2. Describing Univariate Distributions (I)

LPO.8800: Statistical Methods for Education Research

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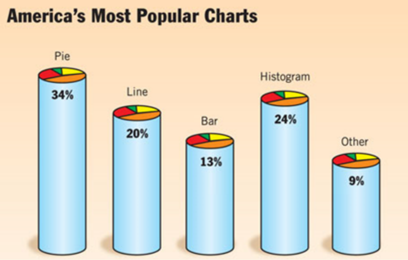
Last time

- Descriptive vs. inferential statistics
- Basic concepts: outcomes, variables, unit of observation, population, sample
- Measurement scales
  - quantitative or categorical
  - nominal, ordinal, interval, ratio
  - discrete vs. continuous
- Sampling methods

Today

- Stata introduction (in brief—see video for more)
- Describing univariate distributions: categorical and quantitative data
- Choice of statistical tools used to describe a variable depends in part on how it is measured



Today

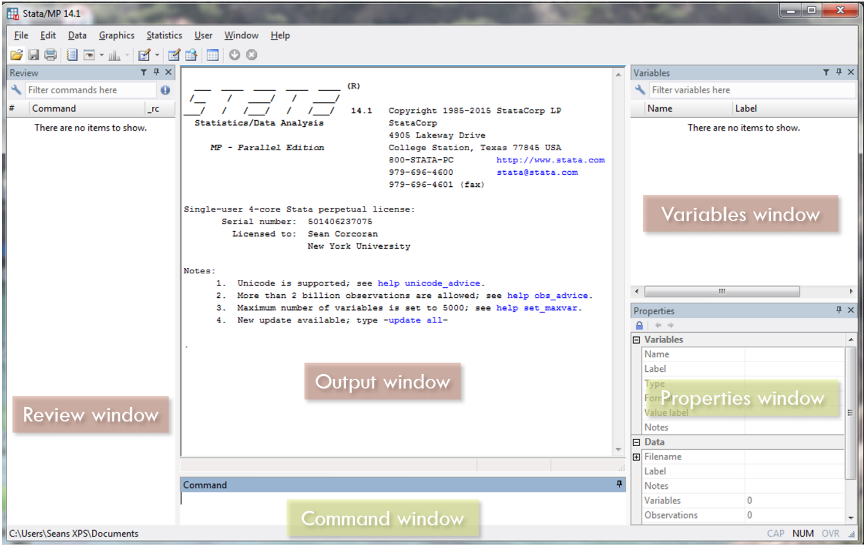
- Describing categorical variables
  - Frequency (and relative frequency) distributions
  - Bar graphs
  - Pie graphs
- Describing quantitative variables
  - Frequency (and relative frequency) distributions, possibly grouped
  - Histograms
  - Stem-and-leaf plot
  - Measures of central tendency

Stata introduction

Interacting with Stata

- Command window (*interactive* mode) vs. **do-file editor** (*batch* mode)
- Review window
- Variables and properties windows
- Results window
- Menu and task bar commands

Stata windows



Basic Stata tasks

- Opening and saving data (.dta) files
- Removing a data file from memory
- Data browser and editor
- Using Stata as a calculator
- Getting help
- *Variables* vs. cases/observations
- *Variable* labels vs. *value* labels

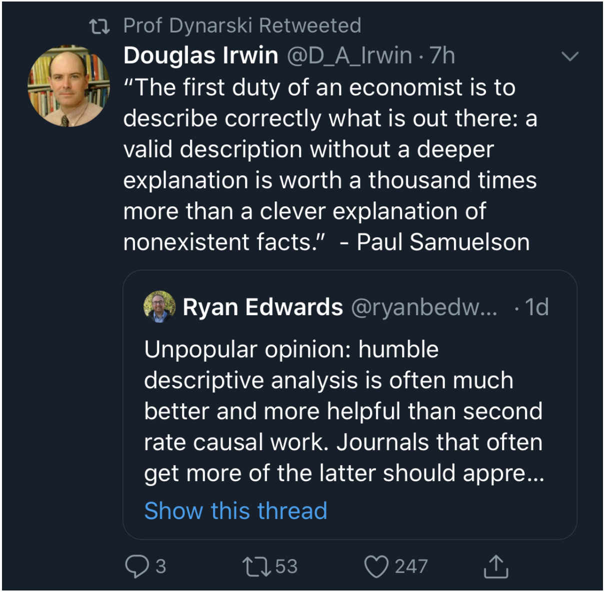
Stata command syntax

Example:

`summarize` *`varlist`* `[if] [in] [weight] [,options]`

- `summarize`: the command
- *`varlist`*: terms in italics are things you provide
- Syntax in brackets [] is *optional*
- Most commands have other options that are specified *after* a comma, at the very end
- `[if]`: execute the command only if a certain condition is true
- `[in]`: execute the command for a certain subset of observation numbers

Importance of descriptive analyses



Importance of descriptive analyses

See the IES report by Loeb et al. (2017), which describes the role of descriptive analysis in education and social science.

Key Themes

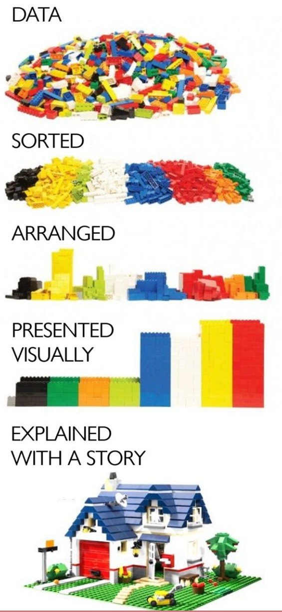
- Descriptive analysis characterizes the world or a phenomenon—answering questions about who, what, where, when, and to what extent. Whether the goal is to identify and describe trends and variation in populations, create new measures of key phenomena, or describe samples in studies aimed at identifying causal effects, description plays a critical role in the scientific process in general and education research in particular.
- Descriptive analysis stands on its own as a research product, such as when it identifies socially important phenomena that have not previously been recognized. In many instances, description can also point toward causal understanding and to the mechanisms behind causal relationships.
- No matter how significant a researcher's findings might be, they contribute to knowledge and practice only when others read and understand the conclusions. Part of the researcher's job and expertise is to use appropriate analytical, communication, and data visualization methods to translate raw data into reported findings in a format that is useful for each intended audience.

Importance of descriptive analyses

Some examples of excellent, highly influential descriptive studies in education:

- Arnold et al. (2009) - on "summer melt"
- Scott-Clayton (2012) - changes over time in undergraduates' propensity to work while in school
- Reardon (2011) - on changes in the academic achievement gap between high- and low-income students
- Lankford, Loeb, & Wyckoff (2002) - on the distribution of teacher qualities across New York State districts and schools
- Hoxby & Avery (2003) - on the "missing one-offs"
- Murnane (2013) - on long-run trends in U.S. HS graduation rates

Importance of descriptive analyses



Frequency distributions

- A **frequency distribution** is a table showing the number (count) of occurrences of each unique outcome in the data.

- The **relative frequency** of an outcome or category is the *proportion* or *percentage* of all observations in that category. (Must sum to one, or 100%).

Frequency distributions

Example: test scores for 16 students

Table: Classroom test scores

Score	Frequency (Count of students)
5	2
10	3
12	1
15	4
20	4
25	2
Total	16

Frequency distributions

Example: test scores for 16 students

Table: Classroom test scores

Score	Frequency (Count of students)	Relative Frequency (Percentage)
5	2	$2/16 * 100 = 12.5\%$
10	3	$3/16 * 100 = 18.75\%$
12	1	$1/16 * 100 = 6.25\%$
15	4	$4/16 * 100 = 25.0\%$
20	4	$4/16 * 100 = 25.0\%$
25	2	$2/16 * 100 = 12.5\%$
Total	16	$16/16 * 100 = 100\%$

Frequency distributions

Side note on terminology:

- The fractions in the above relative frequency distribution: 2/16, 3/16, 1/16—or, 0.125, 0.1875, 0.0625, etc.—are **proportions**: the frequency of cases in a given category divided by the total number of cases in all categories. Ranges between zero and one.
- Multiply by 100 to get **percentages** (12.5%, 18.75%, 6.25%, etc.)
- Relative frequencies can be expressed either way

Pro tip: see handout on **rounding conventions**. Percentages are typically rounded to one decimal place; proportions typically rounded to three.

Frequency distributions

- Because frequency distributions list every distinct value in the data, they are only practical for variables with a limited number of unique values
  - Categorical variables
  - Discrete quantitative variables
- It is possible to *group* variables with many distinct outcomes into smaller categories (shown later)
  - Continuous quantitative variables

Frequency distributions

Easy to generate in Stata using `tabulate`

```
. tabulate region
```

geographic region of school	Freq.	Percent	Cum.
northeast	106	21.20	21.20
north central	151	30.20	51.40
south	150	30.00	81.40
west	93	18.60	100.00
Total	500	100.00	

Frequency distributions

Note the *cumulative* column is not very meaningful here (a categorical, non-ordered variable).

```
. tabulate region

 geographic |
  region of |
     school |      Freq.     Percent        Cum.
------------+-----------------------------------
  northeast |        106       21.20       21.20
north central |      151       30.20       51.40
      south |        150       30.00       81.40
       west |         93       18.60      100.00
------------+-----------------------------------
      Total |        500      100.00
```

Frequency distributions

In SPSS: "valid percent" expresses relative frequency as a percentage of all *non-missing* observations. In this example there is a **missing value code** of 98.

Parents' Marital Status in Eighth Grade

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Divorced	26	5.2	5.5	5.5
	Widowed	6	1.2	1.3	6.7
	Separated	8	1.6	1.7	8.4
	Never Married	5	1.0	1.0	9.4
	Marriage-Like Relationship	5	1.0	1.0	10.5
	Married	427	85.4	89.5	100.0
	Total	477	95.4	100.0	
Missing	98	23	4.6		
Total		500	100.0		

Frequency distributions

To show the count of missing values in Stata, include the `missing` option:
`tabulate` *varname*`, missing`

```
. tabulate parmar18, missing
```

parents' marital status in eighth grade	Freq.	Percent	Cum.
divorced	26	5.20	5.20
widowed	6	1.20	6.40
separated	8	1.60	8.00
never married	5	1.00	9.00
marriage-like relationship	5	1.00	10.00
married	427	85.40	95.40
.	23	4.60	100.00
Total	500	100.00	

Frequency distributions

The `table` command provides a simpler frequency distribution (without the percent or cumulative percent). This command allows for many options for customizing the contents of the table.

```
. table parmar18
```

parents' marital status in eighth grade	Freq.
divorced	26
widowed	6
separated	8
never married	5
marriage-like relationship	5
married	427

Frequency distributions

Note the categories used in each of these variables are mutually exclusive and collectively exhaustive:

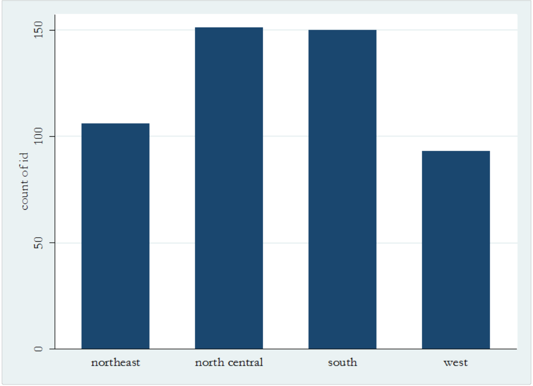
- **mutually exclusive**: being in one category precludes being in another
- **collectively exhaustive**: all possible categories are represented by the defined categories

Bar graphs

- **Bar graphs** are a visual way to display frequency (or relative frequency) distributions
- In Stata: `graph bar` can be used for a frequency (or relative frequency) distributions. An alternative is `histogram` with the `discrete` option.
- Try: *region, parmarl8, advmath8*

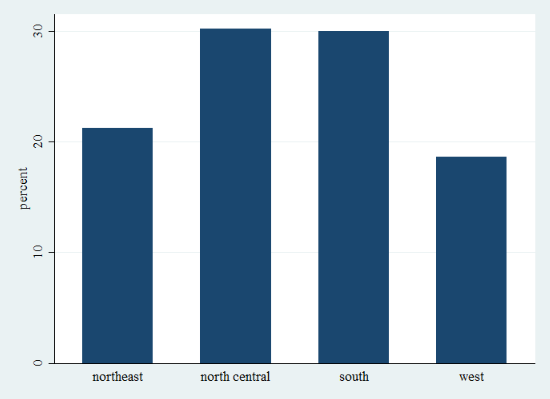
Bar graphs

`graph bar (count), over(`*`region`*`)`



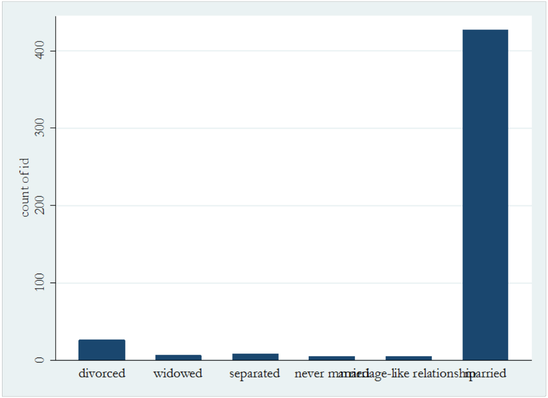
Bar graphs

`graph bar (percent), over(`*`region`*`)`



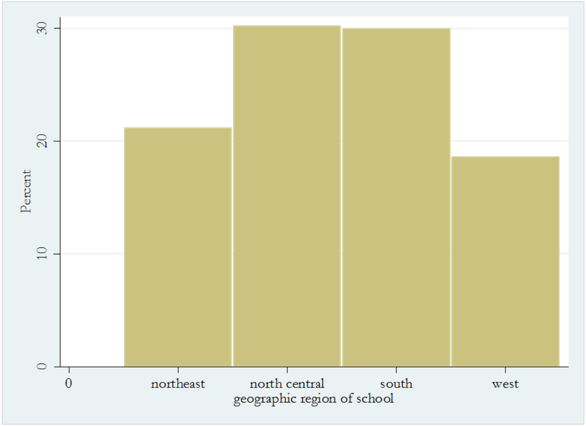
Bar graphs

```
graph bar (count), over(parmar18)
```



Bar graphs

```
histogram region, discrete percent gap(2) xlabel( ,
valuelabel)
```



Bar graphs

- Word of caution with bar graphs: always check where vertical axis begins—does it begin at zero? Avoid misleading scales
- Note "bar graphs" (as opposed to histograms) have gaps between the bars, suggestive of distinct categories
- Note The `gap(2)` option in `histogram` forces a gap of size 2

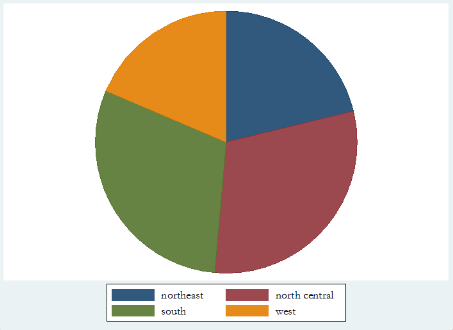
Pie graphs

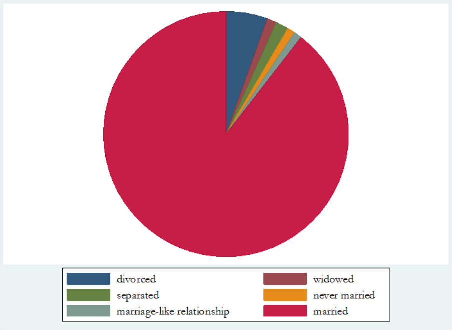
Pie graphs can be used to show the relative frequency of a variable

- These only make sense when the variable has *collectively exhaustive* (and a limited number of) categories
- In Stata: `graph pie`
- Try: *region, parmarl8*

Pie graphs

graph pie, over(*region*)





Pie graphs

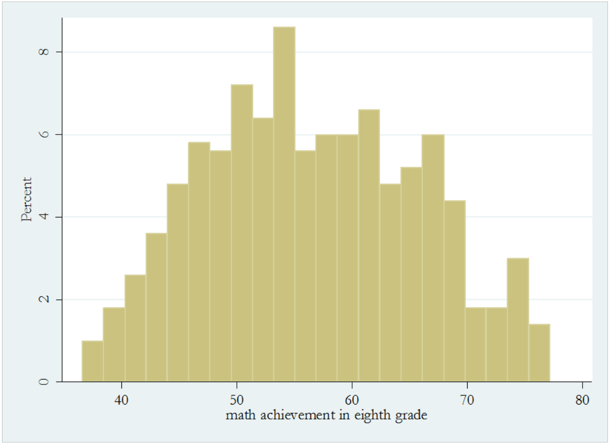
Histograms

A **histogram** is a bar graph where the height of each bar represents the count (or percent) of observations within a given *range* of values, called an **interval** or **bin**

- Number of bins determined by default in Stata, but can be adjusted
- Obviously makes sense only for interval (or ratio) measured variables, where a range of values is meaningful
- In Stata: `histogram`

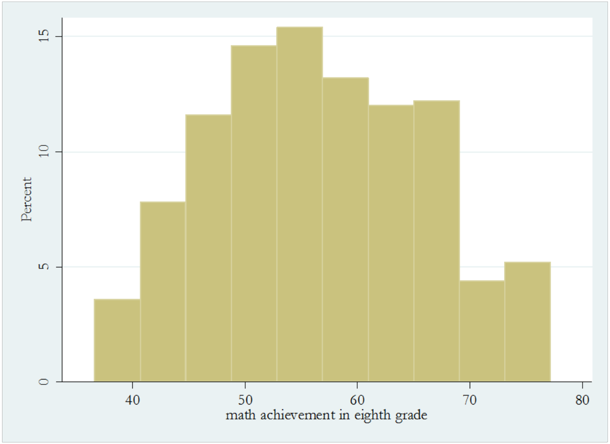
Histograms

`histogram achmat08, percent`



Histograms

`histogram` *`achmat08`*`, percent bin(10)`



Stem-and-leaf plot

A **stem-and-leaf** plot is similar to a histogram, but provides a bit more detail on specific values in the data

- Leading digits are called "stems"
- Trailing digits are called "leaves"
- The number of leaves corresponds to the frequency of a particular value

Stem-and-leaf plot

```
Stem-and-leaf plot for achmat08 (math achievement in eighth grade)

achmat08 rounded to nearest multiple of .1
plot in units of .1

  36* | 6
  37* | 122
  38* | 478
  39* | 0357
  40* | 0225555799
  41* | 34689
  42* | 0335577
  43* | 000133445689
  44* | 00012244467788
  45* | 334467778899
  46* | 3445555667788889
  47* | 0022344456677888
  48* | 00000133667889
  49* | 02223344466889
  50* | 001112356677888899
  51* | 0000111233334455779
  52* | 11111222333445566 8889
  53* | 000022456677889
  54* | 00011223334445555778888999
  55* | 00000003345678899
  56* | 112233467778888889
  57* | 0000112233344444778
  58* | 001133346778888899999
  59* | 034446679
  60* | 0012233345567778 88
  61* | 000112333345689
  62* | 000111123334555889
  63* | 2223677777779
  64* | 011225667888899
  65* | 223455677889
  66* | 0001122333344667799
  67* | 122333456677889
  68* | 00222346666899
  69* | 0122446779
  70* | 0114899
  71* | 389
  72* | 146
  73* | 3335788
  74* | 02233334
  75* | 00014
  76* |
  77* | 222222

.
```

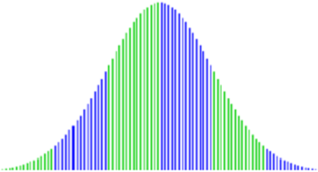
Stem-and-leaf plot

- In the above example, Stata rounded to the nearest 0.1
- First two digits used as stem, and final digit used as leaf
- Leaves correspond to frequency of particular values
- Compare the shape of the stem-and-leaf plot for *actmat08* to that of the histogram

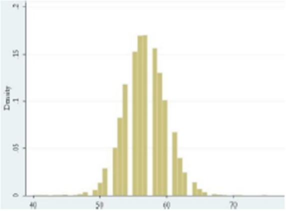
Shape of distributions

- The histogram and stem-and-leaf plots are revealing about the *shape* of a distribution—i.e. which values tend to be more or less frequent
- A distribution is **symmetric** if the distribution of outcomes is identical (or approximately identical) on either side of its central value
- Examples: normal distribution (bell curve), U-shaped distribution, bi-modal distribution, uniform distribution

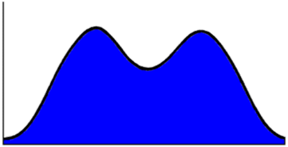
Shape of distributions

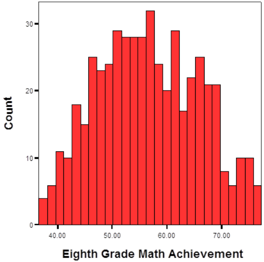


Normal-shaped distribution



Height in inches – age 10-11 in NYC



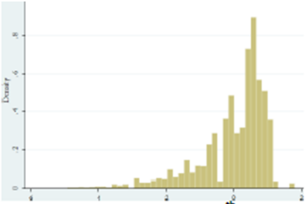


Shape of distributions

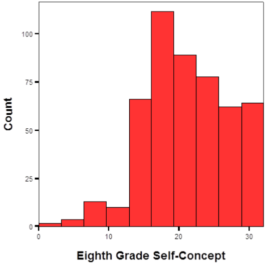
- A distribution is **skewed left** (or **negatively skewed**) if the distribution has a long tail to the left of its central value
- A distribution is **skewed right** (or **positively skewed**) if the distribution has a long tail to the right of its central value

Left-skewed distributions

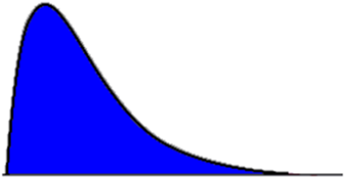
Left-skewed distribution



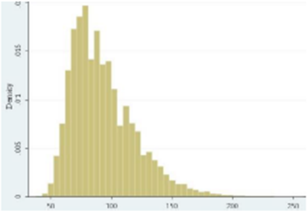
Texas standardized math scores, 5th grade (2000)



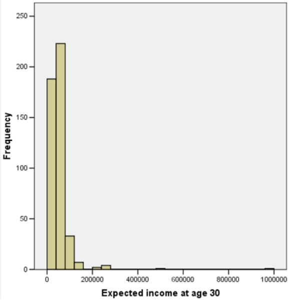
Right-skewed distributions



Right-skewed distribution



Weight in pounds – age 10-11 in NYC



Grouped frequency distributions

- Again, frequency distributions are less useful for variables with *many* distinct possible values (e.g. continuous variables)

- For continuous variables, one could create a smaller number of equal width groups (bins, or intervals) and then create a frequency distribution for this grouped ("re-coded") variable. (This is what the histogram does).

Grouped frequency distributions

- NELS example: *unitmath* is the number of units of high school math taken, and ranges from 1-6. Includes many fractional units.

- Can set up groups or intervals, for example:
  - $1 \leq x < 2$
  - $2 \leq x < 3$
  - $3 \leq x < 4$, and so on

- Lecture 3 will show how to re-code variables in this way (a kind of data transformation)

Measures of central tendency

Measures of central tendency characterize the “center” or “location” of a distribution, its “typical” or “expected” value. Examples:

- Mean
- Median
- Mode

Mean

The **mean** (or *average*) adds all of the observed values and divides by the number of observations n

- Let $x_1, x_2, x_3, ..., x_n$ represent the n values of a variable x (x_i is the ith observation, and i is the *index*)

- Then the **mean** is: $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

Mean

Example: calculating the mean number of wins for baseball teams

Table 2.2: American League Standings, July 28, 2013

East	W	L	PCT
Tampa Bay	62	42	0.596
Boston	62	43	0.590
Baltimore	58	47	0.552
NY Yankees	54	50	0.519
Toronto	47	56	0.456
Central	**W**	**L**	**PCT**
Detroit	58	45	0.563
Cleveland	55	48	0.534
Kansas City	50	51	0.495
Minnesota	45	56	0.446
Chi White Sox	40	61	0.396
West	**W**	**L**	**PCT**
Oakland	61	43	0.587
Texas	56	48	0.538
Seattle	49	55	0.471
LA Angels	48	54	0.471
Houston	35	68	0.340
$\sum W$	780		

Mean

Example: calculating the mean number of wins for baseball teams

$$\overline{W} = \frac{\sum W_i}{n} = \frac{780}{15} = 52$$

Mean

The mean in Stata can be calculated using several commands, including `summarize` (or `sum`).

```
. sum achmat08

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
    achmat08 |       500    56.59102    9.339608      36.61       77.2
```

```
. sum expinc

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
    expinc30 |       459    51574.73    58265.76          0    1000000
```

Mean

The mean of a categorical variable is usually meaningless, except in the case of a *dichotomous* variable coded 0-1, in which case the mean is the proportion equal to 1:

```
. sum advmath8

    Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
    advmath8 |        491    .4602851    .4989286          0          1
```

Mean

The mean is highly influenced by extreme values or **outliers**: observations that fall well above or well below the bulk of the data.

- Example 1 ($n = 15$)
  - 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5
  - mean $= (55/15) = 3.67$

- Example 2 ($n = 15$)
  - 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5000000
  - mean $= (5000050/15) = 333,336.67$

Mean

The mean can be characterized as the "center of gravity," or balance point, of the distribution. It is the point at which the sum of the distances to the mean from observations *above* the mean equal the sum of the distances to the mean from observations *below* the mean

Mean

Deviations from the mean:

Table 2.3: Deviations above and below the mean

	W	$W - \bar{W}$	Totals
Tampa Bay	62	10	
Boston	62	10	
Oakland	61	9	
Baltimore	58	6	
Detroit	58	6	
Texas	56	4	
Cleveland	55	3	
NY Yankees	54	2	50
Kansas City	50	-2	
Seattle	49	-3	
LA Angels	48	-4	
Toronto	47	-5	
Minnesota	45	-7	
Chi White Sox	40	-12	
Houston	35	-17	-50

Mean

The **least squares principle**:

- The *average deviation* of x from its mean will always be zero. That is, the sum of negative deviations from the mean will always equal the sum of positive deviations from the mean.

- The mean is the point in a distribution around which the variation is minimized (as indicated by the *squared* differences): $\sum (x_i - \bar{x})^2$

Median

The **median** is the observation that falls in the middle of the data, when the observations are ordered from lowest to highest values.

- When n is *odd*: a single value will fall in the middle

- When n is *even*: the median is the midpoint of the two middle values

- Alternatively, index the ordered n values from 1 to n. The median will be the value with index $(n+1)/2$

Median

- Example 1 ($n = 15$):
  - 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5
  - median $= 4$
- Example 2 ($n = 6$):
  - 0, 1000, 1000, 5000, 6000, 8000
  - median $= (1000 + 5000)/2 = 3000$

Median

The median in Stata can be calculated using several commands, including `summarize` (or `sum`) with the `detail` option:

```
. sum achmat08, detail

                math achievement in eighth grade
-------------------------------------------------------------
      Percentiles      Smallest
 1%        38.55          36.61
 5%        41.89          37.14
10%       44.185           37.2       Obs                 500
25%        49.42          37.24       Sum of Wgt.         500

50%        56.18                      Mean           56.59102
                        Largest       Std. Dev.      9.339608
75%        63.74           77.2
90%       68.935           77.2       Variance       87.22827
95%        73.33           77.2       Skewness       .1133238
99%         77.2           77.2       Kurtosis       2.242742
```

Median

Because the median is simply the middle value, it is insensitive to extreme values or outliers in the distribution

- Example 1 ($n = 15$):
  - 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5000000
  - median $= 4$

Mode

The **mode** is the outcome that occurs most often in a distribution

- Most appropriate for highly discrete variables, such as categorical variables

- Variables with lots of unique values (such as continuous variables) tend to have few repeats, and thus the mode is not that meaningful

- There is no command in Stata specifically for obtaining the mode. However, can use a frequency distribution or other combinations of commands (like `egen`).

Mode

```
. ** Find mode using tabulate with sort (for descending sort)

. tabulate famsize, sort
```

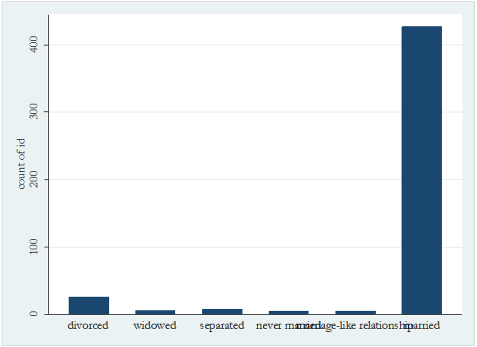
family size	Freq.	Percent	Cum.
4	199	39.80	39.80
5	142	28.40	68.20
6	55	11.00	79.20
3	52	10.40	89.60
7	21	4.20	93.80
9	13	2.60	96.40
2	9	1.80	98.20
8	9	1.80	100.00
Total	500	100.00	

```
. ** Find mode using egen (note you will get a message if >1 mode)

. egen mode=mode(famsize)

. table mode
```

mode	Freq.
4	500

Mode

"Married" is the modal parents' marital status. One might say the "typical" 8th grade student in the NELS has married parents.



Mode

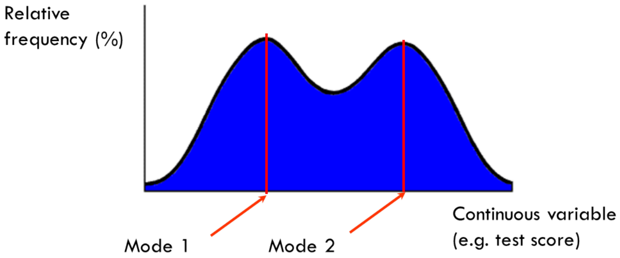
North Central is (technically) the modal region. But region here is more accurately described as *bimodal*—it has a distribution with two values that occur most often (North Central and South).

```
. tabulate region
```

geographic region of school	Freq.	Percent	Cum.
northeast	106	21.20	21.20
north central	151	30.20	51.40
south	150	30.00	81.40
west	93	18.60	100.00
Total	500	100.00	

Mode

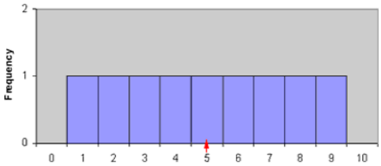
Another *bimodal* distribution



Mode

Some problems with using the mode:

- It is not very useful for "flat" distributions (e.g. the *uniform* distribution)
- Example 1: 1, 1, 2, 2, 3, 3, 4, 4, 5, 5
- Example 2:



Comparing measures of central tendency

How will the mode, median, and mean usually compare? It depends on the shape of the distribution.

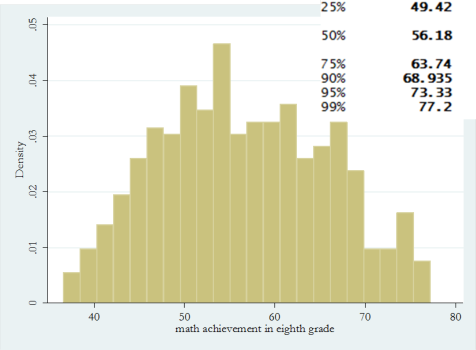
- For *symmetric* distributions the median $\approx$ mean

- If symmetric and *unimodal*, median $\approx$ mean $\approx$ mode (when the mode is meaningful, distributions with a limited number of unique values)

Comparing measures of central tendency

Example: note the mode of this distribution is 72 (not shown). Illustrates the problem of using the mode with a continuous variable.

```
. sum achmat08, detail

                 math achievement in eighth grade
-------------------------------------------------------------
      Percentiles      Smallest
 1%        38.55          36.61
 5%        41.89          37.14
10%       44.185           37.2       Obs                 500
25%        49.42          37.24       Sum of Wgt.         500

50%        56.18                      Mean           56.59102
                        Largest       Std. Dev.      9.339608
75%        63.74           77.2
90%       68.935           77.2       Variance       87.22827
95%        73.33           77.2       Skewness       .1133238
99%         77.2           77.2       Kurtosis       2.242742
```



Comparing measures of central tendency

How will the mode, median, and mean usually compare? It depends on the shape of the distribution.

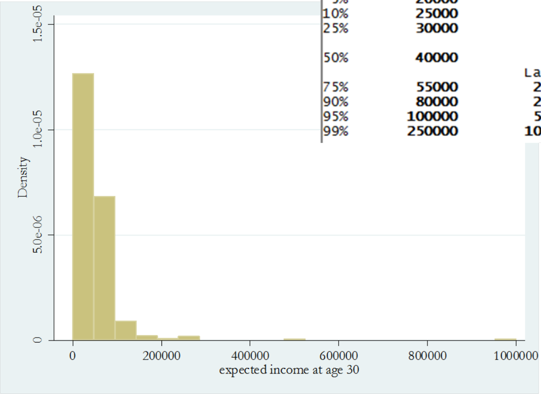
- For *right-skewed* distributions the mean $>$ median

- For *left-skewed* distributions the mean $<$ median

Comparing measures of central tendency

A right-skewed variable with some large positive outliers:

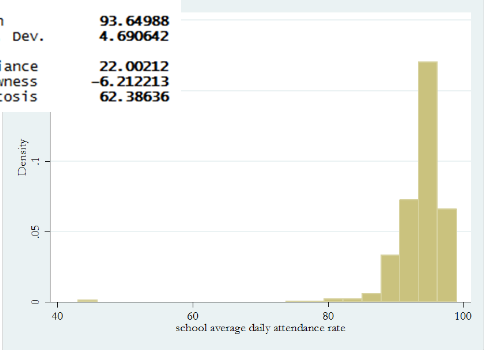
```
. sum expinc, detail

                   expected income at age 30
-------------------------------------------------------------
      Percentiles      Smallest
 1%            1              0
 5%        20000              0
10%        25000              0       Obs                 459
25%        30000              0       Sum of Wgt.         459

50%        40000                      Mean           51574.73
                        Largest       Std. Dev.      58265.76
75%        55000         250000
90%        80000         250000       Variance       3.39e+09
95%       100000         500000       Skewness       10.89864
99%       250000        1000000       Kurtosis       162.8027
```



Comparing measures of central tendency

A left-skewed variable with some low-value outliers:

```
. sum schattrt, detail

              school average daily attendance rate
-------------------------------------------------------------
      Percentiles      Smallest
 1%           80             43
 5%           88             45
10%           90             75       Obs                 417
25%           93             78       Sum of Wgt.         417

50%           95                      Mean           93.64988
                        Largest       Std. Dev.      4.690642
75%           96             98
90%           97             99       Variance       22.00212
95%           98             99       Skewness      -6.212213
99%           98             99       Kurtosis       62.38636
```



Measuring the Black-white wealth gap

The True Cost of Closing the Racial Wealth Gap

https://www.nytimes.com/2021/04/30/business/racial-wealth-gap.html

Using data from the 2019 Survey of Consumer Finances:

- Median Black household wealth: $24,100
- Median white household wealth: $188,200
- Gap: **$164,100**

Measuring the Black-white wealth gap

The True Cost of Closing the Racial Wealth Gap

`https://www.nytimes.com/2021/04/30/business/racial-wealth-gap.html`

Using data from the 2019 Survey of Consumer Finances:

- Median Black household wealth: $24,100
- Median white household wealth: $188,200
- Gap: **$164,100**
- Mean Black household wealth: $142,500
- Mean white household wealth: $983,400
- Gap: **$840,000**
- 97% of white households' total wealth is held by households above the median

Alternative command

An alternative command in Stata for measures of central tendency (and other statistics):

- `tabstat` *achrdg**, `stat(mean p50 n)`
- `tabstat` *achrdg**, `stat(mean p50 n) col(stat)`

```
. tabstat achrdg* , stat(mean p50 n)

   stats |  achrdg08  achrdg10  achrdg12
---------+------------------------------
    mean |  56.04906  56.11404  55.60188
     p50 |    56.445    57.545    57.005
       N |       500       500       500
----------------------------------------

. tabstat achrdg* , stat(mean p50 n) col(stat)

    variable |      mean       p50         N
-------------+------------------------------
    achrdg08 |  56.04906    56.445       500
    achrdg10 |  56.11404    57.545       500
    achrdg12 |  55.60188    57.005       500
--------------------------------------------
```

A bit more on the summation operator

The mean is written as: $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

- $\sum$ is the summation operator
- i is the index of summation
- 1 and n are the lower and upper limit of the summation (i.e., summing the numbers x_i for all values of i from 1 to n)

Three properties of the summation operator

The summation operator has three properties:

- For any constant c: $\sum_{i=1}^{n} c = nc$
- For any constant c: $\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$
- For any constants a and b: $\sum_{i=1}^{n}(ax_i + by_i) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i$

What not to do with the summation operator

Note the following, which are **not** properties of the summation operator:

$$\sum_{i=1}^{n} \frac{x_i}{y_i} \neq \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i}$$

$$\sum_{i=1}^{n} x_i^2 \neq \left(\sum_{i=1}^{n} x_i \right)^2$$

Example of a double summation

Consider two sets of numbers $x_1, ..., x_n$ and $y_1, ..., y_n$. Use the index of summation i for x and the index j for y. The following is an example of a double summation:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j$$

This can be written:

$$\sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j = x_1(y_1 + ... + y_n) + x_2(y_1 + ... + y_n) + ...$$

Or:

$$x_1 y_1 + x_1 y_2 + x_1 y_3 + ... + x_2 y_1 + x_2 y_2 + x_2 y_3 + ...$$

Next lecture

- Univariate descriptive statistics, continued: measures of variability/dispersion, and skewness
- Measures of position in a distribution (e.g., percentiles, z-scores)
- Data transformations, and effects on descriptive statistics