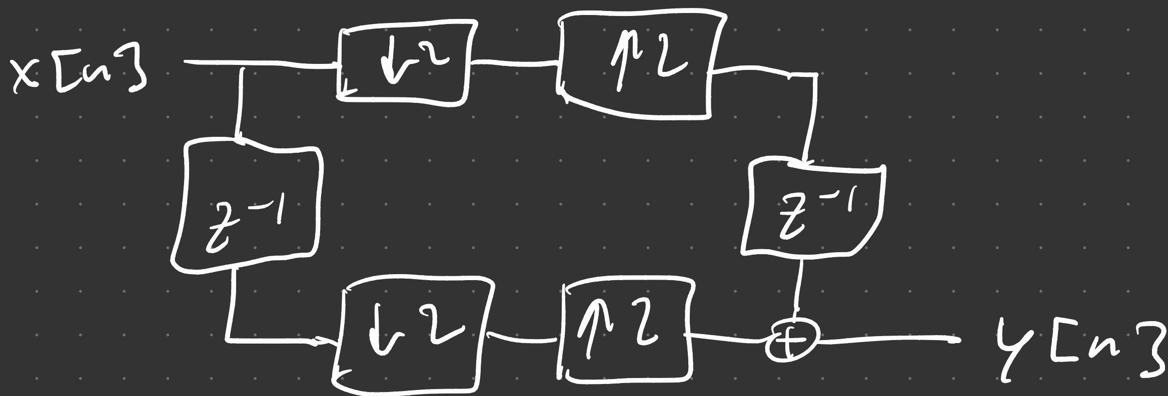
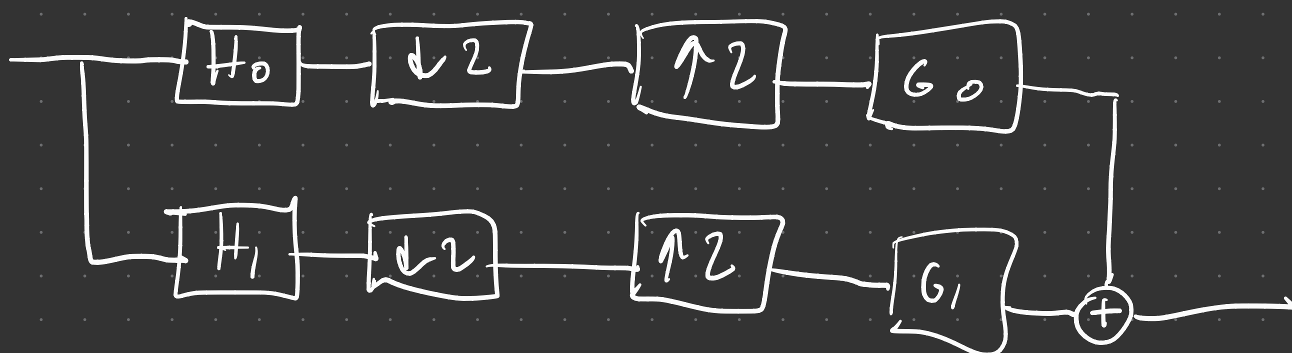
Polyphase Representation of Filter Banks

Exercise:



Show that $y[n] = x[n-1]$.

Consider:



Goal: Write down all of these filters in their respective polyphase representations.

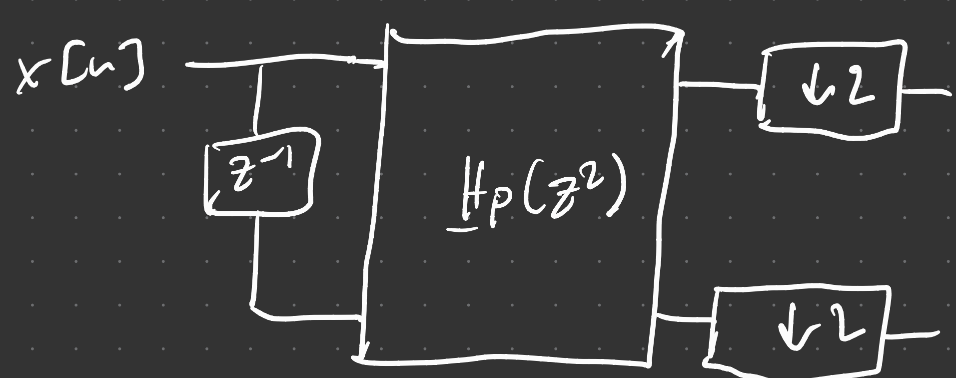
Analysis Bank:

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} H_{0,even}(z^2) + z^{-1} H_{0,odd}(z^2) \\ H_{1,even}(z^2) + z^{-1} H_{1,odd}(z^2) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} H_{0,even}(z^2) & H_{0,odd}(z^2) \\ H_{1,even}(z^2) & H_{1,odd}(z^2) \end{bmatrix}}_{\text{polyphase matrix}} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$\underline{H}_p(z^2)$

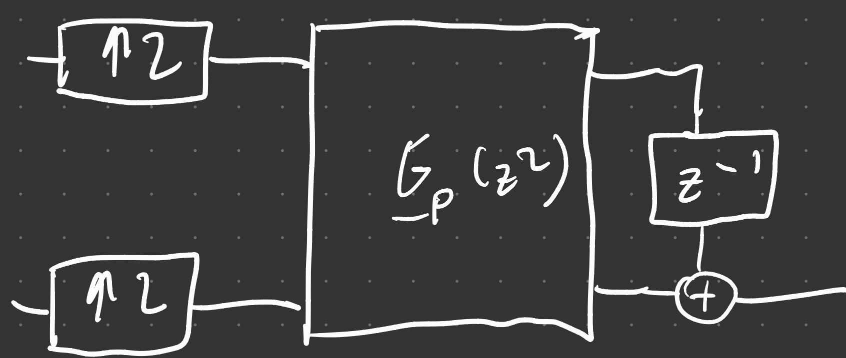
The analysis bank is equivalent to:



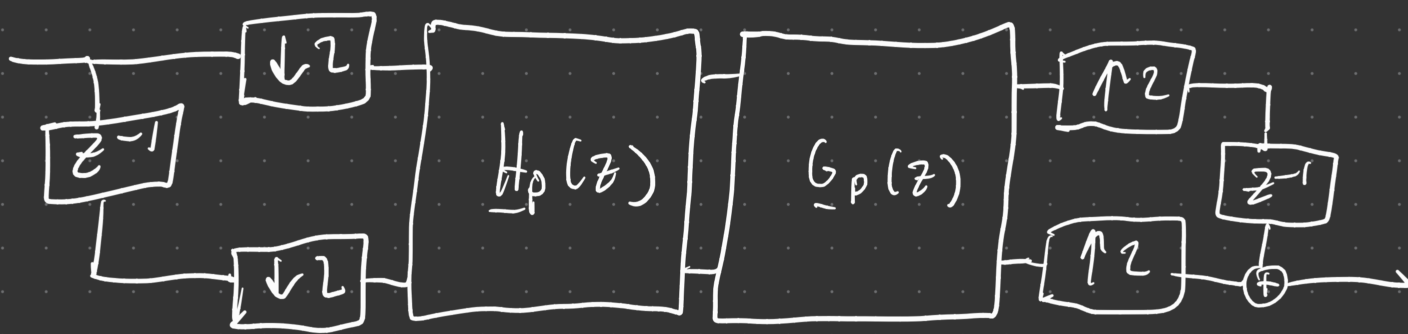
Synthesis Bank:

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} G_{0,odd}(z^2) & G_{1,odd}(z^2) \\ G_{0,even}(z^2) & G_{1,even}(z^2) \end{bmatrix}}_{\substack{\text{polyphase matrix} \\ \underline{F}_p(z^2)}}$$

The synthesis bank is equivalent to:



By applying the multirate identities, we find the equivalent system



polyphase representation of the two-channel filter bank.

Obs: Perfect-reconstruction is guaranteed by:

$$\underline{G}_p(z)\,\underline{H}_p(z) = z^{-k}\,I$$ identity matrix

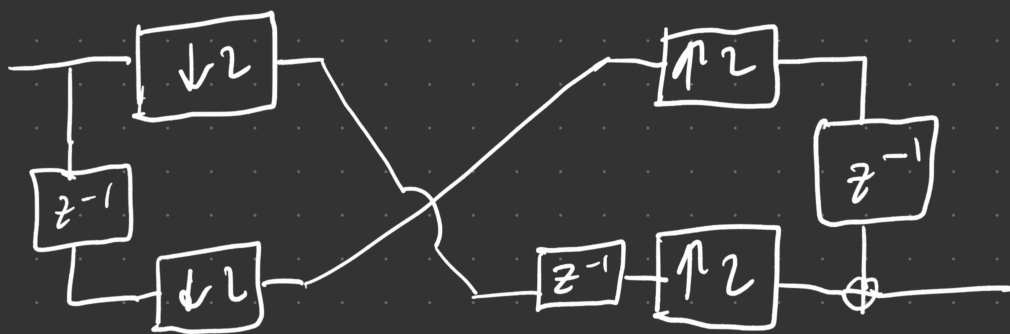
Exercise: Verify that this is true and determine L.

Q: Are there other conditions that guarantee PR?

A: If $\underline{G}_p(z)\,\underline{H}_p(z) = z^{-K}\begin{bmatrix} 0 & 1 \\ z^{-1} & 0 \end{bmatrix}$, then, PR is guaranteed.

First we will ignore z^{-K}.

We have the system:



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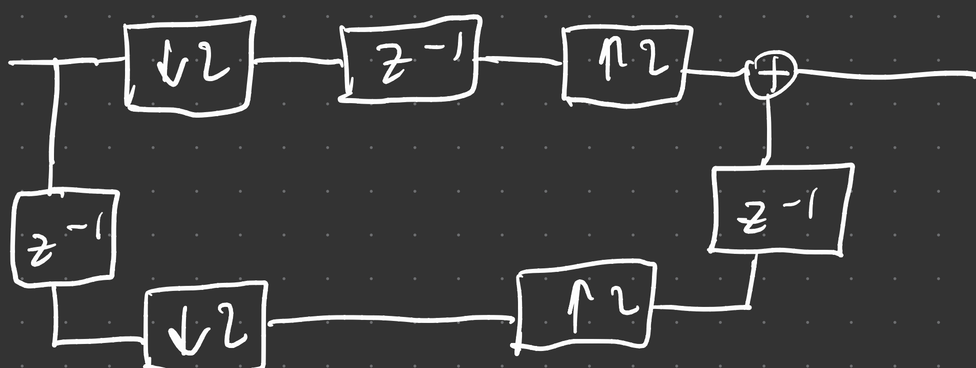

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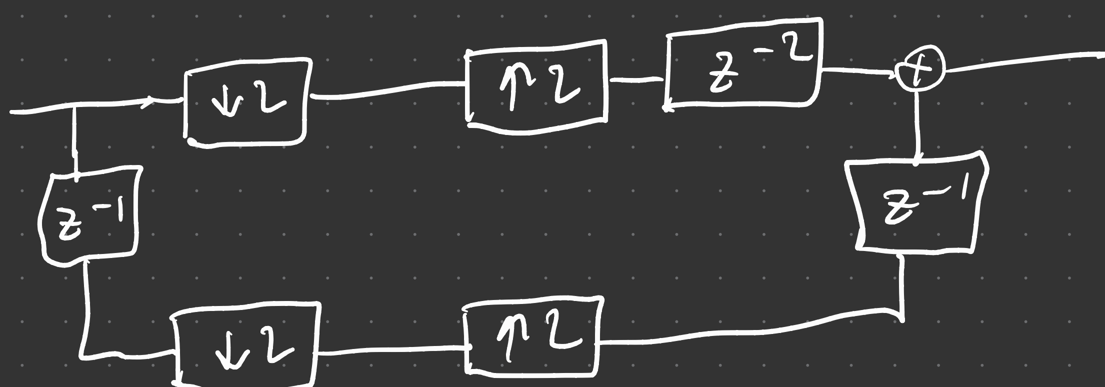

multirate identity

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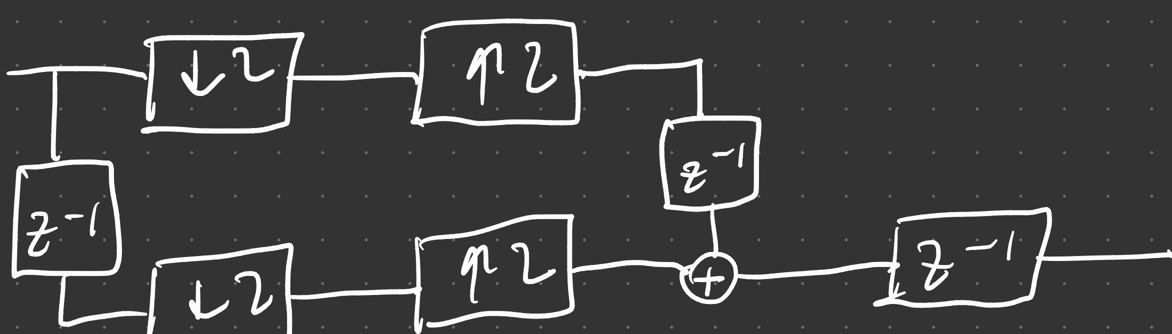


$$\equiv \;-\boxed{z^{-1}}-\boxed{z^{-1}}- \qquad \equiv \;-\boxed{z^{-2}}-$$

Therefore, if $\underline{G}_p(z)\,\underline{H}_p(z) = z^{-k}\begin{bmatrix} 0 & 1 \\ z^{-1} & 0 \end{bmatrix}$,

then the system is PR

<u>Exercise</u>: Determine L.

<u>Exercise:</u> Suppose that $H_1(z) = H_0(-z)$, where H_0 is FIR. Find all PR systems.

<u>Sol$^{\underline{n}}$</u>: Use the polyphase representation.

$$H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

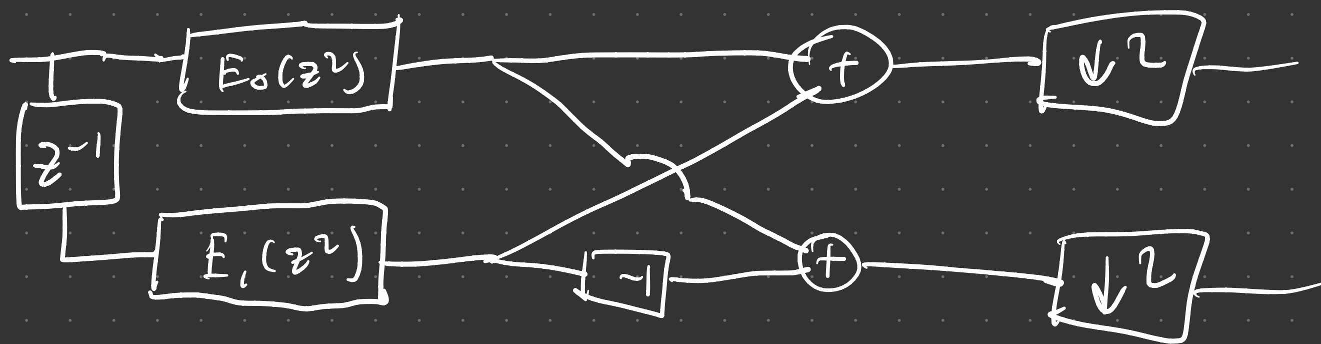
even polyphase   odd polyphase

$$H_1(z) = H_0(-z) = E_0(z^2) - z^{-1}E_1(z^2)$$

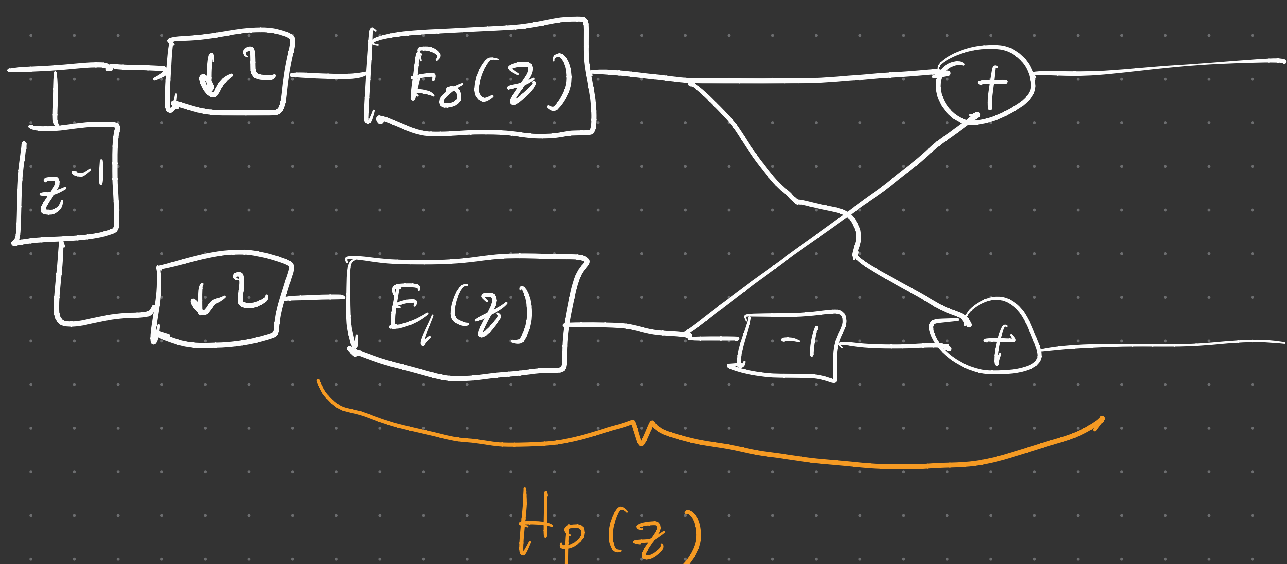
$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} E_0(z^2) & E_1(z^2) \\ E_0(z^2) & -E_1(z^2) \end{bmatrix}}_{\underline{H}_p(z^2)} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{2-point DFT}} \begin{bmatrix} E_0(z^2) & 0 \\ 0 & E_1(z^2) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

Analysis Bank:



By the multirate identities

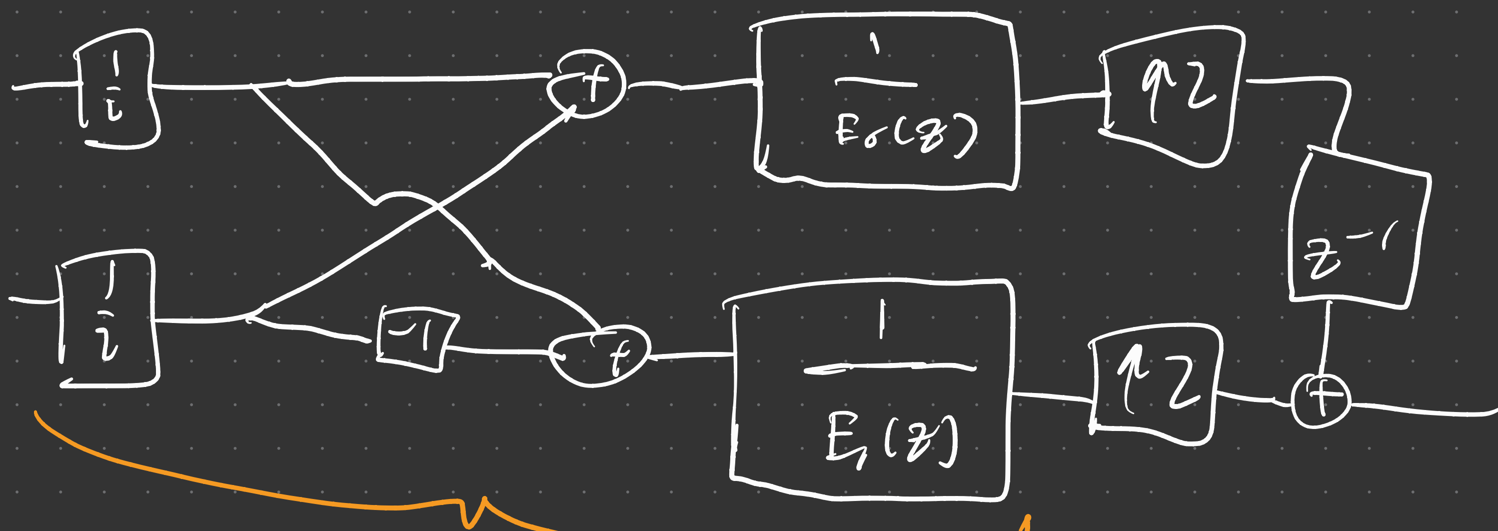


The problem has been reduced to inverting $\underline{H}_p(z)$.
i.e., finding $\underline{F}_p(z)$.

Synthesis Bank:

Observe that

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$\underline{G}_p(z)$

$$\underline{G}_p(z) = \frac{1}{2} \begin{bmatrix} \frac{1}{E_0(z)} & 0 \\ 0 & \frac{1}{E_1(z)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} \frac{1}{E_0(z^2)} & 0 \\ 0 & \frac{1}{E_1(z^2)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} \frac{1}{E_0(z^2)} & \frac{1}{E_0(z^2)} \\ \frac{1}{E_1(z^2)} & -\frac{1}{E_1(z^2)} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \frac{1}{E_1(z^2)} + z^{-1} \frac{1}{E_0(z^2)} & -\frac{1}{E_1(z^2)} + z^{-1} \frac{1}{E_0(z^2)} \end{bmatrix}$$

<u>Remark</u>: So far, we have not assumed <u>anything</u> about the filters,

<u>Q</u>: What if we want FIR filters?

<u>A</u>: Polyphase components must be delays.

i.e.) $E_0(z) = az^{-l_0}$, $E_1(z) = bz^{-l_1}$

The general FIR solution takes the form:

$$\boxed{\begin{aligned} H_0(z) &= E_0(z^2) + z^{-1} E_1(z^2) \\ &= az^{-2l_0} + bz^{-(2l_1+1)} \end{aligned}}$$

Obs: For Haar wavelets, $l_0 = 0$, $l_2 = 0$, $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$

- For IIR solutions, there are infinite possibilities.
- For causal & stable IIR solution, E_0 & E_1 must be min-phase.

(zeros of E_0 & E_1 are inside unit circle
$\Rightarrow$ poles of $\frac{1}{E_0}$ & $\frac{1}{E_1}$ are inside unit circle)