

Last Time:

- Noble Identities
- Polyphase Representation

Recall:



③ If $\gcd\{M, L\} = 1$:



Exercise:



ideal

low-pass

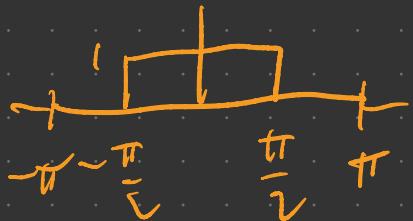
filter



ideal

high-pass

filter



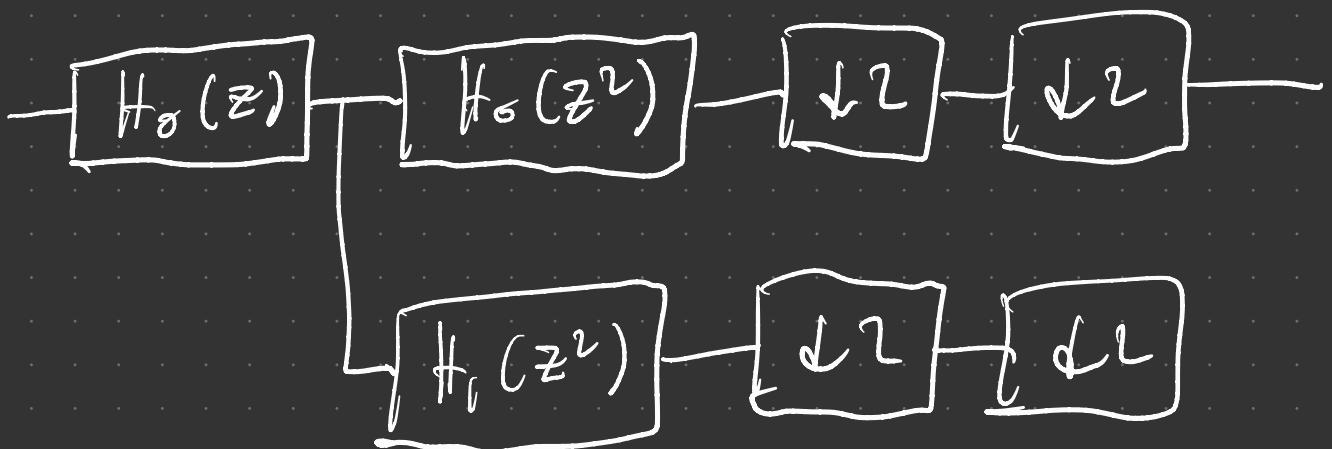
Does there exist $G_o(z)$ and $G_1(z)$ such that



is an equivalent system?

Solution:

By ① :

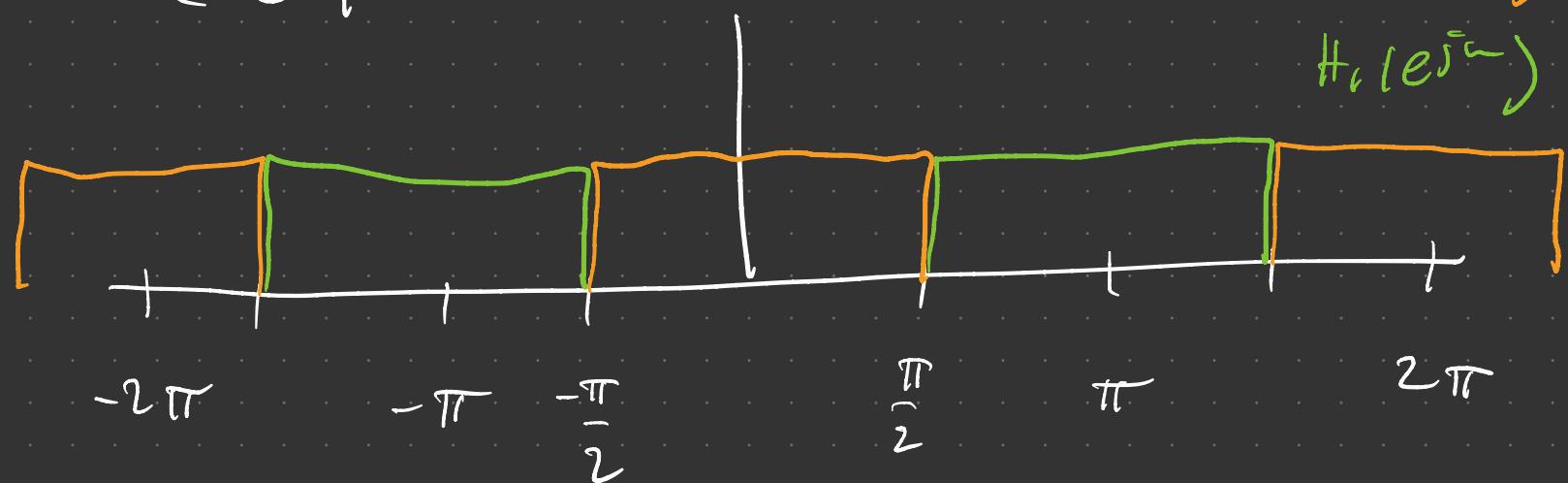


$$\left\{ G_0(z) = H_o(z) H_o(z^2) \right.$$

$$\left. G_1(z) = H_o(z) H_i(z^2) \right.$$

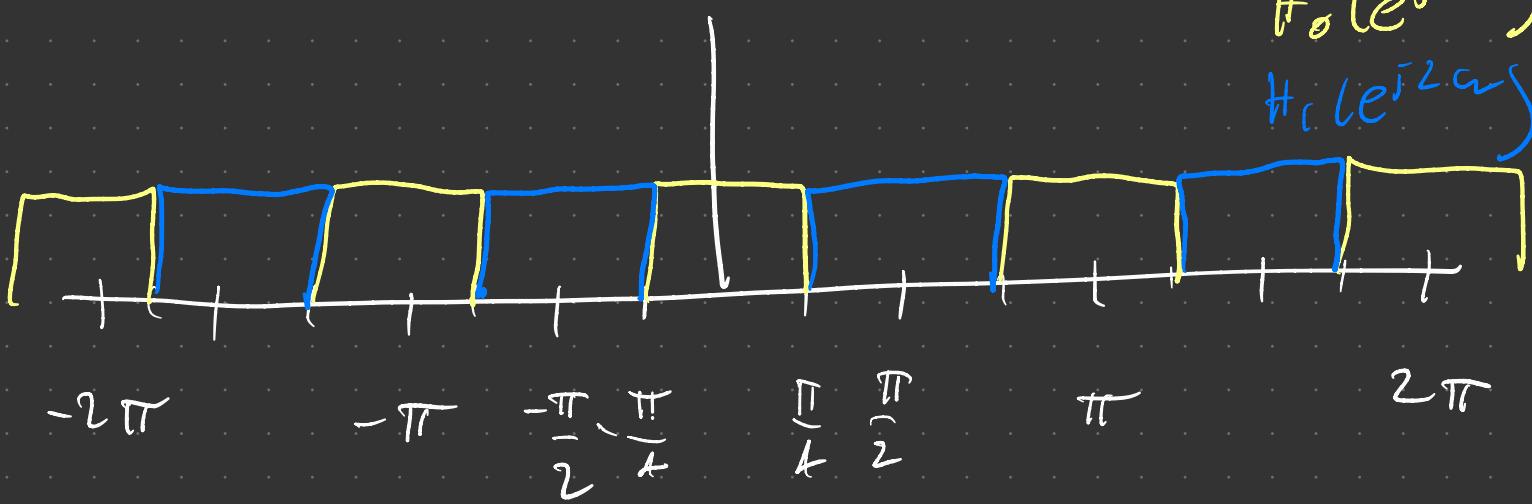
$$H_o(e^{j\omega})$$

$$H_i(e^{j\omega})$$



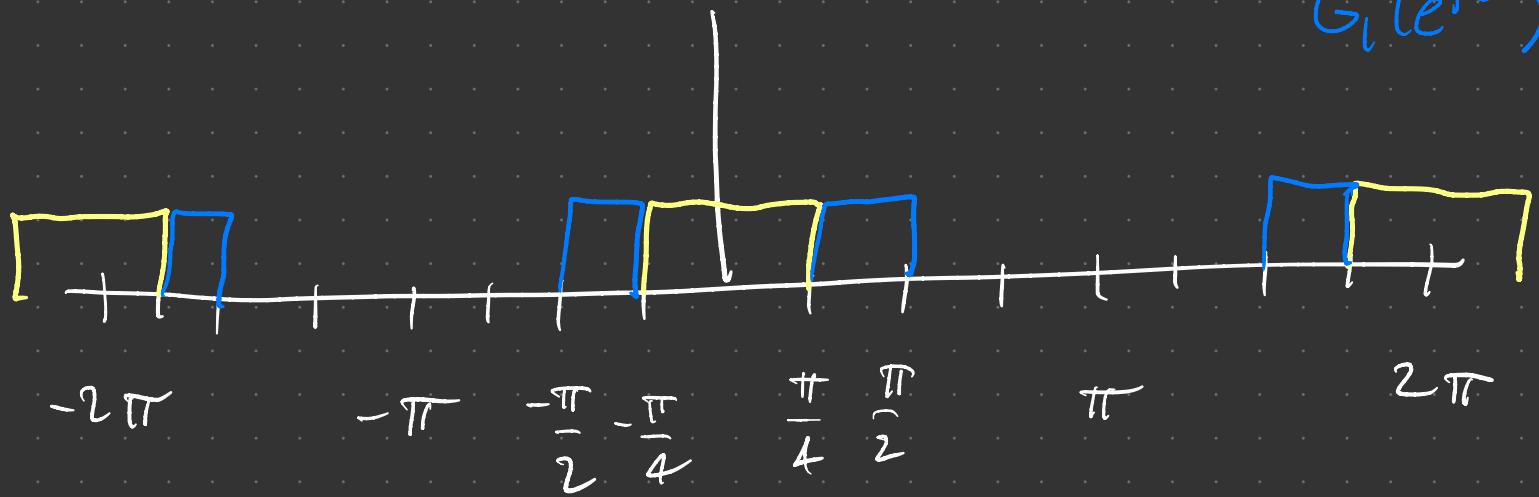
$$H_o(e^{j2\omega})$$

$$H_i(e^{j2\omega})$$



$$G_0(e^{j\omega})$$

$$G_1(e^{j\omega})$$



Obs: • G_0 is low-pass

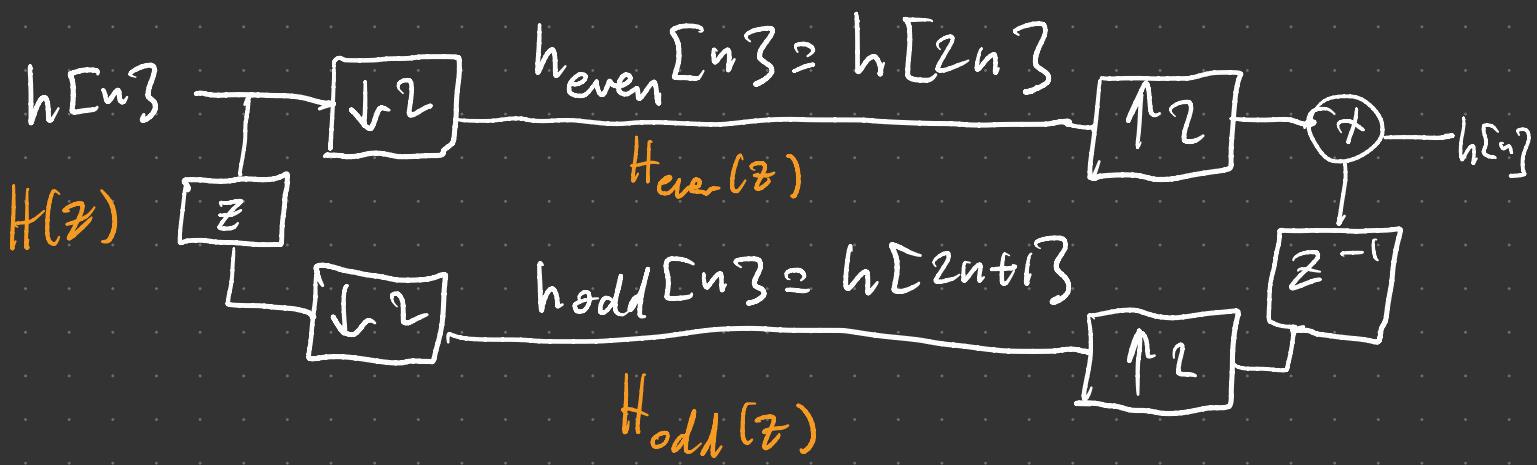
• G_1 is band-pass

we lost the high frequencies.

Polyphase Representations

Recall:

$$H(z) = H_{\text{even}}(z^2) + z^{-1} H_{\text{odd}}(z^2)$$



Ex:

$$H(z) = \frac{1}{1 - az^{-1}}$$

$$= \frac{1}{1 - az^{-1}} \cdot \frac{(1 + az^{-1})}{(1 + az^{-1})} = \frac{1 + az^{-1}}{1 - a^2 z^{-2}}$$

$$= \frac{1}{1 - a^2 z^{-2}} + z^{-1} \frac{a}{1 - a^2 z^{-2}}$$

$$H_{\text{even}}(z) = \frac{1}{1 - a^2 z^{-2}}, \quad H_{\text{odd}}(z) = \frac{a}{1 - a^2 z^{-1}}$$

Time-Domain:

$$h[n] = a^n u[n]$$

$$h_{\text{even}}[n] = a^{2n} u[2n] = (a^2)^n u[n]$$

$$h_{\text{odd}}[n] = a^{2n+1} u[2n+1] = a(a^2)^n u[n]$$

Obs:

$$\left\{ \begin{array}{l} H(z) = H_{\text{even}}(z^2) + z^{-1} H_{\text{odd}}(z^2) \\ H(-z) = H_{\text{even}}(z^2) - z^{-1} H_{\text{odd}}(z^2) \end{array} \right.$$

$$H_{\text{even}}(z^2) = \frac{1}{2} (H(z) + H(-z))$$

Solved α

$$z^{-1} H_{\text{odd}}(z^2) = \frac{1}{2} (H(z) - H(-z))$$

System of
equations

★
$$\begin{bmatrix} H(z) \\ H(-z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} H_{\text{even}}(\tilde{z}) \\ z^{-1} H_{\text{odd}}(\tilde{z}) \end{bmatrix}$$

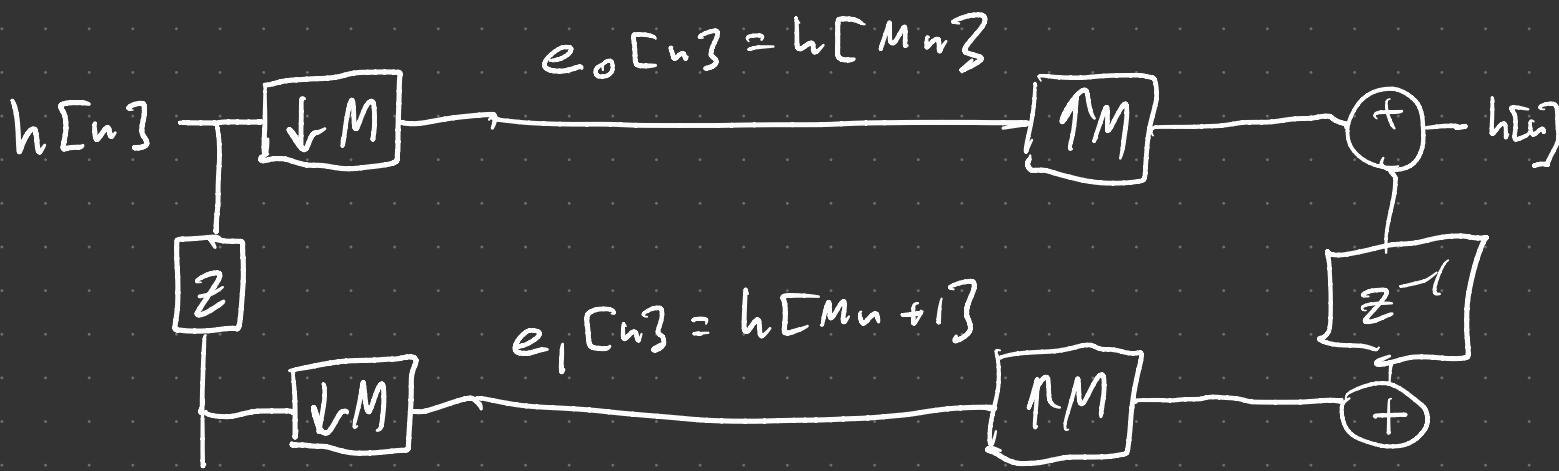
2-point DFT

Exercise: Find $H_{\text{even}}(z)$ and $H_{\text{odd}}(z)$ for the previous example using the system of equations.

Obs: In ★, we have

- 2nd roots of unity ($H((+1)z)$, $H((-1)z)$)
- 2-point DFT

General M-Polyphase



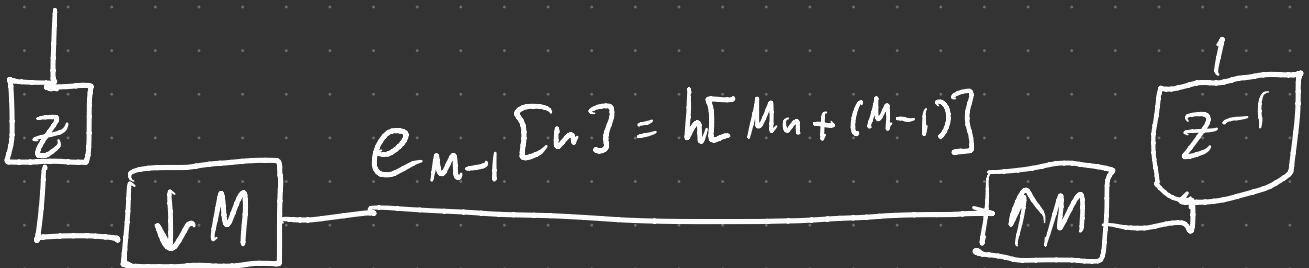
M_{total}

..

branches

..

..
..
..



$$e_k[n] = h[Mn+k], \quad H(z) = \sum_{k=1}^{M-1} z^{-k} E_k(z^M)$$

$$\begin{bmatrix} H(z) \\ H(e^{j\frac{2\pi}{M}} z) \\ \vdots \\ H(e^{j\frac{2(M-1)\pi}{M}} z) \end{bmatrix}$$

$$= F_M$$

M-point
DFT

$$\begin{bmatrix} E_0(z^M) \\ z^{-1} E_1(z^M) \\ \vdots \\ z^{-(M-1)} E_{M-1}(z^M) \end{bmatrix}$$

Exercise: $M=2$

partial fractions

$$H(z) = \frac{3+z^{-1}}{1 + \frac{3}{2}z^{-1} - z^{-2}} = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + 2z^{-1}}$$

\curvearrowright

$$\begin{aligned} -\frac{1}{2} + 2 &= \frac{3}{2} \\ -\frac{1}{2} - 2 &= -1 \end{aligned}$$

A pole-zero plot on the complex plane. The horizontal axis is labeled with -2 and $-\frac{1}{2}$. There are two 'x' marks on the real axis, one at -2 and one at $-\frac{1}{2}$.

$$\left. \begin{aligned} A(1+2z^{-1}) + B(1 - \frac{1}{2}z^{-1}) &= 3 + z^{-1} \\ A+B=3 \\ 2A - \frac{B}{2} = 1 \end{aligned} \right\} \Rightarrow A=1, B=2$$

$$H_{\text{even}}(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - 4z^{-1}}$$

$$H_{\text{odd}}(z) = \frac{\frac{1}{2}}{1 - \frac{1}{4}z^{-1}} + \frac{2(-2)}{1 - 4z^{-1}}$$

Exercise: $M=3$

$$\begin{aligned} H(z) &= \frac{1}{1 - az^{-1}} \\ &= \frac{1}{1 - az^{-1}} \cdot \frac{(1 + az^{-1} + a^2z^{-2})}{(1 + az^{-1} + a^2z^{-2})} \\ &= \frac{1 + az^{-1} + a^2z^{-2}}{1 - a^3z^{-3}} \\ &= \frac{1}{1 - a^3z^{-3}} + z^{-1} \frac{a}{1 - a^3z^{-3}} + z^{-2} \frac{a^2}{1 - a^3z^{-3}} \end{aligned}$$

$$E_0(z) = \frac{1}{1 - a^3z^{-1}}$$

$$E_1(z) = \frac{a}{1 - a^3z^{-1}}$$

$$E_2(z) = \frac{a^2}{1 - a^3z^{-1}}$$

Exercise: Generalize
this to arbitrary
 M .

Exercise: Let

$$H(z) = \frac{3+z^{-1}}{1+\frac{3}{2}z^{-1}-z^{-2}} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1+2z^{-1}}$$

Determine $E_0(z)$, $E_1(z)$, and $E_2(z)$.

$$E_0(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1+8z^{-1}}$$

$$E_1(z) = \frac{\frac{1}{2}}{1-\frac{1}{8}z^{-1}} + \frac{2(-2)}{1+8z^{-1}}$$

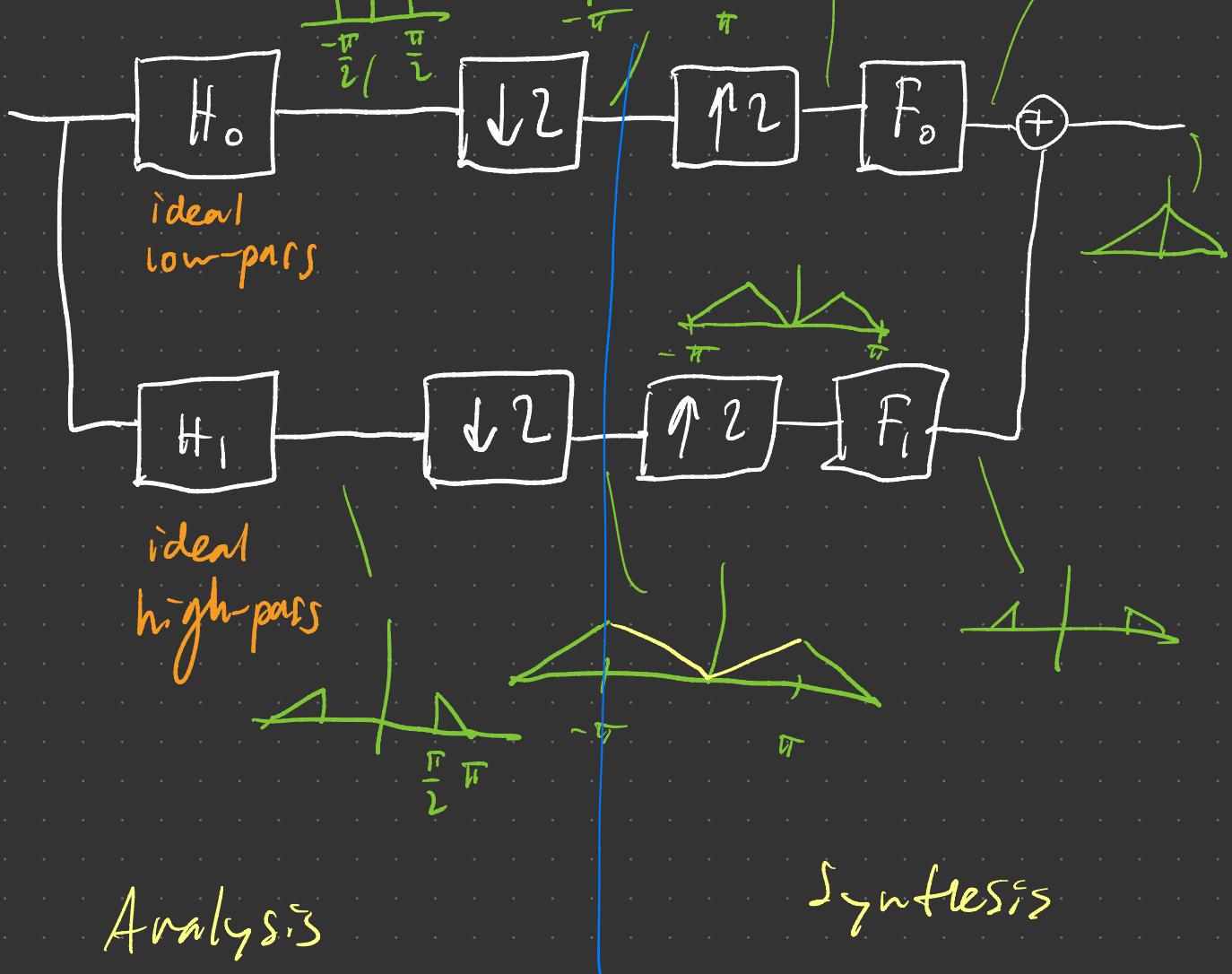
$$E_2(z) = \frac{\left(\frac{1}{2}\right)^2}{1-\frac{1}{8}z^{-1}} + \frac{2(-2)^2}{1+8z^{-1}}$$

Two-channel Filter bank

$x(e^{j\omega})$



ideal
low-pass



Analysis

Bank K

Synthesis
Bank
coding,
compression,
processing,
etc.
(Non linear operations)

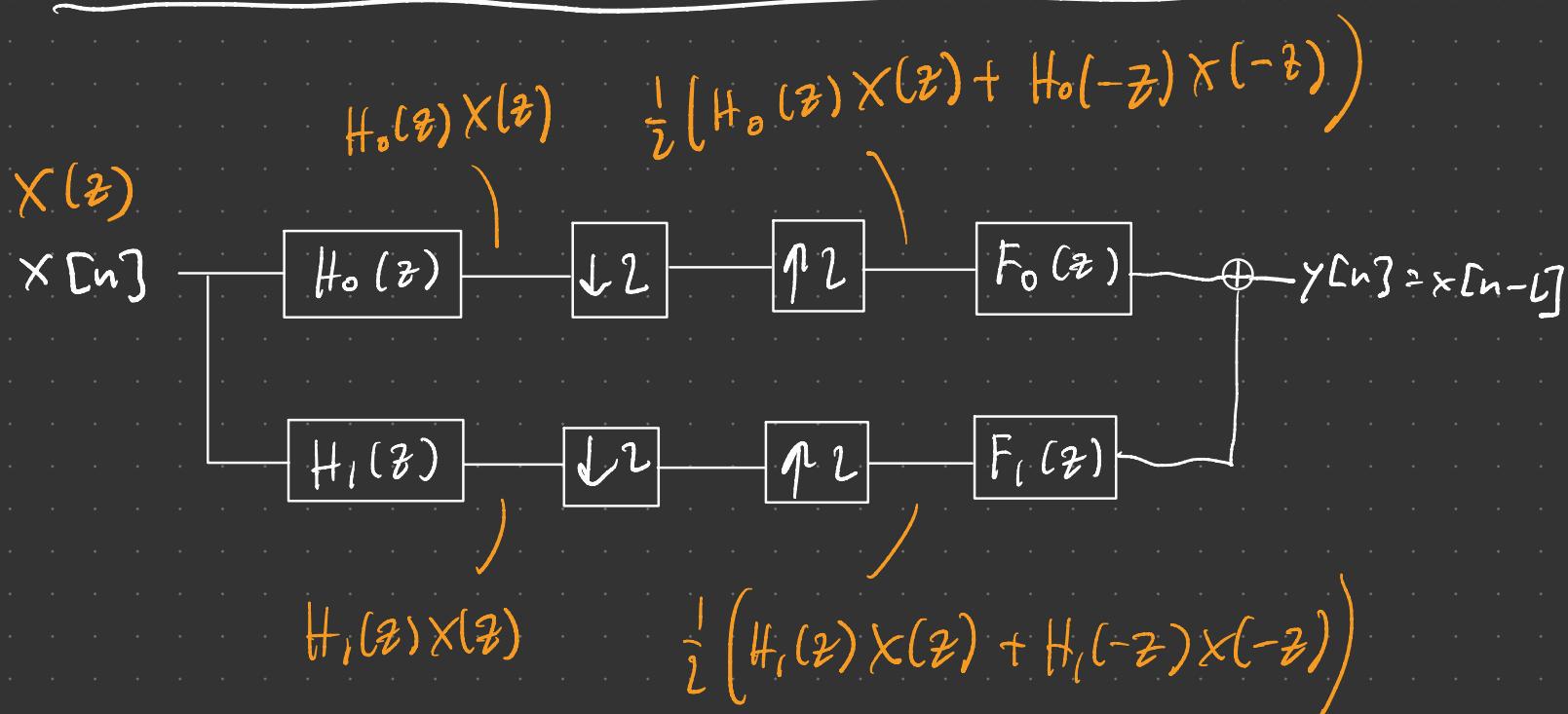
Obs: Ideal Filters \Rightarrow Perfect-Reconstruction (PR)

Practical Filters \Rightarrow Distortion & Aliasing.

Q: How do we design PR filter banks with practical filters to avoid distortion and aliasing?

Answering this question will be the topic of the next few lectures.

PR Two-Channel Filter banks



$$Y(z) = z^{-L} X(z)$$

$$= \frac{1}{2} F_0(z) \left[H_0(z) X(z) + H_0(-z) X(-z) \right]$$

$$+ \frac{1}{2} F_1(z) \left[H_1(z) X(z) + H_1(-z) X(-z) \right]$$

$$= \frac{1}{2} X(z) \left[F_0(z) H_0(z) + F_1(z) H_1(z) \right]$$

Distortion

$$+ \frac{1}{2} X(-z) \left[F_0(z) H_0(-z) + F_1(z) H_1(-z) \right]$$

Aliasing

PR Conditions (Vetterli, 1986)

- $F_0(z) H_0(z) + F_1(z) H_1(z) = 2 z^{-L}$

- $F_0(z) H_0(-z) + F_1(z) H_1(-z) = 0$