Statistical Methods in Education Research
Vanderbilt University
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61 total points

Problem Set 4 *Solutions*

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1. (**16 points—2 each**) In a population of students, the number of absences during the school year ranges from 3 to 7. The probabilities of a randomly drawn student from this population having 3, 4, 5, 6, or 7 absences are shown in the table below. Define the event A as the student being absent *more than* 4 days, and the event B as the student being absent *fewer than* 6 days.

# of Days	3	4	5	6	7
Probability	0.08	0.24	0.41	0.20	0.07

   (a) What is the probability of event A? $\mathbf{P(A) = P(5) + P(6) + P(7) = 0.41 + 0.20 + 0.07 = 0.68}$

   (b) What is the probability of event B? $\mathbf{P(B) = P(3) + P(4) + P(5) = 0.08 + 0.24 + 0.41 = 0.73}$

   (c) What is the probability of $\sim A$? $\mathbf{P(\sim A) = P(3) + P(4) = 0.08 + 0.24 = 0.32}$ or alternatively $\mathbf{1 - P(A) = 1 - 0.68 = 0.32}$

   (d) Are events A and B mutually exclusive? Explain why or why not. **No.** $\mathbf{A \cap B = 5}$ **(i.e. 5 absences appears in both events, so they are not mutually exclusive).**

   (e) What is the probability of $A \cap B$? $\mathbf{P(A \cap B) = P(5) = 0.41}$

   (f) What is the probability of $A \cup B$? $\mathbf{P(A \cup B) = 1.0}$

   (g) Show that $P((A \cap B) \cup (\sim A \cap B)) = P(B)$. **In words, the lefthand side of this equation is the probability that B and A occur *or* B and $\sim$ A occur. In other words, B occurs and either A occurs or it doesn't. This is simply B, and P(B)=0.73. You could also recognize that these are mutually exclusive events—if B and A are true, it cannot be the case that B and $\sim$ A are true. With mutually exclusive events, you can add the two probabilities together:** $\mathbf{P((A \cap B) \cup (\sim A \cap B)) = 0.41 + 0.32 = 0.73}$

   (h) Show that $P(A \cup (\sim A \cap B)) = P(A \cup B)$. **In words, the lefthand side of this equation is the probability that A occurs *or* A doesn't occur and B occurs. In this context (looking at the above table), this is the same as A or B occurring. As seen in part (f), this is 1.**

2. (**6 points—3 each**) Using the probability distribution in Question 1, find the following (and show your work):

   (a) $E(\#$ of absences$)$:

   $\boxed{\sum_{i=1}^{n} X_i * P(X_i) = (3*0.08) + (4*0.24) + (5*0.41) + (6*0.20) + (7*0.07) = 4.94}$

   (b) $Var(\#$ of absences$)$:

   $\boxed{\sum_{i=1}^{n}(X_i - E(X))^2 * P(X_i) = ((3-4.94)^2 * 0.08) + ((4-4.94)^2 * 0.24) + ((5-4.94)^2 * 0.41) + ((6-4.94)^2 * 0.20) + ((7-4.94)^2 * 0.07) = 1.04}$

3. (**8 points—2 each**) Shown below is a 2 x 2 table that reports the fraction of the population in each cell:

		Education level		
		HS	<HS	Totals
Current smoker:	NO	0.614	0.130	0.744
	YES	0.194	0.062	0.256
	Totals	0.808	0.192	1.000

   (a) For a randomly drawn person, what is P(smoker)? **0.256, or 25.6%**

   (b) For a randomly drawn person, what is P(smoker | <HS diploma)? **Here we can use $P(A|B) = P(A \cap B)/P(B)$, or 0.062/0.192 = 0.323, or 32.3%**

   (c) For a randomly drawn person, what is P(smoker | HS diploma+)? **In the same manner as part (b): 0.194/0.808 = 0.240, or 24.0%**

   (d) Are education and smoking status "independent?" Why or why not? **No. The probability of being a current smoker varies depending on one's education level (as shown in parts b and c). Thus they are not independent.**

4. (**5 points**) Shown below is a 2 x 2 table. In Period 1, events A or B can happen. In Period 2, outcome C or D will result. If $P(C|B) = 0.150$ and $P(D|A) = 0.7$, then fill in the missing boxes below:

		Period 1	
		Event A	Event B
Period 2	Event C	**0.240**	0.030
	Event D	**0.560**	**0.170**
		0.800	**0.200**

- First use $P(C|B) = P(C \cap B)/P(B)$ or $0.15 = 0.030/P(B)$ which implies that $P(B) = 0.2$. This provides the first marginal probability shown in the bottom right corner.
- If $P(B \cap C) = 0.03$ and $P(B) = 0.2$ then $P(B \cap D) = 0.2 - 0.03 = 0.17$
- If $P(B) = 0.2$ then $P(A) = 1 - 0.2 = 0.8$
- Now use $P(D|A) = P(D \cap A)/P(A)$ or $0.7 = P(D \cap A)/0.8$ which implies that $P(D \cap A) = 0.56$.
- Finally $P(A \cap C) = 0.80 - 0.56 = 0.24$
- Notice that the four probabilities in the center of the table sum to 1, as they should.

5. (**4 points**) After the attacks of September 11, 2001, the TSA implemented a program called SPOT (Screening of Passengers by Observation Techniques) in which passengers were flagged for suspicious behavior and given additional searching or screening. Suppose that:

   - There are 2 billion plus 100 passenger trips per year (2,000,000,100).
   - 100 of these passengers are terrorists (i.e., less than 0.00000001%).
   - Nearly all (99%) terrorists exhibit the kinds of behaviors that were flagged.
   - Some non-terrorists exhibit these suspicious behaviors, but it is rare (1%).

   The SPOT test has low false negative and false positive rates, suggesting it is an effective way to catch would-be terrorists. Use Bayes' Theorem to calculate the probability that a flagged passenger is, in fact, a terrorist.

   Bayes' Theorem applied here is:

$$
\begin{aligned}
P(\text{terrorist}|\text{flagged}) &= \frac{P(\text{flagged}|\text{terrorist})P(\text{terrorist})}{P(\text{flagged})} \\
&= \frac{\frac{99}{100} * \frac{100}{2{,}000{,}000{,}100}}{\frac{20{,}000{,}099}{2{,}000{,}000{,}100}} \\
&= \frac{99}{20,000,099} \\
&= 0.00000495
\end{aligned}
$$

In other words, very small! While the system involves a test that will "catch" nearly all terrorists, the baseline probability of being a terrorist is very low. Even with a low false positive rate, the SPOT system flags a very large number of innocent passengers.

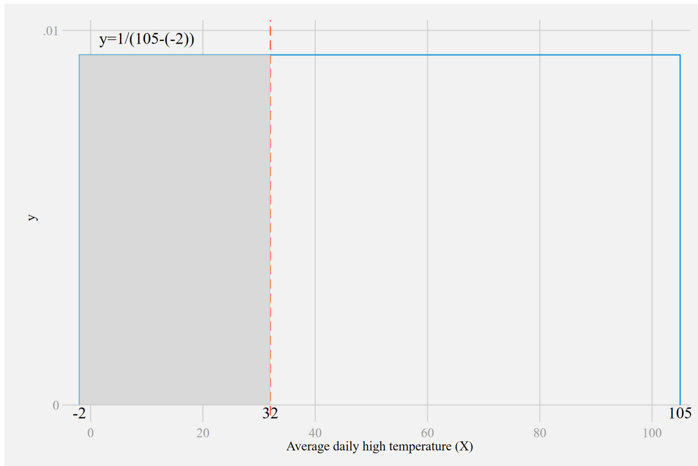
6. (**6 points—3 each**) Paul and Natasha live in Los Angeles. Paul hates cold weather but Natasha has been transferred to a cold Northeastern city. Paul notes that he cannot move go to a city where more than 30% of the days have an average daily high below freezing. Suppose the average daily high temperatures (X) in a city can be described by a uniform distribution where the minimum and maximum average daily highs are -2 and 105, respectively.

   (a) What is the PDF for X, and what is $P(x \leq 32)$? Should Natasha look for a one or a two bedroom apartment? (Hint: you do not need calculus to find the requested probability).

   The PDF for a uniform distribution from $[a, b]$ is: $y = 1/(b - a)$. Or in this case: $y = 1/107$. The PDF is pictured below, and the area under the curve from -2 to 32 is shaded. The probability that this city's daily high temperature is 32 or below is this area, which is easy to calculate given the rectangular distribution: $P(X \leq 32) = 34 * (1/107) = 31.8\%$ Nathsha may want to find a one bedroom apartment! FYI the Stata code I used to produce this graph is below.

```
twoway (function y=1/107, range(-2 105) dropline(-2 105)) (function y=1/107, ///
   range(-2 32) color(gs10*0.5) recast(area)), ylabel(0(0.01)0.01) xline(32, ///
   lpattern(dash)) xtitle(Average daily high temperature (X)) legend(off) ///
   text(-0.0002 -2 "-2") text(-0.0002 32 "32") text(-0.0002 105 "105") ///
   text(0.0098 10 "y=1/(105-(-2))")
```

   (b) What are $E(X)$ and $Var(X)$?



For a uniform distribution, $E(X) = \frac{a+b}{2} = \frac{-2+105}{2} = 51.5$

And $Var(X) = \frac{1}{12}(b-a)^2 = \frac{1}{12}(107^2) = 954.1$. The standard deviation would be: $\sqrt{954.1} = 30.9$

7. (**4 points**) Assume the random variable z has a standard normal distribution. Use Stata, an online calculator, or a textbook table to answer the following:

   (a) The probability is 0.70 that z is less than what number? **Pr(z< 0.5244)=0.70, using `display invnormal(0.70)`**

   (b) The probability is 0.25 that z is less than what number? **Pr(z< -0.6745)=0.25, using `display invnormal(0.25)`**

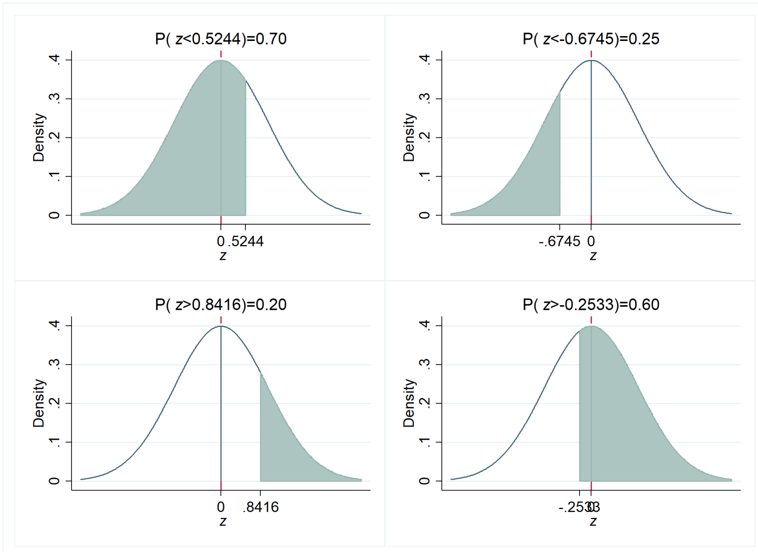
   (c) The probability is 0.20 that z is greater than what number? **Pr(z> 0.8416)=0.20, using `display (-1)*invnormal(0.20)`**

   (d) The probability is 0.60 that z is greater than what number? **Pr(z> -0.2533)=0.60, using `display (-1)*invnormal(0.60)`**

8. (**6 points**) To graduate with honors, you must be in the top 2 percent (*summa cum laude*), 3 percent (*magna cum laude*) or 5 percent (*cum laude*) of your class. Suppose GPAs are distributed normally with a mean of 2.6 and a standard deviation of 0.65. What GPA will you need in order to graduate at each of these three levels?

Under the assumption of a normal distribution, we need to find the GPA cutoff points (x_1, x_2, x_3) such that:

$P(GPA > x_1) = 0.02$ or $P(z > (x_1 - 2.6)/0.65) = 0.02$ (summa)



$P(GPA > x_2) = 0.03$ or $P\left(z > (x_2 - 2.6)/0.65\right) = 0.03$ (magna)

$P(GPA > x_3) = 0.05$ or $P\left(z > (x_3 - 2.6)/0.65\right) = 0.05$ (cum laude)

From the online calculator (or Stata) we find that the values of z for which 2, 3, and 5 percent of outcomes fall above are: 2.054, 1.881, and 1.645. In Stata, the command is `display (-1)*invnormal(p)`, with p=0.02, 0.03, or 0.05. The result of `invnormal` is multiplied by -1 since we are interested in the z value *above* which there is a p probability of falling. Converting z into the original units (GPA points) we find the following GPA cutoffs:

$2.054 = (x_1 - 2.6)/0.65$ or $x_1 = 3.9351$ for summa cum laude

$1.881 = (x_2 - 2.6)/0.65$ or $x_2 = 3.8227$ for magna cum laude

$1.645 = (x_3 - 2.6)/0.65$ or $x_3 = 3.6693$ for cum laude

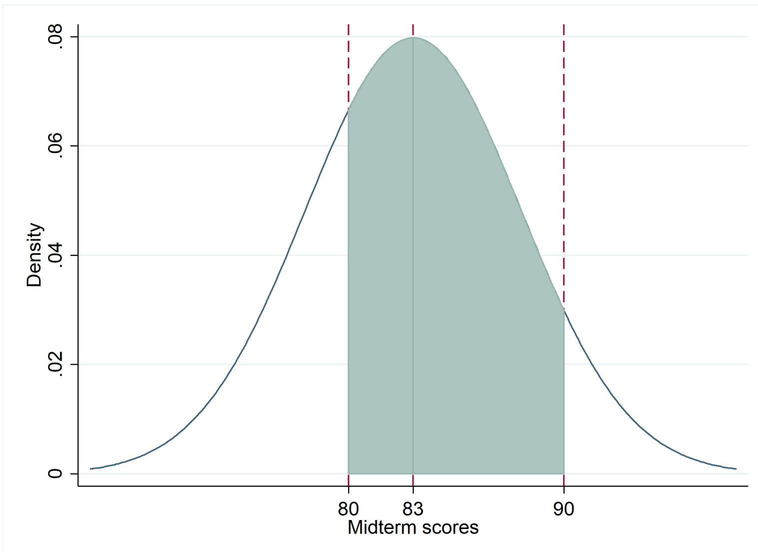
9. (**6 points—3 each**) On the midterm exam in introductory statistics, an instructor always gives a grade of B to students who score between 80 and 90. The scores tend to have a normal distribution with a mean $\mu = 83$ and a standard deviation $\sigma = 5$. About what fraction of the students get a B?

   (a) First, answer this question using what you know about the normal distribution.

   (b) Now use simulated data in Stata. Generate 1,000 student exam scores—this instructor has a big class!—from a normal distribution with the above parameters. Then answer the question based on the data you drew. Are there any differences between your two answers?

This question is asking: $P(80 \leq X \leq 90) = P(\frac{80-83}{5} \leq \frac{X-\mu}{\sigma} \leq \frac{90-83}{5}) = P(-0.6 \leq z \leq 1.4)$. For part (a), we can use Stata to find this probability:

`display normal(1.4)-normal(-0.6)`, this probability is 0.645. Or, about 64.5% of students get a B.



For part (b), can use code like the following:

```
set seed 1989
set obs 1000
gen midterm=rnormal(83,5)
count if midterm>=80 & midterm<=90
display 638/1000
count if midterm>80 & midterm<90
display 638/1000
```

I get 63.8% of students scoring between an 80 or 90. (It doesn't matter whether the inequalities or strict or not in this case). This differs from part (a) because this is random sample of 1,000. 64.5% would be the proportion between 80 and 90 from an infinitely large sample.