CS 156a - Problem Set 2

Samuel Patrone, 2140749

October 10, 2022

The following notebook is publicly available at the following link.

Contents

Problem 1

Answer: [b] 0.01

Code:

[1]:
```
import numpy as np
from IPython.display import clear_output
import matplotlib.pyplot as plt
import pickle
import math as m

def flipcoin_heads(Ntimes):
    heads=0
    for i in range(Ntimes):
        heads+=np.random.randint(0,2)
    return heads

def get_freq(Ncoins,Nflips, c_spec=0):
    c_rand=np.random.randint(0,Ncoins)
    nu_min=1
    nu_spec=0
    for i in range(Ncoins):
        nu_toss=flipcoin_heads(Nflips)/Nflips
        if(i==c_spec): nu_spec=nu_toss
        if(i==c_rand): nu_rand=nu_toss
        if(nu_toss<nu_min): nu_min=nu_toss
    return [nu_spec,nu_rand,nu_min]


def run_exp(Ntimes,Ncoins,Nflips,filename='freq.npy',saveonfile=True):
    freq=[]
    for i in range(Ntimes):
        clear_output(wait=True)
        print("Current progress:",np.round(i/Ntimes*100,2),"%")
        freq.append(get_freq(Ncoins,Nflips, c_spec=0))
    if(saveonfile==True):
        with open('freq.npy', 'wb') as f:
            np.save(f, np.array(freq))
    return np.array(freq)
```

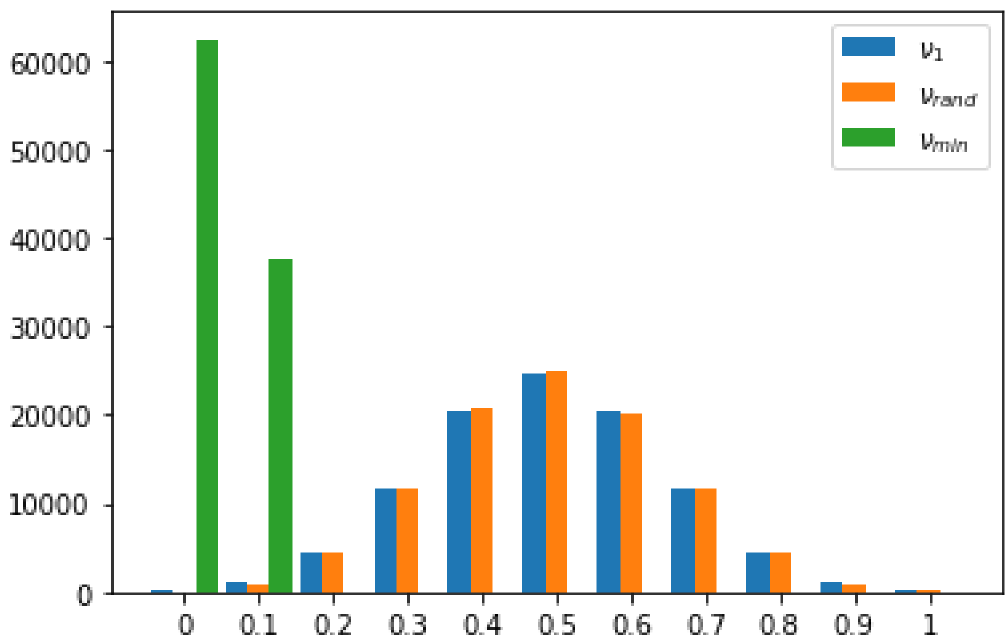
[2]:
```
#The generation of the distribution takes approximately 100 mins. If␣
↪saveonfile=True, the full array is automatically saved on a .npy file.
#freq=run_exp(100000,1000,10)
```

[3]:
```
with open('freq.npy', 'rb') as f:
    freq= np.load(f)
```

[4]:
```python
nu_min_av=np.sum(freq[:,2])/100000
print("The average value of nu_min is",nu_min_av)
```

```
The average value of nu_min is 0.03756
```

[5]:
```python
(freq_1, bins) = np.histogram(freq[:,0], bins=11)
(freq_rand, bins) = np.histogram(freq[:,1], bins=11)
(freq_min, bins) = np.histogram(freq[:,2], bins=3)
freq_min=np.append(freq_min,np.zeros(8),0)

xaxis=np.linspace(0,1,11)

fig=plt.figure()
width=0.03
plt.bar(xaxis,freq_1,width,label=r'$\nu_{1}$')
plt.bar(xaxis+width,freq_rand,width,label=r'$\nu_{rand}$')
plt.bar(xaxis+width*2,freq_min,width,label=r'$\nu_{min}$')
plt.xticks(xaxis+width,['0','0.1','0.2','0.3','0.4','0.5','0.6','0.7','0.8','0.
 ↪9','1'])
plt.legend(loc='upper right')
plt.show()
```



Problem 2

Answer: [d] c_1 and c_{rand}

Derivation:

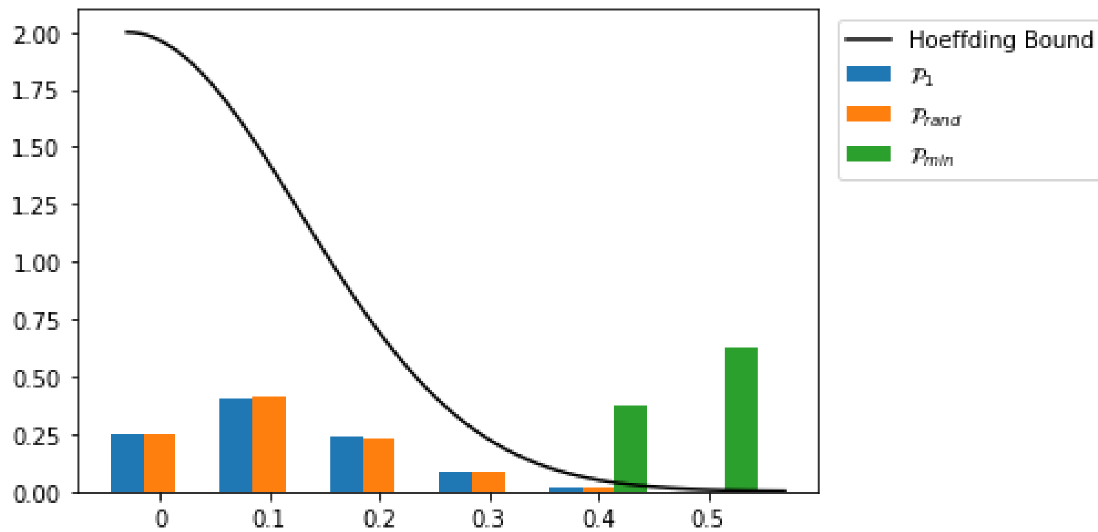
We expect the single bin's Hoeffding Inequality, in formulae

$$\mathcal{P}[|\nu - \mu| > \epsilon] \leq 2e^{-2\epsilon^2 N},$$

to hold only for hypothesis h fixed before the generation of the data set. Clearly, both ν_1 and ν_{rand} are fixed before tossing the 1000 coins, while ν_{min} is decided only after generating the dataset. Here, N is the size of the sample, i.e. the number of times we flip each coin.

We first notice that the probability of heads is $\mu = 0.5$ for the three coins since they are assumed to be fair. Therefore, we can compute $|\nu - \mu|$ in the three cases. We can superimpose Hoeffding bound computed at various ϵ to the previous histogram (once we rescale the frequencies in order to estimate probabilities). The following plot shows explicitly the violation of the Hoeffding Bound by the ν_{min} distribution.

[6]:
```
def Hoeff_bound(N,M,epsilon):
    return 2*M*m.exp(-2*epsilon**2*N)

absdev_1=np.abs(freq[:,0]-0.5)
absdev_rand=np.abs(freq[:,1]-0.5)
absdev_min=np.abs(freq[:,2]-0.5)

(P_1, bins) = np.histogram(absdev_1, bins=6)
(P_rand, bins) = np.histogram(absdev_rand, bins=6)
(P_min, bins) = np.histogram(absdev_min, bins=3)

P_1=P_1/100000
P_rand=P_rand/100000
P_min=np.insert(P_min,0,np.zeros(3),0)/100000

xaxis=np.linspace(0,0.5,6)
xfunc=np.linspace(0,0.6,100)
bound=[Hoeff_bound(10,1,xfunc[i]) for i in range(len(xfunc))]

fig=plt.figure()
width=0.03
plt.bar(xaxis,P_1,width,label=r'$\mathcal{P}_{1}$')
plt.bar(xaxis+width,P_rand,width,label=r'$\mathcal{P}_{rand}$')
plt.bar(xaxis+width*2,P_min,width,label=r'$\mathcal{P}_{min}$')
plt.plot(xfunc,bound,color='black',label='Hoeffding Bound')
plt.xticks(xaxis+width,['0','0.1','0.2','0.3','0.4','0.5'])
plt.legend(loc='upper right',bbox_to_anchor=(1.4, 1))
plt.show()
```



Problem 3

Answer: [e] $(1-\lambda)(1-\mu)+\lambda\mu$

Derivation:

In this problem, there are two binary functions f and h. Let be $P(h(\mathbf{x}) \neq f(\mathbf{x})) = \mu$ the probability of $h(\mathbf{x})$ in making a mistake in approximating $f(\mathbf{x})$. We now consider a noisy version of f for which there is a probability $P(y = f(\mathbf{x})|\mathbf{x}) = \lambda$ that $f(\mathbf{x})$ assign the "correct" value y to the point $\mathbf{x}$.

If we want to compute the probability of error that $h(\mathbf{x})$ makes in approximating y, we have to consider the following two cases in which $h(\mathbf{x})$ gives the wrong estimate for y:

- $h(\mathbf{x}) = f(\mathbf{x})$, but $y \neq f(\mathbf{x})$
- $y = f(\mathbf{x})$, but $h(\mathbf{x}) \neq f(\mathbf{x})$

In formulae,

$$P(h(\mathbf{x}) \neq y) = P(h(\mathbf{x}) = f(\mathbf{x}))P(y \neq f(\mathbf{x})|\mathbf{x}) + P(h(\mathbf{x}) \neq f(\mathbf{x}))P(y = f(\mathbf{x})|\mathbf{x}) = (1-\mu)(1-\lambda) + \mu\lambda\,.$$

Problem 4

Answer: [b] 0.5

Derivation:

We just solve for the derivative of the previous expression with respect to μ to be zero, i.e.

$$\frac{\mathrm{d}P(h \neq y)}{\mathrm{d}\mu} = \frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\lambda)(1-\mu)+\lambda\mu\right] = -1+2\lambda = 0 \rightarrow \lambda = \frac{1}{2}.$$

Problems 5-6

Answers: [c] 0.01 **, [c]** 0.01

Code:

[1]:
```
import numpy as np
import matplotlib.pyplot as plt

def f(x,m,b):
    return m*x+b

def gen_points_2d_bias(N,xleft=-1,xright=1,yleft=-1,yright=1):
    pts=[[1,np.random.uniform(xleft,xright),np.random.uniform(yleft,yright)] for␣
 ↪i in range(N)]
    return np.array(pts)

def gen_line():
    x1,x2,y1,y2=np.random.uniform(-1,1),np.random.uniform(-1,1),np.random.
 ↪uniform(-1,1),np.random.uniform(-1,1)
    m=(y2-y1)/(x2-x1)
    b=(y1*x2-y2*x1)/(x2-x1)
    return m,b

def gen_bias_input(N_pts,m,b,plot=True):
    data=gen_points_2d_bias(N_pts)
    output=[]
    x0=data[:,1]
    y0=data[:,2]
    for i in range(N_pts):
        if(f(x0[i],m,b)>y0[i]): output.append(1.)
        else: output.append(-1.)
    if(plot==True):
        col=[]
        for i in range(len(output)):
            if(output[i]>0): col.append('red')
            else: col.append('green')
        x=np.linspace(-1,1,100)
        plt.plot(x,f(x,m,b),color='blue',label='true')
        plt.scatter(x0,y0,color=col)
        plt.xlim([-1, 1])
        plt.ylim([-1, 1])
    return data, np.array(output)
```

```
def linear_regression_w(X,y):
    return np.dot(np.linalg.pinv(X),y)

def h(w,data):
    return np.sign(np.dot(w,data.T))

def Eout_estimate(m,b,w,Nval=1000):
    data,output=gen_bias_input(Nval,m,b,plot=False)
    g=h(w,data)
    testg=(g==output)
    Eout=len(np.where(testg==False)[0])
    return Eout/Nval

def linreg_run(N_pts,plot=False):

    #Generating linearly separable points
    m,b=gen_line()
    data,y=gen_bias_input(N_pts,m,b,plot=plot)

    #Computing linear regression weights
    w=linear_regression_w(data,y)

    #Ein computation
    g=h(w,data)
    testg=(g==y)
    Ein=len(np.where(testg==False)[0])/N_pts

    #Eout computation
    Eout=Eout_estimate(m,b,w)

    #Plot
    if(plot==True):
        x=np.linspace(-1,1,100)
        plt.plot(x,f(x,-w[1]/w[2],-w[0]/
 ↪w[2]),color='blue',linestyle='dashed',label='lin_reg')
        plt.legend()
        plt.show
    return w, Ein, Eout

def average_linreg(N_pts,N_runs):
    average_Ein=0
    average_Eout=0
    for i in range(N_runs):
        exp=linreg_run(N_pts)
        average_Ein+=exp[1]
        average_Eout+=exp[2]
    print("#######")
```

```
    print("Here's the result for N_runs=",N_runs,"with N_pts=",N_pts)
    print("Average Ein:",average_Ein/N_runs)
    print("Average Eout:",average_Eout/N_runs)
```

[2]:
```
Npts=100
Nruns=1000

average_linreg(Npts,Nruns)
```

```
#######
Here's the result for N_runs= 1000 with N_pts= 100
Average Ein: 0.037800000000000104
Average Eout: 0.04743100000000001
```

Problem 7

Answer: [a] 1

Code:

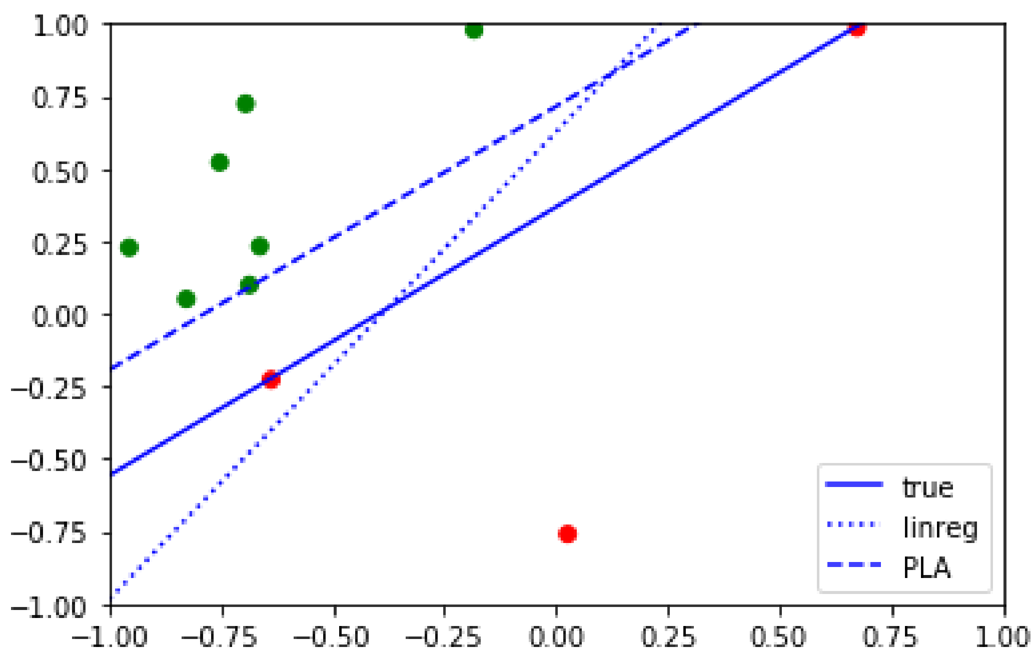
[12]:
```
def PLA_run_linreginit(N_pts, plot=False):

    #Generating linearly separable points
    m,b=gen_line()
    data,y=gen_bias_input(N_pts,m,b,plot)

    #Initialization weights with linear regression
    w=linear_regression_w(data,y)
    if(plot==True):
        x=np.linspace(-1,1,100)
        plt.plot(x,f(x,-w[1]/w[2],-w[0]/
 ↪w[2]),color='blue',linestyle='dotted',label='linreg')

    #Initialization
    converged=False
    iterations=0

    #Learning algorithm
    while(converged==False):
        g=h(w,data)
        testg=(g==y)
        misclassified=np.where(testg==False)[0]
        if(len(misclassified)>0):
            mis_index=np.random.randint(0, len(misclassified))
            i=misclassified[mis_index]
            w+=y[i]*data[i]
            iterations+=1
        g=h(w,data)
```

```
        converged=np.all(g==y)
    if(plot==True):
        x=np.linspace(-1,1,100)
        plt.plot(x,f(x,-w[1]/w[2],-w[0]/
↪w[2]),color='blue',linestyle='dashed',label='PLA')
        plt.legend()
        plt.show
    return w,iterations

def average_PLA(N_pts,N_runs):
    average_iter=0
    for i in range(N_runs):
        average_iter+=PLA_run_linreginit(Npts)[1]
    print("#######")
    print("Here's the result for N_runs=",N_runs,"with N_pts=",N_pts)
    print("Average iterations for convergence:",average_iter/N_runs)
```

[14]:
```
Npts=10
Nruns=1000

average_PLA(Npts,Nruns)
```

```
#######
Here's the result for N_runs= 1000 with N_pts= 10
Average iterations for convergence: 4.65
```

[15]:
```
### Plotted example
ex=PLA_run_linreginit(10, plot=True)
```



Problem 8

Answer: [d] 0.5

Code:

```
[12]: %reset -f
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return x[0]**2+x[1]**2-0.6

def line(x,m,b):
    return m*x+b

def gen_points_2d_bias(N,xleft=-1,xright=1,yleft=-1,yright=1):
    pts=[[1,np.random.uniform(xleft,xright),np.random.uniform(yleft,yright)] for␣
 ↪i in range(N)]
    return np.array(pts)

def add_noise(y,percentage):
    nflips=np.int(len(y)*percentage/100)
    randindex=np.random.choice(range(len(y)), nflips, replace=False)
    for i in range(nflips):
        y[randindex[i]]*=-1
    return y

def gen_bias_input(N_pts,ftarget,noise=0,plot=True):
    data=gen_points_2d_bias(N_pts)
    y=np.sign(ftarget(data[:,1:3].T))
    if(noise!=0):
        y=add_noise(y,noise)
    if(plot==True):
        col=[]
        for i in range(len(y)):
            if(y[i]>0): col.append('red')
            else: col.append('green')
        x0=np.linspace(-1,1,100)
        y0=np.linspace(-1,1,100)
        x0,y0=np.meshgrid(x0,y0)
        plt.contour(x0,y0,ftarget([x0,y0]),[0],colors='blue',linestyles='solid')
        plt.scatter(data[:,1],data[:,2],color=col)
        plt.xlim([-1, 1])
```

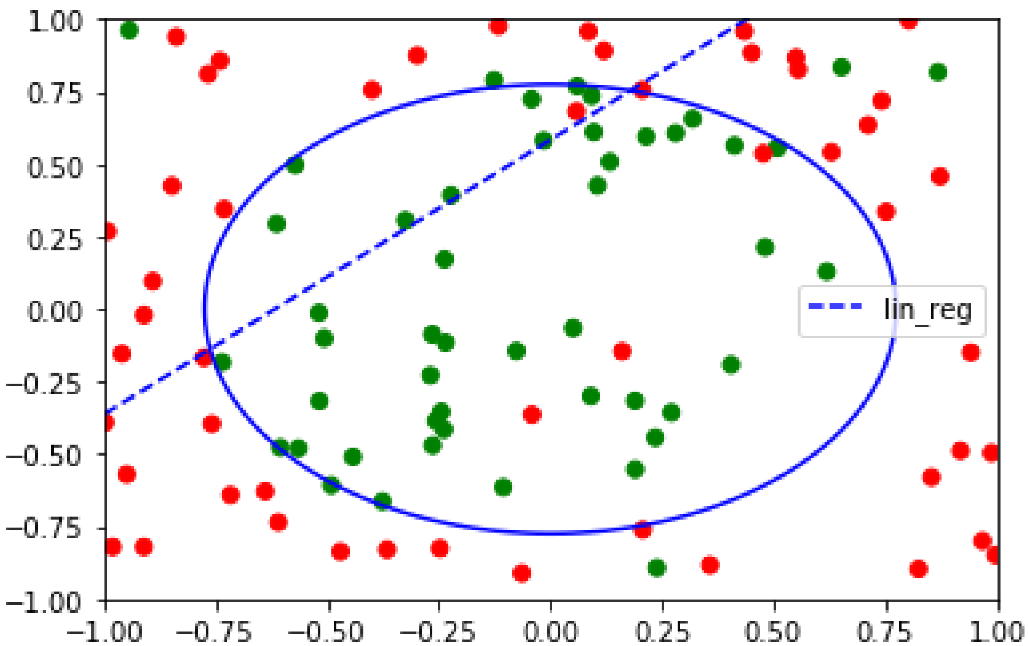
```
        plt.ylim([-1, 1])
    return data, y

def h(w,data):
    return np.sign(np.dot(w,data.T))

def Eout_estimate(ftarget,w,Nval=1000):
    data,output=gen_bias_input(Nval,ftarget,plot=False)
    g=h(w,data)
    testg=(g==output)
    Eout=len(np.where(testg==False)[0])
    return Eout/Nval

def linear_regression_w(X,y):
    return np.dot(np.linalg.pinv(X),y)

def linreg_run_linear(N_pts,ftarget,noise=0,plot=False):

    #Generating non-linearly separable points
    data,y=gen_bias_input(N_pts,ftarget,noise=noise,plot=plot)

    #Computing linear regression weights
    w=linear_regression_w(data,y)

    #Ein computation
    g=h(w,data)
    testg=(g==y)
    Ein=len(np.where(testg==False)[0])/N_pts

    #Eout computation
    Eout=Eout_estimate(ftarget,w)

    #Plot
    if(plot==True):
        x=np.linspace(-1,1,100)
        plt.plot(x,line(x,-w[1]/w[2],-w[0]/
 ↪w[2]),color='blue',linestyle='dashed',label='lin_reg')
        plt.legend()
        plt.show
    return w, Ein, Eout

def average_linreg_linear(N_pts,N_runs,ftarget,noise=0):
    average_Ein=0
    average_Eout=0
    for i in range(N_runs):
        exp=linreg_run_linear(N_pts,ftarget,noise=noise)
        average_Ein+=exp[1]
```

```
        average_Eout+=exp[2]
    print("#######")
    print("Here's the result for N_runs=",N_runs,"with N_pts=",N_pts)
    print("Average Ein:",average_Ein/N_runs)
    print("Average Eout:",average_Eout/N_runs)
```

[13]:
```
average_linreg_linear(1000,1000,f,noise=10)
```

```
#######
Here's the result for N_runs= 1000 with N_pts= 1000
Average Ein: 0.505258
Average Eout: 0.5248650000000001
```

[14]:
```
# Plotted Example

ex=linreg_run_linear(100,f,noise=10,plot=True)
```



Problems 9-10

Answers: [a] $g(x_1, x_2) = \text{sign}(-1 - 0.05x_1 + 0.08x_2 + 0.13x_1x_2 + 1.5x_1^2 + 1.5x_2^2)$ **, [b]** 0.1

Code:

[15]:
```
def nonlin_transform(data):
    return np.column_stack((data[:,0],data[:,1],data[:,2],data[:,1]*data[:
 ↪,2],data[:,1]**2,data[:,2]**2))

def Eout_estimate_nonlinear(ftarget,w,noise=0,Nval=1000):
    data,output=gen_bias_input(Nval,ftarget,noise=noise,plot=False)
    data=nonlin_transform(data)
    g=h(w,data)
    testg=(g==output)
    Eout=len(np.where(testg==False)[0])
    return Eout/Nval

def h(w,data):
    return np.sign(np.dot(w,data.T))

def linreg_run_nonlinear(N_pts,ftarget,noise=0,plot=False):

    #Generating non-linearly separable points
    points,y=gen_bias_input(N_pts,ftarget,noise=noise,plot=plot)

    #Transform data into non-linear feature vector
    data=nonlin_transform(points)

    #Computing linear regression weights
    w=linear_regression_w(data,y)

    #Ein computation
    g=h(w,data)
    testg=(g==y)
    Ein=len(np.where(testg==False)[0])/N_pts

    #Eout computation
    Eout=Eout_estimate_nonlinear(ftarget,w,noise=noise)

    #Plot
    if(plot==True):
        x1=np.linspace(-1,1,100)
        x2=np.linspace(-1,1,100)
        x1,x2=np.meshgrid(x1,x2)
        plt.
 ↪contour(x1,x2,w[0]+w[1]*x1+w[2]*x2+w[3]*x1*x2+w[4]*x1**2+w[5]*x2**2,[0],colors='blue',linesty
        plt.show
```

```
    return w, Ein, Eout

def average_linreg_nonlinear(N_pts,N_runs,ftarget,noise=0):
    sum_Ein=0
    sum_Eout=0
    sum_w=np.zeros(6)
    for i in range(N_runs):
        exp=linreg_run_nonlinear(N_pts,ftarget,noise=noise)
        sum_w+=exp[0]
        sum_Ein+=exp[1]
        sum_Eout+=exp[2]
    av_w=sum_w/N_runs
    av_Ein=sum_Ein/N_runs
    av_Eout=sum_Eout/N_runs
    print("#######")
    print("Here's the result for N_runs=",N_runs,"with N_pts=",N_pts)
    print("Average weights:",av_w)
    print("Average Ein:", av_Ein)
    print("Average Eout:",sum_Eout/N_runs)
    return av_w,av_Ein,av_Eout

def check_weights(av_w,given_w,name='Insert name',Nval=1000):
    points=gen_points_2d_bias(Nval)
    data=nonlin_transform(points)
    g_av=h(av_w,data)
    g_given=h(given_w,data)
    testg=(g_av==g_given)
    P_agree=len(np.where(testg==True)[0])
    print('The function',name,'agrees with the solution of Linear Regression␣
 ↪with a probability of',P_agree/Nval)
    return P_agree/Nval
```

[16]:
```
w,Ein,Eout=average_linreg_nonlinear(1000,1000,f,noise=10)
```

```
#######
Here's the result for N_runs= 1000 with N_pts= 1000
Average weights: [-9.92570638e-01 -7.26949696e-04  3.68746671e-03
9.20842355e-04
  1.56513471e+00  1.55346432e+00]
Average Ein: 0.12367900000000003
Average Eout: 0.125895
```

[17]:
```
g1=np.array([-1,-0.05,0.08,0.13,1.5,1.5])
g2=np.array([-1,-0.05,0.08,0.13,1.5,15])
g3=np.array([-1,-0.05,0.08,0.13,15,1.5])
g4=np.array([-1,-1.5,0.08,0.13,0.05,0.05])
```

```
g5=np.array([-1,-0.05,0.08,1.5,0.15,0.15])

t1=check_weights(w,g1,'[a]')
t2=check_weights(w,g2,'[b]')
t3=check_weights(w,g3,'[c]')
t4=check_weights(w,g4,'[d]')
t5=check_weights(w,g5,'[e]')
```

```
The function [a] agrees with the solution of Linear Regression with a
probability of 0.974
The function [b] agrees with the solution of Linear Regression with a
probability of 0.662
The function [c] agrees with the solution of Linear Regression with a
probability of 0.663
The function [d] agrees with the solution of Linear Regression with a
probability of 0.64
The function [e] agrees with the solution of Linear Regression with a
probability of 0.573
```

[18]:

```
# Plotted Example

ex=linreg_run_nonlinear(100,f,noise=10,plot=True)
```