Last Time: Multirate Identities

Recall: Multirate SP is hard because operations do not commute.

Multirate Identities help to quickly analyze multirate systems.

1. $$\rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow \;\equiv\; \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow$$

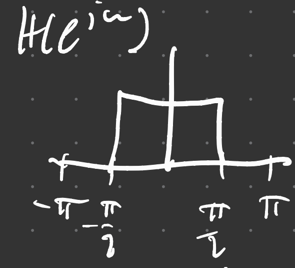
2. $$\rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow \;\equiv\; \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow$$

3. If $\gcd\{M, L\} = 1$:

Ex: • $M=3,\ L=4$ ✓
• $M=2,\ L=4$ ✗

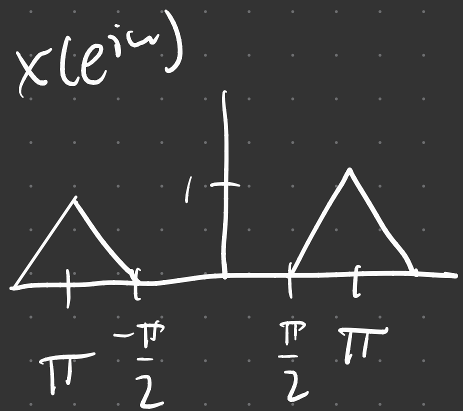
$$\rightarrow \boxed{\downarrow M} \rightarrow \boxed{\uparrow L} \rightarrow \;\equiv\; \rightarrow \boxed{\uparrow L} \rightarrow \boxed{\downarrow M} \rightarrow$$

Exercise:



ideal low-pass filter

$$x[n] \rightarrow \boxed{\uparrow 2} \rightarrow \boxed{\downarrow 4} \rightarrow \boxed{\uparrow 4} \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{H} \rightarrow y[n]$$



What is $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$?

Sol$^{\underline{n}}$:

$$\rightarrow \underbrace{\boxed{\uparrow 2} \rightarrow \boxed{\downarrow 2}}_{\text{identity}} \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{\uparrow 2} \rightarrow \underbrace{\boxed{\uparrow 2} \rightarrow \boxed{\downarrow 2}}_{\text{identity}} \rightarrow \boxed{H} \rightarrow$$

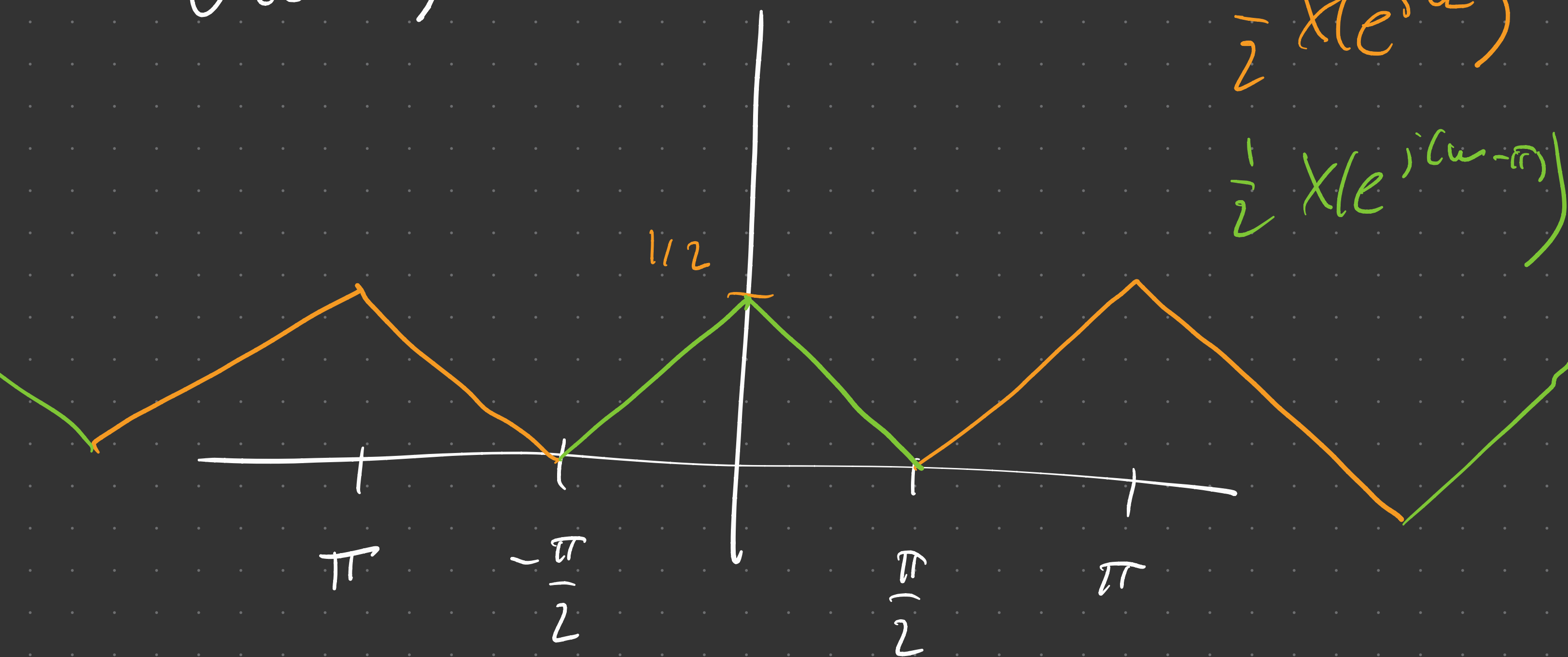
$$x[n] \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{\uparrow 2} \xrightarrow{u[n]} \boxed{H} \rightarrow y[n]$$

$$U(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X(e^{j(\omega-\pi)})\right)$$

$U(e^{j\omega})$

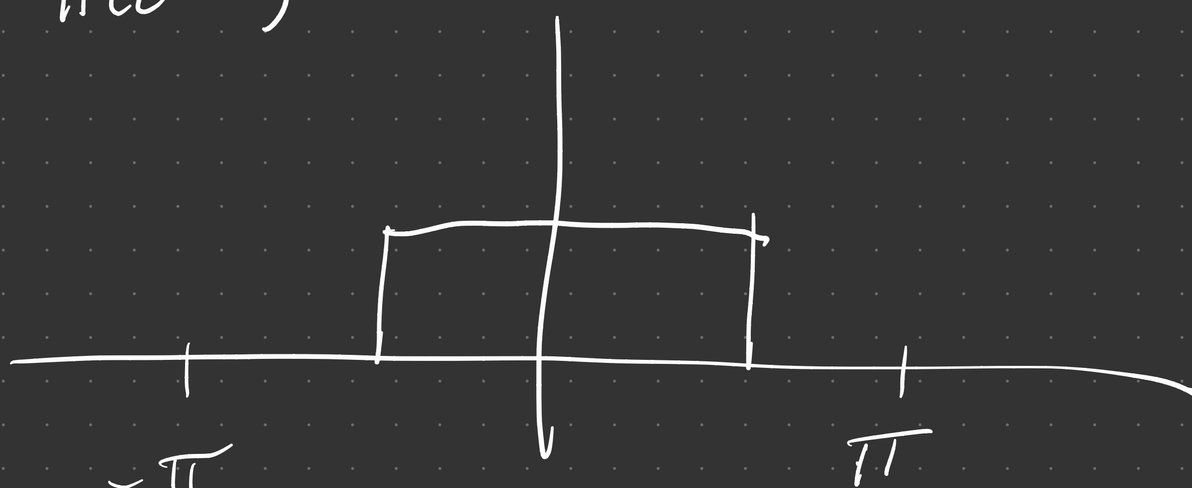
$\frac{1}{2}X(e^{j\omega})$

$\frac{1}{2}X(e^{j(\omega-\pi)})$



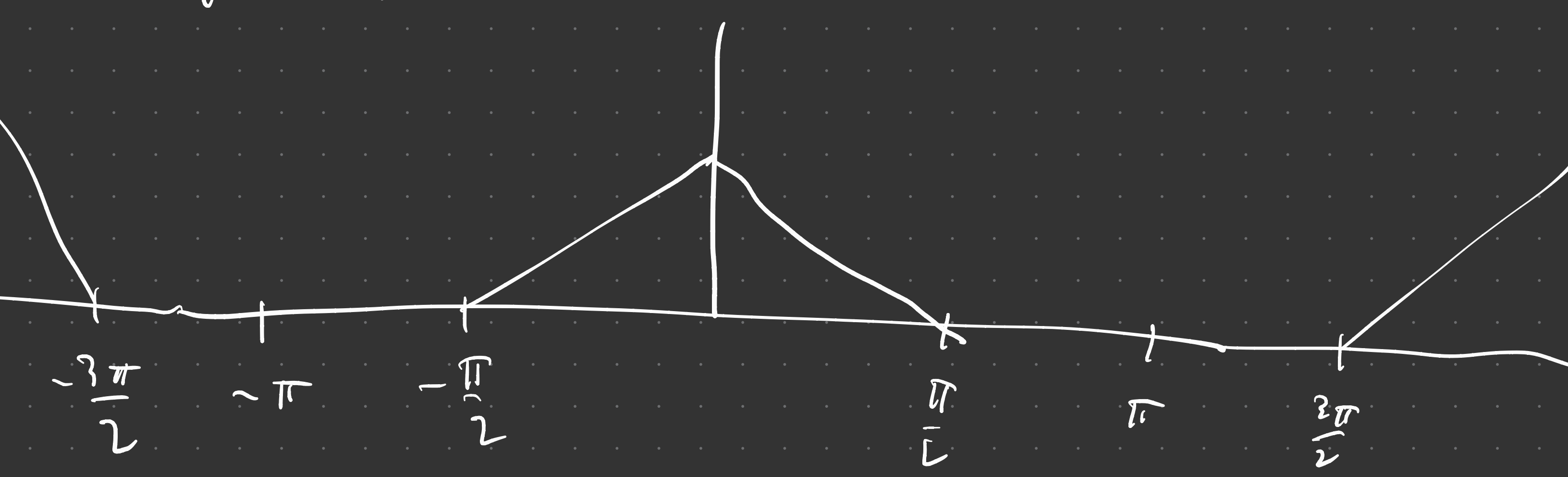
$\cdot$

$H(e^{j\omega})$



$=$

$Y(e^{j\omega})$



Polyphase Representation

Recall:

$$\rightarrow \boxed{\uparrow L} \rightarrow \boxed{F} \rightarrow$$

$$\rightarrow \boxed{H} \rightarrow \boxed{\downarrow M} \rightarrow$$

This is only for theory. Extremely wasteful in practice.

Thought Experiment:

Imagine $L = M = 1024$.

- F is processing a lot of zeros
  - Expensive
- Throwing away most of the computations after H.

Q: Can we filter before upsampling?

Q: Can we filter after downsampling?

Ex: $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

$$= \underbrace{(1 + 3z^{-2})}_{\text{even powers}} + \underbrace{(2z^{-1} + 4z^{-3})}_{\text{odd powers}}$$

$$= (1 + 3z^{-2}) + z^{-1}(2 + 4z^{-2})$$

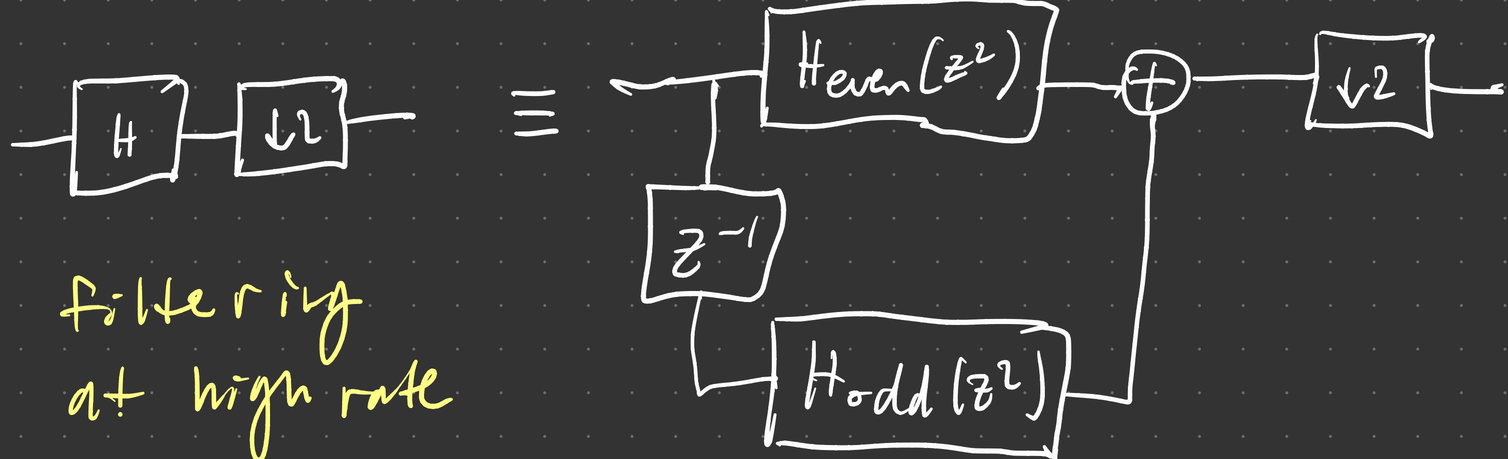
$$= H_{even}(z^2) + z^{-1} H_{odd}(z^2),$$

where

$$H_{even}(z) = (1 + 3z^{-1})$$

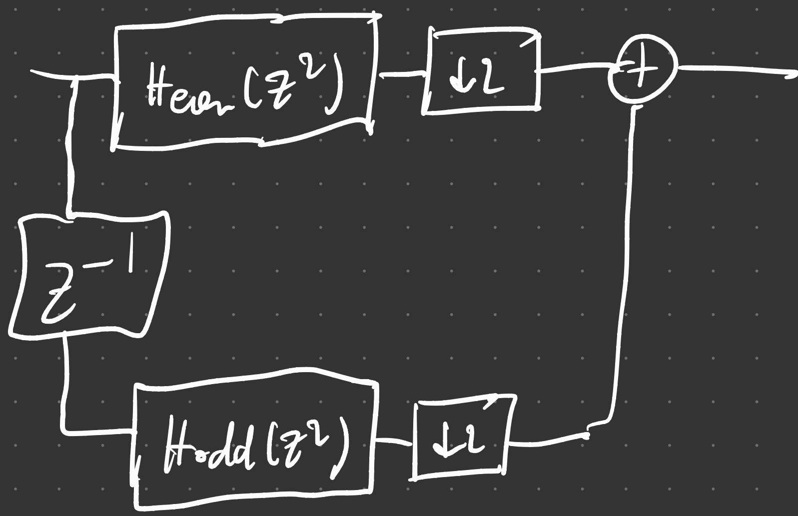
$$H_{odd}(z) = (2 + 4z^{-1})$$

These are the even/odd polyphases.

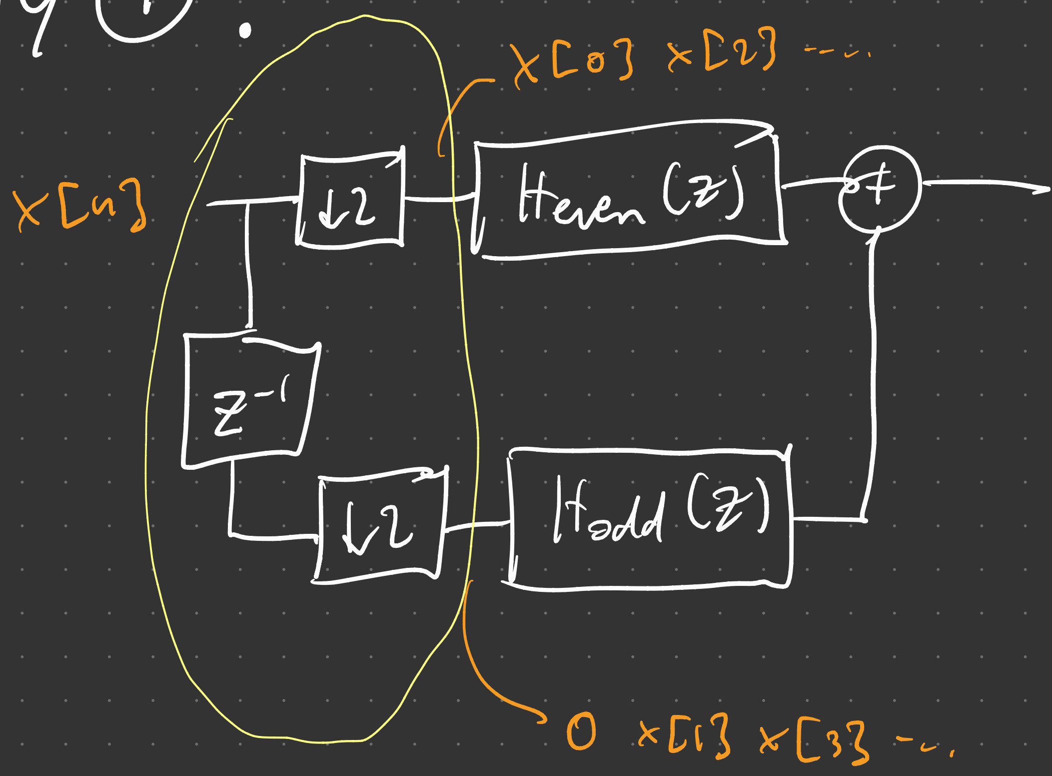


filtering at high rate = slow

Since downsampling is linear:



By ①:



filtering at low rate = fast

Series-to-parallel buffer

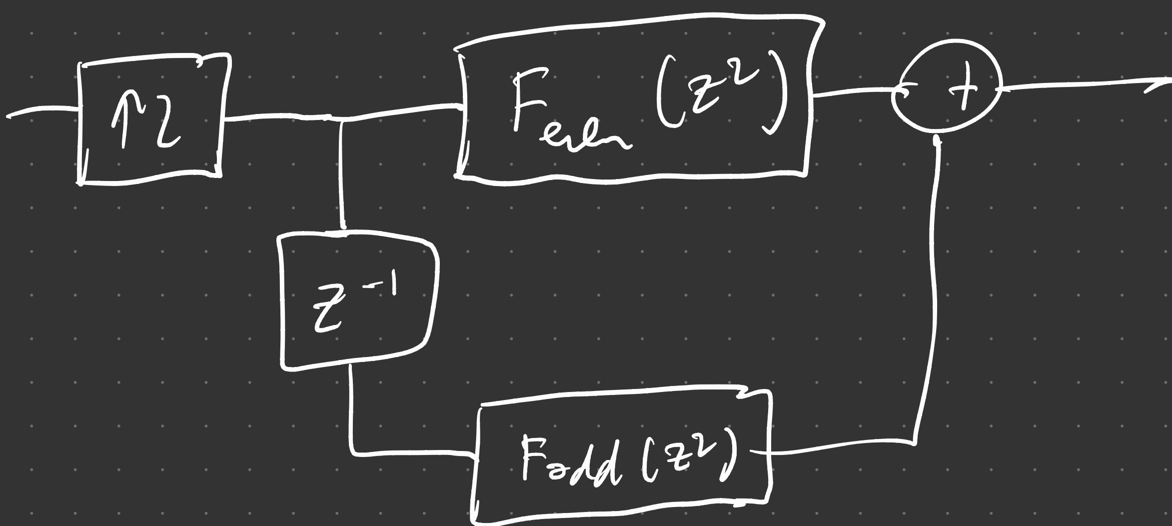
Remark: This is how every multirate system is implemented.

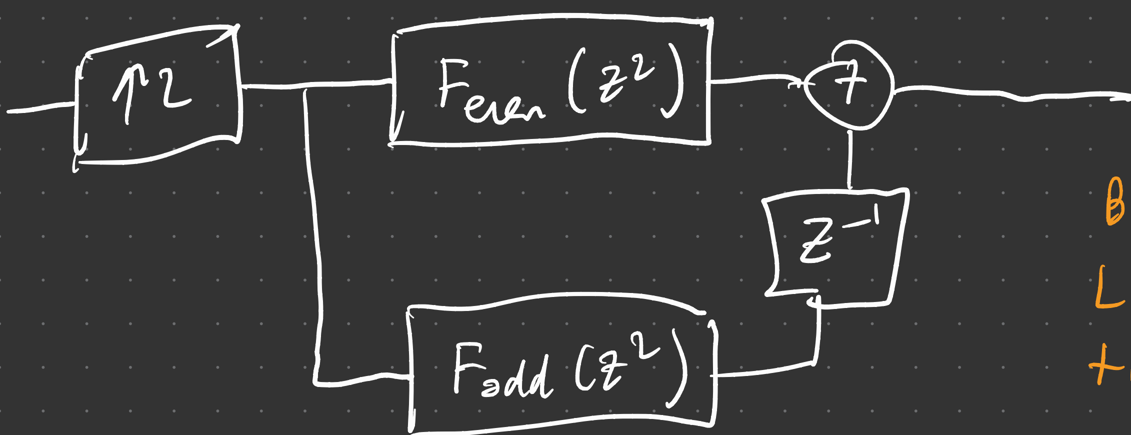
Q: Why is this useful?

Exercise:

Slow

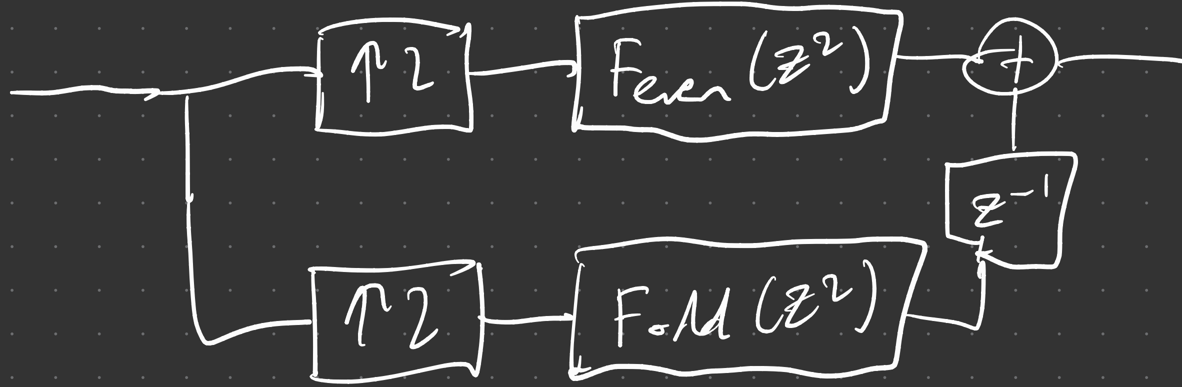
$$F(z) = F_{even}(z^2) + z^{-1} F_{odd}(z^2)$$



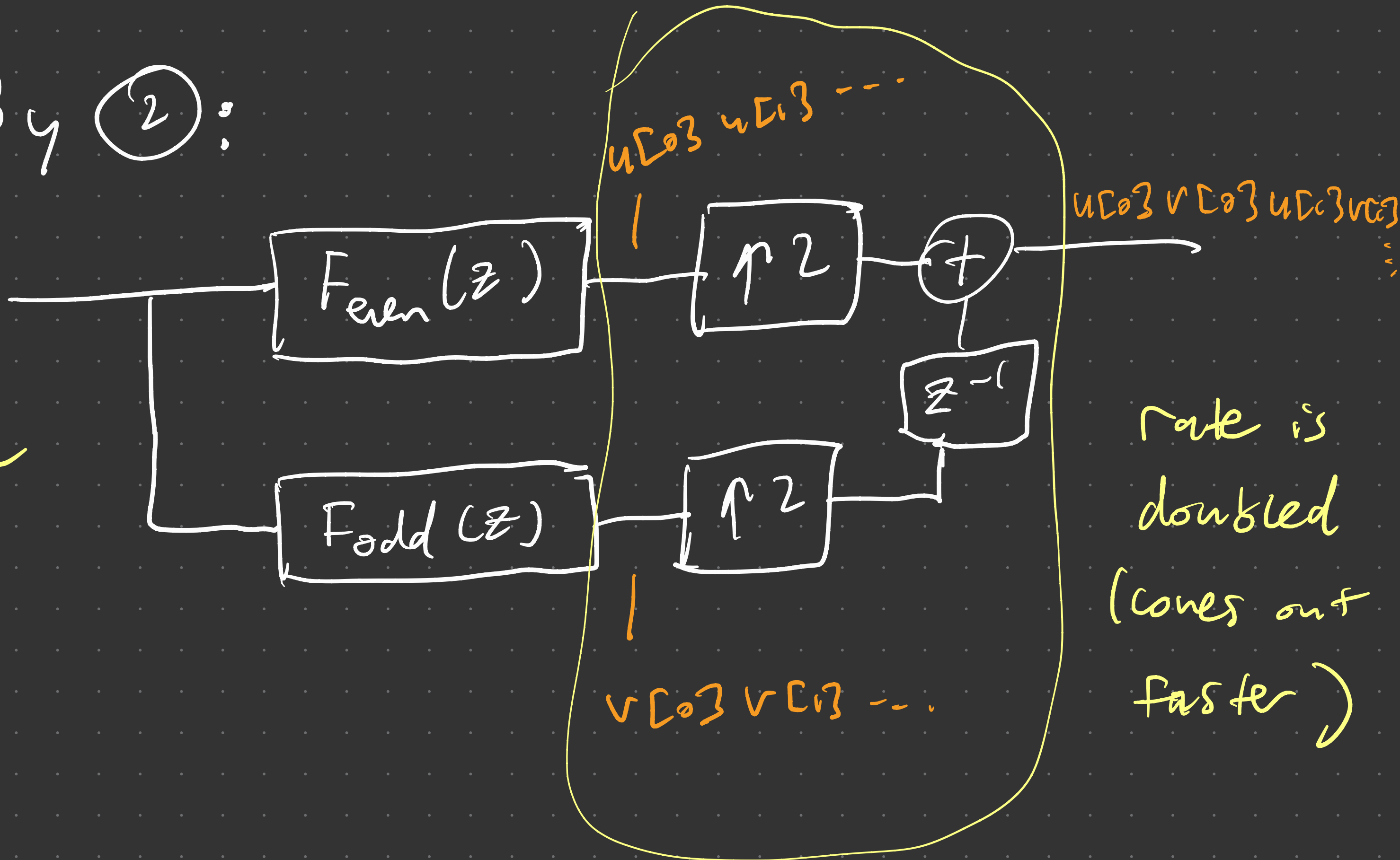


Both are LTI so they commute.

Since upsampling is linear:



By ②:



fast

rate is doubled (comes out faster)

multiplexing

parallel-to-serial buffer

Remark: Polyphase representations efficiently implement multirate operations.

<u>Obs:</u> For FIR filters its very clear.

<u>Q:</u> What about IIR filters?

<u>Exercise:</u>

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

What are the even and odd polyphases?

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \underbrace{\frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}}_{1}$$

$$= \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}}$$

$$= \underbrace{\left(\frac{1}{1-\frac{1}{4}z^{-2}}\right)}_{\text{even}} + z^{-1}\underbrace{\left(\frac{\frac{1}{2}}{1-\frac{1}{4}z^{-2}}\right)}_{\text{odd}}$$

$$H_{even}(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad H_{odd}(z) = \frac{\frac{1}{2}}{1-\frac{1}{4}z^{-1}}$$

Obs: The even/odd polyphases of an IIR filter are both IIR.

Time-Domain Characterization

$$h[n]$$

$$h_{even}[n] = h[2n] \qquad\qquad h_{odd}[n] = h[2n+1]$$

Ex: $H(z) = \dfrac{1}{1-az^{-1}}$

$$h[n] = a^n u[n]$$

unit step

$$h_{even}[n] = a^{2n} u[2n] = (a^2)^n u[n]$$

$$H_{even}(z) = \frac{1}{1 - a^2 z^{-1}}$$

$$h_{odd}[n] = a^{2n+1} u[2n+1] = a\,(a^2)^n u[n]$$

$$H_{odd}(z) = \frac{a}{1 - a^2 z^{-1}}$$