Last Time: "filters from wavelets"

Today: "wavelets from filters"

Remark: We understand how $\varphi$ & $\psi$ determine $g_0$ & $g_1$. We will now see how $g_0$ & $g_1$ determine $\varphi$ & $\psi$.

---

Idea: We can iterate the DWT as many times as we want. Therefore, we should be able to iterate it infinitely many times.

$$\boxed{\text{This procedure needs to converge}}$$

Equivalently, given $g_0[n]$, the iteration

cascade algorithm

$$\varphi^{(k+1)}(t) = \sum_{n \in \mathbb{Z}} \underbrace{g_0[n]}_{h[n]} \sqrt{2}\, \varphi^{(k)}(2t-n)$$

must converge.

Obs: We are looking for the fixed point of the two-scale equation.

Def$^{\underline{n}}$: $x$ is called the fixed point of

$$x_{k+1} = F(x_k)$$

if $x = F(x)$.

Obs: Given an initial value $x_0$, if the sequence $\{x_k\}_{k=0}^{\infty}$ converges to $x$, i.e., $\lim_{k\to\infty} x_k = x$, then $x$ is a fixed point of $F$.

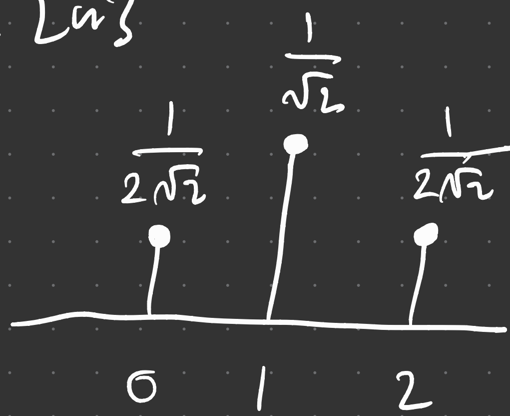
Ex: Does the sequence $x_{k+1} = \frac{x_k}{2}$ converge? What is its fixed point?

$$x_1 = \frac{x_0}{2}, \quad x_2 = \frac{x_1}{2} = \frac{x_0}{4}, \quad \dots, \quad \boxed{x_k = \frac{x_0}{2^k}}$$

This sequence converges to $0$ for any choice of $x_0$.

$$\theta = \frac{\theta}{2}$$ fixed point

<u>Exer:</u> $h[n]$



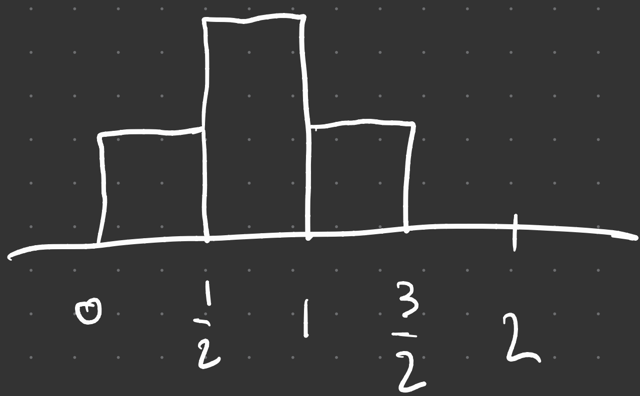

What is the fixed point of the cascade algorithm?

$$\boxed{\varphi(t) = \frac{1}{2}\varphi(2t) + \varphi(2t-1) + \frac{1}{2}\varphi(2t-2)}$$

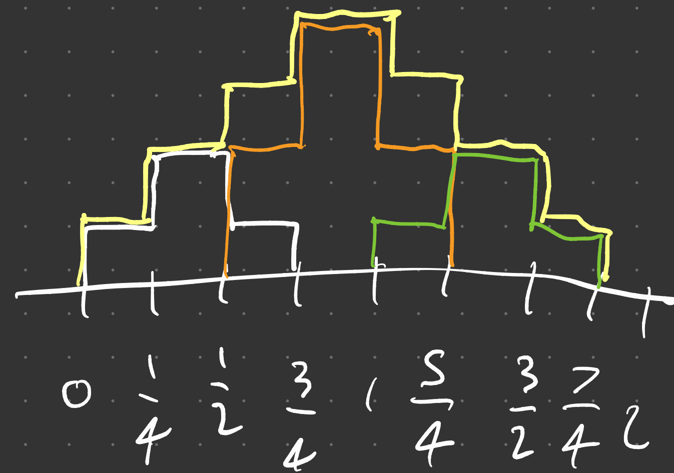
Let us try to run the cascade algorithm with the initial condition

$\varphi^{(0)}(t)$

$\varphi^{(1)}(t)$

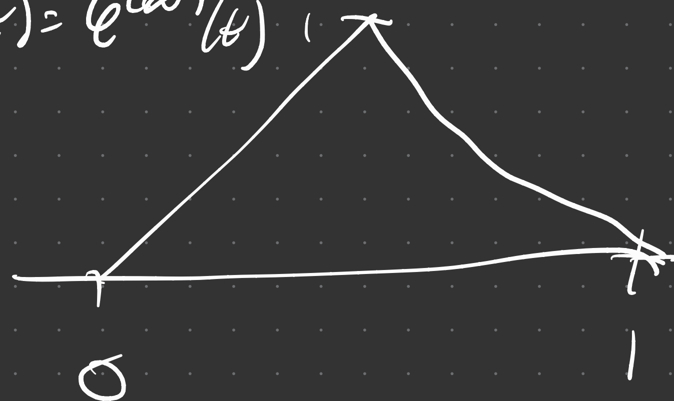



$\varphi^{(2)}(t)$



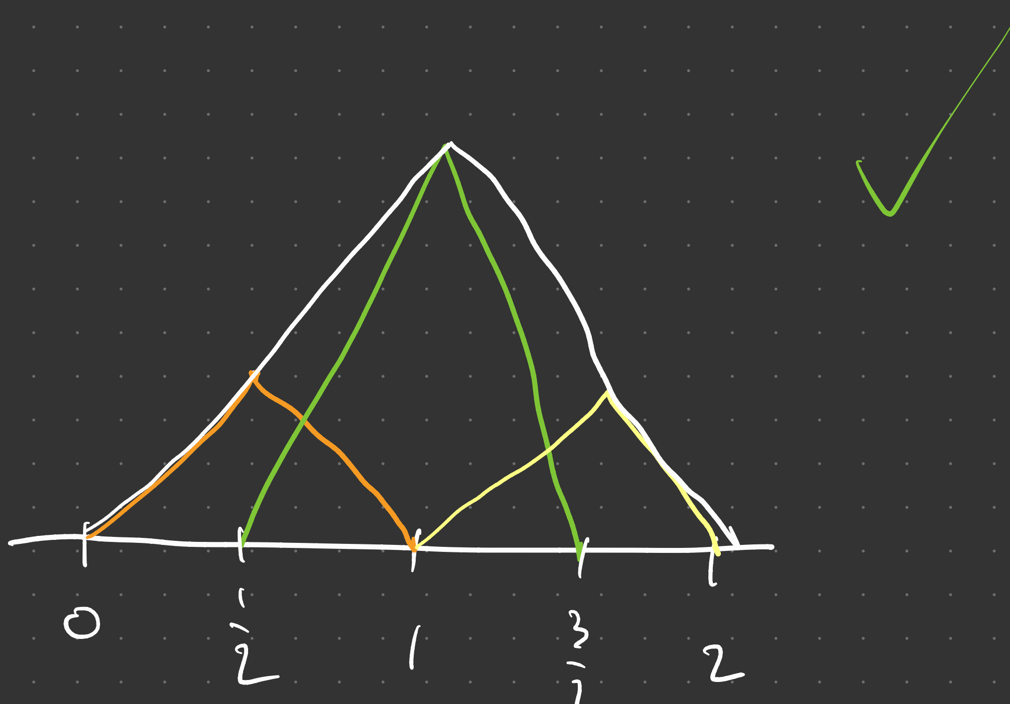

keep iterating ...

$\varphi(t) = \varphi^{(\infty)}(t)$




<u>Exer:</u> Do one more iteration.

<u>Check:</u>



**Q:** How do we understand this in general?

**A:** Fourier transform.

$$\varphi(t) = \sum_{n \in \mathbb{Z}} h[n] \sqrt{2}\, \varphi(2t-n)$$

$$\Big\downarrow \mathcal{F}$$

$$\underbrace{\int_{-\infty}^{\infty} \varphi(t)\, e^{-j\omega t}\, dt}_{\Phi(\omega)} = \int_{-\infty}^{\infty} \Big( \sum_{n \in \mathbb{Z}} h[n] \sqrt{2}\, \varphi(2t-n) \Big) e^{-j\omega t}\, dt$$

$$= \sum_{n \in \mathbb{Z}} h[n] \cdot \sqrt{2} \int_{-\infty}^{\infty} \varphi(2t-n)\, e^{-j\omega t}\, dt$$

$$\begin{bmatrix} u = 2t - n \\ du = 2dt \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h[n] \int_{-\infty}^{\infty} \varphi(u)\, e^{-j\omega\left(\frac{u+n}{2}\right)}\, dt$$

$$= \frac{1}{\sqrt{2}} \underbrace{\left( \sum_{n \in \mathbb{Z}} h[n] e^{-j\omega \frac{n}{2}} \right)}_{H\left(e^{j\frac{\omega}{2}}\right)} \underbrace{\int_{-\infty}^{\infty} \phi(u) e^{-j\omega \frac{u}{2}} dt}_{\Phi\left(\frac{\omega}{2}\right)}$$

$$\Phi(\omega) = \frac{H\left(e^{j\frac{\omega}{2}}\right)}{\sqrt{2}} \Phi\left(\frac{\omega}{2}\right)$$

$$= \frac{H\left(e^{j\frac{\omega}{2}}\right)}{\sqrt{2}} \cdot \frac{H\left(e^{j\frac{\omega}{4}}\right)}{\sqrt{2}} \Phi\left(\frac{\omega}{4}\right)$$

$$= \frac{H\left(e^{j\frac{\omega}{2}}\right)}{\sqrt{2}} \cdot \frac{H\left(e^{j\frac{\omega}{4}}\right)}{\sqrt{2}} \cdot \frac{H\left(e^{j\frac{\omega}{8}}\right)}{\sqrt{2}} \Phi\left(\frac{\omega}{8}\right)$$

$$\vdots$$

$$= \Phi(0) \prod_{i=1}^{\infty} \frac{H\left(e^{j 2^{-i} \omega}\right)}{\sqrt{2}}$$

Recall: $$\Phi(0) = \int_{-\infty}^{\infty} \varphi(t)\, \cancel{e^{-j0t}}^{\,=1}\, dt$$

$$= \int_{-\infty}^{\infty} \varphi(t)\, dt$$

We often impose that $\int \varphi(t)\, dt = 1$

(e.g., this holds for the box )

## Infinite-Product formula:

$$\Phi(\omega) = \prod_{i=1}^{\infty} \frac{H(e^{j2^{-i}\omega})}{\sqrt{2}}$$

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega)\, e^{j\omega t}\, d\omega$$

inverse Fourier transform

"wavelets from filters"

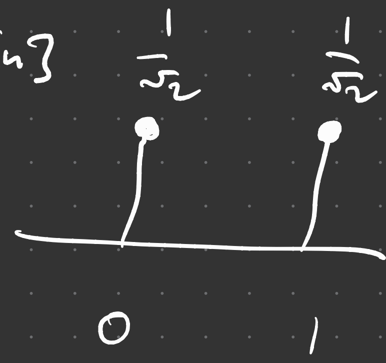
Obs: Convergence of this infinite product is the same as convergence of the cascade algorithm, which is the same as convergence of an infinite-level DWT.

Remark: If $H(e^{i\omega})$ is continuously differentiable at $\omega = 0$, and

$$\min_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |H(e^{i\omega})| > 0,$$

then the infinite product converges.
(Mallat, 1989).

Exer: $\varphi(t)$ $h[n]$


Compute $\Phi(\omega)$ from the infinite product formula.

$$\frac{H(e^{j\omega})}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(1+e^{-j\omega}\right)\right]$$

$$= \frac{1}{2}\left(1+e^{-j\omega}\right)$$

$$H^{(N)}(e^{j\omega}) = \frac{1}{2^N} \underbrace{\prod_{i=1}^{N}\left(1+e^{-j2^{-i}\omega}\right)}_{2^N \text{ terms}}$$

[Check at home]

$$= \frac{1}{2^N}\sum_{k=0}^{2^N-1} e^{-j2^{-N}\omega k}$$

$$\left[\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}\right] = \frac{1}{2^N}\left(\frac{1-e^{-j\omega}}{1-e^{-j2^{-N}\omega}}\right)$$

$$\left[e^{-j\theta} = 1 - j\theta + \frac{\theta^2}{2} - \frac{j\theta^2}{3} + \cdots\right]$$

$$\theta = 2^{-N}\omega$$

$$\xrightarrow[N\to\infty]{} \frac{1-e^{-j\omega}}{j\omega}$$

$$\Phi(\omega) = \frac{1-e^{-j\omega}}{j\omega} = e^{-j\frac{\omega}{2}}\left(\frac{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}{j\omega}\right)$$

modulated sinc

$$= e^{-j\frac{\omega}{2}}\left(\frac{2\sin\left(\frac{\omega}{2}\right)}{\omega}\right)$$

**Remark:** This recovers Euler's celebrated infinite product formula for the sinc.

---

## Fundamental Theorem of Wavelet Analysis (Mallat, 1989)

Let $\varphi \in L^2(\mathbb{R})$ be a <u>valid</u> scaling function. Then, the filter $h[n] = \langle \varphi, \varphi_{1,n} \rangle$ must satisfy

① $\left| H(e^{j\omega}) \right|^2 + \left| H(e^{j(\omega+\pi)}) \right|^2 = 2$

② $H(e^{j0}) = \sum_{n \in \mathbb{Z}} h[n] = \sqrt{2}$

③ $\min_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \left| H(e^{j\omega}) \right| > 0.$

On the other hand, given a filter $h[n]$ such that $H(e^{j\omega})$ satisfies ①, ②, & ③, then the inverse Fourier transform of

$$\Phi(\omega) = \prod_{i=1}^{\infty} \frac{H(e^{j2^{-i}\omega})}{\sqrt{2}}$$

exists and is a valid scaling function.

<u>Obs:</u> This gives a one-to-one correspondance between scaling functions $\varphi$ and filters $h$.

<u>Remark:</u>

$$g_0[n] = h[n]$$

if you allow for complex coeffs.

$$g_1[n] = (-1)^{1-n} h^*[1-n]$$

conjugate mirror filters

<u>Ex:</u> Haar