Student Information

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Answer 1

a)

The characteristic equation is:

$$r^2 = 3r + 4 \implies r^2 - 3r - 4 = 0$$

Then,

$$(r-4)(r+1) = 0$$

The roots are 4 and -1, so;

$$c_1 \cdot 4^n + c_2 \cdot (-1)^n$$

For $n = 0$:

$$c_1 + c_2 = 2$$

For $n = 1$:

$$4c_1 - c_2 = 5$$

Then;

$$c_1 = \frac{7}{5}, \quad c_2 = \frac{3}{5}$$

The solution is:

$$\frac{7}{5} \cdot 4^n + \frac{3}{5} \cdot (-1)^n$$

b)

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Then we can write the recurrence relation as:

$$\sum_{n=2}^{\infty} a_n x^{n-2} = 3\sum_{n=1}^{\infty} a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n.$$

By multiplying and dividing the left-hand side by x^2 and multiplying and dividing $-3\sum_{n=1}^{\infty} a_n x^{n-1}$ by x, we have:

$$\frac{\sum_{n=2}^{\infty} a_n x^n}{x^2} = \frac{3\sum_{n=1}^{\infty} a_n x^n}{x} + 4\sum_{n=0}^{\infty} a_n x^n.$$

Factoring out the first two terms on the left-hand side and the first term of $-3\sum_{n=1}^{\infty} a_n x^n/x$:

$$-\frac{a_0}{x^2} - \frac{a_1}{x} + \frac{\sum_{n=0}^{\infty} a_n x^n}{x^2} = -\frac{3a_0}{x} + \frac{3\sum_{n=0}^{\infty} a_n x^n}{x} + 4\sum_{n=0}^{\infty} a_n x^n.$$

Substituting $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and multiplying by x^2, and $a_0 = 2$ and $a_1 = 5$,

$$-2 - 5x + F(x) = 6x - 3xF(x) + 4x^2 F(x).$$

So,

$$F(x)(4x^2 + 3x - 1) = x - 2.$$

$$F(x) = \frac{x-2}{(4x-1)(x+1)}.$$

Partial fraction decomposition:

$$\frac{A}{4x-1} + \frac{B}{x+1} = F(x) = \frac{x-2}{(4x-1)(x+1)}.$$

$$(A+4B)x + (A-B) = x - 2.$$

Then,

$$A + 4B = 1, \quad -A + B = -2.$$

So,

$$A = -\frac{7}{5}, \quad B = \frac{3}{5}.$$

Substitute A and B in F(x):

$$F(x) = \frac{7}{5} \cdot \frac{1}{1-4x} + \frac{3}{5} \cdot \frac{1}{1+x}.$$

In lecture, we learned that;

$$\langle 1, 1, 1, 1, 1, 1, \ldots \rangle = \frac{1}{1-x}.$$

Rewrite each terms as a power series(like in lecture):

$$F(x) = \frac{7}{5}\sum_{n=0}^{\infty} a_n (4x)^n + \frac{3}{5}\sum_{n=0}^{\infty} a_n (-1)^n x^n.$$

Then;

$$a_n = \frac{7}{5} \cdot 4^n + \frac{3}{5} \cdot (-1)^n \ (similar\ to\ the\ first\ result).$$

Answer 2

Let $b_k = a_{2^k}$. Then the equation becomes:

$$b_n = b_{n-1} + 6b_{n-2}.$$

The characteristic equation is:

$$r^2 - r - 6 = 0 \implies (r-3)(r+2) = 0.$$

The roots are 3 and -2. The general solution is:

$$b_n = c_1 \cdot 3^n + c_2 \cdot (-2)^n.$$

Given $a_1 = 3$, we have $b_0 = 3$, and given $a_2 = 4$, we have $b_1 = 4$. Substituting:

$$b_0 = 3 = c_1 + c_2,$$

$$b_1 = 4 = 3c_1 - 2c_2.$$

So,

$$c_1 = 2, \quad c_2 = 1.$$

Since $n = 2^k \implies k = \log_2 n$, then:

$$a_n = 2 \cdot 3s^{\log_2 n} + (-2)^{\log_2 n}.$$

Answer 3

$$\langle 1, 1, 1, 1, 1, 1, \ldots \rangle = \frac{1}{1-x}.$$

$$\langle 1, 2, 3, 4, 5, 6, \ldots \rangle = \frac{1}{(1-x)^2} \quad \text{(derivative)}.$$

$$\langle 3, 6, 9, 12, 15, \ldots \rangle = \frac{3}{(1-x)^2} \quad \text{(multiply by 3)}.$$

$$\langle 1, 1, 1, 1, 1, 1, \ldots \rangle = \frac{1}{1-x}.$$

$$\langle 1, 3, 9, 27, 81, \ldots \rangle = \frac{1}{1-3x} \quad \text{(geometric series)}.$$

$$\langle 3, 9, 27, 81, \ldots \rangle = \frac{3}{1-3x} \quad \text{(multiply by 3)}.$$

$$\langle 0, 3, 9, 27, 81, \ldots \rangle = \frac{3x}{1-3x} \quad \text{(1 shifting multiple by x)}.$$

Sum of them:

$$\langle 0, 3, 9, 27, 81, \ldots \rangle + \langle 3, 6, 9, 12, 15, \ldots \rangle = \langle 3, 9, 18, 39, 96, \ldots \rangle.$$

Then;

$$\frac{3x}{1-3x} + \frac{3}{(1-x)^2}.$$

Answer 4

a)

If we have a partition $\{A_1, A_2, \ldots, A_k\}$ of S, we can define a relation R like this:

$$xRy \iff x \text{ and } y \text{ are in the same group } A_i.$$

This relation R is an equivalence relation because any x is in some group A_i, so xRx. So R is reflexive. If xRy, then x and y are in the same group, so yRx. So R is symmetric. If xRy and yRz, then x, y, z are all in the same group, so xRz. Then R is transitive. Each group A_i in the partition becomes an equivalence class under R. So, every partition gives us an equivalence relation. Then statement is true.

b)

Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3), (3, 1)\}$. As you can see, R is an antisymmetric relation defined on A. Now, let's find the transitive closure of R, R^T:

$$R^T = \{(1, 2), (2, 3), (3, 1), (1, 3), (2, 1), (3, 2)\}.$$

Observe that R^T is not antisymmetric. Then, the statement is false.

c)

Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$. Since R is reflexive, transitive, and symmetric, it is an equivalence relation. However, it is not antisymmetric, so it is not a partial order. Then, the statement is false.

d)

Firstly;
Assume that R is antisymmetric. Then, for all $x, y \in A$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. Now, consider an arbitrary pair $(x, y) \in R \cap R^{-1}$. This implies that $(x, y) \in R$ and $(y, x) \in R$. Since R is antisymmetric, then we can say that $x = y$. Thus, $(x, y) \in \Delta$, which implies that $R \cap R^{-1} \subseteq \Delta$.
Secondly;
Assume that $R \cap R^{-1} \subseteq \Delta$. We need to show that R is antisymmetric, i.e., for all $x, y \in A$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. Suppose that $(x, y) \in R$ and $(y, x) \in R$. Then, $(x, y) \in R \cap R^{-1}$. Since we assumed that $R \cap R^{-1} \subseteq \Delta$, then $(x, y) \in \Delta$, and $x = y$. Therefore, R is antisymmetric. Then, the statement is true.

e)

Basis: For $n = 1$:

$$R^1 = R \ (correct).$$

Inductive step: Assume that for some positive integer k, $R^k = R$ is true. So, for $R^{k+1} = R$: By definition, $R^{k+1} = R^k \circ R$. By the inductive hypothesis, $R^k = R$:

$$R^{k+1} = R \circ R.$$

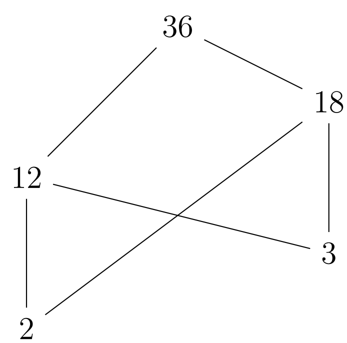
Since R is transitive, for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. Therefore, $R \circ R \subseteq R$. Since R is reflexive, for all $x \in A$, $(x, x) \in R$, and thus $R \subseteq R \circ R$. Therefore:

$$R \circ R = R.$$

So, $R^{k+1} = R$, and by the principle of mathematical induction, we conclude that $R^n = R$ for all positive integers n. The statement is true.

Answer 5

a)



b)

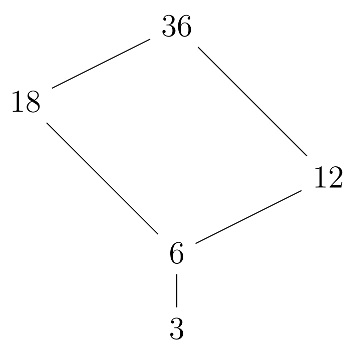
$$R = \{(2, 12), (2, 18), (2, 36), (3, 12), (3, 18), (3, 36), (12, 36), (18, 36)\}$$

So,

$$R_s - R = \{(12, 2), (18, 2), (36, 2), (12, 3), (18, 3), (36, 3), (36, 12), (36, 18)\}$$

c)

Remove element 2 and adding element 6. The new set will be $A' = \{3, 6, 12, 18, 36\}$. The Hasse diagram of this set:



Every pair has both a least upper bound and greatest lower bound, i.e. this is a lattice. It is possible to create a lattice by this way.