20.08.2019

Digital Image Processing (CSE/ECE 478)

Lecture-7: Image Enhancement in Frequency Domain – Preliminary Concepts

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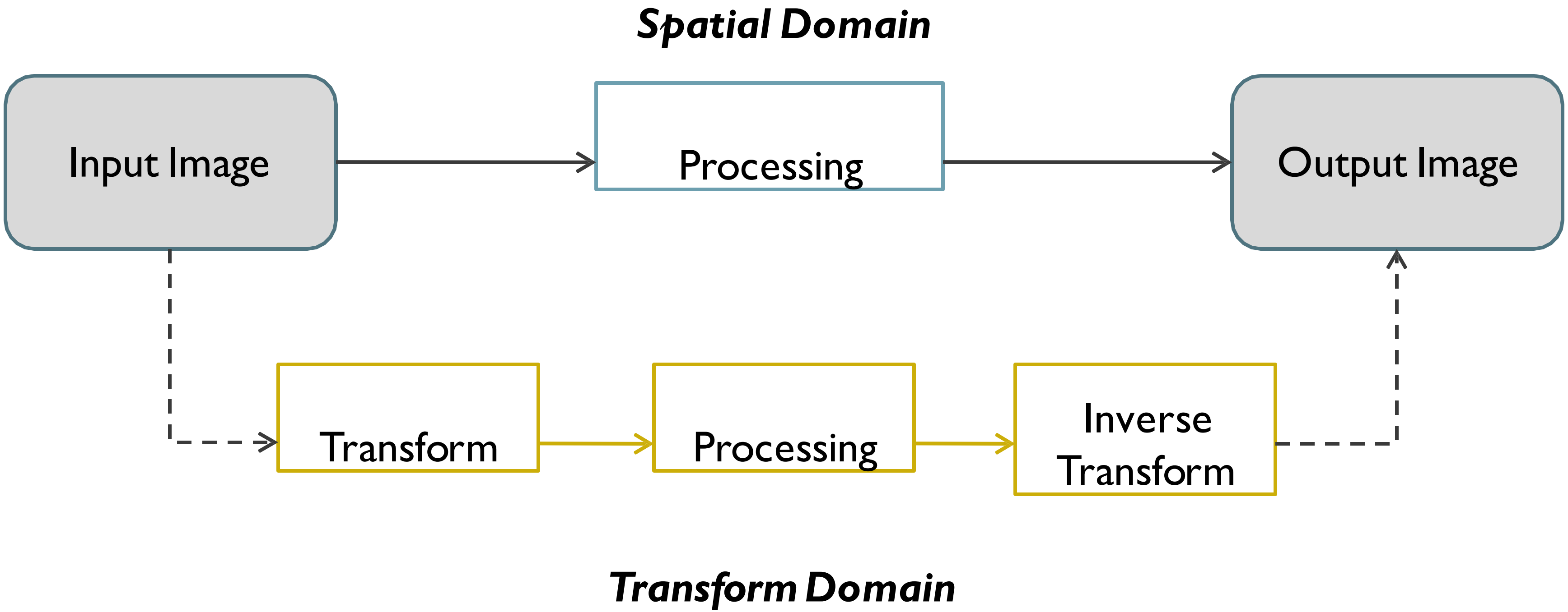
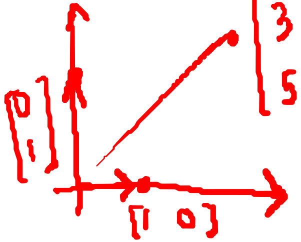
Image Processing – Two Paradigms

- Directly manipulating pixels in spatial domain
- Manipulating in transform domain

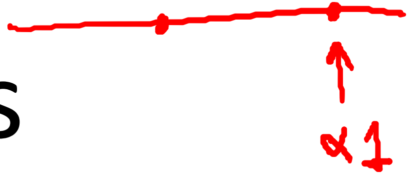
Spatial vs. Transform Domain Processing





$\begin{bmatrix}3\\5\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\end{bmatrix}$

Basis Representations



- 1-D: Points on number line (Scalar multiples of 1)
- 2-D:Points in a plane (Scalar multiples of [1 0],[0 1])
- N-D: Points in R^N space

- Perspective: Coefficients $\longleftrightarrow$ Vector components

Basis Representations

- Def:- Basis (of a vector space):- A linearly independent spanning set of vectors
- *Any* vector v = Unique linear combination of basis vectors
- $v = \propto_1 \ b_1 + \propto_2 \ b_2 + \ldots \propto_N \ b_N$
- $\propto_i$: Extent to which basis vector b_i is present in v
- How do we determine $\propto_i$?

Basis Representations

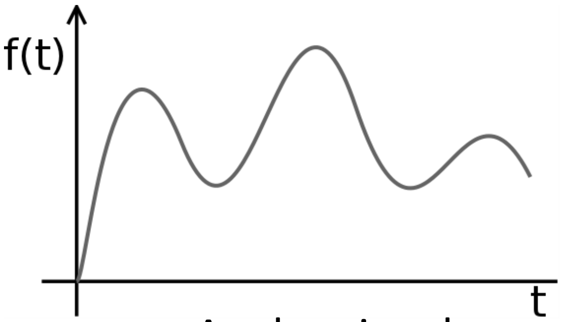
- Transform and Inverse Transform

Basis Representations

- Orthogonal Basis
  - $\propto_i$?
- Orthonormal Basis
  - $\propto_i$?

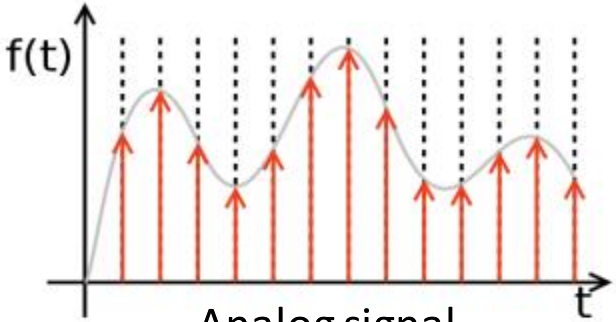
Signal

"Function that conveys information about the behavior or attributes of some phenomenon” (wikipedia)

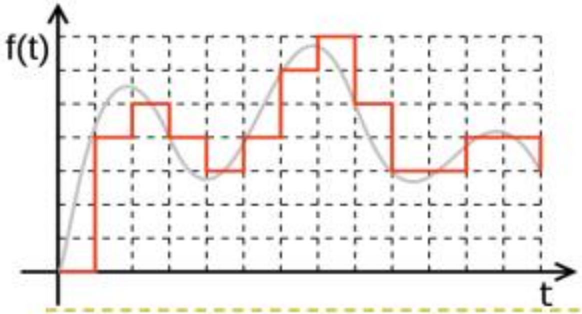


Analog signal

Recap ...



Analog signal



Digital signal

Sampling

256 × 256 | 32 × 32 | 16 × 16

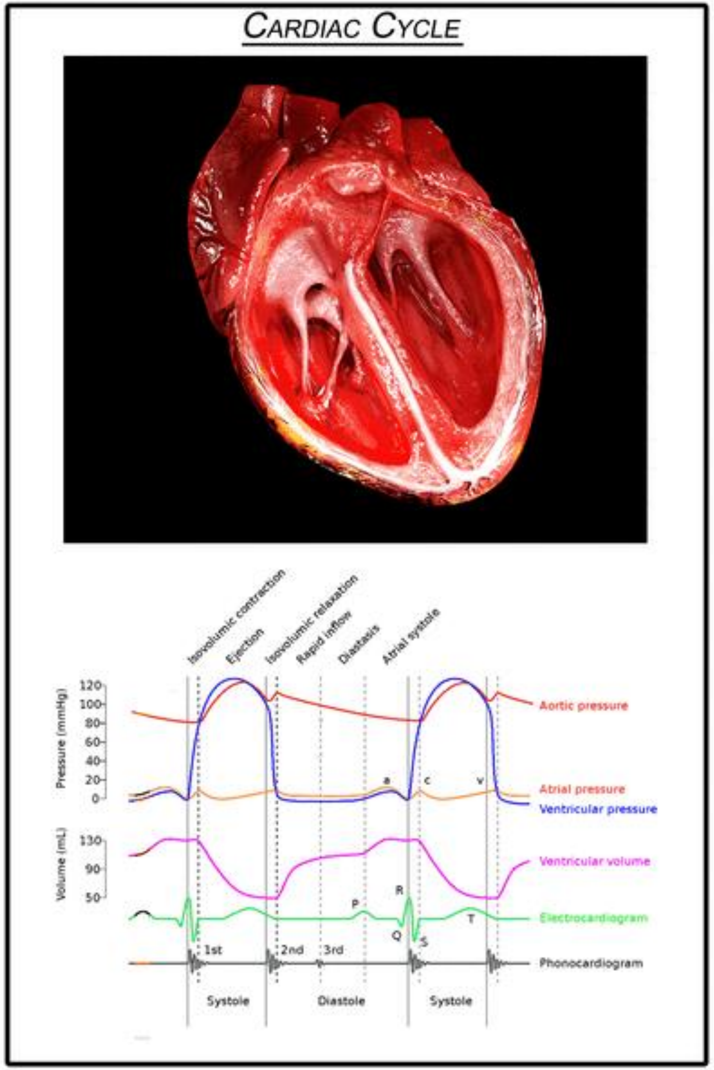
Quantization

8 bits per pixel | 4 bits per pixel | 2 bits per pixel | 1 bit per pixel

Signal - Characteristics

- **Continuous (Analog)** / Discrete (Digital)
- **Periodic** / Non-periodic
- **Temporal** / Spatial / Spatio-temporal

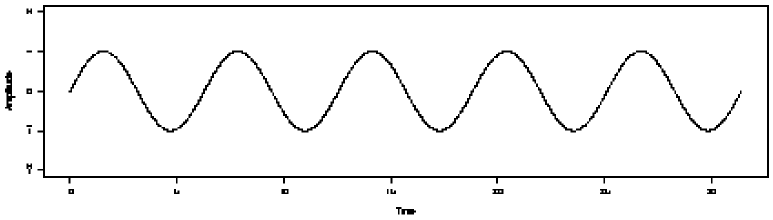
Rhythm of life

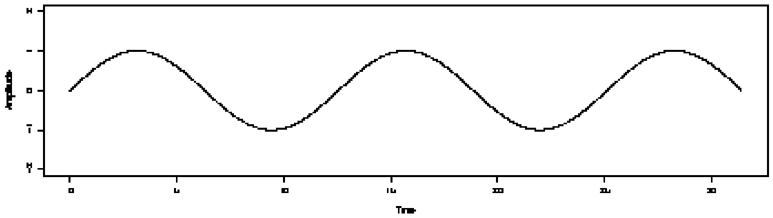


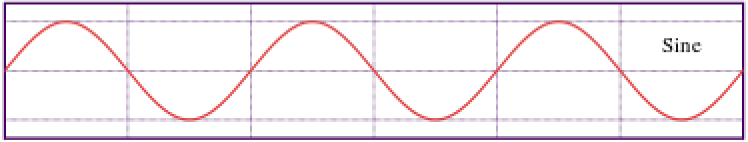
https://commons.wikimedia.org/wiki/File:Cardiac-Cycle-Animated.gif

- Periodic signals
- Periodic →**Frequency** of occurrence
  - Repetitions/<Unit>
    - Unit = {Time, Space}
- Basis ?

Periodic signals



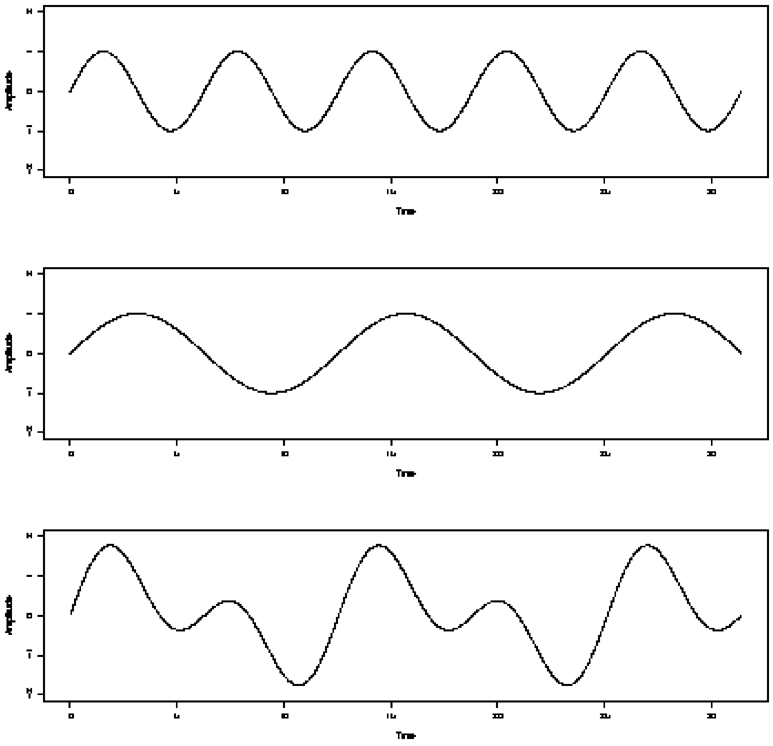


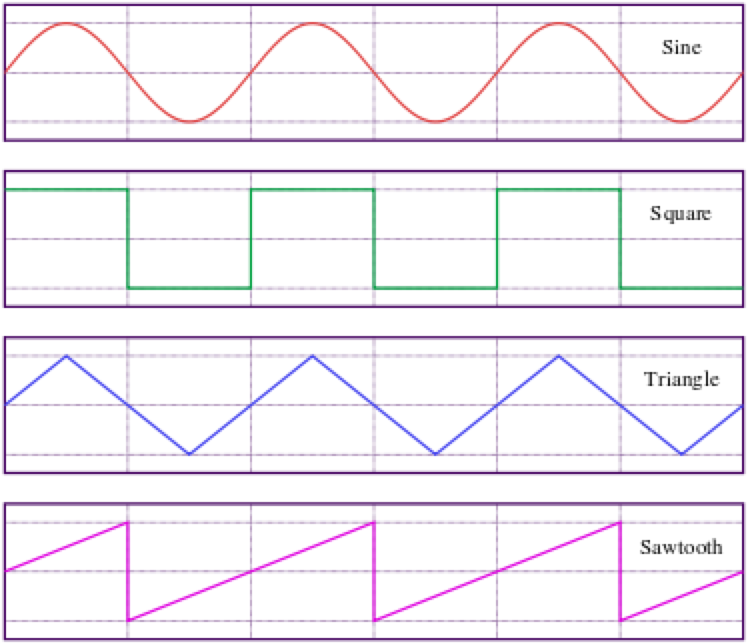


http://web.science.mq.edu.au/~cassidy/comp449/html/ch03s03.html

https://en.wikipedia.org/wiki/Triangle_wave

Periodic signals





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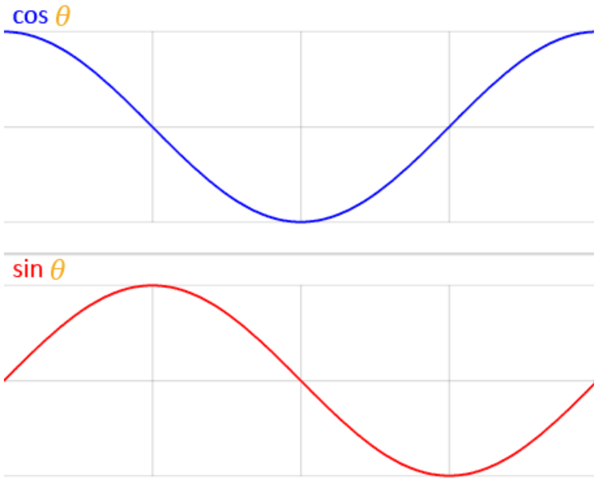
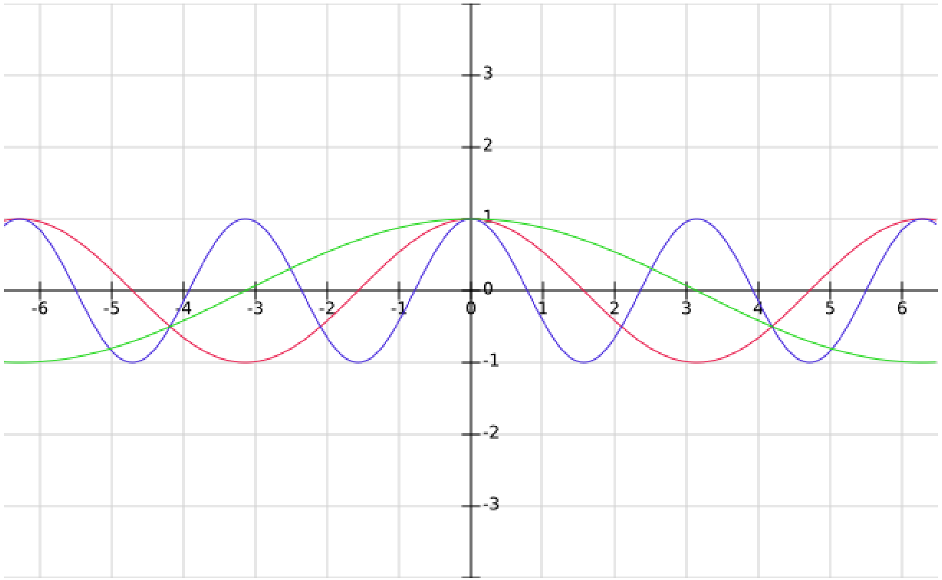
Trigonometric Functions as basis for periodic signals



Simple periodic signals

- $x(t) = A\cos(t)$
- $x(t) = A\cos(2t)$
- $x(t) = A\cos(t/2)$

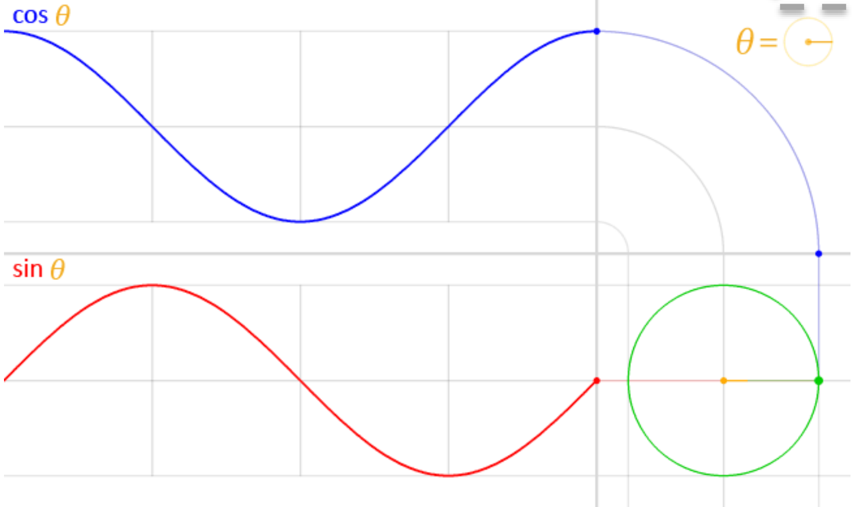


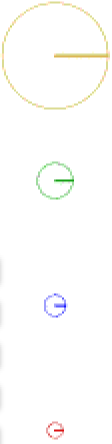
- $x(t) = A\cos(\omega t) = A\,cos(2\pi f t) = A\,cos(\frac{2\pi}{T}t)$

Angular frequency

Periodic Signals

- Periodic → **Frequency** of occurrence
  - Repetitions/<Unit> ($\boxed{\text{cycles}}$/sec = Hz)





$$x(t) = A\cos(\omega t) = A\cos(\boxed{2\pi} f t) = A\cos(\frac{2\pi}{T} t)$$

Angular frequency

Fundamental Period

Periodicity

Definition: A function $f(x)$ is T-*periodic* if

$$f(x+T) = f(x) \text{ for all } x \in \mathbb{R}.$$

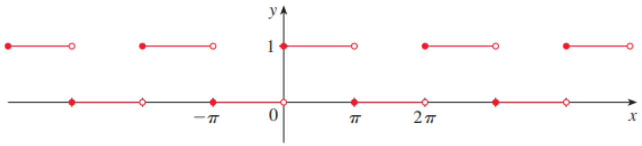
Remarks:

- If $f(x)$ is T-periodic, then $f(x+nT) = f(x)$ for any $n \in \mathbb{Z}$.
- $\displaystyle\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$ for all a.

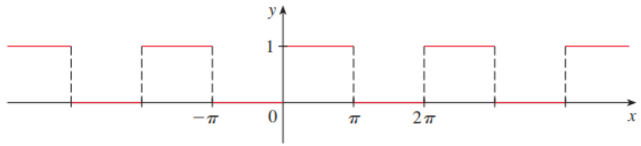
Periodic Signals

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0 \\ 1 & \text{if } 0 \leqslant x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

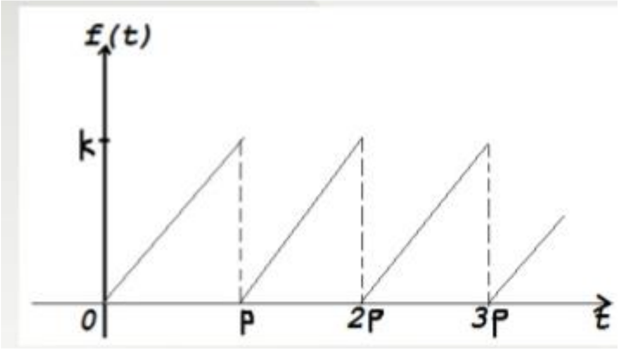
So f is periodic with period 2π and its graph is shown in Figure 1.



(a)



(b)



http://www.stewartcalculus.com/data/CALCULUS%206E%20Early%20Transcendentals/upfiles/topics/6et_at_01_fs_stu.pdf

https://image.slidesharecdn.com/mathesppt-141012085025-conversion-gate02/95/laplace-periodic-function-with-graph-11-638.jpg?cb=1413103924

Periodic Signals

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0 \\ 1 & \text{if } 0 \leqslant x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x)$$

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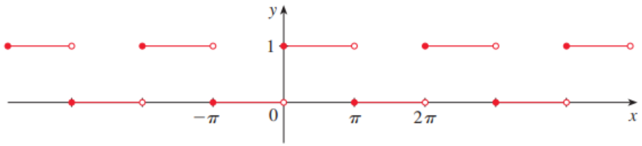
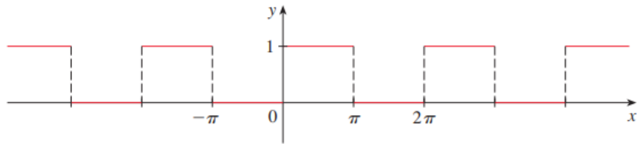


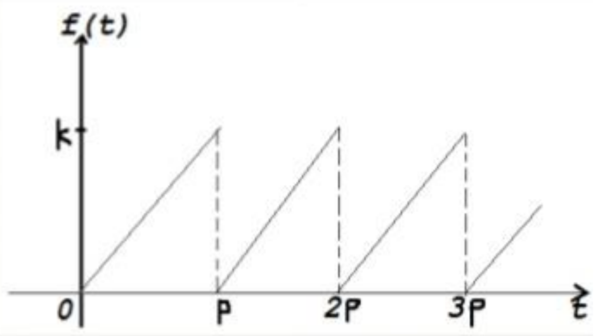
(a)



(b)

$f(t) = \frac{k}{p} t$ if $0 < t < p$, $f(t+p) = f(t)$

➢ ANS.:-



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https://image.slidesharecdn.com/mathesppt-141012085025-conversion-gate02/95/laplace-periodic-function-with-graph-11-638.jpg?cb=1413103924

Definition: Fourier Series

Theorem

If $f(x)$ is a piecewise smooth, 2π-periodic function, then there are (unique) Fourier coefficients $a_0, a_1, a_2, \ldots$ *and* $b_1, b_2, \ldots$ *so that*

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

for all x. This is called the Fourier series *of $f(x)$.*

Trigonometric Functions as basis for periodic signals

$$f(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

Fourier Series in terms of fundamental time period T

http://www.thefouriertransform.com/series/coefficients.php

Trigonometric Functions as basis for periodic signals

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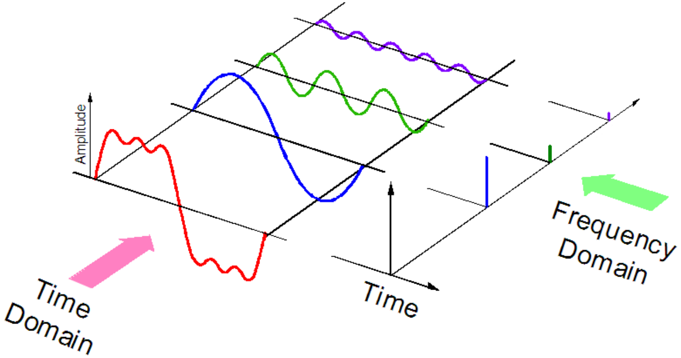
Fourier Series in terms of fundamental time period T



$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi mt}{T}\right)dt$$

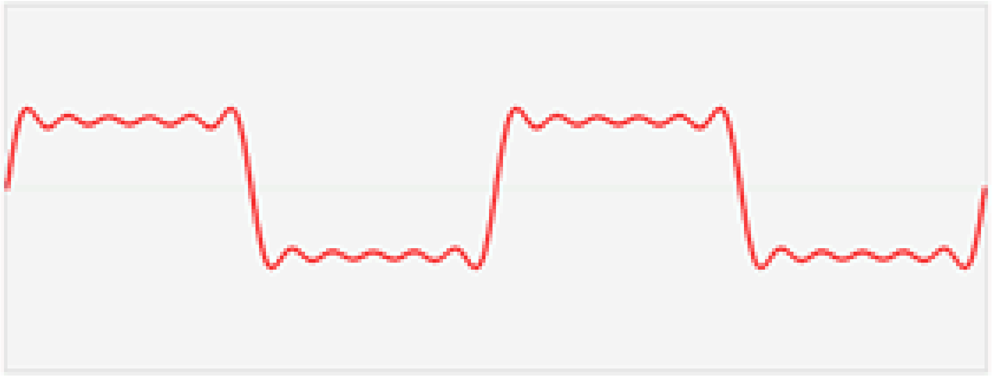
$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

Amplitude of different harmonics whose frequencies are a <u>multiple of base frequency</u>

http://www.thefouriertransform.com/series/coefficients.php

https://ocw.mit.edu/courses/mathematics/18-03sc-differential-equations-fall-2011/unit-iii-fourier-series-and-laplace-transform/fourier-series-basics/MIT18_03SCF11_s21_7text.pdf

Fourier Series, visually



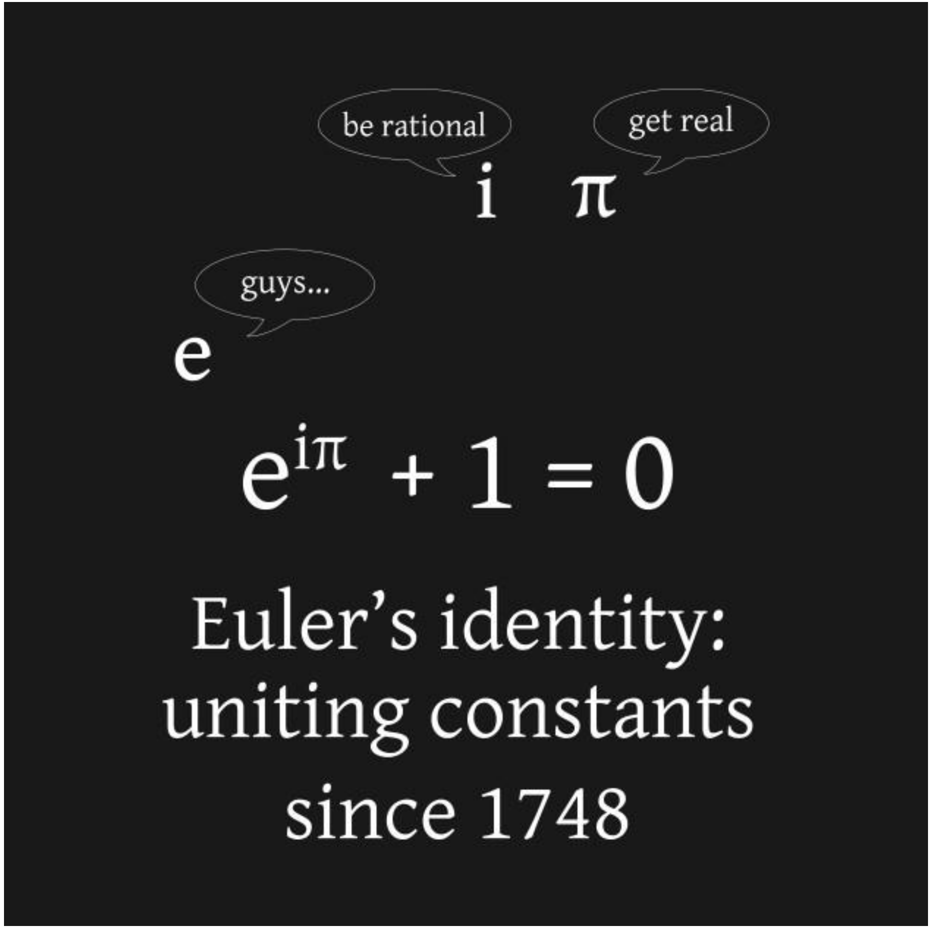
Can we rewrite this as a single summation (instead of two summation series) ?

$$f(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

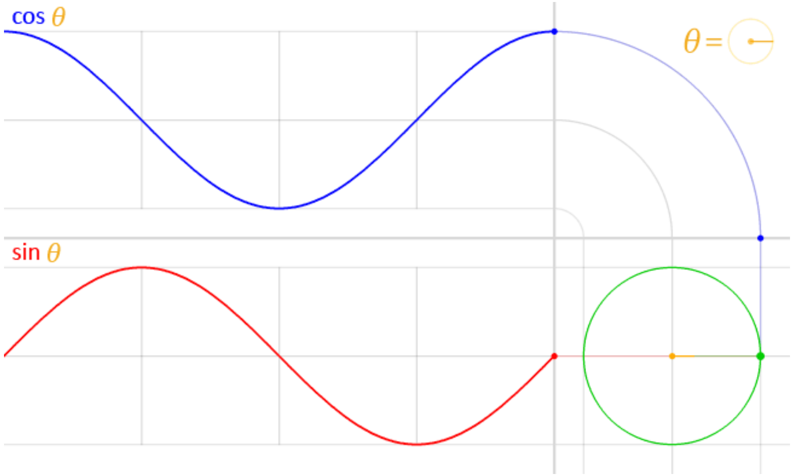
Yes we can … via …

http://www.thefouriertransform.com/series/coefficients.php



$$e^{it} = \cos t + i \sin t$$

$$i = \sqrt{-1}$$

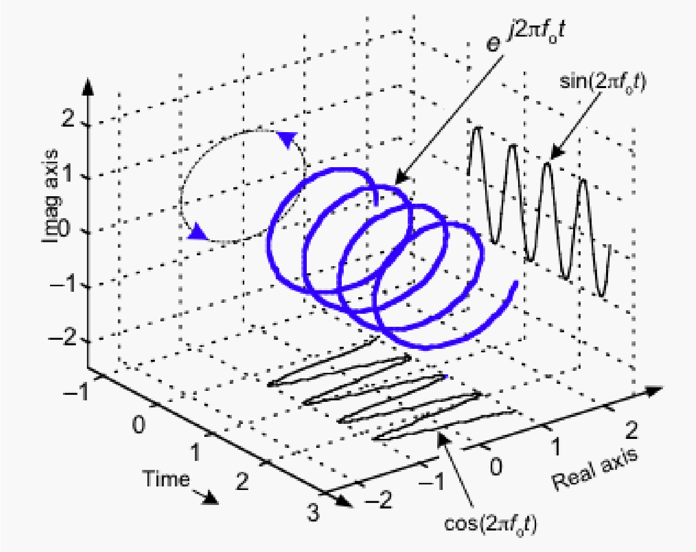


https://res.cloudinary.com/teepublic/image/private/s--s4GX0hG3--/t_Preview/b_rgb:191919,c_limit,f_jpg,h_630,q_90,w_630/v1486157785/production/designs/1177382_1.jpg

Complex sinusoid

$$e^{it} = \cos t + i \sin t$$

$$i = \sqrt{-1}$$



Fourier Series in terms of complex coefficients

$$f(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi mt}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}}$$

$$\boxed{c_n = \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi nt}{T}}\,dt}$$

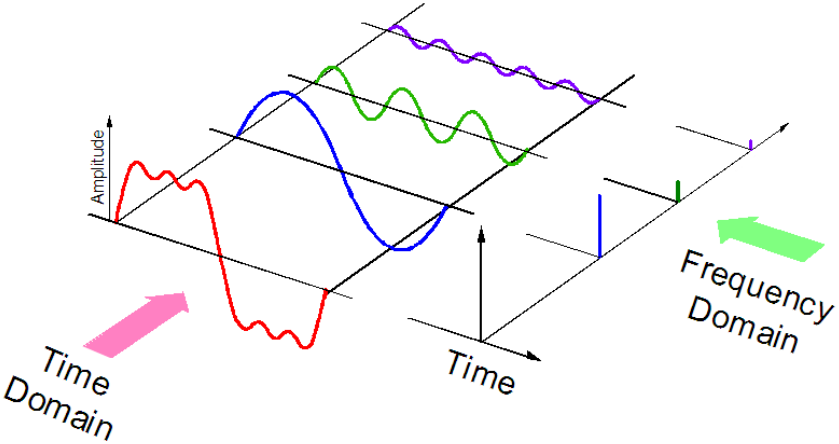
http://www.thefouriertransform.com/series/complexcoefficients.php

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi m t}{T}\right)dt$$

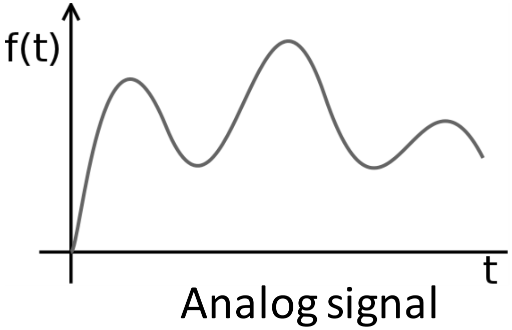
$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right)dt$$

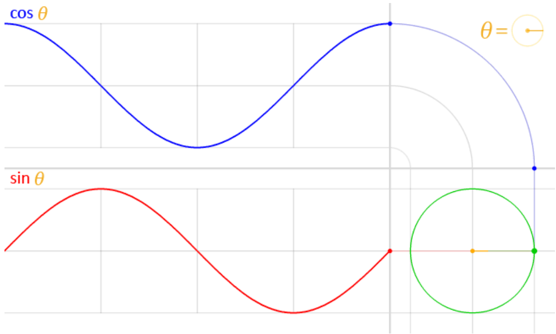
Frequency component

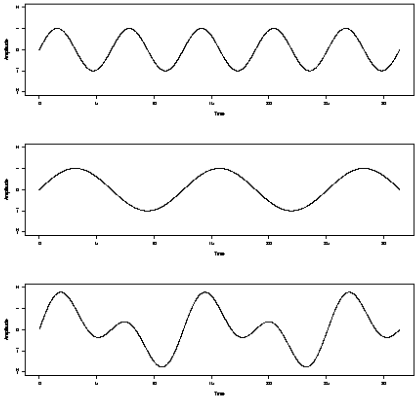


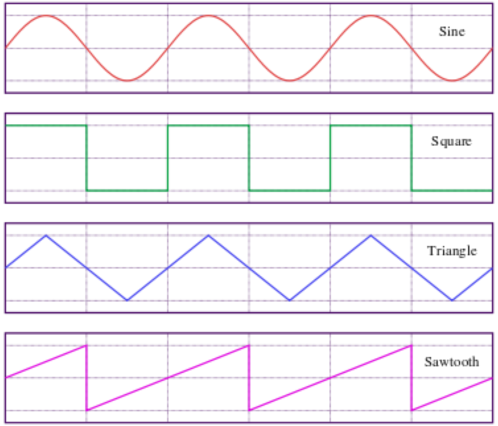
Technically, everything still in temporal domain

So far …

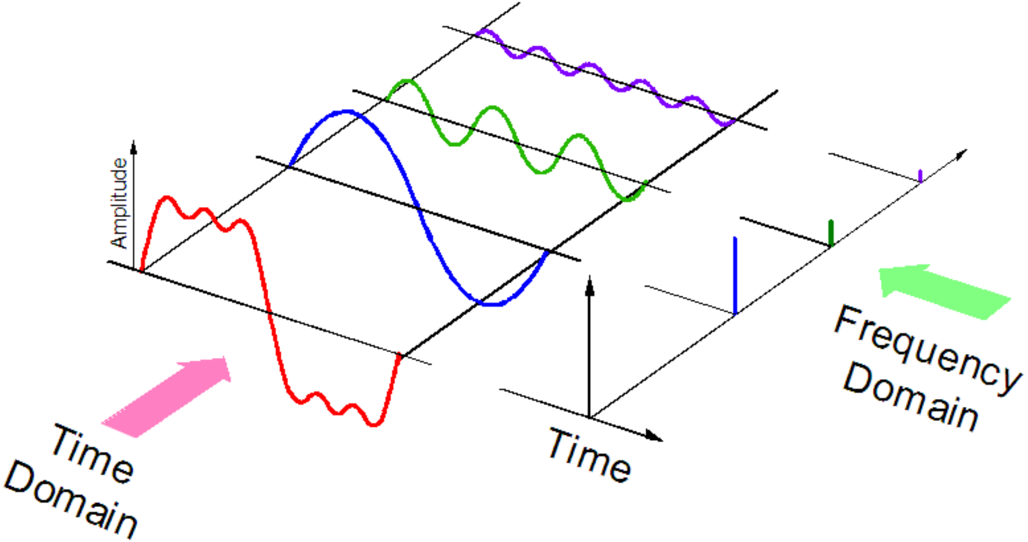

f(t)
t
Analog signal


cos θ
θ=
sin θ




Sine
Square
Triangle
Sawtooth

So far …



$$f(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}}$$

x(t) as sum of simpler periodic functions

- Many phenomena are inherently periodic
  - Astronomical
  - Biological (heart-beat)
  - Physical (tides, vibrating strings)

- Fourier Series → Simpler periodic function harmonics → Natural representation

More natural than .... Taylor series !

"I'm sorry, Taylor, but Fourier had one of the best series of 1807."

$$\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$
$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2$$
$$+ \frac{f'''(c)}{6}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

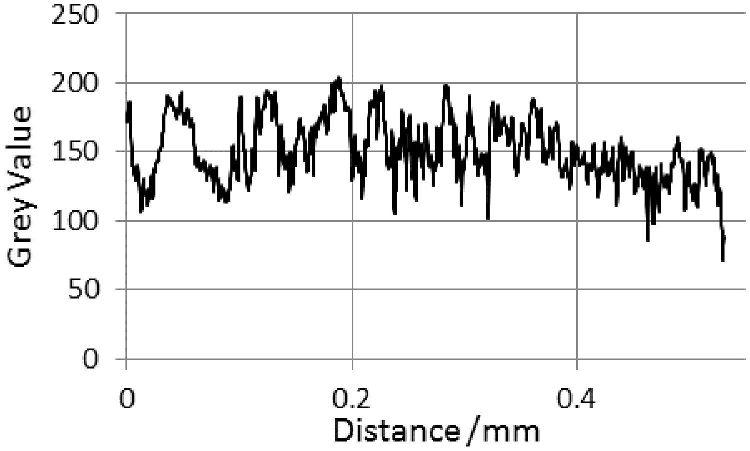
$$f(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

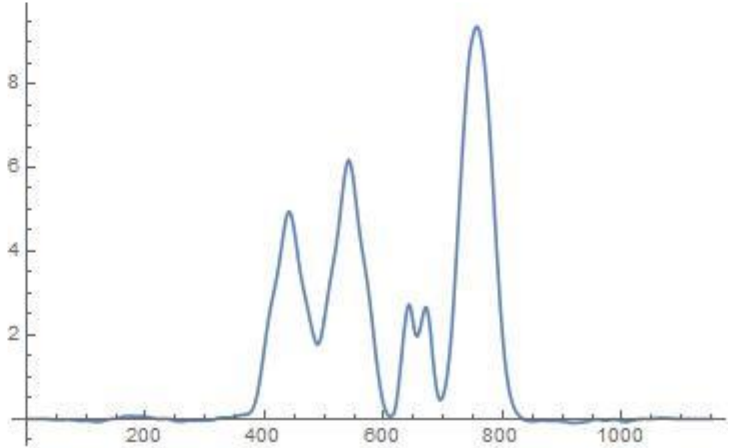
$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

https://www.youtube.com/watch?v=Piq3VgRlkoY

Page-1 (Motivation) : http://www.cse.unsw.edu.au/~ignjat/diff/CD-IEEE.pdf

What if x(t) is non-periodic ?





$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

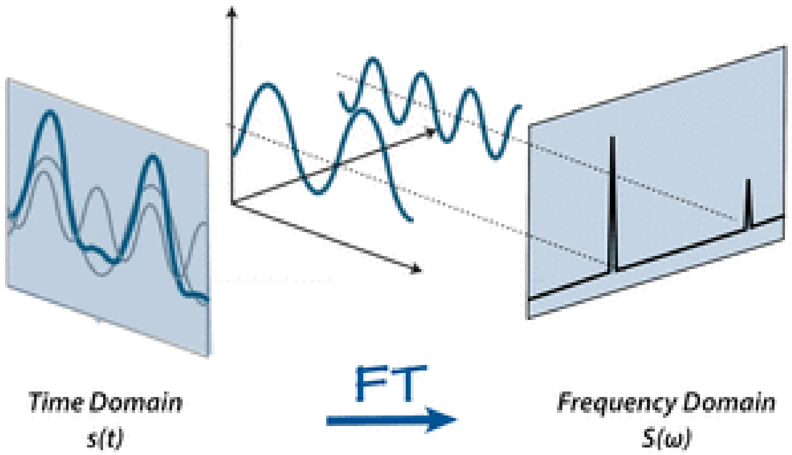
https://i.stack.imgur.com/fKznF.jpg

Fourier Transform(ed) !

http://xkeys.com/PIAboutUs/assets/images/Jacks%20Pages/FourierTransform/JeanFourier_1200.jpg

Fourier Transform

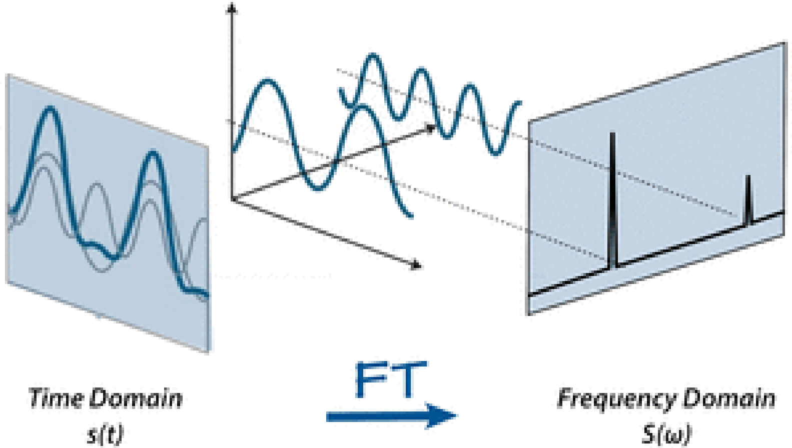
Fourier Transform



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

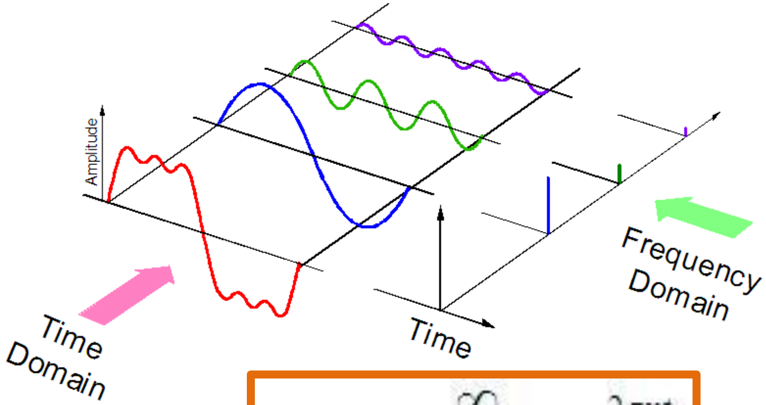
Fourier Transform vs Series

Fourier Transform



$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Fourier Series



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi nt}{T}}\,dt$$

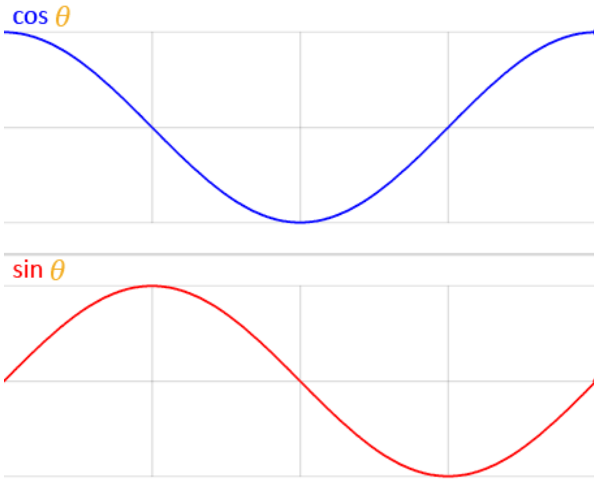
Definition: Fourier Transform

- the Fourier Transform of a function *x(t)* is defined by:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

- The result is a $\boxed{\text{function}}$ of ω (frequency).

Intuition for FT



?

ωt	$\mathrm{Re}(e^{i\omega t})$	$\mathrm{Im}(e^{i\omega t})$	$e^{i\omega t}$

$e^{i\omega t}$ is sort of like a cosine wave with frequency ω

Intuition for FT

- $\frac{64}{16}$?
- $\frac{y}{x}$?
- How much of 30Hz cosine signal in x(t) ?

How much of frequency ω in x(t) ?

How much of frequency ω in x(t) ?

Intuition for FT

- x(t) = Single number
- How much of frequency ω signal is present for all values of t ?

$$X(\omega) = \int_{t=-\infty}^{t=\infty} x(t) e^{-i\omega t} dt$$

Intuition for FT

- Fourier Transform

$$X(\omega) = \int_{t=-\infty}^{t=\infty} x(t) e^{-i\omega t} dt$$

- Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} X(\omega) e^{i\omega t} d\omega$$

https://i.stack.imgur.com/wutf4.jpg

References & Fun Reading/Viewing

- GW DIP textbook, 3rd Ed., (4.1 – 4.2)
- http://www.thefouriertransform.com/
- A visual introduction to Fourier Transform: https://www.youtube.com/watch?v=spUNpyF58BY
- Fourier Transform, Fourier Series and Frequency Spectrum: https://www.youtube.com/watch?v=r18Gi8lSkfM
- https://betterexplained.com/articles/intuitive-understanding-of-sine-waves/