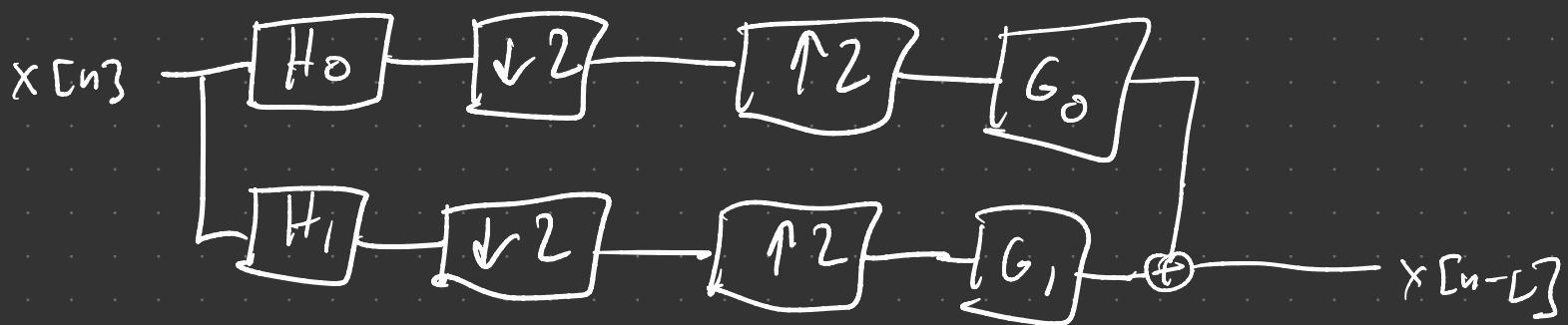


Discrete Wavelet Transform (DWT)

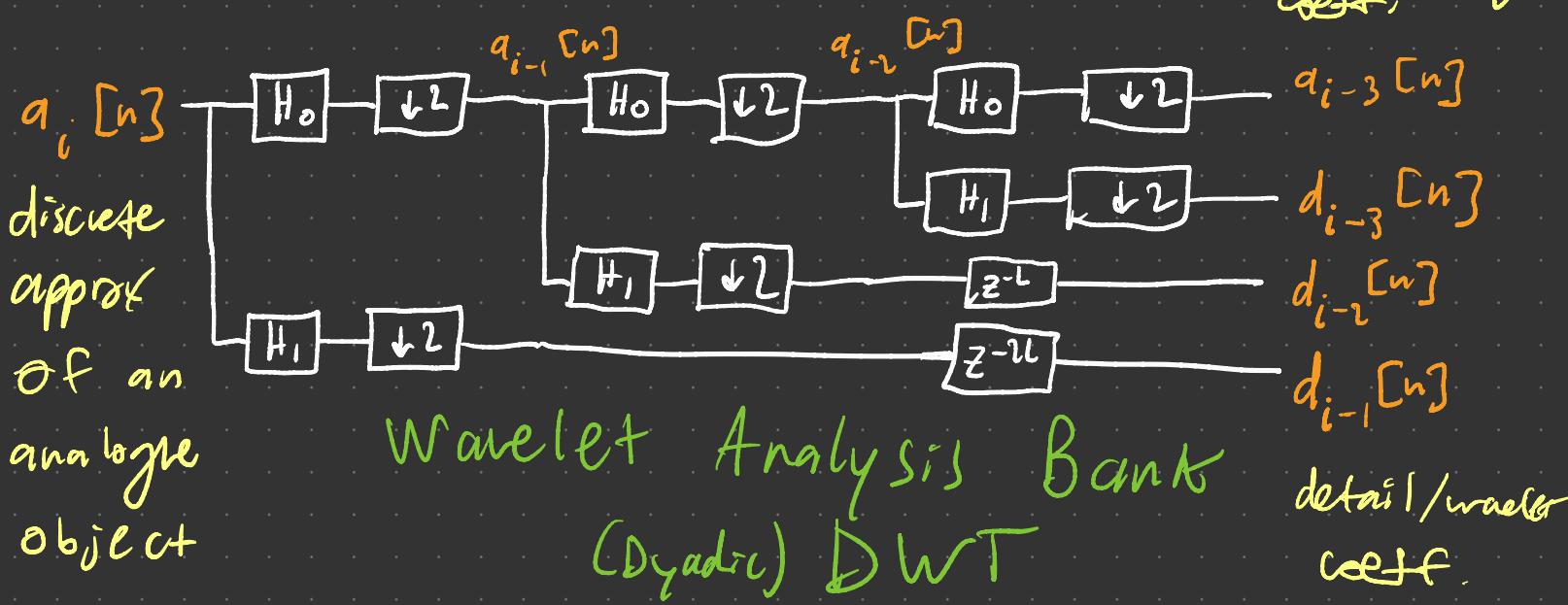
- (Dyadic) DWTs are based on two-channel PR filter banks.



- DWTs are based on iterated

Structures.

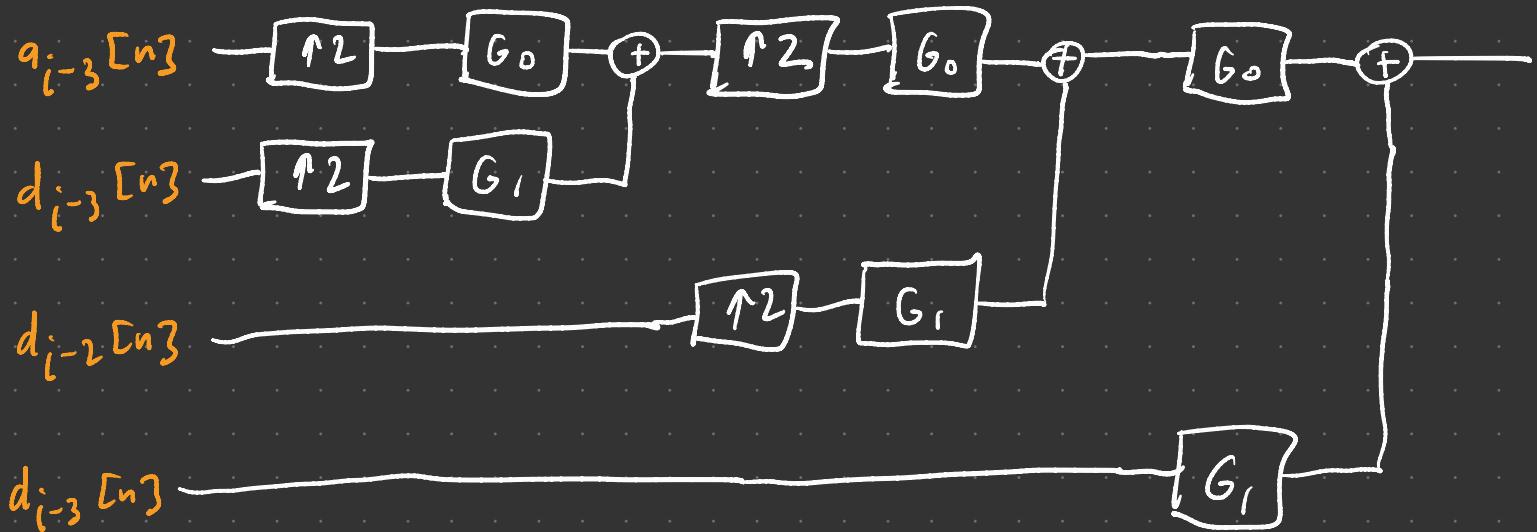
approx/scaling coeff.



Q: Given the approximation and detail coefficients, can we recover the original approx. of the analog signal?

Concretely: given $d_{i-1}[n]$, $d_{i-2}[n]$, $d_{i-3}[n]$, and $a_{i-3}[n]$, can we get back $a_i[n]$?

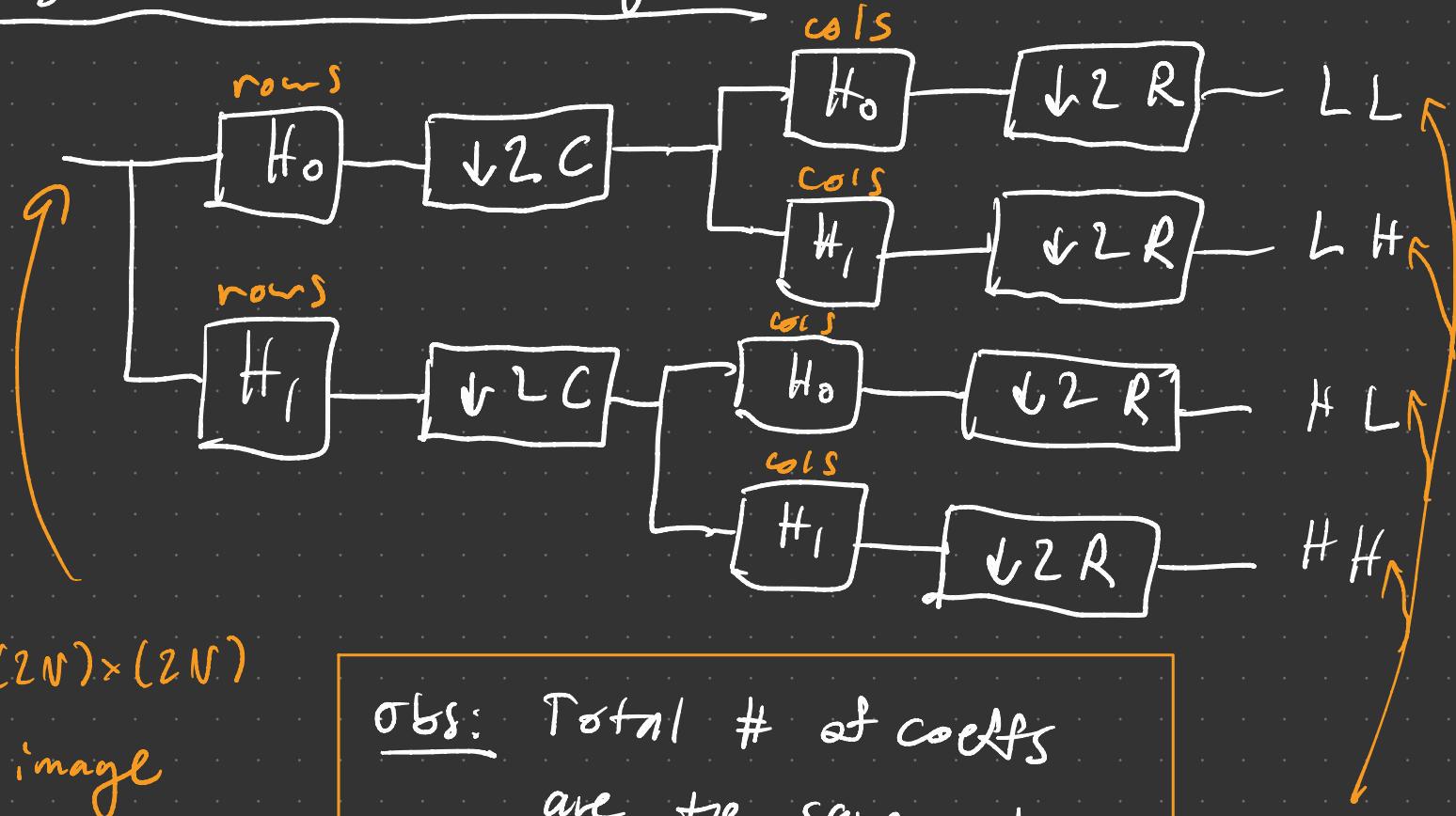
A: Yes.



Wavelet Synthesis Bank
Inverse DWT

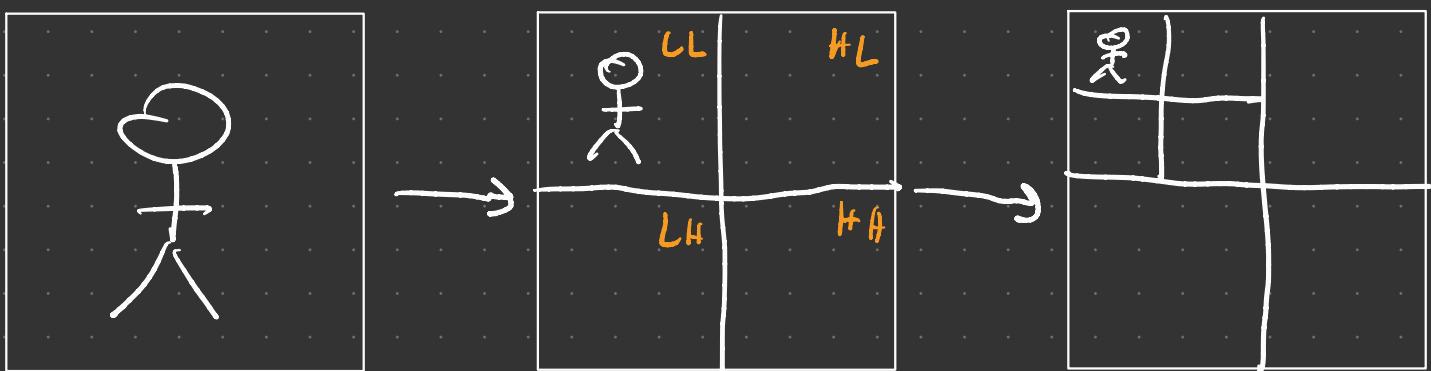
Obs: The DWT is a multiresolution decomp. of the input signal.

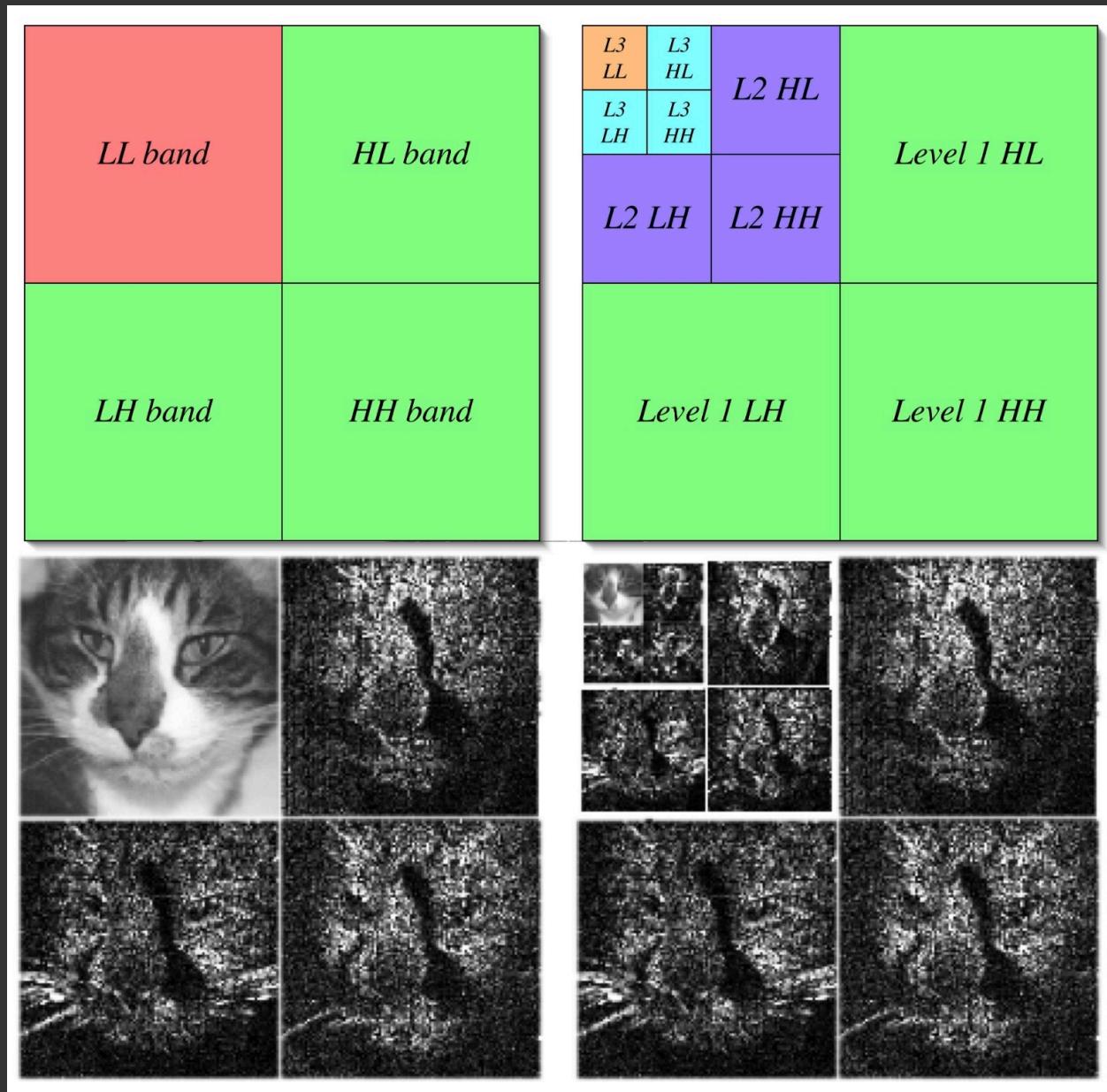
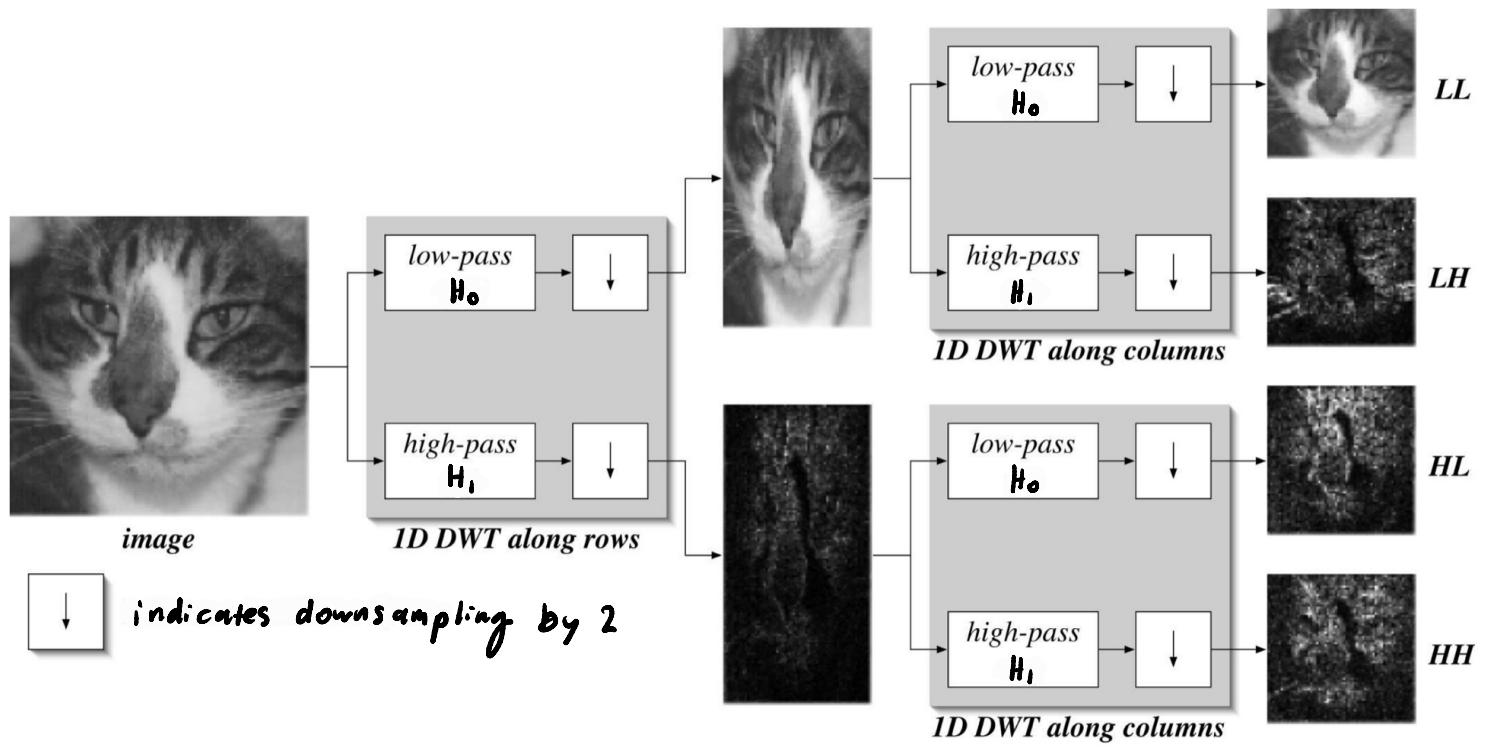
Haar DWT in 2D (Images)



obs.: Total # of coeffs
are the same at
the front and back.

The 2D DWT iterates this procedure on LL.





Remark: For real-world signals many of "sparsity" the scaling/wavelet coefficients are small or zero, even with noise.

→ Easy denoising, compression, etc. via simple thresholding procedures.

→ Thresholding is non linear.

(r)evolution
of
sparsity



- wavelet shrinkage / thresholding algorithms (Donoho et al., 1995)
- compressed sensing (Candes, Romberg, Tao, Donoho, 2006)
- Deep learning (2010 - now)

Q: Where does the discrete approx. of our analog signal come from?

A: Acquisition or sampling device.

(Generalized) Sampling of Analog Signals

$f(t), t \in \mathbb{R}, f: \mathbb{R} \rightarrow \mathbb{R}$

Ideal Sampling:

$$a[n] = f(nT) = \int_{-\infty}^{\infty} f(t) \underbrace{\delta(t-nT)}_{\delta_{nT}(t)} dt$$

approximation

$$= \langle f, \delta_{nT} \rangle \quad \text{inner product}$$

Whittaker - Nyquist - Kotelnikov - Shannon

Sampling Theorem: If $f(t)$ is bandlimited to $[-B, B]$, then

$$f(t) = \sum_{n \in \mathbb{Z}} x[n] \frac{\sin(\pi B(t-nT))}{\pi B(t-nT)}$$

so long as $T < \frac{\pi}{B}$.

Defn: $f(t)$ is band limited to $[-B, B]$ if

its Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

angular frequency

has support in $[-B, B]$, i.e.,

$$\text{supp } F = \overline{\{\omega \in \mathbb{R} : F(\omega) > 0\}}$$

"closure"

satisfies

$$\text{supp } F \subseteq [-B, B].$$

Remarks: We are not indexing freq. in Hz,

If we write $\omega = 2\pi f - \text{Hz}$, we

can define $\tilde{B} = \frac{B}{2\pi}$ in Hz and the

condition becomes $T < \frac{1}{2\tilde{B}}$ or that

the sampling rate satisfies $\tilde{B} < \frac{f_s}{2}$,

where $f_s = \frac{1}{T}$.

Q: How do you invert the Fourier transform?

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Generalized (Non-Ideal) Sampling:

$$q[n] = \underbrace{\langle f, \varphi_n \rangle}_{\text{approximation}} = \int_{-\infty}^{\infty} f(t) \underbrace{\varphi_n(t)}_v dt$$

$\varphi(t-n)$ [i.e., $T=1$]

- φ is modeling the impulse response of the acquisition device

$$\rightarrow \text{if } \tilde{\varphi}(t) = \varphi(-t)$$

$$\langle f, \varphi_n \rangle = (f * \tilde{\varphi})(n)$$

$$= \int_{-\infty}^{\infty} f(t) \tilde{\varphi}(n-t) dt$$

$$= \int_{-\infty}^{\infty} f(t) \varphi(t-n) dt \quad \checkmark$$