Assignment-5

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Download all latex-tikz codes from

https://github.com/spdanda/AI1103/blob/main/Assignment5/Assignment5.tex

CSIR-UGC-NET June-2016 Q50 :
Let X and Y be independent and identically distributed random variables such that $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

1) X and W are independent.
2) Y and W are independent.
3) Z and W are uncorrelated.
4) Z and W are independent.

Solution:
$X, Y \in \{0, 1\} \implies Z \in \{0, 1, 2\}$ and $W \in \{0, 1\}$.
Also,

$$\Pr(X = x) = \begin{cases} \frac{1}{2} & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

$$\Pr(Y = y) = \begin{cases} \frac{1}{2} & y \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

So Probability Generating functions(PGF's) for X and Y are

$$\mathcal{G}_X(z) = E[z^x] \tag{0.0.3}$$
$$= \sum_{i=0}^{1} p_i z^i \tag{0.0.4}$$
$$= \frac{1+z}{2} \tag{0.0.5}$$

Similarly,

$$\mathcal{G}_Y(z) = \frac{1+z}{2} \tag{0.0.6}$$

a) Now calculate the Generating function of W

$$\mathcal{G}_W(z) = E\left[z^{|X-Y|}\right] \tag{0.0.7}$$

$$\mathcal{G}_W(z) = E\left[z^{X-Y} |(X > Y)\right] + E\left[z^{Y-X} |(Y > X)\right] + E\left[z^0 |(X = Y)\right] \tag{0.0.8}$$

$$= \frac{1}{4}z^1 + \frac{1}{4}z^1 + 1 \tag{0.0.9}$$

($\because$ Only possibility for (X, Y) such that $X > Y$ is (1,0), similarly for $Y > X$)

$$= 1 + \frac{z}{2} \tag{0.0.10}$$

$\therefore$ From the generating function obtained, pmf of W is

$$\Pr(W = w) = \begin{cases} \frac{1}{2} & w \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.11}$$

$$\text{Also } E[W] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \tag{0.0.12}$$
$$= \frac{1}{2} \tag{0.0.13}$$

b) PGF of Z:

$$\mathcal{G}_Z(z) = E[z^{X+Y}] \tag{0.0.14}$$
$$= E[z^X \times z^Y] \tag{0.0.15}$$
$$= E[z^X] \times E[z^Y] \tag{0.0.16}$$

($\because$ X and Y are independent)

$$\therefore \mathcal{G}_Z(z) = \left(\frac{1+z}{2}\right)^2 \quad (0.0.17)$$

$$= \frac{1+2z+z^2}{4} \quad (0.0.18)$$

$$= \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^2 \quad (0.0.19)$$

$\therefore$From the PGF of Z; pmf of Z is

$$\Pr(Z = z) = \begin{cases} \frac{1}{4} & z = 0 \\ \frac{1}{2} & z = 1 \\ \frac{1}{4} & z = 2 \end{cases} \quad (0.0.20)$$

$$\text{Also, } E[Z] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} \quad (0.0.21)$$

$$= 1 \quad (0.0.22)$$

Now, for checking each option,

1) Checking if X and W are independent

$$p_1 = \Pr(X = x, W = w) \quad (0.0.23)$$

$$= \Pr(X = x, Y = x \pm w) \quad (0.0.24)$$

$$= \Pr(X = x) \times \Pr(Y = x \pm w) \quad (0.0.25)$$

$$= \frac{1}{2} \times \frac{1}{2} \quad (0.0.26)$$

$$= \begin{cases} \frac{1}{4} & (x \pm w) \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.27)$$

(only one value for Y is obtained for each case when x and w are substituted)

$$\Pr(X = x) \times \Pr(W = w) = \frac{1}{2} \times \frac{1}{2} \quad (0.0.28)$$

$$= \frac{1}{4} \quad (0.0.29)$$

$$\Pr(X = x)\Pr(W = w) = \Pr(X = x, W = w) \quad (0.0.30)$$

$\implies$ X and W are independent and hence Option 1 is true.

2) Checking if Y and W are independent
Solving of this case is identical to the first option except the variable X is replaced by Y. Hence on solving, you get Y and W are independent.
$\therefore$ Option 2 is also true.

3) Checking if W and Z are uncorrelated
Uncorrelated random variables: Two variables are said to be uncorrelated if the expected value of their joint distribution is equal to product of the expected values of their respective marginal distributions.
Also, $WZ \in \{0, 1, 2\}$

$$\Pr(WZ = 2) = \Pr(W = 1, Z = 2) = 0 \quad (0.0.31)$$

($\because$ If $Z = 2$, $X = Y = 1 \implies W = 0$ i.e., $\neq$ to 1)

$$\Pr(WZ = 0) = \frac{1}{2} \quad (0.0.32)$$

($\because$ either $W = 0$ or $Z = 0$)

$$\Pr(WZ = 1) = 1 - \Pr(WZ = 0) - \Pr(WZ = 2) \quad (0.0.33)$$

$$= 1 - \frac{1}{2} \quad (0.0.34)$$

$$= \frac{1}{2} \quad (0.0.35)$$

$\therefore$ Expected value of WZ, $E[WZ]$

$$= \sum_{k=0,1,2} k \Pr(WZ = k) \tag{0.0.36}$$

$$= 1 \times \frac{1}{2} + 0 \tag{0.0.37}$$

$$= \frac{1}{2} \tag{0.0.38}$$

$$\text{Also } E[W] \times E[Z] = \frac{1}{2} \times 1 = \frac{1}{2} \tag{0.0.39}$$

$$= E[WZ] \tag{0.0.40}$$

Hence from the above equation W and Z are uncorrelated random variables.
$\therefore$Option 3 is also true.

4) Let's check for one particular case.

$$\Pr(W = 0 | Z = 0) = 1 \tag{0.0.41}$$

$(\because\ Z = 0 \implies X = Y = 0 \implies W = 0)$

$$\therefore \frac{\Pr(W = 0, Z = 0)}{\Pr(Z = 0)} = 1 \tag{0.0.42}$$

$$\implies \Pr(W = 0, Z = 0) = \Pr(Z = 0)$$
$$\neq \Pr(W = 0) \times \Pr(Z = 0) \tag{0.0.43}$$

$\therefore$ W and Z are not independent random variables
Hence Option 4 is incorrect.
$\therefore$ Answer is Option4.