Assignment 2

Given Date: 06/09/2022

Due Date: 06/16/2022 11:59 pm EDT

Question 1 (Ranking Runtimes)

(20 points) Rank the following functions representing running times from smallest to largest (in terms of growth rate with respect to n). Group functions together if they are in the same equivalence class (i.e. $f(n)$ and $g(n)$ are in the same equivalence class if $f(n) = \Theta(g(n))$).

$2n^3 + 12n^2 + 5$, $8(\log n)^2$, 1.5^n , $n^4 - 12n^3$, $4n^3 \log n$, $4n^3$, $n!$, $7n + 6$

Answer: We first have to identify the asymptotically dominant term in each function. Since the leading term dominates the growth.

$\underline{O(n^3)}$: $2n^3 + 12n^2 + 5 = O(n^3)$ while C=19, k=1 i.e., n>1 $4n^3 = O(n^3)$ while C=4, k=1 i.e., n>1.

we can also show that for n>1, $2n^3 + 12n^2 + 5 = \Theta(4n^3)$ as
$C_1 * 4n^3 < 2n^3 + 12n^2 + 5 < C_2 * 4n^3$, where we can take $C_1 = \frac{1}{2}$, $C_2 = 5$

$\underline{O(n^3 \log n)}$: We know that, $O(n) < O(n \log n)$
Multiplying n^2 on both sides, $O(n^3) < O(n^3 \log n)$ for $n > 1$

where, $4n^3 \log n = O(n^3 \log n)$

$\underline{O(n^4)}$: We know that, $O(n \log n) < O(n^2)$

Multiplying n^2 on both sides, $O(n^3 \log n) < O(n^4)$ for $n > 1$

where, $n^4 - 12n^3 = O(n^4)$

we also know that $O(n^3) < O(n^3 \log n) < O(n^4) < 1.5^n < n!$

Furthermore polynomial grow slowly than the exponential. Thus,

$O(n^2) < O(2^n)$ and if we replace n with $\log n$ where $n > 1$ then same relationship will also hold

$O((\log n)^2) < O(2^{\log n})$ which implies, $O((\log n)^2) < O(n)$

Here, $7n + 6 = O(n)$ and $8(\log n)^2 = O((\log n)^2)$

The final ranking (smallest to largest)

$$\boxed{8(\log n)^2 < 7n + 6 < 2n^3 + 12n^2 + 5 = 4n^3 < 4n^3 \log n < n^4 - 12n^3 < 1.5^n < n!} \tag{1}$$

Question 2 (Growth of Functions)

a. Let $f(x) = x^2 - x + 15$ and $g(x) = x^2 \log(x) - 10$. Prove that $f(x)$ is $O(g(x))$ by:

1. (5 points) Providing witnesses C and k and

   Answer: We can use $C = 1$ and $k = 25$

2. (5 points) Proving that the inequality holds for the value you choose in 1.

   Answer: Now we have to show $|f(x)| \leq C * |g(x)|$ for $C = 1$ and $k = 25$.

   so we have to show, $x^2 - x + 15 \leq C * (x^2 \log(x) - 10)$ where $x > 25$
   $= C * (x^2 \log(x) - 10) - x^2 - x + 15$ for $x > 25$
   $= x^2(C * \log(x) - 1) + x - 25 \geq 0$ for $x > 25$

   for $x > 25$, $(x - 25) > 0$ and, for C=1 and $x > 25$, $(C * \log(x) - 1) > 0$ thus the inequality of $|f(x)| \leq C * |g(x)|$ holds for specific C and k.

b. Let $f(x) = x^5 + 10$ and $g(x) = x^5 + x + 10$

1. (5 points) Prove that f is $O(g)$

   Answer: $x^5 + 10 \leq x^5 + x + 10$ for $x > 0$

   so using C = 1 and k = 0, we can show $|f(x)| \leq C * |g(x)|$, thus f is $O(g)$.

2. (5 points) Prove that f is $\Omega(g)$

   Answer: $x^5 + 10 \geq \frac{1}{2} * (x^5 + x + 10)$ for $x > 1$

   so using C = $\frac{1}{2}$ and k = 1, we can show $|f(x)| \geq C * |g(x)|$, thus f is $\Omega(g)$.

3. (5 points) Given parts a and b, what other relationship can you show about f and g?

   Answer: Combining the upper two answers, we can show that $\frac{1}{2} * (x^5 + x + 10) \leq x^5 + 10 \leq x^5 + x + 10$ where $x > 1$

   If we can show $C_1 * g(x) \leq f(x) \leq C_2 * g(x)$ for $x > k$ then $f(x) = \Theta(g(x))$.

   For our case, $C_1 = \frac{1}{2}$ and $C_2 = 1$ and k = 1. Hence, $\underline{f(x) = \Theta(g(x))}$ (equivalent class).

Question 3 (Proof by Induction)

(15 points) Use induction to show that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$

Answer: BASIS STEP: P(1) is since $1^2(1+1)^2/4 = 1$.
INDUCTIVE STEP: Assume P(k) is true thus
The inductive hypothesis is

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Under this assumption, we have to prove P(k+1) is also true. To prove this, we have to show that $\sum_{i=1}^{k+1} i^3 = (k+1)^2(k+2)^2/4$

LHS $= \sum_{i=1}^{k+1} i^3$
$= 1^3 + 2^3 + ... + k^3 + (k+1)^3$
$= \sum_{i=1}^{k} i^3 + (k+1)^3$
$= \frac{k^2(k+1)^2}{4} + (k+1)^3$
$= (k+1)^2(k^2/4 + (k+1))$
$= (k+1)^2(k^2 + 4k + 4)/4$
$= (k+1)^2(k+2)^2/4$
$= (k+1)^2((k+1)+1)^2/4 =$ RHS

Question 4 (Recursive Set)

(20 points) Let Σ be the alphabet defined as follows: $\Sigma = \{\text{e, l, v}\}$
We now define the set of strings P according to the following:
BASIS STEP: If $x \in \Sigma$ then $x \in P$
RECURSIVE STEP: If $w \in P$ and $x \in \Sigma$, then $xwx \in P$

Show that the string *level* is in P

Answer: since $e \in \Sigma$ then $e \in P$, similarly since $l \in \Sigma$ then $l \in P$, since $v \in \Sigma$ then $v \in P$

Since $v \in P$ and $e \in \Sigma$ so from the RECURSIVE STEP, $\underline{eve \in P}$
Since $eve \in P$ and $l \in \Sigma$ so from the RECURSIVE STEP, $\underline{level \in P}$

Question 5 (Merge Sort Steps)

1. (10 points) How many levels of recursion(level of recursive calls only, not including return steps) are necessary for merge sort to sort a list of size 8?

   Answer: The level of recursion (starting from the initial call to merge sort that is the root of all recursive calls) to sort a list of size 8 is $1 + \lceil \log 8 \rceil = \underline{\mathbf{4}}$

2. (10 points) Write the intermediary ordered list after each recursive merge step from calling merge sort on the following: $\{4, 8, 5, 1, 6, 7, 2, 3\}$

   Answer: In the *splitting* phase, {4,8,5,1} and {6,7,2,3} then will be further split into {4,8},{5,1} and {6,7},{2,3}. Then finally into {4},{8} and {5},{1} and {6},{7} and {2},{3}.

   Now **recursive merge step** is started
   1. {4} and {8} are merged into {4,8}.
   2. {5} and {1} are merged into {5,1}.
   3. {6} and {7} are merged into {6,7}.
   4. {2} and {3} are merged into {2,3}.

$$\boxed{\{4, 8\}\ \{1, 5\} \qquad \{6, 7\}\ \{2, 3\}} \tag{2}$$

   Then, 1.{4,8} and {5,1} are merged into {1,4,5,8}.
   2. {6,7} and {2,3} are merged into {2,3,6,7}.

$$\boxed{\{1, 4, 5, 8\} \quad \{2, 3, 6, 7\}} \tag{3}$$

   Finally,

$$\boxed{\{1, 2, 3, 4, 5, 6, 7, 8\}} \tag{4}$$