Modified Euler-Frobenius Polynomials with application to Sampled Data Model

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Abstract

The broad class of polynomials generally known as Eulerian or Euler-Frobenius polynomials has a rich history in Mathematics and Engineering. In this paper we will see a modified version of these polynomials and how the sampling zero dynamics of a linear and nonlinear system is characterized by these modified polynomials.
In the paper we will see when it's used a Partial Zero-Order Hold or a generalised hold functions to generate an input or when there is a delay in the continuous system.

1 Introduction

The class of polynomials know as the Euler-Frobenius polynomial has been extensively studied in the Mathematics literature.
These kind of polynomials are important for many engineering areas. In particular the modified Euler-Frobenius polynomials appear in sampled-data problems. We show that the sampling zeros can be expressed as function of these polynomials.
After we define the Euler-Frobenius polynomials and their modified version, the definition of multiple integrals that we will use during the proof of some result and then we will see the 3 cases.

2 Standard Euler-Frobenius polynomials

In this section we recall the definitions of standard Euler- Frobenius polynomials. Standard Euler-Frobenius polynomials are denoted $B_r(z)$. There are many definition for these polynomials and we will report the definition we will use in the curse of the paper:

$$B_r(z) = r! \cdot detM_r \tag{1}$$

where

$$M_r = \begin{bmatrix} 1 & \frac{1}{2!} & ... & \frac{1}{(r-1)!} & \frac{1}{r!} \\ 1-z & 1 & \frac{1}{2!} & ... & \frac{1}{(r-1)!} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & ... & 1-z & 1 & \frac{1}{2!} \\ 0 & ... & 0 & 1-z & 1 \end{bmatrix} \tag{2}$$

3 Modified Euler-Frobenius polynomials

As the standard polynomials, we express the modified Euler-Frobenius polynomials as follow

$$B'_r(z,f) = r! \cdot detP_r \tag{3}$$

where

$$P_r = \begin{bmatrix} 1 & \frac{1}{2!} & ... & \frac{1}{(r-1)!} & \frac{1-f}{r!} \\ 1-z & 1 & \frac{1}{2!} & ... & \frac{1-f}{(r-1)!} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & ... & 1-z & 1 & \frac{1-f}{2!} \\ 0 & ... & 0 & 1-z & 1-f \end{bmatrix} \tag{4}$$

with f which is a number greater than 0 and lower than 1. The first few polynomials are

$$B'_0(z,f) = 1 \tag{5}$$

$$B'_1(z,f) = 1-f \tag{6}$$

$$B'_2(z,f) = (1-f)^2 z + (1-f^2) \tag{7}$$

$$B'_3(z,f) = (1-f)^3 z^2 + (1-f)(4+f-2f^2)z + (1-f^3) \tag{8}$$

4 Preliminary definitions and result for sampled-data models

In this Section we introduce definitions and results that will be used in the remainder of the paper. In particular we want to define the multiple integration of a function and the definition of the zero-dynamics of a discrete time-system.

4.1 Multiple integration

We can define the multiple integration of a function $g(t)$ in the following way:

$$I(r, f, \Delta, g) = \int_{f\Delta}^{\Delta} \int_{f\Delta}^{t_{r-1}} ... \int_{f\Delta}^{t_1} g(t) dt ... dt_{r-1} \tag{9}$$

In the case of $g(t) = 1$ and using the Cauchy formula the result will be

$$I(j, f, \Delta, g) = \frac{\Delta^j (1-f)^j}{j!} \tag{10}$$

4.2 Zero-dynamics

For the zero-dynamics of a discrete time system we can say the definition follows the case of the continuous time system.In fact, ther are defined as the internal dynamics that appear in the system when the input and initial conditions are chosen in such a way as to make the output identically zero for all time ($y_k = 0$).

5 Application to sampled data model

In this Section we will develop sampled data models for several non-standard sampled data modelling problems, e.g. for partial zero order holds, for piecewise-constant generalised hold functions and for time-delay systems.
We will show that the Modified Euler-Frobenius polynomials arise in the description of the sampling zeros in all three cases.

5.1 Partial zero-order hold

The PZOH generates a continuos input for the system defined as follow

$$u(t) = \begin{cases} 0 & k\Delta \leq t \leq k\Delta + f\Delta \\ u_k & k\Delta + f\Delta \leq t \leq (k+1)\Delta \end{cases} \tag{11}$$

with $k \in \mathbb{N}$ and $f \in [0, 1]$

5.1.1 System of integrator

Let's start with a system of integrator, taking as example $G(s) = 1/s^3$.

$$\dot{x_1}(t) = x_2(t)$$
$$\dot{x_2}(t) = x_3(t)$$
$$\dot{x_3}(t) = u(t)$$

Now, if we do the exact integration between $k\Delta$ and $(k+1)\Delta$, starting from x_3 and moving up to x_1, we will have

$$x_1((k+1)\Delta) = x_1(k\Delta) + \Delta x_2(k\Delta) + \frac{\Delta^2}{2!} x_3(k\Delta) + I(3, f, \Delta, u(t)) \tag{12a}$$
$$x_2((k+1)\Delta) = x_2(k\Delta) + \Delta x_3(k\Delta) + I(2, f, \Delta, u(t)) \tag{12b}$$
$$x_3((k+1)\Delta) = x_3(k\Delta) + I(1, f, \Delta, u(t)) \tag{12c}$$

In this way we can rewrite the system as follow

$$x_{k+1} = A_d x_k + B_{PZOH} u_k \tag{12}$$

with

$$A_d = \begin{bmatrix} 1 & \Delta & \frac{\Delta^2}{2!} \\ 0 & 1 & \Delta \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$B_{PZOH} = \begin{bmatrix} \frac{\Delta^3(1-f)^3}{3!} \\ \frac{\Delta^2(1-f)^2}{2!} \\ \Delta(1-f) \end{bmatrix} \tag{14}$$

In order to find the transfer function of this system we will use the forward shift operator to write the same system. So

$$\begin{bmatrix} (q-1)x_{1,k} & 0 & 0 \end{bmatrix}^T = D_r \begin{bmatrix} x_{2,k} & x_{3,k} & u_k \end{bmatrix}^T \tag{15}$$

where

$$D_3 = \begin{bmatrix} \Delta & \frac{\Delta^2}{2!} & \frac{\Delta^3(1-f)^3}{3!} \\ -(q-1) & \Delta & \frac{\Delta^2(1-f)^2}{2!} \\ 0 & -(q-1) & \Delta(1-f) \end{bmatrix} \tag{16}$$

Now, using the Cramer's rule, we can express the equation of u_k as follow

$$u_k = \frac{DetN}{DetD_3} \tag{17}$$

where

$$N = \begin{bmatrix} \Delta & \frac{\Delta^2}{2!} & (q-1)x_{1,k} \\ -(q-1) & \Delta & 0 \\ 0 & -(q-1) & 0 \end{bmatrix} \tag{18}$$

Now, from the numerator and the denominator, we will have

$$DetN = (q-1)^r x_{1,k} \tag{19}$$
$$DetD_3 = \Delta^r detP_3 \tag{20}$$

if we mulitply and divide (20) by $r!$,we can sho that it can be rewitten as follow

$$DetD_r = \frac{\Delta^3}{3!} B_3'(z,f) \tag{21}$$

Noting that $y_{1,k} = x_{1,k}$ and $G_q(s) = \frac{y_k}{u_k}$ we can define the trasnfer function of the system

$$G_q(s) = \frac{\Delta^r B_3'(z,f)}{3!(z-1)^3} \tag{22}$$

As we want demostrate in this case and in the other ones, the roots of the modified Euler-Frobenius theorem plays an important role on the zero dynamics of the system (in linear case the zeros of the transfer function).

5.1.2 Linear system

The second system we have dealt with is the linear case, which can be represented,for example,with this transfer function

$$G(s) = 10\frac{s+1}{(s+10)(s+100)(s+1000)} \tag{23}$$

with $r = 2$.
In this case we can write the exact sampled data model as

$$G_q(z) = Z\{L^{-1}\{G(s)G_{PZOH}(s)\}\} \tag{24}$$

where

$$G_{PZOH}(s) = \frac{e^{sf\Delta} - e^{-s\Delta}}{s} \tag{25}$$

Also in this case we want to demonstrate, as $\Delta \to 0$, that r zeros converge to the roots of the modified polynomial.
First of all we use the exact definition of the inverse of the Laplace transform and of the z-transform and we obtain

$$G_q(z) = \frac{1}{2\pi j}\int_{-\gamma+j\infty}^{\gamma+j\infty} \frac{e^{s\Delta}}{z-e^{s\Delta}} G(s) \frac{e^{sf\Delta} - e^{-s\Delta}}{s} ds \tag{26}$$

Now we do a change of coordinates $s = \frac{w}{\Delta}$ to obtain

$$G(\frac{w}{\Delta}) = 10\frac{\Delta^1}{w^1}\frac{1+\frac{\Delta}{w}}{(1+10\frac{\Delta}{w})(1+100\frac{\Delta}{w})(1+1000\frac{\Delta}{w})} \tag{27}$$

Now, if take the limit for $\Delta \to 0$ and we compare the integral for the linear system, so substituting in $G(s)$ the (23), and the integral for the system of integrator, so substituting in the integral $G(s) = \frac{1}{s^2}$, we can see that these 2 integral are very similiar.

$$G_q(z) = \frac{1}{2\pi j}\int_{j\infty}^{j\infty} \frac{e^{s\Delta}}{z-e^{s\Delta}} \frac{10}{w^1} \frac{e^{sf\Delta}-e^{-s\Delta}}{s} ds \tag{28}$$

$$G_q(z) = \frac{1}{2\pi j}\int_{j\infty}^{j\infty} \frac{e^{s\Delta}}{z-e^{s\Delta}} \frac{\Delta}{w^1} \frac{e^{sf\Delta}-e^{-s\Delta}}{s} ds \tag{29}$$

But we know already the result transfer function for (29) (system of integrator), and by comparison we can say that, asymptotically for a linear system, the transfer function will be

$$\lim_{\Delta\to 0} \Delta^{-2} G_q(z) = \frac{10^2 B_2'(z,f)}{2!(z-1)^2} \tag{30}$$

As we can see, also in this case, the zeros of the sampled-data model asymptotically converges to the roots of the modified polynomials,which corresponds to the zeros of the original system.

5.1.3 Affine nonlinear system

In this case we have a system like the follow

$$\dot{x_1}(t) = x_2(t) \tag{31a}$$

$$\dot{x_2}(t) = x_2^2(t)x_1(t) + x_1(t) + u(t) \tag{31b}$$

After we proceed, we need the normal form of this system.
Relative degree in this case is 2 so we choose the change of coordinates $\phi(x)$

$$z(t) = \begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix} = \begin{pmatrix} x1(t) \\ x2(t) \end{pmatrix} \tag{32}$$

Now we have the new system in the new coordinates

$$\dot{z_1}(t) = z_2(t) \tag{33a}$$

$$\dot{z_2}(t) = z_1(t) + z_2^2(t)z_1(t) + u(t) \tag{33b}$$

After that, we need the discrete time system and to obtain it we will use the same method used for the system of integrator, and so doing the integral from the last component of x and moving up to x_1. So we obtain a system similar to (12) but with a difference in the integral

$$z_1((k+1)\Delta) = z_1(k\Delta) + \Delta z_2(k\Delta) + I(2,0,\Delta,(b(z(t)) + a(z(t))u(t))) \tag{34a}$$

$$z_2((k+1)\Delta) = z_2(k\Delta) + I(1,0,\Delta,(b(z(t)) + a(z(t))u(t))) \tag{34b}$$

where

$$b(z) = z_1(t) + z_2^2(t)z_1(t) \qquad\qquad a(z) = 1 \tag{35}$$

Now we have to do some considerations for the integral. First of we can split and so rewrite them as follow

$$I(2,0,\Delta,(b(z(t)) + a(z(t))u(t))) = I(2,0,\Delta,(b(z(t)))) + I(2,f,\Delta,(a(z(t))u(t))) \tag{36}$$

After that we have to consider an approximation of model and it's obtained when the argument of $I(\cdot)$ is a constant value: in this case when $b(z)$ and $a(z)$ are evaluated at the start of the sampling period. With this consideration we can rewrite (37) as

$$b(z(t))I(2,0,\Delta,1) + a(z(t))I(2,f,\Delta,1)u_k \tag{37}$$

After all of this we can rewrite the system again,which has this type of structure

$$\hat{z}_{k+1} = A_d\hat{z}_k + B_d b(\hat{z}_k) + B_{PZOH}a(\hat{z}_k)u_k \tag{38}$$

Where A_d and B_{PZOH} are always the same and

$$B_d = \begin{bmatrix} \frac{\Delta^2}{2!} \\ \Delta \end{bmatrix} \tag{39}$$

(Nothing that, in a general case, a system can have also the η dynamics. In this particular case r is equal to the number of the state and so doesn't appear.) Now we impose another change of coordinates $\tilde{z} = T\hat{z}$, where T will be

$$T = \begin{bmatrix} 1 & 0 \\ -\frac{2}{\Delta(1-f)} & 1 \end{bmatrix} \tag{40}$$

Applying this transformation we obtain

$$\tilde{z}_{1,k+1} = q_{11}\tilde{z}_{1,k} + Q_{12}\tilde{z}_{2,k} + \frac{\Delta^1}{1!}(b(\tilde{z}_k + (1-f)^1 a(\tilde{z}_k)u_k) \tag{41a}$$

$$\tilde{z}_{2,k+1} = Q_{21}\tilde{z}_{1,k} + Q_{22}\tilde{z}_{2,k} + \tilde{B}_2 b(\tilde{z}_k) \tag{41b}$$

Now if we compute the zero dynamics and take the limit for $\Delta \to 0$ we obtain that the sampling zeros converges to the eigenvalue of Q_{22}.

$$\tilde{z}_{2,k+1} = Q_{22}\tilde{z}_{2,k} \tag{42}$$

where

$$Q_{22} = T_{21}A_{12} + A_{22} = -\frac{2}{\Delta(1-f)}\Delta + 1 \tag{43}$$

Now, if I get D_2 as in (16), and I consider

$$D_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\Delta^2(1-f)^2}{2!} & A_{12} \\ -\frac{\Delta^2(1-f)^2}{2!}T_{21} & A_{22} - z \end{bmatrix} \tag{44}$$

and taking both the determinant we can show that

$$B_2'(f,z) = (1-f)^2 det(z - Q_{22}) \tag{45}$$

5.2 Piecewise Constant generalised hold functions

Now we will define the same results but for others type of a system. In this case we will see when the input to the system is done by a generalised order-hold, defined as follow:

$$H(t) = \sum_{j=1}^{m} c_j[\mu(t - \frac{(j-1)\Delta}{m}) - \mu(t - \frac{j\Delta}{m})] \tag{46}$$

In our case we will choose $c_j = 1$ and $m = 2$.
Now, we can see that, following the same steps used for the partial zero order-hold, we obtain a similiar result.

5.2.1 System of integrator

Also in this case we will see the case of $r = 3$, and so a system like $G(s) = \frac{1}{s^3}$.Considering also that

$$u(t) = \begin{cases} c_1 u_k & k\Delta \leq t < k\Delta + \Delta/2 \\ c_2 u_k & k\Delta + (2-1)\Delta/2 \leq t < k\Delta + \Delta \end{cases} \tag{31}$$

With this choice we will obtain a system described as follow:

$$x_{k+1} = A_d x_k + B_{GOH} u_k \tag{47}$$

where A_d is the same described in the previous chapter and B_{GOH} as

$$B_{GOH} = \begin{bmatrix} \sum_{j=1}^{m}(\frac{\Delta^3(1-\frac{j-1}{2})}{3!}) - (\frac{\Delta^3(1-\frac{j}{2})}{3!}) \\ \sum_{j=1}^{m}(\frac{\Delta^2(1-\frac{j-1}{2})}{2!}) - (\frac{\Delta^2(1-\frac{j}{2})}{2!}) \\ \sum_{j=1}^{m}(\frac{\Delta^1(1-\frac{j-1}{2})}{1!}) - (\frac{\Delta^1(1-\frac{j}{2})}{1!}) \end{bmatrix} \tag{48}$$

So, with this system and,as always $y_k = x_{1,k}$ we obtain a transfer function similiar to (22):

$$G_q(s) = \frac{\Delta^3 S_{GOH}(z)}{3!(z-1)^3} \tag{49}$$

where

$$S_{GOH} = \sum_{j=1}^{2}[B_3'(z, \frac{(j-1)}{2}) - B_3'(z, \frac{j}{2})] \tag{50}$$

5.2.2 Linear system

We have seen as the zeros of a linear system are characterized in the case of an input generated by a PZOH. In this case we have a similar result.
In fact, taking (24) and place G_{GOH} instead of G_{PZOH}, we will obtain:

$$G_q(z) = Z\{L^{-1}\{G(s)G_{GOH}(s)\}\} \tag{51}$$

where

$$G_{GOH}(s) = \frac{\sum_{j=1}^{m} c_j(e^{-s\Delta\frac{(j-1)}{m}} - e^{-s\Delta\frac{j}{m}})}{s} \tag{52}$$

As before, to simplify the form of $G_{GOH(s)}$ we will chose $m = 2$ and $c_j = 1$. So (52) becomes:

$$G_{GOH}(s) = \frac{\sum_{j=1}^{2}(e^{-s\Delta\frac{(j-1)}{2}} - e^{-s\Delta\frac{j}{2}})}{s} \tag{53}$$

Now, from the same steps we have done with the partial zero order and usign the same linear system we will obtain:

$$\lim_{\Delta\to 0} \Delta^{-2} G_q(s) = \frac{10^2 S_{GOH}(z)}{2!(z-1)^2} \tag{54}$$

where

$$S_{GOH}(z) = \sum_{j=1}^{2}[B_2'(z, \frac{(j-1)}{2}) - B_2'(z, \frac{j}{2})] \tag{55}$$

5.2.3 Affine nonlinear system

Always using the same nonlinear system and the same steps, we will obtain a system exactly equal to (38), but with B_{GOH} instead of B_{PZOH}, where A_d,B_d and B_{GOH} are defined in (13), (39) and (48), respectively.
The first difference in the proof for nonlinear system is about the integrals. In fact, in the system we have in this case

$$I(j, 0, \Delta, (b(z) + a(z)u)) = I(j, 0, \Delta, (b(z))) + I(j, 0, \Delta, (a(z)u)) \tag{56}$$

with the following approximation:

$$I(j, 0, \Delta, (b(z))) = b(z)I(j, 0, \Delta, 1) \tag{57a}$$
$$I(j, 0, \Delta, (a(z))) = a(z)I(j, 0, \Delta, 1)u_k \tag{57b}$$

After this the proof follows i the same way, applying a similar transformation, where in our case the bottom left block of the matrix T becomes:

$$T_{21} = -\frac{2}{\Delta}\frac{\sum_{j=1}^{2} 1((1-\frac{j-1}{2})^1 - (1-\frac{j}{2})^1)}{\sum_{j=1}^{2} 1((1-\frac{j-1}{2})^2 - (1-\frac{j}{2})^2)} \tag{58}$$

After this we can say that, for $\Delta \to 0$, the approximate sampled-data model (38) has sampling zero dynamics that converge to linear dynamics whose eigenvalues are given by the roots of S_{GOH} (55) with relative degree equal to 2.

5.3 Systems with pure time-delay

Another non-Standard model is the case in which the system has a pure time-delay, when the input is generated by a Zero order-hold (ZOH).

We will see, as before, the case of the system of integrator, the linear and non-linear system. In these case we will consider a delay of the type $D = l\Delta + f\Delta$, where $l \in \mathbb{N}$ and $f \in [0, 1]$. To simplify the formulas we choose $l = 1$, with a generic f.

5.3.1 System of integrator

In this case, using a ZOH we have an input of the form:

$$u(t) = \begin{cases} u_{k-1-1} & k\Delta \leq t < k\Delta + f\Delta \\ u_{k-1} & k\Delta + f\Delta \leq t < (k+1)\Delta \end{cases} \tag{32}$$

The difference in this case is that the system is

$$G(s) = \frac{e^{-sD}}{s^3} \tag{33}$$

Then the exact sampled data model is the following

$$x_{k+1} = A_d x_k + B_{delay} u_{k-1} \tag{34}$$

Where A_d is always the same and

$$B_{delay} = \begin{bmatrix} \frac{\Delta^3}{3!}(q^{-1} - q^{-1}(1-f)^3 + (1-f)^3) \\ \frac{\Delta^2}{2!}(q^{-1} - q^{-1}(1-f)^2 + (1-f)^2) \\ \frac{\Delta^1}{1!}(q^{-1} - q^{-1}(1-f)^1 + (1-f)^1) \end{bmatrix} \tag{35}$$

After this we can describe the transfer function

$$G_r(z) = \frac{\Delta^3 \cdot S_{delay}(z, f)}{3! \cdot z^{1+1}(z-1)^3} \tag{36}$$

with

$$S_{delay} = B_3'(z, 0) - B_3'(z, f) + zB_3'(z, f) \tag{37}$$

As we expected, the sampling zeros are zeros which depends from the Modified Euler-Frobenius polynomials.

5.3.2 Linear system

Now, we will consider a general linear system with a delay. The system we will use has as transfer function

$$G(s) = \frac{e^{-sD}}{s+1} \tag{38}$$

As the previous chapter we will use the exact definition of the inverse of the Laplace transform and of the Z-Transform. So, we will obtain an integral similar to (28) (it's needed the transfer function of the zero order-hold instead of the partial zero order-hold),and from comparison to the integral obtained with a system of integrator (29) we will show that

$$\lim_{\Delta \to 0} \Delta^{-1} G_r(z) = \frac{10^1 \cdot S_{delay}(z, f)}{1! \cdot z^{1+1}(z-1)^1} \tag{39}$$

where S_{delay} is defined in and we can see how, also in this case, the sampling zeros converge to the roots of the polynomial which depends from the standard and Euler-Frobenius polynomial.

5.3.3 Affine non-linear system

As we expected, choosing the same system of the previous chapter and following the same steps, we will obtain an approximated model of the original system with a zero dynamics which eigenvalues converges, as $\Delta \to 0$, to the roots of the polynomial S_{delay}.

6 Conclusion

During this paper we introduced the standard and modified Euler-Frobenius polynomials and their application to sampled data-model. We have seen the case of a partial zero order-hold, a generalised zero order-hold and a system with non-zero delay. In each of these cases we have also seen how these kind of polynomial characterize the sampling zeros and the sampling zero dynamic of a linear and non-linear system.