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DECIDABILITY AND THE FINITE MODEL PROPERTY

It is a familiar fact that the following result can be applied to show decidability for most of the better-known modal logics:

(A) If L is a finitely axiomatized logic with the finite model property, then L is decidable.

It is natural to ask whether (A) can be strengthened to read:

(B) If L is a recursively axiomatized logic with the finite model property, then L is decidable.

The main purpose of this note is to furnish a counter-example to (B).

We use the standard notation and terminology of modal logic, for which see Chellas [1]. The *degree* of a sentence A of modal logic is defined as follows: $d(A) = 0$ if A is atomic, $d(F(A, B)) = \max(d(A), d(B))$, for F a classical connective and $d(\Box A) = d(A) + 1$.

1. THE COUNTER-EXAMPLE

THEOREM. *There is a normal modal logic containing K which is recursively axiomatizable, has the finite model property, but is undecidable.*

Proof. Let X be a set of natural numbers which is recursively enumerable, but not recursive; we assume $0 \in X$, $1 \notin X$. Define L to be the smallest normal logic containing K and all axioms of the following forms:

(A1) $\Box(\Diamond A \to \Box A)$
(A2) $\Diamond(\Box\bot \wedge A) \to \Box(\Box\bot \to A)$
(A3) $\Diamond(\Diamond\top \wedge A) \to \Box(\Diamond\top \to A)$
(A4k) $(\Diamond\Box\bot \wedge \Diamond\Diamond\top) \to \Box^k\Diamond\top, \quad k \in X.$

Since the theorems of L are recursively enumerable, L can be recursively axiomatized by Craig [2].

If x is a point in a frame $\langle W, R\rangle$, x is an endpoint (abbreviated Ex) if

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$\neg\exists y(xRy)$. The logic L is validated by any frame $\langle W, R\rangle$ satisfying the conditions:

(C1) $xRy, yRz, yRw \to z = w$
(C2) $xRy, xRz, Ey, Ez \to y = z$
(C3) $xRy, xRz, \neg Ey, \neg Ez \to y = z$
(C4) $xRy, xRz, Ey, \neg Ez \to \forall w(xR^k w \to \neg Ew), k \in X, k > 1.$

That L is determined by the class C of all frames satisfying (C1) to (C4) can be proved by the usual methods involving the canonical model of L.

Before showing that L has the finite model property, we define two classes of frames. Let α be an ordinal $\leqslant \omega$; we identify α with the set of ordinals $\beta < \alpha$. The frame F_α is defined on the set $\alpha \cup \{-1\}$ by setting $R = \{\langle m, m+1\rangle \mid m+1 \in \alpha\} \cup \{\langle 0, -1\rangle\}$. The frame G_α is defined on α by setting $R = \{\langle m, m+1\rangle \mid m+1 \in \alpha\}$. Note that F_{m+1}, G_{m+1} are in C if $m \in \omega - X$.

Let A be any sentence not provable in L. Then there is a frame F in C so that $\models^{F}_{w} \neg A$. Let F^w be the subframe of F generated by w. If F^w is finite, we are done; so we can assume F^w to be infinite. If wRz holds for only one z, then by (C1) F^w is isomorphic to G_ω. If wRz holds for more than one z, then by (C1) again, F^w is isomorphic to F_ω. In either case, let m be an element of $\omega - X$ which is greater than the degree of A. Then A is refutable in G_{m+1} (F_{m+1}). In either case, the appropriate refuting frame is in C, so L has the finite model property.

Finally, we must show that L is undecidable. We claim that for any k, (A4k) is provable in L if and only if $k \in X$. Let $k \notin X$; if $k = 1$, then (A41) is falsifiable in any $F_m, m > 1$. If $k > 1$ then (A4k) is false at 0 in F_{k+1}. Since X is not recursive, L is undecidable.

The counter-example given above does not really have much to do specifically with modal logic and similar examples can be given in (for example) equational logic with unary operators (see Urquhart [4]).

PROBLEM. Find a counter-example where L is a normal logic containing $S4$.

2. POSSIBLE GENERALIZATIONS

What is the correct generalization of (A)? An examination of a proof of (A) (see e.g., Chellas [1], p. 62) shows that the following more general result is true:

(C) If L is a recursively axiomatized logic which is determined by a recursively enumerable class of finite frames, then L is decidable.

The usual proof relies, of course, on the fact that if Σ is a finite set of formulas, then the set of finite frames validating Σ is recursive. The counter-example given above together with (C) show that the more general statement:

(D) If Σ is a recursive set of formulas, then the set of finite frames validating Σ is recursively enumerable

is false. (Let Σ be a recursive set of axioms for the L of the example.)

One is inclined to think at first glance that (D) is true. This probably accounts for the fact that mistaken claims have been published in this area. As an example, Hansson and Gärdenfors make the following claim:

> It is also possible to define a somewhat stronger variant of the fmp, where for each non-theorem there exists a computable upper limitation to the size of the model that falsifies the formula in question. At the cost of this complication we no longer need to know that the logic is finitely axiomatizable in order to conclude that it is decidable. For if a logic has this stronger property, we only need to check whether a certain formula is true in all models smaller than the given limitation in order to know whether it is a theorem. In fact, most proofs that have been given that a certain logic has the fmp suffice to show that it has the stronger variant too [3], p. 32–33.

The proof sketched in the above quotation overlooks the fact that even though we may be able to confine our attention to a finite set of frames, given a candidate non-theorem, we are still faced with the problem of which of these frames validate the logic in question.

That the result claimed by Hansson and Gärdenfors is in fact false can be seen by a refinement of the counter-example in Section 1. In the definition of the logic L, let X have the additional property that $\omega - X$ has an infinite recursively enumerable subset (that is, X is not simple). For a given formula A, let $f(A)$ be a number $m \notin X$ such that $m \geqslant d(A) + 1$. By assumption, f is a computable function. We claim that if A is not provable in L, then it is refutable in a frame with at most $f(A)$ points. Now if A is not provable it follows by the argument of the theorem that it is refutable at the point 0 in some F_n or G_n, n finite. If $n \leqslant m$, we are through; if not, it follows that A is also refutable at the point 0 in F_m or G_m. Thus L satisfies the strong form of the finite model property mentioned by Hansson and Gärdenfors.

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