

Modified Euler-Frobenius Polynomials with application to Sampled Data Model

Outline

- Introduction
- Euler-Frobenius Polynomials
 - Standard
 - Modified
- Preliminary definitions
 - Multiple Integrations
 - Zero Dynamics
- Applications
- Conclusions

Why?

Euler-Frobenius Polynomials are widely used in math and they play a key role in many engineering areas.

They appears in in sampled-data models both in linear and non linear systems, polynomial interpolation and splines.

With the evolution of Standard to Modified Euler-Frobenius Polynomials we are trying to find a unified framework to identify the zeroes in the sampled-data models

Standard Euler-Frobenius polynomials

$$B_r(z) = r! \cdot \det M_r$$

$$M_r = \begin{bmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} & \frac{1}{r!} \\ 1-z & 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 1-z & 1 & \frac{1}{2!} \\ 0 & \cdots & 0 & 1-z & 1 \end{bmatrix}$$

Modified Euler-Frobenius polynomials

$$B'_r(z, f) = r! \cdot \det P_r$$

$$P_r = \begin{bmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(r-1)!} & \frac{1-f}{r!} \\ 1-z & 1 & \frac{1}{2!} & \cdots & \frac{1-f}{(r-1)!} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdots & 1-z & 1 & \frac{1-f}{2!} \\ 0 & \cdots & 0 & 1-z & 1-f \end{bmatrix}$$

Preliminar(1): Multiple integration

Define:
$$I(r, f, \Delta, g) = \int_{f\Delta}^{\Delta} \int_{f\Delta}^{t_{r-1}} \dots \int_{f\Delta}^{t_1} g(t) dt \dots dt_{r-1}$$

And in the special case: $g = 1$

$$I(j, f, \Delta, g) = \frac{\Delta^j (1 - f)^j}{j!}$$

Preliminar(2): Zero-dynamics

Internal Dynamics
when input and initial conditions
are chosen to make output
identically zero



$$\dot{z}_1(t) = z_2(t)$$

$$\dot{z}_2(t) = z_3(t)$$

$$\dot{z}_3(t) = b(z, \eta) + a(z, \eta)u(t)$$

$$\dot{\eta}(t) = \eta(z, \eta)$$

$$y(t) = z_1(t)$$

Applications Outline

- Partial zero-order hold
 - System of integrators
 - Linear system
 - Affine nonlinear system
- Piecewise Constant generalised hold
 - System of integrators
 - Linear system
 - Affine nonlinear system
- Pure time-delay
 - System of integrators
 - Linear system
 - Affine nonlinear system

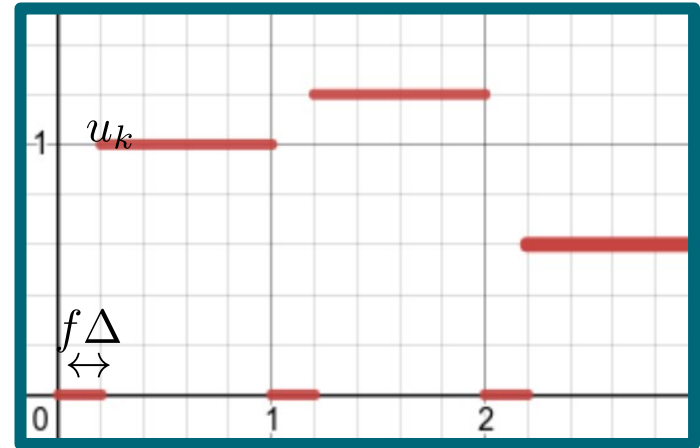
Partial zero-order hold

$$u(t) = \begin{cases} 0 & k\Delta \leq t \leq k\Delta + f\Delta \\ u_k & k\Delta + f\Delta \leq t \leq (k+1)\Delta \end{cases}$$

with $k \in \mathbb{N}$ and $f \in [0, 1]$

And the transfer function

$$G_{PZOH}(s) = \frac{e^{sf\Delta} - e^{-s\Delta}}{s}$$



System of Integrator: $G(s) = 1/s^3$

Discretize the system

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & \Delta & \frac{\Delta^2}{2!} \\ 0 & 1 & \Delta \\ 0 & 0 & 1 \end{bmatrix}}_{A_d} \cdot x_k + \underbrace{\begin{bmatrix} \frac{\Delta^3(1-f)^3}{3!} \\ \frac{\Delta^2(1-f)^2}{2!} \\ \Delta(1-f) \end{bmatrix}}_{B_{PZOH}} \cdot u_k$$

We rewrite the system using the forward shift operator q

$$\begin{bmatrix} (q-1)x_{1,k} & 0 & 0 \end{bmatrix}^T = D_3 \begin{bmatrix} x_{2,k} & x_{3,k} & u_k \end{bmatrix}^T$$

System of Integrator: $G(s) = 1/s^3$

$$\text{Where } D_3 = \begin{bmatrix} \Delta & \frac{\Delta^2}{2!} & \frac{\Delta^3(1-f)^3}{3!} \\ -(q-1) & \Delta & \frac{\Delta^2(1-f)^2}{2!} \\ 0 & -(q-1) & \Delta(1-f) \end{bmatrix}$$

Now we can use the Cramer's rule

$$u_k = \frac{Det N}{Det D_3} \quad \text{Where } N = \begin{bmatrix} \Delta & \frac{\Delta^2}{2!} & (q-1)x_{1,k} \\ -(q-1) & \Delta & 0 \\ 0 & -(q-1) & 0 \end{bmatrix}$$

System of Integrator: $G(s) = 1/s^3$

$$\begin{array}{ccc} \text{Det}N = (q-1)^3 x_{1,k} & \longrightarrow & u_k = \frac{(q-1)^3 x_{1,k}}{\frac{\Delta^3}{3!} B'_3(q, f)} \\ \text{Det}D_3 = \frac{\Delta^3}{3!} B'_3(q, f) & & \end{array}$$

Notice that $y=x_1$ and take y over u_k

$$G_q(z) = \frac{\Delta^3 B'_3(z, f)}{3!(z-1)^3}$$

General Linear Sys

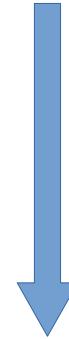
Take a linear sys example

$$G(s) = 10 \frac{s + 1}{(s + 10)(s + 100)(s + 1000)}$$

Use exact definition of the inverse laplace transform and z transform

$$G_q(z) = Z\{L^{-1}\{G(s)G_{PZOH}(s)\}\}$$

$$G_q(z) = \frac{1}{2\pi j} \int_{-\gamma+j\infty}^{\gamma+j\infty} \frac{e^{s\Delta}}{z - e^{s\Delta}} G(s) \frac{e^{sf\Delta} - e^{-s\Delta}}{s} ds$$



General Linear Sys

$$G_q(z) = \frac{1}{2\pi j} \int_{j\infty}^{j\infty} \frac{e^w}{z - e^w} \frac{10}{w^2} \frac{e^{wf} - e^{-w}}{w} dw \quad (1)$$

$$G_q(z) = \frac{1}{2\pi j} \int_{j\infty}^{j\infty} \frac{e^w}{z - e^w} \frac{\Delta^2}{w^2} \frac{e^{wf} - e^{-w}}{w} dw \quad (2)$$

Now compare the results from the linear case (1) and the chain of integrators case(2)

We already knew the result of the first integral, so by comparison we can define

$$\lim_{\Delta \rightarrow 0} \Delta^{-2} G_q(z) = \frac{10B'_2(z, f)}{2!(z - 1)^2}$$

Affine nonlinear System

$$\begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_2^2(t)x_1(t) + x_1(t) + u(t) \end{array} \xrightarrow{\begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix}} \begin{array}{l} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = \underbrace{z_1(t) + z_2^2(t)z_1(t)}_b + \underbrace{1}_a u(t) \end{array}$$

We take the integral from the last variable up to the first one

$$z_1((k+1)\Delta) = z_1(k\Delta) + \Delta z_2(k\Delta) + I(2, 0, \Delta, (b(z(t)) + a(z(t))u(t)))$$

$$z_2((k+1)\Delta) = z_2(k\Delta) + I(1, 0, \Delta, (b(z(t)) + a(z(t))u(t)))$$

We can rewrite the integrals as

$$I(2, 0, \Delta, (b(z(t)) + a(z(t))u(t))) = b(z(t))I(2, 0, \Delta, 1) + a(z(t))I(2, f, \Delta, 1)u_k$$

Affine nonlinear System

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d b(\hat{z}_k) + B_{PZOH} a(\hat{z}_k) u_k \quad B_d = \begin{bmatrix} \frac{\Delta^2}{2!} \\ \frac{\Delta}{1} \end{bmatrix}$$

Then use a coordinates change

$$T = \begin{bmatrix} 1 & 0_{1 \times r-1} \\ -\frac{2}{\Delta(1-f)} & I_{r-1} \end{bmatrix} \quad \downarrow \quad r=2$$

$$\tilde{z}_{1,k+1} = q_{11} \tilde{z}_{1,k} + Q_{12} \tilde{z}_{2,k} + \frac{\Delta^1}{1!} (b(\tilde{z}_k + (1-f)^1 a(\tilde{z}_k) u_k)$$

$$\tilde{z}_{2,k+1} = Q_{21} \tilde{z}_{1,k} + Q_{22} \tilde{z}_{2,k} + \tilde{B}_2 b(\tilde{z}_k)$$

Affine nonlinear System

$$\tilde{z}_{2,k+1} = \underbrace{T_{21}A_{12} + A_{22}}_{\left(-\frac{2}{\Delta(1-f)}\Delta + 1\right)} \cdot \tilde{z}_{2,k} \quad \text{(Where } A_{11}, A_{22} \text{ are partitions from } A_d)$$

Now we can show that the eigenvalues of Q_{22} are the roots of the Modified Euler-Frobenius polynomials

$$D_{r=2} \begin{bmatrix} 0 & I_{r-1} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\Delta^2(1-f)^2}{2!} & A_{12} \\ -\frac{\Delta^2(1-f)^2}{2!}T_{21} & A_{22} - z \end{bmatrix}$$

Take both sides determinant and

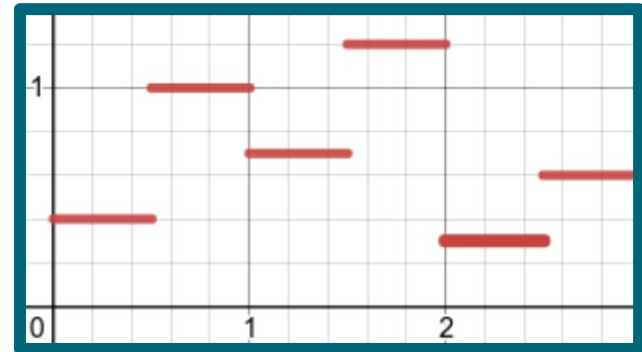
$$B'_2(f, z) = (1-f)^2 \det(z - Q_{22})$$

Piecewise Constant Generalized Hold Functions

$$u(t) = \begin{cases} c_1 u_k & k\Delta \leq t < k\Delta + \Delta/m \\ \dots & \\ \dots & \\ c_m u_k & k\Delta + (m-1)\Delta/m \leq t < k\Delta + \Delta \end{cases}$$

and with the transfer function

$$G_{GOH}(s) = \frac{\sum_{j=1}^m c_j (e^{-s\Delta \frac{(j-1)}{m}} - e^{-s\Delta \frac{j}{m}})}{s}$$



System of Integrator: $G(s) = 1/s^3$

Here we consider the input function with $m=2$
and our system in discrete form becomes

$$x_{k+1} = A_d x_k + B_{GOH} u_k \quad \text{with} \quad B_{GOH} = \begin{bmatrix} \sum_{j=1}^m \left(\frac{\Delta^3 (1 - \frac{j-1}{2})}{3!} \right) - \left(\frac{\Delta^3 (1 - \frac{j}{2})}{3!} \right) \\ \sum_{j=1}^m \left(\frac{\Delta^2 (1 - \frac{j-1}{2})}{2!} \right) - \left(\frac{\Delta^2 (1 - \frac{j}{2})}{2!} \right) \\ \sum_{j=1}^m \left(\frac{\Delta^1 (1 - \frac{j-1}{2})}{1!} \right) - \left(\frac{\Delta^1 (1 - \frac{j}{2})}{1!} \right) \end{bmatrix}$$

$$u(t) = \sum_{j=1}^m c_j \left[\mu\left(t - \frac{(j-1)\Delta}{m}\right) - \mu\left(t - \frac{j\Delta}{m}\right) \right]$$

$$u(t) = \sum_{j=1}^m I\left(i, \frac{(j-1)}{m}, \Delta, c_j u_k\right) - I\left(i, \frac{j}{m}, \Delta, c_j u_k\right)$$

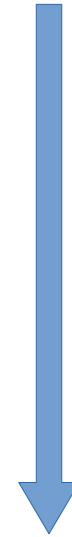
System of Integrator

We can compute the transfer function obtaining

$$G_q(s) = \frac{\Delta^3 S_{GOH}(z)}{3!(z-1)^3}$$

Where S_{GOH} depends from the polynomials

$$S_{GOH} = \sum_{j=1}^{m=2} [B'_3(z, \frac{j-1}{2}) - B'_3(z, \frac{j}{2})]$$



Linear System

Move to z domain

$$G_q(z) = Z\{L^{-1}\{G(s)G_{GOH}(s)\}\}$$

$$\text{with } G_{GOH}(s) = \frac{\sum_{j=1}^m c_j (e^{-s\Delta \frac{(j-1)}{m}} - e^{-s\Delta \frac{j}{m}})}{s}$$

By Comparison with System of Integrator

$$\lim_{\Delta \rightarrow 0} \Delta^{-r} G_q(s) = \frac{k \cdot (z-1)^{|Zeros|} S_{GOH}(z)}{(|Poles| - |Zeros|)! (z-1)^{|Poles|}}$$

$$S_{GOH}(z) = \sum_{j=1}^m [c_j [B'_r(z, \frac{(j-1)}{m}) - B'_r(z, \frac{j}{m})]]$$

Linear System

$$G(s) = 10 \frac{s + 1}{(s + 10)(s + 100)(s + 1000)}$$

$$S_{GOH}(z) = \sum_{j=1}^2 \left[B'_2(z, \frac{j-1}{2}) - B'_2(z, \frac{j}{2}) \right]$$

$$\lim_{\Delta \rightarrow 0} \Delta^{-2} G_q(z) = \frac{10(z-1)^1 S_{GOH}(z)}{2!(z-1)^3}$$

$$S_{GOH}(z) = \sum_{j=1}^m \left[c_j \left[B'_r(z, \frac{j-1}{m}) - B'_r(z, \frac{j}{m}) \right] \right]$$
$$c_j = 1, m = 2$$

$$\frac{k \cdot (z-1)^{|Zeros|} S_{GOH}(z)}{(|Poles| - |Zeros|)! (z-1)^{|Zeros|}}$$

Affine nonlinear System

$$\begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_2^2(t)x_1(t) + x_1(t) + u(t) \end{array} \xrightarrow{\begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix}} \begin{array}{l} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = \underbrace{z_1(t) + z_2^2(t)z_1(t)}_b + \underbrace{1}_a u(t) \end{array}$$

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d b(\hat{z}_k) + B_{GOH} a(\hat{z}_k) u_k$$

$$B_{GOH} = \begin{bmatrix} \sum_{j=1}^m \left(\frac{\Delta^2 (1 - \frac{j-1}{2})}{2!} \right) - \left(\frac{\Delta^2 (1 - \frac{j}{2})}{2!} \right) \\ \sum_{j=1}^m \left(\frac{\Delta^1 (1 - \frac{j-1}{2})}{1!} \right) - \left(\frac{\Delta^1 (1 - \frac{j}{2})}{1!} \right) \end{bmatrix}$$

Affine nonlinear System

We can again move the sys to other coordinates

$$T = \begin{bmatrix} 1 & 0_{1 \times r-1} \\ T_{21} & I_{r-1} \end{bmatrix} \quad T_{21} = -\frac{2 \sum_{j=1}^2 1((1 - \frac{j-1}{2})^1 - (1 - \frac{j}{2})^1)}{\Delta \sum_{j=1}^2 1((1 - \frac{j-1}{2})^2 - (1 - \frac{j}{2})^2)}$$

We can compute the zero dynamics, and as in the previous case, we can see that the eigenvalues of Q_{22} converge to the roots of S_{GOH}

$$S_{GOH}(z) = \sum_{j=1}^m [c_j [B'_2(z, \frac{(j-1)}{2}) - B'_2(z, \frac{j}{2})]]$$

Systems with pure time-delay

Another non standard model we can deal with, is the case in which the system has a pure time-delay, and the input is generated by a Zero order-hold(ZOH)

$$L[f(t - D)] = e^{-sD} F(s)$$

$$G_{ZOH} = \frac{1 - e^{-s\Delta}}{s}$$

System of integrator: $G(s) = \frac{e^{-sD}}{s^3}$

$$D = \Delta l + \Delta f \quad \text{where } l \in N, f \in [0, 1)$$

Discretize as

$$x_{k+1} = A_d x_k + B_{delay} u_{k-l}$$

Using

$$B_{delay} = \begin{bmatrix} \frac{\Delta^3}{3!} (q^{-1} - q^{-1}(1-f)^3 + (1-f)^3) \\ \frac{\Delta^2}{2!} (q^{-1} - q^{-1}(1-f)^2 + (1-f)^2) \\ \frac{\Delta^1}{1!} (q^{-1} - q^{-1}(1-f)^1 + (1-f)^1) \end{bmatrix}$$

System of integrator

$$u(t) = \begin{cases} u_{k-l-1} & k\Delta \leq t < k\Delta + f\Delta \\ u_{k-1} & k\Delta + f\Delta \leq t < (k+1)\Delta \end{cases}$$

Expressed As

$$u(t) = u_{k-l-1}(\mu(t) - \mu(t - f\Delta)) + u_{k-1}\mu(t - f\Delta)$$

Becomes

$$I(i, 0, \Delta, u(t)) = I(i, 0, \Delta, u_{k-l-1}) - I(i, f, \Delta, u_{k-l-1}) + I(i, f, \Delta, u_{k-1})$$

Then as in the previous cases

$$G_r(z) = \frac{\Delta^3 \cdot S_{delay}(z, f)}{3! \cdot z^{l+1}(z-1)^3} \quad S_{delay} = B'_3(z, 0) - B'_3(z, f) + zB'_3(z, f)$$

Linear System

Move the system $G(s)$ from s-domain to z-domain

$$G_q(z) = Z\{L^{-1}\{\bar{G}(s)G_{ZOH}(s)\}\}$$

$$\lim_{\Delta \rightarrow 0} \Delta^{-r} G_q(s) = \frac{k \cdot (z - 1)^{|Zeros|} S_{Delay}(z)}{(|Poles| - |Zeros|)! z^{l+1} (z - 1)^{|Zeros|}}$$

By
Comparison

Having S_{delay} as before

$$S_{\text{delay}}(z, f) = B'_r(z, 0) - B'_r(z, f) + zB'_r(z, f)$$

Affine nonlinear System

$$\begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_2^2(t)x_1(t) + x_1(t) + u(t) \end{array} \xrightarrow{\begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix}} \begin{array}{l} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = \underbrace{z_1(t) + z_2^2(t)z_1(t)}_b + \underbrace{1}_a u(t) \end{array}$$

$$\hat{z}_{k+1} = A_d \hat{z}_k + B_d b(\hat{z}_k) + B_{Delay} a(\hat{z}_k) u_k$$

$$B_{delay} = \begin{bmatrix} \frac{\Delta^2}{2!} (q^{-1} - q^{-1}(1-f)^2 + (1-f)^2) \\ \frac{\Delta^1}{1!} (q^{-1} - q^{-1}(1-f)^1 + (1-f)^1) \end{bmatrix}$$

Affine nonlinear System

We can again move the sys to other coordinates

$$T = \begin{bmatrix} 1 & 0_{1 \times r-1} \\ T_{21} & I_{r-1} \end{bmatrix} \quad T_{21} = - \begin{bmatrix} \frac{r}{\Delta} \cdot \frac{q^{-1}(1-(1-f)^{n-1})+(1-f)^{n-1}}{q^{-1}(1-(1-f)^n)+(1-f)^n} \\ \dots \\ \frac{r(r-1)\dots 2}{\Delta^{r-1}} \cdot \frac{q^{-1}(1-(1-f)^1)+(1-f)^1}{q^{-1}(1-(1-f)^n)+(1-f)^n} \end{bmatrix}$$

We can compute the zero dynamics, and as in the previous case, we can see that the eigenvalues of Q_{22} converge to the roots of S_{GOH}

$$S_{GOH}(z) = \sum_{j=1}^m [c_j [B'_2(z, \frac{(j-1)}{2}) - B'_2(z, \frac{j}{2})]]$$

Conclusions

