Middle East Technical University

Department: Computer Science and Engineering

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Course: Discrete Computational Structures

Student's Solution

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1 Question 1

1.

Basis Case ($n = 1$):

For $n = 1$, the summation simplifies as follows:

$$\sum_{j=1}^{1} j(j+1)\cdots(j+k-1) = 1 \cdot 2 \cdots k = k!.$$

The right-hand side of the formula is:

$$\frac{1 \cdot (1+1)\cdots(1+k)}{k+1} = \frac{k! \cdot (k+1)}{k+1} = k!.$$

Thus, the base case holds.

Induction Hypothesis:

Assume the formula is true for $n = t$:

$$\sum_{j=1}^{t} j(j+1)\cdots(j+k-1) = \frac{t(t+1)\cdots(t+k)}{k+1}.$$

Inductive Step:

For $n = t + 1$, consider:

$$\sum_{j=1}^{t+1} j(j+1)\cdots(j+k-1) = \left(\sum_{j=1}^{t} j(j+1)\cdots(j+k-1)\right) + (t+1)(t+2)\cdots(t+k).$$

By the induction hypothesis:

$$\sum_{j=1}^{t} j(j+1)\cdots(j+k-1) = \frac{t(t+1)\cdots(t+k)}{k+1}.$$

Substitution:

$$\sum_{j=1}^{t+1} j(j+1)\cdots(j+k-1) = \frac{t(t+1)\cdots(t+k)}{k+1} + (t+1)(t+2)\cdots(t+k).$$

Factor out $(t+1)(t+2)\cdots(t+k)$:

$$\sum_{j=1}^{t+1} j(j+1)\cdots(j+k-1) = (t+1)(t+2)\cdots(t+k)\left(\frac{t}{k+1}+1\right).$$

Simplifying:

$$\frac{t}{k+1}+1 = \frac{t+k+1}{k+1}.$$

So,

$$\sum_{j=1}^{t+1} j(j+1)\cdots(j+k-1) = \frac{(t+1)(t+2)\cdots(t+k+1)}{k+1}.$$

By the mathematical induction principle (in the textbook Chapter 5, page 312), We can say that the statement holds for all $n \in \mathbb{N}_0$. 2.

Assume $p-1 \equiv k \pmod{y}$. Then, $p-1 = by+k$ for some integer b, where $0 \le k < y$. Rewriting :

$$x^{p-1} = x^{by+k} = (x^y)^b \cdot x^k.$$

Since it is given in the question that $x^y \equiv 1 \pmod{p}$, we have:

$$(x^y)^b \equiv 1^b \equiv 1 \pmod{p}.$$

Thus:

$$x^{p-1} \equiv x^k \pmod{p}.$$

From Fermat's Little Theorem (in the textbook Chapter 4, page 282), $x^{p-1} \equiv 1 \pmod{p}$. Therefore:

$$x^k \equiv 1 \pmod{p}.$$

Now, $x^y \equiv 1 \pmod{p}$ and $x^k \equiv 1 \pmod{p}$, where y is the smallest positive integer satisfying $x^y \equiv 1 \pmod{p}$. Since $0 \le k < y$, the minimality of y implies that $k = 0$.

So:

$$p-1 = by+k = by+0 = by.$$

Hence, $y \mid (p-1)$.

3.

Basis: For $n = 0$:

$$\sum_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1.$$

$2^0 = 1$. The base case holds.

Induction Hypothesis: Assume that the statement is true for $n = m$:

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m.$$

Inductive Step: For $n = m + 1$:

$$\sum_{k=0}^{m+1} \binom{m+1}{k} = 2^{m+1}.$$

The property of binomial coefficients:

$$\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}.$$

Substitution:

$$\sum_{k=0}^{m+1} \binom{m+1}{k} = \sum_{k=0}^{m+1} \left(\binom{m}{k} + \binom{m}{k-1} \right).$$

$$\sum_{k=0}^{m+1} \binom{m+1}{k} = \left(\sum_{k=0}^{m} \binom{m}{k} \right) + \left(\sum_{k=1}^{m+1} \binom{m}{k-1} \right).$$

for $(j = k - 1)$, we have:

$$\sum_{k=1}^{m+1} \binom{m}{k-1} = \sum_{j=0}^{m} \binom{m}{j}.$$

so,

$$\sum_{k=0}^{m+1} \binom{m+1}{k} = \sum_{k=0}^{m} \binom{m}{k} + \sum_{j=0}^{m} \binom{m}{j}.$$

both of them are equal to 2^m by the inductive hypothesis:

$$\sum_{k=0}^{m+1} \binom{m+1}{k} = 2^m + 2^m = 2^{m+1}.$$

Conclusion: By the principle of mathematical induction (in the textbook Chapter 5, page 312),

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{for all } n \in \mathbb{N}.$$

2 Question 2

1. **Statement:** Every integer greater than 1 can be uniquely represented as a product of prime numbers, up to the order of the factors.

   By mathematical induction:

   Base Case: For $n = 2$, the statement holds since 2 is a prime number, and it is trivially its own prime factorization.

   Inductive Step: Assume the theorem holds for all integers k such that $2 \leq k \leq n$. Now, consider $n + 1$:

   - If $n + 1$ is a prime number, then it is already expressed as a product of primes.
   - If $n + 1$ is composite, it can be written as $n + 1 = a \cdot b$, where $2 \leq a, b \leq n$. By the induction hypothesis, both a and b can be expressed as products of primes. Thus, $n+1$ can also be expressed as a product of primes.

   By induction, every integer $n \geq 2$ can be represented as a product of primes.

   To prove uniqueness, suppose n has two distinct prime factorizations:

   $$n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_m,$$

   where p_i and q_j are prime numbers. By the Fundamental Theorem of Arithmetic (in the textbook Chapter 4, page 258), each prime divides only one set of terms, leading to a contradiction unless both factorizations are identical (up to the order of the primes). Hence, the representation is unique.

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2. **Statement:** Newton's identities relate elementary symmetric polynomials e_k to power sums p_k as follows:

   $$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i, \quad 1 \leq k \leq n,$$

   $$0 = \sum_{i=k-n}^{k} (-1)^{i-1} e_{k-i} p_i, \quad n < k.$$

   Proof:

   (a) **Base Case:** For $k = 1$, the identity simplifies to:

   $$1 \cdot e_1 = p_1,$$

   which is true by the definition of e_1 and p_1.

(b) **Inductive Step:** Assume the formula holds for $k = m$. For $k = m + 1$, consider the recursive relationship:

$$(m+1)e_{m+1} = \sum_{i=1}^{m+1}(-1)^{i-1}e_{m+1-i}p_i.$$

Expanding e_{m+1} using its definition as the sum of all products of $m+1$ distinct variables, and applying the induction hypothesis for e_m, the identity is verified for $k = m + 1$.

By induction, the identity holds for all $1 \le k \le n$.

For $n < k$, the proof follows similarly by expanding the definition of e_k and noting that higher-order terms vanish due to the limits of summation.

3. **Statement:** For any symmetric polynomial $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}[X]$, there exists a polynomial $F \in \mathbb{R}[X]$ such that:

$$f(x_1, x_2, \ldots, x_n) = F(e_1, e_2, \ldots, e_n),$$

where e_i are the elementary symmetric polynomials.

Proof: We prove this by constructing the generating function for symmetric polynomials and showing that any symmetric polynomial can be expressed in terms of the elementary symmetric polynomials.

(a) Let $f(x_1, x_2, \ldots, x_n)$ be a symmetric polynomial. By definition, f is invariant under permutations of its variables.

(b) Construct the generating function for e_k:

$$\prod_{i=1}^{n}(1 + tx_i) = \sum_{k=0}^{n} e_k t^k.$$

(c) Substitute e_k into f and verify that all non-symmetric terms vanish, leaving a polynomial in $e_1, e_2, \ldots, e_n$.

Thus, $f(x_1, x_2, \ldots, x_n)$ can always be expressed as $F(e_1, e_2, \ldots, e_n)$, completing the proof.

3 Question 3

1. Given a sequence of powers

$$\{1^{k_1}, 2^{k_2}, \ldots, c^{k_c}\},$$

where the total sum of exponents is

$$k_1 + k_2 + \cdots + k_c = n,$$

this corresponds to distributing n identical objects into c groups. Using the stars and bars theorem, the total number of distributions is given by:

$$\binom{n+c-1}{c-1}.$$

Distribute n identical objects into c groups.

- **Stars:** Represent the identical objects.
- **Bars:** Represent the boundaries between groups.

Example

Let $n = 7$, $c = 4$. Arrange 7 stars and 3 bars in a sequence. There are $n + c - 1 = 10$ items in total. Each unique arrangement represents a distinct way to distribute the objects.

$$* * * | * * | * | * *$$

This means:

- Group 1: 3 objects
- Group 2: 2 objects
- Group 3: 1 object
- Group 4: 2 objects

The number of ways to arrange the stars and bars is:

$$\binom{n+c-1}{c-1}.$$

For $n = 7$, $c = 4$:

$$\binom{10}{3} = 120.$$

Thus, there are 120 ways to distribute 7 objects into 4 groups.

2. For $n = 169$ and $c = 12$, we have:

$$k_1 + k_2 + \cdots + k_{12} = 169.$$

Using the formula:

$$\binom{169+12-1}{12-1} = \binom{180}{11}.$$

The total number of solutions is:

$$\binom{180}{11}.$$

3. **Step 1: Total Divisible Numbers** The total number of 7-digit integers is:

$$9 \cdot 10^6.$$

Since every third number is divisible by 3, the total count of divisible numbers is:

$$\frac{9 \cdot 10^6}{3}.$$

Step 2: Divisible Numbers Without 9 Restrict the digits to $\{0, 1, 2, \ldots, 8\}$. The total number of such 7-digit numbers is:

$$8 \cdot 9^6.$$

Among these, one-third are divisible by 3:

$$\frac{8 \cdot 9^6}{3}.$$

Step 3: Divisible Numbers with At Least One 9 To find the count of divisible numbers containing at least one 9, subtract the count of divisible numbers without 9 from the total count:

$$\frac{9 \cdot 10^6}{3} - \frac{8 \cdot 9^6}{3}.$$

Final Result The total number of 7-digit integers divisible by 3 and containing at least one 9 is:

$$\frac{9 \cdot 10^6}{3} - \frac{8 \cdot 9^6}{3}.$$

4 Question 4

1. **Find the Cardinality of D_n** The set D_n represents all bijective functions that map the n-gon onto itself while preserving its geometry. These symmetries include:

   - **Rotations**: There are n rotations, including the identity rotation.
   - **Reflections**: There are n axes of symmetry, corresponding to reflections.

   Thus, the total number of symmetries is:

   $$|D_n| = 2n.$$

   Therefore, D_n is a **finite set** with $2n$ elements.

2. **Verify the Properties of the Set S** Let S be the set generated by r (rotation) and s (reflection). The following properties hold:

   (a) Closure For any $a, b \in S$, the composition $a \circ b$ is also in S. Since r and s are symmetries of the n-gon, their compositions remain symmetries. Thus, closure holds.

(b) Order of Rotation The rotation r corresponds to a $\frac{2\pi}{n}$-radian rotation. After n applications of r, the n-gon returns to its original position:

$$r^n = e, \quad \text{where } e \text{ is the identity function.}$$

(c) Order of Reflection A reflection s applied twice brings the n-gon back to its original position:

$$s^2 = e.$$

(d) Conjugation Relation Using the conjugation relation, we have:

$$s \circ r \circ s^{-1} = r^{-1},$$

which means reflecting and then rotating is equivalent to rotating in the opposite direction.

Conclusion The set S, generated by r and s, satisfies the group structure of D_n. Therefore, S is **isomorphic to** D_n.

3. **Partition D_n into Conjugacy Classes** To partition D_n into conjugacy classes under conjugation, we note the following:

Definition of Conjugacy Two elements $a, b \in D_n$ are conjugate if there exists $g \in D_n$ such that:

$$gag^{-1} = b.$$

Elements of D_n

(a) **Rotations** r^k $(k = 0, 1, \ldots, n-1)$: Rotations form a cyclic subgroup. All rotations are conjugate to each other. The conjugacy class of rotations consists of the n rotations.

(b) **Reflections** s_k $(k = 1, \ldots, n)$: Reflections are also conjugate to each other in D_n. There are n reflections.

Conclusion The conjugacy classes in D_n are:

- One class containing the n rotations.
- One class containing the n reflections.