# General Case of Upsampling

$\frac{n}{L} \in \mathbb{Z}$

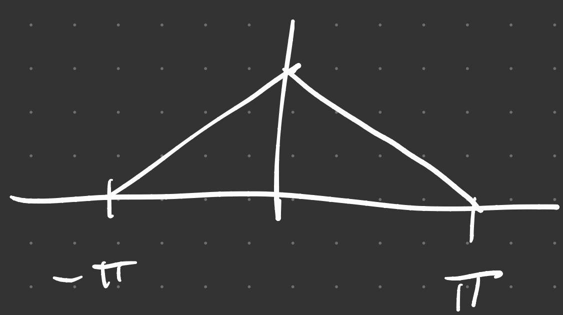
$v[n] \rightarrow \boxed{\uparrow L} \rightarrow$ 
$$u[n] = \begin{cases} v\left[\frac{n}{L}\right], & n \text{ is mult. of } L \\ 0, & \text{else} \end{cases}$$

insert $(L-1)$ zeros between samples

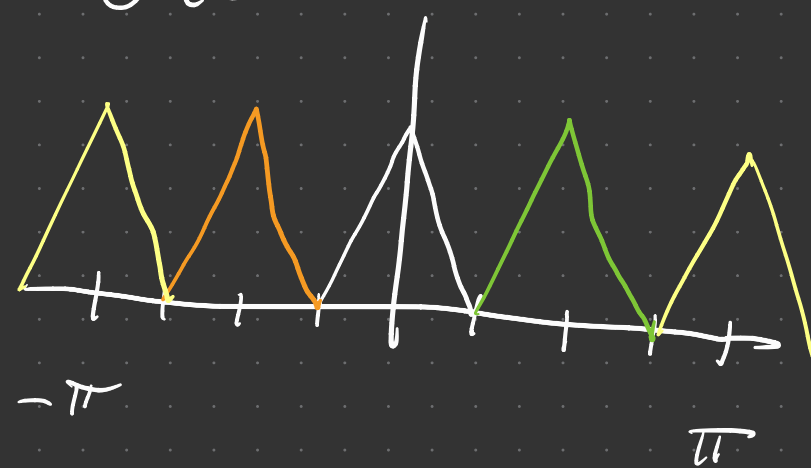
$$U(z) = V(z^L) \Rightarrow U(e^{j\omega}) = V(e^{jL\omega})$$

Ex: $L = 4$

$V(e^{j\omega})$



$U(e^{j\omega})$



- 3 copies of the spectrum
- 4 total spectrums

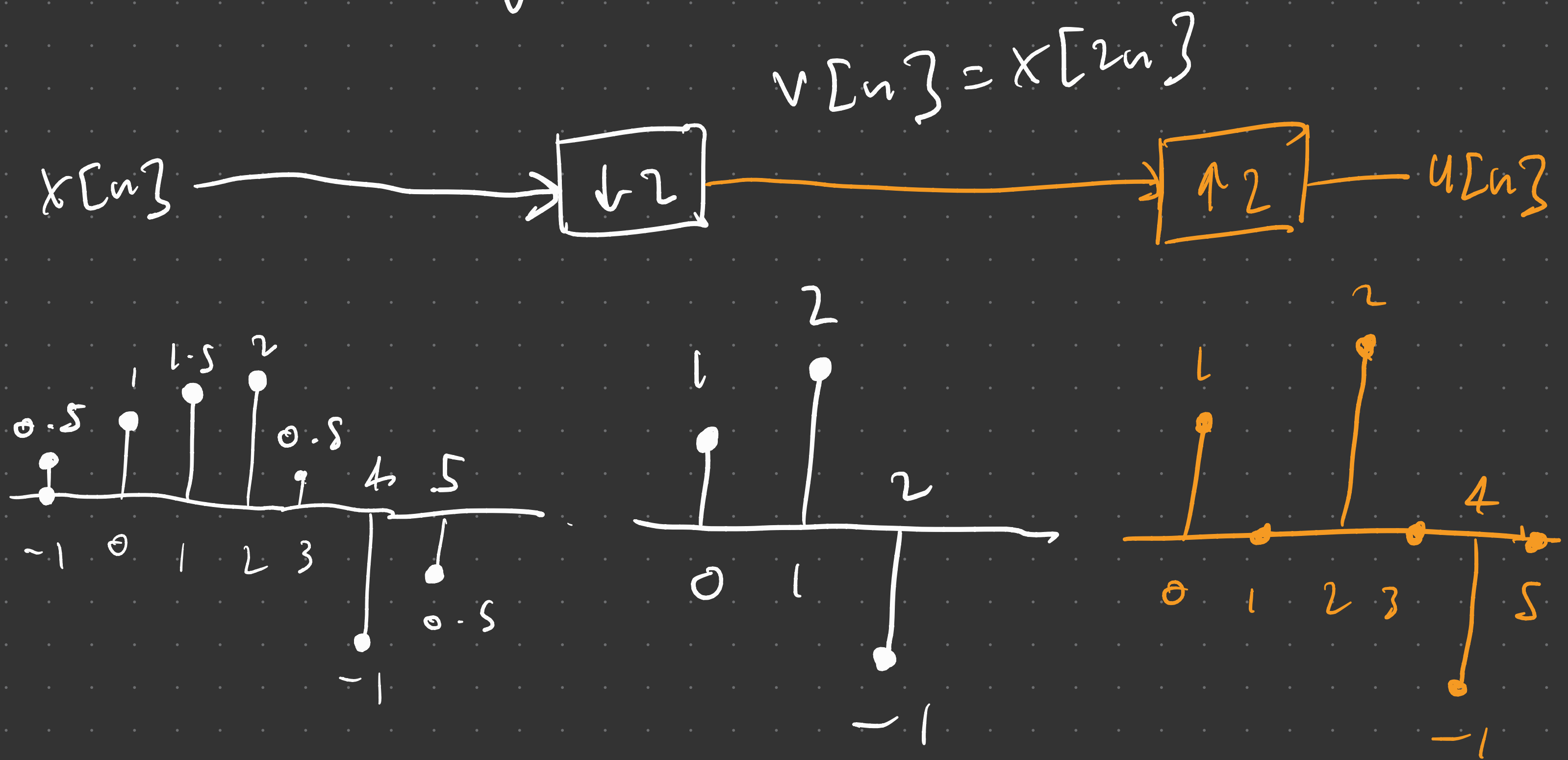
Obs: 4 options for the interp. filter

General! $L$ interp. filters for the $L$ spectrums

# Downsampling by Factor 2

**Q:** What is downsampling?

**A:** Deleting samples




**Q:** Is downsampling linear?

**A:** Yes

**Q:** Is it time-invariant?

**A:** No

**Q:** Is it invertible?

**A:** No.

Q: How are the spectrums $U(e^{j\omega})$ and $V(e^{j\omega})$ related?

A: $U(e^{j\omega}) = V(e^{j2\omega})$

Q: How are $u[n]$ and $x[n]$ related?

A:

| | | | | | | |
|---|---|---|---|---|---|---|
| $x[n]$ : | $x[0]$ | $x[1]$ | $x[2]$ | $x[3]$ | $x[4]$ | ... |
| $+\ (-1)^n x[n]$ : | $x[0]$ | $-x[1]$ | $x[2]$ | $-x[3]$ | $x[4]$ | |
| $2u[n]$ : | $2x[0]$ | $0$ | $2x[2]$ | $0$ | $2x[4]$ | ... |

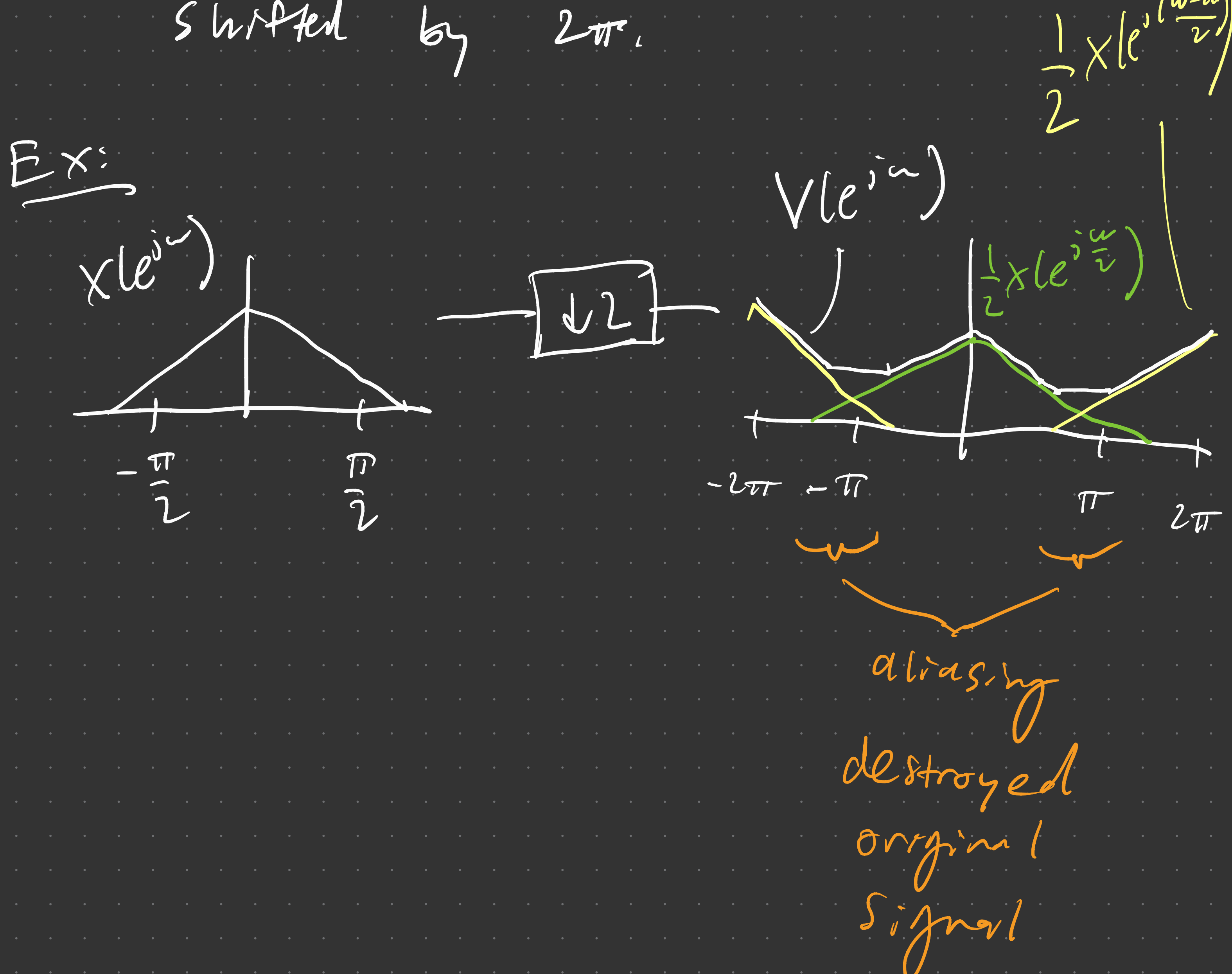
$$\Rightarrow u[n] = \frac{1}{2}\left(x[n] + (-1)^n x[n]\right)$$

$$\Rightarrow U(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X\left(e^{j(\omega-\pi)}\right)\right)$$

$$\Rightarrow \boxed{V(e^{j\omega}) = \frac{1}{2}\left(X\left(e^{j\frac{\omega}{2}}\right) + X\left(e^{j\left(\frac{\omega-2\pi}{2}\right)}\right)\right)}$$

Obs: Downsampled spectrum is the avg. of original spectrum stretched by factor 2 and that same spectrum shifted by $2\pi$.

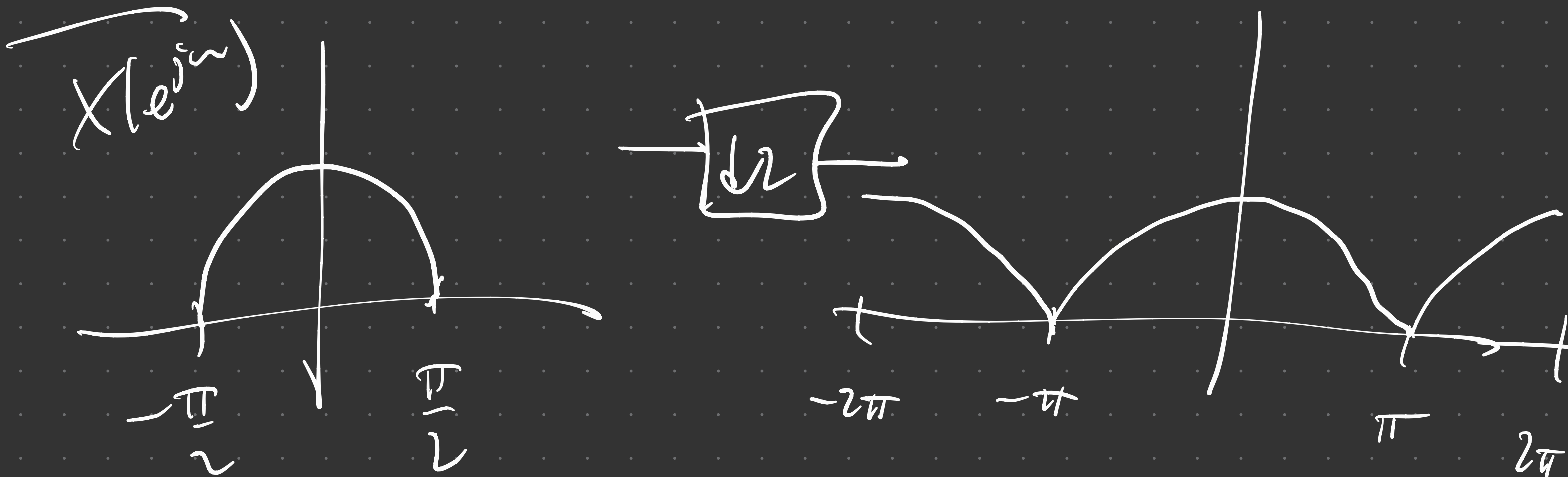
Ex:




Q: What conditions on $X(e^{j\omega})$ for no aliasings?

A: Band limited $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Ex:




Remark: Downsampling is always preceded by filtering

Decimation

$$\longrightarrow \boxed{H} \longrightarrow \boxed{\downarrow 2} \longrightarrow$$

antialiasing filter

Obs: Reverse of upsampling.

# Downsampling by factor 3

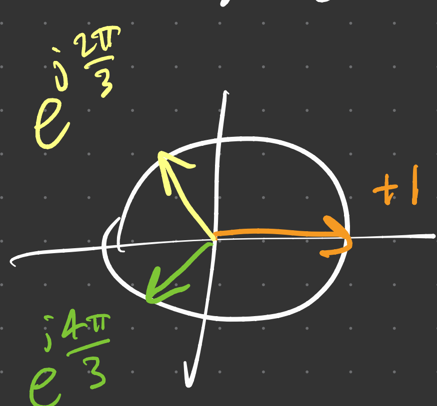
$$x[n] \longrightarrow \boxed{\downarrow 3} \xrightarrow{v[n] = x[3n]} \boxed{\uparrow 3} \longrightarrow u[n]$$

How do we generalize: $(\downarrow 2)$

$$u[n] = \frac{1}{2}\left( (+1)^n x[n] + (-1)^n x[n] \right)$$

Obs: $\pm 1$ are the 2nd roots of unity

Claim: 3rd roots of unity




$$\Rightarrow \quad u[n] = \frac{1}{3}\left( x[n] + e^{j\frac{2\pi}{3}n} x[n] + e^{j\frac{4\pi}{3}n} x[n] \right)$$

- $$U(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j\left(\omega - \frac{2k\pi}{3}\right)}\right)$$

- $$U(e^{j\omega}) = V(e^{j3\omega})$$

$$\boxed{V(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j\left(\frac{\omega - 2k\pi}{3}\right)}\right)}$$

<u>Remark:</u> Before downsampling by factor 3, band limit signal to $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ to avoid aliasing.

# Downsample by factor M

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow v[n] = x[Mn]$$

$$\boxed{V(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega - 2k\pi}{M}\right)}\right)}$$

$$\boxed{V(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} e^{-j\frac{2k\pi}{M}}\right)}$$

**Exercise:** Compare

$$\longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow M} \longrightarrow \quad \text{vs.} \quad \xrightarrow{x[n]} \boxed{\uparrow M} \longrightarrow \boxed{\downarrow M} \xrightarrow{x[n]}$$

keeps every Mth coeff. and sets the rest to 0.

identity

# Multirate / Noble Identities

**Recall:** Multirate DSP is hard because operations don't commute.

- up/down sampling is time-varying

Multirate Identities help to quickly analyze multirate systems.

(1) $$\rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow \;\equiv\; \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow$$

(2) $$\rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow \;\equiv\; \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow$$

(3) If $\gcd\{M, L\} = 1$:

Ex: • $M=3,\ L=4$ ✓
• $M=2,\ L=4$ ✗

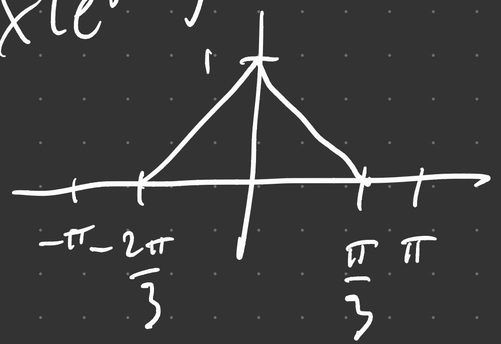
$$\rightarrow \boxed{\downarrow M} \rightarrow \boxed{\uparrow L} \rightarrow \;\equiv\; \rightarrow \boxed{\uparrow L} \rightarrow \boxed{\downarrow M} \rightarrow$$

**Proof:** Do it at home.

## Exercise:

$$x[n] \to \boxed{z^{-3}} \to \boxed{\uparrow 2} \to \boxed{\downarrow 3} \to \boxed{\uparrow 3} \to y[n]$$

$X(e^{j\omega})$




What is $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$?

Sol$^{\underline{n}}$: Don't do it with brute force!

$$X(z) \to \boxed{z^{-3}} \to z^{-3}X(z)$$

annoying to deal with...

By ③: $\to \boxed{z^{-3}} \to \boxed{\downarrow 3} \to \boxed{\uparrow 2} \to \boxed{\uparrow 3} \to$

By ①: $\to \boxed{\downarrow 3} \to \boxed{z^{-1}} \to \boxed{\uparrow 2} \to \boxed{\uparrow 3} \to$

By ②: $\to \boxed{\downarrow 3} \to \boxed{\uparrow 2} \to \boxed{z^{-2}} \to \boxed{\uparrow 3} \to$

By ②: $x[n] \to \boxed{\downarrow 3} \to \boxed{\uparrow 2} \to \boxed{\uparrow 3} \xrightarrow{v[n]} \boxed{z^{-6}} \to y[n] = v[n-6]$

$$x[n] \to \boxed{\downarrow 3} \to \boxed{\uparrow 3} \xrightarrow{u[n]} \boxed{\uparrow 2} \to v[n]$$

$$U(z) = \frac{1}{3} \sum_{k=0}^{2} X\left(z\, e^{-j\frac{2k\pi}{3}}\right)$$

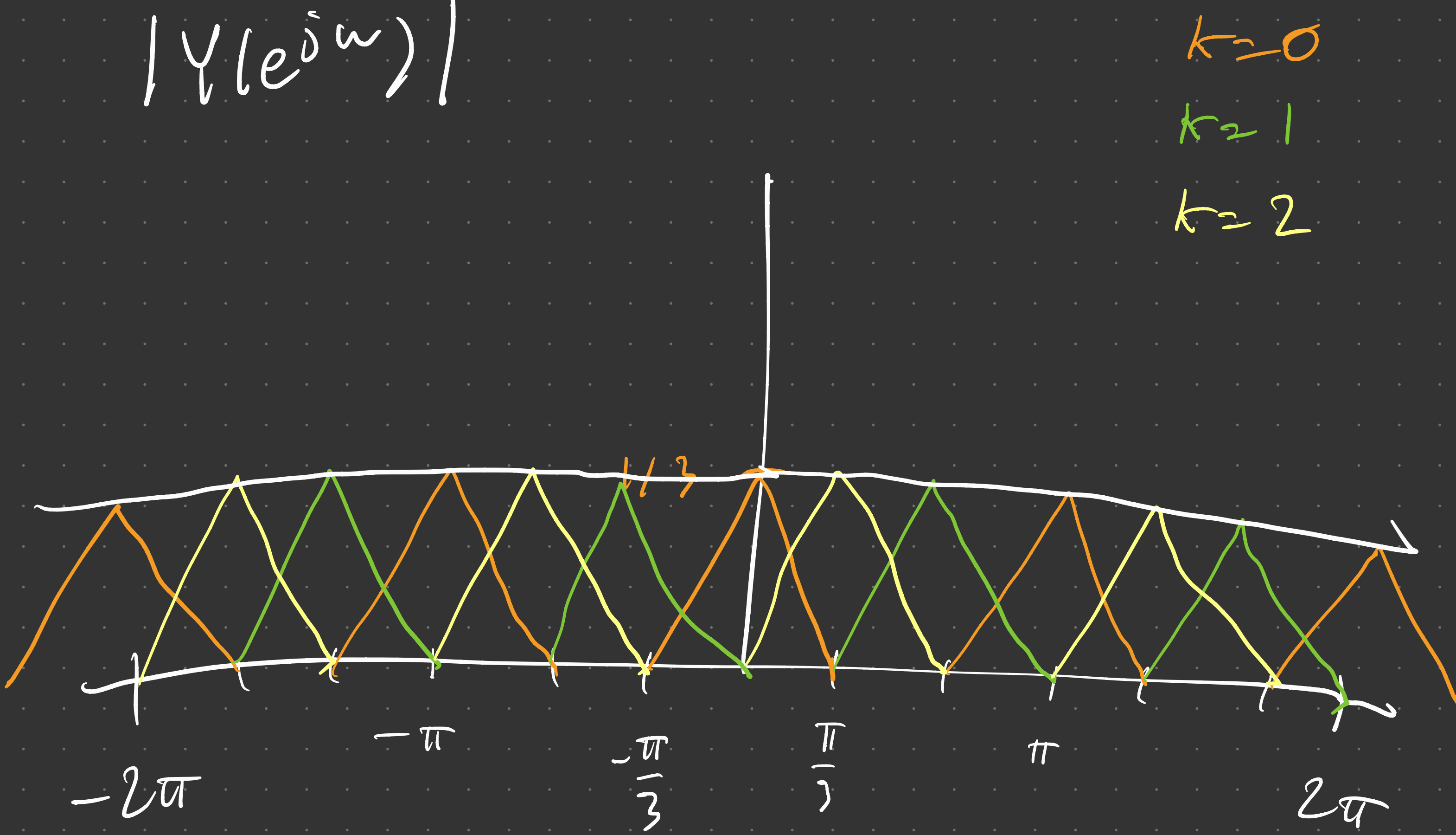
$$V(z) = \frac{1}{3} \sum_{k=0}^{2} X\left(z^2 e^{-j\frac{2k\pi}{3}}\right)$$

$$V(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j2\left(\omega - \frac{k\pi}{3}\right)}\right)$$

$$Y(e^{j\omega}) = e^{-j6\omega} \cdot \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j2\left(\omega - \frac{k\pi}{3}\right)}\right)$$

Exer: Try different orders of the Noble identities.

$|Y(e^{j\omega})|$




**Obs:** Severe aliasing?

**Q:** How could we see this before?

**A:** Input **NOT** bandlimited to $[-\frac{\pi}{3}, \frac{\pi}{3}]$ and we immediately downsample by 3.