MRA (Mallat, Meyer, ca. 1988)

A nested sequence of subspaces $\{V_i\}_{i\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ is called a MRA if:

1. $\overline{\bigcup_{i\in\mathbb{Z}} V_i} = L^2(\mathbb{R})$

2. $\bigcap_{i\in\mathbb{Z}} V_i = \{0\}$

3. $f \in V_0$ if and only if $f(2^i\cdot) \in V_i$

4. $f \in V_0$ implies $f(\cdot - n) \in V_0 \quad \forall n \in \mathbb{Z}$

5. There exists a scaling function $\varphi \in V_0$ such that $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is an $\underbrace{\underline{\text{orthonormal}}\ \underline{\text{basis}}}_{\text{can be relaxed}}$ for V_0.

Can be relaxed

Riesz basis, frames, etc.

Haar system

Q: Is $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ an ortho basis? Yes

For $n, m \in \mathbb{Z}$

$$\langle \varphi(\cdot - n), \varphi(\cdot - m)\rangle = \int_{-\infty}^{\infty} \varphi(t-n)\,\varphi(t-m)\,dt$$

$$= \delta[n-m] \checkmark$$

Exercise: Check the other properties.

Recall: $V_1 = V_0 \oplus W_0$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$

$$\vdots$$

$$V_i = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{i-1}$$

Notation: $\varphi_{i,n}(t) = 2^{\frac{i}{2}}\varphi(2^i t - n)$

$$\psi_{i,n}(t) = 2^{\frac{i}{2}}\psi(2^i t - n)$$

Obs: The functions $\{\varphi_{0,n}\}_{n\in\mathbb{Z}} \cup \{\psi_{i,n}\}_{i\geq 0, n\in\mathbb{Z}}$ form an orthobasis for $L^2(\mathbb{R})$.

Exercise: Show that

$$\langle \psi_{i,n}, \psi_{i',n'} \rangle = \delta[i-i']\,\delta[n-n']$$

$$\langle \psi_{i,n}, \varphi_{0,m} \rangle = 0$$

Given $f \in L^2(\mathbb{R})$, we can write

$$f(t) = \underbrace{\sum_{n\in\mathbb{Z}} \langle f, \varphi_{0,n} \rangle \varphi_{0,n}(t)}_{\substack{\text{coarse approximation} \\ \text{projection of } f \text{ onto } V_0}} + \underbrace{\sum_{i=0}^{\infty} \sum_{n\in\mathbb{Z}} \langle f, \psi_{i,n} \rangle \psi_{i,n}(t)}_{\text{details over all resolutions}}$$

Synthesis procedure

$$= \sum_{n\in\mathbb{Z}} a_0[n]\, \varphi_{0,n}(t) + \sum_{i=0}^{\infty} \sum_{n\in\mathbb{Z}} d_i[n]\, \psi_{i,n}(t)$$

Approximation and Wavelet Spaces:

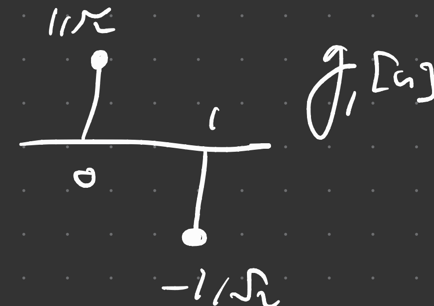
$$V_i = \text{Span}\{\varphi_{i,n}\}_{n\in\mathbb{Z}}$$

$$W_i = \text{Span}\{\psi_{i,n}\}_{n\in\mathbb{Z}}$$

Two-Scale Equations:

$$\varphi(t) = \sum_{n\in\mathbb{Z}} g_0[n]\, \varphi_{1,n}(t)$$

$$\psi(t) = \sum_{n\in\mathbb{Z}} g_1[n]\, \varphi_{1,n}(t)$$



<u>Obs:</u> $g_0[n]$ & $g_1[n]$ are the Haar FB filters.

Last time, we did this by inspection.

Q: Is there a more direct way to do this? (i.e., for other wavelets?)

A: Projections.

$$\varphi \in V_0 \subset V_1 \Rightarrow \varphi(t) = \mathrm{Proj}_{V_1} \varphi(t)$$

$$= \sum_{n \in \mathbb{Z}} \underbrace{\langle \varphi, \varphi_{1,n} \rangle}_{g_0[n]} \varphi_{1,n}(t)$$

$$\psi \in W_0 \subset V_1 \Rightarrow \psi(t) = \mathrm{Proj}_{V_1} \psi(t)$$

$$= \sum_{n \in \mathbb{Z}} \underbrace{\langle \psi, \varphi_{1,n} \rangle}_{g_1[n]} \varphi_{1,n}(t)$$

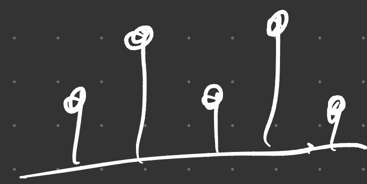
General: Given arbitrary scaling and wavelet function φ and ψ, we can derive two filters:

Low-pass : $g_0[n] = \langle \varphi, \varphi_{1,n} \rangle$

High-pass : $g_1[n] = \langle \psi, \varphi_{1,n} \rangle$

Analysis Procedure (Haar Wavelets)

Ex: finest resolution is $j=3$:



$$a_3[n] = \langle f, \varphi_{3,n} \rangle$$

$$\underbrace{f_3(t)}_{V_3} = \sum_{n \in \mathbb{Z}} a_3[n]\, \varphi_{3,n}(t)$$

How do we get $a_2[n]$ & $d_2[n]$ from $a_3[n]$?

$$V_3 = V_2 \oplus W_2$$

project onto these subspaces

Project onto V_2

$$\mathrm{Proj}_{V_2} f_3(t) = f_2(t) = \sum_{m \in \mathbb{Z}} \langle f_3, \varphi_{2,m} \rangle\, \varphi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \left\langle \sum_{n \in \mathbb{Z}} a_3[n]\, \varphi_{3,n},\ \varphi_{2,m} \right\rangle \varphi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \underbrace{\left[\sum_{n \in \mathbb{Z}} a_3[n] \langle \varphi_{3,n}, \varphi_{2,m} \rangle \right]}_{a_2[m]} \varphi_{2,m}(t)$$

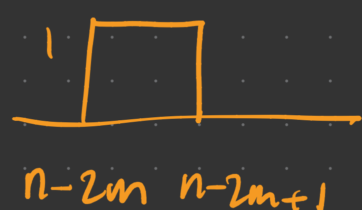
$$\langle \varphi_{3,n}, \varphi_{2,m} \rangle = \int_{-\infty}^{\infty} 2^{\frac{3}{2}} \varphi(2^3 t - n)\, 2\varphi(2^2 t - m)\, dt$$

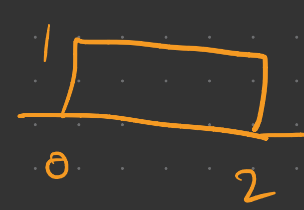
$$\begin{bmatrix} u = 2^3 t \\ du = 2^3\, dt \end{bmatrix}$$

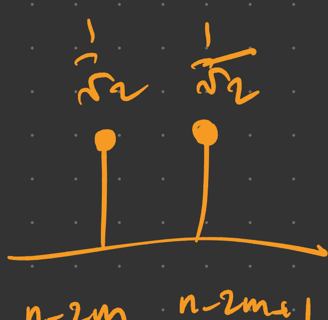
$$= \frac{2^{\frac{5}{2}}}{2^3} \int_{-\infty}^{\infty} \varphi(u-n)\, \varphi\left(\frac{u}{2} - m\right) du$$

$$\begin{bmatrix} t = u - 2m \\ dt = du \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \varphi(t - (n-2m))\, \varphi\left(\frac{t}{2}\right) dt$$







this integral is zero unless $n-2m = 0$ or $n-2m = 1$ in which case it is 1.

$$= g_0[n-2m]$$