Assignment-5

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Download all latex-tikz codes from

https://github.com/spdanda/AI1103/blob/main/Assignment5/Assignment5.tex

CSIR-UGC-NET June-2016 Q50 :
Let X and Y be independent and identically distributed random variables such that $\Pr(X=0) = \Pr(X=1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

1) X and W are independent.
2) Y and W are independent.
3) Z and W are uncorrelated.
4) Z and W are independent.

Solution:
$X, Y \in \{0,1\} \implies Z \in \{0,1,2\}$ and $W \in \{0,1\}$.
Also,

$$\Pr(X = x) = \begin{cases} \frac{1}{2} & x \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

$$\Pr(Y = y) = \begin{cases} \frac{1}{2} & y \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

a)

$$\Pr(W = 0) = \Pr(X = 0, Y = 0) + \Pr(X = 1, Y = 1) \tag{0.0.3}$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \tag{0.0.4}$$

$$= \frac{1}{2} \tag{0.0.5}$$

$$\Pr(W = 1) = 1 - \Pr(W = 0) \tag{0.0.6}$$

$$= \frac{1}{2} \tag{0.0.7}$$

$$\therefore \Pr(W = w) = \begin{cases} \frac{1}{2} & w \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.8}$$

$$\text{Also } E[W] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \tag{0.0.9}$$

$$= \frac{1}{2} \tag{0.0.10}$$

b)

$$\Pr(Z = z) = \Pr(X + Y = z) \tag{0.0.11}$$

$$= \sum_{x=0}^{z} \Pr(X = x) \times \Pr(Y = z - x) \tag{0.0.12}$$

$$= (2 - |z - 1|) \times \frac{1}{2} \times \frac{1}{2} \tag{0.0.13}$$

$$= \frac{2 - |z - 1|}{4} \tag{0.0.14}$$

$$\therefore \Pr(Z = z) = \begin{cases} \frac{2-|z-1|}{4} & z \in \{0,1,2\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.15}$$

$$\text{And } E[Z] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} \tag{0.0.16}$$

$$= 1 \tag{0.0.17}$$

Now, for checking each option,

1) Checking if X and W are independent

$$p_1 = \Pr(X = x, W = w) \tag{0.0.18}$$
$$= \Pr(X = x, Y = x \pm w) \tag{0.0.19}$$
$$= \Pr(X = x) \times \Pr(Y = x \pm w) \tag{0.0.20}$$
$$= \frac{1}{2} \times \frac{1}{2} \tag{0.0.21}$$
$$= \begin{cases} \frac{1}{4} & (x \pm w) \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.22}$$

(only one value for Y is obtained for each case when x and w are substituted)

$$\Pr(X = x) \times \Pr(W = w) = \frac{1}{2} \times \frac{1}{2} \tag{0.0.23}$$
$$= \frac{1}{4} \tag{0.0.24}$$

$\Pr(X = x)\Pr(W = w) = \Pr(X = x, W = w)$

$\implies$ X and W are independent and hence Option 1 is true.

2) Checking if Y and W are independent
Solving of this case is identical to the first option except the variable X is replaced by Y. Hence on solving, you get Y and W are independent.
$\therefore$ Option 2 is also true.

3) Checking if W and Z are uncorrelated
Uncorrelated random variables: Two variables are said to be uncorrelated if the expected value of their joint distribution is equal to product of the expected values of their respective marginal distributions.
Also, $WZ \in \{0, 1, 2\}$

$$\Pr(WZ = 2) = \Pr(W = 1, Z = 2) = 0 \tag{0.0.25}$$

($\because$ If $Z = 2$, $X = Y = 1 \implies W = 0$ i.e., $\neq$ to 1)

$$\Pr(WZ = 0) = \frac{1}{2} (\because \text{ either } W = 0 \text{ or } Z = 0) \tag{0.0.26}$$

$$\Pr(WZ = 1) = 1 - \Pr(WZ = 0) - \Pr(WZ = 2) \tag{0.0.27}$$
$$= 1 - \frac{1}{2} \tag{0.0.28}$$
$$= \frac{1}{2} \tag{0.0.29}$$

$\therefore$ Expected value of WZ, $E[WZ]$

$$= \sum_{k=0,1,2} k \Pr(WZ = k) \tag{0.0.30}$$
$$= 1 \times \frac{1}{2} + 0 \tag{0.0.31}$$
$$= \frac{1}{2} \tag{0.0.32}$$

$$\text{Also } E[W] \times E[Z] = \frac{1}{2} \times 1 = \frac{1}{2} \tag{0.0.33}$$
$$= E[WZ] \tag{0.0.34}$$

Hence from the above equation W and Z are uncorrelated random variables.
$\therefore$Option 3 is also true.

4) Let's check for one particular case.

$$\Pr(W = 0 | Z = 0) = 1 \tag{0.0.35}$$

($\because$ $Z = 0 \implies X = Y = 0 \implies W = 0$)

$$\therefore \frac{\Pr(W = 0, Z = 0)}{\Pr(Z = 0)} = 1 \tag{0.0.36}$$

$$\implies \Pr(W = 0, Z = 0) = \Pr(Z = 0)$$
$$\neq \Pr(W = 0) \times \Pr(Z = 0) \tag{0.0.37}$$

$\therefore$ W and Z are not independent random variables

Hence Option 4 is incorrect.
$\therefore$ Answer is Option4.