

Welcome to ECE 251C

Filter Banks and Wavelets

Course will primarily focus on:

- ① Multirate Signal Processing
- ② Filter Banks
- ③ Wavelets

Lectures: Theory

Quizzes : Based on lectures & Hw
&

Midterm

(uncollected)

Project : Theory and/or applications
(open ended)

Filters / LTI Systems (Review)

Two kinds of filters:

- ① Finite Impulse Response (FIR)
- ② Infinite Impulse Response (IIR)

Q: What is the difference?

Q: What is an impulse response?

Systems:

Sampled
Signal
 $n \in \mathbb{Z}$

$x[n]$

Linear
Time / Translation
Invariant



Linear Systems:

Q: What does it mean for a system
to be linear?



$$A: \text{If } x_1[n] \rightarrow \boxed{H} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{H} \rightarrow y_2[n]$$

Then:

$$\forall \alpha, \beta \in \mathbb{R}$$

$$\alpha x_1[n] + \beta x_2[n] \rightarrow \boxed{H} \rightarrow \alpha y_1[n] + \beta y_2[n]$$

Linear Time / Translation Invariant Systems:

The system does not change when it is turned on.

$$x[n] \rightarrow \boxed{H} \rightarrow y[n]$$

$$\begin{array}{l} \text{delayed} \\ \text{input} \end{array} x[n-k] \rightarrow \boxed{H} \rightarrow y[n-k] \quad \begin{array}{l} \text{delayed} \\ \text{output} \end{array}$$

Impulse Response:

$$\delta[n] \rightarrow \boxed{H} \rightarrow h[n]$$

Kronecker delta

impulse response

$$x[n] \rightarrow \boxed{H} \rightarrow y[n] = (h * x)[n]$$

$$= \sum_{k \in \mathbb{Z}} h[k] x[n-k]$$

convolution

The sequence $h[n], n \in \mathbb{Z}$, is called a filter. The support of $h[n]$ is

$$\text{Supp}(h) = \{n \in \mathbb{Z} : h[n] \neq 0\}$$

Defⁿ: A filter h is FIR if $|\text{Supp}(h)| < \infty$.

Def⁼: A filter h is IIR if $|\text{Supp}(h)| = \infty$.

Ex: Consider the system

$$y[n] = \frac{1}{2} (x[n] + x[n-1]).$$

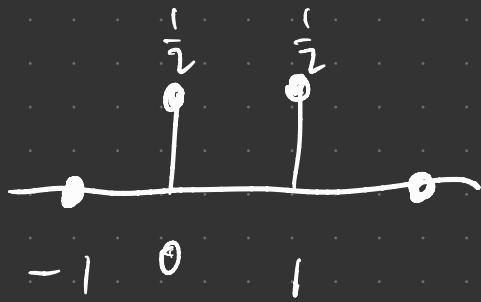
Q: What does this system do?

A:

- Moving average
- Smoothing
- Low-pass filter

Ex: What is the impulse response of this system?

$$h[n] = \frac{1}{2} (\delta[n] + \delta[n-1])$$



2-tap FIR
filter

Q: How do we know this is low pass?

A: Look at its frequency response.

$$\begin{aligned}
 e^{j\omega n} &\xrightarrow{\text{H}} \sum_{k \in \mathbb{Z}} h[k] e^{j\omega(n-k)} \\
 &= e^{j\omega n} \left(\sum_{k \in \mathbb{Z}} h[k] e^{-j\omega k} \right) \\
 &= \underline{e^{j\omega n}} \quad \underline{H(e^{j\omega})}
 \end{aligned}$$

DTFT of $h[n]$
frequency response

Obs: Pure frequencies (complex exponentials)
are eigen functions of LTI systems.

Convolution Theorem:

$$Y[n] = (h * x)[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

Obs: Fourier transforms turn convolutions
into multiplications.

Z-Transform

$$X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n}, \quad z \in \mathcal{C}$$

$$X(e^{j\omega}) = X(z) \Big|_{z=e^{j\omega}}$$



$$\text{Ex: } h[n] = \frac{1}{2} (\delta[n] + \delta[n-1])$$

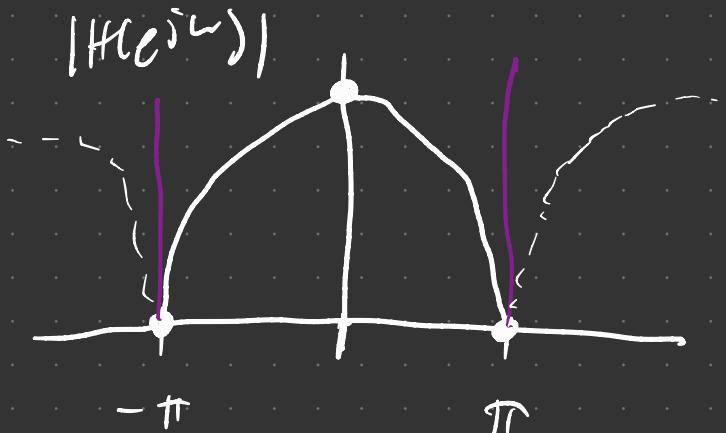
$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{2} + \frac{1}{2} e^{-j\omega} \\ &= e^{-\frac{j\omega}{2}} \left[\frac{e^{\frac{j\omega}{2}} + e^{-\frac{j\omega}{2}}}{2} \right] \end{aligned}$$

$$= \underbrace{e^{-\frac{j\omega}{2}}}_{\text{phase}} \underbrace{\cos\left(\frac{\omega}{2}\right)}_{\text{amplitude}}$$

$$\hookrightarrow e^{j\phi(\omega)} \Rightarrow \phi(\omega) = -\frac{\omega}{2}$$

phase response

$|H(e^{j\omega})| = |\cos(\frac{\omega}{2})|$ is the magnitude response.



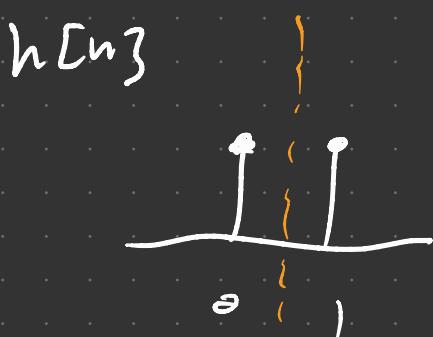
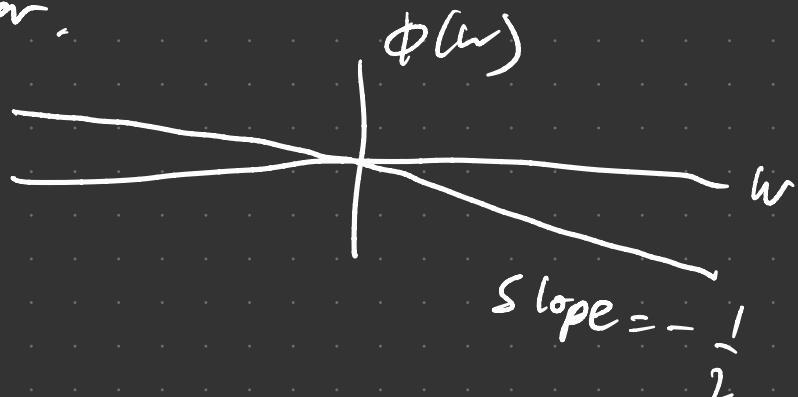
Attenuates high frequencies \Rightarrow low-pass

Linear-Phase Filters

Defn: $H(e^{j\omega}) = e^{j\phi(\omega)} H_{\text{amp}}(\omega)$

is called linear-phase if
 $\phi(\omega)$ is linear.

Ex: $\phi(\omega) = -\frac{\omega}{2}$



$\frac{1}{2} = \text{center of symmetry}$

Symmetric / Antisymmetric filters provide a complete characterization of linear-phase filters.

Type I: odd length / symmetric

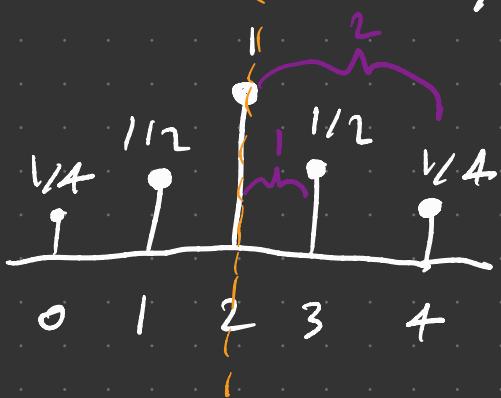
Type II: even length / symmetric

Type III: odd length / anti-symmetric

Type IV: even length / anti-symmetric

Type I: Center of symmetry $L \in \mathbb{Z}$

Ex:



$$L=2 \in \mathbb{Z}$$

$$H(e^{j\omega}) = e^{-j2\omega} \left[1 + 2\left(\frac{1}{2}\right) \cos(\omega) + 2\left(\frac{1}{4}\right) \cos(2\omega) \right]$$

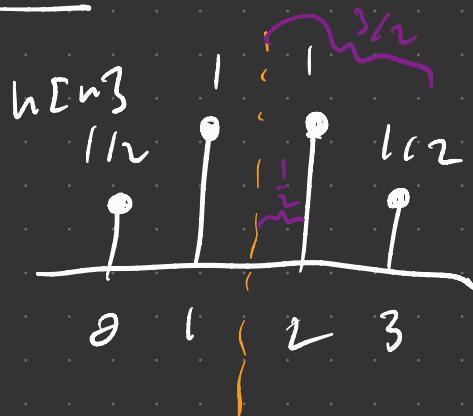
General:

$$H(e^{j\omega}) = e^{-jL\omega} \sum_{n=0}^L a[n] \cos(n\omega)$$

Exercise: Express $a[n]$ in terms of $h[n]$.

Type II: Center of Symmetry $L - \frac{1}{2} \in \mathbb{Z}$

Ex:



$$L = \frac{3}{2} \Rightarrow L - \frac{1}{2} = 1 \in \mathbb{Z}$$

$$H(e^{j\omega}) = e^{-j\frac{3}{2}\omega} \left[2(1) \cos\left(\frac{\omega}{2}\right) + 2\left(\frac{1}{i}\right) \cos\left(\frac{3\omega}{2}\right) \right]$$

General:

$$H(e^{j\omega}) = e^{-jL\omega} \sum_{n=0}^{L-\frac{1}{2}} b[n] \cos\left(n\omega + \frac{1}{2}\right)$$

Exercise: Express $b[n]$ in terms of $h[n]$.

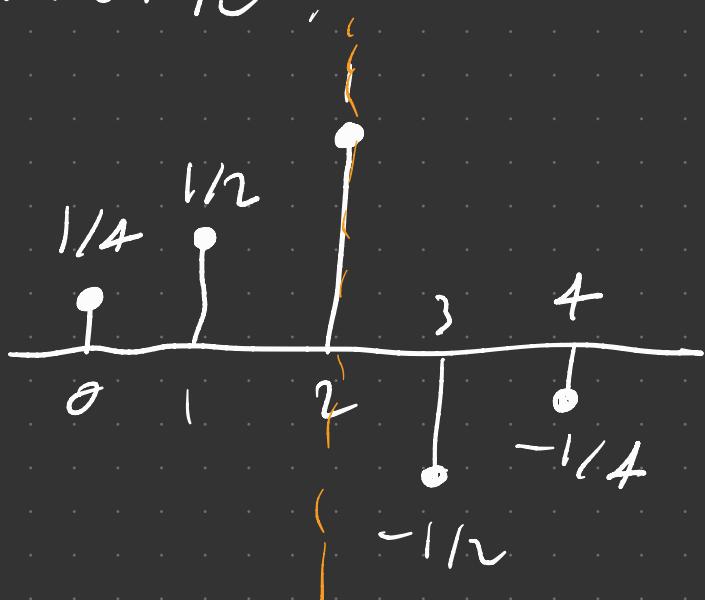
Q: What makes a filter $h[n]$ symmetric?

A: $h[n] = h[2L-n]$

Q: What makes a filter $h[n]$ anti-symmetric?

A: $h[n] = -h[2L-n]$

Exercise: Is the following filter anti-symmetric?



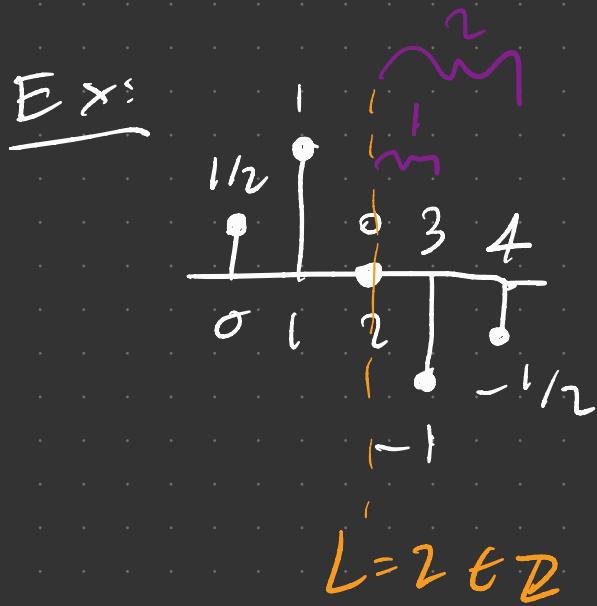
Not
anti-symmetric!

$$L=2$$

Obs: $h[n] = -h[2L-n]$, $\forall n \in \mathbb{Z}$

$$\Rightarrow h[L] = -h[L] \Rightarrow \boxed{h[L] = 0}$$

Type III: Center of anti-symmetry $L \in \mathbb{Z}$



$$H(e^{j\omega}) = e^{-j2\omega} \left[2(1)j \sin(\omega) + 2\left(\frac{1}{2}\right) \sin(2\omega) \right]$$

$$= e^{j(-2\omega + \frac{\pi}{2})} \left[2 \sin(\omega) + \sin(2\omega) \right]$$

$$j = e^{j\frac{\pi}{2}}$$

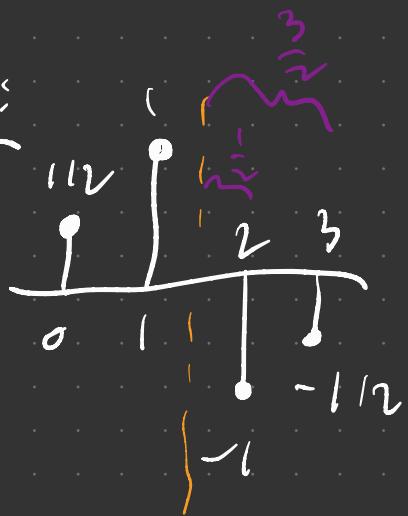
General:

$$H(e^{j\omega}) = e^{j(-L\omega + \frac{\pi}{2})} \sum_{n=0}^L C[n] \sin(n\omega)$$

Exercise: Express $\{C[n]\}$ in terms of $\{h[n]\}$.

Type IV : Center of anti-symmetry $L - \frac{1}{2} \in \mathbb{Z}$

Ex:



$$L = \frac{3}{2} \Rightarrow L - \frac{1}{2} = 1 \in \mathbb{Z}$$

$$H(e^{i\omega}) = e^{i(-\frac{3}{2}\omega + \frac{\pi}{2})} \left[2c_1 \sin\left(\frac{\omega}{2}\right) + 2\left(\frac{1}{2}\right) \sin\left(\frac{3\omega}{2}\right) \right]$$

General:

$$H(e^{i\omega}) = e^{i(-L\omega + \frac{\pi}{2})} \sum_{n=0}^{L-\frac{1}{2}} d[n] \sin((n+\frac{1}{2})\omega)$$

Exercise: Express $d[n]$ in terms of $h[\omega]$,

Summary of Linear-Phase Filters

$$H(e^{j\omega}) = e^{j(-L\omega + \beta)} H_{\text{imp}}(\omega)$$

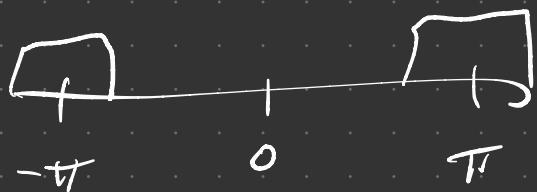
Phase response is linear $\Rightarrow \phi(\omega) = -L\omega + \beta$

$H_{\text{imp}}(\omega)$	$L \in \mathbb{Z}$	$L - \frac{1}{2} \in \mathbb{Z}$
Symmetric $\beta = 0$ $h[n] = h[L-n]$	I $\sum_{n=0}^L a[n] \cos(n\omega)$	$\sum_{n=0}^{L-\frac{1}{2}} b[n] \cos\left((n+\frac{1}{2})\omega\right)$
Anti-sym. $\beta = \frac{\pi}{2}$ $h[n] = -h[L-n]$	III $\sum_{n=0}^L c[n] \sin(n\omega)$	IV $\sum_{n=0}^{L-\frac{1}{2}} d[n] \sin\left((n+\frac{1}{2})\omega\right)$

Remark: Many wavelet filters are linear-phase.

Exercise: Can a high-pass filter be Type II?

$$|H(e^{j\omega})|$$



No. Proof:

$$\cos((n+\frac{1}{2})\pi) = 0 \text{ for } n \in \mathbb{Z}.$$

Obs: Type II filters necessarily have a zero at π.