Assignment-5

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Download all latex-tikz codes from

https://github.com/spdanda/AI1103/blob/main/Assignment5/Assignment5.tex

CSIR-UGC-NET June-2016 Q50 :
Let X and Y be independent and identically distributed random variables such that $\Pr(X=0) = \Pr(X=1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

1) X and W are independent.
2) Y and W are independent.
3) Z and W are uncorrelated.
4) Z and W are independent.

Solution:
$X, Y \in \{0,1\} \implies Z \in \{0,1,2\}$ and $W \in \{0,1\}$.
Also,

$$\Pr(X = x) = \begin{cases} \frac{1}{2} & x \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

$$\Pr(Y = y) = \begin{cases} \frac{1}{2} & y \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

a)

$$\Pr(W = 0) = \Pr(X = 0, Y = 0) + \Pr(X = 1, Y = 1) \tag{0.0.3}$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \tag{0.0.4}$$

$$= \frac{1}{2} \tag{0.0.5}$$

$$\Pr(W = 1) = 1 - \Pr(W = 0) \tag{0.0.6}$$

$$= \frac{1}{2} \tag{0.0.7}$$

$$\therefore \Pr(W = w) = \begin{cases} \frac{1}{2} & w \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.8}$$

$$\text{Also } E[W] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \tag{0.0.9}$$

$$= \frac{1}{2} \tag{0.0.10}$$

b) Characteristic function of random variable is defined as

$$\phi_X(\omega) = E[e^{iwX}] \tag{0.0.11}$$

For discrete random variable X

$$\phi_X(\omega) = \sum_k e^{iwk} P_X(k) \tag{0.0.12}$$

$$\phi_X(\omega) = \frac{1 + e^{iw}}{2} \tag{0.0.13}$$

Similarly, For discrete random variable Y

$$\phi_Y(\omega) = \sum_k e^{iwk} P_Y(k) \tag{0.0.14}$$

$$\phi_Y(\omega) = \frac{1 + e^{iw}}{2} \tag{0.0.15}$$

if X and Y are independent random variables, then

$$\phi_{X+Y}(\omega) = E[e^{iw(X+Y)}] \tag{0.0.16}$$

$$= E[e^{iwX}.e^{iwY}] \tag{0.0.17}$$

$$= \phi_X(\omega) \times \phi_Y(\omega) \tag{0.0.18}$$

$$\phi_Z(\omega) = \left(\frac{1+e^{iw}}{2}\right)^2 \tag{0.0.19}$$

$$= \frac{1+e^{2iw}+2e^{iw}}{4} \tag{0.0.20}$$

$$= \frac{1}{4}e^{0iw} + \frac{1}{2}e^{iw} + \frac{1}{4}e^{2iw} \tag{0.0.21}$$

From the definition of characteristic function of a discrete random variable as in (0.0.11), we get
PMF of Z is

$$p_Z(z) = \Pr(Z = z) = \begin{cases} \frac{1}{4} & z = 0 \\ \frac{1}{2} & z = 1 \\ \frac{1}{4} & z = 2 \end{cases} \tag{0.0.22}$$

$$\text{Also, } E[Z] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} \tag{0.0.23}$$

$$= 1 \tag{0.0.24}$$

Now, for checking each option,

1) Checking if X and W are independent

$$p_1 = \Pr(X = x, W = w) \tag{0.0.25}$$

$$= \Pr(X = x, Y = x \pm w) \tag{0.0.26}$$

$$= \Pr(X = x) \times \Pr(Y = x \pm w) \tag{0.0.27}$$

$$= \frac{1}{2} \times \frac{1}{2} \tag{0.0.28}$$

$$= \begin{cases} \frac{1}{4} & (x \pm w) \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.29}$$

(only one value for Y is obtained for each case when x and w are substituted)

$$\Pr(X = x) \times \Pr(W = w) = \frac{1}{2} \times \frac{1}{2} \tag{0.0.30}$$

$$= \frac{1}{4} \tag{0.0.31}$$

$$\Pr(X = x)\Pr(W = w) = \Pr(X = x, W = w) \tag{0.0.32}$$

$\implies$ X and W are independent and hence Option 1 is true.

2) Checking if Y and W are independent
Solving of this case is identical to the first option except the variable X is replaced by Y. Hence on solving, you get Y and W are independent.
$\therefore$ Option 2 is also true.

3) Checking if W and Z are uncorrelated
Uncorrelated random variables: Two variables are said to be uncorrelated if the expected value of their joint distribution is equal to product of the expected values of their respective marginal distributions.
Also, $WZ \in \{0, 1, 2\}$

$$\Pr(WZ = 2) = \Pr(W = 1, Z = 2) = 0 \tag{0.0.33}$$

($\because$ If $Z = 2$, $X = Y = 1 \implies W = 0$ i.e., $\neq$ to 1)

$$\Pr(WZ = 0) = \frac{1}{2} \tag{0.0.34}$$

($\because$ either $W = 0$ or $Z = 0$)

$$\Pr(WZ = 1) = 1 - \Pr(WZ = 0) - \Pr(WZ = 2) \tag{0.0.35}$$

$$= 1 - \frac{1}{2} \tag{0.0.36}$$

$$= \frac{1}{2} \tag{0.0.37}$$

$\therefore$ Expected value of WZ, $E[WZ]$

$$= \sum_{k=0,1,2} k \Pr(WZ = k) \tag{0.0.38}$$

$$= 1 \times \frac{1}{2} + 0 \tag{0.0.39}$$

$$= \frac{1}{2} \tag{0.0.40}$$

$$\text{Also } E[W] \times E[Z] = \frac{1}{2} \times 1 = \frac{1}{2} \tag{0.0.41}$$

$$= E[WZ] \tag{0.0.42}$$

Hence from the above equation W and Z are uncorrelated random variables.
$\therefore$Option 3 is also true.

4) Let's check for one particular case.

$$\Pr(W = 0 | Z = 0) = 1 \tag{0.0.43}$$

$(\because\ Z = 0 \implies X = Y = 0 \implies W = 0)$

$$\therefore \frac{\Pr(W = 0,\ Z = 0)}{\Pr(Z = 0)} = 1 \tag{0.0.44}$$

$$\implies \Pr(W = 0, Z = 0) = \Pr(Z = 0)$$

$$\neq \Pr(W = 0) \times \Pr(Z = 0) \tag{0.0.45}$$

$\therefore$ W and Z are not independent random variables
Hence Option 4 is incorrect.
$\therefore$ Answer is Option4.