23.08.2019

# Digital Image Processing (CSE/ECE 478)

# Lecture-8: Image Enhancement in Frequency Domain – FT of sampled functions, DFT

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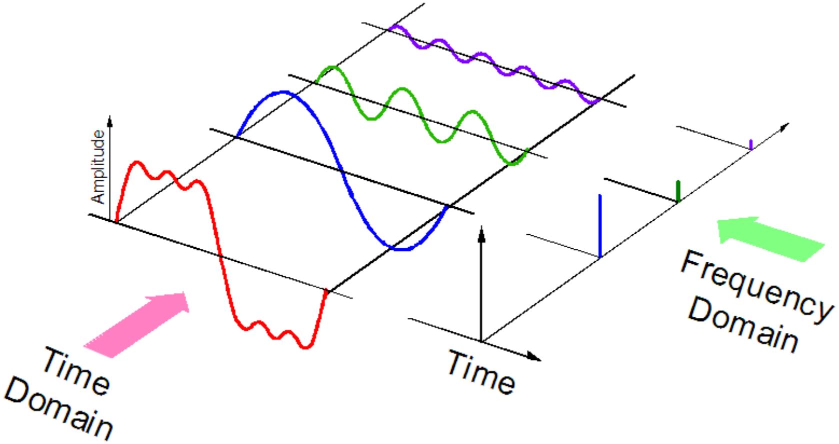
# Announcements

- Tutorial : Fourier Series and Transforms
- Quiz-1 : Wednesday, 9.00 am – 9.45 am, Room: H-205
  - Syllabus: Up to today's lecture
  - Format: **Derivations and Numericals**

# Fourier Series

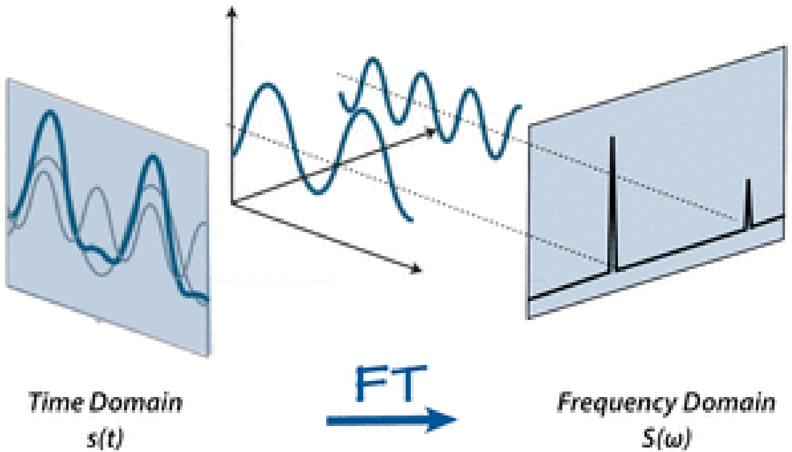
$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-i\frac{2\pi nt}{T}} dt$$



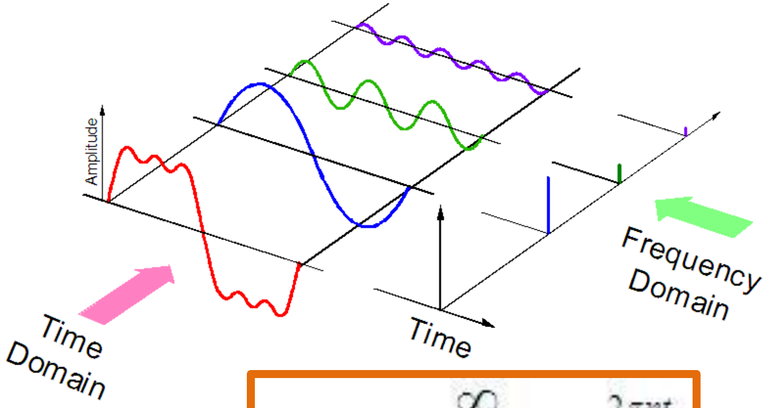
# Fourier Transform vs Series

## Fourier Transform




$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

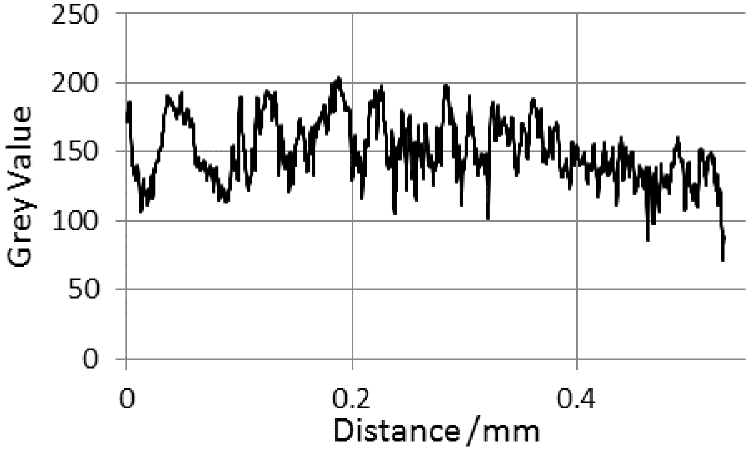
## Fourier Series

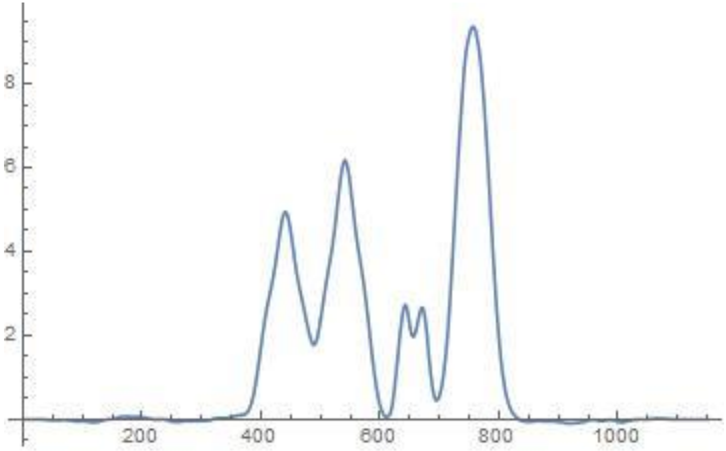



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-i\frac{2\pi nt}{T}} dt$$

# What if x(t) is non-periodic ?




$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

https://i.stack.imgur.com/fKznF.jpg

# Fourier Transform

- Fourier Transform

$$X(\omega) = \int_{t=-\infty}^{t=\infty} x(t) e^{-i\omega t} dt$$

angular frequency

$\omega = 2\pi f$

- Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} X(\omega) e^{i\omega t} d\omega$$

# Fourier Transform (G&W version)

- Fourier Transform

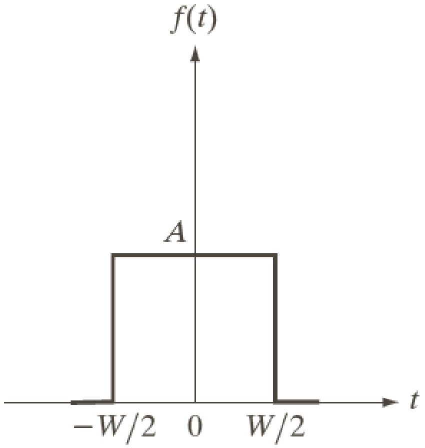
$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

- Inverse Fourier Transform

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

https://i.stack.imgur.com/wutf4.jpg

# FT of box function




$$f(t) = \begin{cases} A & \text{if } -\frac{W}{2} \le t \le \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\cancel{2}A \sin(\pi u w)}{\cancel{2}\pi u}$$

# FT of box function

$$AW\frac{sin(\pi\mu W)}{\pi\mu W}$$

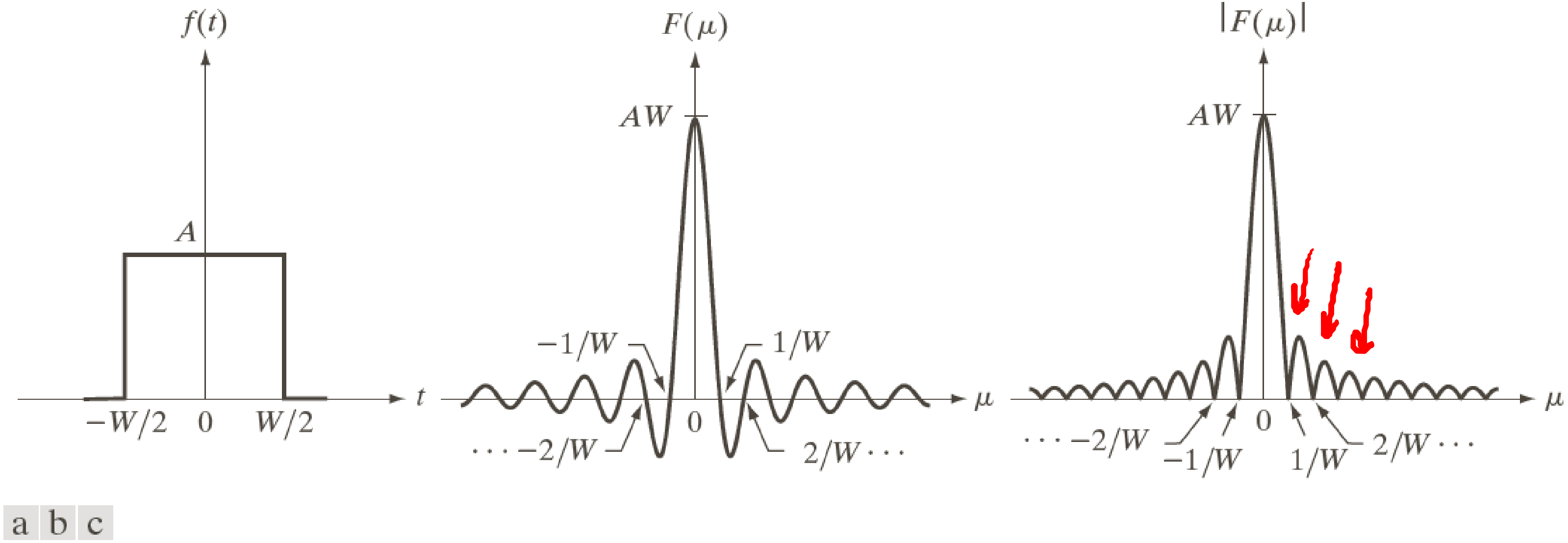



**FIGURE 4.4** (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

# (Some) Properties of FT

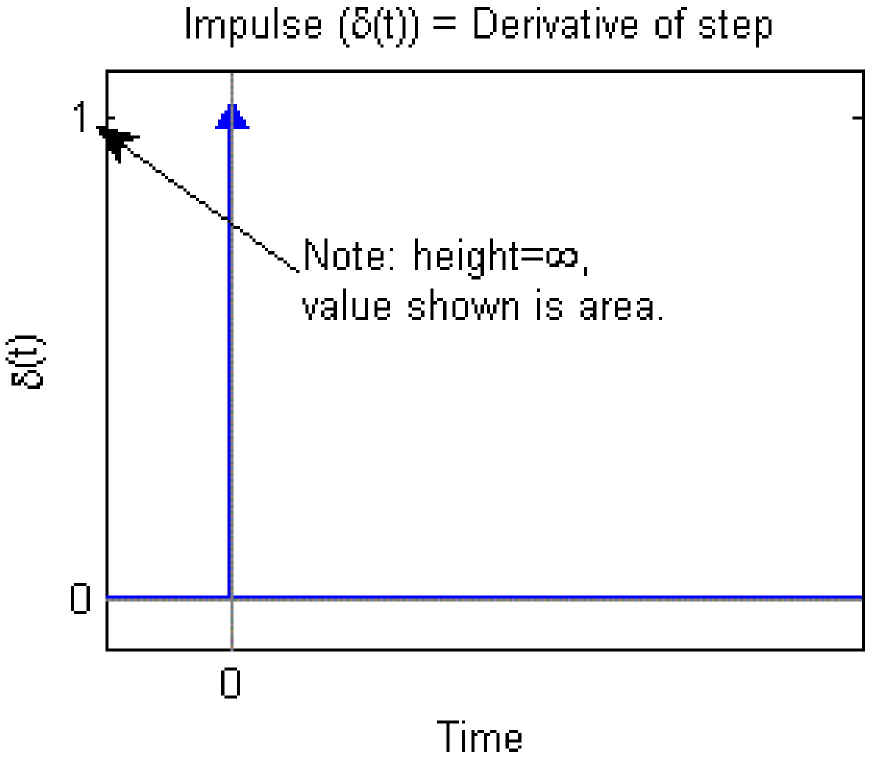
| | Name: | Condition: | Property: |
|---|---|---|---|
| 1 | Amplitude scaling | $f(t) \leftrightarrow F(\omega)$, constant $K$ | $Kf(t) \leftrightarrow KF(\omega)$ |
| 2 | Addition | $f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$, $\cdots$ | $f(t)+g(t)+\cdots \leftrightarrow F(\omega)+G(\omega)+\cdots$ |
| 3 | Hermitian | Real $f(t) \leftrightarrow F(\omega)$ | $F(-\omega) = F^*(\omega)$ |
| 4 | Even | Real and even $f(t)$ | Real and even $F(\omega)$ |
| 5 | Odd | Real and odd $f(t)$ | Imaginary and odd $F(\omega)$ |
| 6 | Symmetry | $f(t) \leftrightarrow F(\omega)$ | $F(t) \leftrightarrow 2\pi f(-\omega)$ |
| 7 | Time scaling | $f(t) \leftrightarrow F(\omega)$ , real $s$ | $f(st) \leftrightarrow \frac{1}{\lvert s\rvert}F(\frac{\omega}{s})$ |
| 8 | Time shift | $f(t) \leftrightarrow F(\omega)$ | $f(t-t_o) \leftrightarrow F(\omega)e^{-j\omega t_o}$ |
| 9 | Frequency shift | $f(t) \leftrightarrow F(\omega)$ | $f(t)e^{j\omega_o t} \leftrightarrow F(\omega-\omega_o)$ |
| 10 | Modulation | $f(t) \leftrightarrow F(\omega)$ | $f(t)\cos(\omega_o t) \leftrightarrow \frac{1}{2}F(\omega-\omega_o)+\frac{1}{2}F(\omega+\omega_o)$ |
| 11 | Time derivative | Differentiable $f(t) \leftrightarrow F(\omega)$ | $\frac{df}{dt} \leftrightarrow j\omega F(\omega)$ |
| 12 | Freq derivative | $f(t) \leftrightarrow F(\omega)$ | $-jtf(t) \leftrightarrow \frac{d}{d\omega}F(\omega)$ |
| 13 | Time convolution | $f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$ | $f(t)*g(t) \leftrightarrow F(\omega)G(\omega)$ |
| 14 | Freq convolution | $f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$ | $f(t)g(t) \leftrightarrow \frac{1}{2\pi}F(\omega)*G(\omega)$ |
| 15 | Compact form | Real $f(t)$ | $f(t) = \frac{1}{2\pi}\int_0^\infty 2\lvert F(\omega)\rvert\cos(\omega t+\angle F(\omega))d\omega$ |
| 16 | Parseval, Energy W | $f(t) \leftrightarrow F(\omega)$ | $W \equiv \int_{-\infty}^\infty \lvert f(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty \lvert F(\omega)\rvert^2 d\omega$ |

$f(x)=f(-x)$

$f(x)=-f(x)$

https://hkn.illinois.edu/wiki/_media/sidebar:screen_shot_2016-04-11_at_12.35.36_pm.png?w=800&tok=1f0b2e

# Unit Impulse Function



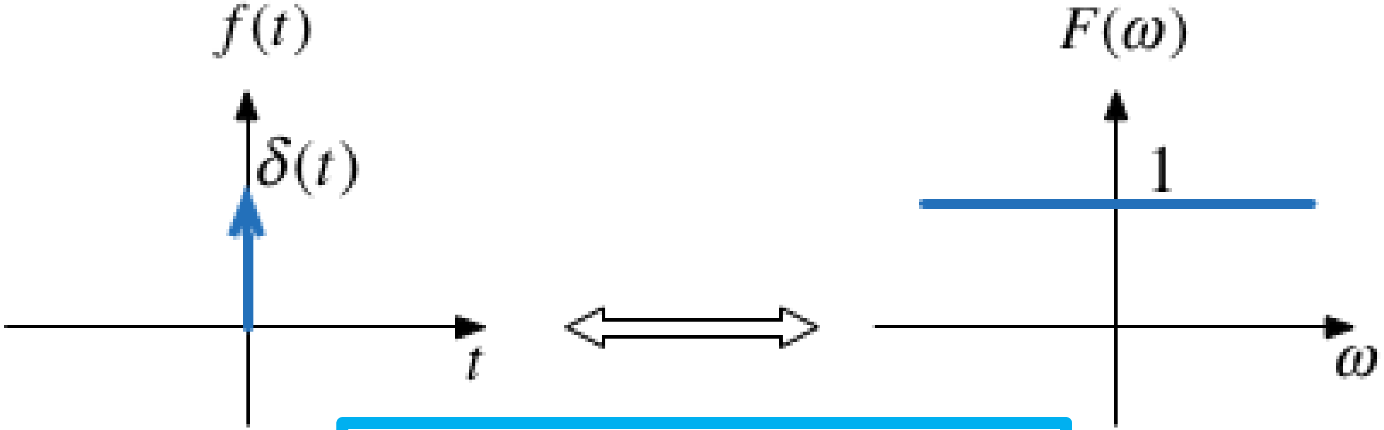

$$\delta(t) = 0, \; for \;\; t \neq 0.$$

$$\delta(0) = +\infty$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$

http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

# FT of impulse function




$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

# Impulse: Sifting Property

$$\int_a^b \delta(t)\,dt = \;?\quad \begin{array}{c} a < 0 < b \\ \text{otherwise} \end{array}$$

http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

# Impulse: Sifting Property

$$\int_a^b \delta(t)dt = \begin{cases} 1, & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$$

?

$$\int_a^b \delta(t) \cdot f(t)dt$$

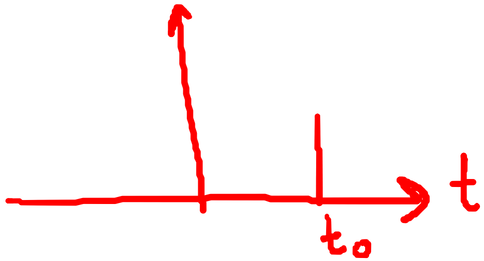
$a < 0 < b$

otherwise

http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

# Time-shifted Impulse: Sifting Property

Time-shifted impulse $\delta(t - t_0)$




http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

# Time-shifted Impulse: Sifting Property

Time-shifted impulse

$$\int_a^b \delta(t-T)\cdot f(t)dt = \; ? \quad \begin{matrix} a < T < b \\ \text{otherwise} \end{matrix}$$

$$\begin{aligned} \int_a^b \delta(t)\cdot f(t)dt &= \int_a^b \delta(t)\cdot f(0)dt \\ &= f(0)\cdot \int_a^b \delta(t)dt \\ &= \begin{cases} f(0), & a < 0 < b \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

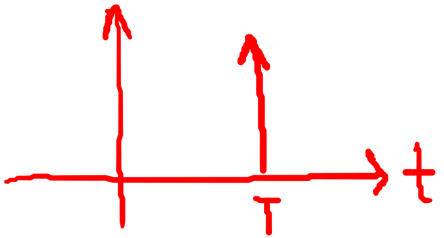
http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

# FT of impulse function

$$\int_a^b \delta(t)\cdot f(t)dt = \int_a^b \delta(t)\cdot f(0)dt$$

$$= f(0)\cdot\int_a^b \delta(t)dt$$

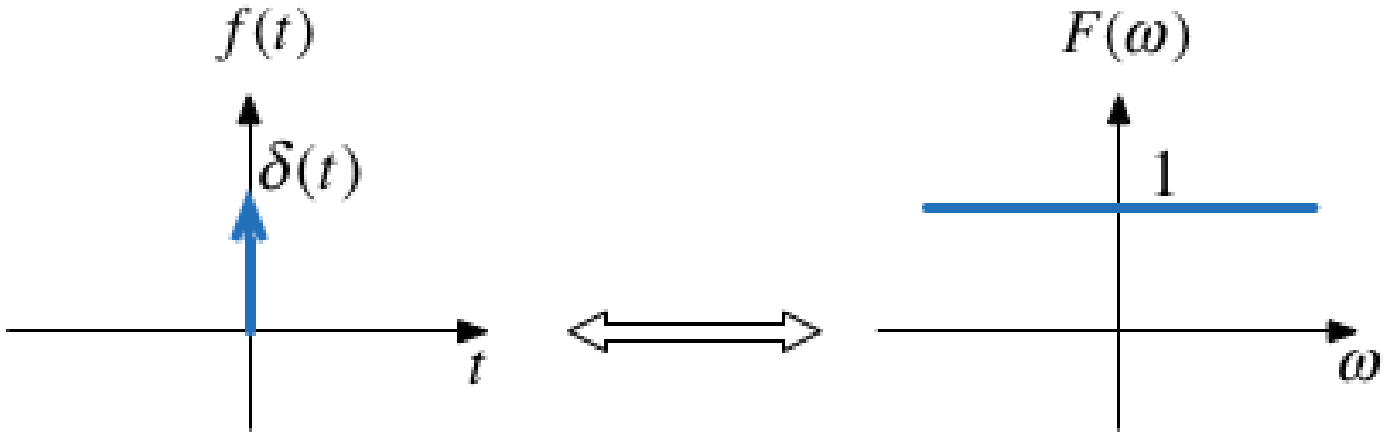
$$= \begin{cases} f(0), & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$$




$$f(t) = \delta(t-T)$$

$$X(\omega) = e^{-j\omega T}$$

# FT of impulse function



# Time-shifted Impulse

Time-shifted impulse

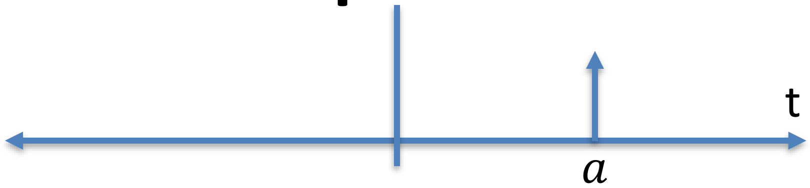
Sifting Property

$$\int_a^b \delta(t - T) \cdot f(t)dt = \begin{cases} f(T), & a < T < b \\ 0, & \text{otherwise} \end{cases}$$

http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

# FT of time-shifted impulse

$$f(t) = \delta(t - a)$$




$$\mathcal{F}[\delta(t - a)] = F(\mu) = e^{-j2\pi\mu a}$$

# Symmetry property of FT

- $f(t) \leftrightarrow F(\mu)$
- $F(t) \leftrightarrow f(-\mu)$

# FT of complex exponential

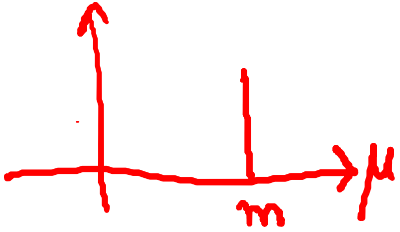
$$\delta(t-a) \longleftrightarrow e^{-j\omega a} \qquad e^{-j2\pi\mu a}$$

$$\delta(t-a) \longleftrightarrow e^{-j2\pi\underline{\mu} a}$$

$$e^{-j2\pi t a} \longleftrightarrow \delta(-\mu-a)$$

$$e^{j2\pi t m} \longleftrightarrow \delta(-\mu+m)$$

$$\delta(\mu-m)$$



# FT of a periodic function

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j2\pi nt}{\Delta T}}$$

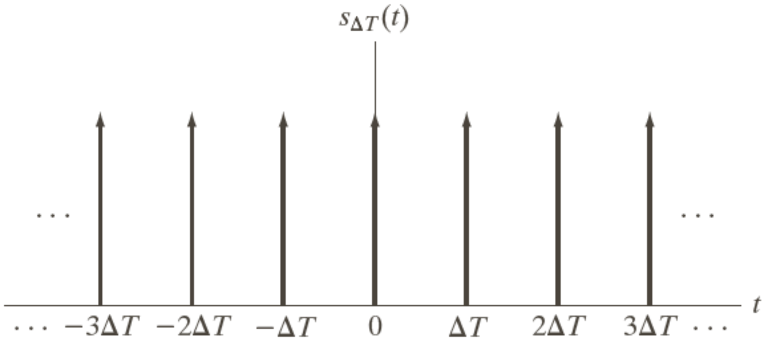
$$c_n = \frac{1}{\Delta T} \int_{-\frac{\Delta T}{2}}^{\frac{\Delta T}{2}} s_{\Delta T}(t) e^{\frac{-j2\pi nt}{\Delta T}} dt$$

$$\mathcal{F}(s_{\Delta T}(t)) = \sum_{n=-\infty}^{n=\infty} c_n \delta(\mu - \frac{n}{\Delta T})$$

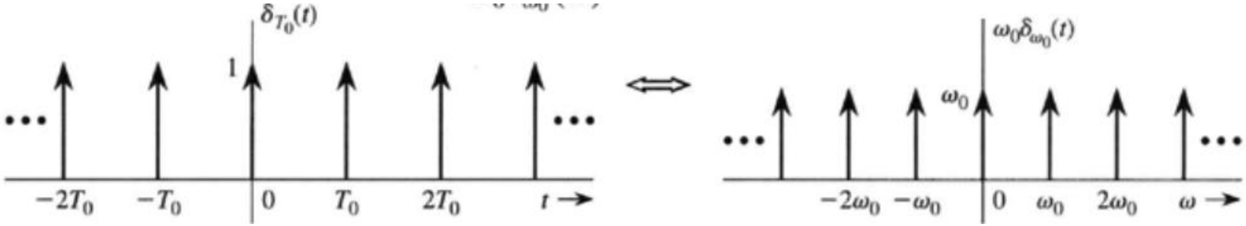
http://lpsa.swarthmore.edu/Fourier/Xforms/FXPeriodic.html

http://devonbryant.github.io/blog/2013/03/02/fourier-transforms/

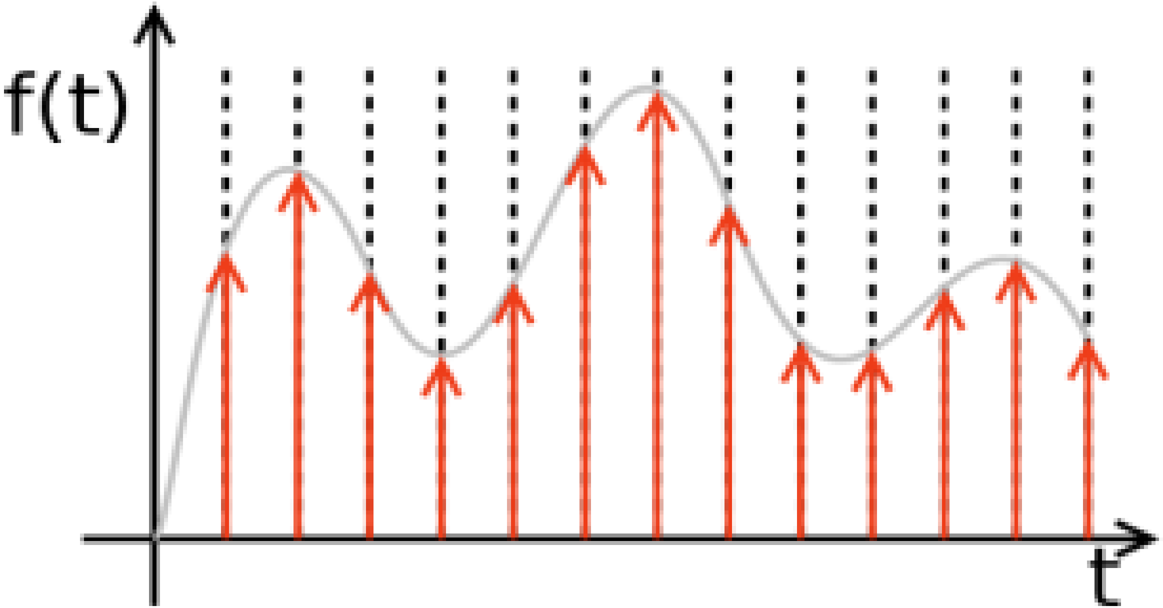
# FT of impulse train



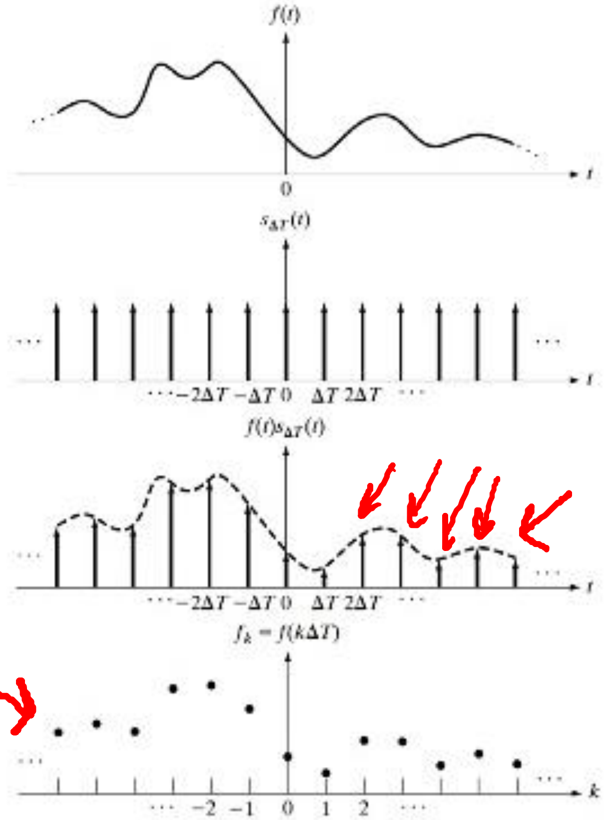

$$\mathcal{F}(s_{\Delta T}(t)) = \frac{1}{\Delta T} \sum_{n=-\infty}^{n=\infty} \delta(\mu - \frac{n}{\Delta T})$$



# Sampling



# Sampling = f(t) x Impulse Train




$\tilde{f}(t)$

$f_k = f(k\Delta T)$

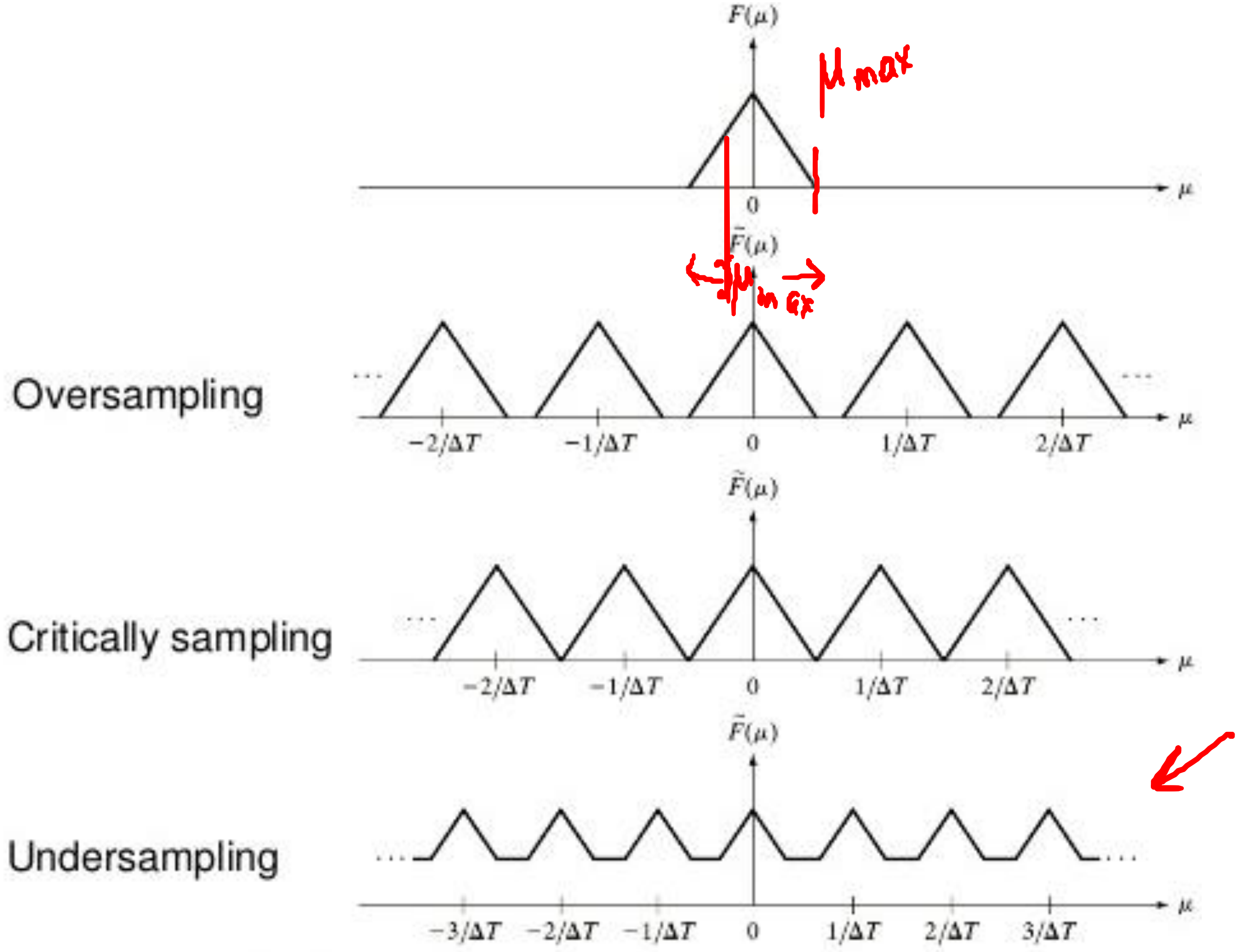
# FT of sampled function

$f(t) \leftrightarrow F(\mu)$

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

# FT of sampled function : properties

- Continuous
- Periodic (copies of f(t)’s FT)
- NOTE: FT is continuous



F(μ)
μ max
μ
0
F̃(μ)
2μ max
Oversampling
−2/ΔT
−1/ΔT
1/ΔT
2/ΔT
Critically sampling
Undersampling
−3/ΔT
3/ΔT

# References & Fun Reading/Viewing

- GW DIP textbook, 3rd Ed.,
  - 4.2.4
  - 4.2.5
  - 4.3.1
  - 4.3.2 (FT of sampled functions)
- http://www.thefouriertransform.com/
- A visual introduction to Fourier Transform: https://www.youtube.com/watch?v=spUNpyF58BY
- Fourier Transform, Fourier Series and Frequency Spectrum: https://www.youtube.com/watch?v=r18Gi8lSkfM