Assignment 1

Given Date: 06-02-2022
Due Date: 06-09-2022 11:59 pm EDT

Question 1 (Logic)

1. For each of the following **implications**, write the negation and contraposition. If the statement is in English, write your response in English as well.

   (a) If it is raining then I will bring my umbrella. (5 points)

   Answer: Lets consider, p and q are propositions where $p =$ *"it is raining"* and $q =$ *"I will my umbrella"*

   $p \to q$ also implies $(\neg p \vee q)$. Thus, Negation of $p \to q$ or $\neg(\neg p \to q)$ also implies $p \wedge \neg q$

   Negation: "It is not the case that if it is raining then I will bring my umbrella."

   Alternatively negation also be "it is raining and I will not bring umbrella".

   Contrapositive: If I will not bring my umbrella, then it is not raining.

   (b) If I water my plants then they will grow. (5 points)

   Answer: Lets consider, p and q are propositions where $p =$ *"I water my plants"* and $q =$ *"they will grow"*

   Negation: "It is not the case that if I water my plants then they will grow."

   Alternatively negation also be "I water my plants and they will not grow".

   Contrapositive: If my plants will not grow, then I don't water them.

   (c) $(\neg p \vee q) \to r$ (5 points)

   Answer: **Negation** of $(\neg p \vee q) \to r$
   $= \neg((\neg p \vee q) \to r)$
   $= \neg(\neg(\neg p \vee q) \vee r)$
   $= \neg\neg(\neg p \vee q) \wedge \neg r$, using de-morgan's law
   $= (\neg p \vee q) \wedge \neg r$

   Contrapositive: $\neg r \to \neg(\neg p \vee q)$ After simplification, $\neg r \to (p \wedge \neg q)$

2. Using a truth table, show that the following statements are equivalent.
   $(p \wedge \neg q) == \neg(p \rightarrow q)$ (5 points)

p	q	¬ q	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Table 1: Truth table for $p \wedge \neg q$

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Table 2: Truth table for $\neg(p \rightarrow q)$

Two propositions are equivalent if they have same truth values. Thus, Table 1 and 2 show that propositions $p \wedge \neg q$ and $\neg(p \rightarrow q)$ are equivalent.

3. Show using equivalences that the following statements are equivalent.

   (a) $(p \to q) \wedge (p \to r) == p \to (q \wedge r)$ (5 points)
   Answer: $(p \to q) \equiv \neg p \vee q$ (from definition)

$$
\begin{aligned}
LHS &\equiv (p \to q) \wedge (p \to r) \\
&\equiv (\neg p \vee q) \wedge (\neg p \vee r) \\
&\equiv (\neg p \vee (q \wedge r)), using\ Distributive\ Law \\
&\equiv p \to (q \wedge r), by\ definition \\
&\equiv RHS
\end{aligned}
$$

   (b) $(p \wedge q) \to (p \vee q) == T(true)$ (5 points)
   Answer: $(p \to q) \equiv \neg p \vee q$ (from definition)

$$
\begin{aligned}
LHS &\equiv (p \wedge q) \to (p \vee q) \\
&\equiv (\neg(p \wedge q)) \vee (p \wedge q) \\
&\equiv \neg r \vee r, where\ r = (p \wedge q) \\
&\equiv T(true), using\ Negation\ Law \\
&\equiv RHS
\end{aligned}
$$

Question 2 (Quantifiers and Predicate Logic)

1. Translate the following from English to predicate logic

   (a) For every positive real number x, there is some real number y that is the square root of x (optional: extend this statement to show that there are two distinct square roots of x) (5 points)

   Answer: A(x) = "x > 0 (x is a positive real number)"
   For the second part, we have to define 2-place predicate, B(x,y): "y^2 = x (x has y as a square root)"

   The predicate logic will be $\forall$x (A(x) $\rightarrow$ $\exists$y B(x,y))

   optional part $\forall$x [A(x) $\rightarrow$ [($\exists$y)($\exists$z) $\rightarrow$ (x $\neq$ y) $\wedge$ B(x,y) $\wedge$ B(x,z)]]

   (b) There is a real number that is its own square root (5 points)

   Answer: C(x) = "x is a real number"
   For the second part, we use the same 2-place predicate, B(x,y): "y^2 = x (x has y as a square root)"

   The predicate logic will be $\exists$x (C(x) $\rightarrow$ B(x,x))

   (c) For any two distinct real numbers there is a third real number with a value between the first two. (5 points)

   Answer: using the same predicate, C(x) = "x is a real number"
   The predicate logic will be $\forall$x,y [(x < y) $\wedge$ C(x) $\wedge$ C(y) $\rightarrow$ $\exists$z (x < z < y) $\wedge$ C(z)]

Question 3 (Proofs)

1. Use a direct proof to show that the sum of two even integers is even. (5 points)

   Answer: Assume that x & y are two even integers. This means x = 2k for some integer k and y = 2n for some integer.

   Adding those two equations, x+y = 2k + 2n = 2(k+n), where k+n is an integer (because sum of integers is an integer).

   This means (x+y) is even.

2. Prove the following by contradiction:
   If x is an irrational number, then $-x$ is an irrational number
   (Recall that a rational number is any number that can be written as a fraction with integers in the numerator and denominator. If this cannot be done, it is irrational.) (5 points)

   Answer: To start a proof by contradiction, we assume that this proposition is false; which means $\exists\ x$ is an irrational number for which $-x$ is an rational number.

   Since, 0 is a rational number and rational numbers are closed under subtraction thus 0 - (-x) is also a rational number, i.e., x is a rational number which contradicts our assumption. Therefore the proposition is not false.

   Alternative Proof To start a proof by contradiction, we assume that this proposition is false; which means $\exists\ x$ is an irrational number for which $-x$ is an rational number.

   Since, -x is a rational number, it can be written as $\frac{p}{q}$ where, p and q are integers and q $\neq$ 0. Thus, x can be written as $\frac{-p}{q}$, where -p and q both are integers (since p and q are integers). This implies x is a rational number which contradicts our assumption. Therefor, if x is an irrational number, then $-x$ is an irrational number.

3. Use an existence proof to show that the cubed root of x is can be equal to the square root of x. (5 points)

   Answer: Here we have to find an explicit value c for which cubed root of c is can be equal to the square root of c.

   1 is such a number since, cubed root and square root of 1 are both 1.

Question 4 (Sets)

1. For the following problems, use these sets:
   A = { n | n $\in \mathbb{Z}$ and n $\leq$ 14}
   B = $\{2, 3, 5, 7, 11, 13\}$

   (a) A $\wedge$ B (5 points)
   Answer: A $\wedge$ B = x | x $\in$ A and x $\in$ B
   Here, A $\wedge$ B = **A =** $\{2, 3, 5, 7, 11, 13\}$, since A is a proper subset of B.

   (b) A - B (5 points)
   Answer: A - B = x | x $\in$ A and x $\notin$ B Here, A - B = $\emptyset$, since A is a proper subset of B.

   (c) (A $\wedge$ B) $\vee$ B (5 points)
   = (A $\wedge$ B) $\vee$ B
   = (A $\vee$ B) $\wedge$ (B $\vee$ B), using Distributive law
   = (A $\vee$ B) $\wedge$ B, using Idempotent law
   = B $\wedge$ B, since A is a proper subset of B
   = **B = { n | n $\in \mathbb{Z}$ and n $\leq$ 14}**

2. Let $f(x) = x^2$ and g(x) = x + 2. Domain of f(x) is all real numbers. Domain of g(x) is also all real numbers.

   (a) What is $f \circ g$? Simplify the function as much as possible. (5 points)
   Answer: f∘g(x) = f(g(x)) = f(x+2) = $(x+2)^2 = x^2 + 4x + 4$

   (b) What are the inverses of these two functions. If the inverse does not exist, explain why. (5 points)

   Answer: **The function f can't be invertible** because it is not one to one. We can understand this with a simple example, a value y = 4 from the domain of f^{-1} (if it exists) can possibly be mapped to two different values -2 and +2 which violates the function definition.

   While, The function g is invertible because it is a one-to-one correspondence. The inverse function g^{-1} reverses the correspondence so $\mathbf{g^{-1}(y) = y - 2}$.

Question 5 (Matrices)

1. What is the cardinality of the set of all prime factors of 120? (5 points)

   Answer: $120 = 2^3 \times 3 \times 5$ and Hence the prime factors are the set S={2,3,5} Thus **the cardinality of the set of all prime factors of 120 = 3.**

2. Let $M = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 8 & 6 \end{bmatrix}$. What is MN^TN? (10 points)

   Answer: $N^T = \begin{bmatrix} 0 & 4 \\ 1 & 8 \\ 2 & 6 \end{bmatrix}$

   $MN^T = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

   $MN^TN = \begin{bmatrix} 8 & 17 & 14 \\ -8 & -17 & -14 \end{bmatrix}$