Last Time: Polyphase Repesentations

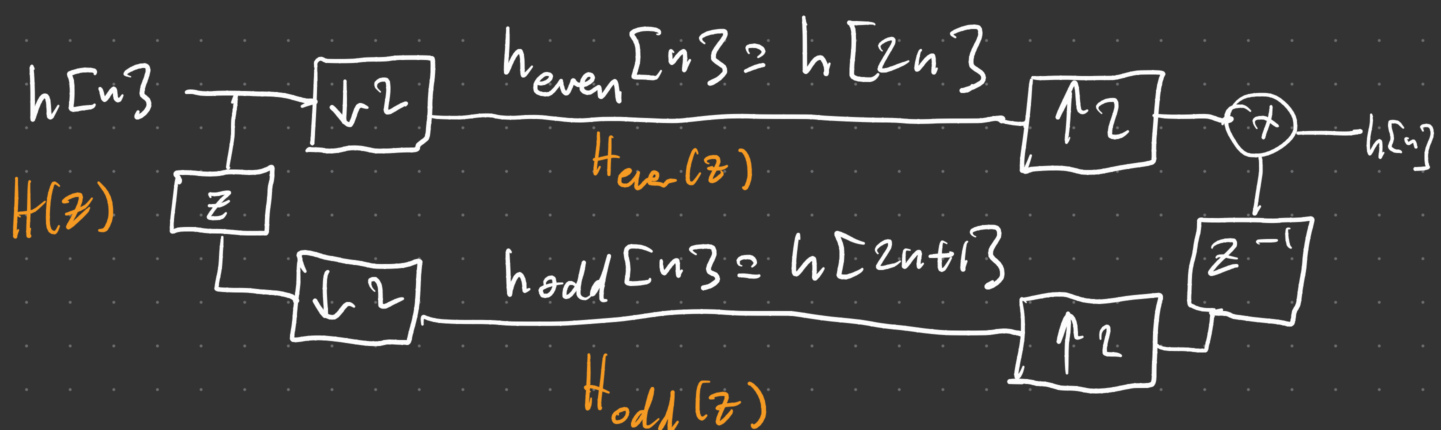
z-Domain:

$$H(z) = H_{even}(z^2) + z^{-1} H_{odd}(z^2)$$

even coeffs odd coeffs

Time-Domain:

$$h_{even}[n] = h[2n], \quad h_{odd}[n] = h[2n+1]$$



Obs:

$$\begin{cases} H(z) = H_{even}(z^2) + z^{-1} H_{odd}(z^2) \\ H(-z) = H_{even}(z^2) - z^{-1} H_{odd}(z^2) \end{cases}$$

$$\Rightarrow \begin{cases} H_{even}(z^2) = \frac{1}{2}\left(H(z) + H(-z)\right) \\ z^{-1} H_{odd}(z^2) = \frac{1}{2}\left(H(z) - H(-z)\right) \end{cases}$$

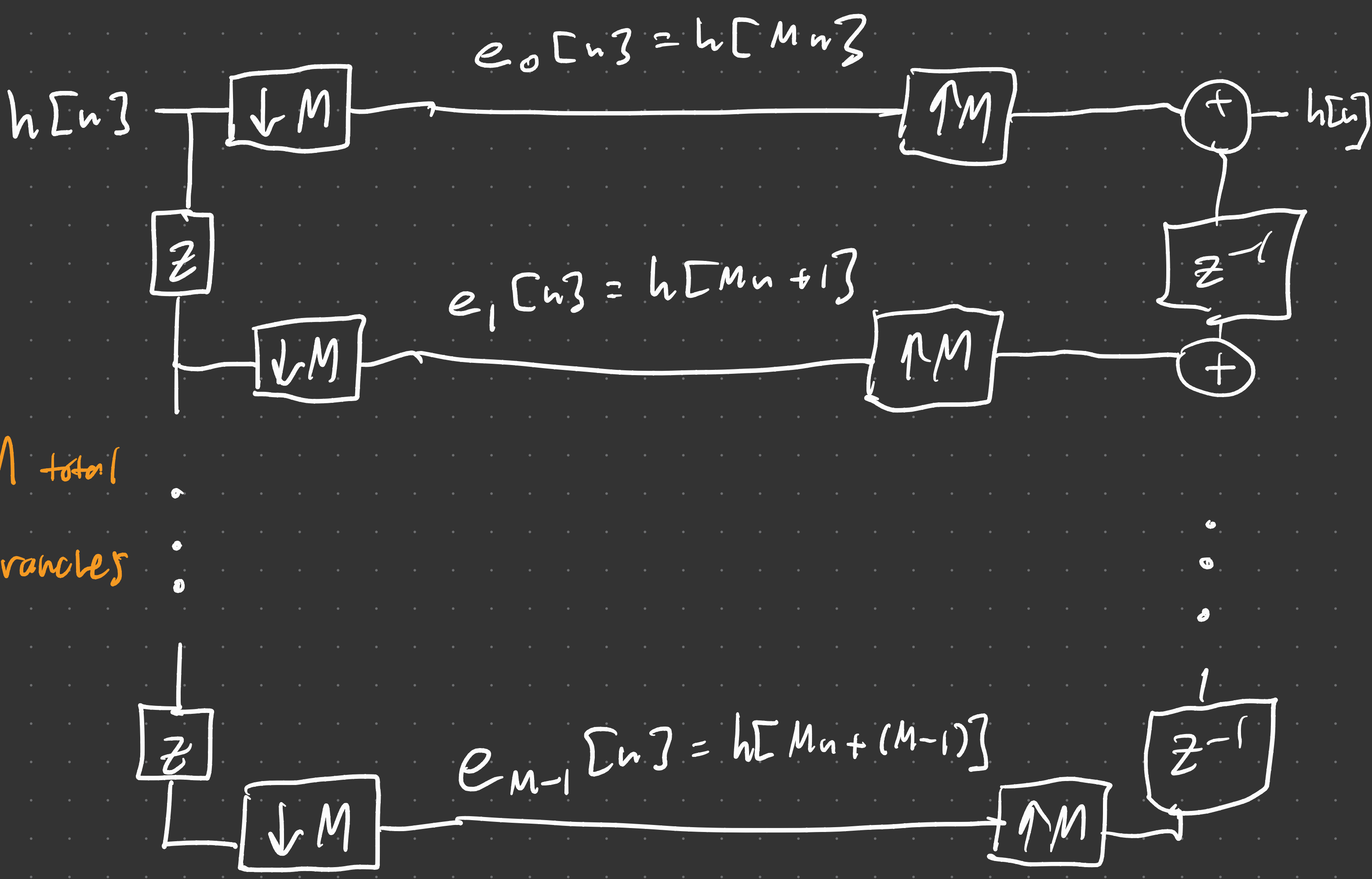
$$★ \quad \begin{bmatrix} H(z) \\ H(-z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} H_{even}(\tilde{z}) \\ z^{-1} H_{odd}(\tilde{z}) \end{bmatrix}$$

(unnormalized)
2-point DFT

Obs: In ★, we have

- 2nd roots of unity $(H((+1)z), H((-1)z))$
- 2-point DFT

General M-Polyphase



M total branches

$$e_k[n] = h[Mn+k], \quad H(z) = \sum_{k=1}^{M-1} z^{-k} E_k(z^M)$$

$$\begin{bmatrix} H(z) \\ H(e^{j\frac{2\pi}{M}} z) \\ \vdots \\ H(e^{j\frac{2(M-1)\pi}{M}} z) \end{bmatrix} = F_M \begin{bmatrix} E_0(z^M) \\ z^{-1} E_1(z^M) \\ \vdots \\ z^{-(M-1)} E_{M-1}(z^M) \end{bmatrix}$$

M-point DFT

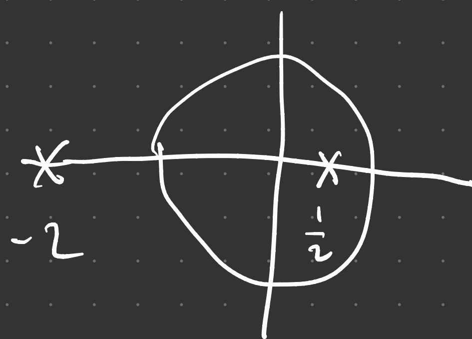
Exercise: $M=2$

partial fractions

$$H(z) = \frac{3+z^{-1}}{1+\frac{3}{2}z^{-1} - z^{-2}} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+2z^{-1}}$$

$$-\frac{1}{2} + 2 = \frac{3}{2}$$

$$-\frac{1}{2} \cdot 2 = -1$$



$$\left[\begin{array}{l} A(1+2z^{-1}) + B(1-\frac{1}{2}z^{-1}) = 3+z^{-1} \\ \left.\begin{array}{l} A+B=3 \\ 2A - \frac{B}{2} = 1 \end{array}\right\} \Rightarrow A=1,\ B=2 \end{array}\right.$$

$$H_{even}(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-4z^{-1}}$$

$$H_{odd}(z) = \frac{\frac{1}{2}}{1-\frac{1}{4}z^{-1}} + \frac{2(-2)}{1-4z^{-1}}$$

Exercise: $M = 3$

$$H(z) = \frac{1}{1 - az^{-1}}$$

$$= \frac{1}{1 - az^{-1}} \cdot \frac{(1 + az^{-1} + a^2 z^{-2})}{(1 + az^{-1} + a^2 z^{-2})}$$

$$= \frac{1 + az^{-1} + a^2 z^{-2}}{1 - a^3 z^{-3}}$$

$$= \frac{1}{1 - a^3 z^{-3}} + z^{-1} \frac{a}{1 - a^3 z^{-3}} + z^{-2} \frac{a^2}{1 - a^3 z^{-3}}$$

$$E_0(z) = \frac{1}{1 - a^3 z^{-1}}$$

$$E_1(z) = \frac{a}{1 - a^3 z^{-1}}$$

$$E_2(z) = \frac{a^2}{1 - a^3 z^{-1}}$$

Exercise: Generalize this to arbitrary M.

Exercise: Let

$$H(z) = \frac{3 + z^{-1}}{1 + \frac{3}{2} z^{-1} - z^{-2}} = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{2}{1 + 2 z^{-1}}$$

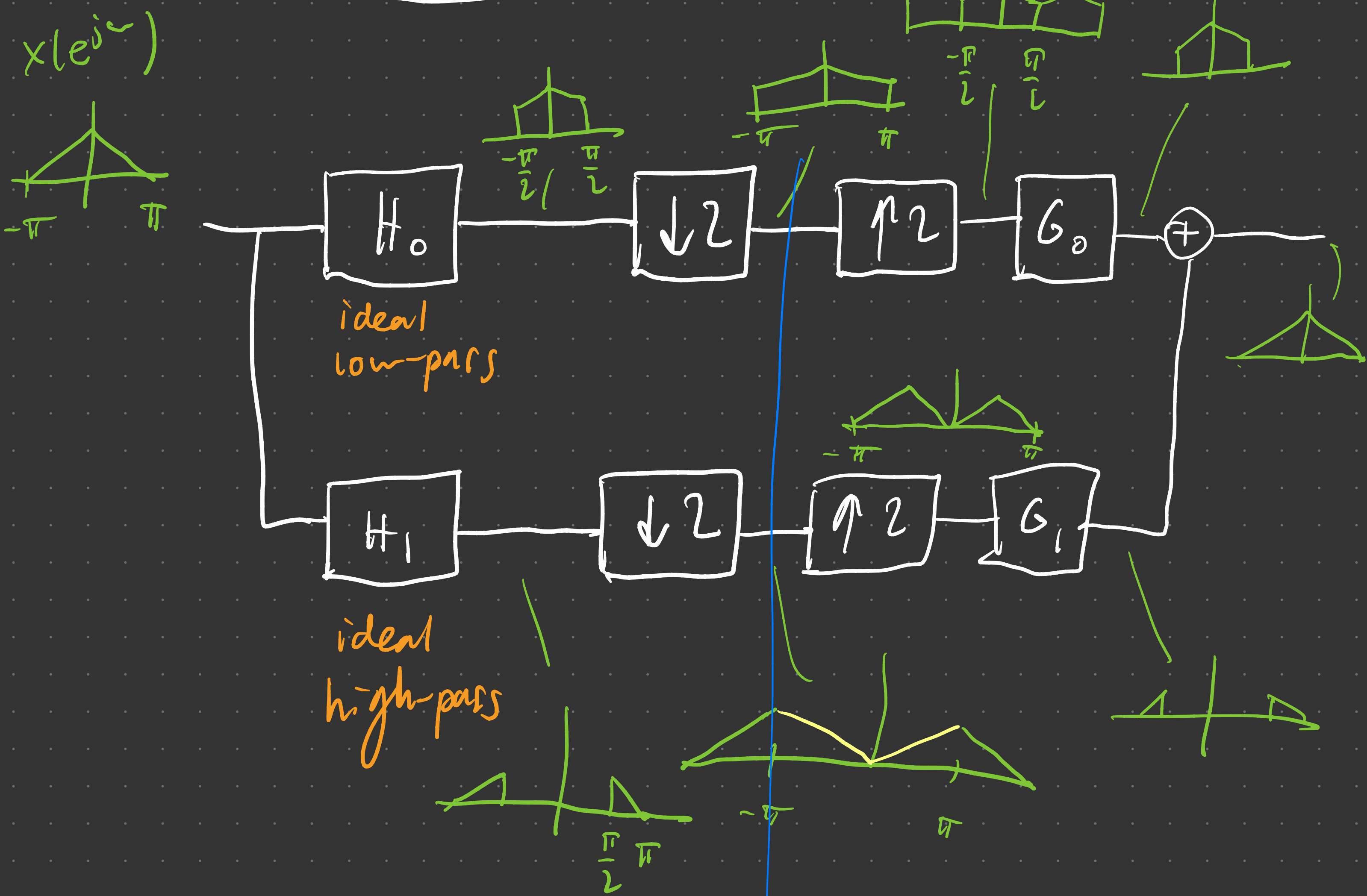
Determine $E_0(z)$, $E_1(z)$, and $E_2(z)$.

$$E_0(z) = \frac{1}{1 - \frac{1}{8} z^{-1}} + \frac{2}{1 + 8 z^{-1}}$$

$$E_1(z) = \frac{\frac{1}{2}}{1 - \frac{1}{8} z^{-1}} + \frac{2(-2)}{1 + 8 z^{-1}}$$

$$E_2(z) = \frac{\left(\frac{1}{2}\right)^2}{1 - \frac{1}{8} z^{-1}} + \frac{2(-2)^2}{1 + 8 z^{-1}}$$

Two-channel Filter bank



Analysis Bank

Synthesis Bank

coding, compression, processing, etc.

(Nonlinear operations)

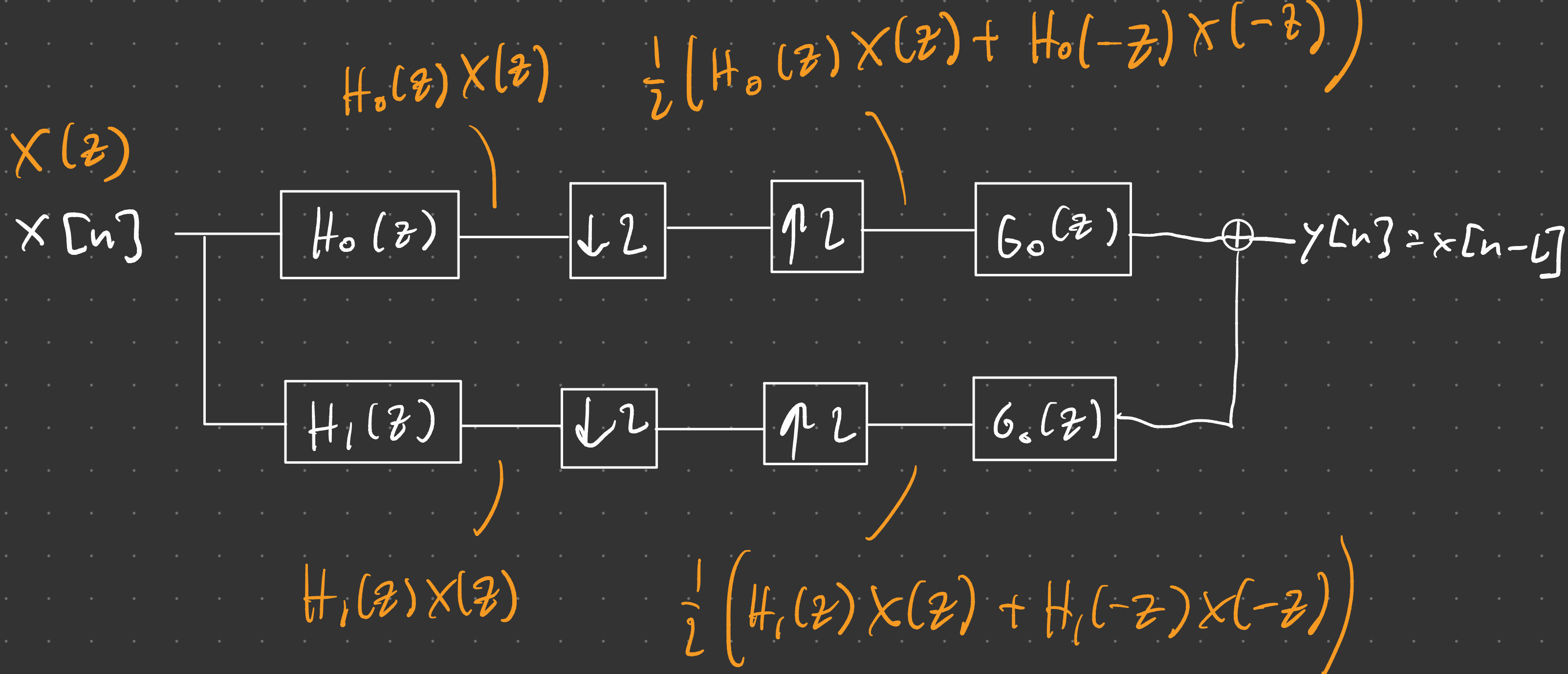
Obs: Ideal Filters $\Rightarrow$ Perfect-Reconstruction (PR)

Practical Filters $\Rightarrow$ Distortion & Aliasing

Q: How do we design PR filter banks with practical filters to avoid distortion and aliasing?

Answering this question will be the topic of the next few lectures.

PR Two-Channel Filterbanks



$$Y(z) = z^{-L} X(z)$$

$$= \frac{1}{2} G_0(z) \left[H_0(z) X(z) + H_0(-z) X(-z) \right]$$

$$+ \frac{1}{2} G_1(z) \left[H_1(z) X(z) + H_1(-z) X(-z) \right]$$

$$= \frac{1}{2} X(z) \underbrace{\left[G_0(z) H_0(z) + G_1(z) H_1(z) \right]}_{\text{Distortion}}$$

$$+ \frac{1}{2} X(-z) \underbrace{\left[G_0(z) H_0(-z) + G_1(z) H_1(-z) \right]}_{\text{Aliasing}}$$

PR Conditions (Vetterli, 1986)

- $G_0(z) H_0(z) + G_1(z) H_1(z) = 2 z^{-L}$
- $G_0(z) H_0(-z) + G_1(z) H_1(-z) = 0$