Modified Euler-Frobenius Polynomials with application to Sampled Data Model

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Outline

- Introduction
- Euler-Frobenius Polynomials
  - Standard
  - Modified
- Preliminary definitions
  - Multiple Integrations
  - Zero Dynamics
- Applications
- Conclusions

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Why?

Euler-Frobenius Polynomials are widely used in math and they play a key role in many engeneering areas.

They appears in in sampled-data models both in linear and non linear systems, polynomial interpolation and splines.

With the evolution of Standard to Modified Euler-Frobenius Polynomials we are trying to find a unified framework to identify the zeroes in the sampled-data models

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Standard Euler-Frobenius polynomials

$$B_r(z) = r! \cdot det M_r$$

$$M_r = \begin{bmatrix} 1 & \frac{1}{2!} & \dots & \frac{1}{(r-1)!} & \frac{1}{r!} \\ 1-z & 1 & \frac{1}{2!} & \dots & \frac{1}{(r-1)!} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & \dots & 1-z & 1 & \frac{1}{2!} \\ 0 & \dots & 0 & 1-z & 1 \end{bmatrix}$$

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Modified Euler-Frobenius polynomials

$$B'_r(z, f) = r! \cdot det P_r$$

$$P_r = \begin{bmatrix} 1 & \frac{1}{2!} & \dots & \frac{1}{(r-1)!} & \frac{1-f}{r!} \\ 1-z & 1 & \frac{1}{2!} & \dots & \frac{1-f}{(r-1)!} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & \dots & 1-z & 1 & \frac{1-f}{2!} \\ 0 & \dots & 0 & 1-z & 1-f \end{bmatrix}$$

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Preliminar(1): Multiple integration

Define: $I(r, f, \Delta, g) = \int_{f\Delta}^{\Delta} \int_{f\Delta}^{t_{r-1}} \dots \int_{f\Delta}^{t_1} g(t) dt \dots dt_{r-1}$

And in the special case: $g = 1$

$$I(j, f, \Delta, g) = \frac{\Delta^j (1-f)^j}{j!}$$

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Preliminar(2): Zero-dynamics

Internal Dynamics
when input and initial conditions
are chosen to make output
identically zero

$$
\begin{aligned}
\dot{z}_1(t) &= z_2(t) \\
\dot{z}_2(t) &= z_3(t) \\
\dot{z}_3(t) &= b(z,\eta) + a(z,\eta)u(t) \\
\dot{\eta}(t) &= \eta(z,\eta) \\
y(t) &= z_1(t)
\end{aligned}
$$

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Applications Outline

- Partial zero-order hold
  - System of integrators
  - Linear system
  - Affine nonlinear system
- Piecewise Constant generalised hold
  - System of integrators
  - Linear system
  - Affine nonlinear system
- Pure time-delay
  - System of integrators
  - Linear system
  - Affine nonlinear system

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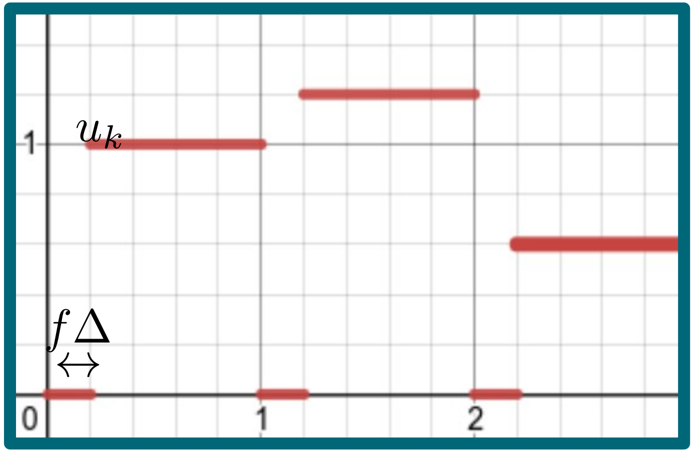
Partial zero-order hold

$$u(t) = \begin{cases} 0 & k\Delta \leq t \leq k\Delta + f\Delta \\ u_k & k\Delta + f\Delta \leq t \leq (k+1)\Delta \end{cases}$$

with $k \in \mathbb{N}$ and $f \in [0, 1]$



And the transfer function

$$G_{PZOH}(s) = \frac{e^{sf\Delta} - e^{-s\Delta}}{s}$$

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System of Integrator: $G(s) = 1/s^3$

Discretize the system

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & \Delta & \frac{\Delta^2}{2!} \\ 0 & 1 & \Delta \\ 0 & 0 & 1 \end{bmatrix}}_{A_d} \cdot x_k + \underbrace{\begin{bmatrix} \frac{\Delta^3(1-f)^3}{3!} \\ \frac{\Delta^2(1-f)^2}{2!} \\ \Delta(1-f) \end{bmatrix}}_{B_{PZOH}} \cdot u_k$$

We rewrite the system using the forward shift operator q

$$\begin{bmatrix} (q-1)x_{1,k} & 0 & 0 \end{bmatrix}^T = D_3 \begin{bmatrix} x_{2,k} & x_{3,k} & u_k \end{bmatrix}^T$$

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System of Integrator: $G(s) = 1/s^3$

Where $D_3 = \begin{bmatrix} \Delta & \frac{\Delta^2}{2!} & \frac{\Delta^3(1-f)^3}{3!} \\ -(q-1) & \Delta & \frac{\Delta^2(1-f)^2}{2!} \\ 0 & -(q-1) & \Delta(1-f) \end{bmatrix}$

Now we can use the Cramer's rule

$u_k = \frac{DetN}{DetD_3}$ Where $N = \begin{bmatrix} \Delta & \frac{\Delta^2}{2!} & (q-1)x_{1,k} \\ -(q-1) & \Delta & 0 \\ 0 & -(q-1) & 0 \end{bmatrix}$

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System of Integrator: $G(s) = 1/s^3$

$$DetN = (q-1)^3 x_{1,k}$$

$$DetD_3 = \frac{\Delta^3}{3!} B_3'(q,f)$$

$$\Rightarrow$$

$$u_k = \frac{(q-1)^3 x_{1,k}}{\frac{\Delta^3}{3!} B_3'(q,f)}$$

Notice that $y=x_1$ and take y over u_k

$$G_q(z) = \frac{\Delta^3 B_3'(z,f)}{3!(z-1)^3}$$

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General Linear Sys

Take a linear sys example

$$G(s) = 10\frac{s+1}{(s+10)(s+100)(s+1000)}$$

Use exact definition of the inverse laplace transform and z transform

$$G_q(z) = Z\{L^{-1}\{G(s)G_{PZOH}(s)\}\}$$

$$\Big\downarrow$$

$$G_q(z) = \frac{1}{2\pi j}\int_{-\gamma+j\infty}^{\gamma+j\infty}\frac{e^{s\Delta}}{z-e^{s\Delta}}G(s)\frac{e^{sf\Delta}-e^{-s\Delta}}{s}ds$$

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General Linear Sys

$$G_q(z) = \frac{1}{2\pi j}\int_{j\infty}^{j\infty} \frac{e^w}{z-e^w}\frac{10}{w^2}\frac{e^{wf}-e^{-w}}{w}dw \qquad (1)$$

$$G_q(z) = \frac{1}{2\pi j}\int_{j\infty}^{j\infty} \frac{e^w}{z-e^w}\frac{\Delta^2}{w^2}\frac{e^{wf}-e^{-w}}{w}dw \qquad (2)$$

Now compare the results from the linear case (1) and the chain of integrators case(2)

We already knew the result of the first integral, so by comparison we can define

$$\lim_{\Delta\to 0}\Delta^{-2}G_q(z) = \frac{10B_2'(z,f)}{2!(z-1)^2}$$

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Affine nonlinear System

$$\begin{aligned} \dot{x_1}(t) &= x_2(t) \\ \dot{x_2}(t) &= x_2^2(t)x_1(t) + x_1(t) + u(t) \end{aligned} \xrightarrow{\begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix}} \begin{aligned} \dot{z_1}(t) &= z_2(t) \\ \dot{z_2}(t) &= \underbrace{z_1(t) + z_2^2(t)z_1(t)}_{b} + \underbrace{1}_{a} u(t) \end{aligned}$$

We take the integral from the last variable up to the first one

$$z_1((k+1)\Delta) = z_1(k\Delta) + \Delta z_2(k\Delta) + I(2, 0, \Delta, (b(z(t)) + a(z(t))u(t)))$$
$$z_2((k+1)\Delta) = z_2(k\Delta) + I(1, 0, \Delta, (b(z(t)) + a(z(t))u(t)))$$

We can rewrite the integrals as

$$I(2, 0, \Delta, (b(z(t)) + a(z(t))u(t))) = b(z(t))I(2, 0, \Delta, 1) + a(z(t))I(2, f, \Delta, 1)u_k$$

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Affine nonlinear System

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d b(\hat{z}_k) + B_{PZOH} a(\hat{z}_k) u_k \qquad B_d = \begin{bmatrix} \frac{\Delta^2}{2!} \\ \frac{\Delta}{1} \end{bmatrix}$$

Then use a coordinates change

r=2

$$T = \begin{bmatrix} 1 & 0_{1\times r-1} \\ -\frac{2}{\Delta(1-f)} & I_{r-1} \end{bmatrix}$$

$$\tilde{z}_{1,k+1} = q_{11}\tilde{z}_{1,k} + Q_{12}\tilde{z}_{2,k} + \frac{\Delta^1}{1!}(b(\tilde{z}_k + (1-f)^1 a(\tilde{z}_k)u_k)$$

$$\tilde{z}_{2,k+1} = Q_{21}\tilde{z}_{1,k} + Q_{22}\tilde{z}_{2,k} + \tilde{B}_2 b(\tilde{z}_k)$$

Affine nonlinear System

$$\tilde{z}_{2,k+1} = \underbrace{T_{21}A_{12} + A_{22} = -\frac{2}{\Delta(1-f)}(\Delta+1)}\cdot \tilde{z}_{2,k}$$

(Where A_{11},A_{22} are partitions from A_d)

Now we can show that the eigenvalues of Q_{22} are the roots of the Modified Euler-Frobenius polnomials

$$D_{r=2}\begin{bmatrix} 0 & I_{r-1} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\Delta^2(1-f)^2}{2!} & A_{12} \\ -\frac{\Delta^2(1-f)^2}{2!}T_{21} & A_{22}-z \end{bmatrix}$$

Take both sides determinant and

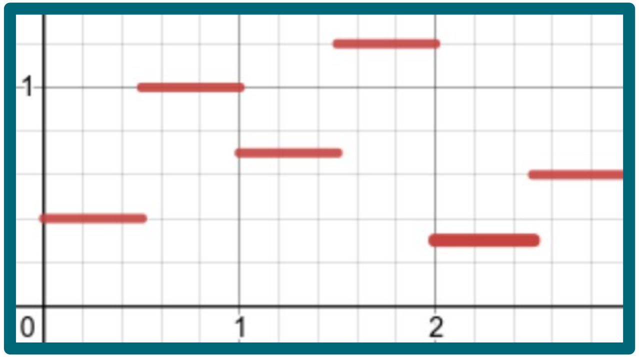
$$B_2'(f,z) = (1-f)^2 det(z - Q_{22})$$

Piecewise Constant Generalized Hold Fuctions

$$u(t) = \begin{cases} c_1 u_k & k\Delta \leq t < k\Delta + \Delta/m \\ \dots & \\ \dots & \\ c_m u_k & k\Delta + (m-1)\Delta/m \leq t < k\Delta + \Delta \end{cases}$$

and with the transfer function

$$G_{GOH}(s) = \frac{\sum_{j=1}^{m} c_j (e^{-s\Delta\frac{(j-1)}{m}} - e^{-s\Delta\frac{j}{m}})}{s}$$



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System of Integrator: $G(s) = 1/s^3$

Here we consider the input function with m=2
and our system in discrete form becomes

$$x_{k+1} = A_d x_k + B_{GOH} u_k$$

$$with \quad B_{GOH} = \begin{bmatrix} \sum_{j=1}^{m} \left(\frac{\Delta^3 (1 - \frac{j-1}{2})}{3!} \right) - \left(\frac{\Delta^3 (1 - \frac{j}{2})}{3!} \right) \\ \sum_{j=1}^{m} \left(\frac{\Delta^2 (1 - \frac{j-1}{2})}{2!} \right) - \left(\frac{\Delta^2 (1 - \frac{j}{2})}{2!} \right) \\ \sum_{j=1}^{m} \left(\frac{\Delta^1 (1 - \frac{j-1}{2})}{1!} \right) - \left(\frac{\Delta^1 (1 - \frac{j}{2})}{1!} \right) \end{bmatrix}$$

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System of Integrator

We can compute the transfer function obtaining

$$G_q(s) = \frac{\Delta^3 S_{GOH}(z)}{3!(z-1)^3}$$

Where S_{GOH} depends from the polynomials

$$S_{GOH} = \sum_{j=1}^{m=2} [B_3'(z, \frac{(j-1)}{2}) - B_3'(z, \frac{j}{2})]$$

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Linear System

Write something her

Move to z domain

$$G_q(z) = Z\{L^{-1}\{G(s)G_{GOH}(s)\}\}$$

$$with \quad G_{GOH}(s) = \frac{\sum_{j=1}^{m} c_j(e^{-s\Delta\frac{(j-1)}{m}} - e^{-s\Delta\frac{j}{m}})}{s}$$

By Comparison with System of Integrator

$$\lim_{\Delta\to 0} \Delta^{-r} G_q(s) = \frac{k \cdot (z-1)^{|Zeros|} S_{GOH}(z)}{(|Poles| - |Zeros|)!(z-1)^{|Poles|}}$$

$$S_{GOH}(z) = \sum_{j=1}^{m} [c_j[B'_r(z, \frac{(j-1)}{m}) - B'_r(z, \frac{j}{m}])$$

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Linear System

$$G(s) = 10\frac{s+1}{(s+10)(s+100)(s+1000)}$$

$$S_{GOH}(z) = \sum_{j=1}^{m}[c_j[B_r'(z, \frac{(j-1)}{m}) - B_r'(z, \frac{j}{m}])$$

$$c_j = 1, m = 2$$

$$S_{GOH}(z) = \sum_{j=1}^{2}[B_2'(z, \frac{(j-1)}{2}) - B_2'(z, \frac{j}{2})$$

$$\frac{k \cdot (z-1)^{|Zeros|} S_{GOH}(z)}{(|Poles| - |Zeros|)!(z-1)^{|Zeros|}}$$

$$\lim_{\Delta \to 0} \Delta^{-2} G_q(z) = \frac{10(z-1)^1 S_{GOH}(z)}{2!(z-1)^3}$$

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Affine nonlinear System

$$\dot{x_1}(t) = x_2(t)$$
$$\dot{x_2}(t) = x_2^2(t)x_1(t) + x_1(t) + u(t)$$

$$\xrightarrow{\begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix}}$$

$$\dot{z_1}(t) = z_2(t)$$
$$\dot{z_2}(t) = \underbrace{z_1(t) + z_2^2(t)z_1(t)}_{b} + \underbrace{1}_{a} u(t)$$

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d b(\hat{z}_k) + B_{GOH} a(\hat{z}_k) u_k$$

$$B_{GOH} = \begin{bmatrix} \sum_{j=1}^{m} \left(\frac{\Delta^2 (1-\frac{j-1}{2})}{2!}\right) - \left(\frac{\Delta^2 (1-\frac{j}{2})}{2!}\right) \\ \sum_{j=1}^{m} \left(\frac{\Delta^1 (1-\frac{j-1}{2})}{1!}\right) - \left(\frac{\Delta^1 (1-\frac{j}{2})}{1!}\right) \end{bmatrix}$$

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Affine nonlinear System

We can again move the sys to other coordinates

$$T = \begin{bmatrix} 1 & 0_{1\times r-1} \\ T_{21} & I_{r-1} \end{bmatrix} \qquad T_{21} = -\frac{2}{\Delta}\frac{\sum_{j=1}^{2} 1((1-\frac{j-1}{2})^1 - (1-\frac{j}{2})^1)}{\sum_{j=1}^{2} 1((1-\frac{j-1}{2})^2 - (1-\frac{j}{2})^2)}$$

We can compute the zero dynamics, and as in the previous case, we can see that the eigenvalues of Q_{22} converge to the roots of S_{GOH}

$$S_{GOH}(z) = \sum_{j=1}^{m}[c_j[B_2'(z, \frac{(j-1)}{2}) - B_2'(z, \frac{j}{2}])$$

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Systems with pure time-delay

Another non standard model we can deal with,
is the case in which the system has a pure time-delay,
and the input is generated by a Zero order-hold(ZOH)

$$L[f(t-D)] = e^{-sD}F(s)$$

$$G_{ZOH} = \frac{1-e^{-s\Delta}}{s}$$

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System of integrator: $G(s) = \dfrac{e^{-sD}}{s^3}$

$$D = \Delta l + \Delta f \quad where\ l \in N\ , f \in [0,1)$$

Discretize as

$$x_{k+1} = A_d x_k + B_{delay} u_{k-l}$$

Using

$$B_{delay} = \begin{bmatrix} \frac{\Delta^3}{3!}(q^{-1} - q^{-1}(1-f)^3 + (1-f)^3) \\ \frac{\Delta^2}{2!}(q^{-1} - q^{-1}(1-f)^2 + (1-f)^2) \\ \frac{\Delta^1}{1!}(q^{-1} - q^{-1}(1-f)^1 + (1-f)^1) \end{bmatrix}$$

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System of integrator

$$u(t) = \begin{cases} u_{k-l-1} & k\Delta \leq t < k\Delta + f\Delta \\ u_{k-1} & k\Delta + f\Delta \leq t < (k+1)\Delta \end{cases}$$

Expressed As

$$u(t) = u_{k-l-1}(\mu(t) - \mu(t - f\Delta)) + u_{k-l}\mu(t - f\Delta)$$

Becomes

$$I(i, 0, \Delta, u(t)) = I(i, 0, \Delta, u_{k-l-1}) - I(i, f, \Delta, u_{k-l-1}) + I(i, f, \Delta, u_{k-1})$$

Then as in the previous cases

$$G_r(z) = \frac{\Delta^3 \cdot S_{delay}(z, f)}{3! \cdot z^{l+1}(z-1)^3} \qquad S_{delay} = B_3'(z, 0) - B_3'(z, f) + zB_3'(z, f)$$

Linear System

Move the system G(s) from s-domain to z-domain

$$G_q(z) = Z\{L^{-1}\{\bar{G}(s)G_{ZOH}(s)\}\}$$

By Comparison

$$\lim_{\Delta\to 0}\Delta^{-r}G_q(s) = \frac{k\cdot(z-1)^{|Zeros|}S_{Delay}(z)}{(|Poles|-|Zeros|)!z^{l+1}(z-1)^{|Zeros|}}$$

Having S_{delay} as before

$$S_{delay}(z,f) = B_r'(z,0) - B_r'(z,f) + zB_r'(z,f)$$

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Affine nonlinear System

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_2^2(t)x_1(t) + x_1(t) + u(t)$$

$$\xrightarrow{\begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix}}$$

$$\dot{z}_1(t) = z_2(t)$$
$$\dot{z}_2(t) = \underbrace{z_1(t) + z_2^2(t)z_1(t)}_{b} + \underbrace{1}_{a} u(t)$$

$$\hat{z}_{k+1} = A_d\hat{z}_k + B_d b(\hat{z}_k) + B_{Delay}a(\hat{z}_k)u_k$$

$$B_{delay} = \begin{bmatrix} \frac{\Delta^2}{2!}(q^{-1} - q^{-1}(1-f)^2 + (1-f)^2) \\ \frac{\Delta^1}{1!}(q^{-1} - q^{-1}(1-f)^1 + (1-f)^1) \end{bmatrix}$$

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Affine nonlinear System

We can again move the sys to other coordinates

$$T = \begin{bmatrix} 1 & 0_{1\times r-1} \\ T_{21} & I_{r-1} \end{bmatrix} \qquad T_{21} = -\begin{bmatrix} \frac{r}{\Delta}\cdot\frac{q^{-1}(1-(1-f)^{n-1})+(1-f)^{n-1}}{q^{-1}(1-(1-f)^{n})+(1-f)^{n}} \\ \dots \\ \frac{r(r-1)\dots 2}{\Delta^{r-1}}\cdot\frac{q^{-1}(1-(1-f)^{1})+(1-f)^{1}}{q^{-1}(1-(1-f)^{n})+(1-f)^{n}} \end{bmatrix}$$

We can compute the zero dynamics, and as in the previous case, we can see that the eigenvalues of Q_{22} converge to the roots of S_{GOH}

$$S_{GOH}(z) = \sum_{j=1}^{m}[c_j[B_2'(z, \frac{(j-1)}{2}) - B_2'(z, \frac{j}{2}])$$

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Conclusions

Modified Eulero-Frobinius Polynomials

System Integrator → PZOH

Linear System → GOH

Affine nl sys → Pure Delay with ZOH

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