I16 KB mirror tripod calculations

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1 Introduction

The purpose of the tripod assembly is to provide a stable mount for a KB mirror system, providing six degrees of freedom (three positions and three angles). These are accomplished by changing the positions of the three leg bases, each in two orthogonal directions. The leg bases are allowed to hinge while the leg tops are completely free to pivot in any direction. Such a device is optimally constrained and, within constraints, there is a finite number of solutions for each of the six base translations to give the required angles and positions (in fact, there are two solutions for each leg, giving a total of six). The purpose of this document is to give clear definitions for the required parameters and outline the steps required to perform calculations of (1) the top plate position and orientation for a given set of base translations and (2) the base translations for a given position and orientation of the top plate.

2 Geometry and definitions

The layout of the tripod, viewed from the top, is shown in Figure 1. We define two coordinate systems. The first (XYZ) is referenced to the base plate plane, defined as Z=0, which contains the three base points $\mathbf{B}_1$, $\mathbf{B}_2$ $\mathbf{B}_3$, which we refer to as $\mathbf{B}_i$ $(i = 1, 2, 3)$. The second (xyz) coordinate system is attached to the top plate. The $z = 0$ plane contains $\mathbf{T}_i$. z is up and y is parallel to $\mathbf{T_1} - \mathbf{T_2}$ (see Figure 1). The main thrust of the calculations is to transform between these two coordinate frames.

We define a 'tooling point' $\mathbf{C}$ which is located at a pre-defined point (C_x, C_y, C_z) in the (xyz) frame. The leg bottoms are free to pivot about a vector in the $Z = 0$ plane, the orientation of which is given by rotation of the Y axis about the Z axis by and angle ψ. Three ψ angles thus define the orientation of the three base hinges. The orientations of each leg (i.e. unit vectors along $\mathbf{T}_i - \mathbf{B}_i$ is given by a rotation of the Z axis about Y by an angle θ_i about the hinge vector.

We define the orientation of the (xyz) system with respect to (XYZ) in terms of rotations by three angles, α_1, α_2 and α_3, about X, Y and Z.

The goal of the calculation is to find C in the (XYZ) coordinate system (i.e. (C_X, C_Y, C_Z)) and $(\alpha_1, \alpha_2, \alpha_3)$ in terms of $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ and *vice versa*.

For convenience the base vectors $\mathbf{B}_i$ are written as sums of their centred values and translations along X and Y provided by two slides on each leg, *i.e.*, $\mathbf{B}_i = \mathbf{B}_i^C + B_i^X \hat{\mathbf{X}} + B_i^Y \hat{\mathbf{Y}}$.

Finally, the (fixed) leg lengths are given by l_i and the fixed distances between the top pivots are t_i (see Figure 1).

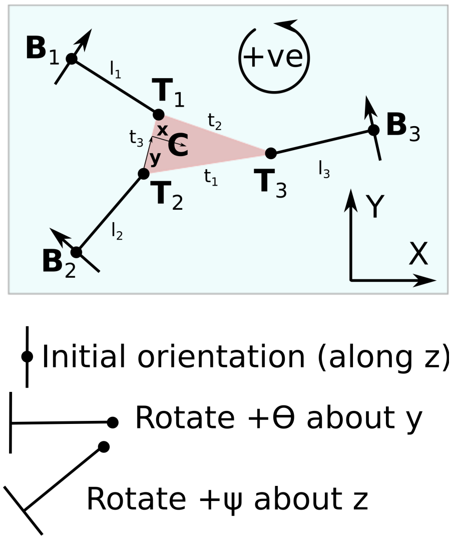

Figure 1: A sketch showing the tripod assembly from the top and some of the vectors and lengths used for the present calculations

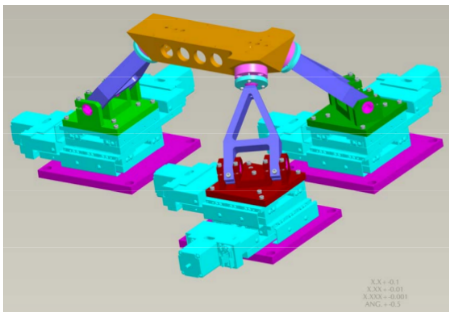

Figure 2: A 3d model of the Diamond implementation of the tripod

3 Calculations 1: Top plate position and orientation

First, we calculate the top base vectors $\mathbf{T}_i$ from known bottom base vectors $\mathbf{B}_i$ by ensuring that the distances t_i are correct. Before doing this, it is convenient to write the top vectors as the sum of the bottom vector and a vector linking the two, which in turn depends on the rotation angles ψ_i (known) and θ_i (unknown):

$$\mathbf{T}_i = \mathbf{B}_i + \hat{\mathbf{v}}_i l_i \tag{1}$$

where

$$\begin{aligned}
\hat{\mathbf{v}}_i &= R_Z(\psi_i) R_Y(\theta_i)\,\hat{\mathbf{Z}} \\
&= \begin{pmatrix} \cos\psi_i & -\sin\psi_i & 0 \\ \sin\psi_i & \cos\psi_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_i & 0 & \sin\theta_i \\ 0 & 1 & 0 \\ -\sin\theta_i & 1 & \cos\theta_i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\psi_i \sin\theta_i \\ \sin\psi_i \sin\theta_i \\ \cos\theta_i \end{pmatrix} \qquad (2)
\end{aligned}$$

Ensuring that the distances between the top vectors are correct allows us to write down a set of three quadratic equations,

$$|\mathbf{T}_2 - \mathbf{T}_1|^2 - t_3^2 \quad = \quad 0$$

$$
\begin{aligned}
|\mathbf{T}_1 - \mathbf{T}_3|^2 - t_2^2 &= 0 \\
|\mathbf{T}_3 - \mathbf{T}_2|^2 - t_1^2 &= 0
\end{aligned} \tag{3}
$$

the solutions of which give θ_i, which in turn gives $\hat{\mathbf{v}}_i$ and $\mathbf{T}_i$. It important to note that there are two solutions for each leg, corresponding to $\pm\theta$, so it is essential to ensure that the preferred solution is adopted.

With $\mathbf{T}_i$ we can now calculate (C_X, C_Y, C_Z) and $\alpha_{1,2,3}$. As C is already known in the xyz frame, finding C in the XYZ frame simply requires a knowledge of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ since (see Figure 1),

$$
C_{XYZ} = \mathbf{T}_2 + C_x\hat{\mathbf{x}} + C_y\hat{\mathbf{y}} + C_z\hat{\mathbf{z}} \tag{4}
$$

where

$$
\begin{aligned}
\hat{\mathbf{y}} &= (\mathbf{T}_1 \widehat{-} \mathbf{T}_2) \\
\hat{\mathbf{z}} &= \{(\mathbf{T}_3 - \widehat{\mathbf{T}_1}) \times \hat{\mathbf{y}}\} \\
\hat{\mathbf{x}} &= \hat{\mathbf{y}} \times \hat{\mathbf{z}}
\end{aligned} \tag{5}
$$

Finally, we calculate the rotation angles of the top plate with respect to the bottom plate. To do this we note that any unit vector attached to the top plate can be obtained by rotating the same vector in the XYZ base frame by angles $\alpha_{1,2,3}$ about X, Y and Z:

$$
\hat{\mathbf{w}}_{xyz} = R_Z(\alpha_3)R_Y(\alpha_2)R_X(\alpha_1)\,\hat{\mathbf{w}}_{XYZ} \tag{6}
$$

By considering the rotations of $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$, $\hat{\mathbf{Z}}$ onto $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, and writing the transformations as a single expression in matrix form, we have (in the XYZ coordinate system)

$$
\begin{aligned}
(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) &= R_Z(\alpha_3)R_Y(\alpha_2)R_X(\alpha_1)\left(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}\right) \\
&= R_Z(\alpha_3)R_Y(\alpha_2)R_X(\alpha_1)
\end{aligned} \tag{7}
$$

which can be written,

$$
\begin{aligned}
\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} &= \begin{pmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & -\sin\alpha_1 \\ 0 & \sin\alpha_1 & \cos\alpha_1 \end{pmatrix}
\end{aligned} \tag{8}
$$

Since all the components of the matrix on the left hand side (*i.e.* $\hat{\mathbf{x}} = (x_1, x_2, x_3)$) have already been calculated, it is straightforward to multiply out the matrices in the right hand side and extract the angles $\alpha_{1,2,3}$

4 Calculations 2: Base translations

The reverse calculations, *i.e.* calculation of the base translations from known tooling point coordinates and angles, is simpler. We can make use of Eqn. 8 to calculate the vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ from the three angles α_i where $\hat{\mathbf{x}} = (x_1, x_2, x_3)$ *etc.* Armed with these unit vectors, it is straightforward to calculate $\mathbf{T}_i$ as follows: Eqn. 4 provides $\mathbf{T}_2$. Next,

$$\mathbf{T}_1 = \mathbf{T}_2 + t_3\,\hat{\mathbf{y}} \tag{9}$$

and the remaining vector can be calculated via the cosine-rule:

$$\begin{aligned}
\cos\chi_2 &= (t_1^2 + t_3^2 - t_2^2)/(2t_1t_3) \\
\sin\chi_2 &= \sqrt{1 - \cos\chi_2^2} \\
\mathbf{T}_3 &= \mathbf{T}_2 + \hat{\mathbf{x}}\,t_1\sin\chi_2 + \hat{\mathbf{y}}\,t_1\cos\chi_2
\end{aligned} \tag{10}$$

with χ_2 being the angle between t_1 and t_3. The tilt angles, θ_i, of the legs is straightforward to calculate using

$$\cos\theta_i = \mathbf{T}_i \cdot \hat{\mathbf{Z}}/l_i \tag{11}$$

Here, θ_i can be chosen to be either positive or negative, depending on which solution is required. Finally, Eqn. 2 gives the vectors $\mathbf{v}_i$ which can be inserted in to Eqn. 1 to give the base vectors $\mathbf{B}_i$.

5 Implementation

These calculations have been implemented in a Python class tripod_class.py. The definitions are consistent with this document and the variable names should be recognizable. Eqns. 3 require an iterative numerical approach which is implemented using only basic matrix methods to provide a rapidly-converging solution. If it is inconvenient to import these methods from standard libraries then they can be built from basic scalar mathematical functions with a few lines of code.