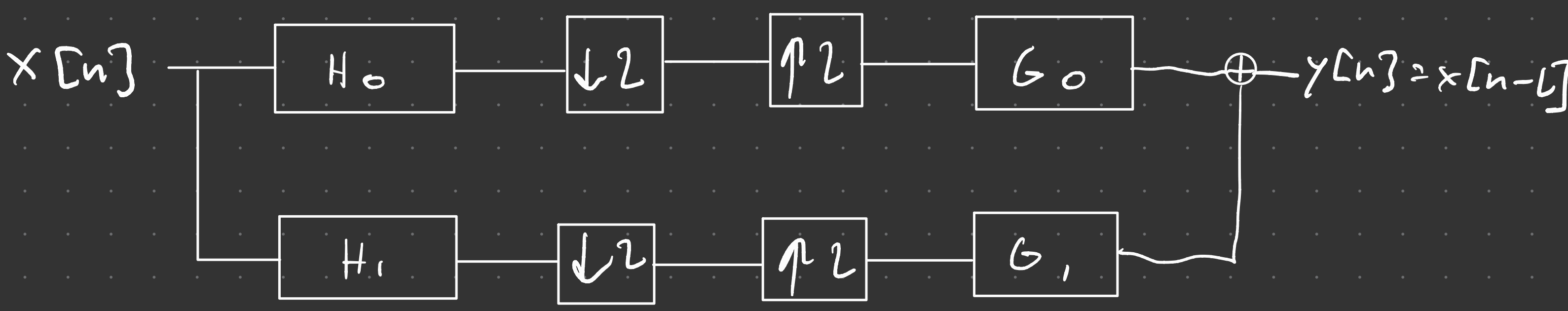
# Last Time: Two-Channel PR Filter Banks




$$Y(z) = z^{-L} X(z)$$

$$= \frac{1}{2} X(z) \underbrace{\left[ G_0(z) H_0(z) + G_1(z) H_1(z) \right]}_{\text{Distortion}}$$

$$+ \frac{1}{2} X(-z) \underbrace{\left[ G_0(z) H_0(-z) + G_1(z) H_1(-z) \right]}_{\text{Aliasing}}$$

## PR Conditions (Vetterli, 1986)

- $G_0(z) H_0(z) + G_1(z) H_1(z) = 2 z^{-L}$
- $G_0(z) H_0(-z) + G_1(z) H_1(-z) = 0$

Obs: The PR conditions are two equations with 4 unknowns $(H_0, H_1, G_0, G_1)$.

$$\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} 2z^{-L} \\ 0 \end{bmatrix}$$

Given the analysis filters, we can find the synthesis filters by inverting this matrix.

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2z^{-L}}{\Delta(z)} \begin{bmatrix} H_1(-z) & -H_1(z) \\ -H_0(-z) & H_0(z) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Synthesis Filters

$$= \frac{2z^{-L}}{\Delta(z)} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$

Analysis Filters

$$\Delta(z) = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$

Q: Suppose that $H_0$ & $H_1$ are FIR.

Are $G_0$ & $G_1$ FIR?

A: Not necessarily. The issue is $\Delta(z)$.

Might not even have causal or stable

IIR $G_0$ & $G_1$...

## FIR PR Filter Bank Design

Q: How do we force $G_0$ & $G_1$ to be FIR?

A: Force $\Delta(z) = 2z^{-L} \Rightarrow$ FIR sol$^{\underline{n}}$.

could put any gain and delay

Remark: This is one possible design choice.

$$2z^{-L} = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$

$$= \underbrace{H_0(z)G_0(z)}_{P_0(z)} - \underbrace{H_0(-z)G_0(-z)}_{P_0(-z)}$$

"product"

# Design Procedure:

1. Find $P_0(z)$ that satisfies

$$P_0(z) - P_0(-z) = 2z^{-L}$$

How do we do this?

2. Factorize $P_0(z)$ into $H_0(z) \cdot G_0(z)$

3. Define $\left.\begin{aligned} H_1(z) &= G_0(-z) \\ G_1(z) &= -H_0(-z) \end{aligned}\right\}$ one possible choice.

Different filter banks / wavelets / properties

$$P_0(z) = P_{0,even}(z^2) + z^{-1} P_{0,odd}(z^2)$$

$$- \quad P_0(-z) = P_{0,even}(z^2) - z^{-1} P_{0,odd}(z^2)$$

$$P_0(z) - P_0(-z) = \underbrace{2z^{-1}}_{\text{odd power}} \underbrace{P_{0,odd}(z^2)}_{\text{only even powers}} = 2z^{-L}$$

$$= 2z^{-(2k+1)}$$

$\Rightarrow$ L must be odd

$P_{0,odd}(z^2) = z^{-2K} \Rightarrow P_{0,odd}(z) = z^{-K}$

pure delay

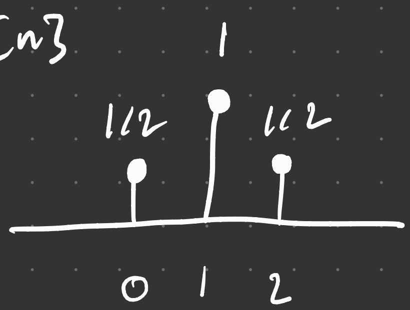
<u>Obs:</u> $P_{0,even}(z)$ is the design choice — even coeff.

$P_{0,odd}(z)$ <u>must</u> be a delay — odd coeff.

<u>Ex:</u> Let

Is $p_0[n]$ valid? Yes

$p_0[n]$



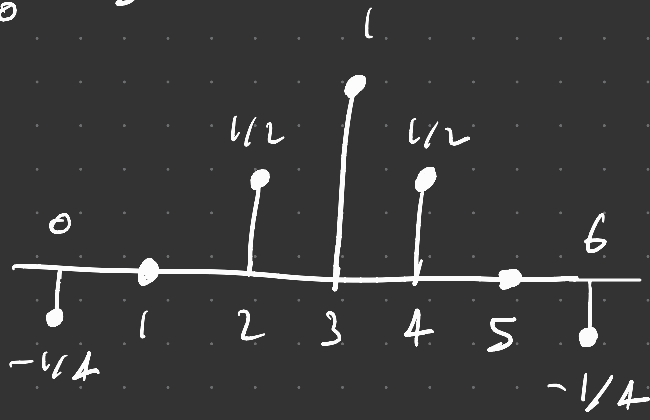

$P_{0,even}[n]$


$P_{0,odd}[n] = \delta[n]$


$K=0, L=1$

Ex: Let

$p_0[n]$

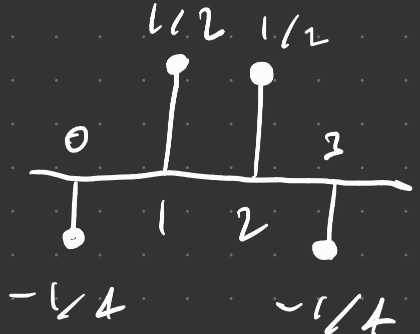



Is $p_0[n]$ valid?

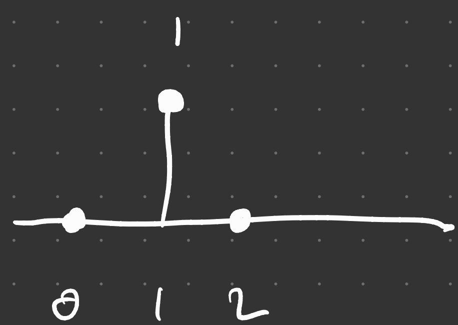
Yes

Type I Linear-Phase

$p_{0,even}[n]$




$p_{0,odd}[n] = \delta[n-1]$




$$K = 1, \quad L = 3$$

Q: What kind of filter is this?

A: It is a (shifted) half-band filter:

$$p_0[2n-L] = \delta[n]$$

**Def$^{n}$**: A filter $p_0[n]$ of the form

$$p_0[2n-L] = \delta[n]$$

is called a half-band filter centered at L.

**Remark**: Interpolation filters satisfy half-band cond.

$$h[2n] = \delta[n].$$

**Obs**: We have now reduced the problem to designing half-band (interpolation) filters.

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## Half-Band Filter Design

If $P_0$ is a half-band filter centered at $L$, then

$$P_0(z) - P_0(-z) = 2z^{-L}$$

Q: What do people typically do?

A: Assume $P_0(z)$ is Type I Linear-Phase

why not Type II? $L$ is odd.

$$P_0(e^{j\omega}) = e^{-jL\omega} P_{ampl}(\omega)$$

real function

center of symmetry

$P_0(-z)$

$$P_0(e^{j(\omega-\pi)}) = e^{-jL(\omega-\pi)} P_{ampl}(\omega-\pi)$$

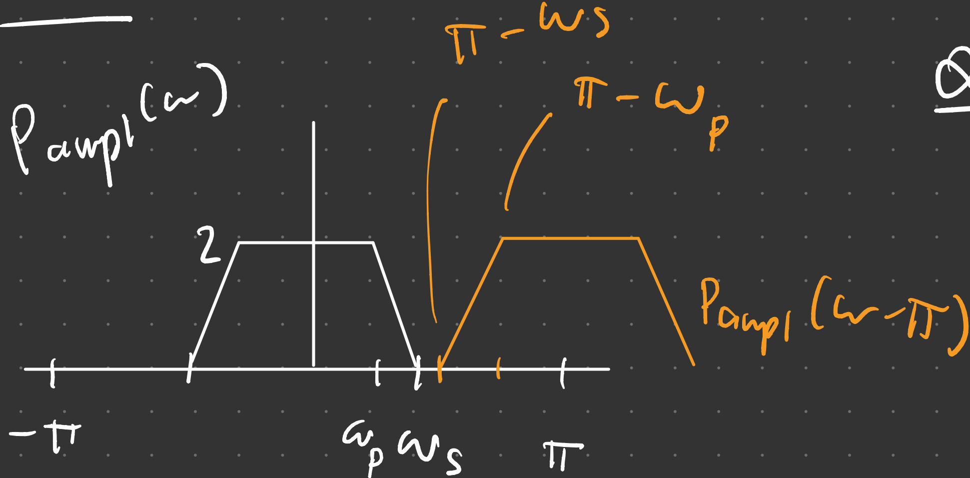
$$= e^{jL\pi} e^{-jL\omega} P_{ampl}(\omega-\pi)$$

$$= -e^{jL\omega} P_{ampl}(\omega-\pi)$$

$$e^{-jL\omega}P_{ampl}(\omega)+e^{-jL\omega}P_{ampl}(\omega-\pi)=2e^{-jL\omega}$$

$$\boxed{P_{ampl}(\omega)+P_{ampl}(\omega-\pi)=2}$$

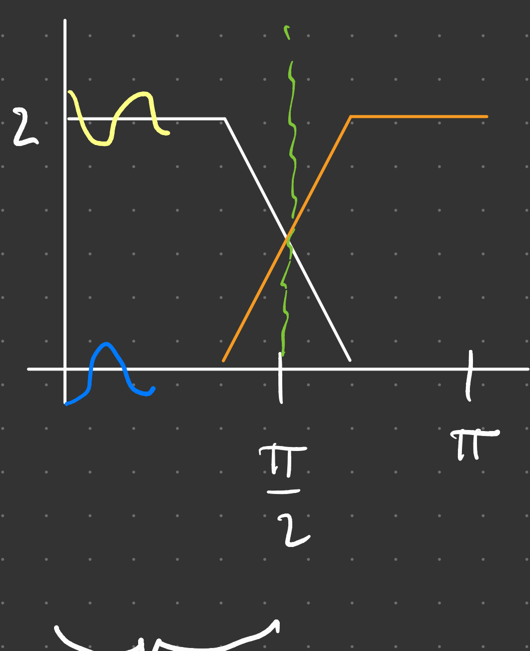
Ex:




Q: How do we guarantee that these add to 2?

$$\pi-\omega_s=\omega_p$$

$$\omega_p+\omega_s=\pi$$



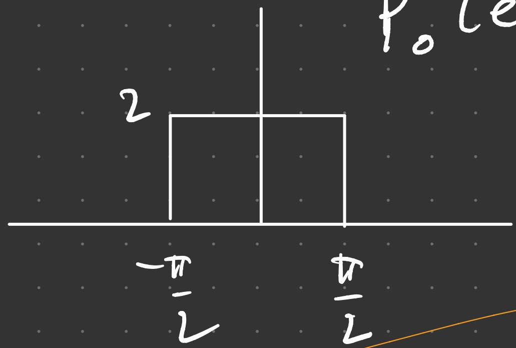

half-band

$$\boxed{\omega_s-\frac{\pi}{2}=\frac{\pi}{2}-\omega_p}$$

ripples need to cancel out

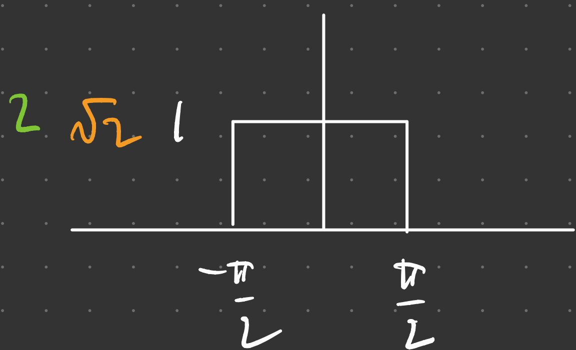
Ex: Ideal low-pass filter $(L=0)$

$$P_0(e^{j\omega}) = H_0(e^{j\omega})\, G_0(e^{j\omega})$$




guarantee PR

$H_0(e^{j\omega})$




$G_0(e^{j\omega})$


Q: Why the gain factor of 2?

A: To account for the magnitude being halved from downsampling.

**Q:** If $P_0$ is of length $2L+1$, how many coeffs do we have to design?

**A:** We only need to design $P_{0,even}$, which is itself symmetric.

| $L$ | # coeffs |
|---|---|
| 1 | 1 |
| 3 | 2 |
| 5 | 3 |
| 7 | 4 |

$$\boxed{\frac{L+1}{2} \text{ coeffs}}$$

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**Q:** What is $P_{0,even}$?

**A:** Type II Linear-Phase with Center of symmetry $\frac{L}{2}$.

☆ $P_{0,even}(e^{j\omega}) = e^{-j\frac{L}{2}\omega} P_{even,ampl}(\omega)$

Putting everything together,

$$P_0(z) = P_{0,even}(z^2) + z^{-1} P_{0,odd}(z^2)$$

$$P_0(e^{j\omega}) = P_{0,even}(e^{j2\omega}) + e^{-jL\omega}$$

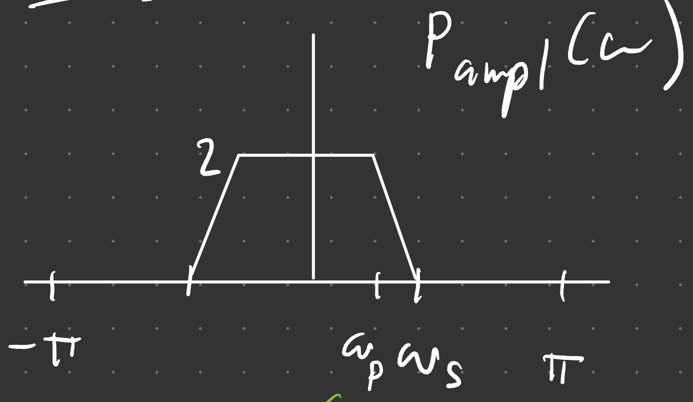
By $\star$ and $\star$,

$$e^{-jL\omega} P_{ampl}(\omega) = e^{-jL\omega} P_{even,ampl}(2\omega) + e^{-jL\omega}$$

$$P_{ampl}(\omega) = P_{even,ampl}(2\omega) + 1$$

$$\boxed{P_{ampl}(\omega) - 1 = P_{even,ampl}(2\omega)}$$

**Ex:**




pass-band freq

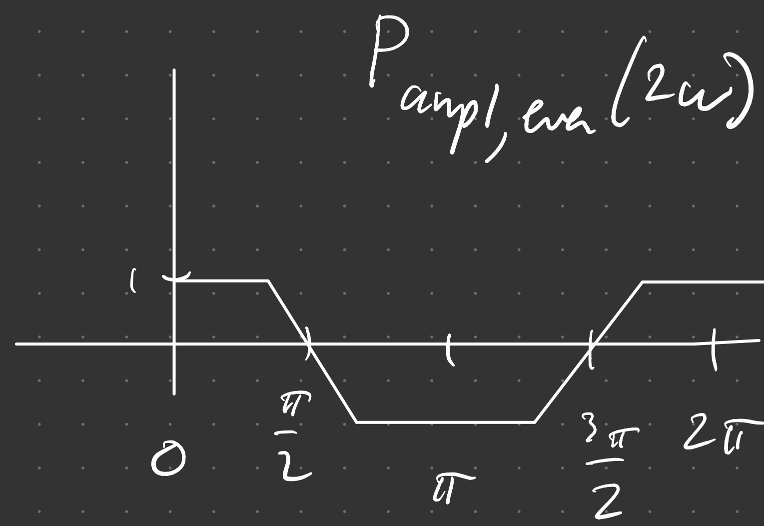
stop-band freq

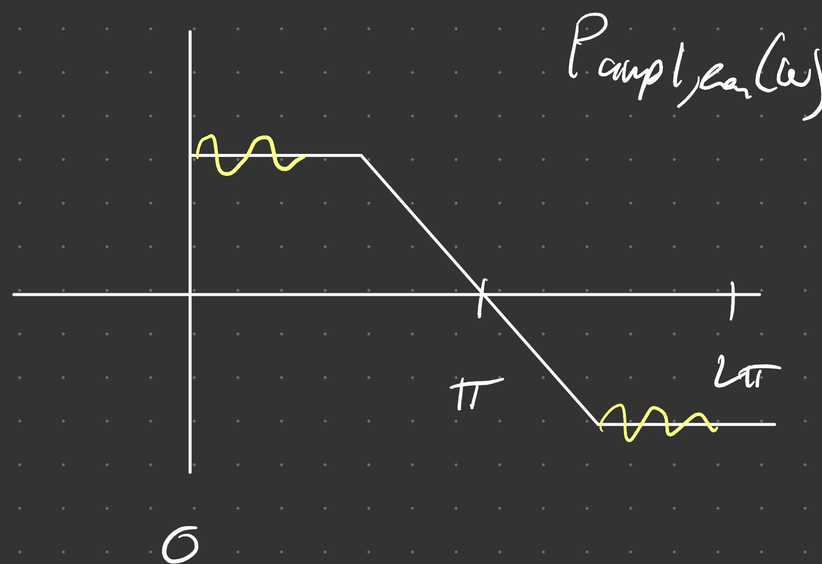
Type II

linear-phase

zero @ $\pi$

One band filter




ripples will automatically cancel by symmetry

**Remark:** We've reduced the problem to designing a Type II linear-phase filter.