University of Colorado - Boulder

APPM 5630 - Convex Optimization Final Project

Image Optimization

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Abstract

Ensuring proper and consistent lighting of images can be challenging, especially when the color and brightness of the light under which an image is taken is unknown. This can cause images of an object to imperfectly represent its fundamental characteristics, which is a significant hurdle to many imaging applications, e.g. training vision guided autonomous systems reliably. Ensuring color constancy through algebraic manipulation of pixel values has become a popular solution to this issue. This paper approaches color constancy through enforcing the gray world balancing assumptions alongside a constraint of the intensity of the resulting image. This is accomplished through the traditional optimization techniques of Newton's method with a quadratic penalty term and sequential quadratic programming. Additionally, a novel iterative approach to an underlying system of equations is considered. These methods are compared, and demonstrated on several examples.

1 Introduction

The "color" of an object is, in a sense, only meaningful when recorded by a particular light sensor, be it biological or mechanical. The most common physical model of color has it defined by the interaction of several physical components across each wavelength of light. These components are the reflective character of an object $S(\lambda)$, the energy emitted by the ambient light source $E(\lambda)$, and the sensitivity of each class of sensor to light $Q_k(\lambda)$. [1] These classes of sensors typically reflect a trichromatic standard, having red, blue, and green channels for color. These components interact to produce ρ_k, the color response of an imaging device exposed to a given surface, given by

$$\rho_k = \int E(\lambda)S(\lambda)Q_k(\lambda)\,d\lambda$$

Color constancy is the problem of ensuring that the perceived color of an object remains invariant to the illumination under which it is recorded. Yet as we can see from the above equation, the influence of the illuminant on the color response cannot be easily extracted. This means that color constancy can be very difficult or even impossible to guarantee in a set of images. For this reason, simple color correction algorithms are often embedded into autonomous systems. Many approaches, such as the one employed here, identify certain color-invariant properties of the image, and ensuring constancy through algebraic manipulation of these properties. Our algorithms operate on the gray world assumption, which assumes that each color in an image should be represented fairly equally. We enforce this by manipulating each of the three primary color channels in an image, the red, blue, and green channels, to meet this assumption.

2 Gray World Assumption and Intensity Model

As outlined in [3], we consider $U \times V$ pixel images in the RGB format,

$$\mathcal{I} = \{R_{uv}, G_{uv}, B_{uv}\},$$

where R_{uv}, G_{uv}, and B_{uv} are the magnitude of a single pixel in one of three color channels. While most image file formats take the color of a pixel to be an 8-bit integer having discrete values bound by $[0, 255]$, but we instead consider pixel values to be doubles in the interval $[0, 1]$. This allows for simpler algebraic manipulation of pixel values, as well as the use of calculus operations that would otherwise require continuity. While converting back to a standard image format necessarily involves rounding pixels back to 8-bit integers, this has an imperceptible difference to the human eye.

To numerically assess the degree to which an image satisfies the gray world assumption, we measure the maximum difference between the average of each color channel and total average of the three channels. This is represented by the color difference criterion, often referred to simply as the color difference of an image:

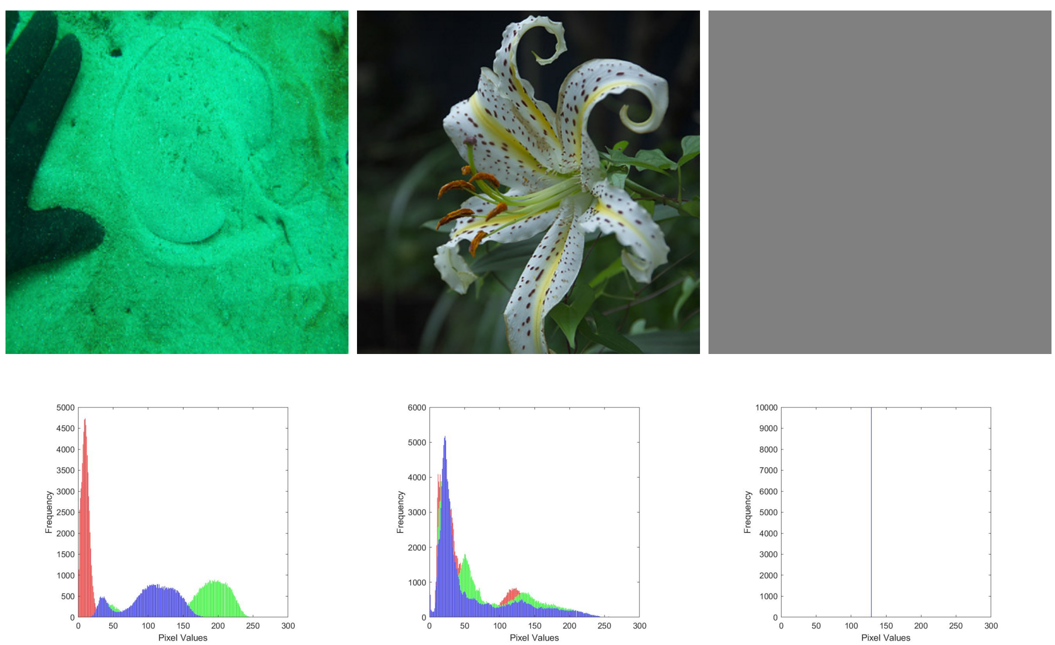
$$\Delta(\mathcal{I}) = \max\{|\overline{R} - \overline{M}|, |\overline{G} - \overline{M}|, |\overline{B} - \overline{M}|\}$$

where

$$\overline{R} = \frac{1}{UV}\sum_{u,v} R_{uv}, \quad \overline{G} = \frac{1}{UV}\sum_{u,v} G_{uv}, \quad \overline{B} = \frac{1}{UV}\sum_{u,v} B_{uv}$$

are the average pixel values of each color channel, and $\overline{M} = \frac{1}{3}(\overline{R} + \overline{G} + \overline{B})$ is the aggregated mean pixel value over all three channels. Importantly, this measure does not in depend on the physical layout of an image. This has important benefits for computation, as channels can be considered a "bag" of pixels.

While images that satisfy the gray world assumption have a close to zero color difference, images with low color difference are not necessarily ideal in a practical sense. For example, any algorithm that sets the image to be a single color will necessarily set its color difference to zero, despite the image becoming meaningless.



(a) High Color Difference (b) Low Color Difference (c) Zero Color Difference

To compensate for this, we introduce the mean intensity of the image as another important characteristic. The intensity channel of an image, Y, is defined as the weighted average of its red, green, and blue color channels.

$$Y = 0.299R + 0.587G + 0.114B$$

We are often concerned with the mean of the intensity channel, denoted $\overline{Y}$, which is roughly equivalent to the "brightness" of an image. We would like our algorithm to produce an image with a low color difference, while also preserving its original intensity.

3 Linear Color Correction

The simplest manipulation of an image is a linear scaling of channel values. Given linear scaling factors α_r, α_g, and α_b, the corresponding transform is given by changing each pixel according to

$$\widetilde{R}_{uv} = \alpha_r R_{uv}, \quad \widetilde{G}_{uv} = \alpha_g G_{uv}, \quad \widetilde{B}_{uv} = \alpha_b B_{uv}$$

A standard approach to color correction, taken for now as arbitrary, computes α_r and α_b as functions of the mean of each channel according to

$$\alpha_r = \frac{\overline{G}}{\overline{R}}, \quad \alpha_b = \frac{\overline{G}}{\overline{B}},$$

keeping $\alpha_g = 1$ fixed. This simple approach dramatically lowers the color difference, is incredibly fast to compute, and is quite easy of implementation within image capturing devices. Because the green channel typically contributes the most to an image's intensity, this choice reflects an effort to somewhat match the intensity of the original image.

A more sophisticated approach to the linear color correction problem involves using the difference criterion as an objective function to minimize. Written as a function in terms of these linear scaling factors, the unconstrained minimization problem is

$$\text{minimize} \quad \Delta(\alpha_r, \alpha_g, \alpha_b) \tag{LU}$$

where

$$\Delta(\alpha_r, \alpha_g, \alpha_b) = \max\left\{\left|\overline{\widetilde{R}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{G}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{B}} - \overline{\widetilde{M}}\right|\right\}$$

This construction makes the implicit assumption that the individual scaling factors are nonnegative. Using a linear transformation of each channel means that applying the scaling factor to each pixel is equivalent to applying it to the average. For example, the average of the scaled red channel is

$$\overline{\widetilde{R}} = \frac{1}{UV}\sum_{u,v} \alpha_r R_{uv} = \frac{\alpha_r}{UV}\sum_{u,v} R_{uv} = \alpha_r \overline{R}.$$

This means that the correction procedure only needs to operate on three pieces of data, the average of each channel. Furthermore, this means that the objective function can be simplified as

$$\begin{aligned}\Delta(\alpha_r, \alpha_g, \alpha_b) &= \max\left\{\left|\overline{\widetilde{R}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{G}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{B}} - \overline{\widetilde{M}}\right|\right\} \\ &= \frac{1}{3}\max\left\{\left|2\alpha_r\overline{R} - \alpha_g\overline{G} - \alpha_b\overline{B}\right|, \left|-\alpha_r\overline{R} + 2\alpha_g\overline{G} - \alpha_b\overline{B}\right|, \left|-\alpha_r\overline{R} - \alpha_g\overline{G} + 2\alpha_b\overline{B}\right|\right\}\end{aligned}$$

Written as such, we can see that this is in fact a convex optimization problem in three dimensions. Each of the three terms in the maximum is the composition of the convex absolute value function and an affine map, and the pointwise maximum of convex functions is convex. What also becomes apparent, however, is that minimizers of (LU) is not unique. For any vector of linear scaling factors $\vec{\alpha} = [\alpha_r, \alpha_g, \alpha_b]^T$, $\Delta(c\vec{\alpha}) = c\Delta(\vec{\alpha})$ for any nonnegative c. In particular, this holds for $c = 0$, meaning that the optimal value of the above optimization problem is necessarily zero. This can happen only if each argument to the maximum is identically zero, which we can represent with the following linear system of equations:

$$\begin{bmatrix} 2\overline{R} & -\overline{G} & -\overline{B} \\ -\overline{R} & 2\overline{G} & -\overline{B} \\ -\overline{R} & -\overline{G} & 2\overline{B} \end{bmatrix} \begin{bmatrix} \alpha_r \\ \alpha_g \\ \alpha_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix in this system is singular, and has a one dimensional null space composed of solutions to (LU). This solution space is spanned by $(1/\overline{R} \quad 1/\overline{G} \quad 1/\overline{B})^T$. In fact, the previously discussed standard approach is generated by scaling this vector by $\overline{G}$. This means that although minimizing the color difference is framed here as an optimization problem, any iterative algorithm to solve the problem would be necessarily slower than simply selecting a point along this line of solutions.

A unique solution to the color constancy problem can be recovered by placing an additional constraint on the optimization problem. The most natural constraint is on the mean intensity of the resulting image. We can define the mean intensity $\overline{Y}$ as a function of the linear scaling factors such that

$$\overline{Y}(\alpha_r, \alpha_g, \alpha_b) = 0.299\alpha_r\overline{R} + 0.587\alpha_g\overline{G} + 0.114\alpha_b\overline{B}$$

and $\overline{Y}_0 = \overline{Y}(1, 1, 1)$ is the original mean intensity of the image. Written as a constrained optimization problem, we now have

$$\begin{aligned} \text{minimize} \quad & \Delta(\alpha_r, \alpha_g, \alpha_b) \\ \text{subject to} \quad & \overline{Y}(\alpha_r, \alpha_g, \alpha_b) = \overline{Y}_0. \end{aligned} \tag{LC}$$

Because this equality constraint is a linear function of α_r, α_g, and α_b, the constraint is affine and the minimization problem remains convex. But again, there is a direct solution to this problem that can be found much faster than one supplied by optimization techniques. The set of points $(\alpha_r, \alpha_g, \alpha_b)$ that satisfy this equality constraint form a hyperplane, and we now know that the set of points that minimize $\Delta(\vec{\alpha})$ form a line. This means that finding the unique minimizer to the constrained optimization problem is equivalent to finding the intersection of this line and hyperplane, given directly by

$$\alpha_r = \frac{\overline{Y}_0}{\overline{R}} \quad \alpha_g = \frac{\overline{Y}_0}{\overline{G}} \quad \alpha_b = \frac{\overline{Y}_0}{\overline{B}}$$

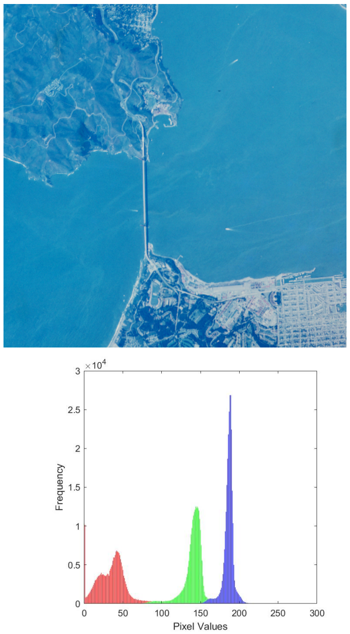
Despite being exceptionally simple to compute, this procedure for generating optimal scaling factors while preserving image intensity is, to our knowledge, both novel and superior to existing methods, while having roughly the same speed.

However, it must be noted that in the preceding analysis of this approach, we have made the somewhat problematic assumption that pixel values can extend beyond their maximum of 1 in a double representation. This can have important consequences for the convexity of the problem. As an example, the true transformation of the red channel induced by a linear scaling factor is given by

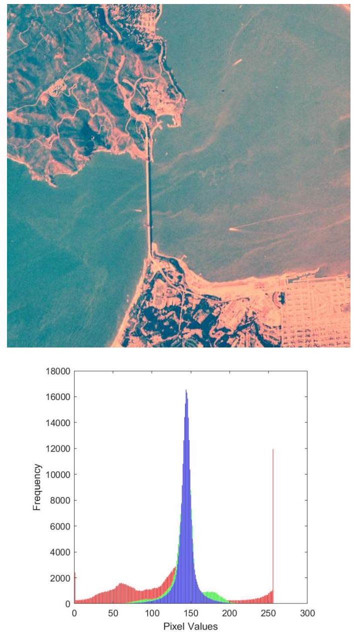
$$\widetilde{R}_{uv} = \min\{\alpha_r R_{uv},\, 1\},$$

which is no longer a linear function of α_r.

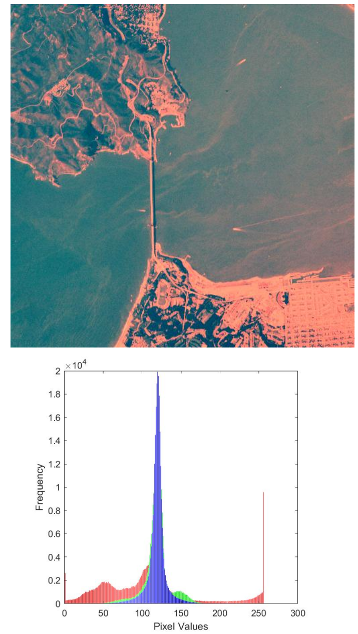
Additionally, scaling factors greater than 1 can cause the brightest pixels in an image to oversaturate, with multiple distinct pixel values each being assigned to their maximum value. This can make the resulting image highly undesirable, both visually and with respect to the above optimization problem. For example, if the "optimal" point described by the above method causes certain points to reach past their maximum value, then the intensity will necessarily be less than its original value. While color correcting in this manner is remarkably fast, and imperfections in the color difference and intensity are typically "good enough" to a human observer, other color correcting techniques bypass these issues entirely.



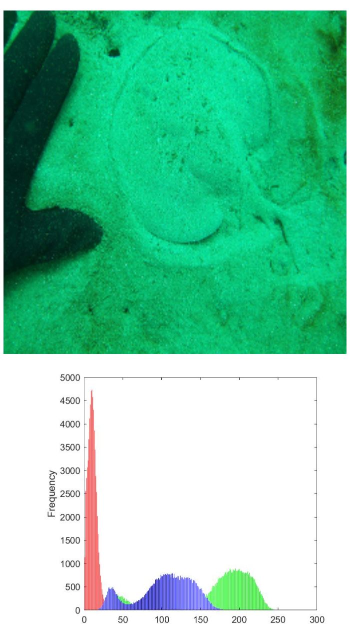
(a) Original



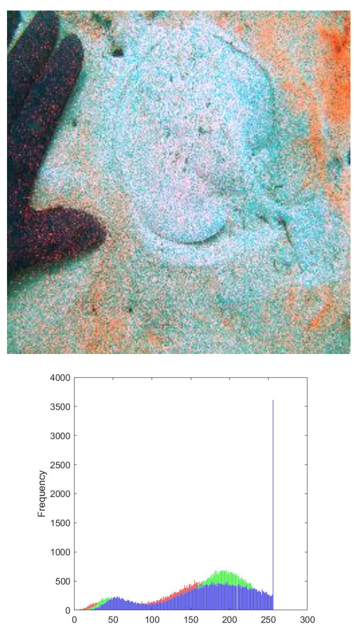
(b) Linear Scaling



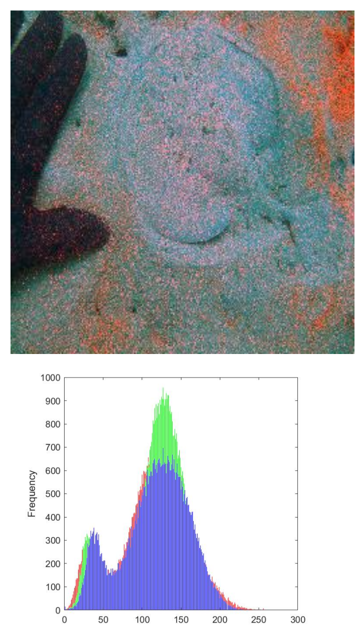
(c) Intensity Preservation



(a) Original



(b) Linear Scaling



(c) Intensity Preservation

4 Nonlinear Color Correction

Nonlinear scaling factors represent a more sophisticated image transformation technique. Typically, a gamma correction is applied to each color channel, where the scaling factors are exponents to pixel values:

$$\widetilde{R}_{uv} = R_{uv}^{\gamma_r}, \quad \widetilde{G}_{uv} = G_{uv}^{\gamma_g}, \quad \widetilde{B}_{uv} = B_{uv}^{\gamma_b}.$$

This typically causes brighter regions in an image to remain bright, and dimmer regions to remain dim. [2] Mathematically, there are significant benefits to the use of nonlinear scaling factors. When pixel values R_{uv} are normalized to lie in the interval $[0, 1]$, the gamma corrected pixel value $\widetilde{R}$ will lie in the same interval for nonnegative gamma values. This means pixels cannot "oversaturate" an image by extending beyond the upper limit. Such a nonlinear problem will be necessarily more difficult to solve, meaning that applications that demand speed such as correcting each frame in a video may be better off using linear scaling. However, the algorithms we describe herein are still quite efficient for large images, and represent a more careful approach to color correction.

4.1 Problem Statement

The unconstrained, nonlinear optimization problem seeks to minimize the color difference given a set of three gamma correction constants, and is represented by

$$\text{minimize} \quad \Delta(\gamma_r, \gamma_g, \gamma_b) \tag{NU}$$

where as before,

$$\Delta(\gamma_r, \gamma_g, \gamma_b) = \max\left\{ \left|\overline{\widetilde{R}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{G}} - \overline{\widetilde{M}}\right|, \left|\overline{\widetilde{B}} - \overline{\widetilde{M}}\right| \right\}$$

Unique to the nonlinear scaling case, we can no longer represent the mean of a transformed channel in terms of its original mean. This means we can represent $\overline{\widetilde{R}}$, $\overline{\widetilde{G}}$ and $\overline{\widetilde{B}}$ only as

$$\overline{\widetilde{R}} = \frac{1}{UV} \sum_{u,v} R_{uv}^{\gamma_r}, \quad \overline{\widetilde{G}} = \frac{1}{UV} \sum_{u,v} G_{uv}^{\gamma_g}, \quad \overline{\widetilde{B}} = \frac{1}{UV} \sum_{u,v} B_{uv}^{\gamma_b}$$

respectively. With these definitions, we can further simplify and rewrite the color difference function:

$$\Delta(\gamma_r, \gamma_g, \gamma_b) = \frac{1}{3UV} \max\left\{ \left|\sum_{u,v} 2R_{uv}^{\gamma_r} - G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b}\right|, \left|\sum_{u,v} -R_{uv}^{\gamma_r} + 2G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b}\right|, \left|\sum_{u,v} -R_{uv}^{\gamma_r} - G_{uv}^{\gamma_g} + 2B_{uv}^{\gamma_b}\right| \right\}$$

However, there are immediate issues with uniqueness and convexity. Just as in the linear case, setting each of the gamma values to zero will completely minimize the color difference, as the image becomes entirely white. Furthermore, while there is no straight line of solutions, taking each gamma coefficient to infinity will also cause the color difference to become zero as the image becomes entirely black. This shows that the function $\Delta\vec{\gamma}$ cannot be convex on any ray emanating from the origin, as it is/approaches zero at either end, and is necessarily nonzero at some point between.

In an attempt to recover a unique solution, we add an equality constraint to the optimization problem based on the mean intensity of the resultant image. The constrained, nonlinear optimization problem then becomes

$$\begin{aligned} \text{minimize} \quad & \Delta(\gamma_r, \gamma_g, \gamma_b) \\ \text{subject to} \quad & \overline{Y}(\gamma_r, \gamma_g, \gamma_b) = \overline{Y}_0. \end{aligned} \tag{NC}$$

where the mean intensity $\overline{Y}$ is defined in terms of the nonlinear scaling factors,

$$\overline{Y}(\gamma_r, \gamma_g, \gamma_b) = \frac{1}{UV} \sum_{u,v} 0.299 R_{uv}^{\gamma_r} + 0.578 G_{uv}^{\gamma_g} + 0.114 B_{uv}^{\gamma_b}$$

and $\overline{Y}_0 = \overline{Y}(1,1,1)$. For pixel values between 0 and 1, this constraint equation is convex, but not affine. As a brief proof of this fact, consider that $\overline{Y}$ is essentially a nonnegative weighted sum of convex functions $f(x) = C^x$ where $C \in [0,1]$. This means that the constrained optimization problem is best characterized as a generalized non-linear program, and our solution techniques reflect this.

To produce an equivalent differentiable objective function, we handle each point of non-differentiability, the max and absolute value functions, individually. To simplify notation, we first define functions $R, G, B : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$\begin{aligned} R(\vec{\gamma}) &= \frac{1}{3UV} \sum_{u,v} 2R_{uv}^{\gamma_r} - G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b} \\ G(\vec{\gamma}) &= \frac{1}{3UV} \sum_{u,v} -R_{uv}^{\gamma_r} + 2G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b} \\ B(\vec{\gamma}) &= \frac{1}{3UV} \sum_{u,v} -R_{uv}^{\gamma_r} - G_{uv}^{\gamma_g} + 2B_{uv}^{\gamma_b} \end{aligned}$$

so that our objective function $\Delta(\vec{\gamma})$ can be rewritten as

$$\Delta(\vec{\gamma}) = \max\left\{|R(\vec{\gamma})|, |G(\vec{\gamma})|, |B(\vec{\gamma})|\right\}.$$

Because the value of the objective function is largely irrelevant to the overall quality of the image beyond reaching its minimum, we can safely replace each absolute value with a square. As we will see in the following section, the color difference reaches its minimum when each function, square or absolute value, is equal to zero, making the problems equivalent. We can use a similar argument to replace the maximum with a differentiable, smooth maximum function. Because each term in the maximum is positive, we use the normalized LogSumExp function, given by

$$\max\{x_1, x_2, x_3\} \approx \mathrm{LSE}(x_1, x_2, x_3) = \ln(\exp(x_1) + \exp(x_2) + \exp(x_3) - 2).$$

This additional -2 term ensures that $\mathrm{LSE}(0,0,0) = \ln(1) = 0$. Combining these two expressions, we obtain a smooth approximation of the color difference criterion, given by

$$\hat{\Delta}(\vec{\gamma}) = \ln\left(\exp\left(R^2(\vec{\gamma})\right) + \exp\left(G^2(\vec{\gamma})\right) + \exp\left(B^2(\vec{\gamma})\right) - 2\right)$$

The intensity constraint is already sufficiently smooth, so these techniques need not be applied to it. While this problem is still not convex, constructing a smooth approximation of the color difference criterion allows us to compute both its gradient and Hessian. These calculations are critically important for many of the optimization techniques discussed in the following sections. Analytic forms of both the gradient and Hessian can be found in the appendix.

Analogous to the linear case, we can introduce an equality constraint defined by the mean intensity of the scaled image. The problem then becomes

$$\begin{aligned} \text{minimize} \quad & \hat{\Delta}(\gamma_r, \gamma_g, \gamma_b) \\ \text{subject to} \quad & \overline{Y}(\gamma_r, \gamma_g, \gamma_b) = \overline{Y}_0, \end{aligned} \tag{SNC}$$

where $\hat{\Delta}(\vec{\gamma})$ and $\overline{Y}(\vec{\gamma})$ are defined as above.

4.2 Existence and Uniqueness

In the following analysis, we make the assumption that each color channel of the original image has no pixels with values exactly equal to 0 or exactly equal to 1. While a somewhat unrealistic condition, the existence of such pixels interferes with the conclusions drawn in the most adversarial of test cases. Moreover, these potential issues can be bypassed entirely through the use of subpixel adjustments to zero or unit pixels, possible using the double representation of pixel values. For small enough adjustments, any scaled image is visually indistinguishable for all except unrealistically large values of $\vec{\gamma}$ once the image is rounded back to integer values for display.

With this assumption in mind, we proceed to establish the existence and uniqueness of nontrivial solutions to (NC). We first consider its minimizers when a single scaling factor fixed.

We prove that when only a single variable is fixed, say γ_r, the unconstrained minimization problem

$$\text{minimize} \quad \Delta(x_r, \gamma_g, \gamma_b) \tag{FNU}$$

has a unique minimizer, and the optimal value is equal to zero. The choice of fixing the red channel is entirely arbitrary, and could be taken to be any of the three channels, as the unconstrained optimization problem is symmetric with respect to its variables. To uniqueness in this case, consider the objective function

$$\Delta(x_r, \gamma_g, \gamma_b) = \frac{1}{3UV} \max\left\{ \left|\sum_{u,v} 2R_{uv}^{x_r} - G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b}\right|, \left|\sum_{u,v} -R_{uv}^{x_r} + 2G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b}\right|, \left|\sum_{u,v} -R_{uv}^{x_r} - G_{uv}^{\gamma_g} + 2B_{uv}^{\gamma_b}\right| \right\}.$$

Suppose there exist scaling factors ${\gamma_g}^*$ and ${\gamma_b}^*$ defined as solutions to the nonlinear equations

$$\sum_{u,v} G_{uv}^{{\gamma_g}^*} = \sum_{u,v} R_{uv}^{x_r}, \qquad \sum_{u,v} B_{uv}^{{\gamma_b}^*} = \sum_{u,v} R_{uv}^{x_r}$$

We consider the conditions under which such solutions exist momentarily. Plugging these equations into our objective function causes each difference to cancel. For the first difference

$$\sum_{u,v} 2R_{uv}^{x_r} - G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b} = \sum_{u,v} 2R_{uv}^{x_r} - R_{uv}^{x_r} - R_{uv}^{x_r} = 0.$$

The other two differences follow similarly. This means that when a solution to the above nonlinear equations exists, the optimal value of (FNU) is necessarily zero. This proves that when one variable is fixed, there is at least a single nontrivial solution. To prove the uniqueness of this solution, we consider that because $\Delta(\vec{\gamma}) \geq 0$ for all $\vec{\gamma}$, any minimizer of the unconstrained optimization problem must set each term in the maximum equal to zero as well. In other words, any minimizer $\vec{\gamma}^*$ must solve the following system of nonlinear equations, obtained by rewriting each difference in the maximum:

$$\begin{aligned} \sum_{u,v} G_{uv}^{\gamma_g} + \sum_{u,v} B_{uv}^{\gamma_b} &= 2\sum_{u,v} R_{uv}^{x_r} \\ 2\sum_{u,v} G_{uv}^{\gamma_g} - \sum_{u,v} B_{uv}^{\gamma_b} &= \sum_{u,v} R_{uv}^{x_r} \\ -\sum_{u,v} G_{uv}^{\gamma_g} + 2\sum_{u,v} B_{uv}^{\gamma_b} &= \sum_{u,v} R_{uv}^{x_r} \end{aligned}$$

where $\sum_{u,v} R_{uv}^{x_r}$ is constant. Adding the first two equations, we have

$$3\sum_{u,v} G_{uv}^{\gamma_g} + 0 = 3\sum_{u,v} R_{uv}^{x_r} \implies \sum_{u,v} G_{uv}^{\gamma_g} = \sum_{u,v} R_{uv}^{x_r}$$

and adding the first and third equations, we have

$$0 + 3\sum_{u,v} B_{uv}^{\gamma_b} = 3\sum_{u,v} R_{uv}^{x_r} \implies \sum_{u,v} B_{uv}^{\gamma_b} = \sum_{u,v} R_{uv}^{x_r}$$

which is exactly the pair of nonlinear equations from before. In order for the system of nonlinear equations to be true, these equations must hold. We now show that these equations have a unique solution for an arbitrary fixed value of x_r.

We consider first the equation for γ_g, noting that the argument for γ_b follows identically. Because each pixel value is in the interval $[0, 1]$, the left-hand side of the equation is strictly monotonically decreasing as a function of γ_g. When $\gamma_g = 0$, the left-hand side is equal to UV, as every pixel value (assumed to be nonzero) goes to 1. As $\gamma_g \to \infty$, the left-hand side approaches 0 as every pixel (assumed to not equal one) approaches zero. On the right-hand side of the equation is a constant, whose maximum value also occurs when x_r approaches zero, and is strictly less than UV. Otherwise, the constant value is strictly greater than zero. This means that by the intermediate value theorem, the equation has a solution ${\gamma_g}^*$ whenever the left-hand side is greater than the constant value at $\gamma_g = 0$ and is eventually less than the constant value as $\gamma_g \to \infty$, which is necessarily true for a fixed value of x_r. And because the left-hand side of the function is strictly monotonically decreasing, this solution is unique.

In conclusion, for any fixed scaling factor for any of the three channels, the other two that perfectly minimize (FNU), i.e. produce an objective function value of zero, are uniquely determined. However, this means that for the unconstrained minimization problem in all three variables, there are infinitely many solutions, one for each value of a single channel's nonlinear scaling factor.

In order to generate a unique meaningful solution, we now reconsider the constrained problem (NU) given by

$$\begin{aligned} \text{minimize} \quad & \Delta(\gamma_r, \gamma_g, \gamma_b) \\ \text{subject to} \quad & \overline{Y}(\gamma_r, \gamma_g, \gamma_b) = \overline{Y}_0. \end{aligned} \tag{NC}$$

As we have shown, the unconstrained version of this problem has infinitely many points for which $\Delta(\vec{\gamma}) = 0$. We now show that only a single of these points satisfies the equality constraint. From the previous discussion, we know when any single scaling factor is fixed, the other two that minimize $\Delta(\vec{\gamma})$ are uniquely determined. In the following example, we assume that the fixed scaling factor is γ_r, but an analogous demonstration carries over to any of the three channels. The mean intensity can then be written as a function of γ_r such that

$$\begin{aligned} \overline{Y}_r(\gamma_r) &= \frac{1}{UV}\sum_{u,v} 0.299 R_{uv}^{\gamma_r} + 0.587 G_{uv}^{{\gamma_g}^*} + 0.114 B_{uv}^{{\gamma_b}^*} \\ &= \frac{0.299}{UV}\sum_{u,v} R_{uv}^{\gamma_r} + \frac{0.587}{UV}\sum_{u,v} G_{uv}^{{\gamma_g}^*} + \frac{0.114}{UV}\sum_{u,v} B_{uv}^{{\gamma_b}^*}. \end{aligned}$$

By the construction of this optimal solution, the points ${\gamma_g}^*$ and ${\gamma_b}^*$ satisfy

$$\sum_{u,v} G_{uv}^{{\gamma_g}^*} = \sum_{u,v} R_{uv}^{\gamma_r}, \qquad \sum_{u,v} B_{uv}^{{\gamma_b}^*} = \sum_{u,v} R_{uv}^{\gamma_r},$$

which means that the mean intensity at the optimal point defined by γ_g can be written as

$$\begin{aligned} \overline{Y}_r(\gamma_r) &= \frac{0.299}{UV}\sum_{u,v} R_{uv}^{\gamma_r} + \frac{0.587}{UV}\sum_{u,v} R_{uv}^{\gamma_r} + \frac{0.114}{UV}\sum_{u,v} R_{uv}^{\gamma_r} \\ &= \frac{(0.299 + 0.587 + 0.114)}{UV}\sum_{u,v} R_{uv}^{\gamma_r} = \frac{1}{UV}\sum_{u,v} R_{uv}^{\gamma_r}. \end{aligned}$$

While this result seems unexpected, the intensity channel is by construction a convex combination of the three primary color channels. Additionally, any optimal point of the color difference criterion occurs when each color channel is equal to one another. Therefore when the color difference is zero, the mean intensity is simply the mean value of any of the three channels, in particular the one whose gamma coefficient we vary to minimize the objective function.

In this way, the question of whether or not a unique solution to the (NU) exists depends entirely on whether or not $\overline{Y}_r(\gamma_r) = \overline{Y}_0$ has a unique solution. If an optimal $\gamma_r{}^*$ is found as a root to the above equation, the optimal values γ_b and γ_g can be found as their own root finding problems. As before, $\overline{Y}_r(\gamma_r)$ is a continuous, strictly monotonically decreasing function of γ_r, reaching its maximum value of 1 when $\gamma_r = 0$ and approaching its infimum of 0 as $\gamma_r \to \infty$. Because the original mean intensity is simply the average of a convex combination of the three color channels, its value must be strictly between 0 and 1. Again by the intermediate value theorem on the strictly monotonically decreasing function of γ_r, there must be a unique $\gamma_r{}^*$ that satisfies $\overline{Y}_r(\gamma_r) = \overline{Y}_0$, which itself uniquely determines optimal points $\gamma_g{}^*$ and $\gamma_b{}^*$.

This analysis also reveals that this unique minimizer of the constrained optimization problem can in fact be found through a sequence of one dimensional root finding problems, a concept which is explored in more detail in section 5. However, with a unique solution guaranteed, we can continue to approach the problem from the perspective of optimization.

4.3 Proof of Strong Duality

In many of the following optimization procedures, it is necessary that strong duality holds for (SNC), which we can investigate directly. The Lagrangian for (SNC) is given by

$$\mathcal{L}(\vec{\gamma}, \nu) = \hat{\Delta}(\vec{\gamma}) + \nu \left(\overline{Y}(\vec{\gamma}) - \overline{Y}_0 \right)$$

and the associated dual function is given by

$$g(\nu) = \inf_{\vec{\gamma}} \hat{\Delta}(\vec{\gamma}) + \nu \left(\overline{Y}(\vec{\gamma}) - \overline{Y}_0 \right) = \inf_{\vec{\gamma}} \left(\hat{\Delta}(\vec{\gamma}) + \nu \overline{Y}(\vec{\gamma}) \right) - \nu \overline{Y}_0$$

where there is an implicit assumption that the components of $\vec{\gamma}$ are nonnegative. By construction, $\hat{\Delta}(\vec{\gamma}) \geq 0$, and is in fact approaches zero as $\vec{\gamma}$ approaches infinity from any direction under the conditions necessary for a unique solution to the primal problem. Additionally, $\hat{\Delta}(\vec{0}) = 0$. By contrast, $\overline{Y}(\vec{\gamma})$ is contained in the interval $(0, 1]$, being equal to 1 when $\vec{\gamma} = 0$ and approaching 0 as $\vec{\gamma} \to \infty$, again from any direction.

This means that the infimum within the dual function has one of two values, depending on the sign of ν. If $\nu \geq 0$, then the infimum is obtained as $\vec{\gamma} \to \infty$ in any direction, as both terms approach zero. However, if $\nu < 0$, then the infimum is obtained when $\vec{\gamma} = \vec{0}$, at which point the first term is zero and the second reaches its maximum of 1. Because ν is assumed to be negative in this case, the infimum is therefore ν itself. Because of this, the dual problem is given by

$$\text{maximize} \quad g(\nu) = \begin{cases} -\nu \overline{Y}_0 & \text{if } \nu \geq 0, \\ \nu(1 - \overline{Y}_0) & \text{if } \nu < 0. \end{cases} \qquad \text{(Dual SNC)}$$

Because $\overline{Y}_0 \in (0, 1)$, this dual problem has a simple maximum of zero, obtained when $\nu = 0$. Not only does this mean we have strong duality in our problem, as the maximum of the dual problem and the minimum of the primal problem are both zero for an arbitrary image, we have an exact solution for the optimal dual variable $\nu^* = 0$. This has important consequences for our choice of solver for the constrained optimization problem.

4.4 Solving the Constrained Optimization Problem

Although this problem is not convex in any meaningful way, many standard optimization procedures can be applied to the smooth objective function $\hat{\Delta}(\vec{\gamma})$. However, the objective function is troublesome in many ways. Although numerical experiments indicate that there are no spurious stationary points beyond the minima established in the previous section, taking each coefficient in $\vec{\gamma}$ to infinity will necessarily approach both a minimum and a stationary point. This can cause many descent methods to occasionally veer off towards these minima at infinity rather than the continuous line of exact solutions.

One quick remedy to this is the incorporation of the constraint as a quadratic penalty to the objective function. Each step of the relevant minimization procedure is given by

$$\vec{\gamma}_{k+1} = \text{argmin } \hat{\Delta}(\vec{\gamma}) + \frac{\mu_k}{2}\left(\overline{Y}(\vec{\gamma}) - \overline{Y}_0\right)^2,$$

where the smoothness of the penalty function and small dimension of the Hessian allows for the use of Newton's method to efficiently solve the subproblem. Importantly, the unconstrained quadratic penalty function exactly recovers the unique solution to the constrained optimization problem for any positive value of μ_k. This owes to the simple fact that at each of the infinite solutions to the unconstrained problem $\hat{\Delta}(\vec{\gamma}^*) = 0$, and only one point among these sets the quadratic penalty equal to zero as well: the point at which the mean intensity matches the original exactly. Because of this, only a single iteration of the quadratic penalty method is necessary to find the solution.

However, the success of Newton's method in finding the unique solution to the penalized problem is highly dependant on both the initial value of $\vec{\gamma}_0$ and μ supplied. In particular, the penalty term $\overline{Y}(\vec{\gamma}) - \overline{Y}_0$ always takes values between -1 and 1 before being squared, meaning that squaring it can make the penalty quite small, even when the intensity is far from the original in a relative sense. For smaller values of μ with poorly chosen initial conditions, the penalty term often does not contribute enough to force terms to the true minimum, and solutions will drift towards infinity. The most potent solution to this issue is to begin the Newton's method with an initial condition very close to zero. This means that the solution is more likely to approach the stationary point closest to the origin, while avoiding being pulled towards infinity. This technique has been shown to work even with values of μ as small as $\mu = 1$.

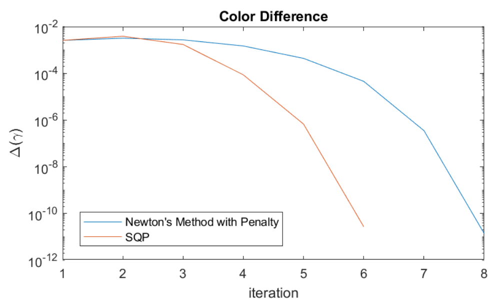
Recognizing the potential of Newton's method to effectively minimize the quadratic penalty subproblem, combined with our knowledge of the optimal dual variable $\nu^* = 0$, we also consider an approach using Sequential Quadratic Programming which removes the dependency on any external parameter like μ. In this method, we consider the relevant KKT conditions:

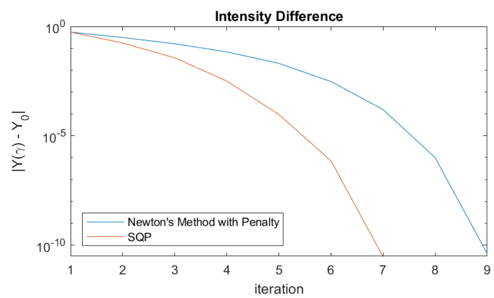
$$\begin{aligned} &\text{Stationarity:} && \nabla\hat{\Delta}(\vec{\gamma}) + \nu\nabla\overline{Y}(\vec{\gamma}) = 0 \\ &\text{Primal Feasibility:} && \overline{Y}(\vec{\gamma}) - \overline{Y}_0 = 0 \end{aligned}$$

Because an optimal point $\vec{\gamma}^*$, ν^* is known to exist, and strong duality holds for our problem, these KKT conditions are necessarily true at this saddle point. Similar to the quadratic penalty case above, this method can be solved with the following block representation of Newton's method,

$$\begin{bmatrix} \vec{\gamma}_{k+1} \\ \nu_{k+1} \end{bmatrix} = \begin{bmatrix} \vec{\gamma}_{k} \\ \nu_{k} \end{bmatrix} - \begin{bmatrix} \nabla^2\hat{\Delta}(\vec{\gamma}_k) + \nu_k\nabla^2\overline{Y}(\vec{\gamma}_k) & \nabla\overline{Y}(\vec{\gamma}_k) \\ \nabla\overline{Y}(\vec{\gamma}_k)^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nabla\hat{\Delta}(\vec{\gamma}_k) + \nu_k\nabla\overline{Y}(\vec{\gamma}_k) \\ \overline{Y}(\vec{\gamma}_k) - \overline{Y}_0 \end{bmatrix}$$

where the matrix to invert is only 4×4, and can be done so with relative ease. As is also the case in the quadratic penalty approach, this method is relatively sensitive to a choice in initial condition. ν_0 can always be chosen to be its optimal value of 0, and experimental results indicate that an initial value of $\vec{\gamma}_0$ close to zero similarly performs the best. Both methods are able to find the unique minimum in remarkably few iterations. Plots of the objective function per iteration, as well as primal feasibility, are presented below, although it should be noted that the color difference at the final iteration of each method is actually zero to machine precision. While convergence depends on the specific input image, the SQP method in general performs faster than the quadratic penalty method. A full demonstration of the comparative cost and speed of both algorithms is presented in section (6).





5 FIPCO - Fast Intensity Preserving Color Optimization

From the above discussion of the uniqueness of minimizers to the intensity constrained problem, it is clear that we can significantly reduce the complexity of the problem by solving it as a series of one-dimensional root finding problems. When any single gamma is fixed, γ_g in this case, we can color correct any image to have zero color difference by finding solutions to these equations, which produces optimal ${\gamma_r}^*$ and ${\gamma_b}^*$.

$$\sum_{u,v} G_{uv}^{{\gamma_g}^*} = \sum_{u,v} R_{uv}^{\gamma_r}, \qquad \sum_{u,v} B_{uv}^{{\gamma_b}^*} = \sum_{u,v} R_{uv}^{\gamma_r} \tag{1}$$

The problem then becomes finding the optimal value for the fixed gamma ${\gamma_g}^*$, such that when taken with its pair of optimal ${\gamma_r}^*$, ${\gamma_b}^*$, the intensity of the scaled image matches the original. This value of ${\gamma_g}^*$ can be found with a one dimensional Newton's method as well using the simple observation that when the color difference is zero, the intensity depends only on any single component of $\vec{\gamma}$, as the averages of each channel are necessarily equal to one another. When taken as a function of γ_g, the mean intensity can then be reduced to the average value of the green channel:

$$\bar{Y}(\gamma_g) = \frac{1}{UV} \sum_{u,v} G_{uv}^{\gamma_g} \tag{2}$$

The algorithm is then simple: Solve Eq. 2 in order to find the ${\gamma_g}^*$ for a specified intensity, which is then used to compute ${\gamma_r}^*$ and ${\gamma_b}^*$ through Eq. 1, giving us the optimal gammas. Using Newton's method on each of these smooth, one dimensional subproblems converges remarkably quickly.

6 Numerical Results and Comparison of Methods

We now compare the speed and computational complexity of each of the three algorithms discussed: Newton with Quadratic Penalty with $\mu = 1$, Sequential Quadratic Programming, and the novel Fast-Intensity-Preserving-Color-Optimization algorithm (FIPCO). As a baseline point of comparison, we also include the fastest results obtained using the MATLAB function `fmincon`, which applied an interior point method with a provided gradient. We also compare the resultant image from the optimal solution to the linear scaling problem, which can be obtained in near constant time. We test each of the four algorithms on three images of the same size, each having a variable initial color difference.

In order to accurately assess the computational complexity of each algorithm, we count not just the number of iterations in each, but also the number of times each algorithm needs to do operate on an entire channel. These $O(UV)$ operations are the most expensive in each function evaluation, but only occur a fixed number of times in each iteration of each method. The implemented penalized Newton's and SQP methods can somewhat mitigate this cost, as this computation is repeated in the calculation of both gradients and Hessians of $\Delta(\vec{\gamma})$ and $\overline{Y}$, giving these methods a significant computational edge over the built-in method. Additionally, the number of iterations for the FIPCO can be broken down between the number of iterations needed to find ${\gamma_g}^*$, then ${\gamma_r}^*$ and ${\gamma_b}^*$. The following tests were each run to ensure the color difference and primal feasibility are less than 10^{-10}.

From the results on the following page, we can clearly see that the novel FIPCO algorithm is vastly superior to the two optimization strategies used, both in terms of operation count and runtime. However, the implemented optimization algorithms are themselves superior to the built-in `fmincon`, owing primarily to a more efficient implementation of function evaluations. While the cost per iteration between the two implemented methods is roughly equal, SQP typically takes fewer iterations. Additionally, it can be seen that the initial color difference is only loosely related to the number of iterations required to converge, with the dominance of any single color channel more likely to increase the necessary number of iterations for convergence. However, the number of iterations required in any of the methods is roughly constant across image size.

7 Conclusions and Synthesis of Class Concepts

Within this paper, we have proven the existence of a unique solution to both the linear and nonlinear color correction and intensity preservation problem, and have presented several ways to efficiently find them. Solutions to the linear scaling problem can be found directly in near constant time, making this method ideal for situations in which speed is the priority. However, the optimization techniques applied to the nonlinear analogue of the problem can still find its unique minimzer to near machine precision accuracy exceptionally quickly. Moreover, the novel FIPCO algorithm can do so as simply a series of one dimensional root-finding problems.

For better or for worse, the FIPCO algorithm is the most novel result of this project, as it is in several ways strictly superior to the method described in [2]. That being said, formulating the color constancy and intensity preservation problem in an optimization framework was critically important in the discovery of this technique. In our initial investigations into the convexity of the problem, we discovered that there existed nontrivial solutions to the problem, but that the lack of differentiability in the objective function Δ would make it difficult to apply even the simplest descent methods. By using techniques learned in class, we were able to generate a sufficiently smooth approximation and compute its analytic gradient and Hessian.

However, it was also necessary to apply knowledge of both the problem and various methods that are even applicable to the nonlinear-program. For example, applying the simple technique of adding a quadratic penalty method worked reasonably well, but an augmented Lagrangian technique performed no better. This led us to investigate the duality of the problem, and we discovered that the optimal dual variable is always equal to zero, and an augmented Lagrangian technique does not add any additional information to the minimization problem. However, knowing the true optimal dual variable led us to consider SQP as a viable method as well. Applying computational tricks to each of these methods, such as taking advantage of the overlapping computation in computing gradients and Hessians, allowed us to find the optimal points very quickly, even for large images. However, it was ultimately the hunt for a proof of existence and uniqueness to solutions led us to our simple, and incredibly fast approach. So while our best performing algorithm is very simple, and not strictly an optimization technique in of itself, building up to it required ample use of theory and concepts learned in class.

Optimization Method	Runtime (s)	Iterations	$O(UV)$ Operations
`fmincon` (Interior Point)	2.154	33	396
Penalized Newton	0.304	8	96
SQP	0.207	6	72
FIPCO	0.125	6 + 6	37

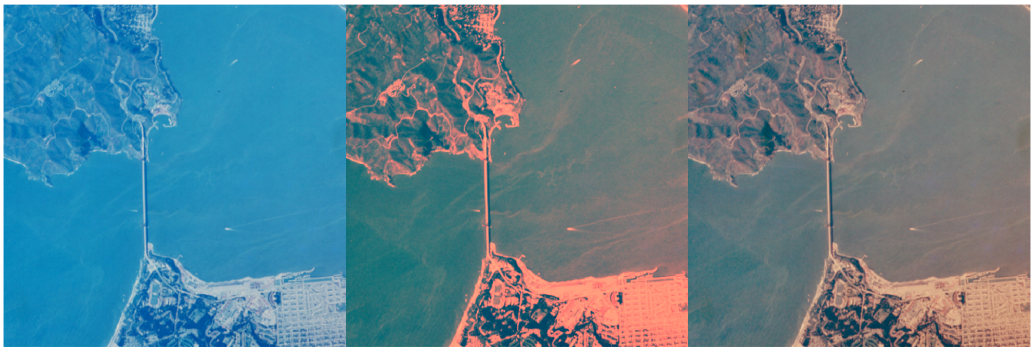


Figure 4: Initial Color Difference: 1.523×10^{-1}. Left: Initial image. Center: Linearly scaled, intensity preserved image. Right: Non-linearly scaled, intensity preserved image

Optimization Method	Runtime (s)	Iterations	$O(UV)$ Operations
`fmincon` (Interior Point)	2.819	45	540
Penalized Newton	0.305	9	108
SQP	0.238	7	84
FIPCO	0.141	7 + 6	39

Figure 5: Initial Color Difference: 9.245×10^{-2}. Left: Initial image. Center: Linearly scaled, intensity preserved image. Right: Non-linearly scaled, intensity preserved image

Optimization Method	Runtime (s)	Iterations	$O(UV)$ Operations
`fmincon` (Interior Point)	1.602	23	276
Penalized Newton	0.328	10	120
SQP	0.235	7	84
FIPCO	0.145	7 + 7	43

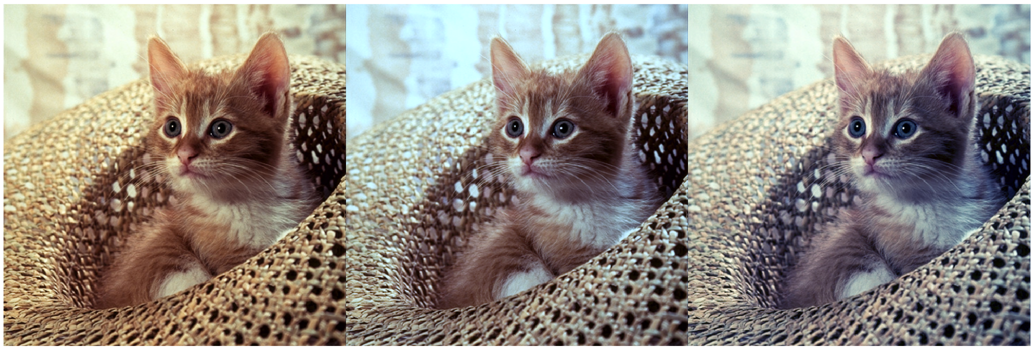


Figure 6: Initial Color Difference: 1.759×10^{-2}. Left: Initial image. Center: Linearly scaled, intensity preserved image. Right: Non-linearly scaled, intensity preserved image

Optimization Method	Runtime (s)	Iterations	$O(UV)$ Operations
`fmincon` (Interior Point)	1.597	23	276
Penalized Newton	0.340	10	120
SQP	0.236	7	84
FIPCO	0.165	7 + 7	43

Figure 7: Initial Color Difference: 1.611×10^{-3}. Left: Initial image. Center: Linearly scaled, intensity preserved image. Right: Non-linearly scaled, intensity preserved image

8 Appendix of Formulas

8.1 Analytic Derivatives of Smooth Difference Function

Define the smooth approximation of the color difference criterion as

$$\hat{\Delta}(\vec{\gamma}) = \ln\left(\exp\left(R^2\left(\vec{\gamma}\right)\right) + \exp\left(G^2\left(\vec{\gamma}\right)\right) + \exp\left(B^2\left(\vec{\gamma}\right)\right) - 2\right)$$

where

$$\begin{aligned} R(\vec{\gamma}) &= \frac{1}{3UV}\sum_{uv} 2R_{uv}^{\gamma_r} - G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b} \\ G(\vec{\gamma}) &= \frac{1}{3UV}\sum_{uv} -R_{uv}^{\gamma_r} + 2G_{uv}^{\gamma_g} - B_{uv}^{\gamma_b} \\ B(\vec{\gamma}) &= \frac{1}{3UV}\sum_{uv} -R_{uv}^{\gamma_r} - G_{uv}^{\gamma_g} + 2B_{uv}^{\gamma_b} \end{aligned}$$

We can rewrite $\hat{\Delta}$ as the composition of functions $f : \mathcal{R}^3 \to \mathcal{R}^3$ and $g : \mathcal{R}^3 \to \mathcal{R}$ such that $\hat{\Delta}(\vec{\gamma}) = g(f(\vec{\gamma}))$, where

$$f(\vec{\gamma}) = \begin{bmatrix} R(\vec{\gamma}) \\ G(\vec{\gamma}) \\ B(\vec{\gamma}) \end{bmatrix}, \qquad g(x_1, x_2, x_3) = \ln(\exp(x_1^2) + \exp(x_2^2) + \exp(x_3^2) - 2).$$

The multi-dimensional chain rule can then be applied to the problem, in which the gradient of $\hat{\Delta}$ is given by

$$\nabla\hat{\Delta}(\vec{\gamma}) = (Jf(\vec{\gamma}))^T \nabla g(f(\vec{\gamma}))$$

where $Jf(\vec{\gamma})$ is the Jacobian of f evaluated at $\vec{\gamma}$, given by

$$J(\vec{\gamma}) = \frac{1}{3UV}\begin{bmatrix} 2\sum_{u,v} R_{uv}^{\gamma_r}\ln(R_{uv}) & -\sum_{u,v} G_{uv}^{\gamma_g}\ln(R_{uv}) & -\sum_{u,v} B_{uv}^{\gamma_b}\ln(R_{uv}) \\ -\sum_{u,v} R_{uv}^{\gamma_r}\ln(G_{uv}) & 2\sum_{u,v} G_{uv}^{\gamma_g}\ln(G_{uv}) & -\sum_{u,v} B_{uv}^{\gamma_b}\ln(G_{uv}) \\ -\sum_{u,v} R_{uv}^{\gamma_r}\ln(B_{uv}) & -\sum_{u,v} G_{uv}^{\gamma_g}\ln(B_{uv}) & 2\sum_{u,v} B_{uv}^{\gamma_b}\ln(B_{uv}) \end{bmatrix}$$

where if any pixel value is zero, its contribution to the sum is zero as well. Similarly, $\nabla g(f(\vec{\gamma}))$ is the gradient of g evaluated at $f(\vec{\gamma})$, given by

$$\nabla g(f(\vec{\gamma})) = \frac{1}{\exp(R^2(\vec{\gamma})) + \exp(G^2(\vec{\gamma})) + \exp(B^2(\vec{\gamma})) - 2}\begin{bmatrix} 2R(\vec{\gamma})\exp(R^2(\vec{\gamma})) \\ 2G(\vec{\gamma})\exp(G^2(\vec{\gamma})) \\ 2B(\vec{\gamma})\exp(B^2(\vec{\gamma})) \end{bmatrix}$$

Put together, the gradient of the smooth color difference criterion is given by

$$\nabla\hat{\Delta}(\vec{\gamma}) = \frac{\begin{bmatrix} 2\sum_{u,v} R_{uv}^{\gamma_r}\ln(R_{uv}) & -\sum_{u,v} R_{uv}^{\gamma_r}\ln(R_{uv}) & -\sum_{u,v} R_{uv}^{\gamma_r}\ln(R_{uv}) \\ -\sum_{u,v} G_{uv}^{\gamma_g}\ln(G_{uv}) & 2\sum_{u,v} G_{uv}^{\gamma_g}\ln(G_{uv}) & -\sum_{u,v} G_{uv}^{\gamma_g}\ln(G_{uv}) \\ -\sum_{u,v} B_{uv}^{\gamma_b}\ln(B_{uv}) & -\sum_{u,v} B_{uv}^{\gamma_b}\ln(B_{uv}) & 2\sum_{u,v} B_{uv}^{\gamma_b}\ln(B_{uv}) \end{bmatrix}\begin{bmatrix} 2R(\vec{\gamma})\exp(R^2(\vec{\gamma})) \\ 2G(\vec{\gamma})\exp(G^2(\vec{\gamma})) \\ 2B(\vec{\gamma})\exp(B^2(\vec{\gamma})) \end{bmatrix}}{3UV(\exp(R^2(\vec{\gamma})) + \exp(G^2(\vec{\gamma})) + \exp(B^2(\vec{\gamma})) - 2)}$$

8.2 Analytic Derivatives of Intensity Constraint and Quadratic Penalty

We can similarly compute the Hessian of the color difference, albeit with a bit more tedium. We compute seconds derivatives manually, beginning with an unmixed second derivative with respect to an arbitrary scaling factor γ_k. This results in

$$\begin{aligned}\frac{\partial^2\hat{\Delta}}{\partial\gamma_k\partial\gamma_k} =& \frac{-(2RR_{\gamma_k}e^{R^2}+2GG_{\gamma_k}e^{G^2}+2BB_{\gamma_k}e^{B^2})^2}{(e^{R^2}+e^{G^2}+e^{B^2}-2)^2}\\ &+\frac{2e^{R^2}(R_{\gamma_k}^2+RR_{\gamma_k\gamma_k}+2R^2R_{\gamma_k}^2)+2e^{G^2}(G_{\gamma_k}^2+GG_{\gamma_k\gamma_k}+2G^2G_{\gamma_k}^2)+2e^{B^2}(B_{\gamma_k}^2+BB_{\gamma_k\gamma_k}+2B^2B_{\gamma_k}^2)}{e^{R^2}+e^{G^2}+e^{B^2}-2}\end{aligned}$$

Next, we consider the mixed partial derivatives of $\hat{\Delta}$, noting that mixed partials of the functions $R(\vec{\gamma}), G(\vec{\gamma})$, and $B(\vec{\gamma})$ are equal to zero. Taking the partial derivative with respect to arbitrary scaling factors γ_k and γ_j, we have

$$\begin{aligned}\frac{\partial^2\hat{\Delta}}{\partial\gamma_k\partial\gamma_j} =& \frac{-(2RR_{\gamma_k}e^{R^2}+2GG_{\gamma_k}e^{G^2}+2BB_{\gamma_k}e^{G^2})(2RR_{\gamma_j}e^{R^2}+2GG_{\gamma_j}e^{G^2}+2BB_{\gamma_j}e^{G^2})}{(e^{R^2}+e^{G^2}+e^{B^2}-2)^2}\\ &+\frac{2(R_{\gamma_k}R_{\gamma_j}e^{R^2}(1+2R^2)+G_{\gamma_k}G_{\gamma_j}e^{G^2}(1+2G^2)+B_{\gamma_k}B_{\gamma_j}e^{B^2}(1+2B^2))}{e^{R^2}+e^{G^2}+e^{B^2}-2}.\end{aligned}$$

These terms can then be combined in the full Hessian of the smooth difference criterion $\nabla^2\hat{\Delta}(\vec{\gamma})$.

Similarly, analytic derivatives of the mean intensity function and the derived quadratic penalty term can be constructed as well. Given the mean intensity function

$$\overline{Y}(\vec{\gamma}) = \frac{1}{UV}\sum_{u,v} 0.299R_{u,v}^{\gamma_r} + 0.587G_{u,v}^{\gamma_g} + 0.114B_{u,v}^{\gamma_b},$$

its gradient is given by

$$\nabla\overline{Y}(\vec{\gamma}) = \frac{1}{UV}\begin{bmatrix}0.299\sum_{u,v} R_{uv}^{\gamma_r}\ln(R_{uv})\\ 0.587\sum_{u,v} G_{uv}^{\gamma_r}\ln(G_{uv})\\ 0.114\sum_{u,v} B_{uv}^{\gamma_r}\ln(B_{uv})\end{bmatrix}$$

where as before, any pixel value equal to zero is not considered in the above sum. The Hessian is defined as

$$\nabla^2\overline{Y}(\vec{\gamma}) = \frac{1}{UV}\begin{bmatrix}0.299\sum_{u,v} R_{uv}^{\gamma_r}\ln^2(R_{uv}) & 0 & 0\\ 0 & 0.587\sum_{u,v} G_{uv}^{\gamma_r}\ln^2(G_{uv}) & 0\\ 0 & 0 & 0.114\sum_{u,v} B_{uv}^{\gamma_r}\ln^2(B_{uv})\end{bmatrix}.$$

The derived quadratic penalty term is given by

$$Q(\vec{\gamma};\mu) = \frac{\mu}{2}(\overline{Y}(\vec{\gamma}) - \overline{Y}_0)^2$$

with a gradient and Hessian given by

$$\begin{aligned}\nabla Q(\vec{\gamma};\mu) &= \mu(\overline{Y}(\vec{\gamma}) - \overline{Y}_0)\nabla\overline{Y}(\vec{\gamma})\\ \nabla^2 Q(\vec{\gamma};\mu) &= \mu\nabla\overline{Y}(\vec{\gamma})\nabla\overline{Y}(\vec{\gamma})^T + \mu(\overline{Y}(\vec{\gamma}) - \overline{Y}_0)\nabla^2\overline{Y}(\vec{\gamma}).\end{aligned}$$

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