

Last Time: Linear-Phase Filters

$$H(e^{j\omega}) = e^{j\phi(\omega)} H_{\text{amp}}(\omega)$$

Defⁿ: $h[n]$ is said to be linear-phase, if the phase response $\phi(\omega)$ is linear.

$$\phi(\omega) = -L\omega + \beta$$

L : center of the filter

β : controls the symmetry of the filter

$H_{\text{amp}}(\omega)$	$L \in \mathbb{Z}$	$L - \frac{1}{2} \in \mathbb{Z}$
Symmetric $\beta = 0$	I $\sum_{n=0}^L a[n] \cos(n\omega)$	II $\sum_{n=0}^{L-1} b[n] \cos\left((n+\tfrac{1}{2})\omega\right)$
$h[n] = h[2L-n]$		
Anti-sym. $\beta = \frac{\pi}{2}$	III $\sum_{n=0}^L c[n] \sin(n\omega)$	IV $\sum_{n=0}^{L-1} d[n] \sin\left((n+\tfrac{1}{2})\omega\right)$
$h[n] = -h[2L-n]$		

Group Delay: Delay of each frequency when filtering with $h[n]$. Mathematically, this is captured by $-\phi'(w)$.

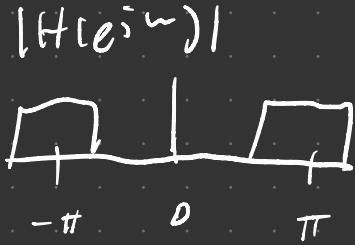
For linear-phase filters:

$$-\phi'(w) = L$$

center of $h[n]$

Obs: Linear-phase filters have constant group delay.

Last Time: High-pass filters cannot be Type II.

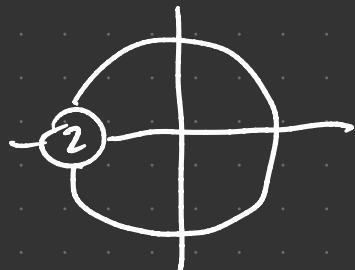


Proof: $\cos((n+\frac{1}{2})\pi) = 0 \quad \forall n \in \mathbb{Z}$.

Remark: Different Types of linear-phase filters put different constraints on the kinds of filters that are realizable. [More exploration in HW]

Exercise: What happens if you cascade two linear-phase filters of the same Type?

Ex: $H(e^{j\omega})$ has two zeros @ π .

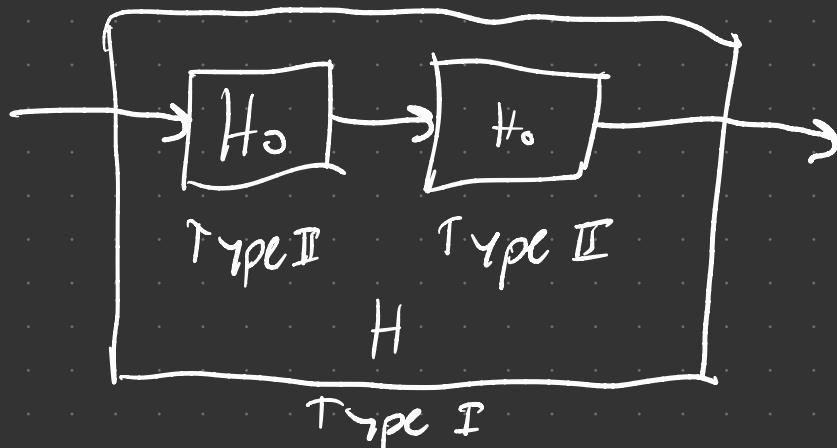
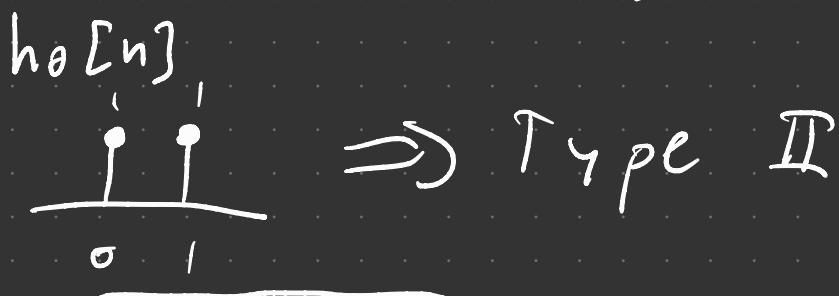


Q: What Type is this?

$$H(z) = (1 + z^{-1})^2 = 1 + 2z^{-1} + z^{-2}$$



Obs: $H(z) = H_0(z) H_0(z)$, $H_0(z) = (1 + z^{-1})$



Cascade of two Type II filters is Type I.

Claim: The cascade of two linear-phase filters of the same Type is always Type I.

Proof: Consider the cascade

$$H(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$$

with

$$H_k(e^{j\omega}) = e^{j(-L_k \omega + \beta)} \text{Hampl}_{k,K}(\omega) \quad k=1, 2.$$

$$\beta = 0 \text{ or } \frac{\pi}{2}$$

Then,

$$\begin{aligned} H(e^{j\omega}) &= e^{j(-(L_1+L_2)\omega + 2\beta)} \text{Hampl}_{1,1}(\omega) \text{Hampl}_{1,2}(\omega) \\ &= e^{j(-(L_1+L_2)\omega)} e^{j2\beta} \underbrace{\text{Hampl}_{1,1}(\omega) \text{Hampl}_{1,2}(\omega)}_{\text{Hampl}(\omega) \text{ real function}} \end{aligned}$$

Observe that $L_1+L_2 \in \mathbb{Z}$ & $(h_1 * h_2)$ is always odd length & symmetric.

∴ $H(e^{j\omega})$ is Type I. □

Remark: Linear-phase filters are tightly constrained. If $h[n]$ is linear-phase with real coefficients and z_0 is a zero of $H(z)$, then

$$(z_0, z_0^{-1}, z_0^*, z_0^{-1*}) \text{ are all zeros}$$

[More explanation in b/w]

All-Pass Filters

Defⁿ: $h[n]$ is said to be all-pass if

$$|H(e^{j\omega})| = 1.$$

Q: What is the point of an all-pass filter?

A: To manipulate the phase response.

- Minimum-phase filters
- Maximum-phase filters

Ex: Simple examples of all-pass filters:

- $H(z) = 1$ identity filter / do nothing

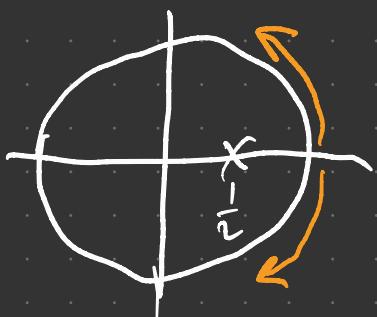
$$\Rightarrow h[n] = \delta[n]$$

- $H(z) = z^{-N}$ delay

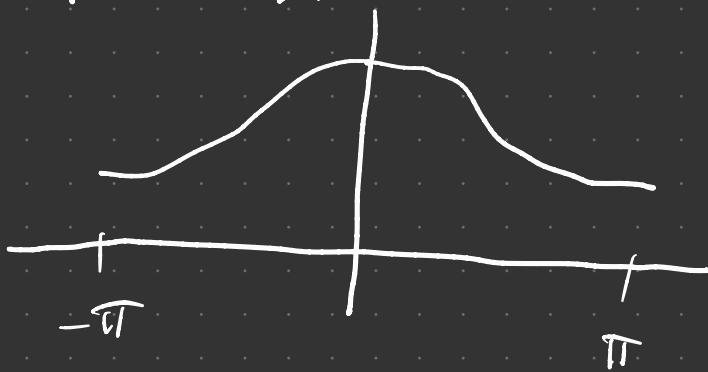
$$\Rightarrow h[n] = \delta[n-N]$$

(Non)

Ex:



$|H(e^{j\omega})|$



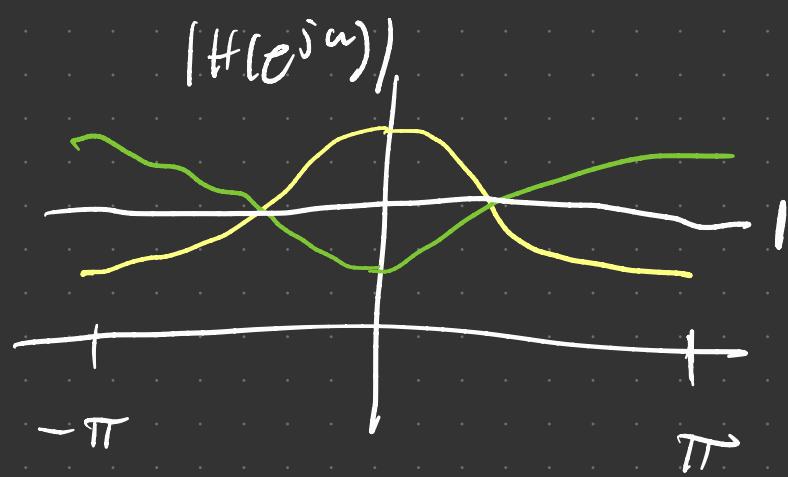
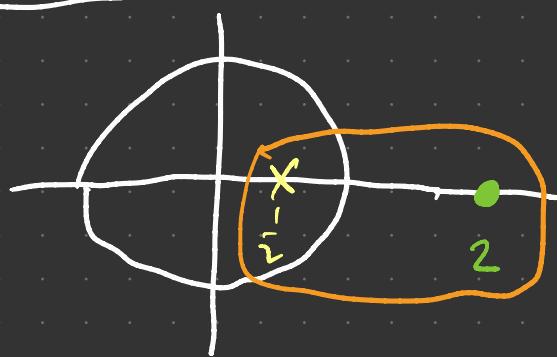
The pole "pushes" you with
"Strength" proportional to the
distance from the pole.

Obs: This is not an all-pass.

Q: How do we make it all-pass?

A: Add a zero to "counteract" the pole.

Ex:



All-pass systems

have pole-zero pairs

$$H(z) = \frac{-\frac{1}{2} + z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

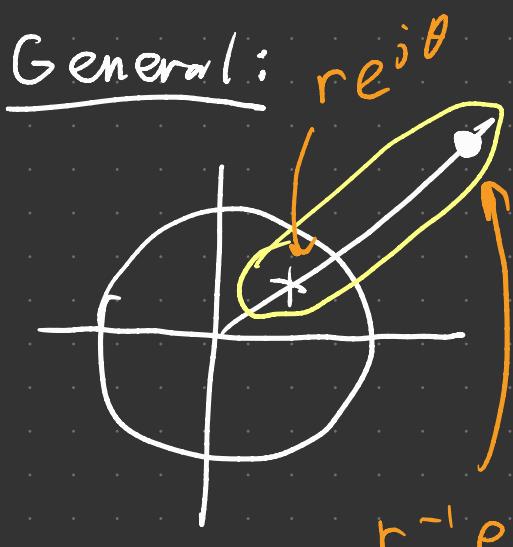
Is this actually
all-pass?

$$H(e^{j\omega}) = \frac{-\frac{1}{2} + e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}} = e^{-j\omega} \begin{bmatrix} -\frac{1}{2}e^{j\omega} + 1 \\ 1 - \frac{1}{2}e^{-j\omega} \end{bmatrix}$$

unit magnitude

complex conj. pairs

$$\Rightarrow |H(e^{j\omega})| = 1.$$



$$H(z) = \frac{-re^{-j\theta} + z^{-1}}{1 - re^{j\theta}z^{-1}}$$

flip coeff
and take
complex
conj. of
denom. to
get num.

Exercise: Where is the zero?

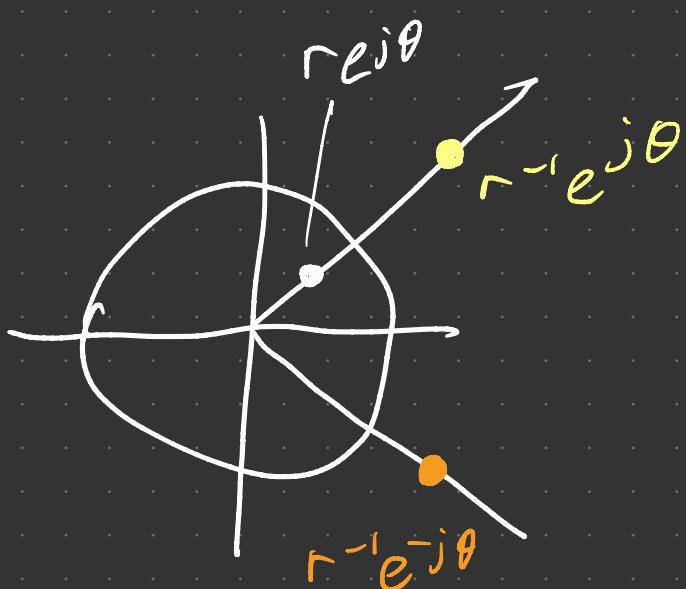
Complex Zero Tricks

$(1 - re^{j\theta} z^{-1}) \Rightarrow$ zero @ $re^{j\theta}$

flip coeffs $(-re^{j\theta} + z^{-1}) \Rightarrow$ zero @ $r^{-1}e^{-j\theta}$

flip coeffs $(-re^{-j\theta} + z^{-1}) \Rightarrow$ zero @ $r^{-1}e^{j\theta}$

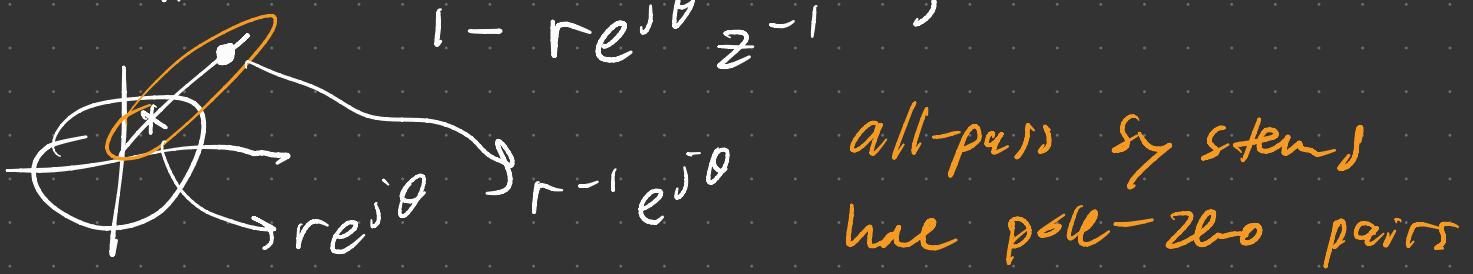
+ complex conj.



Remark: All three have the same magnitude resp.

Defⁿ: A causal and stable system is first-order a li-pass if

$$H(z) = \frac{-re^{-j\theta} + z^{-1}}{1 - re^{j\theta} z^{-1}}, \quad |r| < 1$$



Defⁿ: A causal and stable system is

Nth-order all-pass if

$$H(z) = \prod_{k=1}^N \frac{r_k e^{-j\theta_k} + z^{-1}}{1 - r_k e^{j\theta_k} z^{-1}}, |r_k| < 1$$

Theorem: The phase response $\phi(\omega)$ of a stable and causal all-pass system is monotone nonincreasing



Proof is long and boring...

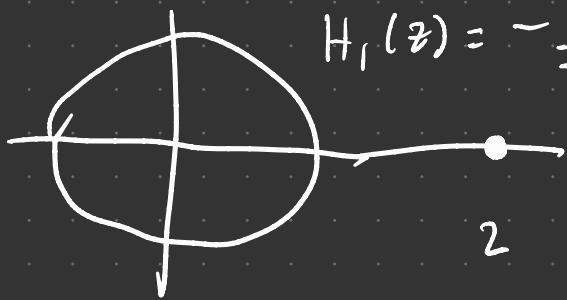
Exercise:



$$H_0(z) = 1 - \frac{1}{z} z^{-1}$$

Find all systems with the same magnitude response and order.

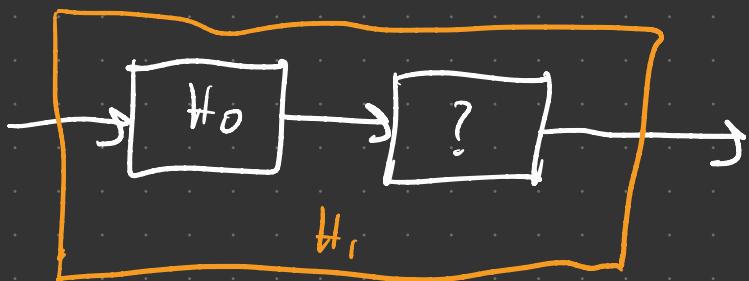
By inspection:



$$H_0(z) = -\frac{1}{2} + z^{-1}$$

Q: What is the difference between H_0 & H_1 ?
A: The phase.

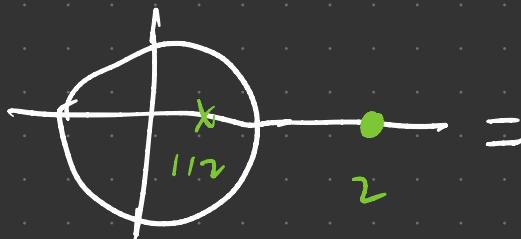
Exercise: How are these two systems related?



All-pass



$$H_0(z)$$



$$H(z)$$



$$H_1(z)$$

$$\left(1 - \frac{1}{2}z^{-1}\right) \cdot \frac{\left(\frac{1}{2} + z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)} = \left(\frac{1}{2} + z^{-1}\right)$$

phase response



Minimum phase



Maximum phase

Obs: All-pass systems can be used to derive many new systems with the same magnitude response without changing the system effect.

- Low-pass remains low-pass
- High-pass remains high-pass
- etc.

Obs: Minimum-phase "act faster"

- smaller group delay

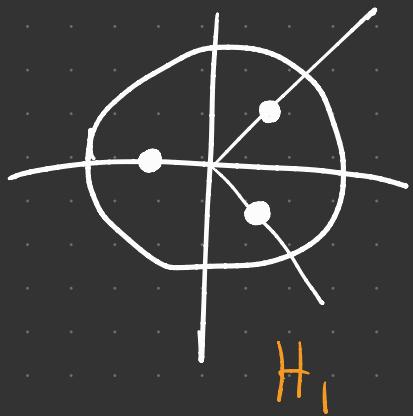
Maximum-phase "act slower"

Obs: You can make a system act faster or slower with an all-pass filter.

Remark: This is how you understand DSP

- Understand how poles/zeros affect the system
- Understand how to manipulate them.

Exercise :



How many other systems have the same order and magnitude response?

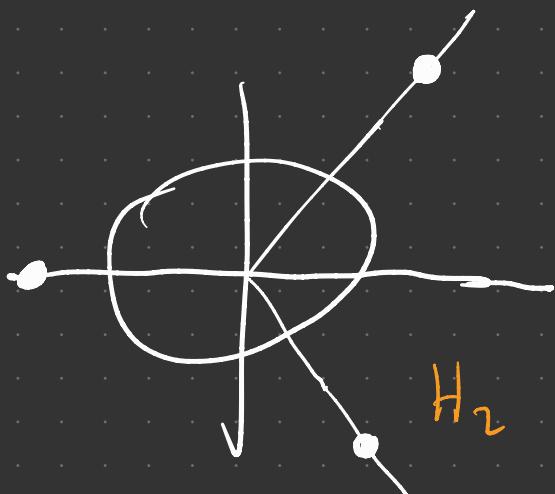
Solⁿ: Flip zeros

$$\Rightarrow 2^3 = 8 \text{ total systems.}$$

General: 2^N , N is the # of zeroes

Q: How many with real coefficients?

A: 4, The two complex zeros need to flip together.



Q: Which one is

- Minimum-phase ?
- Maximum-phase ?

A: Minimum-phase = H_1

Maximum-phase = H_2

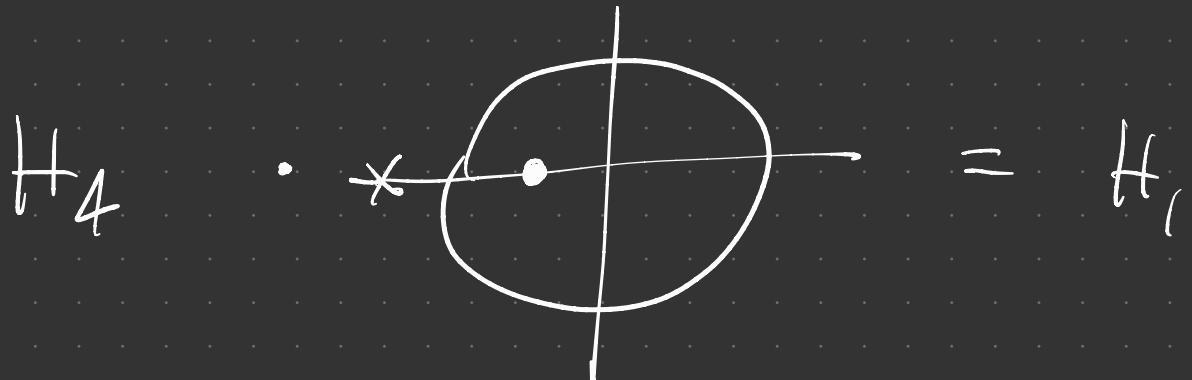
Why? By flipping a zero
inside the unit circle,

you decrease the phase

Proof: Previous exercise.

Obs:

All-pass



Alternative Characterization of Min/Max-Phase

Define the partial energy

$$E[n] = \sum_{k=-\infty}^n |h[k]|^2$$

Theorem: For all systems of same order and magnitude

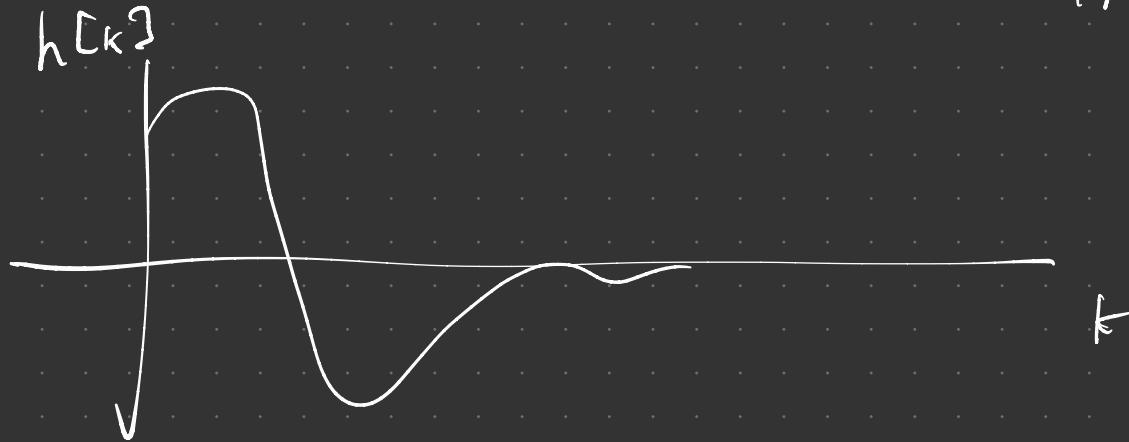
$$\sum_{k=-\infty}^n |h[k]|^2 \leq \sum_{k=-\infty}^n |h_{\min}[k]|^2 \quad \forall n \in \mathbb{Z}$$

where $h_{\min}[n]$ is the min-phase and $h[n]$ is any other system in this family, i.e., h_{\min} has the largest partial energy.

Theorem: The max-phase system $h_{\max}[n]$ has the smallest partial energy.

$$\sum_{k=-\infty}^n |h_{\max}[k]|^2 \leq \sum_{k=-\infty}^n h[k] \quad \forall n \in \mathbb{Z}$$

Obs: • Min-phase has energy concentrated in the "front"



• Max-phase has energy concentrated in the "back"

