Last Time: Linear-Phase Filters

$$H(e^{j\omega}) = e^{j\phi(\omega)} H_{ampl}(\omega)$$

<u>Def</u>: $h[n]$ is said to be <u>linear-phase</u> if the <u>phase response</u> $\phi(\omega)$ is <u>linear</u>.

$$\boxed{\phi(\omega) = -L\omega + \beta}$$

L: center of the filter

β: controls the symmetry of the filter

$H_{ampl}(\omega)$	$L \in \mathbb{Z}$	$L - \frac{1}{2} \in \mathbb{Z}$
Symmetric $\beta = 0$ $h[n] = h[2L-n]$	I $\sum_{n=0}^{L} a[n]\cos(n\omega)$	II $\sum_{n=0}^{L-\frac{1}{2}} b[n]\cos\left((n+\frac{1}{2})\omega\right)$
Antisymm. $\beta = \frac{\pi}{2}$ $h[n] = -h[2L-n]$	III $\sum_{n=0}^{L} c[n]\sin(n\omega)$	IV $\sum_{n=0}^{L-\frac{1}{2}} d[n]\sin\left((n+\frac{1}{2})\omega\right)$

Group Delay: Delay of each frequency when filtering with $h[n]$. Mathematically, this is captured by $-\phi'(\omega)$.

For linear-phase filters:

$$\boxed{-\phi'(\omega) = L} \quad \text{center of } h[n]$$

Obs: Linear-phase filters have constant group delay.

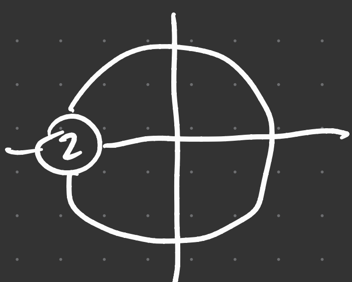
Last Time: High-pass filters cannot be Type II.

Proof: $\cos\left(\left(n+\frac{1}{2}\right)\pi\right) = 0 \quad \forall n \in \mathbb{Z}$.

Remark: Different Types of linear-phase filters put different constraints on the kinds of filters that are realizable. [More exploration in HW]

Exercise: What happens if you cascade two linear-phase filters of the same Type?

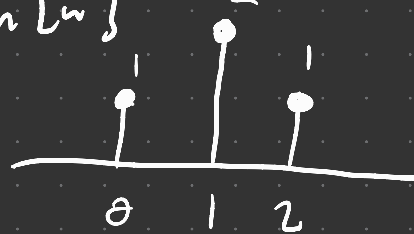
Ex: $H(e^{j\omega})$ has two zeros @ π.



Q: What Type is this?

$$H(z) = (1+z^{-1})^2 = 1 + 2z^{-1} + z^{-2}$$

$h[n]$

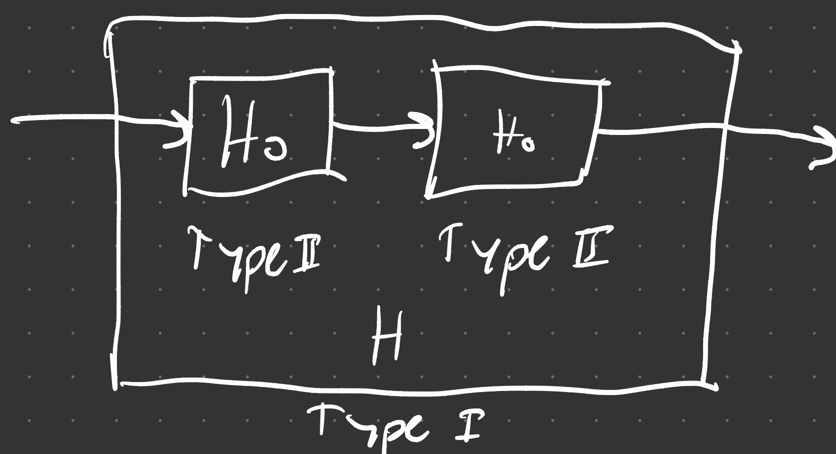


$\Rightarrow$ Type I

Obs: $H(z) = H_0(z)\, H_0(z)$, $H_0(z) = (1+z^{-1})$

$h_0[n]$

$\Rightarrow$ Type II



Cascade of two Type II filters is Type I.

Claim: The cascade of two linear-phase filters of the same Type is **always Type I.**

Proof: Consider the cascade

$$H(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega})$$

with

$$H_k(e^{j\omega}) = e^{j(-L_k \omega + \beta)}\, H_{ampl,k}(\omega)$$

$k=1,2.$

$$\boxed{\beta = 0 \text{ or } \frac{\pi}{2}}$$

Then,

$$H(e^{j\omega}) = e^{j(-(L_1+L_2)\omega + 2\beta)}\, H_{ampl,1}(\omega)\, H_{ampl,2}(\omega)$$

$$= e^{j(-(L_1+L_2)\omega)}\, \underbrace{e^{j2\beta}\, H_{ampl,1}(\omega)\, H_{ampl,2}(\omega)}_{H_{ampl}(\omega) \text{ real function}}$$

Observe that $L_1 + L_2 \in \mathbb{Z}$ & $(h_1 * h_2)$ is always odd length & symmetric.

$\therefore$ $H(e^{j\omega})$ is Type I. $\square$

Remark: Linear-phase filters are tightly constrained. If $h[n]$ is linear-phase with real coefficients and z_0 is a zero of $H(z)$, then

$$\boxed{(z_0,\ z_0^{-1},\ z_0^{*},\ z_0^{-1*}) \text{ are all zeros}}$$

[More exploration in HW1]

All-Pass Filters

Def^{n}: $h[n]$ is said to be all-pass if

$$|H(e^{j\omega})| = 1.$$

Q: What is the point of an all-pass filter?

A: To manipulate the phase response.

- Minimum-phase filters
- Maximum-phase filters

Ex: Simple examples of all-pass filters:

- $H(z) = 1$ identity filter / do nothing

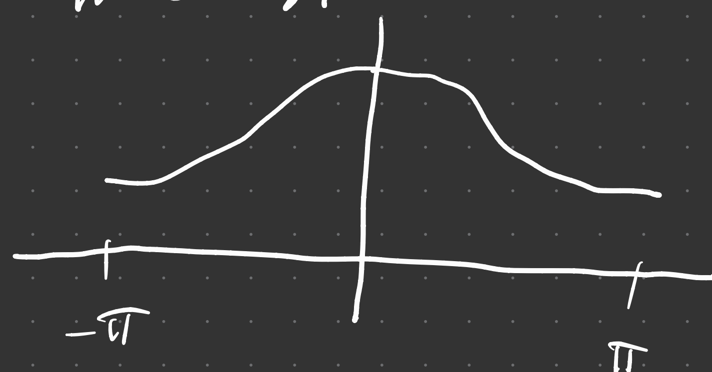
  $\Rightarrow h[n] = \delta[n]$

- $H(z) = z^{-N}$ delay

  $\Rightarrow h[n] = \delta[n-N]$

(Non)
Ex:

$|H(e^{j\omega})|$



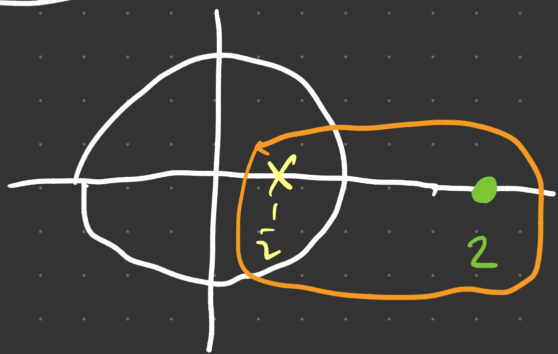
The pole "pushes" you with "strength" proportional to the distance from the pole.

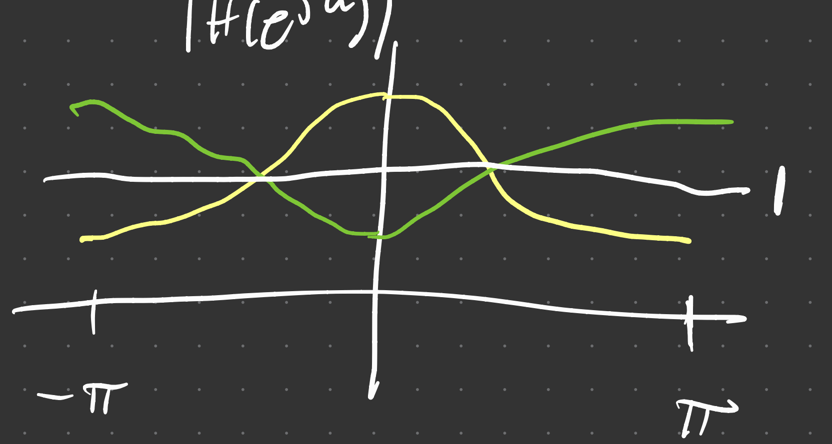
Obs: This is <u>not</u> all-pass.

Q: How do we make it all-pass?

A: Add a zero to "counteract" the pole.

Ex:





All-pass systems have pole-zero pairs

$$H(z) = \frac{-\frac{1}{2} + z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

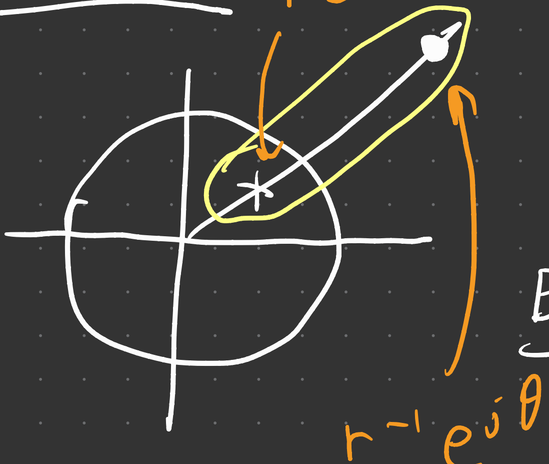
Is this actually all-pass?

$$H(e^{j\omega}) = \frac{-\frac{1}{2} + e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}} = \underbrace{e^{-j\omega}}_{\text{unit magnitude}} \left[\frac{-\frac{1}{2} e^{j\omega} + 1}{1 - \frac{1}{2} e^{-j\omega}} \right]$$

complex conj. pairs

$$\Rightarrow |H(e^{j\omega})| = 1.$$

General: $re^{j\theta}$



$$H(z) = \frac{-re^{-j\theta} + z^{-1}}{1 - re^{j\theta} z^{-1}}$$

flip coeff and take complex conj. of denom. to get num.

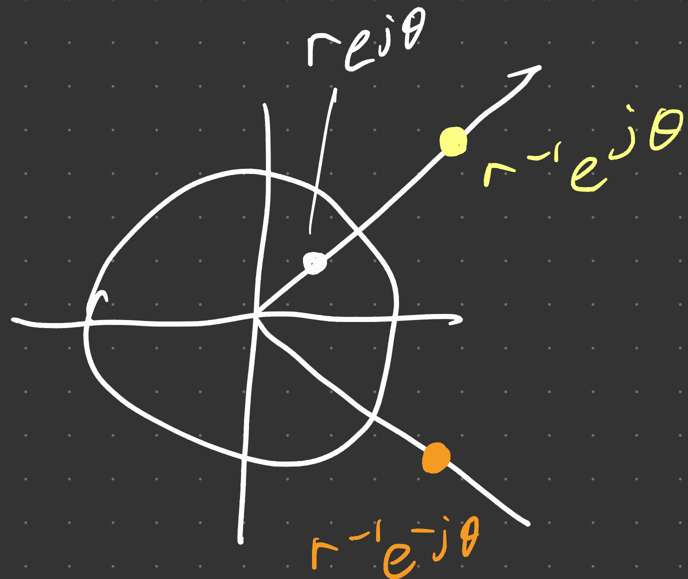
Exersice: where is the zero?

Complex Zero Tricks

$$(1 - re^{j\theta} z^{-1}) \Rightarrow \text{zero} \ @ \ re^{j\theta}$$

flip coeffs $(-re^{j\theta} + z^{-1}) \Rightarrow \text{zero} \ @ \ r^{-1}e^{-j\theta}$

flip coeffs + complex conj. $(-re^{-j\theta} + z^{-1}) \Rightarrow \text{zero} \ @ \ r^{-1}e^{j\theta}$



Remark: All three have the same magnitude resp.

Defn: A causal and stable system is first-order all-pass if

$$H(z) = \frac{-re^{-j\theta} + z^{-1}}{1 - re^{j\theta} z^{-1}}, \quad |r| < 1$$

all-pass systems have pole-zero pairs

Def^{n}: A causal and stable system is <u>Nth-order</u> <u>all-pass</u> if

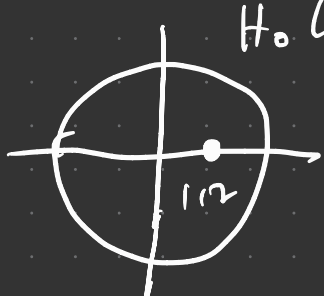
$$H(z) = \prod_{k=1}^{N} \frac{-r_k e^{-j\theta_k} + z^{-1}}{1 - r_k e^{j\theta_k} z^{-1}}, \quad |r_k| < 1$$

Theorem: The phase response $\phi(\omega)$ of a stable and causal all-pass system is monotone nonincreasing

Proof is long and boring...

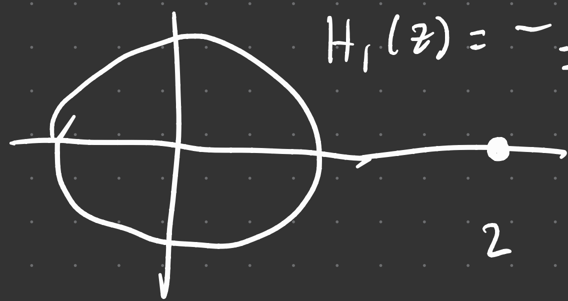
Exercise:

$H_0(z) = 1 - \frac{1}{2} z^{-1}$



Find all systems with the same magnitude response and order.

By inspection:

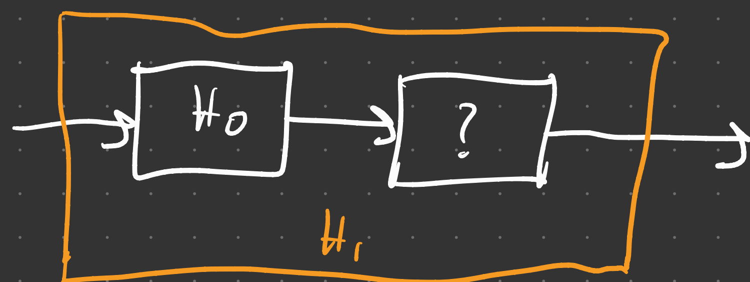


$H_1(z) = -\frac{1}{2} + z^{-1}$

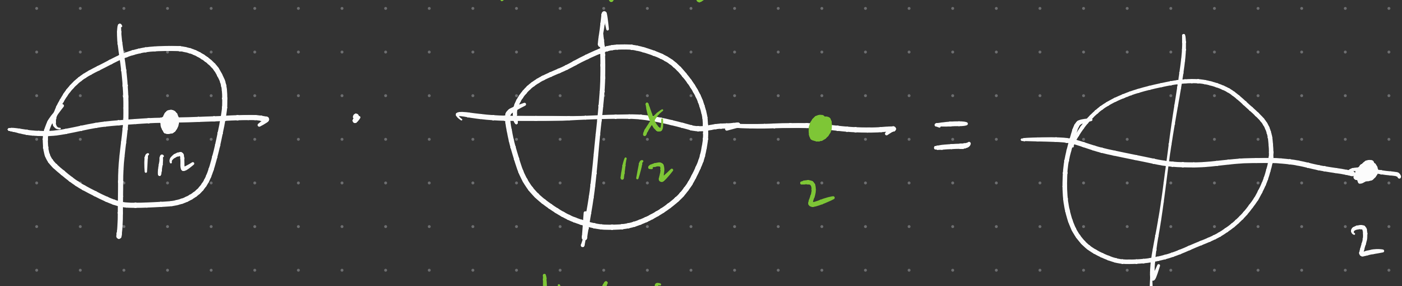
Q: What is the difference between H_0 & H_1?

A: The phase.

Exercise: How are these two systems related?

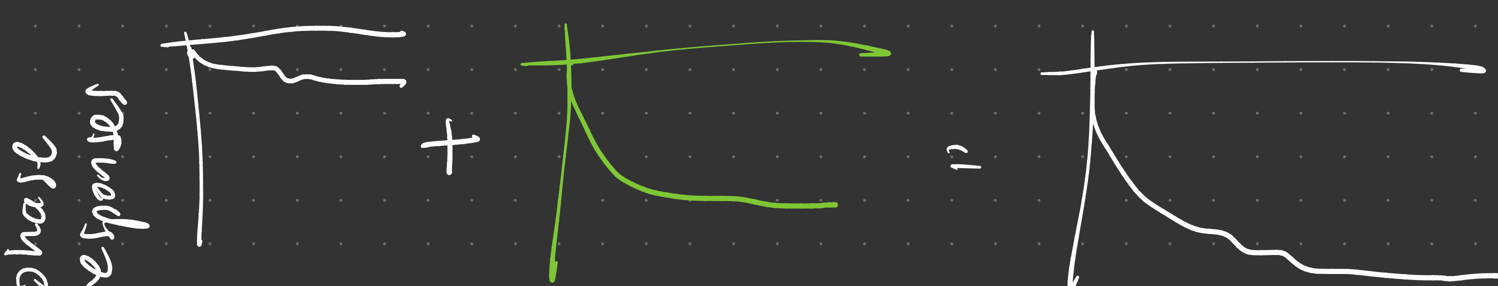


All-pass



$H_0(z)$ $\cdot$ $H(z)$ $=$ $H_1(z)$

$$\left(1 - \frac{1}{2} z^{-1}\right) \cdot \frac{\left(\frac{1}{2} + z^{-1}\right)}{\left(1 - \frac{1}{2} z^{-1}\right)} = \left(\frac{1}{2} + z^{-1}\right)$$



Minimum phase

Maximum phase

Obs: All-pass systems can be used to derive many new systems with the same magnitude response without changing the system effect.

- Low-pass remains low-pass
- High-pass remains high-pass
- etc.

Obs: Minimum-phase "act faster"

- smaller group delay

Maximum-phase "act slower"

Obs: You can make a system act faster or slower with an all-pass filter.

Remark: This is how you understand DSP

- Understand how poles/zeroes affect the system
- Understand how to manipulate them.

Exercise:

H_1

How many other systems have the same order and magnitude response?

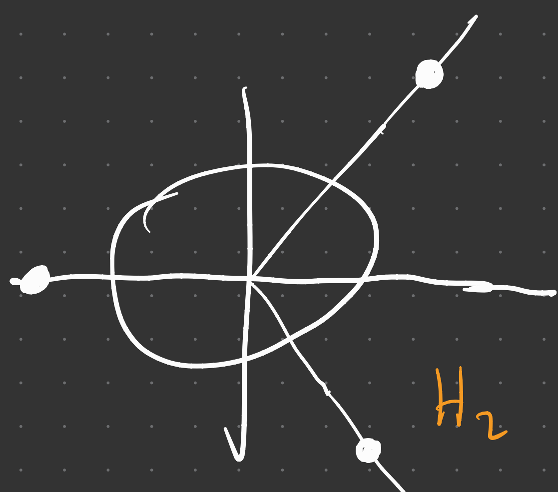
Sol^n: Flip zeros

$\Rightarrow 2^3 = 8$ total systems.

General: 2^N, N is the # of poles/zeros

Q: How many with real coefficients?

A: 4, The two complex zeros need to flip together.



H_2

H_3

H_4

Q: Which one is

- Minimum-phase ?
- Maximum-phase ?

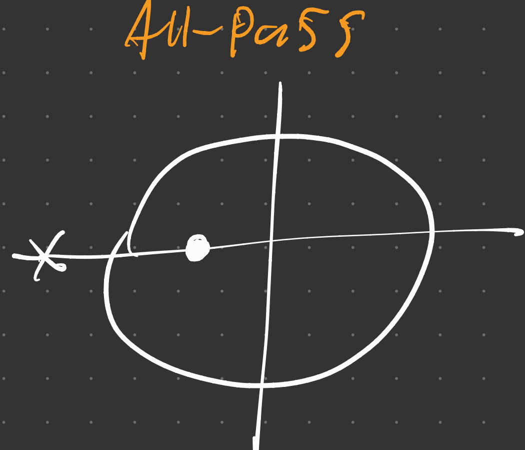
A: Minimum-phase = H_1

Maximum-phase = H_2

Why? By flipping a zero inside the unit circle, you decrease the phase

Proof: Previous exercise.

Obs:

H_4 ·  = H_1

Alternative Characterization of Min/Max-Phase

Define the <u>partial energy</u>

$$E[n] = \sum_{k=-\infty}^{n} |h[k]|^2$$

<u>Theorem:</u> For all systems of same order and magnitude

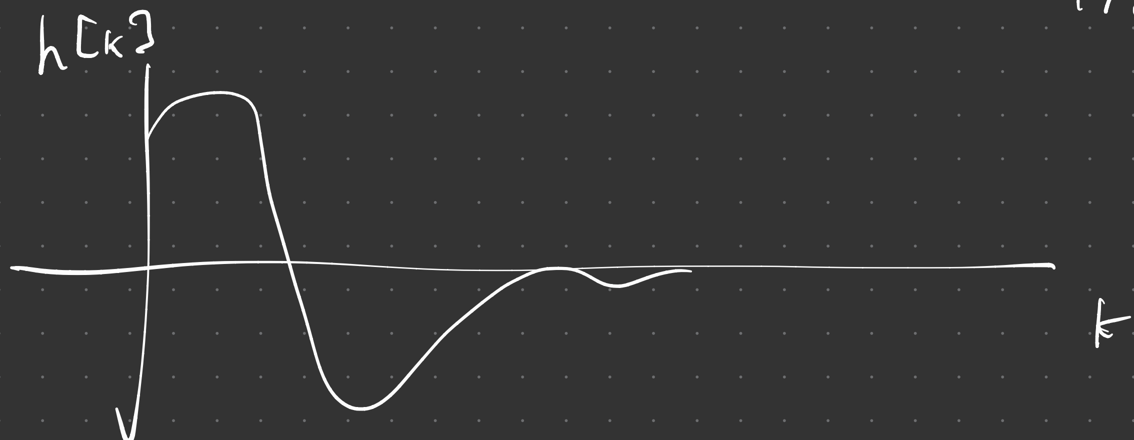
$$\sum_{k=-\infty}^{n} |h[k]|^2 \leq \sum_{k=-\infty}^{n} |h_{min}[k]|^2, \quad \forall n \in \mathbb{Z}$$

where $h_{min}[n]$ is the min-phase and $h[n]$ is any other system in this family, i.e., h_{min} has the <u>largest</u> partial energy.

Theorem: The max-phase system $h_{max}[n]$ has the smallest partial energy.

$$\sum_{k=-\infty}^{n} |h_{max}[k]|^2 \leq \sum_{k=-\infty}^{n} h[k] \quad \forall n \in \mathbb{Z}$$

Obs: • Min-phase has energy concentrated in the "front"



• Max-phase has energy concentrated in the "back"