# Generalized Sampling

Given an analog signal $f: \mathbb{R} \to \mathbb{R}$, we can construct a discrete approximation

$$a[n] = \langle f, \varphi_n \rangle = \int_{-\infty}^{\infty} f(t) \underbrace{\varphi_n(t)}_{\varphi(t-n)} \, dt$$

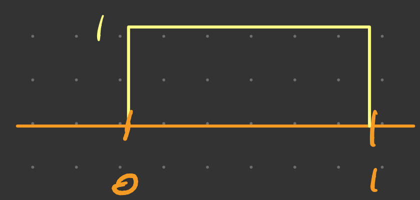
$n \in \mathbb{Z}$

- $\varphi$ models the impulse response of the acquisition system.

Q: What's the simplest choice of $\varphi$?

A: Box (or rect) function:

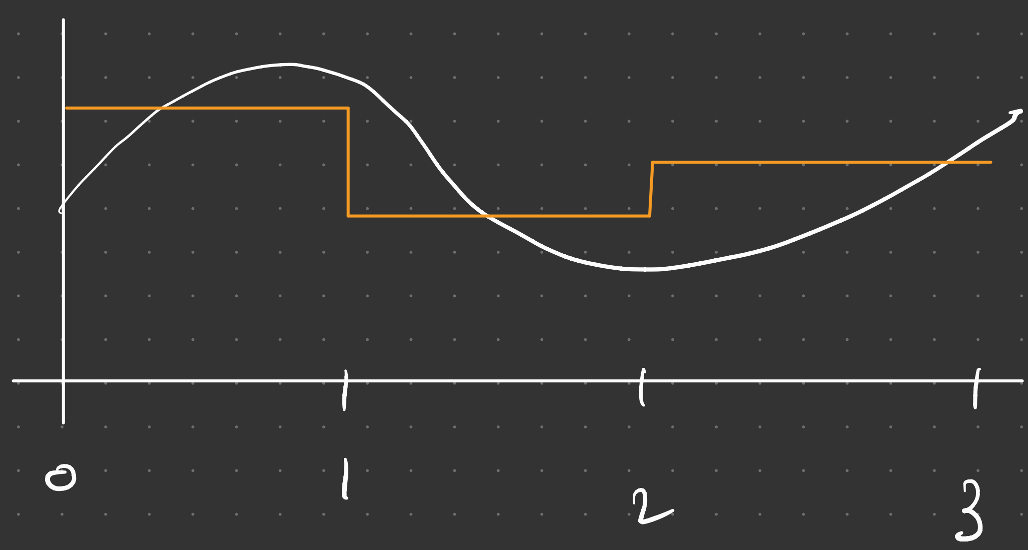
$$\varphi(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{else} \end{cases}$$




Q: What happens if we sample an analog signal with $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$?

$i=0$

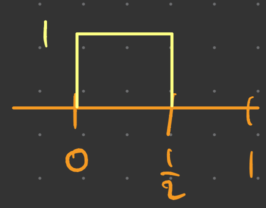
A: Piecewise constant approx.



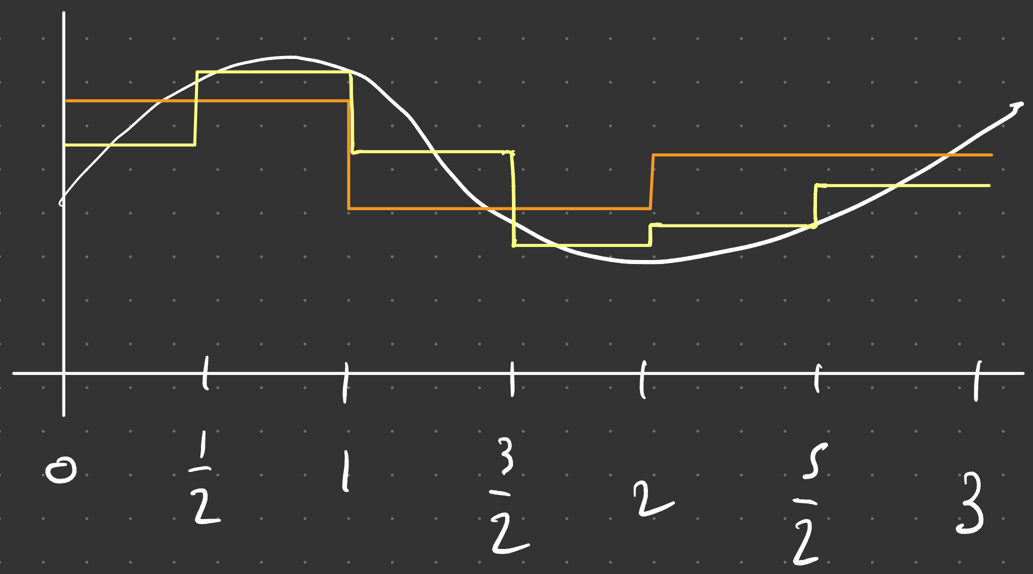

Q: What if we want a higher resolution? How do we double the resolution?

$i=1$

A: Sample with $\{\varphi(2t-n)\}_{n\in\mathbb{Z}}$

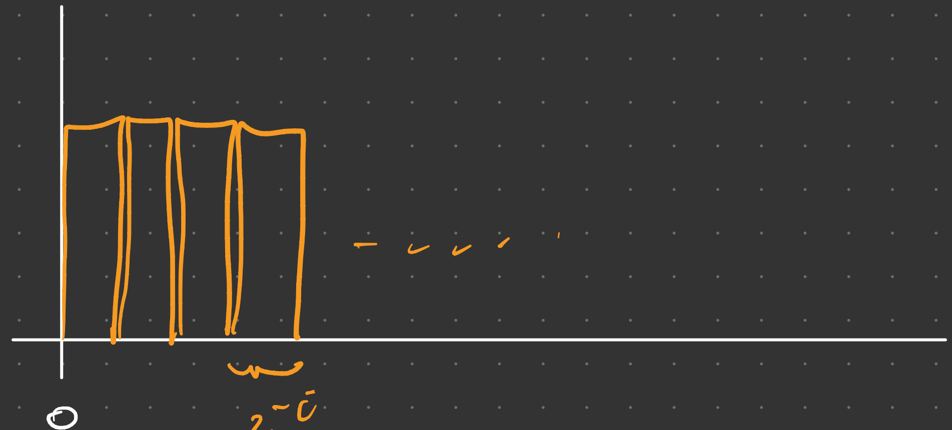


We get a higher-resolution approximation




**Obs:** If we keep doubling the resolution, we can recover the original analog signal !!!

**Remark:** For a general resolution $i$, we sample with $\{\varphi(2^i t - n)\}_{n\in\mathbb{Z}}$



Q: Is there a problem?

What if $i \to \infty$?

$$\varphi(2^i t - n) \to 0 \text{ for almost every } t$$

Fix: Normalize the sampling functions to have unit energy:

$$\left\{ 2^{\frac{i}{2}} \varphi(2^i t - n) \right\}_{n \in \mathbb{Z}, i \in \mathbb{Z}}$$

$$\int_{-\infty}^{\infty} \left[ 2^{\frac{i}{2}} \varphi(2^i t - n) \right]^2 dt$$

$$= 2^i \int_{-\infty}^{\infty} \varphi(2^i t - n)^2 \, dt$$

$u = 2^i t - n$
$du = 2^i dt$

$$= \underbrace{\int_{-\infty}^{\infty} \varphi(u)^2 \, dt}_{\|\varphi\|_{L^2}^2 = \langle \varphi, \varphi \rangle} = 1$$

"Lebesgue space"

Def$^{\underline{n}}$: The space of finite-energy signals is

$$L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} : \|f\|_{L^2} < \infty \}$$

# PR FBs vs. DWTs

**PR FB:**

**DWT:**

- Iterated structure
- Multiscale decomposition of the input signal.

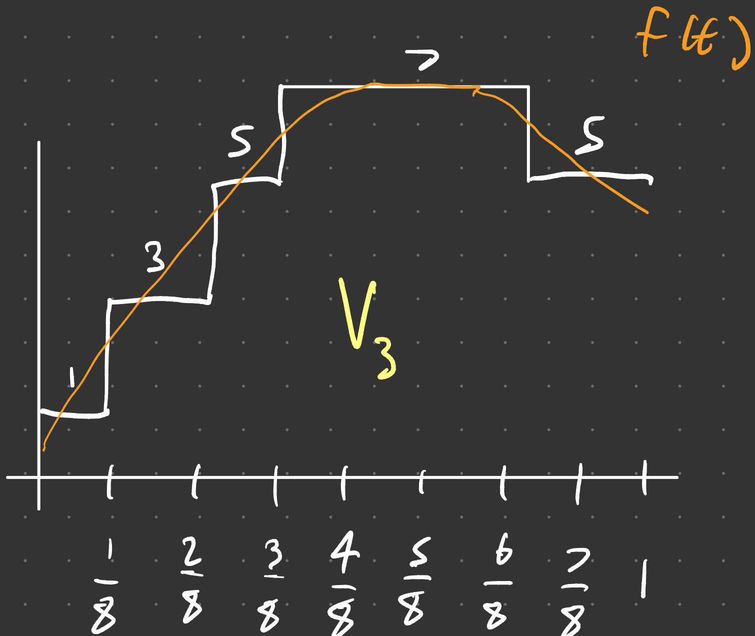
**Remark:** All of this processing is in the discrete domain.

→ There is an underlying sampling procedure.

**Ex:** Suppose we have a resolution 3 approx. of an analog signal.

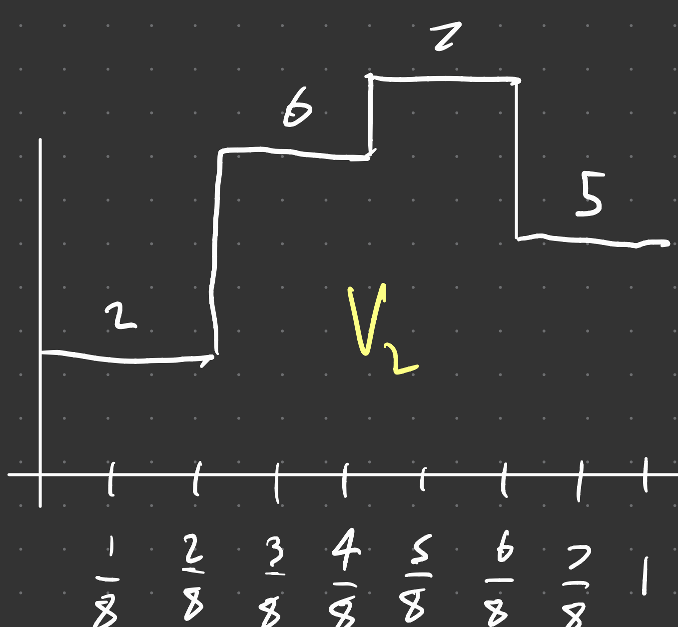
**Q:** What is the bin width?

**A:** $2^{-3} = \frac{1}{8}$




$i = 3$ approximation

Basis: $\{2^{\frac{3}{2}}\varphi(2^3 t - n)\}_{n \in \mathbb{Z}}$

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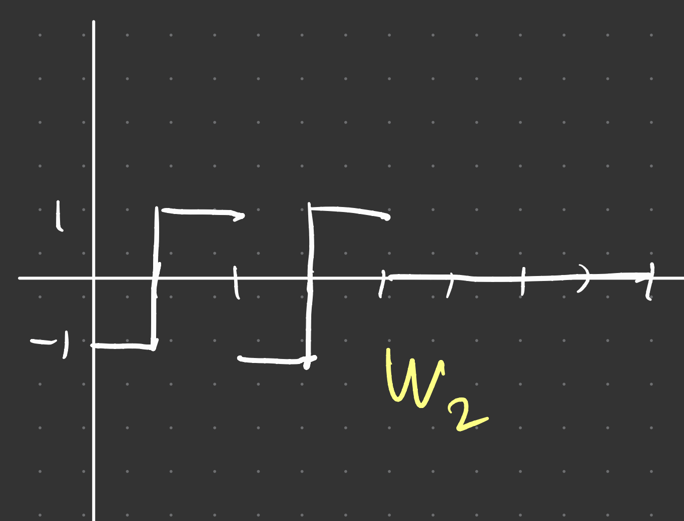
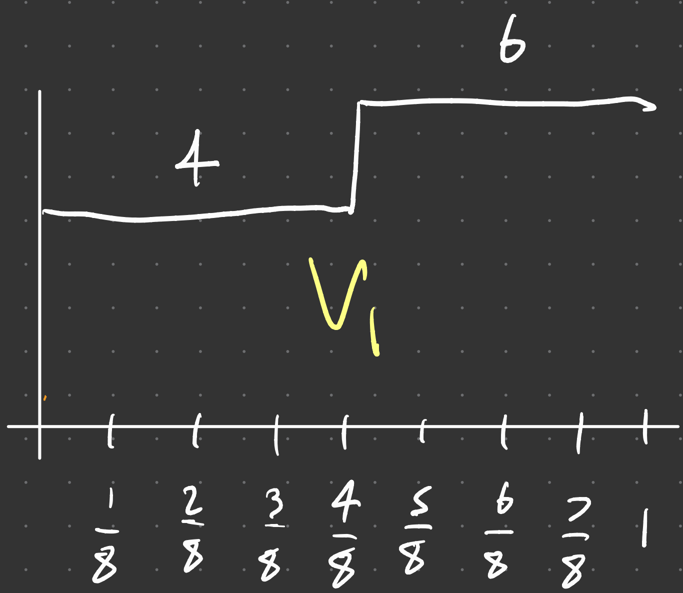



$i = 2$ approximation

Basis: $\{2\varphi(2^2 t - n)\}_{n \in \mathbb{Z}}$

$\boxed{i = 2 \quad \text{detail}}$

Basis: $\{2\psi(2^2 t - n)\}_{n \in \mathbb{Z}}$

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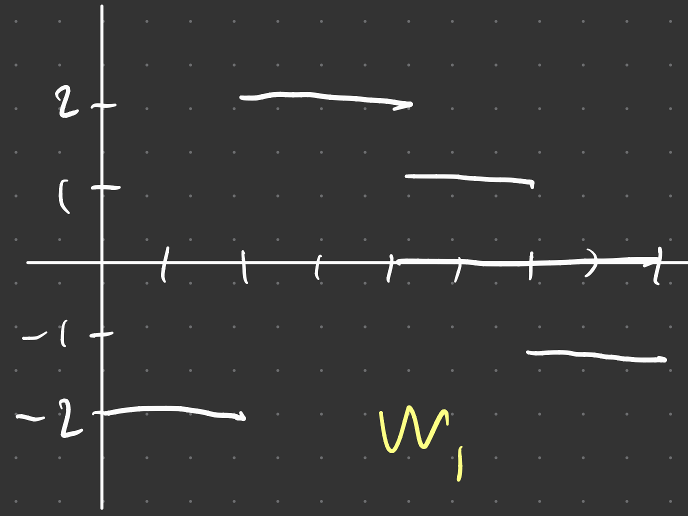
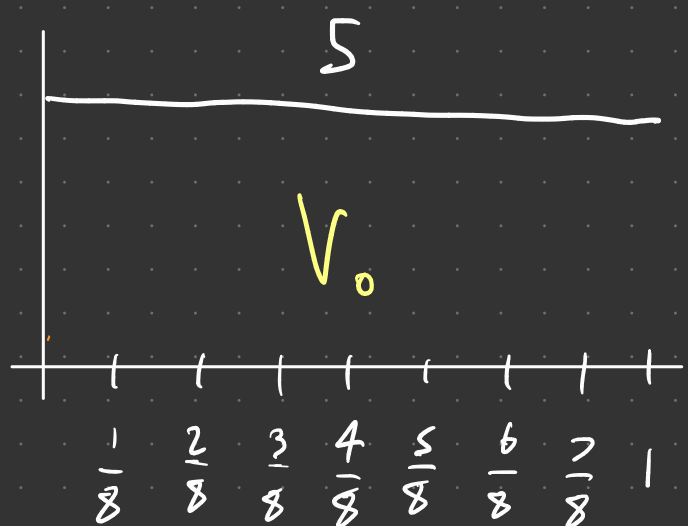



$i = 1$ approximation

Basis: $\{2^{1/2}\varphi(2t-n)\}_{n\in\mathbb{Z}}$

$\boxed{i = 1 \text{ detail}}$

Basis: $\{2^{1/2}\psi(2t-n)\}_{n\in\mathbb{Z}}$

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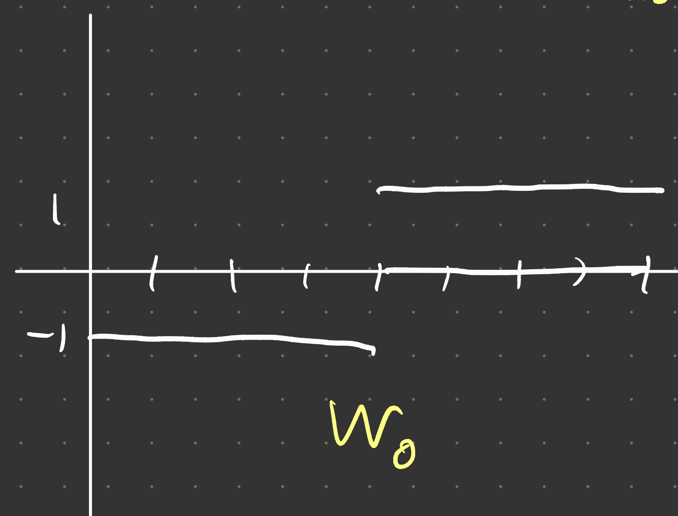



$\boxed{i = 0 \text{ approximation}}$

Basis: $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$

$\boxed{i = 0 \text{ detail}}$

Basis: $\{\psi(t-n)\}_{n\in\mathbb{Z}}$

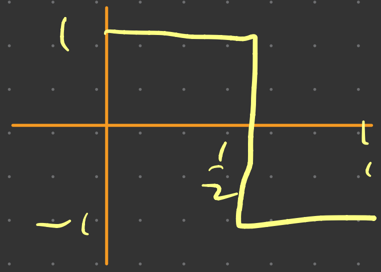
<u>Obs:</u> This is a <u>multiresolution</u> <u>decomp</u>.

<u>Obs:</u> The approximation bases are <u>multiscale</u> versions of $\varphi(t)$.

**Q:** What basis functions are we using for the details?

**A:** Multiscale versions of

$$\psi(t) = \begin{cases} +1, & 0 \le t \le \frac{1}{2} \\ -1, & \frac{1}{2} < t \le 1 \\ 0, & \text{else} \end{cases}$$




**Remark:** $\psi(t)$ is the Haar (mother) wavelet
$\varphi(t)$ is the Haar scaling function (father wavelet).

**Defn:** The approximation space at resolution $i$ is the space

$$V_i = \overline{\text{span}}\left\{2^{\frac{i}{2}}\varphi(2^i t - n)\right\}_{n\in\mathbb{Z}}$$

**Defn:** The wavelet (detail) space at resolution $i$ is the space

$$W_i = \overline{\text{span}}\left\{2^{\frac{i}{2}}\psi(2^i t - n)\right\}_{n\in\mathbb{Z}}$$

**Q:** Which is bigger: $V_0$ or $V_1$?

**A:** $V_0 \subset V_1$

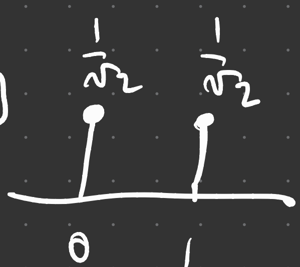
**Q:** Which is bigger: $W_0$ or $V_1$?

**A:** $W_0 \subset V_1$

What does this mean?

- $$\underbrace{\varphi(t)}_{V_0} = \sum_{n \in \mathbb{Z}} g_0[n] \underbrace{\sqrt{2}\,\varphi(2t-n)}_{V_1}$$


$g_0[n]$

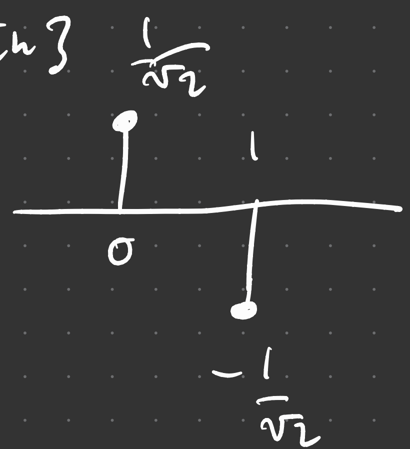



Haar FB

low-pass filter

- $\psi(t) = \sum_{n \in \mathbb{Z}} g_1[n] \sqrt{2}\, \varphi(2t-n)$


$g_1[n]$




Haar FB high-pass filter

Obs: $g_0[n]$ & $g_1[n]$ are the Haar wavelet filters!

Remark: These equations are called the two-scale equations

Q: How do we get $V_1$ from $V_0$ & $W_0$?

A: $V_1 = V_0 \oplus W_0$.

direct sum

- $V_1 = \{ f + g_0 : f \in V_0 \text{ and } g_0 \in W_0 \}$ "sum"

- $V_0 \cap W_0 = \{0\}$ "direct"

Q: What happens to $V_i$ as $i$ becomes large?

A: $$V_i \longrightarrow L^2(\mathbb{R}) \quad \text{as} \quad i \to \infty$$

Q: What happens to $V_i$ as $i$ becomes small?

A: $$V_i \longrightarrow \{0\} \quad \text{as} \quad i \to -\infty$$

Obs: We have constructed a nested sequence of subspaces of $L^2(\mathbb{R})$:

$$\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$$

Remark: This sequence is called a multiresolution analysis (MRA) of $L^2(\mathbb{R})$.