Applications of Optimization to Color Correction

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Introduction

Color Perception and Constancy

- How a color is perceived depends on three variables dependent on the color wave length, λ.
  - The reflective character of an object, $S(\lambda)$.
  - The energy emitted by the ambient light source, $E(\lambda)$.
  - The sensitivity of each class of sensor to light, $Q_k(\lambda)$.
- Multiplying these variables and integrating over the spectrum gives the perceived color, ρ_k [1].

$$\rho_k = \int E(\lambda)S(\lambda)Q_k(\lambda)\,d\lambda$$

- Color constancy is the problem of ensuring that the perceived color of an object remains invariant to the illumination under which it is recorded.

Gray World Assumption

- Without knowing the reflective character or lighting of an object, it is impossible to ensure perfect color constancy.
- We use realistic assumptions about the image in order to compensate for this lack of knowledge.
- Our methods are based on the the gray world assumption.
    - We assume that all colors should be equally represented within an image.

Image Definition

As is standard, we consider a $U \times V$ pixel images in the RGB format

$$\mathcal{I} = \{R_{uv}, G_{uv}, B_{uv}\}$$

R_{uv}, G_{uv}, and B_{uv} represent the magnitude of a single pixel in one of the three color channels [3].

- Most image file formats take the color of a pixel to be an 8-bit integer having discrete values bound by [0,255], but we instead consider pixel values to be doubles in the interval [0,1].

Color Difference

Let

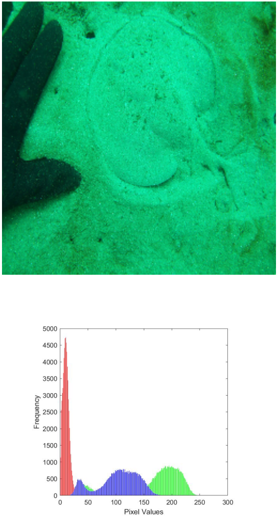
$$\overline{R} = \frac{1}{UV}\sum_{u,v} R_{uv} \qquad \overline{G} = \frac{1}{UV}\sum_{u,v} G_{uv}$$

$$\overline{B} = \frac{1}{UV}\sum_{u,v} B_{uv} \qquad \overline{M} = \frac{1}{3}(\overline{R} + \overline{G} + \overline{B})$$

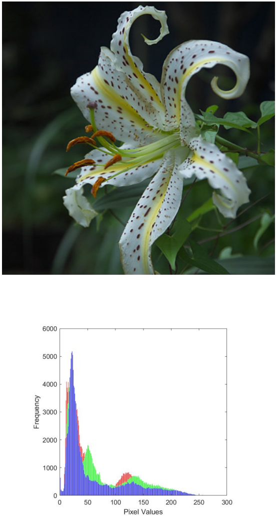
We ensure that the gray world assumption is satisfied in the color correction image by minimizing the color difference,

$$\Delta(\mathcal{I}) = \max\{|\overline{R} - \overline{M}|, |\overline{G} - \overline{M}|, |\overline{B} - \overline{M}|\}$$

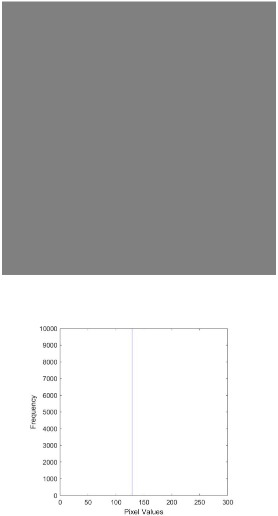
Color Difference Examples



(a) High Color Difference



(b) Low Color Difference



(c) Zero Color Difference

Linear Problem

Linear, Unconstrained Color Correction

The simplest manipulation of an image is a linear scaling of channel values. Given linear scaling factors α_r, α_g, and α_b, the corresponding transform is given by changing each pixel according to

$$\widetilde{R}_{uv} = \alpha_r R_{uv} \qquad \widetilde{G}_{uv} = \alpha_g G_{uv} \qquad \widetilde{B}_{uv} = \alpha_b B_{uv}$$

We use these definitions to define the linear, unconstrained problem

$$\text{minimize} \quad \Delta(\alpha_r, \alpha_g, \alpha_b) \tag{LU}$$

where

$$\Delta(\alpha_r, \alpha_g, \alpha_b) = \max\left\{\left|\overline{\widetilde{R}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{G}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{B}} - \overline{\widetilde{M}}\right|\right\}$$

- Has infinitely many solutions for which $\Delta(\alpha_r^*, \alpha_g^*, \alpha_b^*) = 0$

Linear, Constrained Color Correction

Find a unique solution by adding an intensity constraint based on intensity:

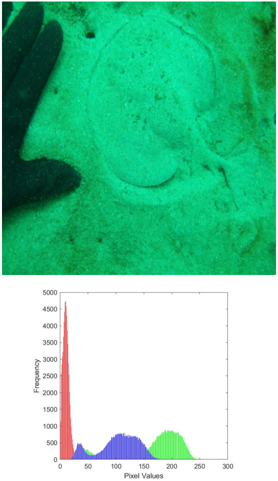
$$\begin{aligned} \text{minimize} \quad & \Delta(\alpha_r, \alpha_g, \alpha_b) \\ \text{subject to} \quad & 0.299\alpha_r\overline{R} + 0.587\alpha_g\overline{G} + 0.114\alpha_b\overline{B} = \overline{Y}_0. \end{aligned} \tag{LC}$$

This equation is solved uniquely by

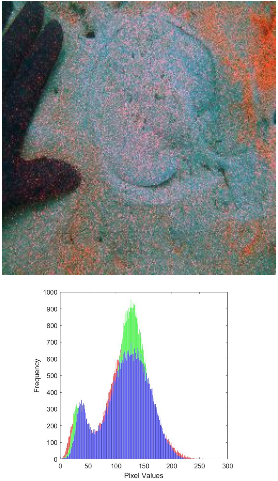
$$\alpha_r^* = \frac{\overline{Y}_0}{\overline{R}} \quad \alpha_g^* = \frac{\overline{Y}_0}{\overline{G}} \quad \alpha_b^* = \frac{\overline{Y}_0}{\overline{B}}$$

Note that this problem is convex. The objective function is the point-wise maximum of convex functions and the constraint is affine. However, there is no point in leveraging this convexity since the problem is so simple.

Corrected Image Quality



(a) Original



(b) Intensity Preservation

Nonlinear Problem

Improving the Color Correction Model

- Linear scaling can overexpose the brightest channel [2].

$$\widetilde{R}_{uv} = R_{uv}^{\gamma_r}, \quad \widetilde{G}_{uv} = G_{uv}^{\gamma_g}, \quad \widetilde{B}_{uv} = B_{uv}^{\gamma_b}.$$

- New constrained optimization problem:

$$\begin{aligned} \text{minimize} \quad & \Delta(\gamma_r, \gamma_g, \gamma_b) \\ \text{subject to} \quad & \overline{Y}(\gamma_r, \gamma_g, \gamma_b) = \overline{Y}_0. \end{aligned} \tag{NC}$$

where

$$\Delta(\gamma_r, \gamma_g, \gamma_b) = \max\left\{ \left|\overline{\widetilde{R}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{G}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{B}} - \overline{\widetilde{M}}\right| \right\}$$

$$\overline{Y}(\gamma_r, \gamma_g, \gamma_b) = \frac{1}{UV} \sum_{u,v} 0.299 R_{uv}^{\gamma_r} + 0.578 G_{uv}^{\gamma_g} + 0.114 B_{uv}^{\gamma_b}$$

Smooth Nonlinear Objective Function

$$\Delta(\gamma_r, \gamma_g, \gamma_b) = \max \left\{ \left| \overline{\widetilde{R}} - \overline{\widetilde{M}} \right|, \left| \overline{\widetilde{G}} - \overline{\widetilde{M}} \right|, \left| \overline{\widetilde{B}} - \overline{\widetilde{M}} \right| \right\},$$

$$\left| \overline{\widetilde{R}} - \overline{\widetilde{M}} \right| = \frac{1}{3UV} \sum_{uv} 2R_{uv}^{\gamma_r} - G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b}$$

- Objective function is neither convex nor smooth
- Minimums are zero, replace absolute values with squares
- Use a normalized smooth maximum function

$$\hat{\Delta}(\vec{\gamma}) = \ln \left(\exp \left(R^2 (\vec{\gamma}) \right) + \exp \left(G^2 (\vec{\gamma}) \right) + \exp \left(B^2 (\vec{\gamma}) \right) - 2 \right)$$

- Can directly compute gradient and Hessian relatively efficiently

Analytic Gradient and Hessian

$$\nabla\hat{\Delta}(\vec{\gamma}) = \frac{\begin{bmatrix} 2\sum_{u,v} R_{uv}^{\gamma_r}\ln(R_{uv}) & -\sum_{u,v} R_{uv}^{\gamma_r}\ln(R_{uv}) & -\sum_{u,v} R_{uv}^{\gamma_r}\ln(R_{uv}) \\ -\sum_{u,v} G_{uv}^{\gamma_g}\ln(G_{uv}) & 2\sum_{u,v} G_{uv}^{\gamma_g}\ln(G_{uv}) & -\sum_{u,v} G_{uv}^{\gamma_g}\ln(G_{uv}) \\ -\sum_{u,v} B_{uv}^{\gamma_b}\ln(B_{uv}) & -\sum_{u,v} B_{uv}^{\gamma_b}\ln(B_{uv}) & 2\sum_{u,v} B_{uv}^{\gamma_b}\ln(B_{uv}) \end{bmatrix}\begin{bmatrix} 2R(\vec{\gamma})\exp(R^2(\vec{\gamma})) \\ 2G(\vec{\gamma})\exp(G^2(\vec{\gamma})) \\ 2B(\vec{\gamma})\exp(B^2(\vec{\gamma})) \end{bmatrix}}{3UV(\exp(R^2(\vec{\gamma})) + \exp(G^2(\vec{\gamma})) + \exp(B^2(\vec{\gamma})) - 2)}$$

$$\begin{aligned}\frac{\partial^2\hat{\Delta}}{\partial\gamma_k\partial\gamma_k} &= \frac{-(2RR_{\gamma_k}e^{R^2} + 2GG_{\gamma_k}e^{G^2} + 2BB_{\gamma_k}e^{B^2})^2}{(e^{R^2} + e^{G^2} + e^{B^2} - 2)^2} \\ &+ \frac{2e^{R^2}(R_{\gamma_k}^2 + RR_{\gamma_k\gamma_k} + 2R^2R_{\gamma_k}^2) + 2e^{G^2}(G_{\gamma_k}^2 + GG_{\gamma_k\gamma_k} + 2G^2G_{\gamma_k}^2) + 2e^{B^2}(B_{\gamma_k}^2 + BB_{\gamma_k\gamma_k} + 2B^2B_{\gamma_k}^2)}{e^{R^2} + e^{G^2} + e^{B^2} - 2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\hat{\Delta}}{\partial\gamma_k\partial\gamma_j} &= \frac{-(2RR_{\gamma_k}e^{R^2} + 2GG_{\gamma_k}e^{G^2} + 2BB_{\gamma_k}e^{G^2})(2RR_{\gamma_j}e^{R^2} + 2GG_{\gamma_j}e^{G^2} + 2BB_{\gamma_j}e^{G^2})}{(e^{R^2} + e^{G^2} + e^{B^2} - 2)^2} \\ &+ \frac{2(R_{\gamma_k}R_{\gamma_j}e^{R^2}(1 + 2R^2) + G_{\gamma_k}G_{\gamma_j}e^{G^2}(1 + 2G^2) + B_{\gamma_k}B_{\gamma_j}e^{B^2}(1 + 2B^2))}{e^{R^2} + e^{G^2} + e^{B^2} - 2}.\end{aligned}$$

Uniqueness

"Uniqueness" for Unconstrained Problem

$$\text{minimize} \quad \Delta(\vec{\gamma}) = \max\left\{ \left|\overline{\widetilde{R}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{G}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{B}} - \overline{\widetilde{M}}\right| \right\}.$$

- Trivial minimums at $\vec{\gamma} = 0$ means optimal value is zero.
- Fixing one scaling constant ($\gamma_r = x_r$) gives a unique minimum over the other two given by solutions to

$$\sum_{u,v} G_{uv}^{\gamma_g{}^*} = \sum_{u,v} R_{uv}^{x_r}, \qquad \sum_{u,v} B_{uv}^{\gamma_b{}^*} = \sum_{u,v} R_{uv}^{x_r}$$

$$R(x_r, \gamma_g{}^*, \gamma_b{}^*) = \sum_{u,v} 2R_{uv}^{x_r} - G_{uv}^{\gamma_g{}^*} - B_{uv}^{\gamma_b{}^*} = \sum_{u,v} 2R_{uv}^{x_r} - R_{uv}^{x_r} - R_{uv}^{x_r} = 0$$

Uniqueness for Reduced Problem

$$\sum_{u,v} G_{uv}^{\gamma_g} = \sum_{u,v} R_{uv}^{x_r}, \qquad \sum_{u,v} B_{uv}^{\gamma_b} = \sum_{u,v} R_{uv}^{x_r}$$

- With mild assumptions on input image, above equations have unique solutions ${\gamma_g}^*$, ${\gamma_b}^*$ for arbitrary x_r.
- With pixel values in $(0, 1)$,
  - Left-hand side is strictly monotonically decreasing from UV to 0.
  - $\sum_{u,v} R_{uv}^{x_r}$ is constant between UV and 0.
  - Solution guaranteed by intermediate value theorem.

Uniqueness for Constrained Problem

- For any fixed value of x_r, there exists a minimizer such that

$$\Delta(x_r, {\gamma_g}^*, {\gamma_b}^*) = 0.$$

- Recover unique solution by enforcing nonlinear equality constraint

$$\overline{Y}(\vec{\gamma}) = \overline{Y}_0.$$

- At optimal solutions, mean intensity is a function of x_r given by

$$\overline{Y}(x_r) = \frac{1}{UV} \sum_{u,v} R_{uv}^{x_r}$$

- $\overline{Y}(\gamma_r) = \overline{Y}_0$ has a unique solution ${\gamma_r}^*$, which defines unique minimizer $({\gamma_r}^*, {\gamma_g}^*, {\gamma_b}^*)$.

Optimization Methods

Newton's Method with Quadratic Penalty

- Solve penalized problem with Newton's method at each step

$$\vec{\gamma}_{k+1} = \text{argmin } \hat{\Delta}(\vec{\gamma}) + \frac{\mu_k}{2}\left(\overline{Y}(\vec{\gamma}) - \overline{Y}_0\right)^2,$$

- Newton's method is well suited for this problem: smooth and low dimensional, has "cheap" analytic Hessians, no finite spurious stationary points
  - $\hat{\Delta}(\vec{\gamma}) \geq 0$ and $\hat{\Delta}(\vec{\gamma}^*) = 0$ means subproblem with any $\mu_k > 0$ will reach unique minimum $\vec{\gamma}^*$.
  - Poor initial condition with small μ can cause solution to drift to stationary points at infinity. Select initial condition close to zero.

Sequential Quadratic Programming

- We can show Strong Duality holds
- Strong Duality means KKT conditions hold at optimal Lagrangian saddle point $(\vec{\gamma}^*, \nu^* = 0)$.
- Can use SQP as root finding method to solve

$$\begin{bmatrix} \vec{\gamma}_{k+1} \\ \nu_{k+1} \end{bmatrix} = \begin{bmatrix} \vec{\gamma}_k \\ \nu_k \end{bmatrix} - \begin{bmatrix} \nabla^2 \hat{\Delta}(\vec{\gamma}_k) + \nu_k \nabla^2 \overline{Y}(\vec{\gamma}_k) & \nabla \overline{Y}(\vec{\gamma}_k) \\ \nabla \overline{Y}(\vec{\gamma}_k)^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nabla \hat{\Delta}(\vec{\gamma}_k) + \nu_k \nabla \overline{Y}(\vec{\gamma}_k) \\ \overline{Y}(\vec{\gamma}_k) - \overline{Y}_0 \end{bmatrix}$$

  - Can be solved without dependence on external parameter like μ.
  - Still sensitive to initial condition $\vec{\gamma}_0$, but $\nu_0 = 0$.

The FIPCO (Fast Intensity Preserving Color Optimization) Algorithm

From our exploration of uniqueness of a solution under one constrained gamma we had the following result:

$$\sum_{u,v} G_{uv}^{\gamma_g} = \sum_{u,v} R_{uv}^{x_r}, \qquad \sum_{u,v} B_{uv}^{\gamma_b} = \sum_{u,v} R_{uv}^{x_r}$$

Just a non-linear equation and can be solved to give us a perfect color balance regardless of the choice of x_r

Intensity constraint can be written as a function of just one of the color channels:

$$\overline{Y}(x_r) = \overline{Y}_0$$

Step 1: Find x_r to match the constraint

Step 2: Solve for γ_g, γ_b

Numerical Results

San Diego

Optimization Method	Runtime (s)	Iterations	$O(UV)$ Operations
`fmincon` (Interior Point)	2.154	33	396
Penalized Newton	0.304	8	96
SQP	0.207	6	72
FIPCO	0.125	$6 + 6$	37

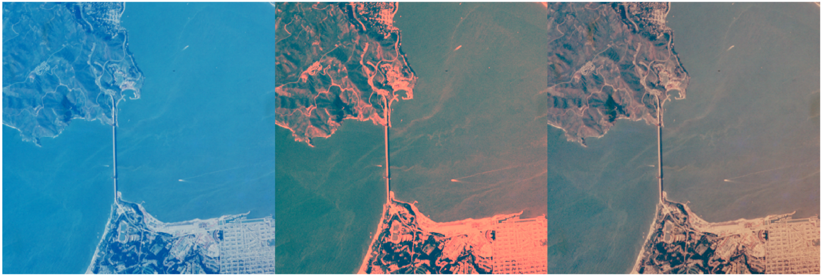


Figure: Initial Color Difference: 1.523×10^{-1}. Left: Initial image. Center: Linearly scaled, intensity preserved image. Right: Non-linearly scaled, intensity

Restaurant Food

Optimization Method	Runtime (s)	Iterations	$O(UV)$ Operations
`fmincon` (Interior Point)	2.819	45	540
Penalized Newton	0.305	9	108
SQP	0.238	7	84
FIPCO	0.141	$7+6$	39

Figure: Initial Color Difference: 9.245×10^{-2}. Left: Initial image. Center: Linearly scaled, intensity preserved image. Right: Non-linearly scaled, intensity

Lily

Optimization Method	Runtime (s)	Iterations	$O(UV)$ Operations
`fmincon` (Interior Point)	1.597	23	276
Penalized Newton	0.340	10	120
SQP	0.236	7	84
FIPCO	0.165	$7+7$	43

Figure: Initial Color Difference: 1.611×10^{-3}. Left: Initial image. Center: Linearly scaled, intensity preserved image. Right: Non-linearly scaled, intensity

The Bibliography

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