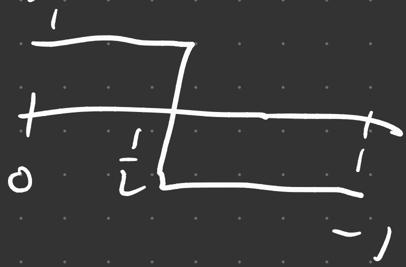
Haar Wavelets

$\varphi(t)$

$\psi(t)$



- $V_0 = \text{span}\{\varphi(t-n)\}_{n\in\mathbb{Z}}$
- $W_i = \text{span}\{\psi_{i,n}\}_{n\in\mathbb{Z}}$ $\quad \psi_{i,n}(t) = 2^{\frac{i}{2}}\psi(2^i t - n)$
- $L^2(\mathbb{R}) = V_0 \oplus \bigoplus_{i=0}^{\infty} W_i$

$\longrightarrow \{\varphi(t-n)\}_{n\in\mathbb{Z}} \cup \{\psi_{i,n}\}_{n\in\mathbb{Z}}$

is an <u>orthobasis</u> for $L^2(\mathbb{R})$.

<u>Q</u>: What if we just want to analyze signals defined on $[0,1]$?

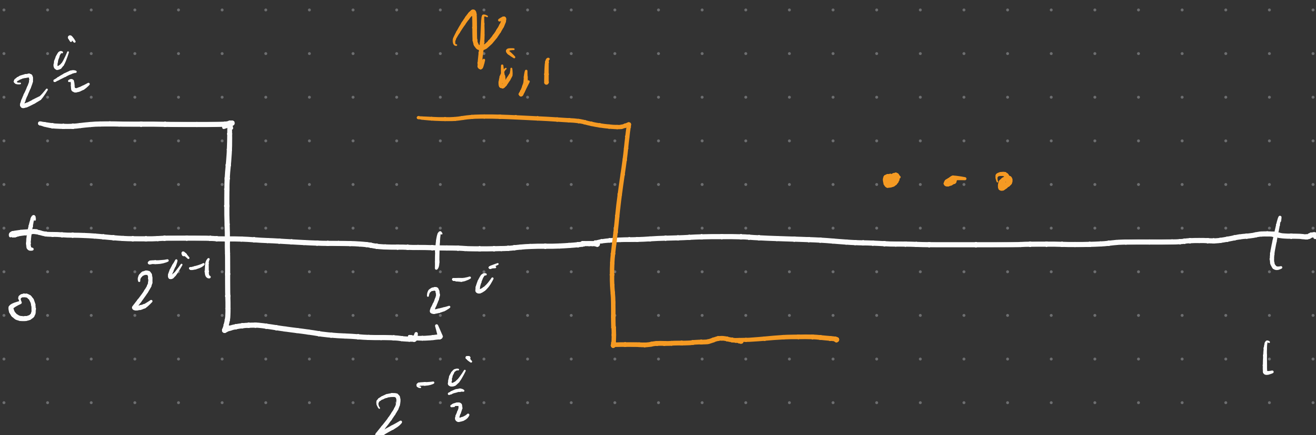
$$L^2[0,1] = \{f:[0,1]\to\mathbb{R} : \int_0^1 |f(t)|^2\, dt < \infty\}$$

finite energy signals on $[0,1]$

Exercise: What is the Haar basis for signals defined on $[0,1]$?

Fix a resolution i. How many Haar wavelets intersect $[0,1]$?

$$\psi_{i,0}(t) = 2^{\frac{i}{2}}\psi(2^i t)$$



- Length of each wavelet is 2^{-i}
- Length of $[0,1]$ is 1

$\Rightarrow$ # of wavelets is $\frac{1}{2^{-i}} = 2^i$

How many scaling functions intersect $[0,1]$?

$\Rightarrow$ only $\varphi(t)$

Haar wavelet basis of $L^2[0,1]$

$$\{\varphi\} \cup \bigcup_{i=0}^{\infty} \bigcup_{n=0}^{2^i-1} \{\psi_{i,n}\}$$ is an orthobasis

for $L^2[0,1]$.

Every $f \in L^2[0,1]$ can be written as

$$f(t) = \underbrace{\langle f, \varphi \rangle \varphi(t)}_{\text{const.}} + \sum_{i=0}^{\infty} \sum_{n=0}^{2^i-1} \langle f, \psi_{i,n} \rangle \psi_{i,n}(t) \qquad t \in [0,1]$$

$\varphi(t) = 1 \ \forall t \in [0,1]$

$$= \underbrace{\langle f, \varphi \rangle}_{\int_0^1 f(t)\,dt} + \sum_{i=0}^{\infty} \sum_{n=0}^{2^i-1} \underbrace{\langle f, \psi_{i,n} \rangle}_{\theta_{i,n}} \psi_{i,n}(t)$$

Q: What is another orthobasis for $L^2[0,1]$?

A: Fourier basis

Fourier Basis of $L^2[0,1]$

$\{e^{j2\pi nt}\}_{n\in\mathbb{Z}}$ is an ortho basis

for $L^2[0,1]$.

Every $f \in L^2[0,1]$ can be written as

$$f(t) = \sum_{n\in\mathbb{Z}} \langle f, e^{j2\pi nt}\rangle e^{j2\pi nt}$$

complex conjugate

$$\langle f, e^{j2\pi nt}\rangle = \underbrace{\int_0^1 f(t)\, e^{-j2\pi nt}\, dt}_{c_n}$$

Fourier coefficients

Fourier series

<u>Exercise</u>: Show that

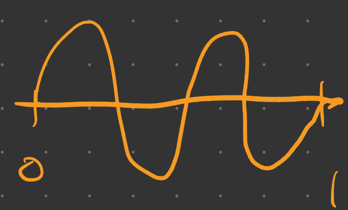
$$\langle e^{j2\pi nt}, e^{j2\pi kt}\rangle = \delta[n-k].$$

$$\int_0^1 e^{j2\pi n t} \, e^{-j2\pi k t} \, dt$$

$$= \int_0^1 e^{j2\pi (n-k) t} \, dt = \delta[n-k]$$

$n = k \Rightarrow e^0 = 1 \Rightarrow \int = 1$

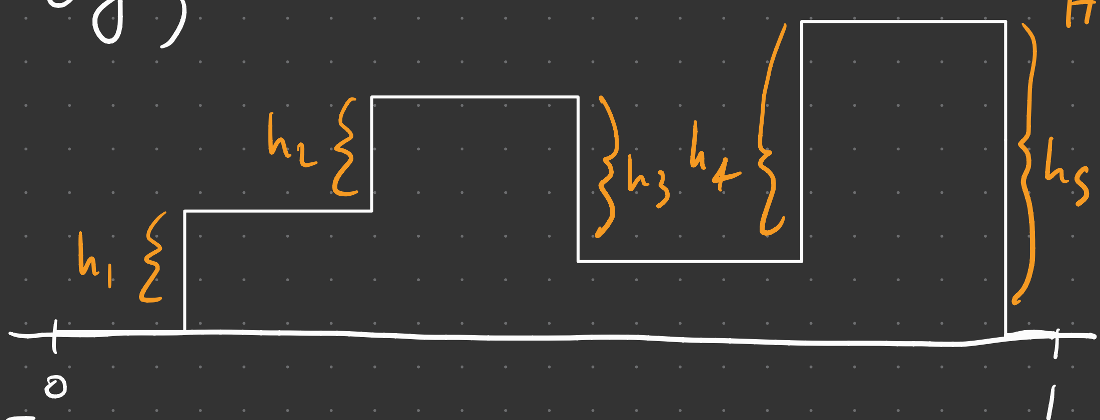
$n \neq k \Rightarrow e^{j2\pi m t} \Rightarrow \int = 0$



Fourier vs. Wavelet

Consider the piecewise constant signal with 5 pieces

e.g.,



$H = h_1 + h_2 + h_3 + h_4 + h_5$

sum of all the jumps

Exercise: Bound the kth largest Fourier coefficient.

$$|c_n| = \left|\int_0^1 f(t)\, e^{-j2\pi n t}\, dt\right|$$

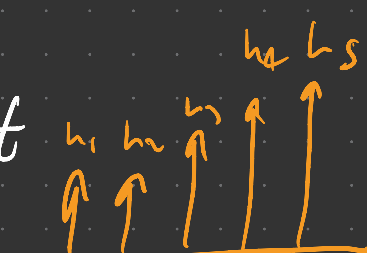
$$\begin{cases} u = f(t) \\ dv = e^{-j2\pi n t} \end{cases}$$

$$\Rightarrow \begin{cases} du = f'(t)\, dt \\ v = \dfrac{e^{-j2\pi n t}}{-j2\pi n} \end{cases}$$

$$= \left| uv\Big|_0^1 - \int_0^1 v\, du \right|$$

$$= \left| \int_0^1 f'(t)\, \frac{e^{-j2\pi n t}}{j2\pi n}\, dt \right|$$

$$\leq \frac{1}{2\pi |n|} \int_0^1 |f'(t)|\, dt$$



$$= \frac{H}{2\pi |n|}$$

Let $|c_{(1)}| \geq |c_{(2)}| \geq \cdots$ be a non decreasing reordering of the Fourier coeffs.

$$\Rightarrow \boxed{|c_{(k)}| \leq H k^{-1}}$$

Exercise: Bound the kth largest wavelet coeff.

Fix the resolution i. Then there are two possible scenarios:

1. The support of $\psi_{i,n}$ does not include a jump

$$|\theta_{i,n}| = \left|\int_0^1 f(t)\,\psi_{i,n}(t)\,dt\right| = 0$$

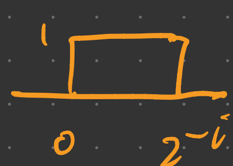
2. The support of $\psi_{i,n}$ does include a jump

$$|\theta_{i,n}| = \left|\int_0^1 f(t)\,\psi_{i,n}(t)\,dt\right|$$

$$\le \int_0^1 \underbrace{|f(t)|}_{\le H}\ \underbrace{|\psi_{i,n}(t)|}_{2^{\frac{i}{2}}\psi(2^i t - n)}\,dt$$

$$\le H\, 2^{\frac{i}{2}} \int_0^1 |\psi(2^i t - n)|\,dt$$

$$= H\, 2^{\frac{i}{2}} \int_0^1 |\psi(2^i t)|\,dt$$



$$= H\, 2^{\frac{i}{2}}\, 2^{-i} = H\, 2^{-\frac{i}{2}}$$

Q: How many wavelets will overlap a jump?

A: S per resolution

At resolution i:

$$\sum_{n=0}^{2^i-1} |\theta_{i,n}| \leq SH 2^{-\frac{i}{2}}$$

Let $\theta_{i,(n)}$ denote the nth largest coeff

at resolution i.

$$SH2^{-\frac{i}{2}} \geq \sum_{n=1}^{2^i} |\theta_{i,(n)}| \geq \sum_{n=1}^{k} |\theta_{i,(k)}|$$

$$\geq k\,|\theta_{i,(k)}|$$

for any $k \in \{1, \ldots, 2^i\}$

$$|\theta_{i,(k)}| \leq SH2^{-\frac{i}{2}} k^{-1}$$

At resolution i, the # of coeffs $\geq T$ is

$$K_i \leq \min\{2^i, SH2^{-\frac{i}{2}}T^{-1}\}$$

The number of coeffs $\geq T$ at at all resolutions is

$$K = \sum_{i=0}^{\infty} K_i \leq \sum_{i=0}^{\infty} \min\{2^i, SH2^{-\frac{i}{2}}T^{-1}\}$$

$$= \sum_{i:2^i \leq SH2^{-\frac{i}{2}}T^{-1}} 2^i + \sum_{i:2^i > SH2^{-\frac{i}{2}}T^{-1}} SH2^{-\frac{i}{2}}T^{-1}$$

$$\leq 6S^{\frac{2}{3}}H^{\frac{2}{3}}T^{-\frac{2}{3}}$$ (check at home)

Let $\theta_{(k)}$ denote the kth largest wavelet coeff: $|\theta_{(1)}| \geq |\theta_{(2)}| \geq \cdots$

Choose $T = |\theta_{(k)}|$

$$\boxed{|\theta_{(k)}| \leq 6^{\frac{3}{2}} S H k^{-\frac{3}{2}}}$$

Approximation in Bases

Q: Given any orthobasis $\{b_k\}_{k=1}^{\infty}$ of $L^2[0,1]$, how do we construct the best N-term approximation?

A: Threshold to only keep the N largest coefficients.

$$f(t) = \sum_{k=1}^{\infty} \underbrace{\langle f, b_k \rangle}_{a_k} b_k(t)$$

Let

$$|a_{(1)}| \geq |a_{(2)}| \geq |a_{(3)}| \geq \cdots$$

be a rearrangement of the $\{a_k\}_{k=1}^{\infty}$ in non-increasing order.

Obs: The best N-term approximation of f is

$$f_M(t) = \sum_{k=1}^{N} a_{(k)} b_{(k)}(t)$$

Alternatively: $f_M(t) = \sum_{k=1}^{\infty} \tau(a_k)\, b_k(t)$,

where

$$\tau(a) = \begin{cases} a, & \text{if } |a| > |a_{k+1}| \\ 0, & \text{else} \end{cases}$$

is the hard-thresholding operator.

Q: What is the approximation error?

$$\|f - f_M\|_{L^2}^2 = \int_0^1 |f(t) - f_M(t)|^2\, dt$$

$$= \int_0^1 \left| \sum_{k=1}^{\infty} a_{(k)} b_{(k)}(t) - \sum_{k=1}^{N} a_{(k)} b_{(k)}(t) \right|^2 dt$$

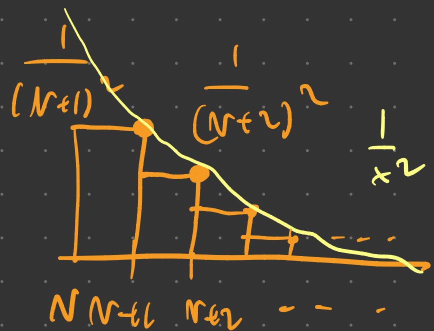
$$= \int_0^1 \left| \sum_{k=N+1}^{\infty} a_{(k)} b_{(k)}(t) \right|^2 dt$$

$$= \sum_{k=N+1}^{\infty} |a_{(k)}|^2 \quad \text{parseval's theorem}$$

Obs: This is the sum of the squares of the tail of the sorted coeffs.

Approximation Errors of Fourier vs. Wavelet

Fourier: $$\|f - f_N^{\text{Fourier}}\|_{L^2}^2 \leq C_F H^2 \sum_{k=N+1}^{\infty} \frac{1}{k^2}$$



$$\leq C_F H^2 \int_N^{\infty} \frac{1}{x^2}\, dx$$

$$= C_F H^2 \left[-\frac{1}{x} \right|_N^{\infty}$$

$$= \frac{C_F H^2}{N}$$

Wavelet: $$\|f - f_N^{\text{wavelet}}\|_{L^2}^2 \leq C_w S^2 H^2 \sum_{k=N+1}^{\infty} \frac{1}{k^3}$$

$$\leq C_w S^2 H^2 \int_N^{\infty} \frac{1}{x^3}\, dx$$

$$= C_w S^2 H^2 \left[-\frac{1}{2} x^{-2} \right|_N^{\infty}$$

$$= \frac{C_w S^2 H^2}{2N^2}$$

Summary:

For piecewise constant signals, the best N-term (squared) approx. error with

- Fourier decays as $\Theta(N^{-1})$
- Wavelet decays as $\Theta(N^{-2})$

and these rates are sharp.

Remark: Story is similar for piecewise poly. signals with higher-order wavelets.

- Story is also similar for signals with certain Besov regularity.