10. Bivariate covariance and correlation

LPO.8800: Statistical Methods in Education Research

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Last time: Tests for comparing two groups

- Confidence interval and test for difference in two population means (μ_1 and μ_2)
- Confidence interval and test for difference in two population proportions (π_1 and π_2)
- Independent vs. dependent samples
- Paired sample t-test
- Statistical power for two-sample tests

Introduction

Most everything we have done thus far relates to *univariate* distributions:

- Descriptive statistics $(e.g., \bar{x}, s^2)$
- Transformations (e.g., z-scores, log)
- Probability distributions (e.g., normal, binomial, uniform) and population parameters $(e.g., \mu, \sigma^2)$
- Random sampling and inferences about population parameters (confidence intervals, hypothesis testing)

Introduction

The most interesting applications of statistics examine the relationship between two or more variables:

- Do women earn less than men?
- Does education increase earnings?
- Does smoking cause lung cancer?
- Do students learn more in small classes than in large ones?
- Does lower birth weight increase the risk of poor health outcomes later in life?

These are examples of **bivariate analyses**. In the last lecture we saw one type of bivariate analysis, where one variable (the group identifier) was binary.

Introduction

Today we will turn to *bivariate* distributions and measures of association, or covariance.

Review of univariate probability distributions

A probability distribution for X assigns probabilities to values of X or intervals of values. PDFs come in discrete and continuous types, depending on the nature of the random variable.

The PDF $f(x_i)$ for a discrete random variable X provides the probability that $X = x_i$ for possible values of X:

$$f(x_i) = P(X = x_i)$$

The PDF $f(x)$ for a continuous random variable X provides the probability that X is between certain values. E.g., between a and b:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Review of univariate probability distributions

The probabilities in a discrete PDF are nonnegative and must sum to one. If there are N possible outcomes indexed by i:

$$\sum_{i=1}^{N} P(X = x_i) = 1$$

The area under a continuous PDF (i.e., the integral) must equal one:

$$P(-\infty \leq X \leq +\infty) = \int_{-\infty}^{+\infty} f(x)dx = 1$$

Review of univariate probability distributions

Two important features of a PDF are its **expected value** (mean) and **variance**, the first and second population *moments*.

For a discrete random variable X with N unique outcomes indexed by i, the expected value of X is:

$$E(X) = \mu_X = \sum_{i=1}^{N} x_i P(x_i)$$

The variance of X is:

$$Var(X) = \sigma_X^2 = E(X - E(X))^2 = \sum_{i=1}^{N} (x_i - E(X))^2 P(x_i)$$

Review of univariate probability distributions

For a continuous random variable X, the expected value of X is:

$$E(X) = \mu_X = \int_{-\infty}^{+\infty} xf(x)dx$$

The variance of X is:

$$Var(X) = \sigma_X^2 = E(X - E(X))^2 = \int_{-\infty}^{+\infty} (x - E(X))^2 f(x)dx$$

Joint probability distributions

A *joint* probability distribution for two random variables X and Y assigns probabilities to values of (X, Y) or to intervals of values. Like univariate PDFs, joint PDFs are discrete or continuous.

The PDF $f(x, y)$ for a pair of discrete random variables X and Y provides the probability that $X = x$ and $Y = y$ for possible values x and y:

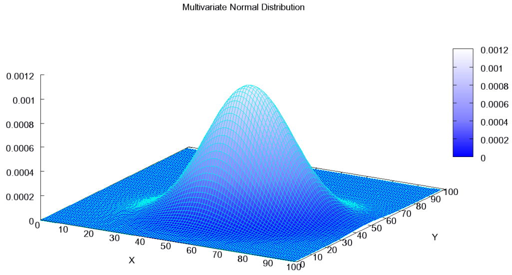
$$f(x, y) = P(X = x \text{ and } Y = y)$$

The PDF for continuous random variables X and Y provides the probability that X is between a and b and Y is between c and d:

$$\int_a^b \int_c^d f(x, y)dxdy$$

Example: bivariate normal distribution

An example joint PDF: bivariate normal distribution



Joint probability distributions

The probabilities in a discrete joint PDF are nonnegative and must sum to one. If there are N possible outcomes for X indexed by i, and M possible outcomes for Y indexed by j:

$$\sum_{i=1}^{N}\sum_{j=1}^{M} P(X = x_i \text{ and } Y = y_i) = 1$$

The area under a continuous joint PDF (i.e., the integral) must equal one:

$$P(-\infty \leq X \leq +\infty \text{ and } -\infty \leq Y \leq +\infty) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)dydx = 1$$

Measures of association

Bivariate probability distributions allow us to think about how two variables are associated:

- *Direction*: are the variables positively correlated, negatively correlation, or uncorrelated?
- *Shape*: is the relationship between the two variables *linear* or *nonlinear*?
- *Strength*: is the relationship between the two variables strong, weak, or moderate?

Covariance

The population covariance between two random variables X and Y is:

$$Cov(X, Y) = \sigma_{XY} = E\left[(X - E(X))(Y - E(Y))\right]$$

For a discrete joint PDF:

$$Cov(X, Y) = \sigma_{XY} = \sum_{x}\sum_{y}(X - E(X))(Y - E(Y))f(x, y)$$

For a continuous joint PDF:

$$Cov(X, Y) = \sigma_{XY} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(X - E(X))(Y - E(Y))f(x, y)dydx$$

Covariance

The covariance is like a weighted average of products: X's deviation from its mean multiplied by Y's deviation from its mean.

- If Y tends to be higher than average when X is higher than average, these products will tend to be positive (a **positive covariance**).
- If Y tends to be *lower* than average when X is higher than average, these products will tend to be negative (a **negative covariance**).

Covariance

Some facts about covariance:

- The magnitude of the covariance depends on the units of X and Y
- $Cov(X, Y) = E(XY) - E(X)E(Y)$
- If X and Y are independent, $Cov(X, Y) = 0$
- Independence implies zero covariance but the converse is not true.
- Covariance is a measure of *linear* association
- If $Y = X$, $Cov(X, X) = Var(X)$. (The covariance of a variable with itself is just its variance).

Correlation

Correlation is a standardized, or unit-free measure of covariance:

$$Corr(X, Y) = \rho_{XY} = E\left[\left(\frac{X - E(X)}{\sigma_X}\right)\left(\frac{Y - E(Y)}{\sigma_Y}\right)\right]$$

$$= \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

Correlation

Some facts about correlation:

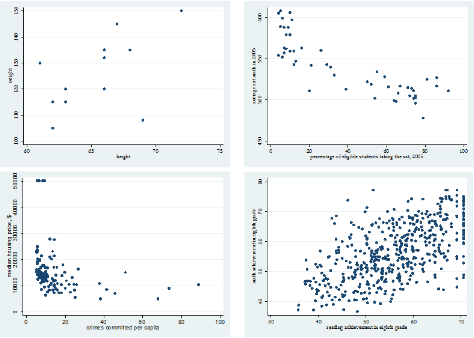
- ρ is a measure of *linear* association
- $-1 \leq \rho_{xy} \leq 1$
  - $\rho_{xy} = 1$ is a perfect positive correlation
  - $\rho_{xy} = -1$ is a perfect negative correlation
  - $\rho_{xy} = 0$ is no correlations
- ρ_{xy} requires both σ_x and σ_y to be positive (i.e., not zero).

Scatter diagrams

The easiest way to see how two variables are associated using data is via a *scatter diagram* or *scatter plot*.

- In Stata: `scatter` *yvar xvar*
- Appropriate for variables that are at least interval measured
- Can provide a sense of *direction* of relationship (if any), *linearity*, and *strength* of association

Scatter diagrams

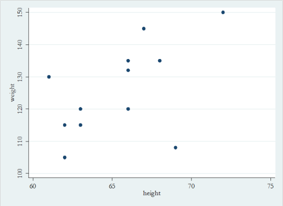


Scatter diagrams

Often with scatter diagrams there is a natural **response** or **outcome**, and **explanatory** variable. We may have in mind a theory in which variation in the response is (at least in part) explained by variation in the explanatory variable.

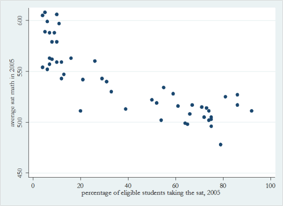
- Denote the outcome as Y and explanatory variable as X.
- Some use the terms "dependent" and "independent" variables. I prefer *not* to use these, given these terms' other meanings in statistics.

Scatter diagram 1: height and weight



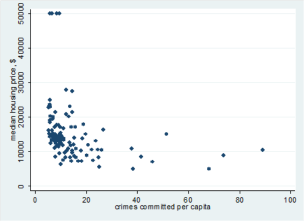
- Positive association
- Mostly linear–a line would fit the points quite well
- Moderately strong association

Scatter diagram 2: percent taking SAT and scores



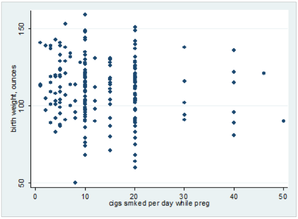
- Negative association
- Quite linear
- Strong association

Scatter diagram 3: crime rate and median house price



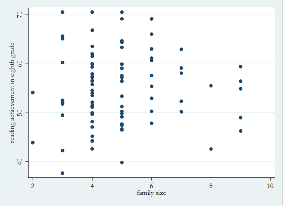
- Negative association
- *Nonlinear*
- Strong nonlinear association

Scatter diagram 4: maternal smoking and birthweight



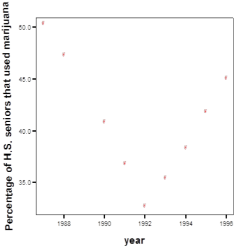
- Cigarettes per day while pregnant, and birthweight (ounces)
- Zero to negative association
- Linearity not obvious
- Weak association

Scatter diagram 5: family size and NELS achievement



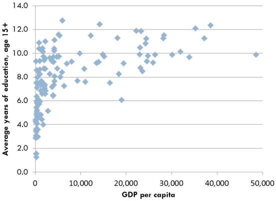
- 8th grade reading scores in West region
- Zero to negative association
- Linearity not obvious
- Weak association

Scatter diagram 6: marijuana usage and time



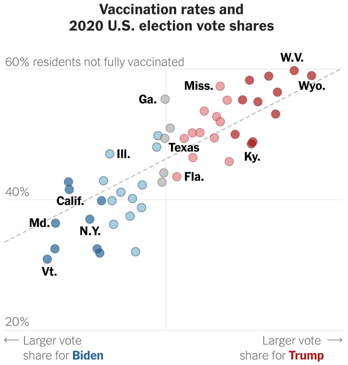
- No *linear* association
- Strong *nonlinear* association / time trend

Scatter diagram 7: GDP per capita and education



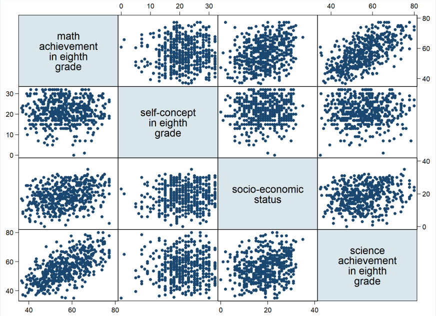
- Positive association
- *Nonlinear* relationship
- Moderately strong association

Scatter diagram 8: Trump vote share and vaccination



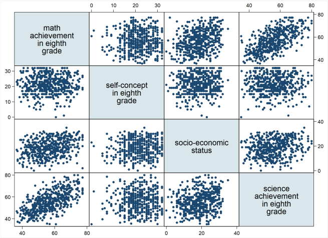
Scatterplot matrix

`graph matrix` *varlist* is a useful command for visualizing the bivariate associations between two or more variables. For example:



Scatterplot matrix

The horizontal axes in each column apply to the variable named in that column; the vertical axes apply to the variable name in that row. Thus, for example, the scatterplot in cell (3,1) is the same as the scatterplot in cell (1,3), but with the axes flipped.



Sample covariance and correlation

The sample analog to the population covariance is:

$$s_{XY} = \frac{\sum_{i=1}^{N}\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)}{n-1}$$

The sample analog to the population correlation is:

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{s_x s_y (n-1)}$$

r_{xy} is known as the sample *correlation coefficient* or the Pearson product moment correlation. The sample s_{xy} and r_{xy} are estimators of the population parameters σ_{xy} and ρ_{xy}.

Covariance and correlation—calculation

	Height (x_i)	Weight (y_i)	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x}) \times (y_i - \bar{y})$
	69	108	3.58	-17.83	-63.90
	61	130	-4.42	4.17	-18.40
	68	135	2.58	9.17	23.68
	66	135	0.58	9.17	5.35
	66	120	0.58	-5.83	-3.40
	63	115	-2.42	-10.83	26.18
	72	150	6.58	24.17	159.10
	62	105	-3.42	-20.83	71.18
	62	115	-3.42	-10.83	37.01
	67	145	1.58	19.17	30.35
	66	132	0.58	6.17	3.60
	63	120	-2.42	-5.83	14.10
Mean	65.42	125.83	0.00	0.00	23.74
Sum			0.00	0.00	284.85
SD	3.32	14.24			

Covariance and correlation—calculation

$$s_{xy} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

$$s_{xy} = \frac{284.85}{12-1} = 25.895$$

$$r_{xy} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{s_x s_y (n-1)}$$

$$r_{xy} = \frac{284.85}{(3.32)(14.24)(12-1)} = 0.548$$

Covariance and correlation in Stata

To obtain correlation coefficients in Stata, use `corr` *yvar xvar*. The result is a correlation matrix.

- Be aware of how Stata handles missing values:
  - **listwise deletion** means observations are not used if *any* of the listed variables in the command are missing.
  - **pairwise deletion** means correlations of pairs of variables are considered in isolation.
- `pwcorr` *yvar xvar* uses pairwise deletion

To obtain the covariance in Stata, use `corr` with `cov` option. The result is a variance-covariance matrix.

Correlation coefficient in Stata

Examples:

```
. corr height weight
(obs=12)

             |   height   weight
-------------+------------------
      height |   1.0000
      weight |   0.5486   1.0000
```

```
. corr satv satm pertak
(obs=51)

             |     satv     satm   pertak
-------------+---------------------------
        satv |   1.0000
        satm |   0.9715   1.0000
      pertak |  -0.8827  -0.8372   1.0000
```

Correlation coefficient in Stata

Examples:

```
. corr famsize achrdg08 if region==4
(obs=93)

             |  famsize achrdg08
-------------+------------------
     famsize |   1.0000
    achrdg08 |   0.0260   1.0000
```

```
. corr year marij
(obs=16)

             |     year    marij
-------------+------------------
        year |   1.0000
       marij |   0.3963   1.0000
```


Correlation coefficient in Stata

Examples:

```
. corr achmat08 achrdg08
(obs=500)

             | achmat08 achrdg08
-------------+------------------
    achmat08 |   1.0000
    achrdg08 |   0.5947   1.0000

. pwcorr achmat08 achrdg08 achmat10 achrdg10

             | achmat08 achrdg08 achmat10 achrdg10
-------------+------------------------------------
    achmat08 |   1.0000
    achrdg08 |   0.5947   1.0000
    achmat10 |   0.8489   0.5919   1.0000
    achrdg10 |   0.5803   0.7538   0.6531   1.0000
```

Correlation coefficient

What is a "strong correlation?" It depends on the context. (How strong would you expect the correlation to be? Is there a theoretical reason why the correlation should be particularly strong or weak?)

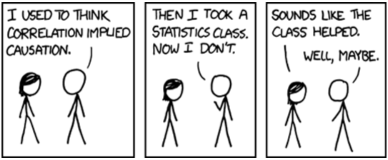
- Rule of thumb ("Cohen's scale") based on the absolute value $|r_{xy}|$:
  - $|r_{xy}| < 0.1$: zero to weak correlation
  - $0.1 < |r_{xy}| < 0.3$: weak to moderate correlation
  - $0.3 < |r_{xy}| < 0.5$: moderately strong correlation
  - $|r_{xy}| > 0.5$: strong correlation

The correlation coefficient itself is *ordinal*. An increase in correlation from 0.1 to 0.2 is not equivalent to an increase from 0.4 to 0.5.

Correlation vs. causation

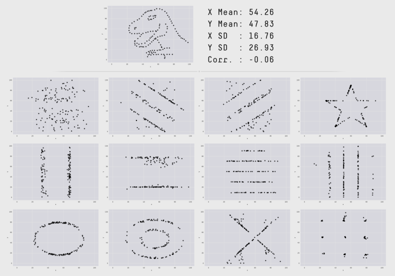
Important: *correlation does not imply causation!*

- *Correlation* means two variables move together.
- *Causation* means that change in one variable is causing change in the other.



The importance of visualizing your data

Never trust summary statistics alone! All of the datasets used below have the same $\bar{x}, \bar{y}, s_x, s_y$, and r_{xy}.



Source:
`https://www.autodesk.com/research/publications/same-stats-different-graphs`

Correlation coefficient—special cases

The Pearson product moment correlation can be applied to any pair of interval-measured variables. However, when one or both of the variables is dichotomous, the correlation coefficient can be expressed in alternative ways.

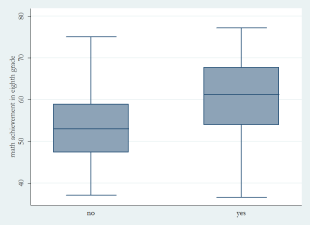
- **"Point biserial" correlation**: when one variable is dichotomous (x) and the other is continuous (y).
- **"Phi coefficient"**: when both variables are dichotomous.

The following slides simply show how r_{xy} can be written when one or both variables are dichotomous. In practice, continue to use `corr` or `pwcorr` in Stata.

Point biserial correlation

$$r_{xy} = \frac{(\bar{y}_1 - \bar{y}_0)s_x}{s_y}$$

$\bar{y}_1$ is the mean of y for observations where $x = 1$, and $\bar{y}_0$ is the mean of y for observations where $x = 0$. Consider the relationship between enrollment in advanced math and math achievement, in the NELS:



Point biserial correlation

```
. sum achmat08 advmath

    Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
    achmat08 |        500    56.59102    9.339608      36.61       77.2
    advmath8 |        491    .4602851    .4989286          0          1

. sum achmat08 advmath if advmath~=.

    Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
    achmat08 |        491    56.73473    9.342372      36.61       77.2
    advmath8 |        491    .4602851    .4989286          0          1
```

```
. tabstat achmat08, by(advmath) stat(mean sd n)

Summary for variables: achmat08
     by categories of: advmath8 (advanced math t

advmath8 |      mean        sd         N
---------+------------------------------
      no |  53.57491  8.098492       265
     yes |  60.43982  9.358113       226
---------+------------------------------
   Total |  56.73473  9.342372       491
```

Point biserial correlation

$$r_{xy} = \frac{(\bar{y}_1 - \bar{y}_0)s_x}{s_y}$$

$$r_{xy} = \frac{(60.45 - 53.57)(0.499)}{9.342} = 0.367$$

```
. corr achmat08 advmath
(obs=491)

             | achmat08 advmath8
-------------+------------------
    achmat08 |   1.0000
    advmath8 |   0.3666   1.0000
```


Phi coefficient

When the two variables x and y are dichotomous, we can calculate the Pearson correlation coefficient (also referred to as a "Phi coefficient") from a 2 x 2 frequency table:

	Variable 2:	
Variable 1:	0	1
0	A	B
1	C	D

$$r_{xy} = \phi = \frac{AD - BC}{\sqrt{(A+B)(C+D)(A+C)(B+D)}}$$

Phi coefficient

Example:

	Gender:	
Advanced math:	Male	Female
No	25	36
Yes	22	21

$$\phi = \frac{AD - BC}{\sqrt{(A+B)(C+D)(A+C)(B+D)}}$$

$$\phi = \frac{(25*21)-(36*22)}{\sqrt{(25+36)(22+21)(25+21)(36+22)}} = -0.101$$

Phi coefficient

advanced math taken in eighth grade	gender male	female	Total
no	25 53.19	36 63.16	61 58.65
yes	22 46.81	21 36.84	43 41.35
Total	47 100.00	57 100.00	104 100.00

```
. corr advmath gender if region==1
(obs=104)

             | advmath8   gender
-------------+------------------
    advmath8 |   1.0000
      gender |  -0.1007   1.0000
```

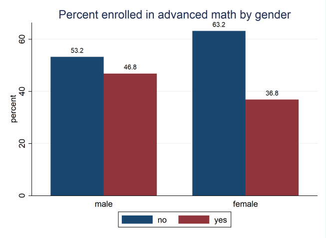
Phi coefficient

Other ways to observe an association between two dichotomous variables:

- **Clustered bar graph** (a bar graph by group), where the height of the bar is the percentage of cases equal to one within each group
- Contingency table (or **crosstabulation**) with row and column percentages

Sample clustered bar graph

```
graph bar if region==1, over(advmath8) over(gender) asyvars percentages
blabel(bar, format(%3.1f)) title(Percent enrolled in advanced math by gender)
graphregion(fcolor(white))
```



The `asyvars percentages` options ensure the bar heights represent percentages within group (and sum to 100). Here, girls appear less likely to be enrolled in advanced math.

2x2 crosstabulation

```
tabulate gender advmath if region==1, row
```

Key
frequency
row percentage

gender	advanced math taken in eighth grade: no	yes	Total
male	25 53.19	22 46.81	47 100.00
female	36 63.16	21 36.84	57 100.00
Total	61 58.65	43 41.35	104 100.00

The `row` option reports percentages that sum to 100 in each row. Here again, girls appear less likely to be enrolled in advanced math.

Spearman rank correlation

Spearman's rank correlation (sometimes called "rho") can be used with ordinal-measured variables—where the size of the difference between x and its mean is not meaningful—or in cases where the underlying relationship is nonlinear.

Spearman rank correlation

Spearman's correlation depends on how variables *rank* in their respective distributions.

- Rank each variable x and y in its respective distribution, in ascending order $1, ..., n$
- For an observation i, d_i is the *difference* between i's ranking for x and i's ranking for y: $d_i = rank(x_i) - rank(y_i)$

$$r_{s,xy} = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n^3 - n}$$

Spearman rank correlation

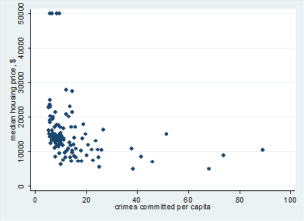
A few notes about Spearman's correlation:

- When x_i and y_i are identically ranked, $d_i = 0$. If $d_i = 0$ for *all cases*, then $r_{s,xy} = 1$ (a perfect positive correlation).
- The further apart x_i and y_i are in their ranks, the larger is d_i and $r_{s,xy}$ gets closer to -1 (a perfect negative correlation).

$$r_{s,xy} = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n^3 - n}$$

Spearman rank correlation in Stata

The Spearman rank correlation is obtained in Stata with the command `spearman` *yvar xvar*. For example, consider the nonlinear relationship between median house price and crime:



Spearman rank correlation in Stata

```
. spearman price crime

 Number of obs =       506
Spearman's rho =       -0.5564

Test of Ho: price and crime are independent
    Prob > |t| =        0.0000

. sum price crime

    Variable |        Obs        Mean    Std. Dev.       Min        Max
-------------+---------------------------------------------------------
       price |        506    22511.51    9208.856       5000      50001
       crime |        506    3.611536    8.590247       .006     88.976

. corr price crime
(obs=506)

             |    price    crime
-------------+------------------
       price |   1.0000
       crime |  -0.3879   1.0000
```

Variable transformations and correlation

Consider the linear transformations of variables x and y:

$$x_1 = a + (b * x_0)$$

$$y_1 = c + (d * y_0)$$

What happens to the correlation between two variables x and y when one (or both) is transformed by a linear function?

Variable transformations and correlation

Re-write the correlation coefficient as follows.

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{s_x s_y (n-1)} = \frac{\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)}{n-1} = \frac{\sum_{i=1}^{n} z_{xi} z_{yi}}{n-1}$$

How is a z-score affected by a linear transformation?

$$z_{x_1} = \frac{a + (b * x_0) - a - (b * \bar{x_0})}{|b| * s_{x_0}} = \frac{b * (x_0 - \bar{x_0})}{|b| * s_{x_0}} = \frac{b}{|b|} z_{x_0}$$

When b is positive, the z-score of the transformed variable x_1 is the same as the original z-score. When b is *negative*, the new z-score is the inverse of the original.

Variable transformations and correlation

How transformations affect the correlation between x and y thus depend only on the multiplicative factor applied to x, y, or both:

- If both multiplicative factors are positive ($b > 0$ and $d > 0$) or both multiplicative factors are negative ($b < 0$ and $d < 0$), then the correlation between the new, transformed variables is the same as the correlation between the original variables.

- If one multiplicative factor is positive and the other is negative ($b > 0$ and $d < 0$ or ($b < 0$ and $d > 0$), then the correlation between the new, transformed variables is the negative of the correlation between the original variables.

Hypothesis tests about ρ

The sample correlation coefficient r_{xy} can be used to estimate the population correlation coefficient ρ. If we know the sampling distribution of r_{xy}, we can construct confidence intervals for ρ or conduct hypothesis tests.

Under $H_0 : \rho = 0$, the following test statistic has a t-distribution with $n - 2$ degrees of freedom:

$$t = \frac{|r_{xy}|\sqrt{n-2}}{\sqrt{1 - r_{xy}^2}}$$

Find p and then determine whether $p < \alpha$. In Stata, use `corr` with the `sig` option.

Simulating draws from a bivariate normal in Stata

Example of 100 draws of X and Y from a bivariate normal distribution with $\mu_x = \mu_y = 0$, $\sigma_x^2 = \sigma_y^2 = 1$ and $\rho_{xy} = 0.5$:

```
matrix C = (1 0.5 \ 0.5 1)
drawnorm x y, n(100) corr(C)
```

Note the covariance is also 0.5 since $\rho = \sigma_{xy}/\sigma_x\sigma_y$