J Optim Theory Appl (2016) 168:625–645
DOI 10.1007/s10957-015-0759-z

Motion Controller Design of Wheeled Inverted Pendulum with an Input Delay Via Optimal Control Theory

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Received: 2 December 2014 / Accepted: 15 May 2015 / Published online: 27 May 2015

Abstract This paper investigates the control of Back-and-Forth motion to fetch object and Lower-Raise-Head motion to avoid obstacle of the wheeled inverted pendulum system with an input delay. In controlling the Back-and-Forth motion, the linear optimal control theory is used because the tilt angle of the pendulum is forced to be small enough by minimizing a quadratic performance criterion with large weight of tilt angle error, and the controller is represented by using predictor-based feedback. In controlling the Lower-Raise-Head motion, linearized models do not work due to the strong nonlinearity caused by big tilt angle for avoiding obstacle. Without considering yaw movement, the wheeled inverted pendulum system is decoupled, the subsystem governing the state of the tilt angle is transformed into a simple linear system by using feedback linearization. With a properly chosen trajectory tracking target, the control of Lower-Raise-Head motion is solved on the basis of optimal trajectory tracking control for linear subsystems with an input delay. Numerical simulations illustrate the effectiveness of the proposed approaches.

Keywords Wheeled inverted pendulum · Motion control · Input delay · Optimal trajectory tracking control · Feedback linearization · Predictor-based feedback

Mathematics Subject Classification 49N05 · 49N35

Communicated by Felix L. Chernousko.

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1 Introduction

Wheeled inverted pendulum (WIP, for short) robot is a kind of human-like mobile robots, a common form in many kinds of mechanical devices [1,2]. It is driven by two wheels, while the actuator does not directly drive the pendulum mounted on chassis, but generates indirect action to the pendulum by dynamic coupling. This kind of self-balancing robot is completely different from animal-type robots in mechanisms and has some unique advantages in practical application, such as: simple structure, small and flexible shape, low cost and low energy consumption. Having all these advantages, the Segway human transporter (http://www.segway.com) is the most successful and popular commercial application of WIP systems. Even though WIP robots have been applied in complex terrain environments, the dynamics analysis and motion control of WIP systems are challenging due to two special characteristics: The controlled system is under-actuated, and the constraints are nonholonomic when the wheels can only move by rolling rather than slipping.

For the controller design of WIP systems, linear control methods including LQR control [3] and pole assignment method [4] are usually used when the angle signals involved are small enough. The control methods based on linearized models do not work if control range and applicable range are not small. In this case, nonlinear control methods are required. Feedback linearization and partial feedback linearization have been used frequently to design nonlinear controllers [5]. This approach requires precise system model. The presence of uncertainties and disturbances could disrupt the performance of the controller and even lead to unstable balance. In order to eliminate the effects from uncertainties and disturbances, sliding mode controllers were proposed to stabilize WIP systems and eliminate the steady tracking error [6]. With perfect robustness, the sliding mode control works effectively on one hand, but results in inevitable chattering on the other hand. Adaptive control is theoretically a robust control strategy for nonlinear systems [7], it is not commonly used in practice, due to the complexity and high cost in implementation.

Trajectory tracking problems are frequently encountered in mobile robot fields [8,9]. In many applications, they are subjected to some additional requirements, such as to track the target within a shorter time, to consume less energy, or to ensure small error sum [10]. Thus, optimal control is a natural choice in solving such tracking problems, with a performance criterion, containing energy consumption and the sum of error of the system model, to be minimized. The weight matrices of the quadratic performance criterion can be conveniently adjusted to meet the requirements of the motion task. Taking the WIP as an example, the tilt angle of the pendulum can be forced small enough in the whole motion process by choosing a large enough weight of the tilt angle error.

Input delay exists commonly in controlled systems from different sources, such as transportation and communication lags, feedback delays, computation delays, measure delays, and especially the inevitable delay induced by the digital filters [11]. The main negative impact of input delay is that the feedback state uses the delayed state. The delayed feedback controllers can be designed by using Lyapunov–Krasovskii functional and linear matrix inequality (LMI) [12,13], which usually yield conservative results. One can also design the controller in the form of predictor-based feedbacks. In this way, an integral transformation [14,15] is used to transform the input delay system into a delay-free one. To overcome the difficulties in implementing the predictor state feedback laws, the truncated predictor feedback (TPF) method is proposed and described in detail in [16]. One disadvantage of this method is its slow rate of convergence. In addition, some numerical algorithms [17,18] can be used to implement the predictor feedback law by calculating its unfeasible integral term. In a previous work [14], we proved that with the quadratic performance criterion in quantity same as that of the delay-free one, the input delay does not affect the optimal control quantity of linear systems, but postpones the action of the optimal control. This result is also correct for optimal trajectory tracking control and has been proved in the "Appendix".

This paper aims at designing controllers to implement two kinds of motions of a delayed WIP robot by using optimal trajectory tracking control theory and predictor-based feedback method. The model description of the WIP is given in Sect. 2, the controllers for the two motions are designed in Sects. 3 and 4, respectively, and finally, some concluding remarks are made in Sect. 5.

2 Dynamics Equation of the WIP

Figure 1 shows the model of the WIP, where the pendulum is anchored to a base chassis with two wheels mounted on each side. It is composed of two parts: two wheels and the intermediate body. The intermediate body is the center portion standing between the left and right wheels, and it consists of the rod of pendulum and the chassis. The parameters and variables of the WIP are described in Table 1.

Let $\mathbf{q} = (X_0, Y_0, \theta, \varphi, \theta_\mathrm{r}, \theta_\mathrm{l})$ be the generalized coordinates of the WIP. If the wheels run under the conditions of pure rolling and nonslipping, then the mobile chassis is subject to the following nonholonomic constraints

Fig. 1 Schematic diagram of the WIP

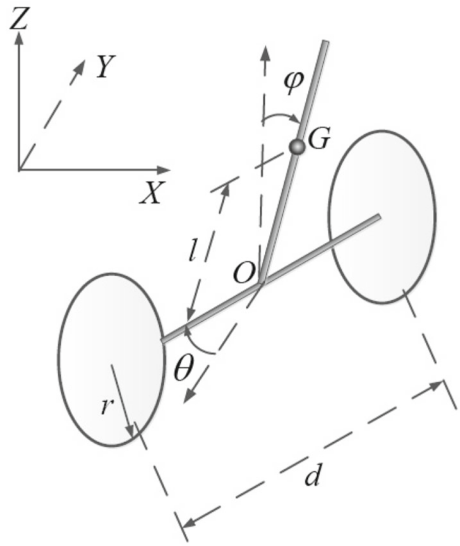

Table 1 The parameters and variables of the WIP

Notation	Definition
T_l, T_r	Torques provided by wheel actuators acting on left and right wheels
θ_l, θ_r	Rotation angels of the left and right wheels
X_0, Y_0	The coordinates of point O in $X-Y$ plane
φ	Tilt angle of the pendulum
θ	Heading angle of the WIP
x	Displacement of the WIP in $X-Y$ plane
M_w	Mass of the wheel
r	Radius of the wheel
M	Mass of the intermediate body
g	Gravity acceleration
l	Distance from the point O to the center of gravity of the intermediate body
H	Height of the intermediate body
d	Distance between the two wheels along the axle center
I_w	Moment of inertia of the wheel along the wheel axis direction
I_wd	Moment of inertia of the wheel about the Z-axis through the center of wheel
I_B	Moment of inertia of the intermediate body along the wheel axis direction
I_Z	Moment of inertia of the intermediate body about the Z-axis through point O
v	the forward velocity of the WIP, and $\dot{x}=v$

$$
\begin{aligned}
&-\dot{X}_0\sin\theta+\dot{Y}_0\cos\theta=0,\\
&\dot{X}_0\cos\theta+\dot{Y}_0\sin\theta+d\frac{\dot{\theta}}{2}-r\dot{\theta}_\mathrm{r}=0,\\
&\dot{X}_0\cos\theta+\dot{Y}_0\sin\theta-d\frac{\dot{\theta}}{2}-r\dot{\theta}_\mathrm{l}=0.
\end{aligned}
$$

Let $\boldsymbol{v}=(\dot{\varphi},v,\dot{\theta})$, and

$$
\mathbf{F}(\mathbf{q})=\begin{bmatrix}-\sin\theta & \cos\theta & 0 & 0 & 0 & 0\\ \cos\theta & \sin\theta & \frac{d}{2} & 0 & -r & 0\\ \cos\theta & \sin\theta & -\frac{d}{2} & 0 & 0 & -r\end{bmatrix},
$$

$$
\mathbf{S}(\mathbf{q})=\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\\ \cos\theta & \sin\theta & 0 & 0 & \frac{1}{r} & \frac{1}{r}\\ 0 & 0 & 1 & 0 & \frac{d}{2r} & -\frac{d}{2r}\end{bmatrix}^\mathrm{T}.
$$

The constraints can be rewritten as $\mathbf{F}(\mathbf{q})_{3\times 6}\dot{\mathbf{q}}=\mathbf{0}_{3\times 1}$, and $\dot{\mathbf{q}}$ stays in the null-space of $\mathbf{F}(\mathbf{q})$, represented by

$$
\dot{\mathbf{q}}=\mathbf{S}(\mathbf{q})\boldsymbol{v}. \tag{1}
$$

The sum of the translational kinetic and rotational kinetic energy of the two wheels is given by

$$T_{\mathrm{w}} = \frac{1}{2}M_{\mathrm{w}}r^2\dot{\theta}_{\mathrm{l}}^2 + \frac{1}{2}M_{\mathrm{w}}r^2\dot{\theta}_{\mathrm{r}}^2 + \frac{1}{2}I_{\mathrm{w}}\dot{\theta}_{\mathrm{r}}^2 + \frac{1}{2}I_{\mathrm{w}}\dot{\theta}_{\mathrm{l}}^2 + I_{\mathrm{wd}}\dot{\theta}^2.$$

The translational kinetic energy of the intermediate body is computed as

$$T_{\mathrm{B}}^{\mathrm{T}} = \frac{1}{2}M(\dot{\theta}l\sin\varphi)^2 + \frac{1}{2}M\left[\dot{\varphi}l\cos\varphi + \frac{1}{2}r(\dot{\theta}_l + \dot{\theta}_r)\right]^2 + \frac{1}{2}M(-\dot{\varphi}l\sin\varphi)^2.$$

The rotational kinetic energy of the intermediate body is given by

$$T_{\mathrm{B}}^{\mathrm{R}} = \frac{1}{2}I_{\mathrm{B}}\dot{\varphi}^2 + \frac{1}{2}I_{\mathrm{Z}}\dot{\theta}^2.$$

Here, the moment of inertia I_{Z} depends on the angular position φ of the pendulum, but it is usually assumed to be constant when the angular position φ is small. Moreover, the gravitational potential energy of the system is

$$P = Mgl\cos\varphi.$$

Then, the Lagrangian function is

$$L = T_{\mathrm{w}} + T_{\mathrm{B}}^{\mathrm{T}} + T_{\mathrm{B}}^{\mathrm{R}} - Mgl\cos\varphi.$$

Using the Euler–Lagrange equations, the dynamic equation of the WIP reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{E}(\mathbf{q})\mathbf{T} + \mathbf{F}^{\mathrm{T}}(\mathbf{q})\boldsymbol{\lambda}, \tag{2}$$

where $\mathbf{T}$ is the input motor-torque vector and $\mathbf{E}(\mathbf{q})$ is the matched matrix, given by

$$\mathbf{T} = \begin{bmatrix} T_{\mathrm{r}} \\ T_{\mathrm{l}} \end{bmatrix}, \qquad \mathbf{E}(\mathbf{q}) = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}^{\mathrm{T}},$$

respectively, and the vector $\boldsymbol{\lambda}$ is the so-called Lagrange multiplier. By calculating (2) and collecting the first-order items and second-order items of the state variable, the dynamic equation becomes

$$\mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{E}(\mathbf{q})\mathbf{T} + \mathbf{F}^{\mathrm{T}}(\mathbf{q})\boldsymbol{\lambda}, \tag{3}$$

where

$$\mathbf{A}(\mathbf{q}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{bmatrix}, \quad \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 \\ 0 \\ v_{31} \\ v_{41} \\ v_{51} \\ v_{61} \end{bmatrix},$$

and $a_{33} = Ml^2\sin^2\varphi + 2I_{\mathrm{wd}} + I_{\mathrm{Z}}, a_{44} = Ml^2 + I_{\mathrm{B}}, a_{45} = a_{54} = a_{46} = a_{64} = \frac{1}{2}Mrl\cos\varphi, a_{55} = a_{66} = \frac{1}{4}Mr^2 + I_{\mathrm{w}} + M_{\mathrm{w}}r^2, a_{56} = a_{65} = \frac{1}{4}Mr^2, v_{31} = Ml^2\dot{\theta}\dot{\varphi}\sin 2\varphi$, $v_{41} = -Ml\sin\varphi(\dot{\theta}^2 l\cos\varphi + g)$, $v_{51} = v_{61} = -\frac{1}{2}Mrl\dot{\varphi}^2\sin\varphi$.

By differentiating at the both sides of (1) with respect to t and substituting it into (3), we have

$$\mathbf{A}\mathbf{S}\dot{\boldsymbol{v}} + \mathbf{A}\dot{\mathbf{S}}\boldsymbol{v} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{E}(\mathbf{q})\mathbf{T} + \mathbf{F}^{\mathrm{T}}(\mathbf{q})\boldsymbol{\lambda}. \tag{4}$$

Multiplying with $\mathbf{S}^{\mathrm{T}}(\mathbf{q})$ to both sides of (4) from the left, together with $\mathbf{S}^{\mathrm{T}}\mathbf{F} = 0$, we find

$$\mathbf{S}^{\mathrm{T}}\mathbf{A}\mathbf{S}\dot{\boldsymbol{v}} + \mathbf{S}^{\mathrm{T}}(\mathbf{A}\dot{\mathbf{S}}\boldsymbol{v} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}})) = \mathbf{S}^{\mathrm{T}}\mathbf{E}(\mathbf{q})\mathbf{T}. \tag{5}$$

Using $\mathbf{S}^{\mathrm{T}}\mathbf{A}\dot{\mathbf{S}} = \mathbf{0}_{3\times 3}$, obtained by detailed calculation, (5) can be simplified to

$$\dot{\boldsymbol{v}} = -\bar{\mathbf{A}}^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) + \bar{\mathbf{A}}^{-1}\mathbf{S}^{\mathrm{T}}\mathbf{E}(\mathbf{q})\mathbf{T}, \tag{6}$$

where $\bar{\mathbf{A}} = \mathbf{S}^{\mathrm{T}}\mathbf{A}(\mathbf{q})\mathbf{S}$. Therefore, the dynamic equation (6) can be calculated and rewritten in the following form

$$\left.\begin{aligned} \ddot{\varphi} &= \frac{-Ml\left(\dot{\theta}^2 l\cos\varphi + g\right)\left[Mr^2\sin\varphi + 2I_{\mathrm{w}}\sin\varphi + 2M_{\mathrm{w}}r^2\sin\varphi\right] + M^2l^2r^2\dot{\varphi}^2\sin\varphi\cos\varphi}{\Delta + M^2l^2r^2\cos^2\varphi} \\ &\quad + \frac{Mr^2 + 2I_{\mathrm{w}} + 2M_{\mathrm{w}}r^2 + Mlr\cos\varphi}{\Delta + M^2l^2r^2\cos^2\varphi}u_2, \\ \dot{v} &= \frac{Mlr^2\left[Ml\sin\varphi\cos\varphi(\dot{\theta}^2 l\cos\varphi + g) - \dot{\varphi}^2\sin\varphi Ml^2 - \dot{\varphi}^2\sin\varphi I_{\mathrm{B}}\right]}{\Delta + M^2l^2r^2\cos^2\varphi} \\ &\quad - \frac{Mlr^2\cos\varphi + Mrl^2 + I_{\mathrm{B}}r}{\Delta + M^2l^2r^2\cos^2\varphi}u_2, \\ \ddot{\theta} &= \frac{2Mr^2l^2\dot{\theta}\dot{\varphi}\sin 2\varphi}{\Omega + 2Ml^2r^2\cos^2\varphi} + \frac{rd}{\Omega + 2Ml^2r^2\cos^2\varphi}u_1, \end{aligned}\right\} \tag{7}$$

where

$$\begin{aligned} \Delta &= -M^2l^2r^2 - 2Ml^2I_{\mathrm{w}} - 2Ml^2M_{\mathrm{w}}r^2 - I_{\mathrm{B}}Mr^2 - 2I_{\mathrm{B}}I_{\mathrm{w}} - 2M_{\mathrm{w}}r^2I_{\mathrm{B}}, \\ \Omega &= -M_{\mathrm{w}}d^2r^2 - 4I_{\mathrm{wd}}r^2 - 2I_{\mathrm{Z}}r^2 - d^2I_{\mathrm{w}} - 2Ml^2r^2, \; u_1 = T_{\mathrm{l}} - T_{\mathrm{r}}, u_2 = T_{\mathrm{l}} + T_{\mathrm{r}}. \end{aligned}$$

In this paper, the Back-and-Forth motion to fetch object and the Lower-Raise-Head motion to avoid obstacle of a WIP robot under a delayed input will be investigated.

The two kinds of motion are all planar motion, which means that the turning motion of the WIP will not be taken into account in this paper. Then, $\theta = 0$, and $u_1 = T_\mathrm{l} - T_\mathrm{r} = 0$, $X_0 = x$, $Y_0 = 0$. Equation (7) can be simplified to

$$\left.\begin{aligned} \ddot{\varphi} &= \frac{-Mlg\left[Mr^2\sin\varphi + 2I_\mathrm{w}\sin\varphi + 2M_\mathrm{w}r^2\sin\varphi\right] + M^2l^2r^2\dot{\varphi}^2\sin\varphi\cos\varphi}{\Delta + M^2l^2r^2\cos^2\varphi} \\ &\quad + \frac{Mr^2 + 2I_\mathrm{w} + 2M_\mathrm{w}r^2 + Mlr\cos\varphi}{\Delta + M^2l^2r^2\cos^2\varphi}u_2, \\ \ddot{x} &= \frac{Mlr^2\left[Ml\sin\varphi\cos\varphi g - \dot{\varphi}^2\sin\varphi Ml^2 - \dot{\varphi}^2\sin\varphi I_\mathrm{B}\right]}{\Delta + M^2l^2r^2\cos^2\varphi} \\ &\quad - \frac{Mlr^2\cos\varphi + Mrl^2 + I_\mathrm{B}r}{\Delta + M^2l^2r^2\cos^2\varphi}u_2. \end{aligned}\right\} \tag{8}$$

3 Control of the Back-and-Forth Motion

The control task for the Back-and-Forth motion to fetch object is shown in Fig. 2. While the WIP goes ahead, fetches the object, and returns back to the starting point, the inverted pendulum is kept stabilized. The key idea of our method is to introduce a quadratic performance criterion with a large weight of tilt angle error to force the tilt angle of the pendulum small enough. Thus, the control task can be solved using a linearized model, and the controller can be designed as an application of the optimal trajectory tracking control theory for linear systems with an input delay. Moreover, the input delay can be well handled by using predictor-based feedback method through the linearized model.

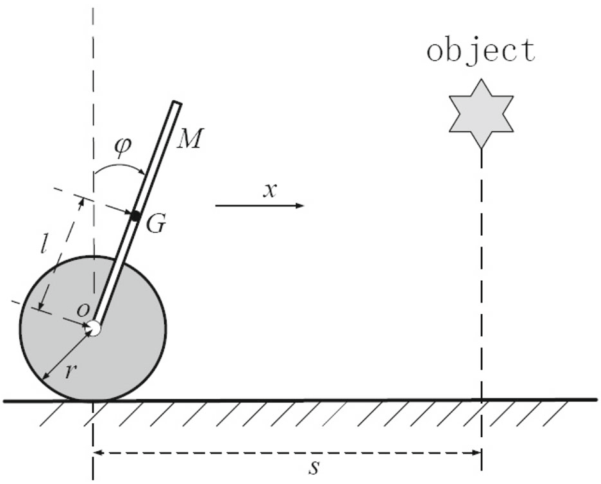

Fig. 2 The Back-and-Forth motion of the WIP

The linearized equation of system (8) with an input delay is given by

$$\left.\begin{aligned}\ddot{\varphi} &= \frac{-Mlg\left(Mr^2+2I_{\mathrm{w}}+2M_{\mathrm{w}}r^2\right)}{\varDelta+M^2l^2r^2}\varphi+\frac{Mr^2+2I_{\mathrm{w}}+2M_{\mathrm{w}}r^2+Mlr}{\varDelta+M^2l^2r^2}u_2(t-\tau)\\ \ddot{x} &= \frac{M^2l^2r^2g}{\varDelta+M^2l^2r^2}\varphi-\frac{Mlr^2+Mrl^2+I_{\mathrm{B}}r}{\varDelta+M^2l^2r^2}u_2(t-\tau),\end{aligned}\right\} \tag{9}$$

where τ is the input delay. The control takes place only when $t \geq \tau$. This fact should be taken into account when one calculates the quadratic performance criterion. Let $\mathbf{X} = [x_1, x_2, x_3, x_4]^{\mathrm{T}} = [\varphi, \dot{\varphi}, x, \dot{x}]^{\mathrm{T}}$, and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-Mlg\left(Mr^2+2I_{\mathrm{w}}+2M_{\mathrm{w}}r^2\right)}{\varDelta+M^2l^2r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{M^2l^2r^2g}{\varDelta+M^2l^2r^2} & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{Mr^2+2I_{\mathrm{w}}+2M_{\mathrm{w}}r^2+Mlr}{\varDelta+M^2l^2r^2} \\ 0 \\ -\frac{Mlr^2+Mrl^2+I_{\mathrm{B}}r}{\varDelta+M^2l^2r^2} \end{bmatrix},$$

then, the corresponding state equation of (9) is written as

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}u_2(t-\tau).$$

3.1 Controller Design

For implementing the control task, the trajectory tracking target can be chosen as $[\bar{\varphi}(t), \bar{x}(t)]^{\mathrm{T}} = [0, (at-t^2)\mathrm{e}^{-\alpha t}]^{\mathrm{T}}$, where the parameters a and α are determined by the distance s and the weight matrices of the performance criterion. Hence, the trajectory tracking target vector is given by

$$\bar{\mathbf{X}} = [\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4]^{\mathrm{T}} = \left[\bar{\varphi}(t), \dot{\bar{\varphi}}(t), \bar{x}(t), \dot{\bar{x}}(t)\right]^{\mathrm{T}} = \Big[0, 0, (at-t^2)\mathrm{e}^{-\alpha t}, (a-2t)$$
$$\mathrm{e}^{-\alpha t} - \alpha(at-t^2)\mathrm{e}^{-\alpha t}\Big]^{\mathrm{T}}.$$

Let $\mathbf{Y}(t) = \mathbf{X}(t) - \bar{\mathbf{X}}(t)$, $\boldsymbol{\omega}(t) = \mathbf{A}\bar{\mathbf{X}} - \dot{\bar{\mathbf{X}}}$, then the corresponding system governing the tracking error takes the form

$$\dot{\mathbf{Y}}(t) = \mathbf{A}\mathbf{Y}(t) + \mathbf{B}u_2(t-\tau) + \boldsymbol{\omega}(t). \tag{10}$$

A controller will be designed to minimize the quadratic performance criterion

$$J = \frac{1}{2}\mathbf{Y}^{\mathrm{T}}(t_f)\mathbf{Q}_0\mathbf{Y}(t_f) + \frac{1}{2}\int_0^{t_f}\left[\mathbf{Y}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{Y}(t) + u_2^{\mathrm{T}}(t-\tau)\mathbf{R}u_2(t-\tau)\right]\mathrm{d}t, \tag{11}$$

where $\mathbf{Q}_0, \mathbf{Q}$ are nonnegative definite symmetric matrices, $\mathbf{R}$ is a positive definite symmetric matrix, and t_f $(>2\tau)$ is the terminal time of the control. As shown in the "Appendix", the input delay does not change the optimal control quantity, but postpones the action of the optimal control. Thus, the delayed optimal controller of system (10) that minimizes (11) is given by

$$u_2^*(t-\tau) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}[\mathbf{P}_0(t)\mathbf{Y}(t) + \mathbf{b}_0(t)], \tag{12}$$

where $\mathbf{P}_0(t) \in \mathbb{R}^{n\times n}$, $\mathbf{b}_0(t) \in \mathbb{R}^n$ are the solutions of the Riccati differential equations

$$\begin{aligned}
\dot{\mathbf{P}}_0 &= -\mathbf{P}_0\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{P}_0 + \mathbf{P}_0\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0 - \mathbf{Q}, \quad \mathbf{P}_0(t_f) = \mathbf{Q}_0,\\
\dot{\mathbf{b}}_0 &= -[\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0]^{\mathrm{T}}\mathbf{b}_0 - \mathbf{P}_0\boldsymbol{\omega}, \quad \mathbf{b}_0(t_f) = 0.
\end{aligned}$$

The delayed optimal trajectory tracking controller (12) consists of two parts: The first part $-\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0(t)\mathbf{Y}(t)$ is a state feedback that is determined by the current state, and the other part $-\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{b}_0(t)$ is a feedforward that is determined by the tracking target. However, due to the input delay, only the delayed state $\mathbf{Y}(t-\tau)$ rather than the current state $\mathbf{Y}(t)$ is available in implementation. Hence, it is necessary to use a predictor state to replace the current state, so the delayed optimal trajectory tracking controller can be rewritten in the conventional way

$$u_2^*(t-\tau) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}[\mathbf{P}_0(t)\bar{\mathbf{Y}}(t) + \mathbf{b}_0(t)], \tag{13}$$

where $\bar{\mathbf{Y}}(t)$ is the predictor state of $\mathbf{Y}(t)$, given by

$$\bar{\mathbf{Y}}(t) = \mathrm{e}^{\mathbf{A}\tau}\mathbf{Y}(t-\tau) + \int_{t-\tau}^{t} \mathrm{e}^{\mathbf{A}(t-s)}[\mathbf{B}u_2^*(s-\tau) + \boldsymbol{\omega}(s)]\mathrm{d}s. \tag{14}$$

The control implementation can be carried out numerically, as done in [18]. If the order of the system is relatively high, a common case in practice, then one can choose the special case $t_f \to +\infty$ to design the optimal trajectory tracking controller, simply by solving algebraic Riccati equations.

3.2 Simulation Results

For simplicity, we consider the case of $t_f = +\infty$, with fixed parameter values and initial values: $M = 8\,\mathrm{kg}$, $M_{\mathrm{w}} = 2\,\mathrm{kg}$, $l = 0.5\,\mathrm{m}$, $r = 0.25\,\mathrm{m}$, $g = 10\,\mathrm{m/s}^2$, $\tau = 0.1\,\mathrm{s}$, $s = 2.6\,\mathrm{m}$, $I_{\mathrm{B}} = 4\,\mathrm{kg\,m}^2$, $I_{\mathrm{w}} = \frac{1}{16}\,\mathrm{kg\,m}^2$, $\mathbf{R} = 1$, $\mathbf{Q}_0 = 0, \mathbf{Q} = \mathrm{diag}(10000, 0, 5, 0)$, $\varphi(0) = 0$, $\dot{\varphi}(0) = 0$, $x(0) = 0$, $\dot{x}(0) = 0$. Then, the matrices $\mathbf{A}$ and $\mathbf{B}$ in (10) become

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{140}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{40}{17} & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -\frac{15}{34} \\ 0 \\ \frac{7}{17} \end{bmatrix}.$$

Under this parameter combination, the values of a and α in the trajectory tracking target can be chosen carefully to be $a = 20$, $\alpha = 0.5$, in order to meet the requirements in

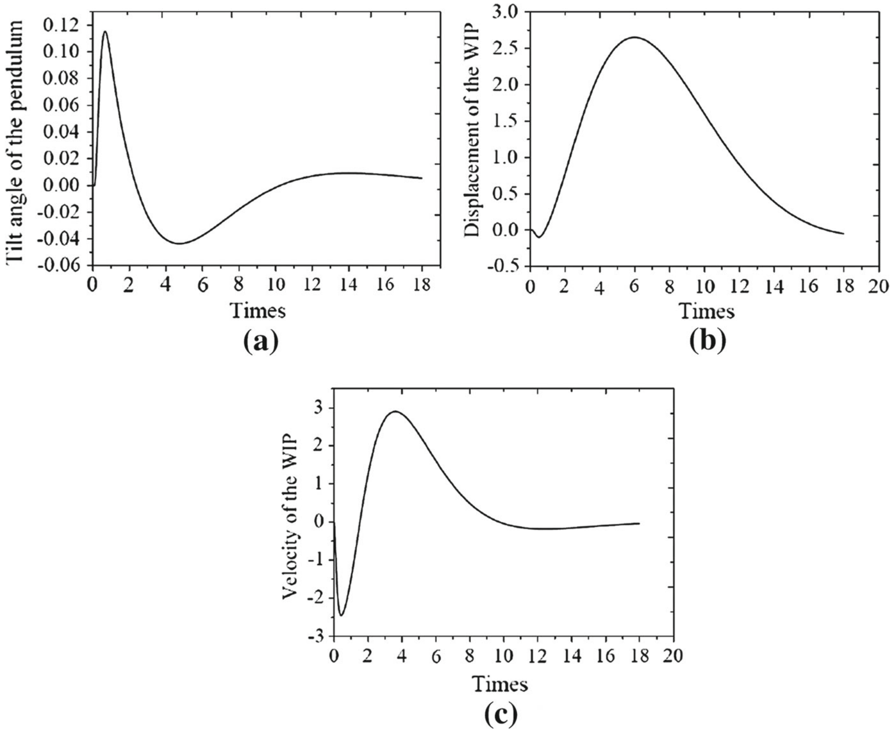

Fig. 3 The time histories of the WIP during the Back-and-Fourth motion process. **a** Tilt angle of the pendulum, **b** the displacement of the WIP, **c** the velocity of the WIP

realization of the Back-and-Forth motion. The MATLAB command `lqr` returns the solutions of (40) and (41) as follows:

$$\mathbf{P}_0 = \begin{bmatrix} 2778.3 & 566.43 & 75.261 & 280.99 \\ 566.43 & 270.53 & 50.274 & 208.12 \\ 75.261 & 50.274 & 22.008 & 48.435 \\ 280.99 & 208.12 & 48.435 & 199.08 \end{bmatrix},$$

$$\mathbf{b}_0(t) = \begin{bmatrix} (80.340t^2 - 2134.4t + 4996.8)\mathrm{e}^{-0.5t} \\ (63.476t^2 - 1638.4t + 3771.5)\mathrm{e}^{-0.5t} \\ (10.578t^2 - 284.25t + 758.47)\mathrm{e}^{-0.5t} \\ (60.605t^2 - 1564.6t + 3604.3)\mathrm{e}^{-0.5t} \end{bmatrix}.$$

Thus, all the quantities required in the delayed optimal trajectory tracking controller (13) are available in hand.

Figure 3a shows that the tilt angle of the pendulum becomes small enough and is stabilized during the Back-and-Forth motion. Figure 3b shows that the WIP moves forward 2.6 m to fetch the object and returns back to the starting point within 18 seconds, the speed is low, and Fig. 3c shows that this motion process consists of four stages: reversed accelerating $\longrightarrow$ forward accelerating $\longrightarrow$ reversed accelerating $\longrightarrow$

forward accelerating. This is in full agreement with the actual situation of the motion control.

The weight matrix **Q** plays an important role in the implementation of the Back-and-Forth motion and must be carefully chosen. Actually, it restrains the motion range of the rod of pendulum and makes the WIP to track the reference displacement as far as possible. If the weight of the tilt angle error is increased, the maximum value of the tilt angle of the pendulum will be decreased, and a better performance of the whole motion is obtained, but the maximal displacement of the WIP will be decreased. In order to fetch the object, the parameter value a should be increased. If one increases the weight of the WIP displacement error only, the motion range of the tilt angle will be increased, and this may lead to system collapse under the optimal trajectory tracking controller $u_2^*(t-\tau)$.

This proposed controller has a simple structure and can be easily implemented. It has some merits over the available methods. For example, the controller design of the Back-and-Forth motion to fetch object can be carried out also by using conventional tracking control method or by using adaptive control method. In the application of the conventional tracking control method, linear control strategies do not work because the tilt angle of the pendulum cannot be kept small while the WIP moves forward without additional control measurement. In addition, the use of adaptive control method requires timely and constantly distance signals of the robot and object, which results in complexity and high cost. One disadvantage of the proposed control method is the low speed in carrying out the control task.

4 Control of the Lower-Raise-Head Motion

As shown in Fig. 4, the control task for the Lower-Raise-Head Motion to avoid obstacle is to pass through an obstacle like "gates" that are lower than the WIP, via firstly lowering the "head", and then raising the "head", of the pendulum. In the whole motion

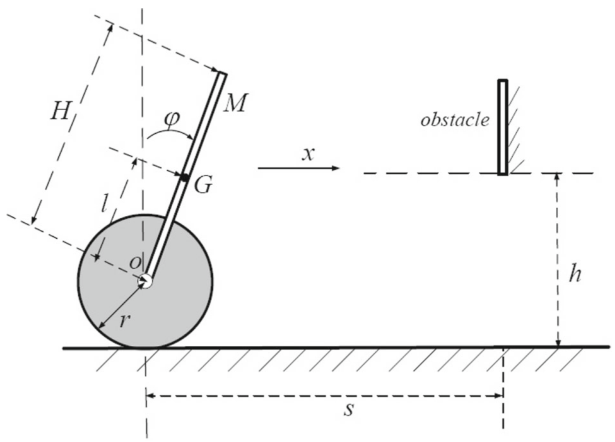

Fig. 4 The Lower-Raise-Head motion of the WIP

process, the tilt angle of the pendulum should be large enough in a time period, so that the pendulum can pass through the "gate". In this paper, the yaw movement is not taken into consideration, then, the WIP system is decoupled, the subsystem governing the state of the tilt angle is transformed into a simple linear system by using feedback linearization. Thus, the input delay can be dealt with by using the predictor-based feedback method.

Let $[x_1, x_2, x_3, x_4]^{\mathrm{T}} = [\varphi, \dot{\varphi}, x, \dot{x}]^{\mathrm{T}}$; then, Eq. (8) with an input delay can be rewritten in state equation

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= \frac{-Mlg\left[Mr^2\sin x_1 + 2I_{\mathrm{w}}\sin x_1 + 2M_{\mathrm{w}}r^2\sin x_1\right] + M^2l^2r^2x_2^2\sin x_1\cos x_1}{\Delta + M^2l^2r^2\cos^2 x_1}\\
&\quad + \frac{Mr^2 + 2I_{\mathrm{w}} + 2M_{\mathrm{w}}r^2 + Mlr\cos x_1}{\Delta + M^2l^2r^2\cos^2 x_1}u_2(t-\tau),\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= \frac{Mlr^2\left[Ml\sin x_1\cos x_1 g - x_2^2\sin x_1 Ml^2 - x_2^2\sin x_1 I_{\mathrm{B}}\right]}{\Delta + M^2l^2r^2\cos^2 x_1}\\
&\quad - \frac{Mlr^2\cos x_1 + Mrl^2 + I_{\mathrm{B}}r}{\Delta + M^2l^2r^2\cos^2 x_1}u_2(t-\tau),
\end{aligned}$$

or in the form composed of two subsystems as follows:

$$\left.\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= \tfrac{-Mlg\left[Mr^2\sin x_1+2I_{\mathrm{w}}\sin x_1+2M_{\mathrm{w}}r^2\sin x_1\right]+M^2l^2r^2x_2^2\sin x_1\cos x_1}{\Delta+M^2l^2r^2\cos^2 x_1}\\
&\quad + \tfrac{Mr^2+2I_{\mathrm{w}}+2M_{\mathrm{w}}r^2+Mlr\cos x_1}{\Delta+M^2l^2r^2\cos^2 x_1}u_2(t-\tau),
\end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned}
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= \tfrac{Mlr^2\left[Ml\sin x_1\cos x_1 g-x_2^2\sin x_1 Ml^2-x_2^2\sin x_1 I_{\mathrm{B}}\right]}{\Delta+M^2l^2r^2\cos^2 x_1}\\
&\quad - \tfrac{Mlr^2\cos x_1+Mrl^2+I_{\mathrm{B}}r}{\Delta+M^2l^2r^2\cos^2 x_1}u_2(t-\tau).
\end{aligned}\right\} \tag{16}$$

Note that the subsystem (15) is unrelated to x_3, x_4 and the subsystem (16) is determined by the state and controller of subsystem (15). Therefore, we can design optimal trajectory tracking controller to implement Lower-Raise-Head motion of the pendulum by using subsystem (15) and to implement forward motion of the WIP through subsystem (16) by adjusting the parameters of the trajectory tracking target.

4.1 Controller Design

For the controller design of the Lower-Raise-Head motion by using the feedback linearization approach [19], let

$$
\begin{aligned}
E &= \frac{Mr^2 + 2I_{\mathrm{w}} + 2M_{\mathrm{w}}r^2 + Mlr\cos x_1}{\Delta + M^2l^2r^2\cos^2 x_1},\\
F &= \frac{-Mlg[Mr^2\sin x_1 + 2I_{\mathrm{w}}\sin x_1 + 2M_{\mathrm{w}}r^2\sin x_1] + M^2l^2r^2x_2^2\sin x_1\cos x_1}{\Delta + M^2l^2r^2\cos^2 x_1},\\
u_2(t-\tau) &= \frac{1}{E}(-F + \nu(t-\tau)).
\end{aligned}
$$

Then, subsystem (15) becomes

$$
\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}\nu(t-\tau), \tag{17}
$$

where $\mathbf{X} = [x_1, x_2]^{\mathrm{T}}$, and

$$
\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
$$

In passing through the obstacle, the tilt angle of the pendulum should be greater than the critical angle value in a certain range, where the critical tilt angle is $\varphi_0 = \pm\arccos\frac{h-r}{H}$. In addition, let the trajectory tracking target vector and the quadratic performance criterion for subsystem (17) be chosen as follows:

$$
\begin{aligned}
\bar{\mathbf{X}}(t) &= \left[k(\alpha t - t^2)\mathrm{e}^{-\beta t}, k(\alpha - 2t)\mathrm{e}^{-\beta t} - k\beta(\alpha t - t^2)\mathrm{e}^{-\beta t}\right]^{\mathrm{T}},\\
J &= \frac{1}{2}\mathbf{Y}^{\mathrm{T}}(t_f)\mathbf{Q}_0\mathbf{Y}(t_f) + \frac{1}{2}\int_0^{t_f}\left[\mathbf{Y}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{Y}(t) + \nu^{\mathrm{T}}(t-\tau)\mathbf{R}\nu(t-\tau)\right]\mathrm{d}t,
\end{aligned}
$$

where $\mathbf{Y}(t) = \mathbf{X}(t) - \bar{\mathbf{X}}(t)$ is the tracking error, and k, α, β are parameters to be determined by $\varphi_0, s, \dot{x}(0) = x_4(0)$ and weight matrices $\mathbf{Q}, \mathbf{R}$. Therefore, the original Lower-Raise-Head motion control problem is converted into an optimal trajectory tracking control problem of a linear system with an input delay.

Moreover, let $\boldsymbol{\omega}(t) = \mathbf{A}\bar{\mathbf{X}} - \dot{\bar{\mathbf{X}}}$, then, the corresponding system governing the tracking error takes the form

$$
\dot{\mathbf{Y}}(t) = \mathbf{A}\mathbf{Y}(t) + \mathbf{B}\nu(t-\tau) + \boldsymbol{\omega}(t),
$$

According to the "Appendix", the delayed optimal controller for implementing the Lower-Raise-Head motion control task can be designed as follows:

$$\begin{aligned} u_2(t-\tau) &= \frac{1}{E}\left[-F + \nu^*(t-\tau)\right] \\ &= \frac{1}{E}\left[-F - \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0(\mathbf{X}(t) - \bar{\mathbf{X}}(t)) - \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{b}_0(t)\right], \end{aligned} \tag{18}$$

where $\mathbf{P}_0(t) \in \mathbb{R}^{n\times n}$ and $\mathbf{b}_0(t) \in \mathbb{R}^n$ are the solutions of (26) and (27), respectively.

Note that E, F are determined by the current state $\mathbf{X}(t)$, and the key information of the controller (18) is also the current state $\mathbf{X}(t)$. However, due to the same reason in Sect. 3, we can only receive the delayed state $\mathbf{X}(t-\tau)$ rather than the current state $\mathbf{X}(t)$. Thus, the delayed controller can be designed using a predictor state

$$\begin{aligned} u_2(t-\tau) &= \frac{1}{E}\left[-F + \nu^*(t-\tau)\right] \\ &= \frac{1}{E}\left[-F - \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0(\tilde{\mathbf{X}}(t) - \bar{\mathbf{X}}(t)) - \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{b}_0(t)\right], \end{aligned} \tag{19}$$

where $\tilde{\mathbf{X}}(t) = [\tilde{x}_1, \tilde{x}_2]^{\mathrm{T}}$ is the predictor state, denoted by

$$\begin{aligned} \tilde{\mathbf{X}}(t) :=& \bar{\mathbf{X}}(t) + \mathrm{e}^{\mathbf{A}\tau}\left(\mathbf{X}(t-\tau) - \bar{\mathbf{X}}(t-\tau)\right) \\ &- \int_{t-\tau}^{t} \mathrm{e}^{\mathbf{A}(t-s)}\left[\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0(\tilde{\mathbf{X}}(s) - \bar{\mathbf{X}}(s))\right]\mathrm{d}s \\ &+ \int_{t-\tau}^{t} \mathrm{e}^{\mathbf{A}(t-s)}\left[\boldsymbol{\omega}(s) - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{b}_0(s)\right]\mathrm{d}s. \end{aligned}$$

Hence, the Lower-Raise-Head of the pendulum and the forward motion of the chassis can be implemented by designing an optimal trajectory tracking controller for the second-order linear state equation with an appropriate trajectory tracking target. The parameters of the trajectory tracking target and the weights of the quadratic performance criterion should be chosen carefully to meet the control task.

4.2 Simulation Results

For simplicity, the parameter values in Fig. 4 and initial values are given by $M = 8\,\mathrm{kg}$, $M_{\mathrm{w}} = 2\,\mathrm{kg}$, $l = 0.5\,\mathrm{m}$, $H = 1.5\,\mathrm{m}$, $r = 0.25\,\mathrm{m}$, $g = 10\,\mathrm{m/s^2}$, $\tau = 0.1\,\mathrm{s}$, $s = 5\,\mathrm{m}$, $h = 1.58\,\mathrm{m}$, $I_{\mathrm{B}} = 4\,\mathrm{kg\,m^2}$, $I_{\mathrm{w}} = \frac{1}{16}\,\mathrm{kg\,m^2}$, $\varphi(0) = 0$, $\dot{\varphi}(0) = 0$, $x(0) = 0$, $\dot{x}(0) = 0$. For convenient demonstration, the weight matrices of J are given by $\mathbf{Q}_0 = 0, \mathbf{Q} = \mathrm{diag}(1, 0), \mathbf{R} = 1$, and the terminal time is selected as $t_f = +\infty$.

Note that the critical tilt angle $\varphi_0 = \pm\arccos\frac{h-r}{H} = \pm 0.4828$; then, the unknown parameters in the trajectory tracking target vector can be chosen as $k = 0.05, \alpha = 20, \beta = 0.5$, so as to keep the tilt angle of the pendulum greater than the critical

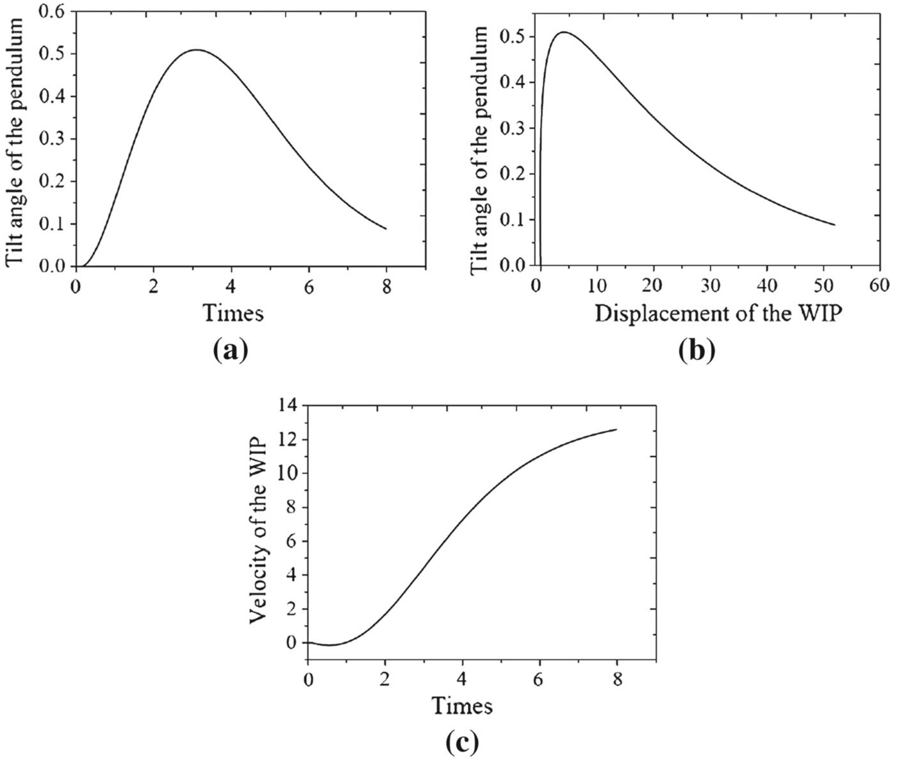

Fig. 5 The time histories of the WIP during the Lower-Raise-Head motion process. **a** The tilt angle of the pendulum, **b** the displacement of the WIP, **c** the velocity of the WIP

angle value in a range. With this parameter combination, the MATLAB command `lqr` returns the solutions of (40) and (41) as follows

$$\mathbf{P}_0 = \begin{bmatrix} 1.4142 & 1 \\ 1 & 1.4142 \end{bmatrix}, \quad \mathbf{b}_0(t) = \begin{bmatrix} \left(0.3403 - 0.0943t + 0.0032t^2\right) \mathrm{e}^{-0.5t} \\ \left(0.8356 - 0.2965t + 0.0109t^2\right) \mathrm{e}^{-0.5t} \end{bmatrix}.$$

Thus, all the quantities required in the delayed optimal trajectory tracking controller (19) have been determined.

In Fig. 5a, b, the tilt angle of the pendulum $\varphi > 0.4828$ when $2.52\,\mathrm{s} < t < 3.75\,\mathrm{s}$, and $\varphi > 0.4828$ when $1.85\,\mathrm{m} < x < 7.85\,\mathrm{m}$. Hence, the WIP can pass through the obstacle as demanded. Figure 5c shows the whole motion process when the WIP passes through the obstacle. This motion consists of two stages only: reversed accelerating $\longrightarrow$ forward accelerating. Again, this is in fully agreement with the actual situation.

The parameter combination of k, α, β plays an important role in the implementation of the Lower-Raise-Head motion. Firstly, it ensures that the tile angle of the pendulum would be increased to an angle value that is bigger than the critical tilt angle, so that the pendulum rod can pass through the "gate" within a time period. Secondly, this parameter combination should ensure that the WIP to move forward fast enough.

The choice of the parameter combination is not unique, and the way of finding such parameter combination is similar to that for the gain parameter combination of PID controllers.

5 Conclusions

The Back-and-Forth motion to fetch object and the Lower-Raise-Head motion to avoid obstacle are two basic functions of WIP robots. Due to the combination impact of input delay and nonlinearity, optimal controller design for such kind of nonlinear systems is a challenging problem. In this paper, in controlling the Back-and-Forth motion to fetch object, a large weight of tilt angle error in the quadratic performance criterion is chosen to force the tilt angle of the pendulum becoming small, so that linear quadratic optimal control theory can be applied. While in controlling the Lower-Raise-Head motion to avoid obstacle, feedback linearization is used to transform the nonlinear delayed system into a linear system with an input delay. In both cases, the delayed optimal controller is designed by using a predictor state, thus, the control can be easily implemented. Numerical simulation shows that the proposed design method works effectively.

A key observation of this paper is that the optimal trajectory tracking control method plays an unique role in implementing some control tasks of nonlinear systems: The angle signals involved can be kept small enough by minimizing the quadratic performance criterion with a large weight matrix, so that to have a simple linearized system in designing controller. It is expected to find other applications elsewhere.

Acknowledgments The authors thank the financial support of NSF of China under Grant 11372354 and Funding of Jiangsu Innovation Program for Graduate Education (CXLX13-129), and thank the anonymous reviewers for their valuable comments that help much in improving the presentation of the manuscript.

Appendix: Derivation of Equation (12)

Consider the optimal control problem of system (10) that minimizes the performance criterion (11). Firstly, let consider the special case of $\tau = 0$, the augmented quadratic performance criterion can be defined as

$$\begin{aligned}\bar{J} = &\frac{1}{2}\mathbf{Y}^{\mathrm{T}}(t_f)\mathbf{Q}_0\mathbf{Y}(t_f) + \frac{1}{2}\int_0^{t_f}\left[\mathbf{Y}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{Y}(t) + u_2^{\mathrm{T}}(t)\mathbf{R}u_2(t)\right]\mathrm{d}t\\ &+ \int_0^{t_f}\left[\boldsymbol{\lambda}^{\mathrm{T}}(t)(\mathbf{A}\mathbf{Y}(t) + \mathbf{B}u_2(t) + \boldsymbol{\omega}(t) - \dot{\mathbf{Y}}(t))\right]\mathrm{d}t,\end{aligned}$$

and the Hamiltonian function is defined by

$$H(\mathbf{Y}, u_2, \boldsymbol{\lambda}, t) = \frac{1}{2}\mathbf{Y}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{Y}(t) + \frac{1}{2}u_2^{\mathrm{T}}(t)\mathbf{R}u_2(t) + \boldsymbol{\lambda}^{\mathrm{T}}(t)(\mathbf{A}\mathbf{Y}(t) + \mathbf{B}u_2(t) + \boldsymbol{\omega}(t)). \tag{20}$$

Then, $\bar{J}$ can be rewritten in the form of

$$\bar{J} = \frac{1}{2}\mathbf{Y}^{\mathrm{T}}(t_f)\mathbf{Q}_0\mathbf{Y}(t_f) - \boldsymbol{\lambda}^{\mathrm{T}}(t_f)\mathbf{Y}(t_f) + \boldsymbol{\lambda}^{\mathrm{T}}(0)\mathbf{Y}(0) + \int_0^{t_f}\left[H(\mathbf{Y}, u_2, \boldsymbol{\lambda}, t) + \dot{\boldsymbol{\lambda}}^{\mathrm{T}}\mathbf{Y}(t)\right]\mathrm{d}t.$$

Consider the variation of $\bar{J}$ due to the variables u_2 and $\mathbf{Y}$, we have

$$\delta\bar{J} = \delta\mathbf{Y}^{\mathrm{T}}(t_f)\mathbf{Q}_0\mathbf{Y}(t_f) - \delta\mathbf{Y}^{\mathrm{T}}(t_f)\boldsymbol{\lambda}(t_f) + \int_0^{t_f}\left[\delta\mathbf{Y}^{\mathrm{T}}\left(\frac{\partial H}{\partial\mathbf{Y}} + \dot{\boldsymbol{\lambda}}\right) + \delta u_2^{\mathrm{T}}\frac{\partial H}{\partial u_2}\right]\mathrm{d}t$$

By setting $\delta\bar{J} = 0$, the necessary conditions for the minimum value problem are

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial\mathbf{Y}}, \quad \dot{\mathbf{Y}} = \frac{\partial H}{\partial\boldsymbol{\lambda}}, \quad \frac{\partial H}{\partial u_2} = 0, \quad \boldsymbol{\lambda}(t_f) = \mathbf{Q}_0\mathbf{Y}(t_f). \tag{21}$$

From (20) and (21), the optimal control of linear system (10) minimizes performance criterion (11) is given by

$$u_2^*(t) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(t), \tag{22}$$

where $\boldsymbol{\lambda}(t)$ is determined by the following differential equations

$$-\dot{\boldsymbol{\lambda}} = \mathbf{Q}\mathbf{Y} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}, \qquad \boldsymbol{\lambda}(t_f) = \mathbf{Q}_0\mathbf{Y}(t_f), \tag{23}$$

$$\dot{\mathbf{Y}} = \mathbf{A}\mathbf{Y} - \mathbf{S}\boldsymbol{\lambda} + \boldsymbol{\omega}, \qquad \mathbf{Y}(0) = \mathbf{X}(0) - \bar{\mathbf{X}}(0), \tag{24}$$

where $\mathbf{S} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}$. According to (22), (23), and (24), we have

Lemma 5.1 When $\tau = 0$, the optimal control of the linear system (10) that minimizes (11) is given by

$$u_2^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}[\boldsymbol{P}_0(t)\boldsymbol{Y}(t) + \boldsymbol{b}_0(t)], \tag{25}$$

where $\boldsymbol{P}_0(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}_0(t) \in \mathbb{R}^n$ are the solutions of the Riccati differential equations

$$\dot{\boldsymbol{P}}_0 = -\boldsymbol{P}_0\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_0 + \boldsymbol{P}_0\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_0 - \boldsymbol{Q}, \quad \boldsymbol{P}_0(t_f) = \boldsymbol{Q}_0, \tag{26}$$

$$\dot{\boldsymbol{b}}_0 = -\left[\boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_0\right]^{\mathrm{T}}\boldsymbol{b}_0 - \boldsymbol{P}_0\boldsymbol{\omega}, \quad \boldsymbol{b}_0(t_f) = 0. \tag{27}$$

Proof Firstly, let

$$\boldsymbol{\lambda}(t) = \mathbf{P}_0\mathbf{Y}(t) + \mathbf{b}_0(t). \tag{28}$$

Differentiating at the both sides of (28), combining (23) and (24) lead to

$$\dot{\boldsymbol{\lambda}} = \dot{\mathbf{P}}_0\mathbf{Y} + \mathbf{P}_0\dot{\mathbf{Y}} + \dot{\mathbf{b}}_0 = \dot{\mathbf{P}}_0\mathbf{Y} + \mathbf{P}_0(\mathbf{A}\mathbf{Y} - \mathbf{S}\boldsymbol{\lambda} + \boldsymbol{\omega}) + \dot{\mathbf{b}}_0 = -\mathbf{Q}\mathbf{Y} - \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}. \tag{29}$$

Equating the corresponding coefficients of both sides of (29), one has the Riccati equations (26) and (27). □

From Lemma 5.1, the optimal control $u_2^*(t)$ consists of two parts, $-\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0(t)\mathbf{Y}(t)$ is a feedback control that is determined by the current state, and the rest $-\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{b}_0(t)$ is a feedforward that is determined by the tracking target.

When $\tau \neq 0$, let us introduce a new integral state transformation as follows:

$$\mathbf{Z}(t) = \mathbf{Y}(t) + \int_{t-\tau}^{t} \mathrm{e}^{-\mathbf{A}(s-t+\tau)}[\mathbf{B}u_2(s) + \boldsymbol{\omega}(s+\tau)]\mathrm{d}s. \tag{30}$$

Then, with $\mathbf{B}_0 = \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}$, $\mathbf{G}(t) = \mathrm{e}^{-A\tau}\boldsymbol{\omega}(t+\tau)$, the system (10) is changed to a delay-free form

$$\dot{\mathbf{Z}}(t) = \mathbf{A}\mathbf{Z}(t) + \mathbf{B}_0 u_2(t) + \mathbf{G}(t). \tag{31}$$

Equation (10) has an equivalent form as follows:

$$\mathbf{Y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{Y}(0) + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}[\mathbf{B}u_2(s-\tau) + \boldsymbol{\omega}(s)]\mathrm{d}s,$$

and the solution $\mathbf{Y}(t)$ satisfies

$$\begin{aligned}\mathbf{Y}(t+\tau) &= \mathrm{e}^{\mathbf{A}\tau}\left(\mathbf{Y}(t) + \int_t^{t+\tau} \mathrm{e}^{\mathbf{A}(t-s)}[\mathbf{B}u_2(s-\tau) + \boldsymbol{\omega}(s)]\mathrm{d}s\right)\\ &= \mathrm{e}^{\mathbf{A}\tau}\left(\mathbf{Y}(t) + \int_{t-\tau}^{t} \mathrm{e}^{-\mathbf{A}(s-t+\tau)}[\mathbf{B}u_2(s) + \boldsymbol{\omega}(s+\tau)]\mathrm{d}s\right) = \mathrm{e}^{\mathbf{A}\tau}\mathbf{Z}(t).\end{aligned} \tag{32}$$

Thus, the initial condition $\mathbf{Y}(0) = \mathbf{Y}_0$ for system (10) is changed into $\mathbf{Z}(0) = \mathrm{e}^{-\mathbf{A}\tau}\mathbf{Y}(\tau)$.

The quadratic performance criterion J for system (11) can be decomposed by $J = J_1 + J_2$, where $J_1 = \frac{1}{2}\int_0^\tau \mathbf{Y}^{\mathrm{T}}\mathbf{Q}\mathbf{Y}\mathrm{d}t$ is fixed because the control does not take effect when $t \in [0, \tau[$, and

$$J_2 = \frac{1}{2}\mathbf{Y}^{\mathrm{T}}(t_f)\mathbf{Q}_0\mathbf{Y}(t_f) + \frac{1}{2}\int_0^{t_f-\tau}\left[\mathbf{Y}^{\mathrm{T}}(t+\tau)\mathbf{Q}\mathbf{Y}(t+\tau) + u_2^{\mathrm{T}}(t)\mathbf{R}u_2(t)\right]\mathrm{d}t. \tag{33}$$

Hence, $J_d = J_1 + J_2 = \min$ if and only if $J_2 = \min$. By substituting (32) into (33), the criterion J_2 takes the form

$$J_2 = \frac{1}{2}\mathbf{Z}^{\mathrm{T}}(t_f-\tau)\tilde{\mathbf{Q}}_0\mathbf{Z}(t_f-\tau) + \frac{1}{2}\int_0^{t_f-\tau}\left[\mathbf{Z}^{\mathrm{T}}(t)\tilde{\mathbf{Q}}\mathbf{Z}(t) + u_2^{\mathrm{T}}(t)\mathbf{R}u_2(t)\right]\mathrm{d}t,$$

where $\tilde{\mathbf{Q}} = \left(\mathrm{e}^{\mathbf{A}\tau}\right)^{\mathrm{T}}\mathbf{Q}\mathrm{e}^{\mathbf{A}\tau}$, $\tilde{\mathbf{Q}}_0 = \left(\mathrm{e}^{\mathbf{A}\tau}\right)^{\mathrm{T}}\mathbf{Q}_0\mathrm{e}^{\mathbf{A}\tau}$. Thus the original optimal control problem can be converted into the optimal control problem for linear delay-free system (31) associated with the performance criterion J_2.

According to Lemma 5.1, the optimal control for linear system (31) that minimizes J_2 is given by

$$u_2^*(t) = -\mathbf{R}^{-1}\mathbf{B}_0^{\mathrm{T}}[\mathbf{P}_d(t)\mathbf{Z}(t) + \mathbf{b}_d(t)], \tag{34}$$

where $\mathbf{P}_d(t) \in \mathbb{R}^{n\times n}$ and $\mathbf{b}_d(t) \in \mathbb{R}^n$ are the solutions of the Riccati differential equations

$$\dot{\mathbf{P}}_d = -\mathbf{P}_d\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{P}_d + \mathbf{P}_d\mathbf{B}_0\mathbf{R}^{-1}\mathbf{B}_0^{\mathrm{T}}\mathbf{P}_d - \tilde{\mathbf{Q}}, \quad \mathbf{P}_d(t_f-\tau) = \tilde{\mathbf{Q}}_0, \tag{35}$$

$$\dot{\mathbf{b}}_d = -[\mathbf{A} - \mathbf{B}_0\mathbf{R}^{-1}\mathbf{B}_0^{\mathrm{T}}\mathbf{P}_d]^{\mathrm{T}}\mathbf{b}_d - \mathbf{P}_d\mathbf{G}(t), \quad \mathbf{b}_d(t_f-\tau) = 0. \tag{36}$$

By substituting (32) into (34), the delayed optimal control of system (10) can be expressed in terms of the state $\mathbf{Y}(t)$, rather than $\mathbf{Z}(t)$, in the form of

$$u_2^*(t-\tau) = -\mathbf{R}^{-1}(\mathrm{e}^{-\mathbf{A}\tau}\mathbf{B})^{\mathrm{T}}\left[\mathbf{P}_d(t-\tau)\mathrm{e}^{-\mathbf{A}\tau}\mathbf{Y}(t) + \mathbf{b}_d(t-\tau)\right]. \tag{37}$$

Lemma 5.2 Let $u_2^*(t-\tau)$ be the delayed optimal control of system (10), $u_{20}^*(t)$ be the optimal control of system (10) with $\tau = 0$, then

$$u_2^*(t-\tau) = u_{20}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}[\boldsymbol{P}_0(t)\boldsymbol{Y}(t) + \boldsymbol{b}_0(t)], \quad (t \in [\tau, t_f]),$$

where $\boldsymbol{P}_0(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}_0(t) \in \mathbb{R}^n$ are the solutions of the Riccati differential equations (26) and (27).

Proof According to Lemma 5.1, the optimal control of the linear system (10) with $\tau = 0$ can be given by (25). Multiplying (26) by $(\mathrm{e}^{\mathbf{A}\tau})^{\mathrm{T}}$ from the left and by $\mathrm{e}^{\mathbf{A}\tau}$ from the right, respectively, one see that $\mathbf{W}(t) = (\mathrm{e}^{\mathbf{A}\tau})^{\mathrm{T}}\mathbf{P}_0(t)\mathrm{e}^{\mathbf{A}\tau}$ is the solution of the following differential equation

$$\dot{\mathbf{W}}(t) = -\mathbf{W}(t)\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{W}(t) + \mathbf{W}(t)\mathbf{B}_0\mathbf{R}^{-1}\mathbf{B}_0^{\mathrm{T}}\mathbf{W}(t) - \tilde{\mathbf{Q}},$$

subject to $\mathbf{W}(t_f) = \tilde{\mathbf{Q}}_0$. Replacing t with $t-\tau$ in (35) leads to

$$\dot{\mathbf{P}}_d(t-\tau) = -\mathbf{P}_d(t-\tau)\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{P}_d(t-\tau) + \mathbf{P}_d(t-\tau)\mathbf{B}_0\mathbf{R}^{-1}\mathbf{B}_0^{\mathrm{T}}\mathbf{P}_d(t-\tau) - \tilde{\mathbf{Q}},$$

subject to $\mathbf{P}_d(t_f-\tau) = \tilde{\mathbf{Q}}_0$. It follows that both $\mathbf{W}(t)$ and $\mathbf{P}_d(t-\tau)$ satisfy the same differential equation under the same terminal condition. Thus, $\mathbf{W}(t)$ and $\mathbf{P}_d(t-\tau)$ must be the same, namely

$$\mathbf{P}_d(t-\tau) = \left(\mathrm{e}^{\mathbf{A}\tau}\right)^{\mathrm{T}}\mathbf{P}_0(t)\mathrm{e}^{\mathbf{A}\tau}. \tag{38}$$

Similarly, multiplying (27) by $(\mathrm{e}^{A\tau})^{\mathrm{T}}$ from the left, $\mathbf{V}(t) = (\mathrm{e}^{\mathbf{A}\tau})^{\mathrm{T}}\mathbf{b}_0$ is the solution of

$$\dot{\mathbf{V}}(t) = -\left[\mathbf{A} - \mathrm{e}^{-\mathbf{A}\tau}\mathbf{S}\mathbf{P}_0\mathrm{e}^{\mathbf{A}\tau}\right]^{\mathrm{T}}\mathbf{V}(t) - \left(\mathrm{e}^{\mathbf{A}\tau}\right)^{\mathrm{T}}\mathbf{P}_0\boldsymbol{\omega}, \quad \mathbf{V}(t_f) = 0.$$

By substituting (38) into (36) and replacing t with $t-\tau$ leads to

$$\dot{\mathbf{b}}_d(t-\tau) = -\left[\mathbf{A} - \mathrm{e}^{-\mathbf{A}\tau}\mathbf{S}\mathbf{P}_0\mathrm{e}^{\mathbf{A}\tau}\right]^{\mathrm{T}}\mathbf{b}_d(t-\tau) - \left(\mathrm{e}^{\mathbf{A}\tau}\right)^{\mathrm{T}}\mathbf{P}_0\boldsymbol{\omega}, \quad \mathbf{b}_d(t_f-\tau) = 0.$$

Hence, $\mathbf{V}(t)$ and $\mathbf{b}_d(t-\tau)$ must be the same, namely

$$\mathbf{b}_d(t-\tau) = \left(\mathrm{e}^{\mathbf{A}\tau}\right)^{\mathrm{T}} \mathbf{b}_0(t). \tag{39}$$

By substituting (38) and (39) into (37), the following simple relationship holds for $t \in [\tau, t_f]$

$$u_2^*(t-\tau) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}[\mathbf{P}_0(t)\mathbf{Y}(t) + \mathbf{b}_0(t)] = u_{20}^*(t).$$

This completes the proof. □

Lemma 5.2 implies that the input delay does not affect the optimal control, when the system has no state delay, the optimal control of the delay-free system fully determines the optimal control of the corresponding system with an input delay, no matter how large the delay is.

When $t_f \to +\infty$, because the control takes effect only when $t \geq \tau$, the J can be simplified as follows:

$$\begin{aligned} J &= \frac{1}{2}\int_0^{+\infty} \left[\mathbf{Y}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{Y}(t) + \nu^{\mathrm{T}}(t-\tau)\mathbf{R}\nu(t-\tau)\right] \mathrm{d}t \\ &= \frac{1}{2}\int_0^{+\infty} \mathbf{Y}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{Y}(t)\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^{\tau} \nu^{\mathrm{T}}(t-\tau)\mathbf{R}\nu(t-\tau)\mathrm{d}t + \frac{1}{2}\int_\tau^{+\infty} \nu^{\mathrm{T}}(t-\tau)\mathbf{R}\nu(t-\tau)\mathrm{d}t \\ &= \frac{1}{2}\int_0^{+\infty} \left[\mathbf{Y}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{Y}(t) + \nu^{\mathrm{T}}(t)\mathbf{R}\nu(t)\right] \mathrm{d}t, \end{aligned}$$

and (26) becomes algebraic equation, then, we have

$$-\mathbf{P}_0\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{P}_0 + \mathbf{P}_0\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0 - \mathbf{Q} = \mathbf{0}, \tag{40}$$

$$\dot{\mathbf{b}}_0 = -\left[\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_0\right]^{\mathrm{T}} \mathbf{b}_0 - \mathbf{P}_0\boldsymbol{\omega}, \quad \mathbf{b}_0(t_f) = 0. \tag{41}$$

In this case, the matrix $\mathbf{P}_0$ is time independent and the Riccati equation becomes easy to solve. That is why the case of infinite time horizon is more often studied in practice.

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