27.08.2019

Digital Image Processing (CSE/ECE 478)

Lecture-9: Image Enhancement in Frequency Domain – 2D DFT

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Announcements

- Quiz on Wednesday, H-205, 9.00 – 9.45am
- Derivations and Numericals
  - Image Enhancement
  - Fourier Transform and properties
  - Basics of Integral Calculus

Announcements

- Projects
  - Your own ideas (check with me first !)
  - Groups : max 3 / group
  - No Deep Learning !

Announcements (contd.)

- Conference websites
  - Image Processing:
    - Intl. Conf. on Image Processing: ICIP
    - Intl. Conf. on Computational Photography: ICCP
    - Intl. Conf. on Computational Creativity: ICCC
  - Computer Vision
    - Winter Conf. on Applications of Computer Vision : WACV
    - British Machine Vision Conference : BMVC
- TIP: Shortlist based on title, refine later based on abstract

Fourier Transform

- Fourier Transform

$$X(\omega) = \int_{t=-\infty}^{t=\infty} x(t) e^{-i\omega t} dt$$

angular frequency

$\omega = 2\pi f$

- Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} X(\omega) e^{i\omega t} d\omega$$

Fourier Transform (G&W version)

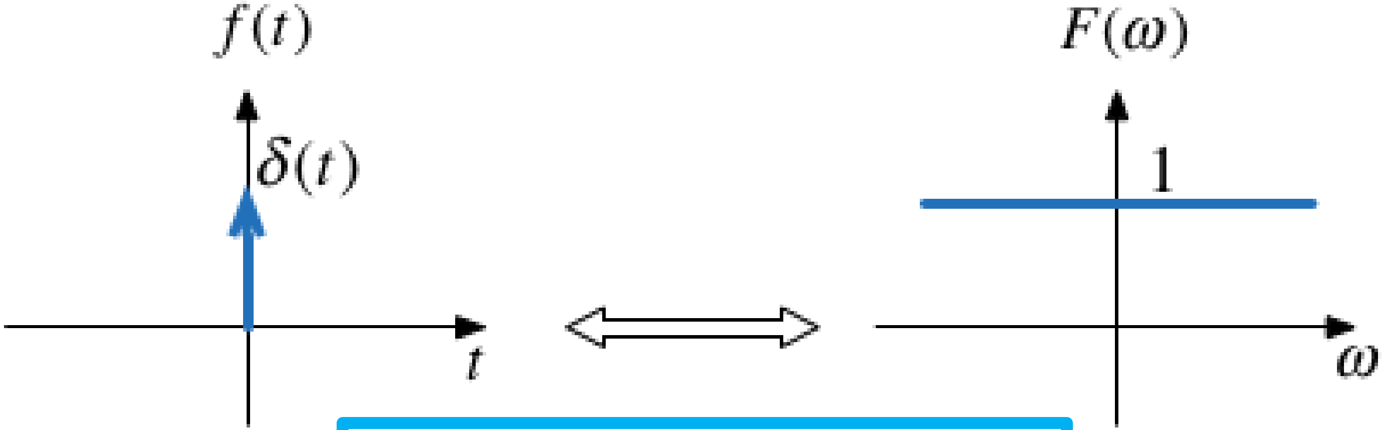
- Fourier Transform

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

- Inverse Fourier Transform

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

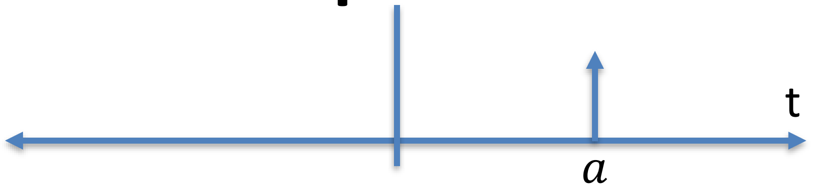
FT of impulse function



$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

FT of time-shifted impulse

$$f(t) = \delta(t - a)$$



$$\mathcal{F}[\delta(t-a)] = F(\mu) = e^{-j2\pi\mu a}$$

FT of complex exponential

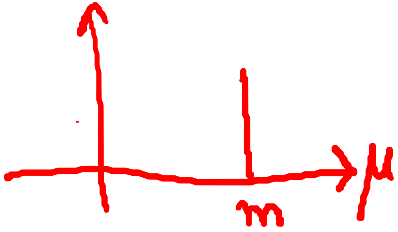
$$\delta(t-a) \longleftrightarrow e^{-j\omega a} \qquad e^{-j2\pi\mu a}$$

$$\delta(t-a) \longleftrightarrow e^{-j2\pi\underline{\mu} a}$$

$$e^{-j2\pi t a} \longleftrightarrow \delta(-\mu-a)$$

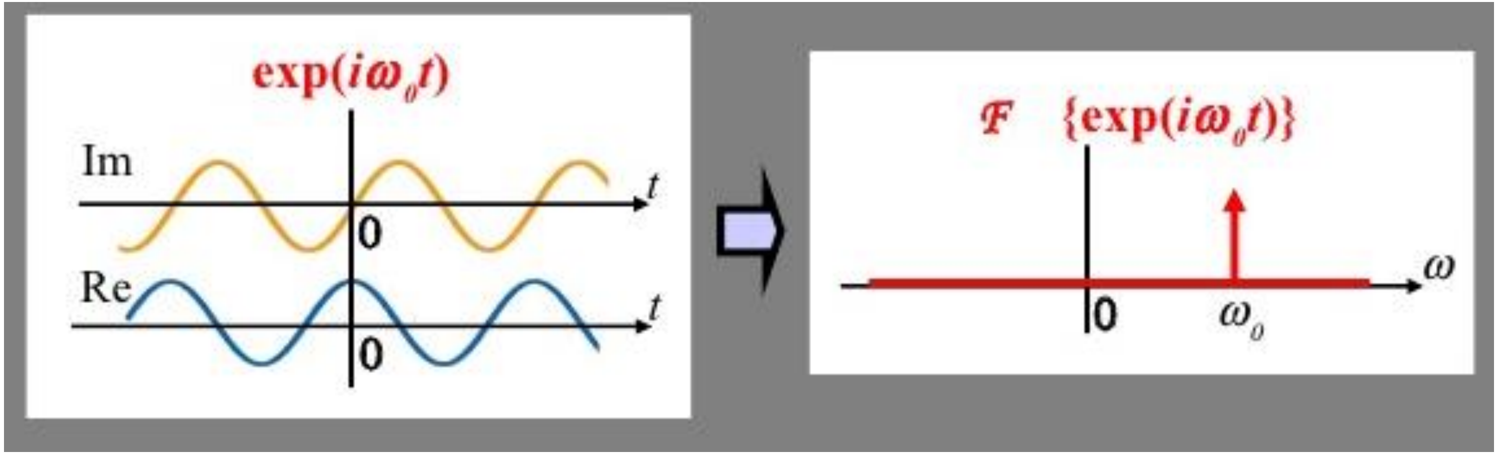
$$e^{j2\pi t m} \longleftrightarrow \delta(-\mu+m)$$

$$\delta(\mu-m)$$



FT of a complex exponential

- $x(t) = e^{\,i\omega_0 t}$

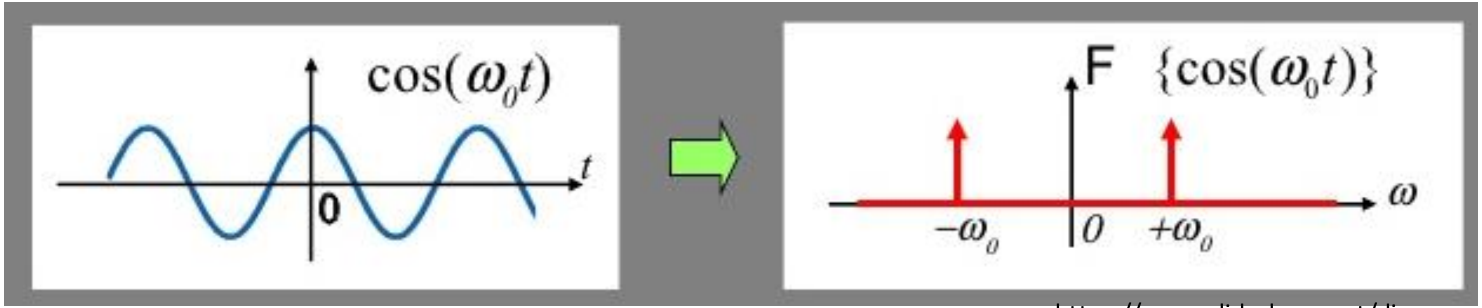


https://www.slideshare.net/diarmseven/optics-fourier-transform-ii

FT of a cosine signal

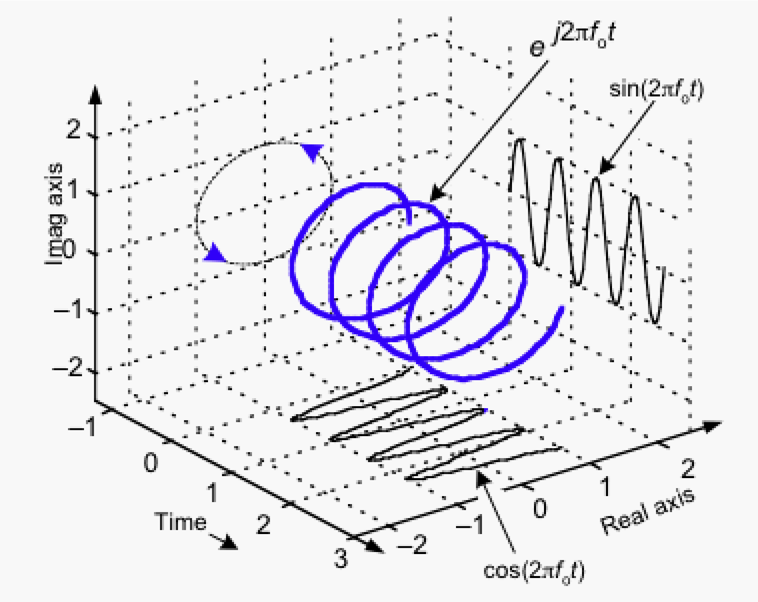
- $x(t) = \cos(\omega_0 t)$

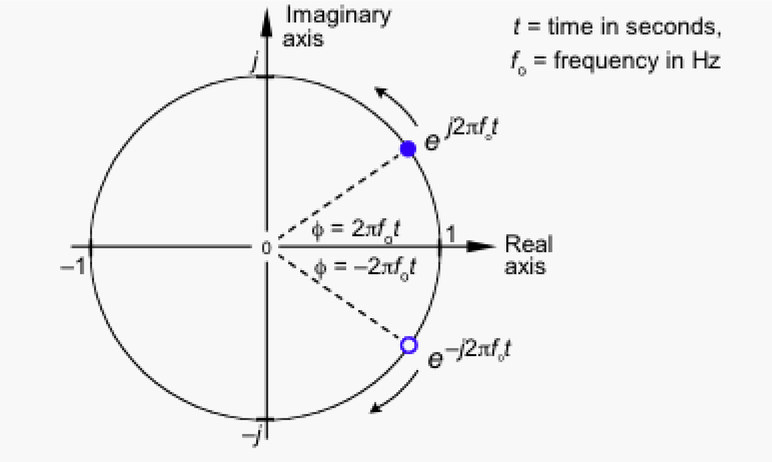
$$\Im\left\{e^{i2\pi at}\right\} = \delta(f - a)$$



https://www.slideshare.net/diarmseven/optics-fourier-transform-ii

“Negative” frequencies





https://dsp.stackexchange.com/questions/431/what-is-the-physical-significance-of-negative-frequencies

https://electronics.stackexchange.com/questions/15539/negative-frequencies-what-is-that

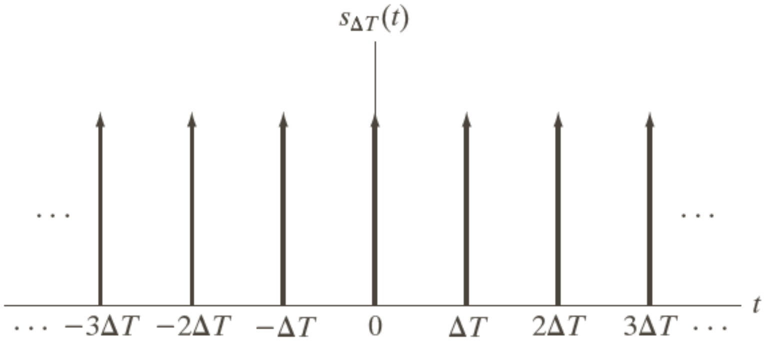
FT of a periodic function

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j2\pi nt}{\Delta T}}$$

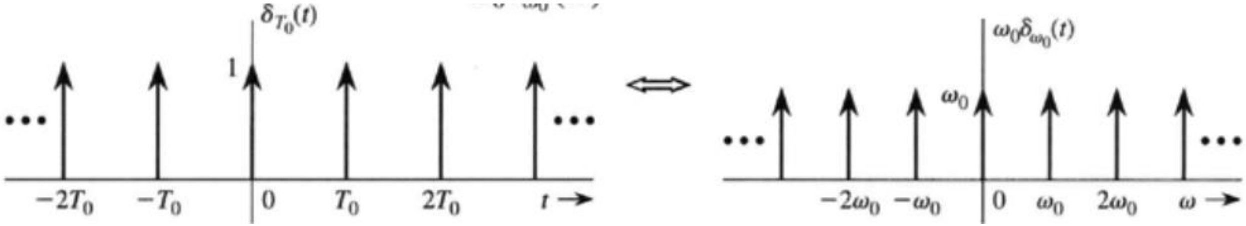
$$c_n = \frac{1}{\Delta T} \int_{-\frac{\Delta T}{2}}^{\frac{\Delta T}{2}} s_{\Delta T}(t) e^{\frac{-j2\pi nt}{\Delta T}} dt$$

$$\mathcal{F}(s_{\Delta T}(t)) = \sum_{n=-\infty}^{n=\infty} c_n \delta(\mu - \frac{n}{\Delta T})$$

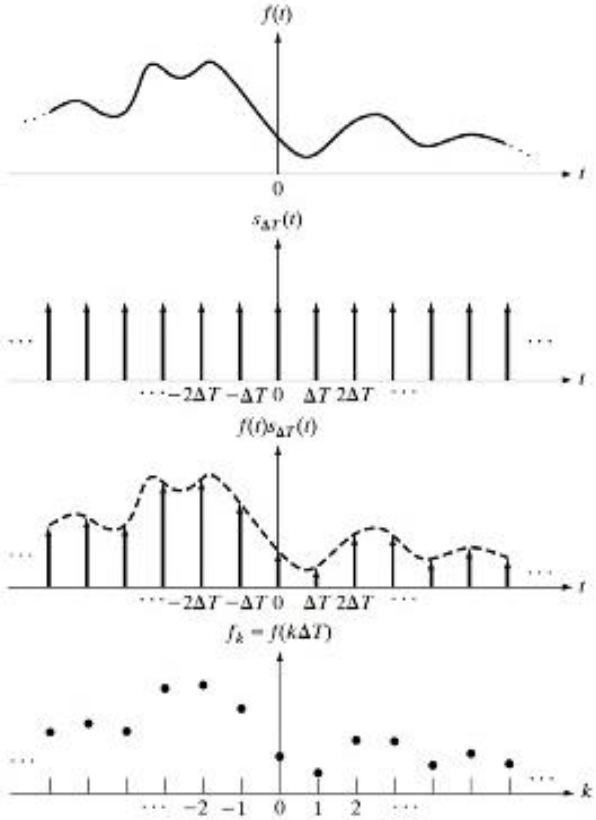
FT of impulse train



$$\mathcal{F}(s_{\Delta T}(t)) = \frac{1}{\Delta T} \sum_{n=-\infty}^{n=\infty} \delta(\mu - \frac{n}{\Delta T})$$



Sampling = f(t) x Impulse Train



$$\tilde{f}(t) = f(t)\, s_{\Delta T}(t)$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{f(t)\,\delta(t - n\Delta T)}_{f_n}$$

FT of sampled function

$$\tilde{F}(\mu) = \mathfrak{F}(\tilde{f}(t)) = \mathfrak{F}(\underline{f(t)}\,\underline{s_{\Delta T}(t)})$$

$$= F(\mu) * S(\mu)$$

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,f(t-\tau)\,d\tau$$

- Fourier Transform of a sampled function is an infinite periodic sequence of copies of the transform of the original continuous function

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

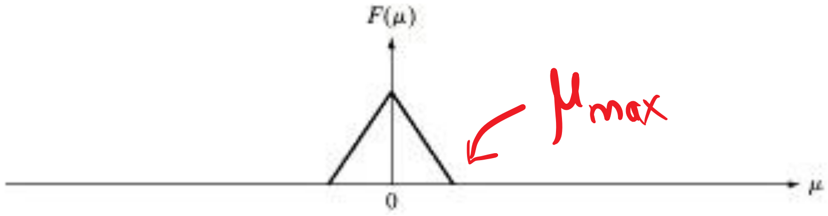
Transform of
original (continuous) function

FT of sampled function

- Periodic (copies of f(t)’s FT)
- NOTE: FT is continuous

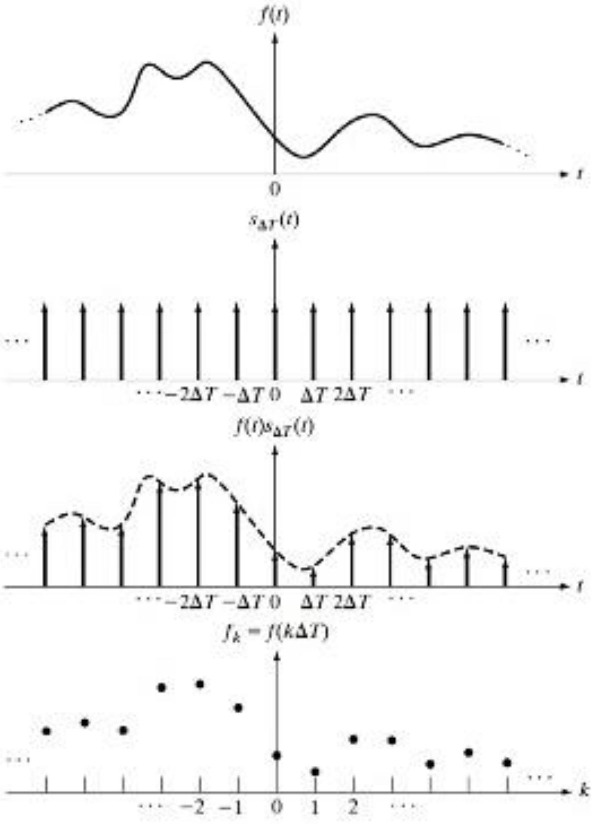
FT of sampled signal

- Characterizing one period is enough



- How do we actually get the frequency 'samples' ?

Discrete Fourier Transform

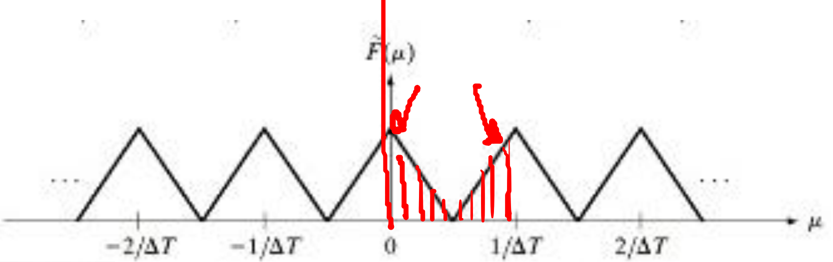


$$F(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt$$

$$F(\mu) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

$$F(\mu) = \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{n=\infty} f(n\Delta T) e^{-j2\pi\mu n\Delta T}$$



- Substituting

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

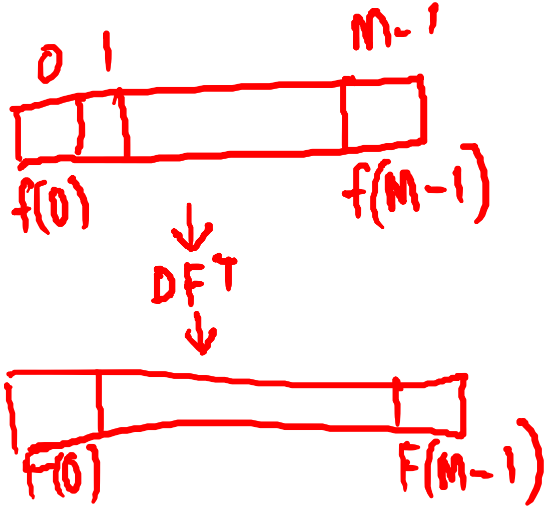
$$\Delta u\,(M-1) = \frac{1}{\Delta T}$$

- Into $\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n\, e^{-i2\pi\mu n\Delta T}$

- yields

$$\boxed{F_m = \sum_{n=0}^{M-1} f_n\, e^{-i2\pi mn/M}}$$

$$m = 0, 1, 2, \ldots, M-1$$



https://slideplayer.com/slide/10719747/

DFT and IDFT

DFT

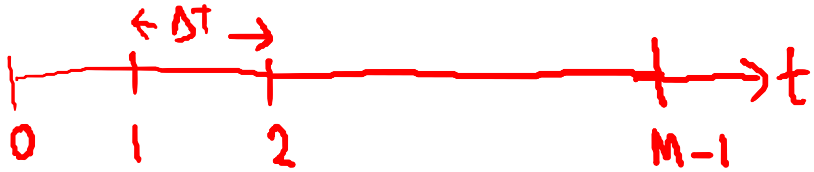
$$F[u] = \sum_{x=0}^{M-1} f[x] e^{-2\pi j x u/M}$$

NOTE: No direct dependence on ΔT

IDFT

$$f[x] = \frac{1}{M} \sum_{u=0} F(u) e^{2\pi j u x/M}$$

Relationship between Sampling and Frequency Intervals



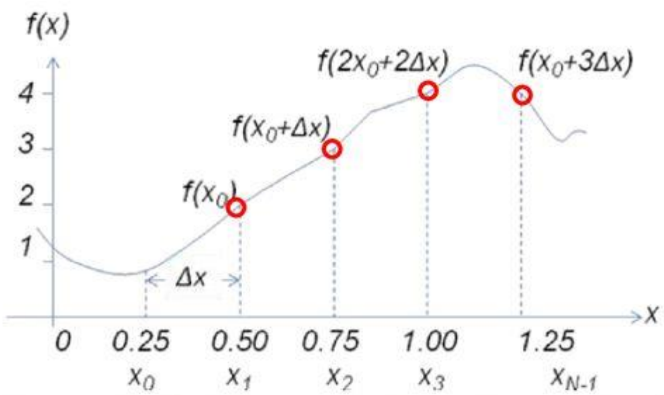
$$T = (M-1)\Delta T$$

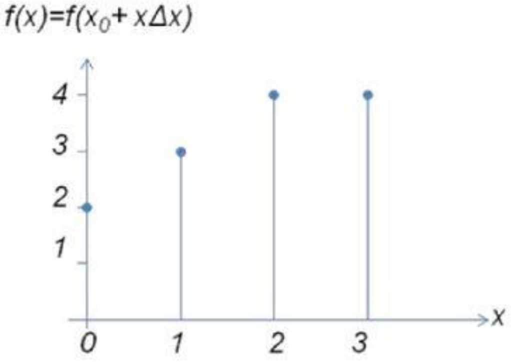
$$\frac{1}{\Delta T} = (M-1)\Delta u$$

- Ω (Range of frequencies) depends inversely on sampling interval ΔT
- Δu (Frequency Resolution of DFT) depends inversely on duration T over which $f(t)$ is sampled

1D DFT by example

Implementation of a Fourier Transform of a 4 samples signal





1D DFT by example

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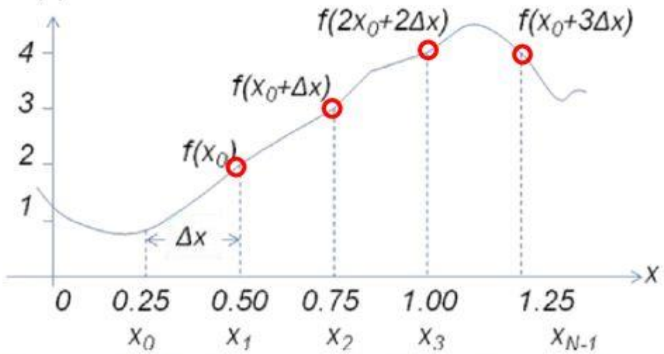

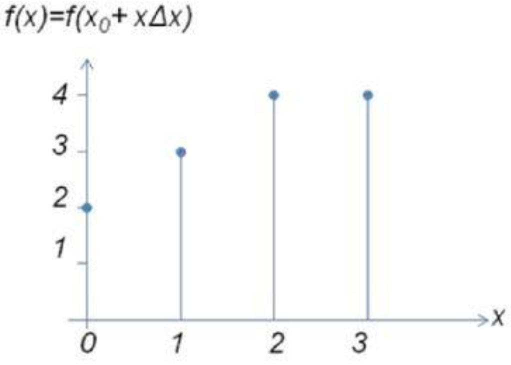

We sample the signal on the left and we end up with the four values on the right:

Signal in Time Domain (integer numbers)			Signal in Frequency Domain (complex numbers)		Power Spectrum in Frequency domain (float numbers)	
f(0)	2	FT	F(0)	3.25	$\lvert F(0)\rvert$	3.25
f(1)	3	->	F(1)	$\frac{1}{4}[-2+j]$	$\lvert F(1)\rvert$	$\sqrt{5}/4 = 0.56$
f(2)	4	iFT	F(2)	$-\frac{1}{4}[1+j0]$	$\lvert F(2)\rvert$	$1/4 = 0.25$
f(3)	4	<-	F(3)	$-\frac{1}{4}[2+j]$	$\lvert F(3)\rvert$	$\sqrt{5}/4 = 0.56$

1D DFT by example

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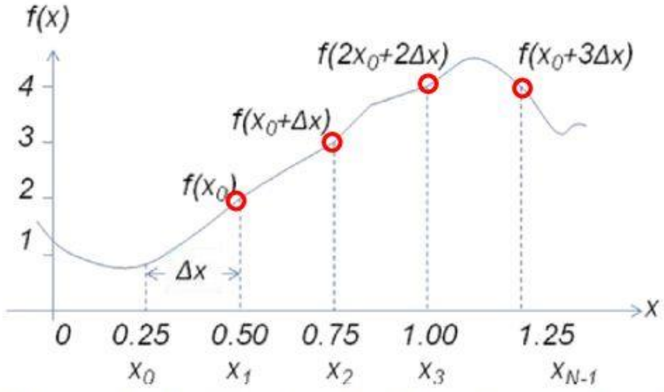

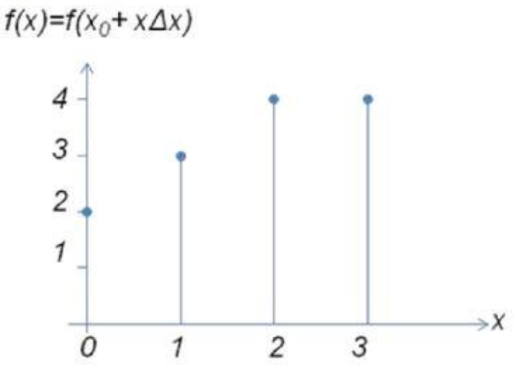

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f(3)	4	<-	F(3)	$-\frac{1}{4}$[2+j]	$\lvert F(3) \rvert$	$\sqrt{5}/4 = 0.56$

Calculations: $F(0) = \frac{1}{4}\sum_{x=0}^{3} f(x) expo[0] = \frac{1}{4}[f(0)+f(1)+f(2)+f(3)] = \frac{1}{4}[2+3+4+4] = 3.25$

$F(1) = \frac{1}{4}\sum_{x=0}^{3} f(x) expo[-j2\pi x/4] = \frac{1}{4}[2e^{0} + 3e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2}] = \frac{1}{4}[-2+j]$ and so on.

Discrete Fourier Transform (DFT) – 1D

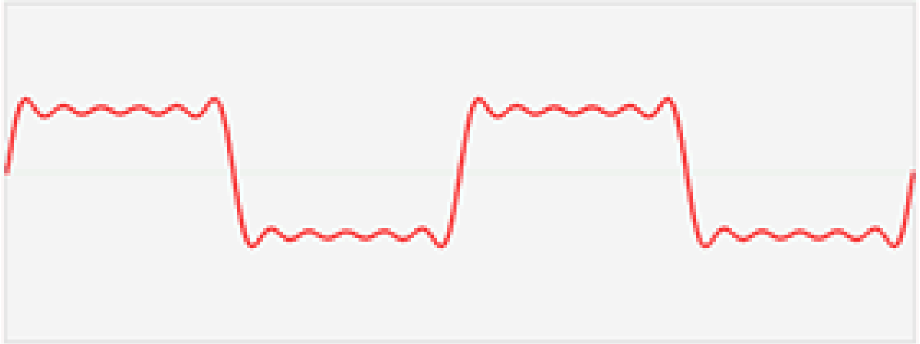
DFT

$$F[u] = \frac{1}{M} \sum_{x=0}^{M-1} f[x] e^{-j2\pi ux/M}$$

Inverse DFT

$$f[x] = \sum_{x=0}^{M-1} F[u] e^{j2\pi ux/M}$$

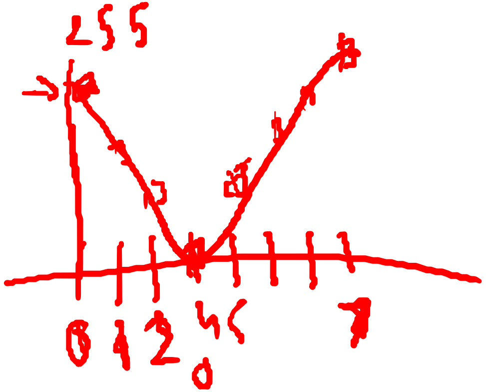
Fourier Analysis



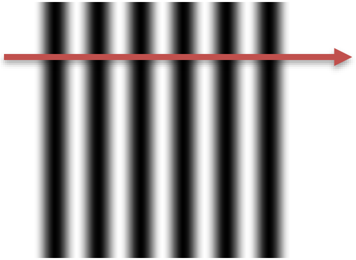
$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}}$$

Digital Images

- Frequency: Amount of x Hz signal in the image
  - For images ?

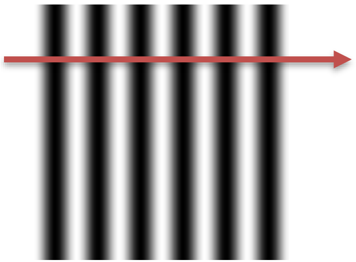


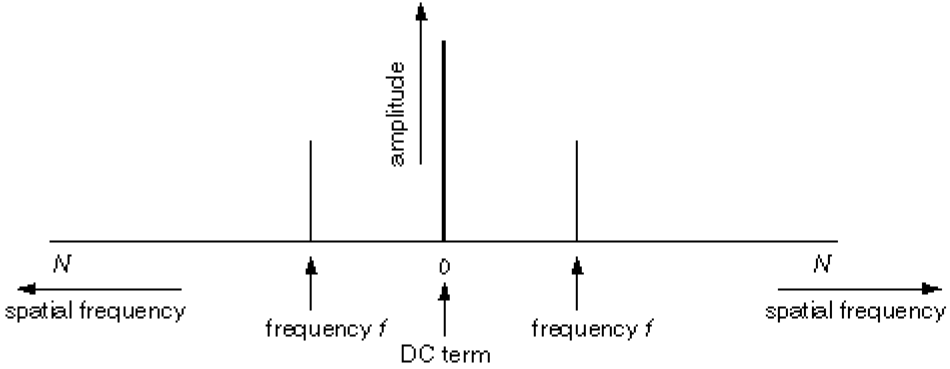
DFT Example



Courtesy: http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

DFT Example





Courtesy: http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

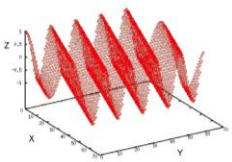
2D DFT

$$F(u,v) = \sum_{y=0}^{N-1} \sum_{x=0}^{M-1} f(x,y)\, e^{-2\pi j \left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

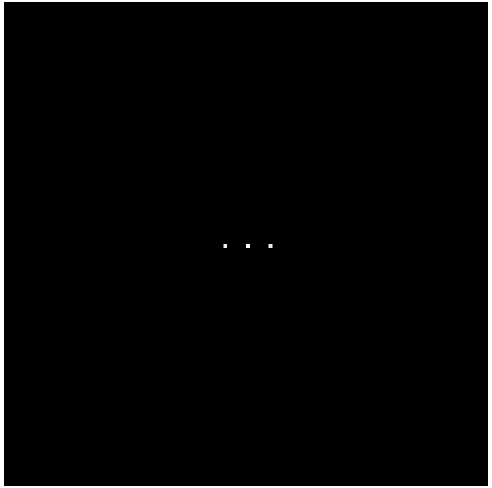
$$f(x,y) = \frac{1}{MN} \sum_{v=0}^{N-1} \sum_{u=0}^{M-1} F(u,v)\, e^{2\pi j \left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

DFT for simple spatial patterns

Brightness Image

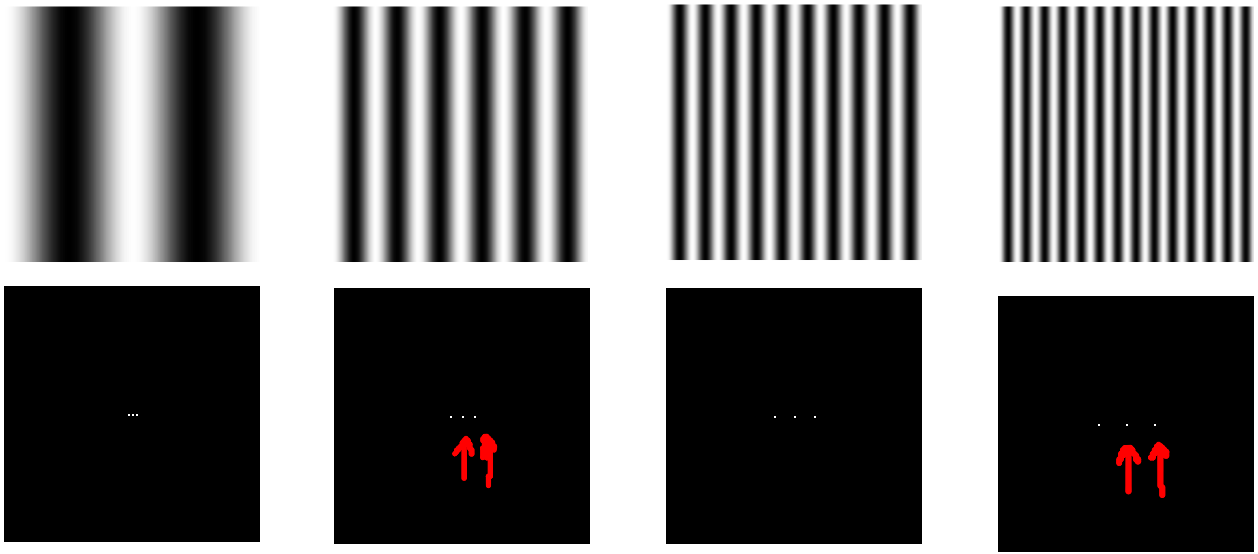


Fourier transform



http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

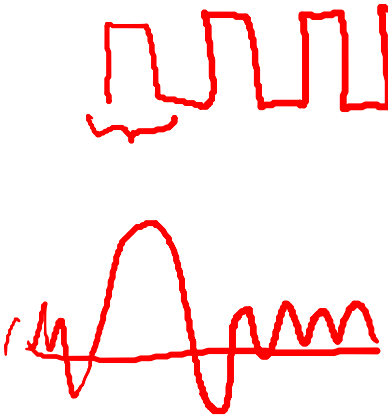
DFT Example

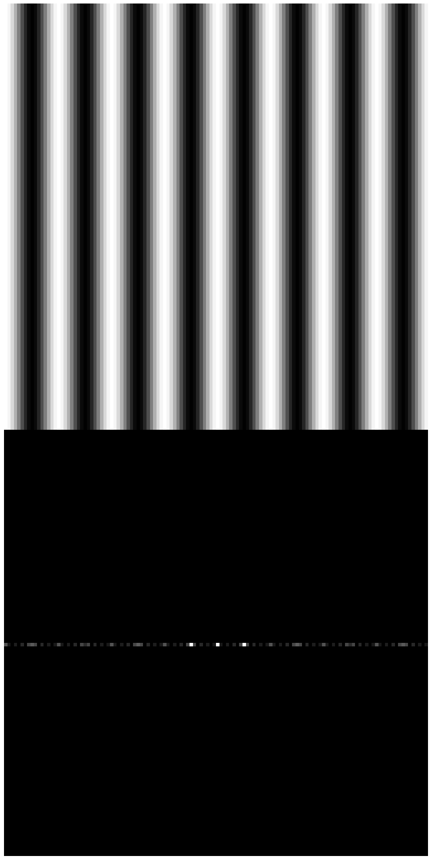


Courtesy: http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

https://www.cs.unm.edu/~brayer/vision/fourier.html

DFT for simple ‘spatial’ patterns



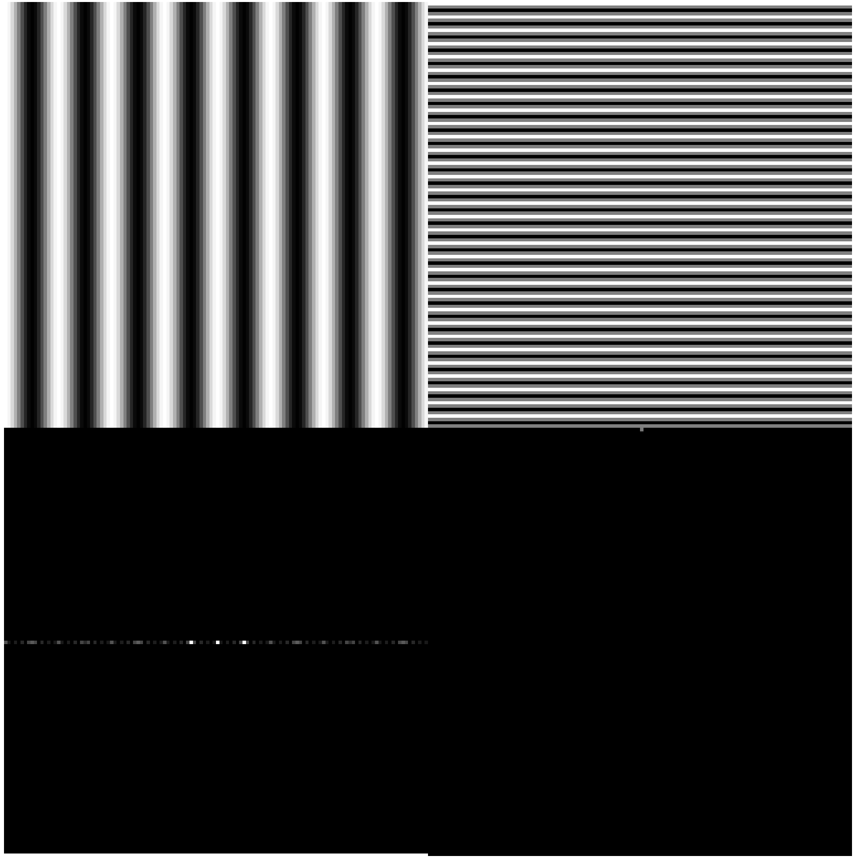


$$F[u] = \frac{1}{M}\sum_{x=0}^{M-1} f[x]e^{-j2\pi ux/M}$$

- Center (DC)
  - Average. Why ?
- Frequency ?
- Cosine v/s step
- Compact !

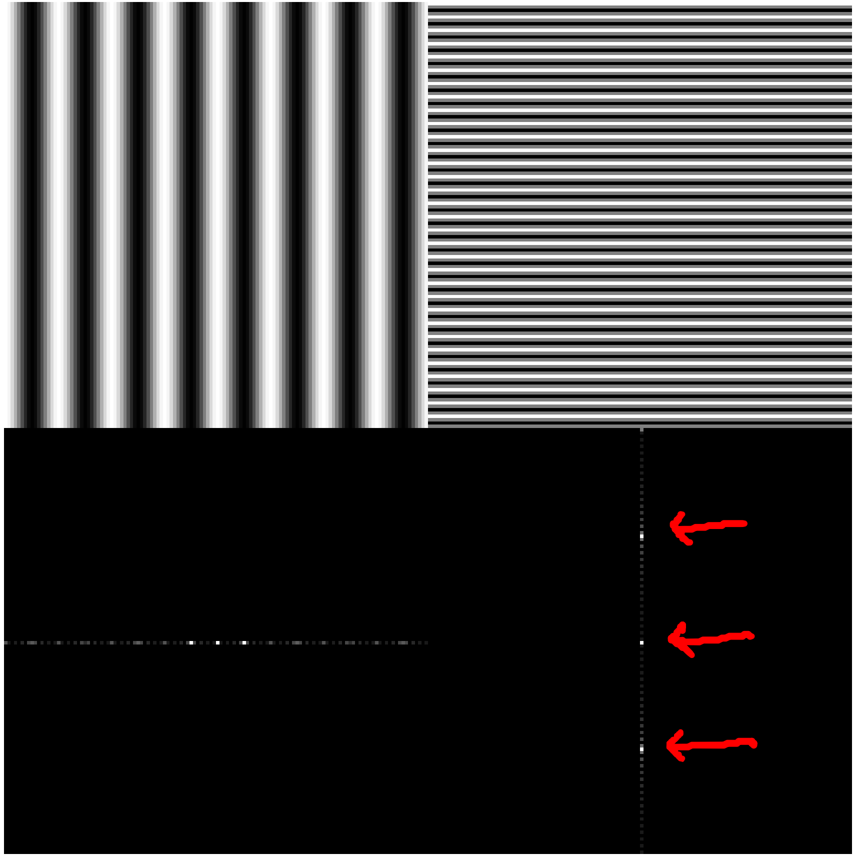
https://www.cs.unm.edu/~brayer/vision/fourier.html

DFT for simple 'spatial' patterns

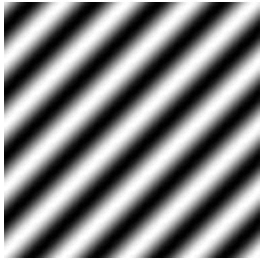


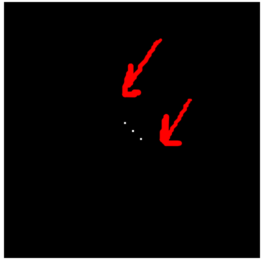
https://www.cs.unm.edu/~brayer/vision/fourier.html

DFT for simple ‘spatial’ patterns



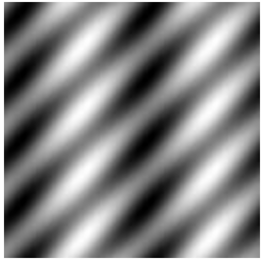
DFT Example (Rotation)

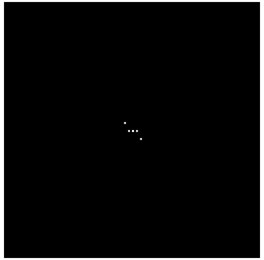




Courtesy: http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

DFT Example (Sum of Signals)





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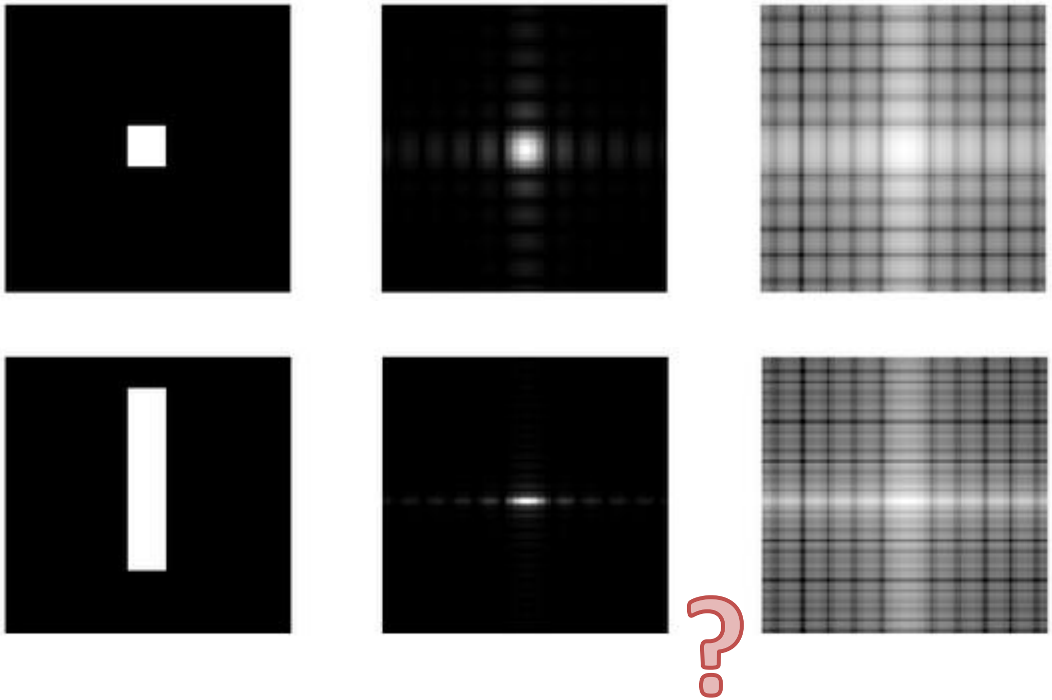
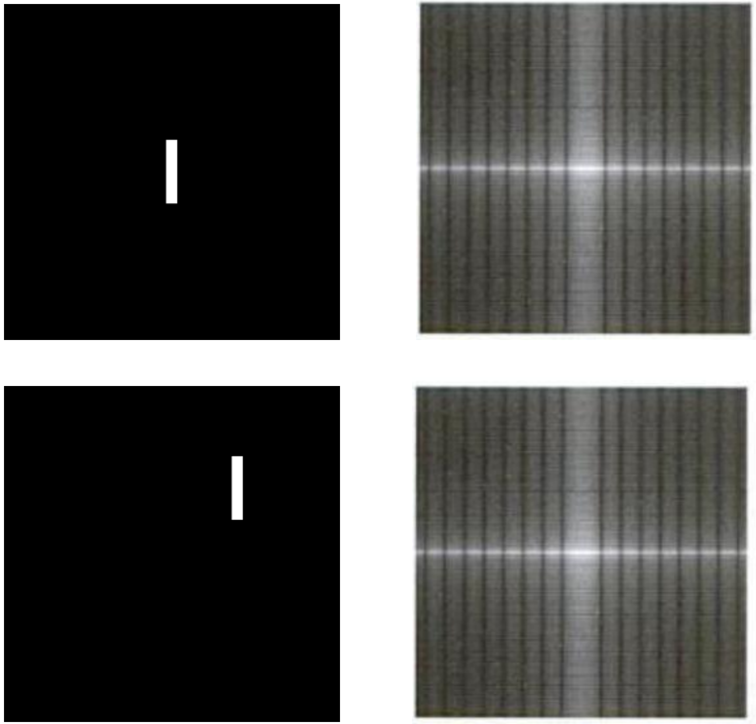
DFT Example (Log Transformation)



DFT Example (Translation - Magnitude)



Important Terms

- Magnitude spectrum

$$|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$$

- Phase Spectrum

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

- Power Spectrum

$$P(u) = |F(u)|^2$$

Discrete Fourier Transform

1-D, Discrete Case:

Fourier Transform: $$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \quad u = 0,\ldots,\text{M-1}$$

Inverse Fourier Transform: $$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \quad x = 0,\ldots,\text{M-1}$$

Discrete Fourier Transform

1-D, Discrete Case:

Fourier Transform: $$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad u = 0,\ldots,\text{M-1}$$

Inverse Fourier Transform: $$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \qquad x = 0,\ldots,\text{M-1}$$

F(*u*) can be written as:

$$F(u) = R(u) + jI(u)$$

Polar coordinate:

$$F(u) = |F(u)|e^{-j\phi(u)}$$

where

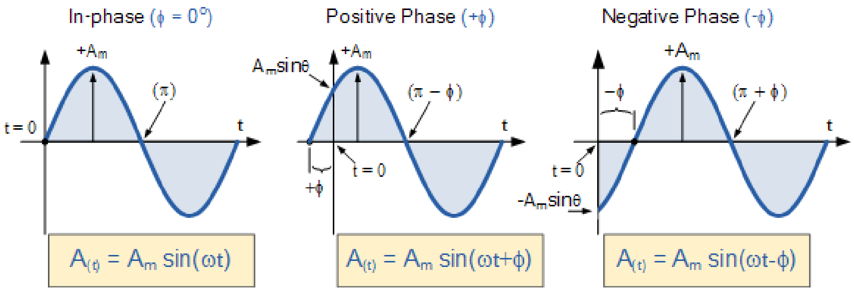
$$|F(u)| = \sqrt{R(u)^2 + I(u)^2}$$

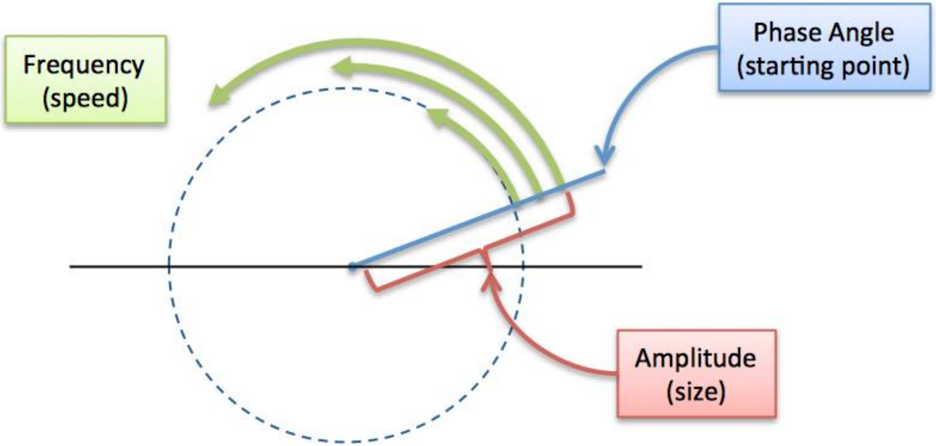
magnitude

$$\phi(u) = \tan^{-1}\left(\frac{I(u)}{R(u)}\right)$$

phase

Phase





https://www.electronics-tutorials.ws/accircuits/phase-difference.html

http://betterexplained.com/wp-content/uploads/images//circular-path-parameters-20121201-203317.png

Magnitude and Phase Spectra

Figure 4a

Original

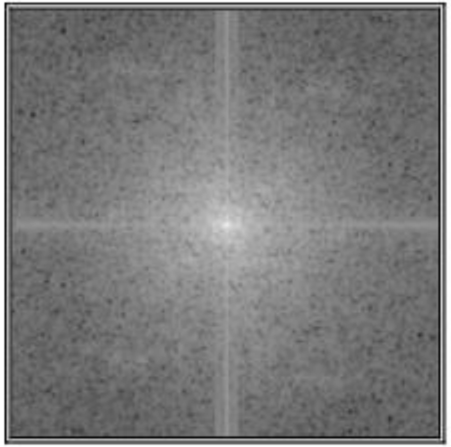


Figure 4b

$\log(|A(\Omega,\Psi)|)$

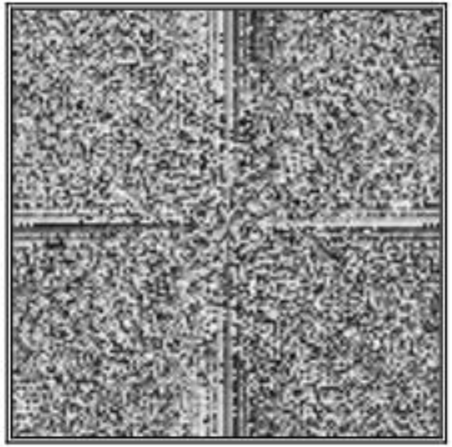


Figure 4c

$\varphi(\Omega,\Psi)$

https://i.imgur.com/YidmGcQ.png

Magnitude and Phase Spectra
Both matter for reconstruction

Figure 4a

Original

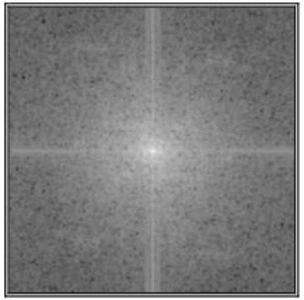


Figure 4b

$\log(|A(\Omega,\Psi)|)$

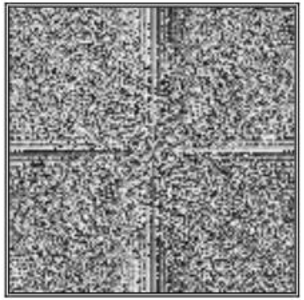


Figure 4c

$\varphi(\Omega,\Psi)$

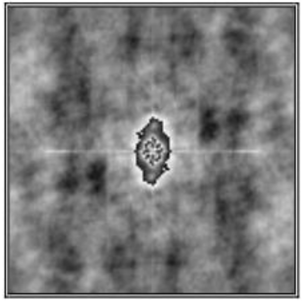


Figure 5a

$\varphi(\Omega,\Psi) = 0$

Figure 5b

$|A(\Omega,\Psi)| = \textit{constant}$

https://i.imgur.com/YidmGcQ.png

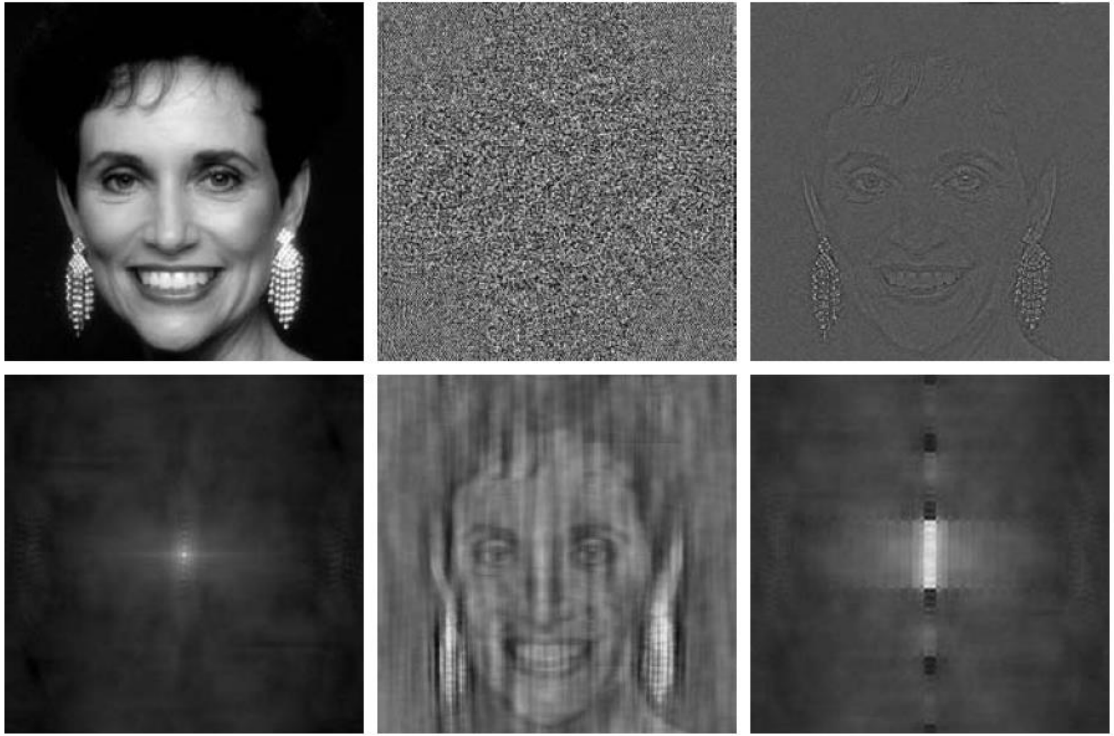


FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.