Modified Euler-Frobenius Polynomials With Application to Sampled Data Modelling

Diego S. Carrasco, *Member, IEEE*, Graham C. Goodwin, *Life Fellow, IEEE*, and Juan I. Yuz, *Member, IEEE*

***Abstract*—The broad class of polynomials generally known as Eulerian or Euler-Frobenius polynomials has a rich history in Mathematics and Engineering. They have been generalised in several directions and find application in diverse areas ranging from sampled-data modelling and polynomial interpolation to wavelets. Here we describe a modified class of Euler-Frobenius polynomials which are functions of two independent variables. We show that the sampling zero (dynamics) of general linear and nonlinear systems can be described using this new class of polynomials when considering several non-standard sampled-data modelling problems, including when generalised hold functions are used to generate the input and when the continuous system has a time delay.**

***Index Terms*—Delay systems, modeling, nonlinear systems, sampled-data control.**

I. Introduction

The class of polynomials generally known as Eulerian or Euler-Frobenius polynomials has been extensively studied in the Mathematics literature. Their history can be traced back to Euler's original work in 1775 [1, Part II, Chapter VII]. There exists a substantial literature on these polynomials, see for example [2]–[5].

The Euler-Frobenius Polynomials are known to play a key role in many engineering areas. For example, they appear in sampled-data models for both linear and nonlinear systems [6]–[11], polynomial interpolation [12], Splines [13] and Wavelets [14]. Important extensions of Euler-Frobenius polynomials appear also in sampled-data problems with first-order and fractional-order holds – see e.g. [15], [16].

In this paper we define a new class of polynomials, namely Modified Euler-Frobenius polynomials. We show that the sampling zeros for both linear and nonlinear systems, having either pure time delays or piecewise constant generalised holds, can be expressed in a unified framework as a function of these polynomials. For the sake of presentation, we introduce what we call a Partial Zero Order Hold (PZOH), which can be considered a generalisation of the usual Zero Order Hold (ZOH). We will show that the Modified Euler-Frobenius Polynomials appear as the sampling zeros of linear and nonlinear systems using a PZOH. The results obtained for a PZOH will be the basis for the other two cases.

The work presented in the current paper is, in part, motivated by the circle of ideas which originated with the seminal work reported in [6] on sampling zeros for linear dynamical systems. There it was shown that, when sampling a continuous time linear system, the resulting discrete time system has extra zeros, their location depending on the sampling period. In the nonlinear context, this problem was first addressed in [17]. Extensions of these known ideas can be found in [7]–[11], [18], [19]. There have also been recent extensions to fractional order systems [20] and stochastic systems [21]. Estimates of how close the discretisation zeros, obtained for small sampling periods, are to their limiting values can be found in [8], [22], [23].

Time-delay systems have been the focus of extensive research [24]–[28]. Of particular relevance to the current paper is the issue of sampling of systems having time delays—see [29]–[33] for the linear case and [34] for the nonlinear case. Indeed, the results presented in the current paper for linear time-delay systems are related to those presented in Hara et. al in [29] for SISO systems, and in [33] for the MIMO case.

Generalised Hold Functions (GHF) have been used extensively in the control literature [35]–[37]. They have been shown to be able to arbitrarily shift the location of the sampling zeros in the linear case [35], while the trade-offs arising from such procedure are explored in [38], [39]. The nonlinear case has been addressed in recent work [40] using Piecewise-Constant GHF (PC-GHF). It is important to note that a PC-GHF is a particular case of, so-called, multirate sampling that was introduced to overcome the instability arising from sampling zeros introduced by a ZOH, see for instance [41], [42].

The key contribution of the current paper is to present a unified framework for the sampling zeros of linear and nonlinear systems, using a new class of Modified Euler-Frobenius Polynomials. These polynomials establish a link between the sampling zero polynomials of general systems with Zero Order Holds (ZOH), Piecewise-constant Generalised Holds and time-delay systems with ZOH.

The layout of the remainder of the paper is as follows: Section II briefly summarises some of the known properties

Manuscript received September 28, 2015; revised March 29, 2016 and September 28, 2016; accepted December 22, 2016. Date of publication January 9, 2017; date of current version July 26, 2017. This work was supported by Conicyt Chile through the Becas Chile program, grant FONDECYT 1151466 and the Basal FB0008 Advanced Center for Electrical and Electronic Engineering (AC3E) and by the Australian Research Council through the Discovery Project DP140104141. Recommended by Associate Editor F. Mazenc.

D.S. Carrasco and G.C. Goodwin are with the Priority Research Centre for Complex Dynamic Systems and Control, The University of Newcastle, Callaghan, NSW 2308, Australia (e-mail: diego.carrascoyanez@newcastle.edu.au; graham.goodwin@newcastle.edu.au).

J.I. Yuz is with the Department of Electronic Engineering, Universidad Técnica Federico Santa María, Chile (e-mail: juan.yuz@usm.cl).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2017.2650784

of the standard Euler-Frobenius polynomials. Section III introduces the Modified Euler-Frobenius polynomials and establishes several properties. Section IV introduces definitions and results that apply to sampled-data models. Application of Modified Euler-Frobenius polynomials to non-standard sampled data modelling problems are explored in Section V. An Illustrative example is given in Section VI. Finally, conclusions are drawn in Section VII.

II. Standard Euler-Frobenius Polynomials

In this section we recall the definitions of standard Euler-Frobenius polynomials.

Standard Euler-Frobenius polynomials are denoted $B_r(z)$. There are many equivalent ways of defining these polynomials [11, Ch. 20]. For example, they can be defined via the following exponential generating function [11, p. 254]

$$\sum_{r=0}^{\infty} B_r(z) \cdot \frac{x^r}{r!} = \frac{z-1}{z-e^{(z-1)x}}$$

The first few of these polynomials are

$$\begin{aligned} B_0(z) &= 1 \\ B_1(z) &= 1 \\ B_2(z) &= z+1 \\ B_3(z) &= z^2+4z+1 \end{aligned}$$

Another definition, useful in the context of the current paper, is as follows.

Definition 1: The Euler-Frobenious polynomials satisfy

$$B_r(z) = r! \cdot \det M_r \tag{1}$$

where $r \in \mathbb{N}_{\geq 0}$ and

$$M_r = \begin{bmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} & \frac{1}{r!} \\ 1-z & 1 & \cdots & \frac{1}{(r-2)!} & \frac{1}{(r-1)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1-z & 1 & \frac{1}{2!} \\ 0 & \cdots & 0 & 1-z & 1 \end{bmatrix} \tag{2}$$

□□□

Definition 1 was first presented in [10], in an equivalent form using the δ operator. A summary of several properties of the Euler-Frobenius polynomials can be found in [11, Ch. 20].

III. Modified Euler-Frobenius Polynomials

In this section we define the Modified Euler-Frobenius polynomials and prove several properties. We denote the Modified Euler-Frobenius polynomials by $B'_r(z,f)$. The following definition is analogous to Definition 1.

Definition 2: The Modified Euler-Frobenius polynomials are defined by

$$B'_r(z,f) = r! \cdot \det P_r \tag{3}$$

where $f \in [0,1)$, $r \in \mathbb{N}_{\geq 0}$ and

$$P_r = \begin{bmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} & \frac{(1-f)^r}{r!} \\ 1-z & 1 & \cdots & \frac{1}{(r-2)!} & \frac{(1-f)^{r-1}}{(r-1)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1-z & 1 & \frac{(1-f)^2}{2!} \\ 0 & \cdots & 0 & 1-z & (1-f) \end{bmatrix} \tag{4}$$

□□□

The first few of these polynomials are

$$\begin{aligned} B'_0(z,f) &= 1 \\ B'_1(z,f) &= 1-f \\ B'_2(z,f) &= (1-f)^2 z + (1-f^2) \\ B'_3(z,f) &= (1-f)^3 z^2 + (1-f)(4+f-2f^2)z + (1-f^3) \end{aligned}$$

Remark 1: Comparing Definitions 1 and 2 we can see that the difference in the polynomials lies in the last column of the matrices M_r and P_r. More specifically, terms depending upon $(1-f)$ appear in (4). Furthermore, it is easily seen that, when $f=0$, the Modified Euler-Frobenius polynomials collapse to the Standard Euler-Frobenius polynomials. □

Remark 2: The authors are unaware of any prior reference to the polynomials defined in (3). □

We next establish several properties of the Modified Euler-Frobenius polynomials. Our results mirror the extensive literature available on properties of the standard Euler-Frobenius polynomials—see [2]–[5], [9].

Theorem 1: The Modified Euler-Frobenius polynomials satisfy the following properties:

i) $B'_r(z,0) = B_r(z)$

ii) $B'_r(z,f) = \sum_{j=1}^{r-1} \binom{r}{j} (z-1)^{j-1} B'_{r-j}(z,f) + (1-f)^r (z-1)^{r-1}$

iii) $B'_r(z,f) = \dfrac{r!}{\Delta^r} \det D_r$, where $\Delta \in \mathbb{R}_{\geq 0}$ and

$$D_r = \begin{bmatrix} \Delta & \frac{\Delta^2}{2} & \cdots & \frac{\Delta^{r-1}}{(r-1)!} & \frac{\Delta^r (1-f)^r}{r!} \\ 1-z & \Delta & \cdots & \frac{\Delta^{r-2}}{(r-2)!} & \frac{\Delta^{r-1}(1-f)^{r-1}}{(r-1)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \Delta & \frac{\Delta^2 (1-f)^2}{2} \\ 0 & 0 & \cdots & 1-z & \Delta(1-f) \end{bmatrix}$$

iv) $B'_r(z,f) = r!(z-1)^r \displaystyle\sum_{\ell=-\infty}^{\infty} \frac{e^{-f(\log z + 2\pi j \ell)} - z^{-1}}{(\log z + 2\pi j \ell)^{r+1}}$

Proof:

i) Immediate from the definitions of $B_r(z)$ and M_r in (1)–(2) and the definitions of $B'_r(z,f)$ and P_r in (3)–(4).

ii) Consider P_r, as defined in (4), which can also be written as

$$P_r = \left[\begin{array}{c|cccc} 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} & \frac{(1-f)^r}{r!} \\ \hline 1-z & & & & \\ \vdots & & & & \\ 0 & & P_{r-1} & & \\ 0 & & & & \end{array}\right]$$

Hence, using the Schur Determinant Lemma [43, Result E.1.2], we obtain

$$\det P_r = \det P_{r-1} \times \det\left(1 - \begin{bmatrix} \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} & \frac{(1-f)^r}{r!} \end{bmatrix} P_{r-1}^{-1} \begin{bmatrix} 1-z \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$$

Note that

$$P_{r-1}^{-1} \begin{bmatrix} 1-z \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\det P_{r-1}} \cdot ([C]_{1,j})^T \cdot (1-z)$$

where C is the cofactor matrix of P_{r-1}, i.e. $C_{ij} = (-1)^{i+j} Q_{ij}$, and Q_{ij} is the (i,j) minor of P_{r-1}, and where $[C]_{1,j}$ represents the first row of C. We have that

$$C_{1j} = (-1)^{1+j} \times \det \begin{bmatrix} 1-z & 1 \cdots \frac{1}{(j-2)!} & \frac{1}{j!} & \cdots & \frac{1}{(r-3)!} & \frac{(1-f)^{r-2}}{(r-2)!} \\ 0 & 1-z \cdots \frac{1}{(j-3)!} & \frac{1}{(j-1)!} & \cdots & \frac{1}{(r-4)!} & \frac{(1-f)^{r-3}}{(r-3)!} \\ \vdots & \vdots \ddots \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 \quad 1-z & \frac{1}{2!} & \cdots & \frac{1}{(r-j-1)!} & \frac{(1-f)^{r-j}}{(r-j)!} \\ 0 & 0 \cdots 0 & 1 & \cdots & \frac{1}{(r-j-2)!} & \frac{(1-f)^{r-j-1}}{(r-j-1)!} \\ \vdots & \vdots \quad \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 \cdots 0 & 0 & \cdots & 1 & \frac{(1-f)^2}{2!} \\ 0 & 0 \cdots 0 & 0 & \cdots & 1-z & (1-f) \end{bmatrix}$$

Calculating the determinant and subsequent subdeterminants along the first column $j-1$ times, we obtain

$$\begin{aligned} C_{1j} &= (-1)^{1+j}(1-z)^{j-1} \det P_{r-j-1} \\ &= (z-1)^{j-1} \det P_{r-j-1} \end{aligned}$$

which, in turn, implies

$$P_{r-1}^{-1} \begin{bmatrix} 1-z \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{-1}{\det P_{r-1}} \cdot \begin{bmatrix} (z-1) \cdot \det P_{r-2} \\ (z-1)^2 \cdot \det P_{r-3} \\ \vdots \\ (z-1)^{r-1} \cdot \det P_0 \end{bmatrix}$$

and therefore

$$\begin{aligned} \det P_r &= \det P_{r-1} + \frac{(z-1)}{2!} \det P_{r-2} + \cdots \\ &\quad + \frac{(z-1)^{r-2}}{(r-1)!} \det P_1 \\ &\quad + (1-f)^r \frac{(z-1)^{r-1}}{r!} \det P_0 \\ &= \sum_{j=1}^{r-1} \frac{1}{j!} (z-1)^{j-1} \det P_{r-j} \\ &\quad + \frac{(1-f)^r}{r!} (z-1)^{r-1} \det P_0 \end{aligned} \tag{5}$$

Using (3) completes the proof.

iii) We note that proving this property is equivalent to proving $\det D_r = \Delta^r \det P_r$, where P_r is as defined in (4). We prove the result by induction. First note that $\det D_1 = \Delta^1 \det P_1$ is true since $\Delta = \Delta \cdot 1$. Then, assuming $\det D_i = \Delta^i \det P_i$ holds for $i = 1, \ldots, r$, we will show $\det D_{r+1} = \Delta^{r+1} \det P_{r+1}$ also holds.
Following exactly the same argument as in the proof of Theorem 1 part (ii) we can write

$$\det D_{r+1} = \det D_r \times \left(\Delta - \begin{bmatrix} \frac{\Delta^2}{2!} & \cdots & \frac{\Delta^r}{r!} & \frac{\Delta^{r+1}(1-f)^{r+1}}{(r+1)!} \end{bmatrix} \times D_r^{-1} \begin{bmatrix} 1-z \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$$

In the same way, we can show that

$$D_r^{-1} \begin{bmatrix} 1-z \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{-1}{\det D_r} \begin{bmatrix} (z-1) \det D_{r-1} \\ \vdots \\ (z-1)^r \det D_0 \end{bmatrix}$$

and therefore

$$\begin{aligned} \det D_{r+1} &= \sum_{j=1}^{r} \frac{\Delta^j}{j!} (z-1)^{j-1} \det D_{r-j+1} \\ &\quad + \frac{\Delta^{r+1}(1-f)^{r+1}}{(r+1)!} (z-1)^r \det D_0 \end{aligned}$$

Since, by the induction hypothesis, we assume $\det D_i = \Delta^i \det P_i, i = 1, \ldots, r$, we have

$$\det D_{r+1} = \Delta^{r+1}\left(\sum_{j=1}^{r} \frac{1}{j!}(z-1)^{j-1} \det P_{r-j+1} + \frac{(1-f)^{r+1}}{(r+1)!}(z-1)^r \det P_0\right) = \Delta^{r+1} \det P_{r+1}$$

where the last equality follows from (5).

iv) We defer the proof of this result to Section V since it is convenient to use sampled-data modelling ideas presented there. See Remark 5. ∎

IV. Preliminary Definitions and Results for Sampled-Data Models

In this Section we introduce definitions and results that will be used in the remainder of the paper.

A. Multiple Integration

Definition 3: The following notation will be used to describe the multiple integration of a function $g(t): \mathbb{R} \to \mathbb{R}$

$$I(r, f, \Delta, g) = \int_{f\Delta}^{\Delta} \int_{f\Delta}^{t_{r-1}} \cdots \int_{f\Delta}^{t_1} g(t) dt \cdots dt_{r-1}$$

□□□

We then have the following lemma.

Lemma 1: The multiple integrals $I(r, f, \Delta, g)$ satisfy the following equality for the special case $g = 1$

$$I(j, f, \Delta, 1) = \frac{\Delta^j (1-f)^j}{j!}$$

Proof: We use the following Cauchy formula [44, pg. 132]

$$\int_{t_0}^{t} \int_{t_0}^{t_{n-1}} \cdots \int_{t_0}^{t_1} g(s) ds dt_1 \cdots dt_{n-1} = \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} g(s) ds$$

For the case $g(s) = 1, t = \Delta, t_0 = f\Delta$, we have that

$$I(j, f, \Delta, 1) = -\frac{(\Delta - s)^j}{j!}\bigg|_{f\Delta}^{\Delta} = \frac{\Delta^j (1-f)^j}{j!}$$

which completes the proof. ∎

B. Truncation Errors

Definition 4: We define $T = N \cdot \Delta$ as the time horizon over which the properties of a sampled-data model will be analysed, where Δ is the length of the time discretisation and N is the number of steps. □□□

Definition 5: For asymptotic analysis as $\Delta \to 0$, we introduce the notation

$$f(\Delta) \in \mathcal{O}(g(\Delta)) \tag{6}$$

for two functions $f(\cdot)$ and $g(\cdot)$, if and only if $|f(\Delta)| \leq C \cdot |g(\Delta)|$, for all $\Delta < \Delta_0$, and where C is a positive constant. □□□

When (6) holds, we say that $f(\Delta)$ is *of the order of* $g(\Delta)$. We also introduce the following definition of truncation error:

Definition 6 (Local Vector Fixed Step Truncation Error [19]): The *local vector fixed step truncation error* of an approximate model is said to be of the order of $(\Delta^{m_1}, \ldots, \Delta^{m_n})$ if and only if, for initial state errors

$$\hat{x}_1[k] - x_1[k] \in \mathcal{O}(\Delta^{\bar{m}_1})$$
$$\vdots$$
$$\hat{x}_n[k] - x_n[k] \in \mathcal{O}(\Delta^{\bar{m}_n})$$

for any $\bar{m}_i \geq m_i, i = 1, \ldots, n$ then after N steps, where N is a finite *fixed* number, we have that

$$\hat{x}_1[k+N] - x_1[k+N] \in \mathcal{O}(\Delta^{m_1})$$
$$\vdots$$
$$\hat{x}_n[k+N] - x_n[k+N] \in \mathcal{O}(\Delta^{m_n})$$

□□□

C. Nonlinear Systems and Normal Form

The characterisation of sampling zeros in linear sampled-data models can be extended to the nonlinear case using normal forms and (sampling) zero dynamics [10]. Consider a nonlinear continuous-time system, described in state-space form, as follows:

$$\dot{x}(t) = \beta(x(t)) + \alpha(x(t))u(t) \tag{7}$$

$$y(t) = h(x(t)) \tag{8}$$

where $x(t)$ is the state evolving in an open subset $\mathcal{M} \subset \mathbb{R}^n$, and where the vector fields $\beta(\cdot), \alpha(\cdot)$, and the output function $h(\cdot)$ are locally Lipschitz [45, Section 110]. In addition, we recall [46]:

Definition 7 (Relative Degree): The nonlinear system (7)–(8) is said to have relative degree r at a point x_o if

i) $L_\alpha L_\beta^k h(x) = 0$ for all x in a neighbourhood of x_o and all $k < r-1$.

ii) $L_\alpha L_\beta^{r-1} h(x_o) \neq 0$.

where $L_\beta h(x) = \partial h / \partial x \cdot \beta(x)$ is the Lie Derivative of h with respect to β. □□□

Definition 8 (Local Coordinate Transformation): Suppose the system (7)–(8) has relative degree r at x_o $(r \leq n)$. Then a *local coordinate transformation* $\Phi(\cdot) = \begin{bmatrix} \phi_1^T(\cdot) & \ldots & \phi_n^T(\cdot) \end{bmatrix}^T$ in a neighbourhood of x_o is given by

$$z_1 = \phi_1(x) = h(x)$$
$$z_2 = \phi_2(x) = L_\beta h(x)$$
$$\vdots$$
$$z_r = \phi_r(x) = L_\beta^{r-1} h(x)$$

□□□

It is well known (see e.g. [46]), that if $r < n$, then it is always possible to find $n - r$ functions $z_{r+1} = \phi_{r+1}(x), \ldots, z_n =$

$\phi_n(x)$ such that

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} = \Phi(x) \tag{9}$$

has a nonsingular Jacobian at x_o and, in addition, $L_\alpha \phi_i(x) = 0$ in a neighbourhood of x_o for all $i = r+1, \dots, n$. This is the basis for the following definition [46]:

Definition 9 (Normal Form): The normal form of the nonlinear system (7)–(8) is given by the state space description in the new coordinates $\Phi(x)$ as described in Definition 8. The model can be written as

$$\dot{\zeta} = \left[\begin{array}{c|c} 0 & \\ \vdots & I_{r-1} \\ 0 & \\ \hline 0 & 0 \ \dots \ 0 \end{array}\right] \zeta + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} (b(\zeta, \eta) + a(\zeta, \eta) \cdot u(t)) \tag{10a}$$

$$\dot{\eta} = c(\zeta, \eta) \tag{10b}$$

where $y = z_1$ and

$$\zeta = \begin{bmatrix} z_1 & \dots & z_r \end{bmatrix}^T \tag{11a}$$

$$\eta = \begin{bmatrix} z_{r+1} & \dots & z_n \end{bmatrix}^T \tag{11b}$$

$$a(\zeta, \eta) = L_\alpha L_\beta^{r-1} h(\Phi^{-1}(z)) \tag{11c}$$

$$b(\zeta, \eta) = L_\beta^r h(\Phi^{-1}(z)) \tag{11d}$$

$$c(\zeta, \eta) = \begin{bmatrix} L_\beta \phi_{r+1}(\Phi^{-1}(z)) \\ \vdots \\ L_\beta \phi_n(\Phi^{-1}(z)) \end{bmatrix} \tag{11e}$$

with $z = \begin{bmatrix} \zeta^T & \eta^T \end{bmatrix}^T$ defined as in (9). □□□

Remark 3: Note that the vector ζ contains the output and its first $r-1$ derivatives. □

Definition 10 (Discrete-Time Zero Dynamics [47]): The zero dynamics of a discrete-time system are defined as the internal dynamics that appear in the system when the input and initial conditions are chosen in such a way as to make the output identically zero for all time, i.e. $y_k = 0, \forall k$. □□□

V. Applications to Sampled Data Models

In this Section we develop sampled data models for several non-standard sampled data modelling problems, e.g. for partial zero order holds, for piecewise-constant generalised hold functions and for time-delay systems. We show that the Modified Euler-Frobenius polynomials arise in the description of the sampling zeros in all three cases.

A. Partial Zero Order Hold

We introduce a Partial Zero Order Hold as a didactic tool, to simplify the exposition of subsequent results. It is defined as follows.

Definition 11 (Partial Zero Order Hold (PZOH)): We say that the sequence $\{u_k\}$ is applied to a continuous time system via a PZOH with sampling period Δ when the continuous input to the system satisfies

$$u(t) = \begin{cases} 0 & , k\Delta \le t < k\Delta + f\Delta \\ u_k & , k\Delta + f\Delta \le t < (k+1)\Delta \end{cases}$$

where $k \in \mathbb{N}$ and $f \in [0, 1)$. □□□

The following theorem describes the discrete-time models of linear and nonlinear systems, and their properties, when a PZOH is used to generate the continuous-time input.

Theorem 2: Let a PZOH be used to generate the input to a system. We then have the following results:

i) When the system is an r-th order integrator with continuous transfer function $G(s) = 1/s^r$, then the *exact* sampled-data model satisfies the following discrete-time state-space equations:

$$x_{k+1} = A_d x_k + B_{pzoh} u_k \tag{12}$$

where

$$x_k = \begin{bmatrix} x_{1,k} & x_{2,k} & \cdots & x_{r,k} \end{bmatrix}^T \tag{13}$$

$$A_d = \begin{bmatrix} 1 & \Delta & \cdots & \frac{\Delta^{r-1}}{(r-1)!} \\ 0 & 1 & \cdots & \frac{\Delta^{r-2}}{(r-2)!} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{14}$$

$$B_{pzoh} = \begin{bmatrix} \frac{\Delta^r (1-f)^r}{r!} \\ \frac{\Delta^{r-1}(1-f)^{r-1}}{(r-1)!} \\ \vdots \\ \Delta(1-f) \end{bmatrix} \tag{15}$$

with output $y_k = x_{1,k}$.

The exact input-output transfer function between the discrete input $\{u_k\}$ and sampled output $\{y_k\}$ is then given by

$$G_q(z) = \frac{\Delta^r \cdot B_r'(z, f)}{r! \cdot (z-1)^r} \tag{16}$$

where $B_r'(z, f)$ are the Modified Euler-Frobenius polynomials.

ii) When the system is a general linear system having continuous transfer function $G = G(s)$, then the exact sampled-data model is given by

$$G_q(z) = \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ G(s) G_{pzoh}(s) \right\} \Big|_{t=k\Delta} \right\} \tag{17}$$

where

$$G_{pzoh}(s) = \frac{e^{-sf\Delta} - e^{-s\Delta}}{s} \tag{18}$$

If the system has n poles and m zeros, then asymptotically, as the sampling period $\Delta \to 0$, the exact sampled-data model (17) has the following limit

$$\lim_{\Delta\to 0} \Delta^{-r} G_q(z) = \frac{K\cdot(z-1)^m \cdot B_r'(z,f)}{(n-m)!\cdot(z-1)^n} \tag{19}$$

where $r = n-m$. Thus n poles converge to $z=1$, m zeros converge to $z=1$, and r zeros converge to the zeros of $B_r'(z,f)$.

iii) When the system is an affine nonlinear system with relative degree r of the form (7)–(8), then an approximate sampled-data model is given by

$$\hat{\zeta}_{k+1} = A_d\hat{\zeta}_k + B_d\cdot b(\hat{\zeta}_k,\hat{\eta}_k) + B_{pzoh}\cdot a(\hat{\zeta}_k,\hat{\eta}_k)u_k \tag{20a}$$

$$\hat{\eta}_{k+1} = \hat{\eta}_k + \Delta c(\hat{\zeta}_k,\hat{\eta}_k) \tag{20b}$$

where $\hat{z} = [\hat{\zeta}^T \, \hat{\eta}^T]^T$, with $\hat{\zeta}_k = [\hat{z}_{1,k}\cdots\hat{z}_{r,k}]^T$, $\hat{\eta}_k = [\hat{z}_{r+1,k}\cdots\hat{z}_{n,k}]^T$, is an approximation of the coordinates $z=\Phi(x)$, given by the normal form of the system. The model output is $\hat{y}_k = \hat{z}_{1,k}$. The functions $a(\cdot), b(\cdot)$ and $c(\cdot)$ are as defined in (11). The matrices A_d and B_{pzoh} are as defined in (14) and (15), respectively, and the vector B_d is defined as follows

$$B_d = \begin{bmatrix} \frac{\Delta^r}{r!} & \frac{\Delta^{r-1}}{(r-1)!} & \cdots & \Delta \end{bmatrix}^T \tag{21}$$

The approximate model (20) has Local Vector Fixed Step Truncation Errors of the order of $(\Delta^{r+1},\cdots,\Delta^2,\Delta^2)$. In addition, the approximate sampled-data model (20) has sampling zero dynamics that, as the sampling period $\Delta\to 0$, converge to linear dynamics whose eigenvalues are given by the roots of the Modified Euler-Frobenius polynomial, $B_r'(z,f)$.

Proof:

i) Consider a state space description of an r-th order integrator with $y(t) = x_1(t)$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ &\vdots\\ \dot{x}_r(t) &= u(t) \end{aligned}$$

Exact integration over the sampling period $[k\Delta,(k+1)\Delta]$, starting with x_r and progressively moving up to x_1, leads to

$$x_1((k+1)\Delta) = x_1(k\Delta) + \Delta x_2(k\Delta) + \cdots + \frac{\Delta^{r-1}}{(r-1)!}x_r(k\Delta) + I(r,0,\Delta,u(t)) \tag{22a}$$

$$x_2((k+1)\Delta) = x_2(k\Delta) + \Delta x_3(k\Delta) + \cdots + \frac{\Delta^{r-2}}{(r-2)!}x_r(k\Delta) + I(r-1,0,\Delta,u(t)) \tag{22b}$$

$$\vdots$$

$$x_r((k+1)\Delta) = x_r(k\Delta) + I(1,0,\Delta,u(t)) \tag{22c}$$

Due to the definition of a PZOH, we know that $I(j,0,\Delta,u(t)) = I(j,f,\Delta,u_k), j=1,\ldots,r$. The result (12)–(15) then follows by using Lemma 1.

On the other hand, the system of equations presented above can be rewritten using the forward shift operator, q, as

$$\begin{aligned} &\begin{bmatrix}(q-1)x_{1,k} & 0 & \cdots & 0 & 0\end{bmatrix}^T\\ &\quad = D_r \begin{bmatrix} x_{2,k} & x_{3,k} & \cdots & x_{n,k} & u_k\end{bmatrix}^T \end{aligned}$$

where

$$D_r = \left[\begin{array}{cccc|c} \Delta & \frac{\Delta^2}{2} \cdots & & \frac{\Delta^{r-1}}{(r-1)!} & \\ -(q-1) & \Delta & \cdots & \frac{\Delta^{r-2}}{(r-2)!} & \\ \vdots & \ddots & \ddots & \vdots & B_{pzoh} \\ 0 & 0 & \ddots & \Delta & \\ 0 & 0 & \cdots & -(q-1) & \end{array}\right]$$

Using Cramer's rule we can solve for u_k and obtain

$$u_k = \frac{\det N}{\det D_r}$$

where

$$N = \begin{bmatrix} \Delta & \frac{\Delta^2}{2} & \cdots & \frac{\Delta^{r-1}}{(r-1)!} & (q-1)x_{1,k} \\ -(q-1) & \Delta & \cdots & \frac{\Delta^{r-2}}{(r-2)!} & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \Delta & 0 \\ 0 & 0 & \cdots & -(q-1) & 0 \end{bmatrix}$$

Computing the determinant of N along the last column we obtain

$$\begin{aligned} \det N &= (-1)^{1+r}\cdot(q-1)x_{1,k}\cdot(-(q-1))^{r-1}\\ &= (q-1)^r x_{1,k} \end{aligned}$$

On the other hand, as shown in Theorem 1 Part (iv), $\det D_r = \Delta^r \det P_r$ and hence

$$\det D_r = \frac{\Delta^r}{r!} \cdot r! \cdot \det P_r = \frac{\Delta^r}{r!} \cdot B_r'(q,f)$$

We can then write

$$u_k = \frac{(q-1)^r}{\frac{\Delta^r}{r!} \cdot B_r'(q,f)} x_{1,k}$$

Noting that $y_k = x_{1,k}$ and that $G_q(z) = y_k/u_k$ the result (16) is obtained.

ii) The expression (17) is obtained directly by using [48, Eq. 12.13.3] and considering that the transfer function of a PZOH is given by (18).

The proof of (19) follows the same lines as [6, Theorem 1]. Specifically, consider a general linear system of the form

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

with relative degree $r = n - m$. Then, using (17), and the definition of the inverse Laplace transform and $\mathcal{Z}$-transform [49], the exact discrete-time system is given by

$$G_q(z) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \frac{e^{s\Delta}}{z-e^{s\Delta}} G(s) \frac{e^{-sf\Delta}-e^{-s\Delta}}{s} ds$$

where $\gamma > Re(p_i), i = 1,\ldots,n$. Using the change of variables $s = w/\Delta$ we have

$$G_q(z) = \frac{1}{2\pi j}\int_{\gamma\Delta-j\infty}^{\gamma\Delta+j\infty} \frac{e^{w}}{z-e^{w}} G\left(\frac{w}{\Delta}\right) \frac{e^{-wf}-e^{-w}}{w} dw$$

where

$$\begin{aligned} G\left(\frac{w}{\Delta}\right) &= \frac{K(w/\Delta)^m(1-z_1\Delta/w)\cdots(1-z_m\Delta/w)}{(w/\Delta)^n(1-p_1\Delta/w)\cdots(1-p_n\Delta/w)} \\ &= K\left(\frac{\Delta}{w}\right)^r \frac{(1-z_1\Delta/w)\cdots(1-z_m\Delta/w)}{(1-p_1\Delta/w)\cdots(1-p_n\Delta/w)} \end{aligned}$$

Hence, taking the limit as $\Delta \to 0$ we obtain

$$\begin{aligned} &\lim_{\Delta\to 0} \Delta^{-r} G_q(z) \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{e^{w}}{z-e^{w}} \left(\frac{K}{w^r}\right) \frac{e^{-wf}-e^{-w}}{w} dw \quad (23) \end{aligned}$$

where the integration path has an infinitesimal detour around the origin. On the other hand, from (16), we know that the exact discrete-time transfer function for an r-th order integrator is given by

$$G_i(z) = \frac{\Delta^r \cdot B_r'(z,f)}{r! \cdot (z-1)^r} \quad (24)$$

We also know, from (17), that $G_i(z)$ can be obtained from

$$\begin{aligned} G_i(z) &= \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s^r}\frac{e^{-sf\Delta}-e^{-s\Delta}}{s}\right\}\Big|_{t=k\Delta}\right\} \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{e^{s\Delta}}{z-e^{s\Delta}} \frac{1}{s^r} \frac{e^{-sf\Delta}-e^{-s\Delta}}{s} ds \end{aligned}$$

Using again the change of variables $s = w/\Delta$ we obtain

$$G_i(z) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{e^{w}}{z-e^{w}} \frac{\Delta^r}{w^r} \frac{e^{-wf}-e^{-w}}{w} dw \quad (25)$$

Equating (24) and (25), and replacing in (23), we finally obtain

$$\lim_{\Delta\to 0} \Delta^{-r} G_q(z) = \frac{K \cdot B_r'(z,f)}{r! \cdot (z-1)^r}.$$

iii) From Section IV, an affine nonlinear system of relative degree r can be written in normal form as shown in (10). Integration over the sampling period $[k\Delta, (k+1)\Delta]$, starting with z_r and progressively moving up to z_1, leads to the following exact representation,

$$\begin{aligned} z_1((k+1)\Delta) &= z_1(k\Delta) + \cdots + \frac{\Delta^{r-1}}{(r-1)!} z_r(k\Delta) \\ &\quad + I(r, 0, \Delta, (b(z(t)) + a(z(t))u(t))) \\ z_2((k+1)\Delta) &= z_2(k\Delta) + \cdots + \frac{\Delta^{r-2}}{(r-2)!} z_r(k\Delta) \\ &\quad + I(r-1, 0, \Delta, (b(z(t)) + a(z(t))u(t))) \\ &\vdots \\ z_r((k+1)\Delta) &= z_r(k\Delta) \\ &\quad + I(1, 0, \Delta, (b(z(t)) + a(z(t))u(t))) \\ \eta((k+1)\Delta) &= \eta_r(k\Delta) + I(1, 0, \Delta, c(z(t))) \end{aligned}$$

Because of the use of a PZOH, we have

$$\begin{aligned} &I(j, 0, \Delta,\ (b(z(t)) + a(z(t))u(t))) \\ &\qquad = I(j, 0, \Delta, b(z(t))) + I(j, f, \Delta, a(z(t))u_k) \end{aligned}$$

for $j = 1, \ldots, r$. The approximate model is obtained when the integrand of the $I(\cdot)$ functions is approximated by a constant value, in this case, when $b(z(t))$ and $a(z(t))$ are evaluated at the beginning of the sampling period, i.e.,

$$\begin{aligned} I(j, 0, \Delta, b(z(t))) &\approx b(z(k\Delta)) \cdot I(j, 0, \Delta, 1) \\ I(j, f, \Delta, a(z(t))u_k) &\approx a(z(k\Delta))u_k \cdot I(j, f, \Delta, 1) \end{aligned}$$

for $j = 1, \ldots, r$. The same idea applies to $I(1, 0, \Delta, c(z(t)))$. Using Lemma 1 and defining $z = \hat{z} + e$, where e is the error of the approximation, the model (20) is finally obtained.

The proof for the Local Fixed Step Truncation Errors follows the exact same lines as Theorem 2 Part (i) in [19]. Note that the only difference between the approximate sampled data model obtained here, when a PZOH is used, and the approximate sampled data model obtained in [19], when a ZOH is used, lies in the factor $(1-f)^r$ multiplying $a(z(k\Delta))$ in the first r states. This fact does not influence the error analysis performed in [19].

Next, consider the following similarity transformation $\tilde{\zeta} = T\hat{\zeta}$, where

$$T = \left[\begin{array}{c|c} 1 & \mathbf{0}_{1\times(r-1)} \\ \hline T_{21} & I_{r-1} \end{array}\right]$$

$$T_{21} = -\left[\begin{array}{cccc} \frac{r}{\Delta(1-f)} & \frac{r(r-1)}{\Delta^2(1-f)^2} & \cdots & \frac{r(r-1)\cdots 2}{\Delta^{r-1}(1-f)^{r-1}} \end{array}\right]^T$$

is nonsingular. Then, the sampled-data model (20), in the new coordinates, is given by

$$\tilde{z}_{1,k+1} = q_{11}\tilde{z}_{1,k} + Q_{12}\tilde{z}_{2:r,k} + \frac{\Delta^r}{r!}\left(b(\tilde{\zeta}_k, \hat{\eta}_k) + (1-f)^r a(\tilde{\zeta}_k, \hat{\eta}_k)u_k\right) \tag{26a}$$

$$\tilde{z}_{2:r,k+1} = Q_{21}\tilde{z}_{1,k} + Q_{22}\tilde{z}_{2:r,k} + \tilde{B}_2 \cdot b(\tilde{\zeta}_k, \hat{\eta}_k) \tag{26b}$$

$$\hat{\eta}_{k+1} = \hat{\eta}_k + \Delta c(\tilde{\zeta}_k, \hat{\eta}_k) \tag{26c}$$

where $\tilde{z}_{2:r,k} = [\tilde{z}_{2,k} \cdots \tilde{z}_{r,k}]^T$ and

$$q_{11} = A_{11} - A_{12}T_{21}$$

$$Q_{12} = A_{12}$$

$$Q_{21} = T_{21}A_{11} + A_{21} - (T_{21}A_{12} + A_{22})T_{21}$$

$$Q_{22} = T_{21}A_{12} + A_{22}$$

$$TB_d = \tilde{B} = [\tilde{B}_1 \tilde{B}_2^T]^T$$

$$\tilde{B}_1 = \frac{\Delta^r}{r!}$$

$$\tilde{B}_2 = \begin{bmatrix} \frac{\Delta^{r-1}}{(r-1)!}\left(1 - \frac{1}{1-f}\right) \\ \frac{\Delta^{r-2}}{(r-2)!}\left(1 - \frac{1}{(1-f)^2}\right) \\ \vdots \\ \frac{\Delta}{1!}\left(1 - \frac{1}{(1-f)^{r-1}}\right) \end{bmatrix}$$

and where the state matrix of the sampled-data model, defined in (14), is partitioned as follows

$$A_d = \left[\begin{array}{c|c} A_{11} & A_{12}^{1\times(r-1)} \\ \hline A_{21}^{(r-1)\times 1} & A_{22}^{(r-1)\times(r-1)} \end{array}\right]$$

From (26), the zero dynamics are given by

$$\tilde{z}_{2:r,k+1} = Q_{22}\tilde{z}_{2:r,k} + \tilde{B}_2 \cdot b(\tilde{\zeta}_k, \hat{\eta}_k)$$
$$\hat{\eta}_{k+1} = \hat{\eta}_k + \Delta c(0, \tilde{\zeta}_{2:r,k}, \hat{\eta}_k)$$

From the structure of $\tilde{B}_2$, it is clear that, as the sampling period $\Delta \to 0$, the sampling zero dynamics converge to

$$\tilde{z}_{2:r,k+1} = Q_{22}\tilde{z}_{2:r,k}$$

We next show that the eigenvalues of the matrix Q_{22} are the roots of the Modified Euler-Frobenius polynomials. First consider the matrix D_r, as defined in Theorem 1 part (iv). Then, consider the following

$$D_r \left[\begin{array}{c|c} 0 & I_{r-1} \\ \hline 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} \frac{\Delta^r(1-f)^r}{r!} & A_{12} \\ \hline -\frac{\Delta^r(1-f)^r}{r!}T_{21} & A_{22} - zI_{n-1} \end{array}\right]$$

Taking the determinant on both sides of the above equation, and using [43, Result E.1.2], we obtain

$$\begin{aligned} &\det D_r \cdot (-1)^{r+1} \\ &= \frac{\Delta^r(1-f)^r}{r!} \cdot \det\left(A_{22} - zI_{r-1} + T_{21}A_{12}\right) \\ &= \frac{\Delta^r(1-f)^r}{r!} \cdot (-1)^{r-1} \det\left(zI_{r-1} - (T_{21}A_{12} + A_{22})\right) \end{aligned}$$

Hence

$$\frac{r!}{\Delta^r} \det D_r = (1-f)^r \cdot \det\left(zI_{r-1} - (T_{21}A_{12} + A_{22})\right)$$

From Theorem 1 part (iv) and the definition of the matrix Q_{22}, we then have

$$B_r'(z, f) = (1-f)^r \cdot \det\left(zI_{r-1} - Q_{22}\right)$$

which completes the proof. ■

Theorem 2 Part (i) shows that, when a PZOH is used, the *exact* discrete-time representation of an r-th order integrator has sampling zeros that correspond to the Modified Euler-Frobenius polynomials. Furthermore, Part (ii) shows that, for a general *linear* system, the *exact* discrete-time representation has sampling zeros that asymptotically converge, as $\Delta \to 0$, to the zeros of the Modified Euler-Frobenius polynomials. Also, Part (iii) shows that, for an affine *nonlinear* system, an *approximate* discrete-time representation with quantified error analysis, has sampling zero dynamics that asymptotically converge, as $\Delta \to 0$, to linear dynamics whose eigenvalues are the zeros of the Modified Euler-Frobenius polynomials.

Remark 4: A special case of Theorem 2 arises when the input is generated by a zero-order hold (ZOH), since it corresponds to the choice $f = 0$. For this special case, results analogous to those presented in Theorem 2 can be found in the literature. Specifically, the results for $f = 0$ can be found in [6] and [10]. The error analysis can be found in [19]. □

Remark 5: We can also use Theorem 2 to establish Part (iv) of Theorem 1. Specifically, from Theorem 2 Part (ii), when $G(s) = 1/s^r$, then the corresponding exact discrete transfer function is

$$G_q(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s^r}\frac{e^{-sf\Delta} - e^{-s\Delta}}{s}\right\}\Big|_{t=k\Delta}\right\}$$

where $\mathcal{Z}\{\cdot\}$ and $\mathcal{L}^{-1}\{\cdot\}$ denote the $\mathcal{Z}$-transform and the inverse Laplace transform, respectively. Using the definitions of the inverse Laplace transform and the $\mathcal{Z}$-transform [49], we obtain

$$G_q(z) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \frac{e^{s\Delta}}{z-e^{s\Delta}}\frac{e^{-sf\Delta}-e^{-s\Delta}}{s^{r+1}}ds$$

where $\gamma \in \mathbb{R}^+, |z| > e^{s\Delta}$. Since the integrand vanishes for $s = \pm\infty$, we can close the integration path to the right of the origin of the complex plane and compute the residues using the same procedure as in [11, p. 28] to obtain

$$G_q(z) = \sum_{\ell=-\infty}^{\infty} \frac{e^{-s_\ell f\Delta}-e^{-s_\ell \Delta}}{\Delta \cdot s_\ell^{r+1}}$$

where $s_\ell = (\log z + 2\pi j\ell)/\Delta$ are the roots of $z - e^{s\Delta} = 0$. Noting that $e^{-s_\ell\Delta} = z^{-1}$, we have

$$G_q(z) = \Delta^r \sum_{\ell=-\infty}^{\infty} \frac{e^{-f(\log z + 2\pi j\ell)} - z^{-1}}{(\log z + 2\pi j\ell)^{r+1}} \tag{27}$$

On the other hand, we know from Theorem 2 Part (i) that $G_q(z)$ is also given by (16). Equating (16) and (27) establishes Theorem 1 Part (iv). □

B. Piecewise Constant Generalised Hold Functions

Next we show how the modified Euler-Frobenius polynomials can be used to describe the asymptotic sampling zeros associated with systems of relative degree r when a piecewise-constant generalised hold function is used.

Definition 12: A piecewise-constant generalised hold function is defined by

$$H(t) = \sum_{j=1}^{m} c_j\left[\mu(t-\frac{(j-1)\Delta}{m}) - \mu(t-\frac{(j)\Delta}{m})\right]$$

where $\mu(t-t_0)$ is a unit step at $t=t_0$. □□□

The following theorem describes the discrete-time models of linear and nonlinear systems, and their properties, when a piecewise-constant generalised hold function is used to generate the continuous-time input.

Theorem 3: Let a piecewise-constant generalised hold function be used to generate the input to a system, we then have the following results:

i) When the system is an r-th order integrator with continuous transfer function $G(s) = 1/s^r$, then the *exact* sampled-data model satisfies the following discrete-time state-space equations:

$$x_{k+1} = A_d x_k + B_{goh} u_k \tag{28}$$

where $x_k = [x_{1,k} x_{2,k} \cdots x_{r,k}]^T$, A_d is defined in (14), and where

$$B_{goh} = \begin{bmatrix} \sum_{j=1}^{m} c_j\left(\frac{\Delta^r(1-\frac{j-1}{m})^r}{r!} - \frac{\Delta^r(1-\frac{j}{m})^r}{r!}\right) \\ \sum_{j=1}^{m} c_j\left(\frac{\Delta^{r-1}(1-\frac{j-1}{m})^{r-1}}{(r-1)!} - \frac{\Delta^{r-1}(1-\frac{j}{m})^{r-1}}{(r-1)!}\right) \\ \vdots \\ \sum_{j=1}^{m} c_j\left(\Delta(1-\frac{j-1}{m}) - \Delta(1-\frac{j}{m})\right) \end{bmatrix} \tag{29}$$

with output $y_k = x_{1,k}$.

The exact input-output transfer function between the discrete input $\{u_k\}$ and sampled output $\{y_k\}$ is given by

$$G_q(z) = \frac{\Delta^r \cdot S_{goh}(z)}{r! \cdot (z-1)^r} \tag{30}$$

where

$$S_{goh}(z) = \sum_{j=1}^{m} c_j\left[B'_r\left(z, \frac{(j-1)}{m}\right) - B'_r\left(z, \frac{j}{m}\right)\right] \tag{31}$$

and where $B'_r(z,f)$ are the Modified Euler-Frobenius polynomials.

ii) When the system is a general linear system having continuous transfer function $G(s)$, then the exact sampled-data model is given by

$$G_q(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{G(s)G_{goh}(s)\right\}\Big|_{t=k\Delta}\right\} \tag{32}$$

where

$$G_{goh}(s) = \frac{\sum_{j=1}^{m} c_j\left(e^{-s\Delta\frac{(j-1)}{m}} - e^{-s\Delta\frac{j}{m}}\right)}{s} \tag{33}$$

If the system has n poles and m zeros, then asymptotically, as the sampling period $\Delta \to 0$, the exact sampled-data model (32) has the following limit

$$\lim_{\Delta\to 0}\Delta^{-r}G_q(z) = \frac{K\cdot(z-1)^m \cdot S_{goh}(z)}{(n-m)!\cdot(z-1)^n}$$

where $S_{goh}(z)$ is as defined in (31), and $r = n-m$. Thus n poles converge to $z=1$, m zeros converge to $z=1$, and r zeros converge to the zeros of $S_{goh}(z)$.

iii) When the system is an affine nonlinear system with relative degree r of the form (7)–(8), then an approximate sampled-data model is given by

$$\hat{\zeta}_{k+1} = A_d\hat{\zeta}_k + B_d\cdot b(\hat{\zeta}_k,\hat{\eta}_k) + B_{goh}\cdot a(\hat{\zeta}_k,\hat{\eta}_k)u_k \tag{34a}$$

$$\hat{\eta}_{k+1} = \hat{\eta}_k + \Delta c(\hat{\zeta}_k,\hat{\eta}_k) \tag{34b}$$

where $\hat{z} = [\hat{\zeta}^T\hat{\eta}^T]^T$, with $\hat{\zeta}_k = [\hat{z}_{1,k}\cdots\hat{z}_{r,k}]^T$, $\hat{\eta}_k = [\hat{z}_{r+1,k}\cdots\hat{z}_{n,k}]^T$, is an approximation of the coordinates $z = \Phi(x)$, given by the normal form of the system. The model output is $\hat{y}_k = \hat{z}_{1,k}$. The functions $a(\cdot), b(\cdot)$

and $c(\cdot)$ are as defined in (11). The matrices A_d, B_d and B_{goh} are as defined in (14), (21) and (29), respectively. The approximate model (34) has Local Vector Fixed Step Truncation Errors of the order of $(\Delta^{r+1}, \cdots, \Delta^2, \Delta^2)$. In addition, the approximate sampled-data model (34) has sampling zero dynamics that, as the sampling period $\Delta \to 0$, converge to linear dynamics whose eigenvalues are given by the roots of $S_{goh}(z)$, as defined in (31).

Proof:

i) Consider the state-space description of an r-th order integrator given in (22). The crucial observation is that the input to the system satisfies

$$u(t) = \begin{cases} c_1 u_k & k\Delta \le t < k\Delta + \Delta/m \\ \vdots & \\ c_m u_k & k\Delta + (m-1)\Delta/m \le t < k\Delta + \Delta \end{cases}$$

which can also be expressed as

$$u(t) = \sum_{j=1}^{m} c_j \left[\mu\left(t - \frac{(j-1)\Delta}{m}\right) - \mu\left(t - \frac{j\Delta}{m}\right)\right] u_k$$

for $k\Delta \le t < (k+1)\Delta$ and therefore

$$\begin{aligned} &I(i, 0, \Delta, u(t)) \\ &= \sum_{j=1}^{m} I\left(i, \frac{(j-1)}{m}, \Delta, c_j u_k\right) - I\left(i, \frac{j}{m}, \Delta, c_j u_k\right) \\ &= \sum_{j=1}^{m} c_j \left[\frac{\Delta^i (1 - \frac{(j-1)}{m})^i}{i!} - \frac{\Delta^i (1 - \frac{j}{m})^i}{i!} \right] u_k \end{aligned} \tag{35}$$

for $i = 1, \ldots, r$, where the last equality was obtained considering Lemma 1. The result (28)–(29) then follows immediately.

Following the same argument as in Theorem 2 Part (i), the system of equations presented above can be rewritten using the forward shift operator, q, as

$$\begin{aligned} &\begin{bmatrix} (q-1)x_{1,k} & 0 & \cdots & 0 & 0 \end{bmatrix}^T \\ &\qquad = V_r \begin{bmatrix} x_{2,k} & x_{3,k} & \cdots & x_{n,k} & u_k \end{bmatrix}^T \end{aligned}$$

where

$$V_r = \left[\begin{array}{cccc|c} \Delta & \frac{\Delta^2}{2} & \cdots & \frac{\Delta^{r-1}}{(r-1)!} & \\ -(q-1) & \Delta & \cdots & \frac{\Delta^{r-2}}{(r-2)!} & \\ \vdots & \ddots & \ddots & \vdots & B_{goh} \\ 0 & 0 & \ddots & \Delta & \\ 0 & 0 & \cdots & -(q-1) & \end{array}\right]$$

From Theorem 2 Part (i) we know that $y_k/u_k = \det V_r/(z-1)^r$. In addition, since the determinant is a linear function of the column vectors [50, Theorem 2.1], when considering the structure of the vector B_{goh}, and the fact that the matrix V_r resembles the structure of P_r, as defined in (4), with $f = (j-1)/m$ or $f = j/m$, we have

$$\det V_r = \frac{\Delta^r}{r!} \sum_{j=1}^{m} c_j \left[B_r'\left(z, \frac{(j-1)}{m}\right) - B_r'\left(z, \frac{j}{m}\right)\right]$$

This completes the proof.

ii) The expression (32) is obtained directly by using [48, Eq. 12.13.3] and considering that the transfer function of a piecewise-constant generalised hold function is given by (33).

The remainder of the proof follows the same lines as the proof of Theorem 2 Part (ii), but now considering that the exact discrete-time system is given by

$$G_q(z) = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} \frac{e^{s\Delta}}{z - e^{s\Delta}} G(s) G_{goh}(s) ds$$

and that the exact discrete-time transfer function of an r-th order integrator is given by (30), and is also equivalent to

$$G_i(z) = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} \frac{e^{s\Delta}}{z - e^{s\Delta}} \frac{1}{s^r} G_{goh}(s) ds$$

where $G_{goh}(s)$ is defined in (33).

iii) The proof follows the same lines as the proof of Theorem 2 Part (iii), but now noticing that

$$\begin{aligned} &I(j, 0, \Delta,\ (b(z(t)) + a(z(t))u(t))) \\ &\qquad = I(j, 0, \Delta, b(z(t))) + I(j, 0, \Delta, a(z(t))u(t)) \end{aligned}$$

for $j = 1, \ldots, r$, where integrals are approximated as follows

$$I(j, 0, \Delta, b(z(t))) \approx b(z(k\Delta)) \cdot I(j, 0, \Delta, 1)$$
$$I(j, 0, \Delta, a(z(t))u(t)) \approx a(z(k\Delta)) \cdot I(j, 0, \Delta, u(t))$$

for $j = 1, \ldots, r$, where $I(j, 0, \Delta, u(t))$ is given by (35). The remainder of the proof follows the same lines as the proof of Theorem 2 Part (iii), but considering that the similarity transformation $\tilde{\zeta} = T\hat{\zeta}$ is now given by

$$T = \left[\begin{array}{c|c} 1 & \mathbf{0_{1\times(r-1)}} \\ \hline T_{21} & I_{r-1} \end{array}\right]$$

$$T_{21} = -\begin{bmatrix} \frac{r}{\Delta} \frac{\sum_{j=1}^m c_j \left(\left(1-\frac{j-1}{m}\right)^{n-1} - \left(1-\frac{j}{m}\right)^{n-1}\right)}{\sum_{j=1}^m c_j \left(\left(1-\frac{j-1}{m}\right)^{n} - \left(1-\frac{j}{m}\right)^{n}\right)} \\ \frac{r(r-1)}{\Delta^2} \frac{\sum_{j=1}^m c_j \left(\left(1-\frac{j-1}{m}\right)^{n-2} - \left(1-\frac{j}{m}\right)^{n-2}\right)}{\sum_{j=1}^m c_j \left(\left(1-\frac{j-1}{m}\right)^{n} - \left(1-\frac{j}{m}\right)^{n}\right)} \\ \vdots \\ \frac{r(r-1)\cdots 2}{\Delta^{r-1}} \frac{\sum_{j=1}^m c_j \left(\left(1-\frac{j-1}{m}\right)^{1} - \left(1-\frac{j}{m}\right)^{1}\right)}{\sum_{j=1}^m c_j \left(\left(1-\frac{j-1}{m}\right)^{n} - \left(1-\frac{j}{m}\right)^{n}\right)} \end{bmatrix}$$

■

Remark 6: Note that when $c_1 = c_2 = \ldots = c_m = 1$, then

$$S_{goh}(z) = B_r'(z, 0)$$

where $B_r'(z, 0)$ is the sampling zero polynomial for systems having relative degree r when a ZOH is used. □

C. Systems With Pure Time-Delay

Here we examine the case where systems having *pure time delays* have their the input generated by a zero-order hold (ZOH).

Given a continuous-time delay D, we write it in the form $D = \ell\Delta + f\Delta$, where $\ell \in \mathbb{N}, f \in [0, 1]$. We will examine the asymptotic sampling zero polynomial as $\Delta \to 0$ along the subsequence given by $\Delta = D/(\ell + f)$, for a given D and fixed f, as $\ell = 1, 2, \ldots$. The reason for this is that, as $\Delta \to 0$ arbitrarily, both f and ℓ will change, and hence, the roots of the asymptotic sampling zero polynomial, which are a function of f, will also change. However, if $\Delta \to 0$ along the subsequence given by $\Delta = D/(\ell + f)$, with D known, fixed f and $\ell = 1, 2, \ldots$, we ensure the roots of the sampling zero polynomial converge to a unique value.

With the above as background, we establish the following theorem.

Theorem 4: Consider the case when a ZOH is used to generate the input to a system having time delay. We then have the following results:

i) When the system is an r-th order integrator with continuous transfer function $G(s) = e^{-sD}/s^r$, with $D = \ell\Delta + f\Delta, \ell \in \mathbb{N}, f \in [0, 1]$, then the *exact* sampled-data model satisfies the following discrete-time state-space equations

$$x_{k+1} = A_d x_k + B_{delay} u_{k-\ell} \tag{36}$$

where $x_k = [x_{1,k} x_{2,k} \cdots x_{r,k}]^T$, A_d is defined in (14), and where

$$B_{delay} = \begin{bmatrix} \frac{\Delta^r}{r!}\left(q^{-1} - q^{-1}(1-f)^r + (1-f)^r\right) \\ \frac{\Delta^{r-1}}{(r-1)!}\left(q^{-1} - q^{-1}(1-f)^{r-1} + (1-f)^{r-1}\right) \\ \vdots \\ \Delta\left(q^{-1} - q^{-1}(1-f) + (1-f)\right) \end{bmatrix} \tag{37}$$

where q is the forward shift operator and $y_k = x_{1,k}$.

The exact input-output transfer function between the discrete input $\{u_k\}$ and discrete output $\{y_k\}$ is given by

$$G_r(z) = \frac{\Delta^r \cdot S_{delay}(z, f)}{r! \cdot z^{\ell+1}(z-1)^r} \tag{38}$$

where

$$S_{delay}(z, f) = B'_r(z, 0) - B'_r(z, f) + zB'_r(z, f) \tag{39}$$

and where $B'_r(z, f)$ are the Modified Euler-Frobenius polynomials.

ii) When the system is a general linear system having continuous transfer function $G(s) = e^{-sD}\bar{G}(s)$, where $\bar{G}(s)$ is the delay free component of $G(s)$, then the exact sampled-data model is given by

$$G(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\left\{e^{-sD}\bar{G}(s)G_{zoh}(s)\right\}\big|_{t=k\Delta}\right\} \tag{40}$$

where

$$G_{zoh}(s) = \frac{1 - e^{-s\Delta}}{s} \tag{41}$$

If the system has n poles and m zeros, then asymptotically, as the sampling period $\Delta \to 0$ along the subsequence $\Delta = D/(\ell + f)$, where f is fixed and $\ell = 1, 2, \ldots$, the exact sampled-data model (32) has the following limit

$$\lim_{\Delta \to 0} \Delta^{-r} G_r(z) = \frac{K \cdot (z-1)^m \cdot S_{delay}(z, f)}{(n-m)! \cdot z^{\ell+1}(z-1)^n}$$

where $S_{delay}(z, f)$ is as defined in (39), and $r = n - m$. Thus, n poles converge to $z = 1$, m zeros converge to $z = 1$, and r zeros converge to the zeros of $S_{delay}(z, f)$.

iii) When the system is an affine nonlinear system with relative degree r of the form (7)–(8), and has a time delay D, then an approximate sampled-data model for $D = \ell\Delta + f\Delta, \ell \in \mathbb{N}, f \in [0, 1]$, is given by

$$\hat{\zeta}_{k+1} = A_d\hat{\zeta}_k + B_d \cdot b(\hat{\zeta}_k, \hat{\eta}_k) + B_{delay} \cdot a(\hat{\zeta}_k, \hat{\eta}_k)u_{k-\ell} \tag{42a}$$

$$\hat{\eta}_{k+1} = \hat{\eta}_k + \Delta c(\hat{\zeta}_k, \hat{\eta}_k) \tag{42b}$$

where $\hat{z} = [\hat{\zeta}^T \hat{\eta}^T]^T$, with $\hat{\zeta}_k = [\hat{z}_{1,k} \cdots \hat{z}_{r,k}]^T$, $\hat{\eta}_k = [\hat{z}_{r+1,k} \cdots \hat{z}_{n,k}]^T$, is an approximation of the coordinates $z = \Phi(x)$, given by the normal form of the system. The model output is $y_k = \hat{z}_{1,k}$. The functions $a(\cdot), b(\cdot)$ and $c(\cdot)$ are as defined in (11). The matrices A_d, B_d and B_{delay} are as defined in (14), (21) and (37), respectively. The approximate model (42) has Local Vector Fixed Step Truncation Errors of the order of $(\Delta^{r+1}, \cdots, \Delta^2, \Delta^2)$. In addition, the approximate sampled-data model (42) has sampling zero dynamics that converge as $\Delta \to 0$, along the subsequence $\Delta = D/(\ell + f)$, for a given D and fixed f, as $\ell = 1, 2, \ldots$, to linear dynamics whose eigenvalues are given by the roots of $S_{delay}(z, f)$, as defined in (39).

Proof:

i) Consider the state-space description of an r-th order integrator given in (22). The crucial observation is that, with a delay $D = \ell\Delta + f\Delta$, the input to the system satisfies

$$u(t) = \begin{cases} u_{k-\ell-1} & k\Delta \le t < k\Delta + f\Delta \\ u_{k-\ell} & k\Delta + f\Delta \le t < (k+1)\Delta \end{cases} \tag{43}$$

The input (43) can be expressed as

$$u(t) = u_{k-\ell-1}(\mu(t) - \mu(t - f\Delta)) + u_{k-\ell}\mu(t - f\Delta)$$

and therefore

$$\begin{aligned} I(i, 0, \Delta, u(t)) &= I(i, 0, \Delta, u_{k-\ell-1}) \\ &\quad - I(i, f, \Delta, u_{k-\ell-1}) + I(i, f, \Delta, u_{k-\ell}) \end{aligned} \tag{44}$$

for $i = 1, \ldots, r$. The result (36)–(37) is obtained using Lemma 1 to evaluate the integrals, and the fact that $u_{k-\ell-1} = q^{-1}u_{k-\ell}$.

Similarly to Theorem 2 Part (i), the system of equations presented in above can be rewritten as

$$\begin{aligned}&\begin{bmatrix}(q-1)x_{1,k} & 0 & \cdots & 0 & 0\end{bmatrix}^T\\ &\qquad = W_r\begin{bmatrix}x_{2,k} & x_{3,k} & \cdots & x_{n,k} & u_{k-\ell}\end{bmatrix}^T\end{aligned}$$

where

$$W_r = \left[\begin{array}{ccccc|c} \Delta & \frac{\Delta^2}{2} & \cdots & & \frac{\Delta^{r-1}}{(r-1)!} & \\ -(q-1) & \Delta & \cdots & & \frac{\Delta^{r-2}}{(r-2)!} & \\ \vdots & \ddots & \ddots & & \vdots & B_{delay} \\ 0 & 0 & \ddots & & \Delta & \\ 0 & 0 & \cdots & & -(q-1) & \end{array}\right]$$

From Theorem 2 Part (i) we have that $y_k/u_k = \det W_r/(z^m(z-1)^r)$. In addition, since the determinant is a linear function of the column vectors [50, Theorem 2.1], when considering the structure of the vector B_{delay}, we have

$$\det W_r = q^{-1}\Delta^r \det M_r - q^{-1}\Delta^r \det P_r + \Delta^r \det P_r$$

where M_r and P_r are defined as in (2) and (4), respectively. We therefore have

$$\det W_r = q^{-1}\frac{\Delta^r}{r!}\left(B_r'(q,0) - B_r'(q,f) + qB_r'(q,f)\right))$$

Note that, for the case $\ell \neq 0$, there are ℓ additional poles at the origin, which correspond to the integer part of the delay. This completes the proof.

ii) The expression (40) is obtained directly by using [48, Eq. 12.13.3] and considering that the transfer function of a ZOH is given by (41) and the fact that $G(s)$ can be separated as $G(s) = e^{-sD}\bar{G}(s)$, where $\bar{G}(s)$ is the delay free part of $G(s)$.
The remainder of the proof follows the same line as the proof of Theorem 2 Part (ii), but now considering that the exact discrete-time system is given by

$$G_q(z) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{e^{s\Delta}}{z-e^{s\Delta}}G(s)\frac{1-e^{-s\Delta}}{s}ds$$

and that the exact discrete-time transfer function of an r-th order integrator is given by (38), and is also equivalent to

$$G_i(z) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{e^{s\Delta}}{z-e^{s\Delta}}\frac{1}{s^r}\frac{1-e^{-s\Delta}}{s}ds$$

iii) The proof follows the same lines as the proof of Theorem 2 Part (iii), but now noticing that

$$\begin{aligned}&I(j,0,\Delta,\ (b(z(t)) + a(z(t))u(t)))\\ &\qquad = I(j,0,\Delta,b(z(t))) + I(j,0,\Delta,a(z(t))u(t))\end{aligned}$$

for $j = 1,\ldots,r$, where the integrals are approximated as follows

$$I(j,0,\Delta,b(z(t))) \approx b(z(k\Delta))\cdot I(j,0,\Delta,1)$$
$$I(j,0,\Delta,a(z(t))u(t)) \approx a(z(k\Delta))\cdot I(j,0,\Delta,u(t))$$

for $j = 1,\ldots,r$, where $I(j,0,\Delta,u(t))$ is given by (44). Considering the fact that $u_{k-\ell-1} = q^{-1}u_{k-\ell}$ the result is obtained.
The remainder of the proof follows the exact same lines as the proof of Theorem 2 Part (iii), but considering that the similarity transformation $\tilde{\zeta} = T\hat{\zeta}$ is now given by

$$T = \left[\begin{array}{c|c} 1 & \mathbf{0_{1\times(r-1)}} \\ \hline T_{21} & I_{r-1}\end{array}\right]$$

$$T_{21} = -\begin{bmatrix} \frac{r}{\Delta}\cdot\frac{q^{-1}\left(1-(1-f)^{n-1}\right)+(1-f)^{n-1}}{q^{-1}(1-(1-f)^n)+(1-f)^n} \\ \frac{r(r-1)}{\Delta^2}\cdot\frac{q^{-1}\left(1-(1-f)^{n-2}\right)+(1-f)^{n-2}}{q^{-1}(1-(1-f)^n)+(1-f)^n} \\ \vdots \\ \frac{r(r-1)\cdots 2}{\Delta^{r-1}}\cdot\frac{q^{-1}\left(1-(1-f)^1\right)+(1-f)^1}{q^{-1}(1-(1-f)^n)+(1-f)^n}\end{bmatrix}$$

■

Remark 7: Note that, Theorem 4 Part (i) is consistent with the result presented in [29, Section 3.1]. Specifically, the zero polynomials $B_m(z,\Delta)$ defined in [29, Eq. 3.2] are compactly expressed as $S_{delay}(z,f)$ defined above. □

Remark 8: Theorem 4 is also connected to work presented in [51] for the nonlinear case. However, the latter work is based on two restrictive approximations, namely that $\ell f \approx \Delta, \ell \in \mathbb{N}$ and $u_{k-1} = \alpha_2/\alpha_1 \cdot u_k$. These restrictions lead to an approximate description of a subset of the sampling zeros, and therefore to a different asymptotic sampling zero polynomial than the one presented in this paper. In fact, it turns out to be a special case of the result presented in Theorem 3 for the piecewise-constant generalised hold function. □

VI. Numerical Example

To exemplify the ideas in the paper we illustrate Theorem 4 for the case of a linear system having transfer function $G(s) = e^{-sD}\overline{G}(s)$, where $D = 1$ and $\overline{G}(s) = 1/(s+1)$. The exact sampled data model, when a ZOH is used to generate the input, can be obtained using Theorem 4 Part (i) which yields

$$G_q(z) = \frac{\left(1-e^{-\Delta(1-f)}\right)z - e^{-\Delta}\left(1-e^{f\Delta}\right)}{z^{1+\ell}(z-e^{-\Delta})} \tag{45}$$

where $\ell = \lfloor 1/\Delta \rfloor$ is the integer part of the delay $D = 1$ with respect to the sampling period. By direct calculation from (45), the exact sampling zero is located at

$$z_r = F_r(\Delta,f) = \frac{e^{-\Delta}\left(1-e^{f\Delta}\right)}{1-e^{-\Delta(1-f)}} \tag{46}$$

On the other hand, we know from Theorem 4 Part (ii) that, for each fixed f, the asymptotic sampling zero polynomial for this

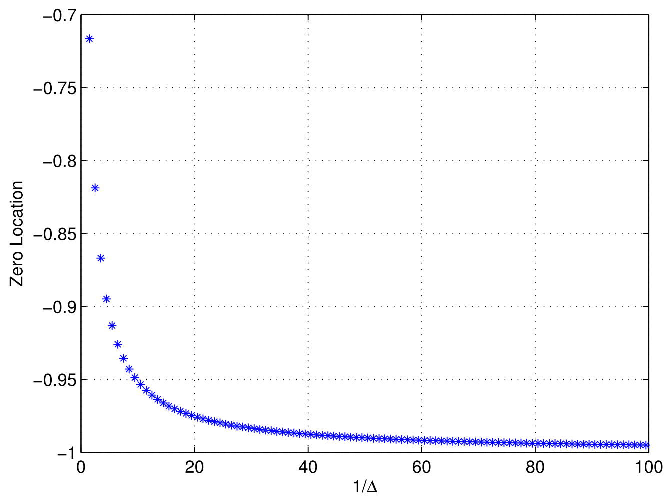


Fig. 1. Convergence of sampling zero for $f = 1/2$.

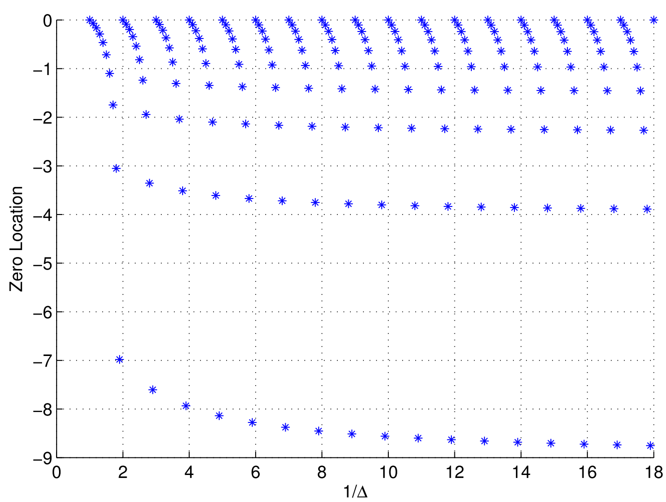


Fig. 2. Convergence of sampling zero with no fixed f.

example is given by

$$\begin{aligned} S_{delay}(z,f) &= B_1'(z,0) - B_1'(z,f) + zB_1'(z,f) \\ &= (1-f)z + f \end{aligned}$$

and thus the asymptotic location of the zero is given by

$$z_a = F_a(f) = \frac{-f}{1-f} \tag{47}$$

To verify the connection between (46) and (47) for this specific case, we apply L'Hôpital's rule to (46) leading to

$$\lim_{\Delta \to 0} z_r = \lim_{\Delta \to 0} \frac{-e^{-\Delta} + (1-f)e^{-\Delta(1-f)}}{(1-f)e^{-\Delta(1-f)}} = \frac{-f}{1-f}$$

as given in (47). Note that the limit $\Delta \to 0$ is a limit along the subsequence given by $\Delta = D/(\ell + f), \ell = 1, 2, \ldots,$ with D and f fixed.

The convergence of the exact sampling zero z_r is shown in Fig. 1, along the subsequence $\Delta = D/(\ell + f), \ell = 1, 2, \ldots$ for $f = 1/2$ and $D = 1$. As predicted by Theorem 4 Part ii, the exact sampling zero converges to the zero of $S_{delay}(z, f)$ which, for the case $r = 1, f = 1/2$ is located at $z = -1$.

Fig. 2 shows the different values that the sampling zero takes as a function of $1/\Delta$, when $\Delta \to 0$, without fixing f. Note that, in this case, $f = 1/\Delta - \ell$. It can be seen that as Δ decreases the location of the sampling zero resets whenever D/Δ is an integer. Also, it can be seen from Fig. 2 that along each subsequence where f is fixed (i.e. every ten data points), the location of the sampling zero converges to a different, but well defined value. For example, for $f = 0.5$, the zero converges to -1, and for $f = 0.9$, the zero converges to -9.

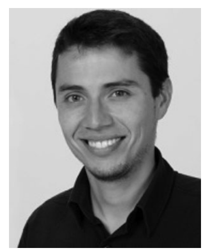
VII. Conclusion

This paper has introduced the Modified Euler-Frobenius polynomials. The polynomials depend upon two independent variables, denoted z and f. We have established several properties of the polynomials. We have also shown that the polynomials are useful to characterise the (asymptotic) sampling zeros and sampling zero dynamics of linear and nonlinear systems for several non-standard sampled-data problems. We have examined the case of partial zero order holds, generalised zero order holds and systems having non-zero delay. In the latter case, convergence of the sampling zeros occurs along subsequences where $\Delta \to 0$ as $\Delta = D/(\ell + f)$, where $\ell = 1, 2, \ldots$, and f is fixed.

References

[1] L. Euler, *Institutiones Calculi Differentialis Cum Eius Vsu in Analysi Finitorum ac Doctrina Serierum*. Academiae Imperialis Scientiarum Petropolitanae, St. Petersbourg, Russia, 1755.
[2] F. Frobenius, "On Bernoulli numbers and Euler polynomials," *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, pp. 809–847, 1910.
[3] S. Sobolev, "On the roots of Euler polynomials," *Soviet Math. Dokl.*, vol. 18, pp. 935–938, 1977.
[4] L. Carlitz, "Eulerian numbers and polynomials," *Math. Mag.*, pp. 247–260, 1959.
[5] D. Foata, "Eulerian polynomials: From Eulers time to the present," in *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*. Springer: New York, 2010, pp. 253–273.
[6] K. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems," *Automatica*, vol. 20, no. 1, pp. 31–38, Jan. 1984.
[7] B. Wahlberg, "Limit results for sampled systems," *Int. J. Control*, vol. 48, no. 3, pp. 1267–1283, 1988.
[8] M. J. Błachuta, "On zeros of pulse transfer functions," *IEEE Trans. Autom. Control*, vol. 44, no. 6, pp. 1229–1234, 1999.
[9] S. Weller, W. Moran, B. Ninness, and A. Pollington, "Sampling zeros and the Euler-Frobenius polynomials," *IEEE Trans. Autom. Control*, vol. 46, no. 2, pp. 340–343, 2001.
[10] J. I. Yuz and G. C. Goodwin, "On sampled-data models for nonlinear systems," *IEEE Trans. Autom. Control*, vol. 50, no. 10, pp. 1477–1489, 2005.
[11] J. I. Yuz and G. C. Goodwin, *Sampled-Data Models for Linear and Nonlinear Systems*. Springer: New York, 2014.
[12] G. Birkhoff and C. R. De Boor, "Piecewise polynomial interpolation and approximation," *Approximation of Functions*, pp. 164–190, 1965.
[13] L. Schumaker, *Spline Functions: Basic Theory*. Cambridge University Press: Cambridge, U.K., 2007.
[14] C. K. Chui, *An Introduction to Wavelets*. Academic Press: New York, 2014, vol. 1.
[15] T. Hagiwara, T. Yuasa, and M. Araki, "Stability of the limiting zeros of sampled-data systems with zero-and first-order holds," *Int. J. Control*, vol. 58, no. 6, pp. 1325–1346, 1993.
[16] M. Ishitobi, "Stability of zeros of sampled system with fractional order hold," *Control Theory Applications, IEE Proceedings-*, vol. 143, no. 3, pp. 296–300, 1996.
[17] S. Monaco and D. Normand-Cyrot, "Zero dynamics of sampled nonlinear systems," *Syst. Control Lett.*, vol. 11, no. 3, pp. 229–234, 1988.

[18] S. Monaco and D. Normand-Cyrot, "Issues on nonlinear digital control," *Eur. J. Control*, vol. 7, no. 2, pp. 160–177, 2001.
[19] D. S. Carrasco, G. C. Goodwin, and J. I. Yuz, "Vector measures of accuracy for sampled data models of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 224–230, Jan. 2013.
[20] E. Yucra, J. Yuz, and G. C. Goodwin, "Sampling zeros of discrete models for fractional order systems," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2383–2388, 2013.
[21] D. S. Carrasco, G. C. Goodwin, and J. I. Yuz, "Sampled data models for nonlinear stochastic systems: Truncation errors and sampling zero dynamics," *IEEE Trans. Autom. Control*, vol. 61, no. 1, pp. 103–115, 2016.
[22] M. J. Blachuta, "On zeros of pulse transfer functions of systems with first-order hold," in *Proc. 37th IEEE Conf. Decision Control*, vol. 1. IEEE, 1998, pp. 307–312.
[23] M. J. Blachuta, "On fast sampling zeros of systems with fractional-order hold," in *Proc. American Control Conf.*, vol. 4. , IEEE, 2001, pp. 3229–3230.
[24] V. L. Kharitonov, "Robust stability analysis of time delay systems: A survey," *Annu. Rev. Control*, vol. 23, pp. 185–196, 1999.
[25] J.-P. Richard, "Time-delay systems: An overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.
[26] K. Gu, J. Chen, and V. L. Kharitonov, *Stability of Time-Delay Systems*. Springer Science & Business Media: New York, 2003.
[27] A. Germani, C. Manes, and P. Pepe, "Linearization and decoupling of nonlinear delay systems," in *Proc. American Control Conf.*, vol. 3, Jun. 1998, pp. 1948–1952.
[28] C. Califano and C. H. Moog, "On the existence of the normal form for nonlinear delay systems," in *Recent Results on Nonlinear Delay Control Systems*. Springer: New York, 2016, pp. 113–142.
[29] S. Hara, R. Kondo, and H. Katori, "Properties of zeros in digital control systems with computational time delay," *Int. J. Control*, vol. 49, no. 2, pp. 493–511, 1989.
[30] M. Ishitobi, "Stable zeros of a discrete system obtained by sampling a continuous-time plant with a time delay," *Int. J. Control*, vol. 59, no. 4, pp. 1053–1062, 1994.
[31] C. Zeng, S. Liang, S. Gan, and X. HU, "Asymptotic properties of zero dynamics of multivariable sampled-data models with time delay," *WSEAS Trans. Syst.*, vol. 13, pp. 23–32, 2014.
[32] V. A. Bondarko, "Zeros of delayed sampled systems," *Cybern. Phys.*, vol. 2, no. 3, pp. 183–188, 2013.
[33] V. A. Bondarko, "Limiting zeros of sampled systems with time delay," *Autom. Remote Control*, vol. 76, no. 8, pp. 1327–1346, 2015.
[34] S. Monaco and D. Normand-Cyrot, "On the sampling of time delayed nonlinear systems," in *Proc. Eur. Control Conf. (ECC-91)*. Ed. Hermes, 1991, pp. 556–559.
[35] P. T. Kabamba, "Control of linear systems using generalized sampled-data hold functions," *IEEE Trans. Autom. Control*, vol. 32, no. 9, pp. 772–783, 1987.
[36] E.-W. Bai and S. Dasgupta, "A note on generalized hold functions," *Syst. Control Lett.*, vol. 14, no. 4, pp. 361–368, 1990.
[37] J. I. Yuz, G. C. Goodwin, H. Gamier et al. *et al.*, "Generalised hold functions for fast sampling rates," in *Proc. 43rd IEEE Conf. Decision Control (CDC'04)*, vol. 2, 2004, pp. 1908–1913.
[38] A. Feuer and G. C. Goodwin, "Generalized sample hold functions-frequency domain analysis of robustness, sensitivity, and intersample difficulties," *IEEE Trans. Autom. Control*, vol. 39, no. 5, pp. 1042–1047, 1994.
[39] J. Zhang and C. Zhang, "Robustness analysis of control systems using generalized sample hold functions," in *Proc. 33rd IEEE Conf. Decision Control*, 1994, pp. 237–242.
[40] C. Zeng and S. Liang, "Sampled-data models for nonlinear systems via the taylor-lie series and multirate input," *WSEAS Trans. Syst. Control*, vol. 9, pp. 160–170, 2014.
[41] K. J. Åström and B. Wittenmark, *Computer-Controlled Systems: Theory and Design*. Prentice-Hall: Englewood Cliffs, NJ, 1984.
[42] S. Monaco and D. Normand-Cyrot, "Multirate sampling and zero dynamics: From linear to nonlinear," in *Nonlinear Synthesis*. Springer: New York, 1991, pp. 200–213.
[43] G. C. Goodwin and R. L. Payne, *Dynamic System Identification: Experiment Design and Data Analysis*. Academic Press: New York, 1977.
[44] P. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*. Springer (Berlin and New York), 1992.
[45] J. C. Butcher, *Numerical Methods for Ordinary Differential Equations*. Wiley: New York, 2008.
[46] A. Isidori, *Nonlinear Control Systems*. Springer Verlag: Berlin, Germany, 1995.
[47] S. Monaco and D. Normand-Cyrot, "Minimum-phase nonlinear discrete-time systems and feedback stabilization," in *Proc. 26th IEEE Conf. Decision Control*, vol. 26, Dec. 1987, pp. 979–986.
[48] G. C. Goodwin, S. F. Graebe, and M. E. Salgado, *Control System Design*. Prentice-Hall: Englewood Cliffs, NJ, 2001.
[49] T. Kailath, *Linear Systems*. Prentice-Hall Englewood Cliffs, NJ, 1980.
[50] S. Lang, *Linear Algebra (Third Edition)*. Springer-Verlag: Berlin, Germany, 2004.
[51] C. Zeng, S. Liang, and H. Li, "Asymptotic properties of zero dynamics for nonlinear discretized systems with time delay via Taylor method," *Nonlinear Dynamics*, vol. 79, no. 2, pp. 1481–1493, 2015.



Diego S. Carrasco (M'14) received his professional title of Ingeniero Civil Electrónico and M.Sc. degree in electronic engineering from Universidad Técnica Federico Santa María, Chile, in 2009 and the Ph.D. degree in electrical engineering from the University of Newcastle, Australia, in 2014.

His research interests include sampled-data control, sampling, numerical analysis and artificial pancreas development.

Graham C. Goodwin (LF'11) received the B.Sc. degree in physics, the B.E. (with honors) in physics and the Ph.D. degree from the University of New South Wales in 1965, 1967, and 1971, respectively.

He is the co-author of nine books, four edited books, 218 international journal papers and 322 refereed international conference papers.

Prof. Goodwin received the IEEE Control Systems Field Award in 2010, the Rufus T. Oldenburger Medal in 2013, the 1999 IEEE Control Systems Society Hendrik Bode Lecture Prize, a Best Paper award by IEEE Transactions on Automatic Control, a Best Paper award by Asian Journal of Control, two Best Engineering Text Book awards from the International Federation of Automatic Control in (1984 and 2005), the 2008 Quazza Medal from the International Federation of Automatic Control, the 2010 Nordic Process Control Award, and the 2011 Asian Control Association Wook Hyun Kwon Education Award. He is an Honorary Fellow of Institute of Engineers, Australia; a Fellow of the International Federation of Automatic Control, a Fellow of the Australian Academy of Science; a Fellow of the Australian Academy of Technology, Science and Engineering; a Member of the International Statistical Institute; a Fellow of the Royal Society, London and a Foreign Member of the Royal Swedish Academy of Sciences. He holds Honorary Doctorates from Lund Institute of Technology, Sweden and the Technion Israel.

Juan I. Yuz (M'06) received the professional title of Ingeniero Civil Electrónico and a M.Sc. degree in electronic engineering from Universidad Técnica Federico Santa María (UTFSM), Chile in 2001 and the Ph.D. degree in electrical engineering from The University of Newcastle, Australia, in 2006.

He is currently Associate Professor in the Departamento de Electrónica, UTFSM. He is Associate Editor of Automatica and Revista Iberoamericana de Automatización e Informática Industrial, and Deputy Director of the Advanced Center for Electrical and Electronic Engineering, UTFSM. His research interests include control and identification of sampled-data systems, and their applications to power electronics and biomedical systems.