BITS Pilani

Pilani|Dubai|Goa|Hyderabad

M.Tech.(Data Science & Engineering)

Introduction to Statistical Methods

Team ISM

BITS Pilani

Session No 3

Introduction to Conditional Probability, independent events, Total Probability

(Session 3: 26^{th}/27^{th} Nov 2022)

Contact Session 3

Contact Session 3: Module 2(Conditional Probability & Bayes theorem)

Contact Session	**List of Topic Title**	**Reference**
CS - 3	Introduction to conditional probability,indepents events, Total probability	T1 & T2
HW	Problems on conditional probability	T1 & T2
Lab		

Agenda

- Conditional Probability
- Independent events
- Total Probability

Text Books

No	Author(s), Title, Edition, Publishing House
T1	Statistics for Data Scientists, An introduction to probability, statistics and Data Analysis, Maurits Kaptein et al, Springer 2022
T2	Probability and Statistics for Engineering and Sciences, 8^{th} Edition, Jay L Devore, Cengage Learning

CONDITIONAL PROBABILITY

The probabilities assigned to various events depend on what is known about the experimental situation when the assignment is made. Subsequent to the initial assignment, partial information relevant to the outcome of the experiment may become available. Such information may cause us to revise some of our probability assignments

CONDITIONAL PROBABILITY

We examine how the information "an event B has occurred" affects the probability assigned to A. For example, A might refer to an individual having a particular disease in the presence of certain symptoms. If a blood test is performed on the individual and the result is negative , then the probability of having the disease will change (it should decrease, but not usually to zero, since blood tests are not infallible). We will use the notation to represent the conditional probability of A given that the event B has occurred. B is the "conditioning event."

CONDITIONAL PROBABILITY

DEFINITION

For any two events A and B with $P(B) > 0$, the **conditional probability of A given that B has occurred** is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Conditional probabilities in a 2 × 2 contingency table

	A	A^c	
B	$\Pr(A \cap B) = \Pr(A \mid B)\Pr(B)$	$\Pr(A^c \cap B) = \Pr(A^c \mid B)\Pr(B)$	$\Pr(B)$
B^c	$\Pr(A \cap B^c) = \Pr(A \mid B^c)\Pr(B^c)$	$\Pr(A^c \cap B^c) = \Pr(A^c \mid B^c)\Pr(B^c)$	$\Pr(B^c)$
	$\Pr(A)$	$\Pr(A^c)$	1

Examples on Conditional Probability

Example: In a housing colony, 70% of the houses are well planned and 60% of the houses are well planned and well built. Find the probability that an arbitrarily chosen house in this colony is well built given that it is well planned.

Solution: Let A be the event that the house is well planned

B be the event that the house is well built

therefore $P(A) = 0.70,\ P(A \cap B) = 0.60$

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{0.60}{0.70} = 0.8571$$

CONDITIONAL PROBABILITY

Example:

If the probability that a communication system will have high fidelity is 0.81 and the probability that it will have high fidelity and high selectivity is 0.18, what is probability that a system with high fidelity will also have high selectivity?

Solution:

If A is the event that a communication system has high selectivity and B is the event that it has high fidelity, we have $P(B)$=0.81 and $P(A \cap B)$=0.18, and substitution into the formula yields

$$P(A \mid B) = \frac{0.18}{0.81} = \frac{2}{9}$$

CONDITIONAL PROBABILITY

Example:

Suppose that of all individuals buying a certain digital camera, 60% include an optional memory card in their purchase, 40% include an extra battery, and 30% include both a card and battery.

a) Given that the randomly selected individual purchased an extra battery, find the probability that an optional card was also purchased.

b) Given that the randomly selected individual purchased an optional card, find the probability that an extra battery was also purchased.

Solution:

let A={ memory card purchased } and B={ battery purchased }. Then $P(A)$=.60,$P(B)$=.40, and P(both purchased)=$P(A \cap B)$=.30.

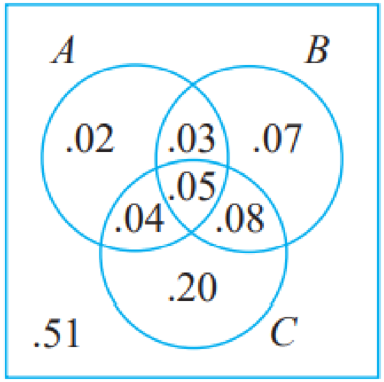
a) $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.30}{.40} = .75$$

b) $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{.30}{.60} = .50$$

Notice that $P(A|B) \neq P(A)$ and $P(B|A) \neq P(B)$.

Example:

A news magazine publishes three columns entitled "Art" (A), "Books" (B), and "Cinema" (C). Reading habits of a randomly selected reader with respect to these columns are



Read regularly	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	.14	.23	.37	.08	.09	.13	.05

Find P(A/B), P(A/BUC), P(A/ reads at least one)

Solution: We thus have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.08}{.23} = .348$$

$$P(A \mid B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)} = \frac{.04 + .05 + .03}{.47} = \frac{.12}{.47} = .255$$

$$P(A \mid \text{reads at least one}) = P(A \mid A \cup B \cup C) = \frac{P(A \cap (A \cup B \cup C))}{P(A \cup B \cup C)}$$

$$= \frac{P(A)}{P(A \cup B \cup C)} = \frac{.14}{.49} = .286$$

and

$$P(A \cup B \mid C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{.04 + .05 + .08}{.37} = .459$$

Examples on Conditional Probability

Example: In a certain college, 25% of the students failed Maths, 15% of the students failed chemistry and 10% of the students failed both maths and chemistry. A student is selected at random, find

a. If he failed chemistry, what is the probability that he failed Maths? $P(M) = 0.25,\ P(C) = 0.15,\ P(M \cap C) = 0.1$

$$P\left(\frac{M}{C}\right) = \frac{P(M \cap C)}{P(C)} = \frac{0.1}{0.15} = 0.6667$$

b. If the failed maths, what is the probability he failed chemistry?

$$P\left(\frac{C}{M}\right) = \frac{P(C \cap M)}{P(M)} = \frac{0.1}{0.25} = 0.4$$

c. What is the probability that the student failed in Maths or chemistry?

Examples on Conditional Probability

Example : The probabilities of a regularly scheduled flight departs on time is 0.83, arrives on time is 0.82 & it departs and arrives on time is 0.78. Find the probability that a plane (i) arrives on time given that it departed on time, (ii) departed on time given that it has arrived on time and (iii) find $\boldsymbol{P\left(\frac{A}{\bar{D}}\right)}$

Ans: Let D and A be the events that the flight departs and arrives on time respectively. Then,

$$\boldsymbol{P\,(D) = 0.83, P(A) = 0.82} \text{ and } \boldsymbol{P\,(D \cap A) = 0.78}$$

(i) Probability that the plane arrives on time given that it departed on time is

$$\boldsymbol{P\left(\frac{A}{D}\right) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.9398}$$

(ii) Probability that the plane departed on time given that it has arrived on time is

$$\boldsymbol{P\left(\frac{D}{A}\right) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.9512}$$

(iii) $\boldsymbol{P\left(\frac{A}{\bar{D}}\right) = \frac{P(A \cap \bar{D})}{P(\bar{D})} = \frac{0.82 - 0.78}{1 - 0.83} = 0.24}$

This is the probability that the flight arrives on time given that it did not depart on time

CONDITIONAL PROBABILITY

Multiplication Rule

Let A and B be two events in sample space.

The **conditional probability** that event A occurs given that event B has occurred and it is denoted by

$$P\left(\frac{A}{B}\right) = \frac{P(A \text{ and } B)}{P(B)} \quad \text{OR} \quad P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

It can also written as $P(A\cap B) = P(B)\,P(A/B) \qquad P(B)\neq 0$

$= P(A)\,P(B/A) \qquad P(A)\neq 0$

Let A,B and C be three events in a sample space S,

then $P(A\cap B\cap C) = P(A)\,P(B/A)\,P(C/A\cap B)$ and it is called Multiplication Rule

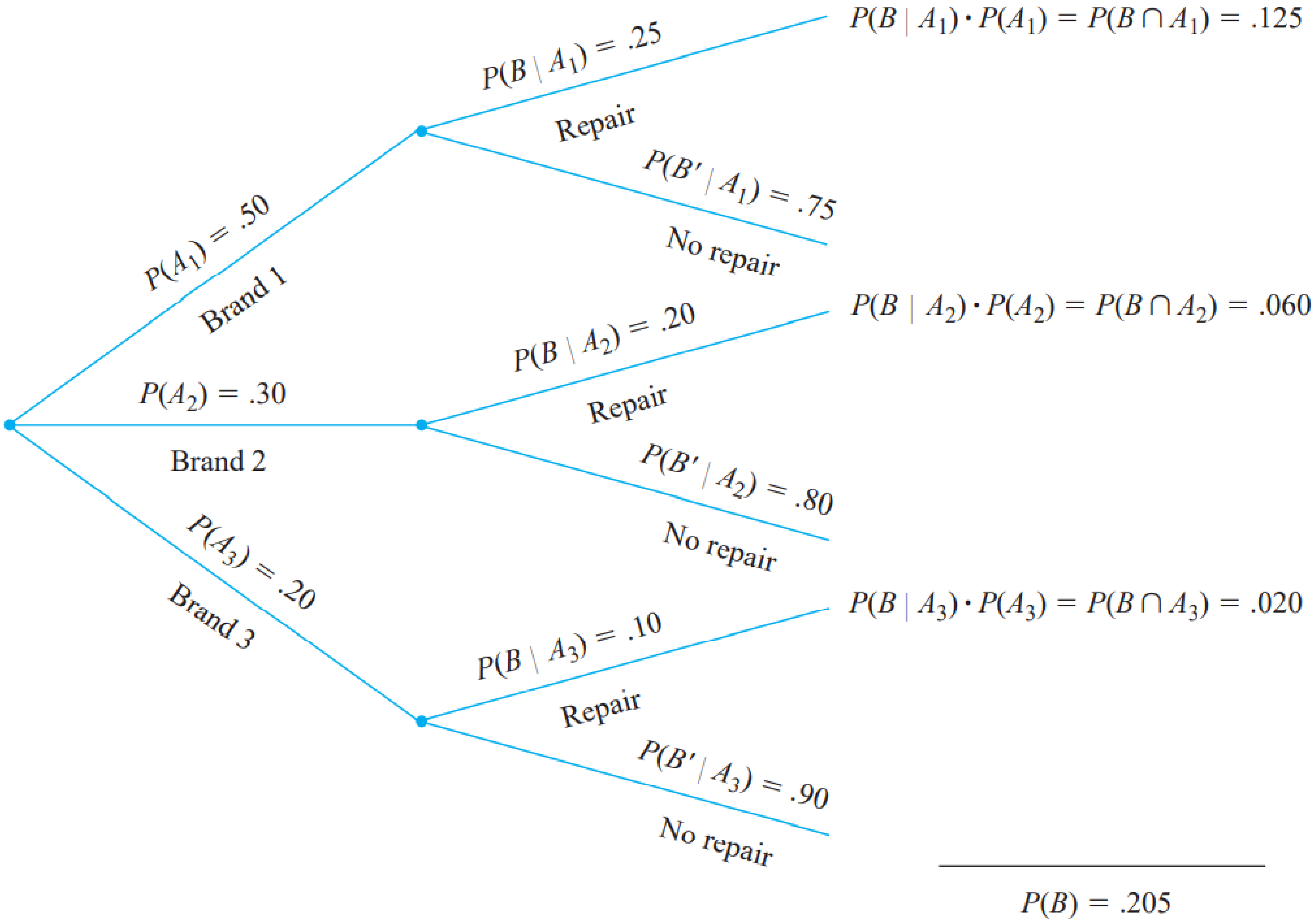
Multiplication Rule

In general, $A_1, A_2, \ldots\ldots. A_n$ are events in S, then

$$P(A_1 \cap A_2 \cap \ldots. \cap A_n) = P(A_1)\, P(A_2/A_1) P(A_3/A_1 \cap A_2) \ldots\ldots. P(A_n/A_1 \cap A_2 \cap \ldots\ldots \cap A_{n-1})$$

A chain of video stores sells three different brands of DVD players. Of its DVD player sales, 50% are brand 1 (the least expensive), 30% are brand 2, and 20% are brand 3. Each manufacturer offers a 1-year warranty on parts and labor. It is known that 25% of brand 1's DVD players require warranty repair work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

1. What is the probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?
2. What is the probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?
3. If a customer returns to the store with a DVD player that needs warranty repair work, what is the probability that it is a brand 1 DVD player? A brand 2 DVD player? A brand 3 DVD player?


P(A1) = .50
Brand 1
P(A2) = .30
Brand 2
P(A3) = .20
Brand 3
P(B | A1) = .25
Repair
P(B′ | A1) = .75
No repair
P(B | A2) = .20
Repair
P(B′ | A2) = .80
No repair
P(B | A3) = .10
Repair
P(B′ | A3) = .90
No repair
P(B | A1) · P(A1) = P(B ∩ A1) = .125
P(B | A2) · P(A2) = P(B ∩ A2) = .060
P(B | A3) · P(A3) = P(B ∩ A3) = .020
P(B) = .205

1. What is the probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?

$$P(A_1 \cap B) = P(B|A_1) \cdot P(A_1) = .125$$

2. What is the probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?

$$\begin{aligned} P(B) &= P[\text{(brand 1 and repair) or (brand 2 and repair) or (brand 3 and repai}\\ &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= .125 + .060 + .020 = .205 \end{aligned}$$

3. If a customer returns to the store with a DVD player that needs warranty repair work, what is the probability that it is a brand 1 DVD player? A brand 2 DVD player? A brand 3 DVD player?

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{.125}{.205} = .61$$

$$P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{.060}{.205} = .29$$

$$P(A_3|B) = 1 - P(A_1|B) - P(A_2|B) = .10$$

INDEPENDENT EVENTS

We can deduce an important result from the conditional probability:

If B has no effect on A, then, $\boldsymbol{P\left(\frac{A}{B}\right) = P(A)}$ Also $\boldsymbol{P\left(\frac{B}{A}\right) = P(B)}$ and we say the events are independent.

i.e., The probability of A does not depend on B.

so, $$\boldsymbol{P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}}$$

becomes, $$\boldsymbol{P(A) = \frac{P(A \cap B)}{P(B)}}$$

or $$\boxed{P(A \cap B) = P(A) \times P(B)}$$

Examples on Independent Events

A box contains 20 fuses of which 5 are defective. If two fuses are choosen at random one after the other. What is probability that both the fuses are defective if (i) the first fuse is replaced, (ii) the first fuse is not replaced.

Solution: Let A be the event that the first fuse is defective and
B be the event that the second fuse is defective

(i) When the first fuse is replaced, the events are independent hence

$$P(A \cap B) = P(A) \times P(B) = \frac{5C_1}{20C_1} \times \frac{5C_1}{20C_1} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

(ii) When first fuse is not replaced, the events are not independent then

$$P(B \cap A) = \mathrm{P(A)} \times P\left(\frac{B}{A}\right) = \frac{5C_1}{20C_1} \times \frac{4C_1}{19C_1} = \frac{1}{19}$$

Examples on Independent Events

A problem in statistics is given to 3 students A,B,C. Their chances of solving it are 1/2,1/3/,1/4. Find the probability that the problem is solved.

Solution: Problem can be solved by either A or B or C

Therefore we have to use

P(AUBUC) = P(A) + P(B) + P(C) - P(A∩B) - P(B∩C) - P(A∩C) + P(A ∩ B ∩ C)

Using complement of the event i.e the problem is not solved . Therefore P(problem solved is)=1-P(not solved)

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$

$$= 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C})$$

$$= 1 - \left[1 - \frac{1}{2}\right]\left[1 - \frac{1}{3}\right]\left[1 - \frac{1}{4}\right]$$

$$= \frac{1}{4}$$

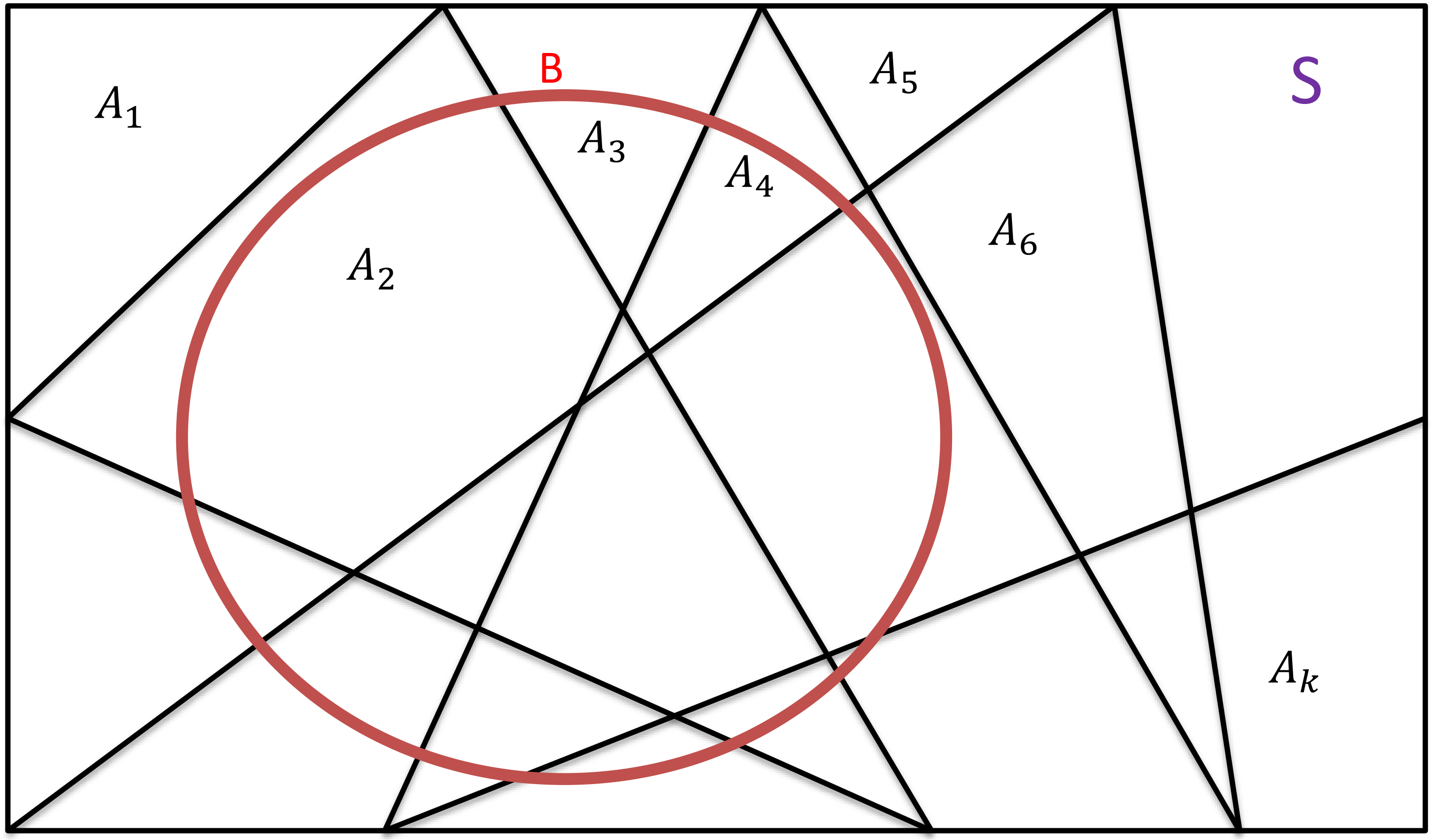
Note: If A, B,C are independent then $\bar{A}, \bar{B}, \bar{C}$ are also independent.

The Law of Total Probability

Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k)$$
$$= \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

$$S = A_1 \cup A_2 \cup \ldots \cup A_n \quad \text{and} \quad A_1 \cap A_2 \cap \ldots \cap A_n = \emptyset$$



$$\therefore \boldsymbol{B} = \boldsymbol{B} \cap \boldsymbol{S} = \boldsymbol{B} \cap \{\boldsymbol{A}_1 \cup \boldsymbol{A}_2 \cup \boldsymbol{A}_3, \ . \quad . \quad . \cup \boldsymbol{A}_n\}$$

Proof:

We have $S = \{A_1 \cup A_2 \cup A_3 \cup \;.\;\;.\;\;.\cup A_n\}$ and $A \subset S$

$$\therefore B = B \cap S = B \cap \{A_1 \cup A_2 \cup A_3,\;.\;\;.\;\;.\cup A_n\}$$

Using distributive law in the R.H.S, we get

Since $B \cap A_i$ (i = 1 to n) are mutually exclusive, we have by applying addition rule of probability,

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots\cdots + P(B \cap A_n)$$

$$i.e., \; P(B) = \sum_{i=1}^{i=n} P(B \cap A_i)$$

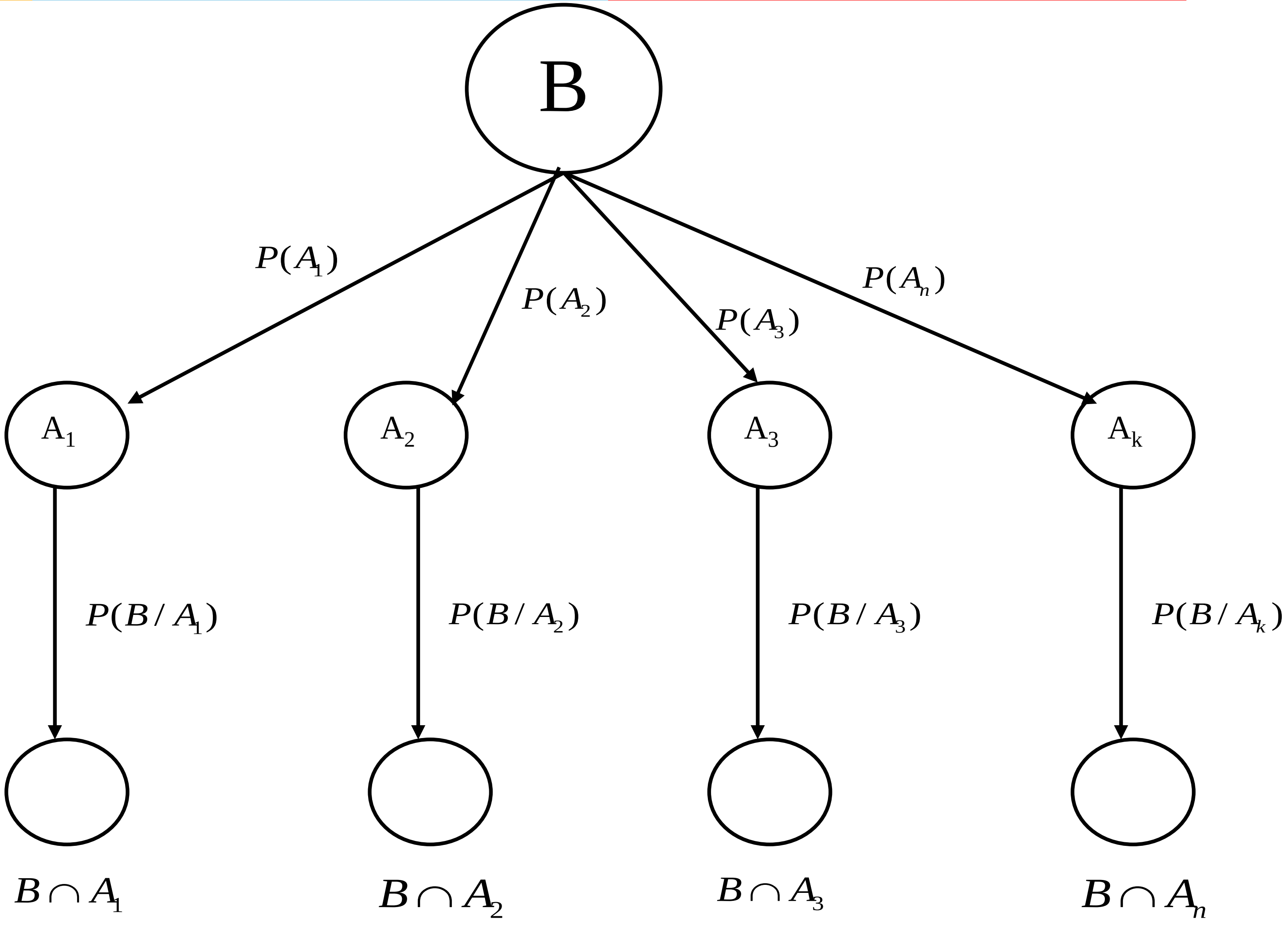
Using multiplication rule on each term on R.H.S, namely

$$P(B \cap A_i) = P(A_i) \cdot P(B \mid A_i) \qquad (1)$$

$$P(B) = \sum_{i=1}^{i=n} P(A_i) P(B \mid A_i)$$

(2)-Total theorem on Probability

The Theorem of Total Probability (tree diagram)



Law of Total Probability

An individual has 3 different email accounts. Most of her messages, in fact 70%, come into account #1, whereas 20% come into account #2 and the remaining 10% into account #3. Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively. What is the probability that a randomly selected message is spam?

$A_i = \{\text{message is from account \# } i\}$ for $i = 1, 2, 3,\quad B = \{\text{message is spam}\}$

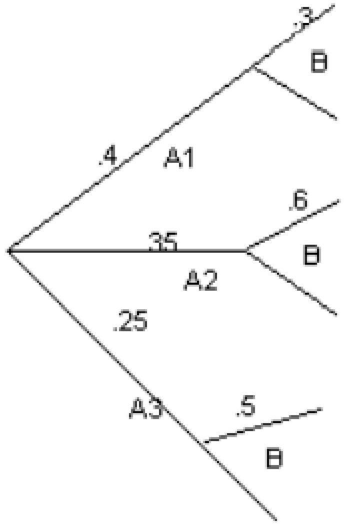
Then the given percentages imply that

$$P(A_1) = .70,\ P(A_2) = .20,\ P(A_3) = .10$$
$$P(B \mid A_1) = .01,\ P(B \mid A_2) = .02,\ P(B \mid A_3) = .05$$

Now it is simply a matter of substituting into the equation for the law of total probability: $P(B) = (.01)(.70) + (.02)(.20) + (.05)(.10) = .016$

Law of Total Probability

At a certain gas station, 40% of the customers use regular gas (A1), 35% use plus gas (A2), and 25% use premium (A3). Of those customers using regular gas, only 30% fill their tanks (event B). Of those customers using plus, 60% fill their tanks, whereas of those using premium, 50% fill their tanks. What is the probability that the next customer fills the tank?



$.4 \times .3 = .12 = P(A_1 \cap B) = P(A_1) \bullet P(B \mid A)$

$.35 \times .6 = .21 = P(A_2 \cap B)$

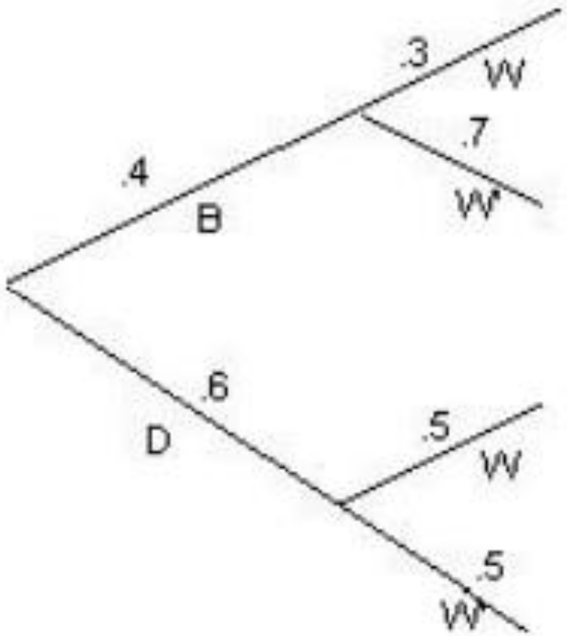
$.25 \times .5 = .125 = P(A_3 \cap B)$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) = .455$$

Law of Total Probability

A company that manufactures video cameras produces a basic model and a deluxe model. Over the past year, 40% of the cameras sold have been of the basic model. Of those buying the basic model, 30% purchase an extended warranty, whereas 50% of all deluxe purchasers do so. What is the probability that that a randomly selected purchaser has an extended warranty?

Using a tree diagram, B = basic, D = deluxe, W = warranty purchase, W' = no warranty



$.4 \times .3 = .12 = P(B \cap W)$

$.4 \times .7 = .28 = P(B \cap W')$

$.6 \times .5 = .30 = P(D \cap W)$

$.6 \times .5 = .30 = P(D \cap W')$

We want $P(W) = .30 + .12 = .42$

HW: Exercise

The population of a particular country consists of three ethnic groups. Each individual belongs to one of the four major blood groups. The accompanying *joint probability table* gives the proportions of individuals in the various ethnic group–blood group combinations.

		Blood Group			
		O	**A**	**B**	**AB**
Ethnic Group	**1**	.082	.106	.008	.004
	2	.135	.141	.018	.006
	3	.215	.200	.065	.020

Suppose that an individual is randomly selected from the population, and define events by $A = \{\text{type A selected}\}$, $B = \{\text{type B selected}\}$, and $C = \{\text{ethnic group 3 selected}\}$.

a. Calculate $P(A)$, $P(C)$, and $P(A \cap C)$.

b. Calculate both $P(A \mid C)$ and $P(C \mid A)$, and explain in context what each of these probabilities represents.

c. If the selected individual does not have type B blood, what is the probability that he or she is from ethnic group 1?

HW: Exercise

If A and B are two events with P(A)= ½, P(B)= 1/3 and $P(A \cap B) = 1/4$

Find

$$P\left(\frac{A}{B}\right), P\left(\frac{B}{A}\right), \mathrm{P}\left(\frac{\overline{\mathrm{A}}}{\overline{B}}\right), \mathrm{P}\left(\frac{\overline{\mathrm{B}}}{\overline{A}}\right), \mathrm{P}\left(\frac{\mathrm{A}}{\overline{B}}\right)$$

HW: Exercise

One card is randomly collected from the deck of 52 cards.

- What is the probability that this card is a heart?
- What is the probability that this card is not a heart?
- What is the probability that it is a heart and a king?
- What is the probability that the card is a heart or a king?
- Are the events that the card is a heart and is a king independent?

HW: Exercise

A card is randomly drawn from an incomplete deck of cards from which the ace of diamonds is missing.

1. What is the probability that the card is “clubs”?
2. What is the probability that the card is a “queen”?
3. Are the events “clubs” and “queen” independent?

HW: Exercise

In a group of children from primary school there are 18 girls and 15 boys. Of the girls, 9 have had measles. Of the boys, 6 have had measles.

1. What is the probability that a randomly chosen child from this group has had measles?
2. If we randomly choose one person from the group of 18 girls, what is the probability that this girl has had measles?
3. Are the events “boy” and “measles” in this example independent?

HW: Exercise

In a Japanese cohort study, 5,322 male non-smokers and 7,019 male smokers were followed for four years. Of these men, 16 non-smokers and 77 smokers developed lung cancer.

1. What is the probability that a randomly chosen non-smoker from this group developed lung cancer?
2. What is the probability that a randomly chosen smoker from this group developed lung cancer?
3. Are the events “smoking” and “lung cancer” in this example independent?
4. What is the conditional probability that the patient is a smoker if he has developed lung cancer?

Thanks