# 11. Bivariate regression

LPO.8800: Statistical Methods in Education Research

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## Last time

- Multivariate distributions
- Measures of association: covariance and correlation
- Scatter diagrams
- Pearson correlation, Spearman correlation
- Effect of variable transformations on correlation
- Hypothesis tests about $\rho$

## Last time

The sample correlation coefficient $r_{xy}$: a measure useful for assessing the strength of a linear relationship between two variables:

- Ranges from -1 to $+1$
- Unit-free and useful for many purposes

We are often interested, however, in quantifying a relationship in the original units (e.g. *how much* of a change in student outcomes is associated with a unit change in class size?)

## Today

**Regression** is another way of quantifying the relationship between two (or more) variables

- Most widely used tool in social science and policy analysis
- Can be used for descriptive or predictive purposes
- Sometimes useful for causal inference, but only under very strict assumptions (more on this in Lecture 12, and in later courses)

## Simple linear regression

Goal: to find the **line of best fit**—the best linear predictor of $y$ given $x$. A line is defined by its *slope* ($b$) and *intercept* ($a$):

$$y = a + bx$$

In regression analysis these are referred to as the *slope coefficient* and the *constant term*.

## Simple vs. multiple linear regression

A **simple regression** involves *two* variables:

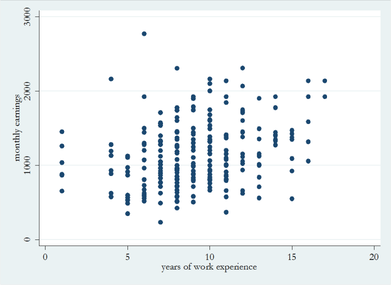
- **response variable** (**Y**): often an outcome you are trying to explain or predict (also known as the “dependent” variable)
- **explanatory variable** (**X**): often a presumed cause of the outcome, or a predictor of it (sometimes called the “independent” variable—I avoid using this term)

**Multiple (multivariate) regression** involves *many* variables:

- **response variable** (**Y**)
- **two or more explanatory variables** ($\mathbf{X_1}, \mathbf{X_2}, ...$): there is often one of particular interest, plus additional *covariates* or *controls*

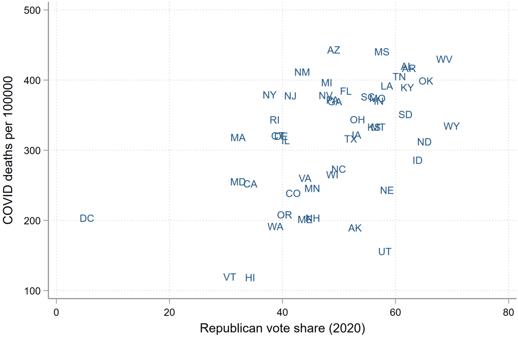
# Example 1

From the *wage2* data (NLSY): monthly earnings vs. years of work experience for men with at least 16 years of education



# Example 2

COVID death rate and Republican vote share



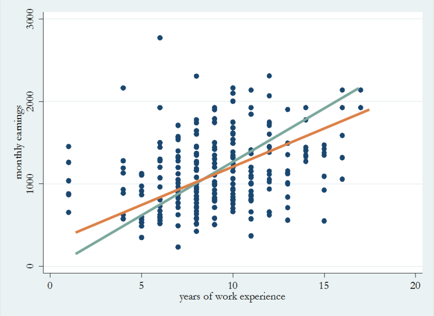
# Line of best fit

What makes a particular line the "best fit" or "best predictor" of $y$ given a value of $x$? Remember, a line is defined by its intercept and slope:

$$y = a + bx$$

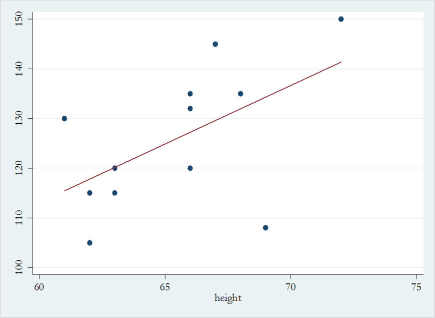
# Example 1, cont.

There are lots of candidates for the best fit line, with different values of $a$ and $b$. Which is the "best"?



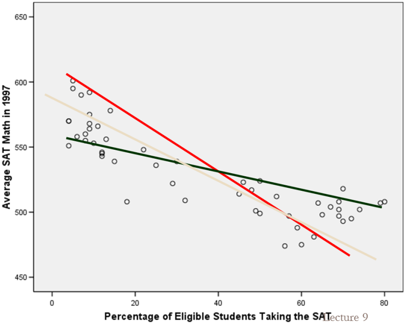
# Example 3

Height and weight data



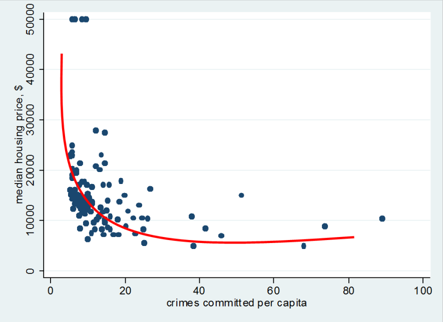
# Example 4

State average math SAT scores



## Example 5

Crime rate and median house price data; a line is probably not appropriate to describe this relationship



## Line of best fit

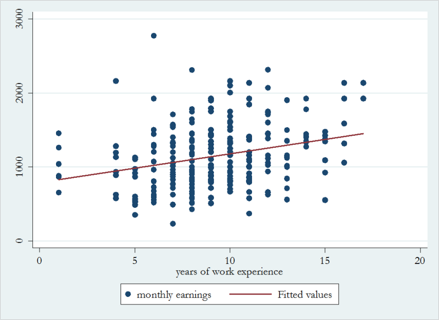
Stata can provide a line of best fit, using two overlaying graphs (`scatter` and `lfit`). Using the *wage2* data:

```
twoway (scatter wage exper) (lfit wage exper) if educ>=16
```

It is conventional to put the response variable on the vertical ($y$) axis, and the explanatory variable on the horizontal ($x$) axis.

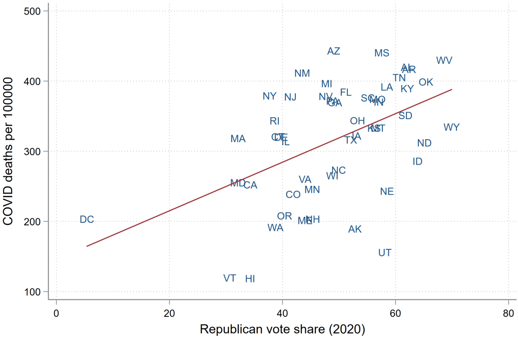
# Example 1, cont.

Overlaid graphs `scatter` and `lfit`:



# Example 2, cont.

Overlaid graphs `scatter` and `lfit`:



## Line of best fit

In Example 1 the best fit line has an intercept of 789.93, and a slope of 38.71: $\widehat{\mathbf{y}} = \mathbf{789.93} + \mathbf{38.71x}$

- The best fit line is also called a **prediction equation**
- $\hat{y}$ is the **predicted value** for $y$, given $x$.

We can use the prediction equation to predict $y$ for a specific $x$ value (here, years of work experience):

- Example: suppose $x = 10$ years
- $\hat{y} = 789.93 + (38.71 * 10) = 1,177.03$
- Predicted monthly earnings is $1,777.03 for a worker with 10 years of experience

## Line of best fit

Interpreting the prediction equation: $\widehat{\mathbf{y}} = \mathbf{789.93} + \mathbf{38.71x}$

- 789.93: the predicted monthly earnings when work experience $x = 0$
- 38.71: the predicted *change* in monthly earnings when work experience increases by one year
- Note *10* additional years of work experience would be predicted to change monthly earnings by: $(38.71 * 10 = 387.10)$

## Least squares

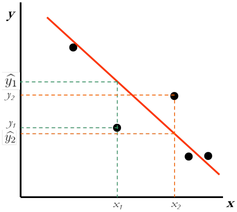
How does one determine the line of "best fit"? Potential criteria:

- Line that passes through as many points as possible?
- Line that minimizes *prediction error*: the gaps (deviations) between the data points and the line?
  - Problem: positive deviations cancel out negative deviations
  - Solution: *squared* deviations (compare to variance formula)
  - Another possibility (rarely used): mean absolute deviation

## Least squares

For a given line, we have a set of predictions for $y$, one for every value of $x$ in the data:

- $\hat{y}_1$ is the predicted value of $y$ when $x$ is $x_1$
- $\hat{y}_2$ is the predicted value of $y$ when $x$ is $x_2$
- ...and so on, up to $\hat{y}_n$



## Least squares

The line that minimizes the *sum of the squared deviations* between the data points $(y, x)$ and the line $(\hat{y}, x)$ is the **least squares** (or ordinary least squares, **OLS**) regression line:

$$\overset{\min}{a, b} \sum_{i=1}^{n} \left(y_i - \widehat{y_i}\right)^2$$

$$\overset{\min}{a, b} \sum_{i=1}^{n} \left(y_i - a - bx\right)^2$$

I.e. choose intercept and slope $(a, b)$ to minimize the sum of the squared deviations $(y_i - \widehat{y_i})$

## Least squares

Using calculus it can be shown that the least squares slope $(b)$ and intercept $(a)$ are as follows:

$$b = \frac{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)}{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}$$

$$a = \bar{y} - b\bar{x}$$

# Least squares

Recall that:

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{s_x s_y (n-1)}$$

Multiply by $s_y/s_x$:

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{s_x s_y (n-1)} \frac{s_y}{s_x} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{s_x^2 (n-1)}$$

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}(n-1)} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = b$$

So $r_{xy}\frac{s_y}{s_x} = b$

# Least squares

$$b = r_{xy}\frac{s_y}{s_x}$$

Notice the slope coefficient will have the same sign $(+/-)$ as the correlation coefficient $r_{xy}$ (re: $s_y$ and $s_x$ are both greater than zero)

## Least squares

Notice the least squares solution for $a$ indicates that the point $(\bar{x}, \bar{y})$ falls on the regression line. That is, line passes through the mean of both variables):

$$\hat{y} = a + b\bar{x}$$

$$\hat{y} = (\bar{y} - b\bar{x}) + b\bar{x} = \bar{y}$$

The best prediction of $y$ when $x = \bar{x}$ is $\bar{y}$.

## Least squares

Notice also if $r = 0$, then $b = 0$, and the best prediction of $y$ for *any* value of $x$ is $\bar{y}$:

$$\hat{y} = a$$

$$a = \bar{y} - b\bar{x} = \bar{y}$$

Put another way, if there is no correlation between $y$ and $x$, the best prediction of $y$ given $x$ is just $\bar{y}$.

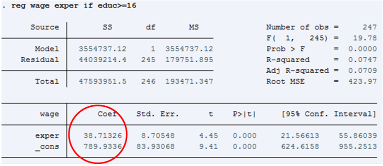
# Regression in practice

To compute the least squares intercept and slope coefficient in Stata use `regress` or `reg` *yvar xvar* (aka "running a regression"). Examples:

- Monthly wage and experience
- Height and weight
- State percent taking SAT and mean math SAT score
- COVID death rates and Republican vote share

# Example 1, cont.

From the *wage2* data (NLSY): monthly earnings vs. years of work experience, for men with at least 16 years of education



```
. reg wage exper if educ>=16

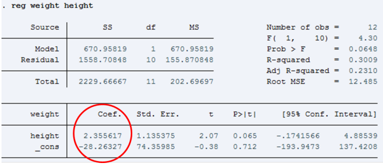
      Source |       SS           df       MS      Number of obs   =       247
-------------+----------------------------------   F(1, 245)       =     19.78
       Model |  3554737.12         1  3554737.12   Prob > F        =    0.0000
    Residual |  44039214.4       245  179751.895   R-squared       =    0.0747
-------------+----------------------------------   Adj R-squared   =    0.0709
       Total |  47593951.5       246  193471.347   Root MSE        =    423.97

------------------------------------------------------------------------------
        wage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       exper |   38.71326    8.70548     4.45   0.000     21.56613    55.86039
       _cons |   789.9336   83.93068     9.41   0.000     624.6158    955.2513
------------------------------------------------------------------------------
```

Note: "_cons" refers to the intercept, or **constant term**.

# Example 3, cont.

Height and weight data:



```
. reg weight height

      Source |       SS       df       MS              Number of obs =      12
-------------+------------------------------           F(  1,    10) =    4.30
       Model |   670.95819     1   670.95819           Prob > F      =  0.0648
    Residual |  1558.70848    10  155.870848           R-squared     =  0.3009
-------------+------------------------------           Adj R-squared =  0.2310
       Total |  2229.66667    11   202.69697           Root MSE      =  12.485

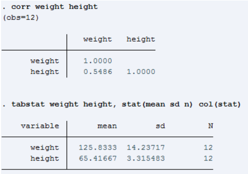
------------------------------------------------------------------------------
      weight |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      height |   2.355617   1.135375     2.07   0.065    -.1741566     4.88539
       _cons |  -28.26327   74.35985    -0.38   0.712    -193.9473    137.4208
------------------------------------------------------------------------------
```

$$\hat{y} = -28.263 + 2.356x$$

$$\widehat{weight} = -28.263 + 2.356 * height$$

# Regression in practice

One can also use descriptive statistics and $r$ to calculate the least squares slope coefficient and intercept:



```
. corr weight height
(obs=12)

             |   weight   height
-------------+------------------
      weight |   1.0000
      height |   0.5486   1.0000


. tabstat weight height, stat(mean sd n) col(stat)

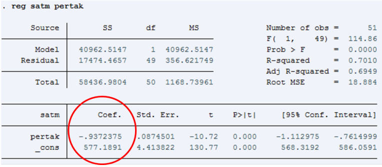
    variable |      mean        sd         N
-------------+------------------------------
      weight |  125.8333  14.23717        12
      height |  65.41667  3.315483        12
--------------------------------------------
```

$$b = 0.549 * (14.237/3.315) = 2.357$$

$$a = 125.833 - 2.357 * (65.417) = -28.406$$

# Example 4, cont.

State average math SAT scores



```
. reg satm pertak

      Source |       SS       df       MS              Number of obs =      51
-------------+------------------------------           F(  1,    49) =  114.86
       Model |  40962.5147     1  40962.5147           Prob > F      =  0.0000
    Residual |  17474.4657    49  356.621749           R-squared     =  0.7010
-------------+------------------------------           Adj R-squared =  0.6949
       Total |  58436.9804    50  1168.73961           Root MSE      =  18.884

------------------------------------------------------------------------------
        satm |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      pertak |  -.9372375   .0874501   -10.72   0.000    -1.112975   -.7614999
       _cons |   577.1891   4.413822   130.77   0.000     568.3192    586.0591
------------------------------------------------------------------------------
```

$$\hat{y} = 577.189 - 0.9372x$$

$$\widehat{SATm} = 577.189 - 0.9372 * pertak$$

# Regression in practice

Using the regression equation to predict $y$ given a specific value of $x$, e.g. percent taking SAT of $x = 40$:

- $\hat{y} \ = 577.189 - 0.9372 * (40) = 539.7$
- Note the value of $x$ used does not have to appear in the actual data.
- What if $x = 40$ *did* appear in the data? Why not use the actual, observed $y$ when $x = 40$ as the prediction?

# Regression in practice

Steps for interpreting a regression slope coefficient:

1. Identify the **explanatory** variable and its units (e.g., height in inches, years of work experience).
2. Describe a **one-unit increase** in the explanatory variable in everyday language (e.g., one additional year of work experience).
3. Identify the **response** or **outcome** variable and its units (e.g., weight in pounds, monthly earnings).
4. Describe the coefficient as the **change in the outcome** predicted for a one-unit change in the explanatory variable (e.g., an additional year of work experience is predicted to increase monthly earnings by $38.71).

Note: Adapted from Remler & Van Ryzin (2011), chapter 8.

# Example 2, cont.

COVID death rate and Republican vote share

```
. reg death_per100k repshare

      Source |       SS           df       MS      Number of obs   =        52
-------------+----------------------------------   F(1, 50)        =     15.73
       Model |  86272.4831         1  86272.4831   Prob > F        =    0.0002
    Residual |  274157.456        50  5483.14912   R-squared       =    0.2394
-------------+----------------------------------   Adj R-squared   =    0.2241
       Total |  360429.939        51  7067.25371   Root MSE        =    74.048

------------------------------------------------------------------------------
death_p~100k |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    repshare |   3.461963    .872773     3.97   0.000     1.708947    5.214979
       _cons |   145.8957   44.12522     3.31   0.002     57.26762    234.5238
------------------------------------------------------------------------------
```

## Example 2, cont.

Presenting regression output in publication form

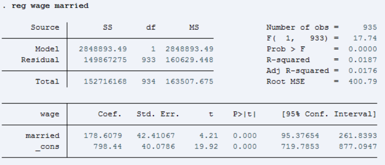
| | (1)<br>Death rate<br>per 100,000 |
|---|---|
| Republican vote | 3.46 |
| share in 2020 | (0.873)*** |
| | |
| Constant | 145.896 |
| | (44.125)*** |
| | |
| R-squared | 0.239 |
| N | 52 |

## Regression on a dummy variable

When the explanatory variable is dichotomous (0-1, a “dummy” variable), $x$ only takes on two values. Example: married vs. unmarried

- $x = 0$: person is unmarried
- $x = 1$: person is married

# Regression on a dummy variable



```
. reg wage married

      Source |       SS       df       MS              Number of obs =     935
-------------+------------------------------           F(  1,   933) =   17.74
       Model |  2848893.49     1  2848893.49           Prob > F      =  0.0000
    Residual |   149867275   933  160629.448           R-squared     =  0.0187
-------------+------------------------------           Adj R-squared =  0.0176
       Total |   152716168   934  163507.675           Root MSE      =  400.79

------------------------------------------------------------------------------
        wage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     married |   178.6079   42.41067     4.21   0.000     95.37654    261.8393
       _cons |     798.44    40.0786    19.92   0.000     719.7853    877.0947
------------------------------------------------------------------------------
```

There are only two possible predictions:

- When $x = 0$: $\hat{y} = a = 798.44$
- When $x = 1$: $\hat{y} = a + b = 798.44 + 178.61 = 977.05$

# Predicted values and residuals

It is possible to have Stata compute the **predicted values** and **residuals** (prediction errors) after `reg`:

- `predict` *`yhat`*`, xb`
- `predict` *`uhat`*`, resid`

These are called **postestimation** commands:

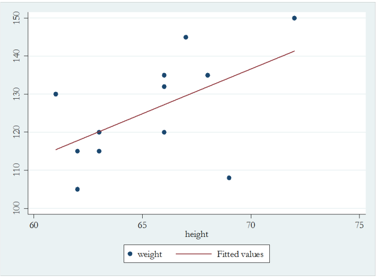
- `xb` refers to the predicted value ($\hat{y}$)
- `resid` refers to the residual ($\hat{u}$, calculated as $y_i - \widehat{y_i}$)

To generate predicted values or residuals only for the estimation sample:

- Add `if e(sample)` to the `predict` command
- Otherwise $\hat{y}$ and $\hat{u}$ will be created for any case in which $x$ is not missing (even if $y$ is missing)

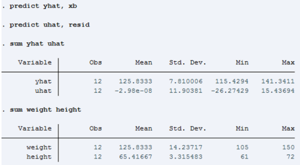
# Example 3, cont.

Height and weight data:



`twoway (scatter weight height) (lfit weight height)`

# Example 3, cont.



```
. predict yhat, xb

. predict uhat, resid

. sum yhat uhat

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        yhat |        12    125.8333    7.810006   115.4294   141.3411
        uhat |        12   -2.98e-08    11.90381  -26.27429   15.43694

. sum weight height

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      weight |        12    125.8333    14.23717        105        150
      height |        12    65.41667    3.315483         61         72
```

Things to note:

- Mean of the residuals ($\hat{u}$) is always zero.
- Mean of the predicted values ($\hat{y}$) is always $\bar{y}$

These follow algebraically from least squares.

# Measuring fit

How well does the regression line fit the data?

- Mechanically, the least squares intercept and slope can be calculated for any set of data points $(x, y)$
- The line of best fit (OLS) is not necessarily a *good* fit
- Least squares minimizes the sum of the squared residuals, but performs better with some data than others

$R^2$, the **coefficient of determination**, is a measure of the goodness of fit.

# $R^2$

$R^2$ is the proportion of the total variation in $y$ from its mean that is "explained" (predicted) by $x$.

Equivalently, it is the proportionate reduction in the variability of $y$ from its mean that can be achieved using the predicted, rather than the actual, $y$.

# $R^2$

The total variation in $y$ around its mean is the **total sum of squares (TSS)**:

$$TSS = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

Using the predicted $y$ instead of the actual, the **model sum of squares (SSM)** is:

$$SSM = \sum_{i=1}^{n} (\widehat{y_i} - \bar{y})^2$$

# $R^2$

The $R^2$ is therefore:

$$R^2 = \frac{SSM}{TSS}$$

$R^2$ is **always between 0 and 1**

The model sum of squares (SSM) is sometimes called the "explained" sum of squares.

## $R^2$

The "unexplained" variation in $y$ is the **sum of squared errors (SSE)**, seen before (a.k.a. the residual sum of squares SSR):

$$SSE = \sum_{i=1}^{n} (y_i - \widehat{y_i})^2$$

It makes sense that the $R^2$ should be related to the SSE, which we aim to minimize when finding the best fit line. In fact, we can write $R^2$ as:

$$R^2 = 1 - \frac{SSE}{TSS}$$

## $R^2$

Or:

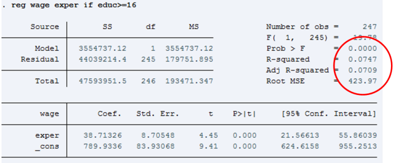
$$R^2 = 1 - \frac{SSE}{TSS} = \frac{TSS - SSE}{TSS}$$

Written this way, it's easy to see why $R^2$ is the proportionate reduction in prediction error:

- TSS would be your SSE if you simply used $\bar{y}$ to predict $y$.
- SSE is your sum of squared errors after fitting the regression line.
- (TSS-SSE) is your reduction in prediction error.
- (TSS-SSE)/TSS is this reduction expressed as a proportion of the original TSS.

# Example 1, cont.

Example using monthly earnings and experience:



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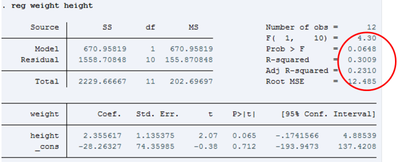
       _cons |   789.9336   83.93068     9.41   0.000     624.6158    955.2513
------------------------------------------------------------------------------
```

$$R^2 = \frac{SSM}{TSS} = 0.0747$$

# Example 3, cont.

Example using height and weight



```
. reg weight height

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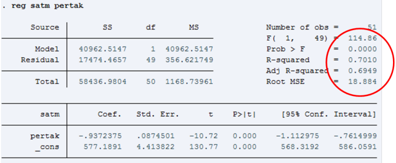
    Residual | 1558.70848    10  155.870848           R-squared     =  0.3009
-------------+------------------------------           Adj R-squared =  0.2310
       Total | 2229.66667    11   202.69697           Root MSE      =  12.485

------------------------------------------------------------------------------
      weight |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      height |   2.355617   1.135375     2.07   0.065    -.1741566     4.88539
       _cons |  -28.26327   74.35985    -0.38   0.712    -193.9473    137.4208
------------------------------------------------------------------------------
```

$$R^2 = \frac{SSM}{TSS} = 0.3009$$

## Example 4, cont.

Example using state SAT average scores



```
. reg satm pertak

      Source |       SS       df       MS              Number of obs =      51
-------------+------------------------------           F(  1,    49) =  114.86
       Model |  40962.5147     1  40962.5147           Prob > F      =  0.0000
    Residual |  17474.4657    49  356.621749           R-squared     =  0.7010
-------------+------------------------------           Adj R-squared =  0.6949
       Total |  58436.9804    50  1168.73961           Root MSE      =  18.884

------------------------------------------------------------------------------
        satm |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      pertak |  -.9372375   .0874501   -10.72   0.000    -1.112975   -.7614999
       _cons |   577.1891   4.413822   130.77   0.000     568.3192    586.0591
------------------------------------------------------------------------------
```

$$R^2 = \frac{SSM}{TSS} = 0.7010$$

## Mean squared error (MSE)

A related measure is the **mean squared error (MSE)**:

$$MSE = \frac{\sum_{i=1}^{n} \left(y_i - \widehat{y_i}\right)^2}{n-2} = \frac{\sum_{i=1}^{n} \widehat{u_i}^2}{n-2} = \frac{SSE}{n-2}$$

The MSE is the *average* squared deviation of the predicted $y$ from the actual $y$ (uses $n-2$ in the denominator). Note the numerator is the sum of squared errors (SSE).

Note: least squares minimizes $\sum \hat{u}_i^2$, so it also minimizes $MSE$

# Root mean squared error (RMSE)

The square root of the MSE is the **root mean squared error (RMSE)** or the "standard error of the estimate":

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \widehat{y}_i)^2}{n-2}} = \sqrt{\frac{\sum_{i=1}^{n}\widehat{u_i}^2}{n-2}} = \sqrt{\frac{SSE}{n-2}}$$

Just as the standard deviation can be interpreted (intuitively, not literally) as the average deviation of $y$ from its mean, the RSME can be interpreted (intuitively, not literally) as the average deviation of $y$ from its *predicted value*

# Other properties of fit measures

Some other mathematical properties of fit measures:

- $R^2$ is the square of the correlation between predicted and actual $y$ (call this $R$).
- In simple linear regression, $R = |r_{xy}|$ (the absolute value of the correlation between $y$ and $x$).
- $R$ is positive—a negative correlation between the predicted and actual y's would not make sense. (You would have a very bad prediction if your predictions moved in the opposite direction of the actual data).

# Side note on notation

- Up to this point I have used *a* and *b* to represent the intercept and slope of the best fit line.
- We may wish to use sample data on $(y, x)$ to make inferences about their relationship in the *population*. In this case we could denote $\alpha$ and $\beta$ as the intercept and slope in the population that we are seeking to estimate. *a* and *b* (or $\hat{\alpha}$ and $\hat{\beta}$) are then estimators of the population intercept and slope.
- "$\hat{y} = a + bx$" or "$\hat{y} = \hat{\alpha} + \hat{\beta}x$" are sometimes used to describe the *fitted* prediction equation, or the results of OLS.

# Conditional mean interpretation

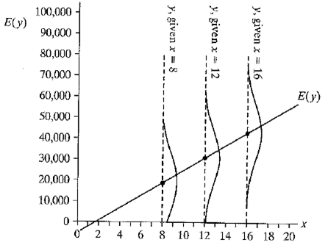
- A linear regression can always be used for description or prediction purposes.
- If the relationship between $x$ and the **mean** of $y$ in the population is **linear**, then the regression line is: $E(y|x) = \alpha + \beta x$
- Under this interpretation, the regression line provides us the **conditional mean** of $y$ for a given level of $x$.
- Again, only makes sense if the mean of $y$ is linearly related to $x$.

# Conditional mean interpretation

- Along the same lines, the standard deviation of $y$ for a given level of $x$ would be the **conditional standard deviation**.
- Assume for now that $y$ has a normal distribution at each level of $x$, and a constant standard deviation ($\sigma$)
- In this context the RMSE is treated as an estimate of the standard deviation of $y$ given $x$ (i.e., $\sigma$)

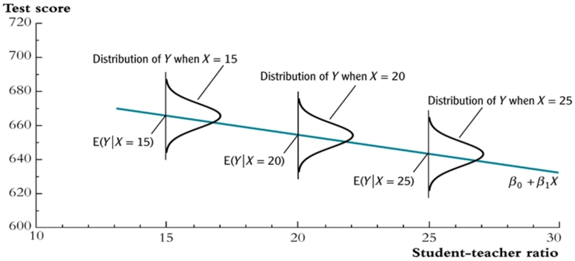
# Conditional mean interpretation

Example from Agresti and Finlay: earnings, conditional on years of education



# Conditional mean interpretation

Test score vs. student-teacher ratio:



# Inferences about $\beta$

If we are using a sample of $y$ and $x$ to *estimate* the intercept and slope coefficient in the population, we need to quantify our uncertainty in the same way we did for $\bar{x}$:

- Confidence intervals for $\beta$
- Hypothesis tests about $\beta$

To do so we'll need some information about the *sampling distribution* of the slope estimator ($b$). As was true for $\bar{x}$, this requires some assumptions:

## Inferences about $\beta$

1. The data points $(x, y)$ represent a **random sample** from the population.
2. In the population, the mean of $y$ conditional on $x$ can be written as: $E(y|x) = \alpha + \beta x$
3. The conditional variance of $y$ given $x$ is **constant** $(\sigma^2)$—called *homoskedasticity*
4. The conditional distribution of $y$ given $x$ is **normal** (this assures our sampling distribution is normal—can be relaxed with a large $n$).

## Inferences about $\beta$

To construct confidence intervals and conduct hypothesis tests for $\beta$, we need the **standard error** of $b$. When assumptions 1-3 above hold, this is:

$$se_b = \frac{RMSE}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

Note: can also be written

$$se_b = \sqrt{\frac{SSE/(n-2)}{(n-1)s_x^2}}$$

## Inferences about $\beta$

$$se_b = \sqrt{\frac{SSE/(n-2)}{(n-1)s_x^2}}$$

This expression makes clear what affects the standard error of $b$:

- $SSE$ (overall fit of the regression line)
- $s_x^2$ (overall variation in $x$)
- $n$ (sample size)

## Inferences about $\beta$

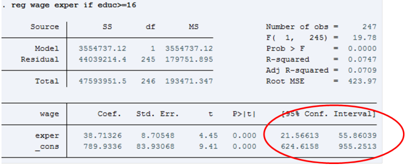
Under the above assumptions, the population slope $\beta$ will lie within $\pm t_{\alpha/2}$ standard errors of $b$ $(1-\alpha)$% of the time, using a $t$-distribution with $n-2$ degrees of freedom. So a $(1-\alpha)$% confidence interval for $\beta$ is:

$$b \pm t_{\alpha/2} \times se_b$$

This is given in the Stata output (for a 95% CI). Note the $\alpha$ significance level is not the same thing as the regression intercept, also sometimes denoted using $\alpha$.

# Example 1, cont.

Example using monthly earnings:



```
. reg wage exper if educ>=16

      Source |       SS       df       MS              Number of obs =     247
-------------+------------------------------           F(  1,   245) =   19.78
       Model |  3554737.12     1  3554737.12           Prob > F      =  0.0000
    Residual |  44039214.4   245  179751.895           R-squared     =  0.0747
-------------+------------------------------           Adj R-squared =  0.0709
       Total |  47593951.5   246  193471.347           Root MSE      =  423.97

------------------------------------------------------------------------------
        wage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       exper |   38.71326    8.70548     4.45   0.000     21.56613    55.86039
       _cons |   789.9336   83.93068     9.41   0.000     624.6158    955.2513
------------------------------------------------------------------------------
```

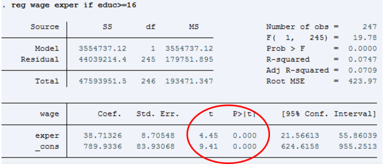
# Inferences about $\beta$

Under $H_0 : \beta = 0$, the test statistic $t = b/se_b$ has a $t$-distribution with $n - 2$ degrees of freedom. We can use this to test the alternative hypothesis that the slope is significantly different from zero.

The $t$-statistic and (two-sided) $p$-value are given in the Stata output. In this case we can reject $H_0$ and conclude that the slope coefficient is statistically different from zero.

## Example 1, cont.

Example using monthly earnings:



```
. reg wage exper if educ>=16

      Source |       SS       df       MS              Number of obs =     247
-------------+------------------------------           F(  1,   245) =   19.78
       Model |  3554737.12     1  3554737.12           Prob > F      =  0.0000
    Residual |  44039214.4   245  179751.895           R-squared     =  0.0747
-------------+------------------------------           Adj R-squared =  0.0709
       Total |  47593951.5   246  193471.347           Root MSE      =  423.97

------------------------------------------------------------------------------
        wage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       exper |   38.71326    8.70548     4.45   0.000     21.56613    55.86039
       _cons |   789.9336   83.93068     9.41   0.000     624.6158    955.2513
------------------------------------------------------------------------------
```

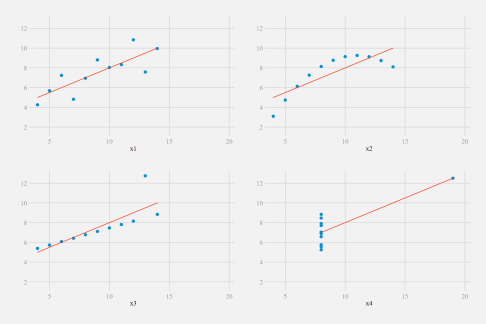
## Regression diagnostics

A few tests of basic assumptions underlying the simple linear regression:

- Does the linearity assumption appear to hold?
- Does the variance in the residuals appear *homoskedastic*?

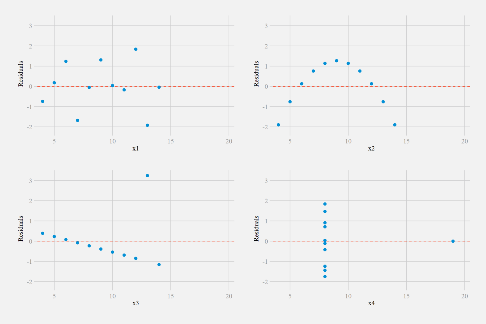
Consider the following four scatterplots (*anscombe.dta*). All four datasets contain $n = 11$ with the same estimated intercept and slope.

# Regression diagnostics



# Regression diagnostics

A **residual plot** can be revealing: plot $\hat{u}$ against $x$:

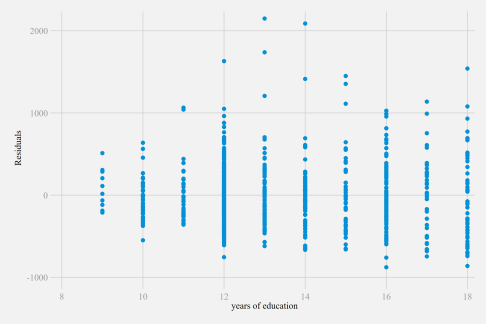


# Regression diagnostics

Observations:

- If the relationship is truly linear, the residuals should form a roughly rectangular band around zero.
- Nonlinear relationships will yield a different pattern (e.g. Case 2 above)
- Cases 3 and 4 above illustrate influence of an outlier
- If the vertical spread of the residuals varies with $x$, the homoskedasticity assumption may be unsupportable (e.g., a "funnel" shape is a common type of *heteroskedasticity*)

# Heteroskedasticity



# Heteroskedasticity

What is the consequence of heteroskedasticity? Most importantly, the standard error calculation for $b$ above is wrong. This means your inference is wrong (confidence intervals, significance tests, etc.) More on this in Regression I.

# Influential observations

The estimated intercept and slope in a regression can be heavily influenced by *outliers*. An "influential observation" is one that has a significant effect on the intercept and/or slope.

- Outlying data points with an $x$ value near $\bar{x}$ may affect the intercept but are not likely to affect the slope much.
- Outlying data points with an $x$ value far from $\bar{x}$ that do not stray far from the regression line are unlikely to affect the slope or intercept.

## Influential observations

Define $x - \bar{x}$ as *leverage*. Data points with large residuals and high leverage have the most influence on the estimated slope and intercept.

Cook's influence values are an index of the extent to which the data point influences the slope and intercept. To obtain these use `predict` ***varname*,** `cook`. As a rule of thumb, values $> 1$ have "large" influence.

## Next time

- When can regression be considered causal?
- Multivariate relationships and "controlling" for other variables (an introduction)