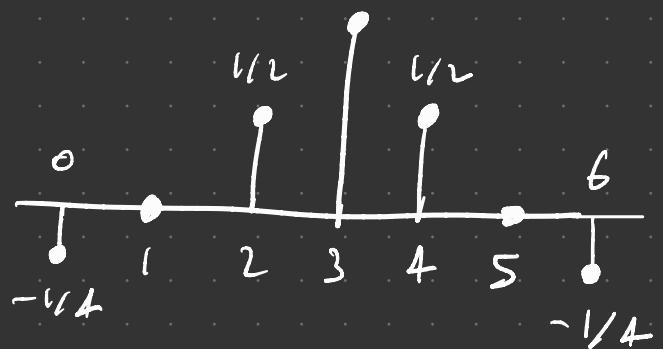


$K=0, L=1$

Ex: Let

$$p_0[n]$$

Is  $p_0[n]$  valid?

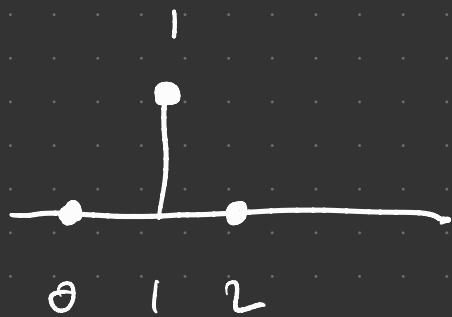
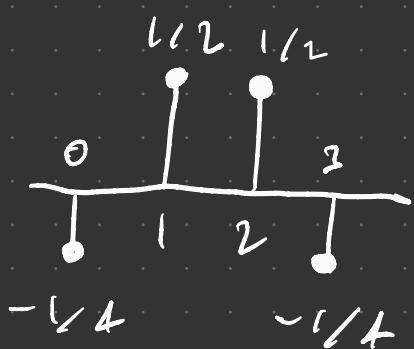


Type I Linear-Phase

Yes

$$p_{0,\text{even}}[n]$$

$$p_{0,\text{odd}}[n] = \delta[n - 1]$$



$$k = 1, L = 3$$

Q: What kind of filter is this?

A: It is a (shifted) half-band filter:

$$p_0[2n-L] = \delta[n]$$

Defn: A filter  $p_o[n]$  of the form

$$p_o[2n+L] = \delta[n]$$

is called a half-band filter  
centered at L.

Remark: Interpolation filters satisfy half-band cond.  
 $h[2n] = \delta[n]$ .

Obs: we have now reduced the problem  
to designing half-band (interpolation)  
filters.

## Half-Band Filter Design

If  $P_o$  is a half-band filter centered  
at  $L$ , then

$$P_o(z) - P_o(-z) = 2z^{-L}$$

Q: What do people typically do?

A: Assume  $P_0(z)$  is Type I Linear-Phase

Why not Type II?  $L$  is odd.

$$P_0(e^{j\omega}) = e^{-jL\omega} P_{\text{amp}}(\omega) \quad \cancel{\star}$$

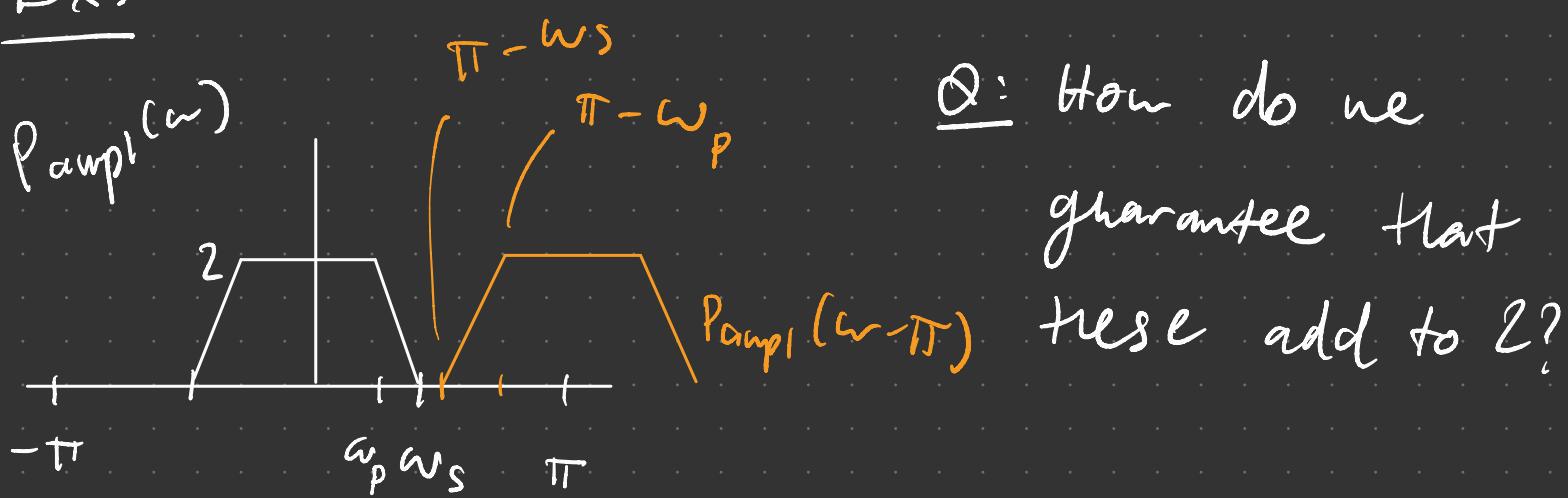
↑  
real function  
center of  
symmetry

$$\begin{aligned} P_0(-z) \\ P_0(e^{j(\omega-\pi)}) &= e^{-jL(\omega-\pi)} P_{\text{amp}}(\omega-\pi) \\ &= e^{jL\pi} e^{-jL\omega} P_{\text{amp}}(\omega-\pi) \\ &= -e^{jL\omega} P_{\text{amp}}(\omega-\pi) \end{aligned}$$

$$e^{-jL\omega} P_{\text{amp}}(\omega) + e^{-jL\omega} P_{\text{amp}}(\omega - \pi) = 2 e^{-jL\omega}$$

$$P_{\text{amp}}(\omega) + P_{\text{amp}}(\omega - \pi) = 2$$

Ex:



$$\pi - \omega_s = \omega_p$$

$$\omega_p + \omega_s = \pi$$

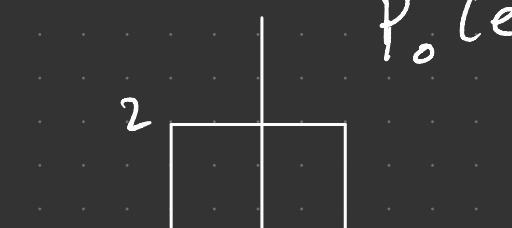
$$\omega_s - \frac{\pi}{2} = \frac{\pi}{2} - \omega_p$$

ripples need to cancel out

$\omega$   
half-band

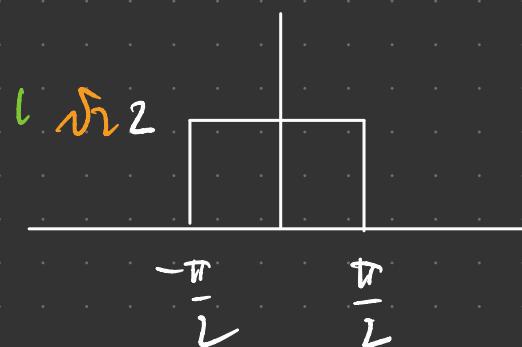
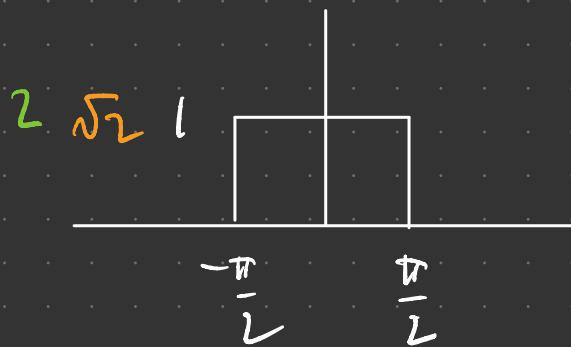
Ex: Ideal low-pass filter ( $L = 0$ )

$$P_o(e^{j\omega}) = H_o(e^{j\omega}) G_o(e^{j\omega})$$



$$H_o(e^{j\omega})$$

$$G_o(e^{j\omega})$$



guarantee  
PR

Q: Why the gain factor of 2?

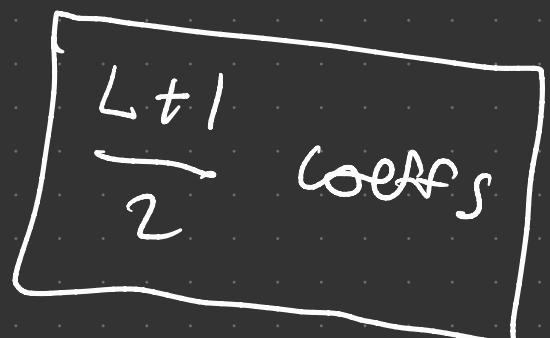
A: To account for the magnitude

being halved from down sampling.

Q: If  $P_0$  is of length  $2L+1$ ,  
how many coeffs do we have  
to design?

A: We only need to design  $P_{0,\text{even}}$ ,  
which is itself symmetric.

$L$	# coeffs
1	1
3	2
5	3
7	4



Q: What is  $P_{0,\text{even}}$ ?

A: Type II Linear-Phase with  
Center of Symmetry  $\frac{L}{2}$ .

\*  $P_{0,\text{even}}(e^{j\omega}) = e^{-j\frac{L}{2}\omega} P_{\text{even, amp}}(\omega)$

Putting everything together,

$$P_o(z) = P_{o,\text{even}}(z^2) + z^{-1} P_{o,\text{odd}}(z^2)$$

$$P_o(e^{j\omega}) = P_{o,\text{even}}(e^{j2\omega}) + e^{-jL\omega}$$

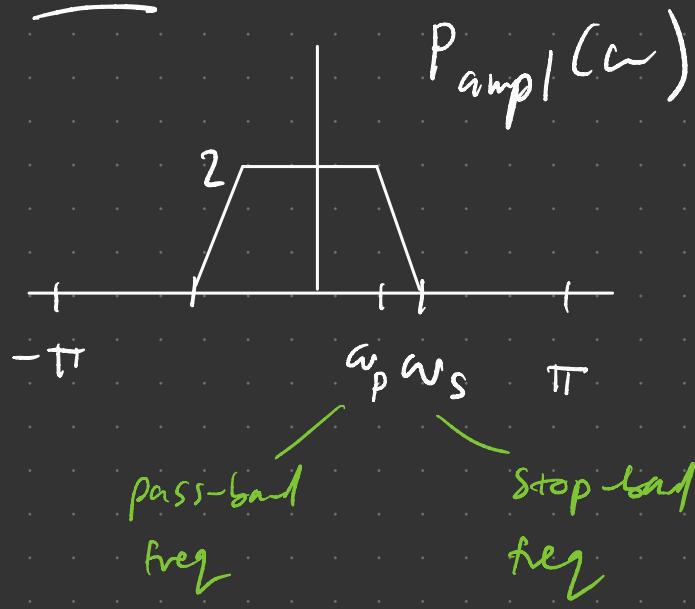
By  $\star$  and  $\star$ ,

$$e^{-jL\omega} P_{\text{amp}}(\omega) = e^{-jL\omega} P_{\text{even, amp}}(2\omega) + e^{-jL\omega}$$

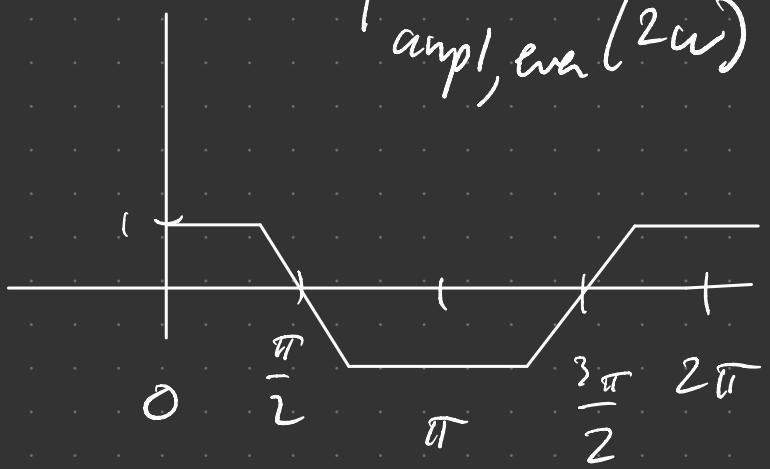
$$P_{\text{amp}}(\omega) = P_{\text{even, amp}}(2\omega) + 1$$

$$P_{\text{amp}}(\omega) - 1 = P_{\text{even, amp}}(2\omega)$$

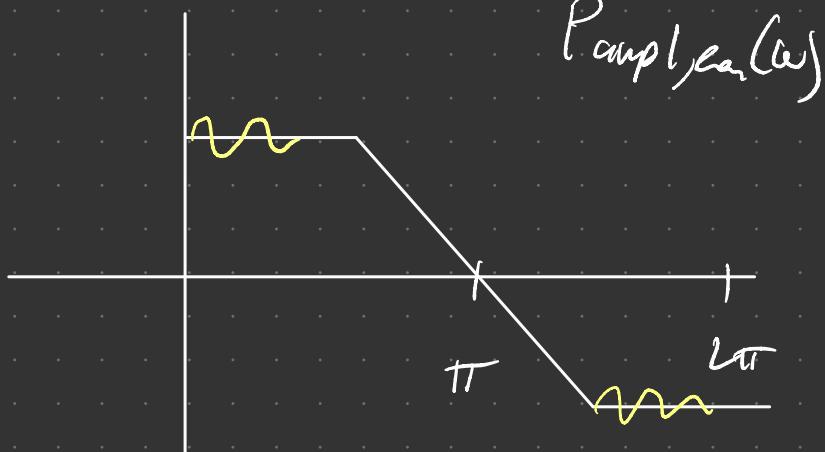
Ex:



$P_{\text{amp}1, \text{even}}(2\omega)$



Type II  
linear-phase  
zero @  $\pi$



One band filter

ripples will  
automatically  
cancel by symmetry

Remark: We've reduced the problem  
to designing a Type II  
linear-phase filter.