Result from Challenging Problem 1

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Download all latex-tikz codes from

https://github.com/spdanda/AI1103/blob/main/Challenging%20Problem1/Result_from_ChallengingProb1/main.tex

Statement :

A non-negative random variable has zero expectation value if and only if it is zero.

Proof :

Let Y be a non-negative random variable defined on the probability space Ω.

Let A_m be the set of all $\omega \in \Omega$ such that the value of Y is greater than $\frac{1}{m}$ where $m \in N$ i.e.,

$$A_m = \left\{\omega \in \Omega \,:\, Y(\omega) \geq \frac{1}{m}\right\}$$

Given

$$E(Y) = 0 \implies \int_{\Omega} Y d_P = 0 \tag{0.0.1}$$

As Y is non-negative,

$$\int_{\Omega} Y d_P \geq \int_{A_m} Y d_p \tag{0.0.2}$$

Also in the set A_m, $Y \geq \frac{1}{m}$

$$\therefore \int_{A_m} Y d_p \geq \int_{A_m} \frac{1}{m} d_p \tag{0.0.3}$$

$$= \frac{1}{m} \int_{A_m} d_p \tag{0.0.4}$$

$$= \frac{1}{m} \Pr(A_m) \tag{0.0.5}$$

$\therefore$ From (0.0.2) and (0.0.5)

$$\int_{\Omega} Y d_P \geq \frac{1}{m} \Pr(A_m) \tag{0.0.6}$$

$$\implies \frac{1}{m} \Pr(A_m) \leq 0 \tag{0.0.7}$$

$$\implies \Pr(A_m) \leq 0 \; (\because m \in N) \tag{0.0.8}$$

$$\implies \Pr(A_m) = 0 \tag{0.0.9}$$

Also,

$$\Pr(Y \neq 0) = \Pr\left(\bigcup A_m\right)$$

$$(\because \text{In the set } A_m,\; Y \geq \frac{1}{m} \implies Y \text{ always} > 0) \tag{0.0.10}$$

$$= \Pr(A_1 + A_2 + A_3 + ...) \tag{0.0.11}$$

$$= 0 \, (\because \Pr(A_m) = 0 \; \forall m \in N) \tag{0.0.12}$$

$$\implies \Pr(Y \neq 0) = 0 \tag{0.0.13}$$

$$\implies \Pr(Y = 0) = 1 - \Pr(Y \neq 0) = 1 \tag{0.0.14}$$

Hence, Probability of the random variable equals to 0 is one i.e., the random variable is always equals to 0 in it's domain.

Conversely, If Y=0 then

$$\int_{\Omega} Y d_P = 0 \tag{0.0.15}$$

$$\implies E(Y) = 0 \tag{0.0.16}$$

Hence proved.