Modified Euler-Frobenius Polynomials with application to Sampled Data Model

Sciamanna Daniele
Rotondi Simone

Outline

Sciamanna Daniele
Rotondi Simone

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Why?

Euler-Frobenius Polynomials are widely used in math and they play a key role in many engeneering areas.

They appears in in sampled-data models both in linear and non linear systems, polynomial interpolation and splines.

With the evolution of Standard to Modified Euler-Frobenius Polynomials we are trying to find a unified framework to identify the zeroes in the sampled-data models

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Standard Euler-Frobenius polynomials

$$B_r(z) = r! \cdot det M_r$$

$$M_r = \begin{bmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} & \frac{1}{r!} \\ 1-z & 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdots & 1-z & 1 & \frac{1}{2!} \\ 0 & \cdots & 0 & 1-z & 1 \end{bmatrix}$$

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Modified Euler-Frobenius polynomials

$$B'_r(z, f) = r! \cdot det P_r$$

$$P_r = \begin{bmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} & \frac{1-f}{r!} \\ 1-z & 1 & \frac{1}{2!} & \cdots & \frac{1-f}{(r-1)!} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & \cdots & 1-z & 1 & \frac{1-f}{2!} \\ 0 & \cdots & 0 & 1-z & 1-f \end{bmatrix}$$

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Preliminar(1): Multiple integration

Define: $I(r, f, \Delta, g) = \displaystyle\int_{f\Delta}^{\Delta} \int_{f\Delta}^{t_{r-1}} ... \int_{f\Delta}^{t_1} g(t)dt...dt_{r-1}$

And in the special case: $g = 1$

$$I(j, f, \Delta, g) = \frac{\Delta^j (1 - f)^j}{j!}$$

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Preliminar(2): Zero-dynamics

Internal Dynamics
when input and initial conditions
are chosen to make output
identically zero

$$\dot{z}_1(t) = z_2(t)$$
$$\dot{z}_2(t) = z_3(t)$$
$$\dot{z}_3(t) = b(z, \eta) + a(z, \eta)u(t)$$
$$\dot{\eta}(t) = \eta(z, \eta)$$
$$y(t) = z_1(t)$$

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Applications Outline

- Partial zero-order hold
  - System of integrators
  - Linear system
  - Affine nonlinear system
- Piecewise Constant generalised hold
  - System of integrators
  - Linear system
  - Affine nonlinear system
- Pure time-delay
  - System of integrators
  - Linear system
  - Affine nonlinear system
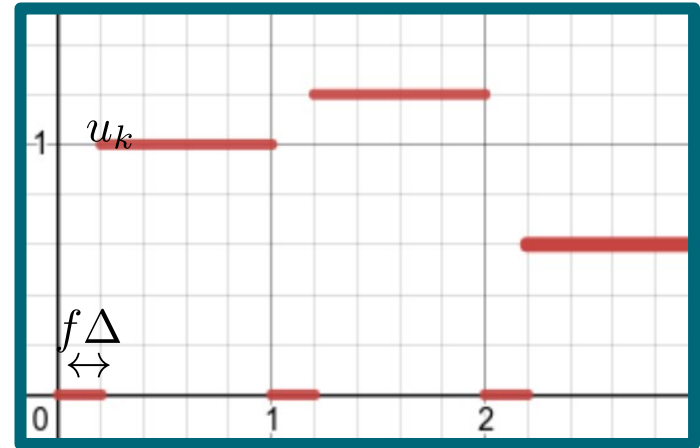
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Partial zero-order hold

$$u(t) = \begin{cases} 0 & k\Delta \leq t \leq k\Delta + f\Delta \\ u_k & k\Delta + f\Delta \leq t \leq (k+1)\Delta \end{cases}$$

with $k \in \mathbb{N}$ and $f \in [0,1]$

And the transfer function

$$G_{PZOH}(s) = \frac{e^{sf\Delta} - e^{-s\Delta}}{s}$$

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System of Integrator: $G(s) = 1/s^3$

Discretize the system

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & \Delta & \frac{\Delta^2}{2!} \\ 0 & 1 & \Delta \\ 0 & 0 & 1 \end{bmatrix}}_{A_d} \cdot x_k + \underbrace{\begin{bmatrix} \frac{\Delta^3(1-f)^3}{3!} \\ \frac{\Delta^2(1-f)^2}{2!} \\ \Delta(1-f) \end{bmatrix}}_{B_{PZOH}} \cdot u_k$$

We rewrite the system using the forward shift operator q

$$\begin{bmatrix} (q-1)x_{1,k} & 0 & 0 \end{bmatrix}^T = D_3 \begin{bmatrix} x_{2,k} & x_{3,k} & u_k \end{bmatrix}^T$$

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System of Integrator: $G(s) = 1/s^3$

Where $\quad D_3 = \begin{bmatrix} \Delta & \frac{\Delta^2}{2!} & \frac{\Delta^3(1-f)^3}{3!} \\ -(q-1) & \Delta & \frac{\Delta^2(1-f)^2}{2!} \\ 0 & -(q-1) & \Delta(1-f) \end{bmatrix}$

Now we can use the Cramer's rule

$$u_k = \frac{Det N}{Det D_3} \quad \text{Where} \quad N = \begin{bmatrix} \Delta & \frac{\Delta^2}{2!} & (q-1)x_{1,k} \\ -(q-1) & \Delta & 0 \\ 0 & -(q-1) & 0 \end{bmatrix}$$

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System of Integrator: $G(s) = 1/s^3$

$$DetN = (q-1)^3 x_{1,k}$$

$$DetD_3 = \frac{\Delta^3}{3!} B_3'(q,f)$$

$\longrightarrow$

$$u_k = \frac{(q-1)^3 x_{1,k}}{\frac{\Delta^3}{3!} B_3'(q,f)}$$

Notice that y=x$_1$ and take y over u$_k$

$$G_q(z) = \frac{\Delta^3 B_3'(z,f)}{3!(z-1)^3}$$

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General Linear Sys
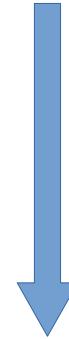
Take a linear sys example

$$G(s) = 10 \frac{s+1}{(s+10)(s+100)(s+1000)}$$

Use exact definition of the inverse laplace transform and z transform

$$G_q(z) = Z\{L^{-1}\{G(s)G_{PZOH}(s)\}\}$$

$$G_q(z) = \frac{1}{2\pi j} \int_{-\gamma+j\infty}^{\gamma+j\infty} \frac{e^{s\Delta}}{z - e^{s\Delta}} G(s) \frac{e^{sf\Delta} - e^{-s\Delta}}{s} ds$$

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General Linear Sys

$$G_q(z) = \frac{1}{2\pi j} \int_{j\infty}^{j\infty} \frac{e^w}{z - e^w} \frac{10}{w^2} \frac{e^{wf} - e^{-w}}{w} dw \quad (1)$$

$$G_q(z) = \frac{1}{2\pi j} \int_{j\infty}^{j\infty} \frac{e^w}{z - e^w} \frac{\Delta^2}{w^2} \frac{e^{wf} - e^{-w}}{w} dw \quad (2)$$

Now compare the results from the linear case (1) and the chain of integrators case(2)

We already knew the result of the first integral, so by comparison we can define

$$\lim_{\Delta \to 0} \Delta^{-2} G_q(z) = \frac{10 B_2'(z, f)}{2!(z - 1)^2}$$

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Affine nonlinear System

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_2^2(t)x_1(t) + x_1(t) + u(t)$$

$$\begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix}$$

$$\dot{z}_1(t) = z_2(t)$$
$$\dot{z}_2(t) = \underbrace{z_1(t) + z_2^2(t)z_1(t)}_{b} + \underbrace{1}_{a} u(t)$$

We take the integral from the last variable up to the first one

$$z_1((k+1)\Delta) = z_1(k\Delta) + \Delta z_2(k\Delta) + I(2, 0, \Delta, (b(z(t)) + a(z(t))u(t)))$$
$$z_2((k+1)\Delta) = z_2(k\Delta) + I(1, 0, \Delta, (b(z(t)) + a(z(t))u(t)))$$

We can rewrite the integrals as

$$I(2, 0, \Delta, (b(z(t)) + a(z(t))u(t))) = b(z(t))I(2, 0, \Delta, 1) + a(z(t))I(2, f, \Delta, 1)u_k$$

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Affine nonlinear System

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d b(\hat{z}_k) + B_{PZOH} a(\hat{z}_k) u_k \qquad B_d = \begin{bmatrix} \frac{\Delta^2}{2!} \\ \frac{\Delta}{1} \end{bmatrix}$$

Then use a coordinates change

$$T = \begin{bmatrix} 1 & 0_{1 \times r-1} \\ -\frac{2}{\Delta(1-f)} & I_{r-1} \end{bmatrix}$$

r=2

$$\tilde{z}_{1,k+1} = q_{11}\tilde{z}_{1,k} + Q_{12}\tilde{z}_{2,k} + \frac{\Delta^1}{1!}(b(\tilde{z}_k) + (1-f)^1 a(\tilde{z}_k) u_k)$$

$$\tilde{z}_{2,k+1} = Q_{21}\tilde{z}_{1,k} + Q_{22}\tilde{z}_{2,k} + \tilde{B}_2 b(\tilde{z}_k)$$

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Affine nonlinear System

$$\tilde{z}_{2,k+1} = \underbrace{T_{21}A_{12} + A_{22}}_{} = \left(-\frac{2}{\Delta(1-f)}\Delta + 1\right) \cdot \tilde{z}_{2,k}$$

(Where A_{11}, A_{22} are partitions from A_d)

Now we can show that the eigenvalues of Q_{22} are the roots of the Modified Euler-Frobenius polnomials

$$D_{r=2}\begin{bmatrix} 0 & I_{r-1} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\Delta^2(1-f)^2}{2!} & A_{12} \\ -\frac{\Delta^2(1-f)^2}{2!}T_{21} & A_{22} - z \end{bmatrix}$$
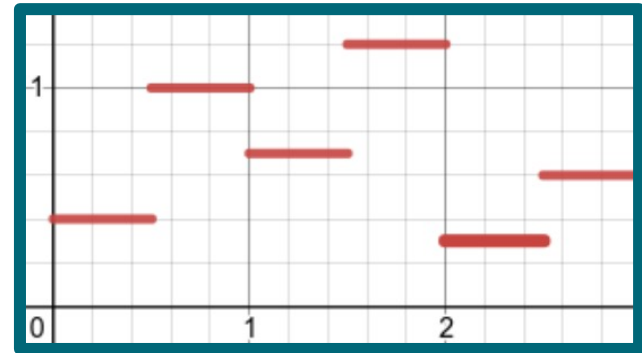
Take both sides determinant and

$$B_2'(f,z) = (1-f)^2 det(z - Q_{22})$$

Piecewise Constant Generalized Hold Fuctions

$$u(t) = \begin{cases} c_1 u_k & k\Delta \leq t < k\Delta + \Delta/m \\ \dots \\ \dots \\ c_m u_k & k\Delta + (m-1)\Delta/m \leq t < k\Delta + \Delta \end{cases}$$

and with the transfer function

$$G_{GOH}(s) = \frac{\sum_{j=1}^{m} c_j (e^{-s\Delta\frac{(j-1)}{m}} - e^{-s\Delta\frac{j}{m}})}{s}$$
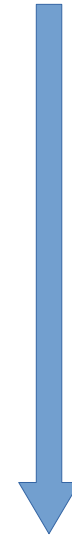
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System of Integrator

We can compute the transfer function obtaining

$$G_q(s) = \frac{\Delta^3 S_{GOH}(z)}{3!(z-1)^3}$$

Where S_{GOH} depends from the polynomials

$$S_{GOH} = \sum_{j=1}^{m=2} [B_3'(z, \frac{(j-1)}{2}) - B_3'(z, \frac{j}{2})]$$

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System of Integrator: $G(s) = 1/s^3$

Here we consider the input function with m=2
and our system in discrete form becomes

$$x_{k+1} = A_d x_k + B_{GOH} u_k$$

$$with \quad B_{GOH} = \begin{bmatrix} \sum_{j=1}^{m}(\frac{\Delta^3(1-\frac{j-1}{2})}{3!}) - (\frac{\Delta^3(1-\frac{j}{2})}{3!}) \\ \sum_{j=1}^{m}(\frac{\Delta^2(1-\frac{j-1}{2})}{2!}) - (\frac{\Delta^2(1-\frac{j}{2})}{2!}) \\ \sum_{j=1}^{m}(\frac{\Delta^1(1-\frac{j-1}{2})}{1!}) - (\frac{\Delta^1(1-\frac{j}{2})}{1!}) \end{bmatrix}$$

$$u(t) = \sum_{j=1}^{m} c_j [\mu(t - \frac{(j-1)\Delta}{m} - \mu(t - \frac{j\Delta}{m})]$$

Linear System

Move to z domain

$$G_q(z) = Z\{L^{-1}\{G(s)G_{GOH}(s)\}\}$$

$$with \quad G_{GOH}(s) = \frac{\sum_{j=1}^{m} c_j(e^{-s\Delta\frac{(j-1)}{m}} - e^{-s\Delta\frac{j}{m}})}{s}$$

By Comparison with System of Integrator

$$\lim_{\Delta\to 0} \Delta^{-r} G_q(s) = \frac{k \cdot (z-1)^{|Zeros|} S_{GOH}(z)}{(|Poles| - |Zeros|)!(z-1)^{|Poles|}}$$

$$S_{GOH}(z) = \sum_{j=1}^{m}[c_j[B_r'(z, \frac{(j-1)}{m}) - B_r'(z, \frac{j}{m})])$$

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Rotondi Simone

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Linear System

$$G(s) = 10\frac{s+1}{(s+10)(s+100)(s+1000)}$$

$$S_{GOH}(z) = \sum_{j=1}^{m} [c_j[B_r'(z, \frac{(j-1)}{m}) - B_r'(z, \frac{j}{m})])$$

$$c_j = 1, m = 2$$

$$S_{GOH}(z) = \sum_{j=1}^{2} [B_2'(z, \frac{(j-1)}{2}) - B_2'(z, \frac{j}{2})]$$

$$\frac{k \cdot (z-1)^{|Zeros|} S_{GOH}(z)}{(|Poles| - |Zeros|)!(z-1)^{|Zeros|}}$$

$$\lim_{\Delta \to 0} \Delta^{-2} G_q(z) = \frac{10(z-1)^1 S_{GOH}(z)}{2!(z-1)^3}$$

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Affine nonlinear System

$$\dot{x}_1(t) = x_2(t)$$
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$$\begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix}$$

$$\dot{z}_1(t) = z_2(t)$$
$$\dot{z}_2(t) = \underbrace{z_1(t) + z_2^2(t)z_1(t)}_{b} + \underbrace{1}_{a} u(t)$$

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d b(\hat{z}_k) + B_{GOH} a(\hat{z}_k) u_k$$

$$B_{GOH} = \begin{bmatrix} \sum_{j=1}^{m} \left(\frac{\Delta^2(1 - \frac{j-1}{2})}{2!} \right) - \left(\frac{\Delta^2(1 - \frac{j}{2})}{2!} \right) \\ \sum_{j=1}^{m} \left(\frac{\Delta^1(1 - \frac{j-1}{2})}{1!} \right) - \left(\frac{\Delta^1(1 - \frac{j}{2})}{1!} \right) \end{bmatrix}$$

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Affine nonlinear System

We can again move the sys to other coordinates

$$T = \begin{bmatrix} 1 & 0_{1 \times r-1} \\ T_{21} & I_{r-1} \end{bmatrix} \qquad T_{21} = -\frac{2}{\Delta} \frac{\sum_{j=1}^{2} 1((1 - \frac{j-1}{2})^1 - (1 - \frac{j}{2})^1)}{\sum_{j=1}^{2} 1((1 - \frac{j-1}{2})^2 - (1 - \frac{j}{2})^2)}$$

We can compute the zero dynamics, and as in the previous case, we can see that the eigenvalues of Q_{22} converge to the roots of S_{GOH}

$$S_{GOH}(z) = \sum_{j=1}^{m} [c_j [B_2'(z, \frac{(j-1)}{2}) - B_2'(z, \frac{j}{2})])$$

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Systems with pure time-delay

Another non standard model we can deal with,
is the case in which the system has a pure time-delay,
and the input is generated by a Zero order-hold(ZOH)

$$L[f(t - D)] = e^{-sD} F(s)$$

$$G_{ZOH} = \frac{1 - e^{-s\Delta}}{s}$$

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System of integrator: $G(s) = \dfrac{e^{-sD}}{s^3}$

$$D = \Delta l + \Delta f \quad where\ l \in N\ , f \in [0, 1)$$

Discretize as

$$x_{k+1} = A_d x_k + B_{delay} u_{k-l}$$

Using

$$B_{delay} = \begin{bmatrix} \frac{\Delta^3}{3!}\left(q^{-1} - q^{-1}(1-f)^3 + (1-f)^3\right) \\ \frac{\Delta^2}{2!}\left(q^{-1} - q^{-1}(1-f)^2 + (1-f)^2\right) \\ \frac{\Delta^1}{1!}\left(q^{-1} - q^{-1}(1-f)^1 + (1-f)^1\right) \end{bmatrix}$$

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System of integrator

$$u(t) = \begin{cases} u_{k-l-1} & k\Delta \leq t < k\Delta + f\Delta \\ u_{k-1} & k\Delta + f\Delta \leq t < (k+1)\Delta \end{cases}$$

Expressed As

$$u(t) = u_{k-l-1}(\mu(t) - \mu(t - f\Delta)) + u_{k-l}\mu(t - f\Delta)$$

Becomes

$$I(i, 0, \Delta, u(t)) = I(i, 0, \Delta, u_{k-l-1}) - I(i, f, \Delta, u_{k-l-1}) + I(i, f, \Delta, u_{k-1})$$

Then as in the previous cases

$$G_r(z) = \frac{\Delta^3 \cdot S_{delay}(z, f)}{3! \cdot z^{l+1}(z-1)^3} \qquad S_{delay} = B'_3(z, 0) - B'_3(z, f) + zB'_3(z, f)$$

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Linear System

Move the system G(s) from s-domain to z-domain

$$G_q(z) = Z\{L^{-1}\{\bar{G}(s)G_{ZOH}(s)\}\}$$

By
Comparison

$$\lim_{\Delta \to 0} \Delta^{-r}G_q(s) = \frac{k \cdot (z-1)^{|Zeros|}S_{Delay}(z)}{(|Poles| - |Zeros|)!z^{l+1}(z-1)^{|Zeros|}}$$

Having S_{delay} as before

$$S_{delay}(z,f) = B'_r(z,0) - B'_r(z,f) + zB'_r(z,f)$$

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Affine nonlinear System

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$$\dot{z}_1(t) = z_2(t)$$
$$\dot{z}_2(t) = \underbrace{z_1(t) + z_2^2(t)z_1(t)}_{b} + \underbrace{1}_{a} u(t)$$

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d b(\hat{z}_k) + B_{Delay} a(\hat{z}_k) u_k$$

$$B_{delay} = \begin{bmatrix} \frac{\Delta^2}{2!} (q^{-1} - q^{-1}(1-f)^2 + (1-f)^2) \\ \frac{\Delta^1}{1!} (q^{-1} - q^{-1}(1-f)^1 + (1-f)^1) \end{bmatrix}$$

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Affine nonlinear System

We can again move the sys to other coordinates

$$T = \begin{bmatrix} 1 & 0_{1 \times r-1} \\ T_{21} & I_{r-1} \end{bmatrix} \qquad T_{21} = - \begin{bmatrix} \frac{r}{\Delta} \cdot \frac{q^{-1}(1-(1-f)^{n-1})+(1-f)^{n-1}}{q^{-1}(1-(1-f)^n)+(1-f)^n} \\ ... \\ \frac{r(r-1)...2}{\Delta^{r-1}} \cdot \frac{q^{-1}(1-(1-f)^1)+(1-f)^1}{q^{-1}(1-(1-f)^n)+(1-f)^n} \end{bmatrix}$$

We can compute the zero dynamics, and as in the previous case, we can see that the eigenvalues of Q_{22} converge to the roots of S_{GOH}

$$S_{GOH}(z) = \sum_{j=1}^{m} [c_j [B_2'(z, \frac{(j-1)}{2}) - B_2'(z, \frac{j}{2})])$$

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Conclusions

Modified Eulero-Frobinius Polynomials

System Integrator → PZOH

Linear System → GOH

Affine nl sys → Pure Delay with ZOH

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