Assignment-5

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Download all latex-tikz codes from

https://github.com/spdanda/AI1103/blob/main/Assignment5/Assignment5.tex

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Let X and Y be independent and identically distributed random variables such that $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

1) X and W are independent.
2) Y and W are independent.
3) Z and W are uncorrelated.
4) Z and W are independent.

Solution :
$X, Y \in \{0, 1\} \implies Z \in \{0, 1, 2\}$ and $W \in \{0, 1\}$.

Definition 0.1. PMF's for the given random variables X and Y are

$$\Pr(X = x) = \begin{cases} \frac{1}{2} & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

$$\Pr(Y = y) = \begin{cases} \frac{1}{2} & y \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

Definition 0.2. Probability Generating functions(PGF's) for the random variables X and Y are

$$\mathcal{G}_X(z) = E[z^x] \tag{0.0.3}$$
$$= \sum_{i=0}^{1} p_i z^i \tag{0.0.4}$$
$$= \frac{1+z}{2} \tag{0.0.5}$$

Similarly,

$$\mathcal{G}_Y(z) = \frac{1+z}{2} \tag{0.0.6}$$

Lemma 0.1. *Generating function for* $W = |X - Y|$ *where* $\mathcal{G}_X(z) = \frac{1+z}{2}$ *and* $\mathcal{G}_Y(z) = \frac{1+z}{2}$ *is (X and Y being independent)*

$$\mathcal{G}_W(z) = \frac{1+z}{2}$$

Proof.

$$\mathcal{G}_W(z) = E\left[z^{|X-Y|}\right] \tag{0.0.7}$$

$$\implies \mathcal{G}_W(z) = E\left[z^{X-Y} |(X > Y)\right] + E\left[z^{Y-X} |(Y > X)\right] + E\left[z^{X-Y} |(X = Y)\right] \tag{0.0.8}$$

$$\implies \mathcal{G}_W(z) = \sum \Pr(X, Y | X > Y) z^{X-Y} + \sum \Pr(X, Y | X < Y) z^{-(X-Y)} + \sum \Pr(X, Y | X = Y) z^{X-Y} \tag{0.0.9}$$

Case	Possibilities for (X, Y)	$\Pr(X, Y)$
$X > Y$	(1,0)	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
$X < Y$	(0,1)	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
$X = Y$	(0,0)	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
	(1,1)	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

TABLE 4: Probability Table for (X, Y) in different cases.

$\therefore$From the table,

$$\begin{aligned}\mathcal{G}_W(z) = [\Pr(X,Y)\, z^{X-Y}]_{(X,Y)=(1,0)} + \\ [\Pr(X,Y)\, z^{Y-X}]_{(X,Y)=(0,1)} + \\ [\Pr(X,Y)\, z^{X-Y}]_{(X,Y)=(0,0)} + \\ [\Pr(X,Y)\, z^{X-Y}]_{(X,Y)=(1,1)}\end{aligned} \tag{0.0.10}$$

$$= \frac{1}{4}z^{(1-0)} + \frac{1}{4}z^{(1-0)} + \left(\frac{1}{4}z^{(0-0)} + \frac{1}{4}z^{(1-1)}\right) \tag{0.0.11}$$

$$= \frac{z}{2} + \frac{1}{2} \tag{0.0.12}$$

$$= \frac{1+z}{2} \tag{0.0.13}$$

□

Lemma 0.2. *Expected value of W with $\mathcal{G}_W(z) = \frac{1+z}{2}$ is $\frac{1}{2}$*

Proof. As $\mathcal{G}_W(z) = \frac{1+z}{2}$, pmf of W is

$$\Pr(W = w) = \begin{cases} \frac{1}{2} & w \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.14}$$

So,

$$E[W] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \tag{0.0.15}$$

$$= \frac{1}{2} \tag{0.0.16}$$

□

Lemma 0.3. *Generating Function for $Z = X + Y$ is (X and Y being independent)*

$$\mathcal{G}_Z(z) = \mathcal{G}_X(z) \times \mathcal{G}_Y(z)$$

Proof.

$$\mathcal{G}_Z(z) = E[z^{X+Y}] \tag{0.0.17}$$

$$= E[z^X z^Y] \tag{0.0.18}$$

$$= E[z^X] \times E[z^Y] \tag{0.0.19}$$

($\because$ X and Y are independent)

$$= \mathcal{G}_X(z) \times \mathcal{G}_Y(z) \tag{0.0.20}$$

□

$\therefore$ From **Lemma 0.2**;

$$\mathcal{G}_Z(z) = \left(\frac{1+z}{2}\right)^2 \tag{0.0.21}$$

$$= \frac{1+2z+z^2}{4} \tag{0.0.22}$$

$$\tag{0.0.23}$$

Lemma 0.4. *Expected value of Z with $\mathcal{G}_Z(z) = \frac{1+2z+z^2}{4}$ is 1*

Proof. As

$$\mathcal{G}_Z(z) = \frac{1+2z+z^2}{4} \tag{0.0.24}$$

$$= \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^2 \tag{0.0.25}$$

PMF of Z is

$$\Pr(Z = z) = \begin{cases} \frac{1}{4} & z = 0 \\ \frac{1}{2} & z = 1 \\ \frac{1}{4} & z = 2 \end{cases} \tag{0.0.26}$$

$\therefore$ Expected value of Z is

$$E[Z] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} \tag{0.0.27}$$

$$= 1 \tag{0.0.28}$$

□

Now, checking for each option,

1) Checking if X and W are independent:

$$p_1 = \Pr(X = x, W = w) \tag{0.0.29}$$

$$= \Pr(X = x, Y = x \pm w) \tag{0.0.30}$$

$$= \Pr(X = x) \times \Pr(Y = x \pm w) \tag{0.0.31}$$

($\because$ X and Y are independent)

Note that here $Y \in \{0, 1\}$.
So Possible values of Y for different values of x and w are

x	w	Possibilities for Y	$\Pr(Y)$
0	0	0	$\frac{1}{2}$
0	1	1	$\frac{1}{2}$
1	0	1	$\frac{1}{2}$
1	1	0	$\frac{1}{2}$

TABLE 1: Probability Table for Y when different x and w's are substituted

So, the value of $\Pr(Y)$ $\forall$ values of x, w is equals to $\frac{1}{2}$
$\implies$ $\Pr(X = x) \times \Pr(Y = x \pm w)$ $\forall$ values of $x, w \in \{0, 1\}$ is equals to $\frac{1}{2} \times \frac{1}{2}$

$$\implies p_1 = \frac{1}{2} \times \frac{1}{2} \tag{0.0.32}$$

$$= \frac{1}{4} \tag{0.0.33}$$

$$\therefore \Pr(X = x, W = w) = \frac{1}{4} \ \forall x, w \in \{0, 1\} \tag{0.0.34}$$

Also,

$$\Pr(X = x) \times \Pr(W = w) = \frac{1}{2} \times \frac{1}{2} \quad \forall x, w \in \{0, 1\} \tag{0.0.35}$$

$$= \frac{1}{4} \tag{0.0.36}$$

$\therefore$ From (0.0.34) and (0.0.36).

$$\Pr(X = x)\Pr(W = w) = \Pr(X = x, W = w) \tag{0.0.37}$$

$\implies$ X and W are independent and hence Option 1 is true.

2) Checking if Y and W are independent : Solving this case is identical to the first option except the variable X is replaced by Y (Note that here W is symmetric wrt to X and Y).
Hence on solving, you get Y and W are independent.
$\therefore$ Option 2 is also true.
4) Checking if W and Z are independent:
Let's check for one particular case.

$$\Pr(W = 0 | Z = 0) = 1 \tag{0.0.38}$$

$(\because\ Z = 0 \implies X = Y = 0 \implies W = 0)$

$$\therefore \frac{\Pr(W = 0, Z = 0)}{\Pr(Z = 0)} = 1 \tag{0.0.39}$$

$$\implies \Pr(W = 0, Z = 0) = \Pr(Z = 0) \tag{0.0.40}$$

$$= \frac{1}{4} \tag{0.0.41}$$

$$\neq \Pr(W = 0) \times \Pr(Z = 0) \tag{0.0.42}$$

$\therefore$ W and Z are not independent random variables
Hence option 4 is false.

3) Checking if W and Z are uncorrelated:
Uncorrelated random variables: Two variables are said to be uncorrelated if the expected value of their joint distribution is equal to product of the expected values of their respective marginal distributions.
Let $U = WZ \implies U \in \{0, 1, 2\}$ ($\because$ $W \in \{0, 1\}$ and $Z \in \{0, 1, 2\}$)
Lemma 0.5. *PMF for the random variable U is*

$$\Pr(U = u) = \begin{cases} \frac{1}{2} & u = 0 \\ \frac{1}{2} & u = 1 \\ 0 & u = 2 \end{cases} \tag{0.0.43}$$

Proof.

$$\Pr(U=2) = \Pr(WZ=2) \quad (0.0.44)$$
$$= \Pr(W=1, Z=2) = 0 \quad (0.0.45)$$

($\because$ If $Z=2$, $X=Y=1 \implies W=0$ i.e., $\neq$ to 1)

$$\therefore \Pr(U=2) = 0$$

$$\Pr(U=0) = \Pr(WZ=0) \quad (0.0.46)$$

So, either $W=0$ or $Z=0$
$\implies X=Y=0$ or $X=Y=1$

$$\Pr(U=0) = \Pr(X=0, Y=0) + \Pr(X=1, Y=1) \quad (0.0.47)$$

$$= \Pr(X=0)\Pr(Y=0) + \Pr(X=1)\Pr(Y=1) \quad (0.0.48)$$

$$= \frac{1}{4} + \frac{1}{4} \quad (0.0.49)$$
$$= \frac{1}{2} \quad (0.0.50)$$

$$\therefore \Pr(U=0) = \frac{1}{2}$$

Now,

$$\Pr(U=1) = 1 - [\Pr(U=0) - \Pr(U=2)] \quad (0.0.51)$$

$$= 1 - \left[\frac{1}{2} + 0\right] \quad (0.0.52)$$
$$= \frac{1}{2} \quad (0.0.53)$$

$$\therefore \Pr(U=1) = \frac{1}{2}$$

Hence PMF of U is

$$\Pr(U=u) = \begin{cases} \frac{1}{2} & u=0 \\ \frac{1}{2} & u=1 \\ 0 & u=2 \end{cases} \quad (0.0.54)$$

□

Expected value of WZ, $E[WZ]$

$$= \sum_{k=0,1,2} k \Pr(WZ=k) \quad (0.0.55)$$
$$= 1 \times \frac{1}{2} + 0 \quad (0.0.56)$$
$$= \frac{1}{2} \quad (0.0.57)$$

$$\text{Also } E[W] \times E[Z] = \frac{1}{2} \times 1 = \frac{1}{2} \quad (0.0.58)$$
$$= E[WZ] \quad (0.0.59)$$

Hence from (0.0.59) W and Z are uncorrelated random variables.
$\therefore$Option 3 is also true.

$\therefore$ Incorrect Option is 4.