

Assignment-5

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Download all latex-tikz codes from

<https://github.com/spdanda/AI1103/blob/main/Assignment5/Assignment5.tex>

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Let X and Y be independent and identically distributed random variables such that $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

- 1) X and W are independent.
- 2) Y and W are independent.
- 3) Z and W are uncorrelated.
- 4) Z and W are independent.

Solution :

$X, Y \in \{0, 1\} \implies Z \in \{0, 1, 2\}$ and $W \in \{0, 1\}$.

- 1) PMF's of X and Y :

$$\Pr(X = x) = \begin{cases} \frac{1}{2} & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.1)$$

$$\Pr(Y = y) = \begin{cases} \frac{1}{2} & y \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.2)$$

Probability Generating functions(PGF's) for X and Y are

$$\mathcal{G}_X(z) = E[z^X] \quad (0.0.3)$$

$$= \sum_{i=0}^1 p_i z^i \quad (0.0.4)$$

$$= \frac{1+z}{2} \quad (0.0.5)$$

Similarly,

$$\mathcal{G}_Y(z) = \frac{1+z}{2} \quad (0.0.6)$$

- 2) Generating function of W ;

$$\mathcal{G}_W(z) = E[z^{|X-Y|}] \quad (0.0.7)$$

$$\begin{aligned} \mathcal{G}_W(z) &= E[z^{X-Y} | (X > Y)] + \\ &E[z^{Y-X} | (Y > X)] + E[z^0 | (X = Y)] \end{aligned} \quad (0.0.8)$$

$$= \frac{1}{4}z^1 + \frac{1}{4}z^1 + 1 \quad (0.0.9)$$

(\because Only possibility for (X, Y) such that $X > Y$ is $(1, 0)$, similarly for $Y > X$)

$$= 1 + \frac{z}{2} \quad (0.0.10)$$

From the generating function obtained, pmf of W is

$$\Pr(W = w) = \begin{cases} \frac{1}{2} & w \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.11)$$

Expected value of W

$$E[W] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \quad (0.0.12)$$

$$= \frac{1}{2} \quad (0.0.13)$$

- 3) PGF of Z :

$$\mathcal{G}_Z(z) = E[z^{X+Y}] \quad (0.0.14)$$

$$= E[z^X \times z^Y] \quad (0.0.15)$$

$$= E[z^X] \times E[z^Y] \quad (0.0.16)$$

($\because X$ and Y are independent)

$$\therefore \mathcal{G}_Z(z) = \left(\frac{1+z}{2}\right)^2 \quad (0.0.17)$$

$$= \frac{1+2z+z^2}{4} \quad (0.0.18)$$

$$= \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^2 \quad (0.0.19)$$

From the PGF of Z ; pmf of Z is

$$\Pr(Z = z) = \begin{cases} \frac{1}{4} & z = 0 \\ \frac{1}{2} & z = 1 \\ \frac{1}{4} & z = 2 \end{cases} \quad (0.0.20)$$

Expected value of Z :

$$E[Z] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} \quad (0.0.21)$$

$$= 1 \quad (0.0.22)$$

Now, for checking each option,

1) Checking if X and W are independent

$$p_1 = \Pr(X = x, W = w) \quad (0.0.23)$$

$$= \Pr(X = x, Y = x \pm w) \quad (0.0.24)$$

$$= \Pr(X = x) \times \Pr(Y = x \pm w) \quad (0.0.25)$$

$$= \frac{1}{2} \times \frac{1}{2} \quad (0.0.26)$$

$$= \begin{cases} \frac{1}{4} & (x \pm w) \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.27)$$

(only one value for Y is obtained for each case when x and w are substituted)

$$\Pr(X = x) \times \Pr(W = w) = \frac{1}{2} \times \frac{1}{2} \quad (0.0.28)$$

$$= \frac{1}{4} \quad (0.0.29)$$

$$\Pr(X = x) \Pr(W = w) =$$

$$\Pr(X = x, W = w) \quad (0.0.30)$$

$\implies X$ and W are independent and hence Option 1 is true.

2) Checking if Y and W are independent
Solving of this case is identical to the first option except the variable X is replaced by Y . Hence on solving, you get Y and W are independent.

\therefore Option 2 is also true.

3) Checking if W and Z are uncorrelated
Uncorrelated random variables: Two variables are said to be uncorrelated if the expected value of their joint distribution is equal to product of the expected values of their respective marginal distributions.

Also, $WZ \in \{0, 1, 2\}$

$$\Pr(WZ = 2) = \Pr(W = 1, Z = 2) = 0 \quad (0.0.31)$$

(\because If $Z = 2, X = Y = 1 \implies W = 0$ i.e., \neq to 1)

$$\Pr(WZ = 0) = \frac{1}{2} \quad (0.0.32)$$

(\because either $W = 0$ or $Z = 0$)

$$\Pr(WZ = 1) = 1 - \Pr(WZ = 0) - \Pr(WZ = 2) \quad (0.0.33)$$

$$= 1 - \frac{1}{2} \quad (0.0.34)$$

$$= \frac{1}{2} \quad (0.0.35)$$

∴ Expected value of WZ , $E[WZ]$

$$= \sum_{k=0,1,2} k \Pr(WZ = k) \quad (0.0.36)$$

$$= 1 \times \frac{1}{2} + 0 \quad (0.0.37)$$

$$= \frac{1}{2} \quad (0.0.38)$$

$$\text{Also } E[W] \times E[Z] = \frac{1}{2} \times 1 = \frac{1}{2} \quad (0.0.39)$$

$$= E[WZ] \quad (0.0.40)$$

Hence from the above equation W and Z are uncorrelated random variables.

∴ Option 3 is also true.

4) Let's check for one particular case.

$$\Pr(W = 0 | Z = 0) = 1 \quad (0.0.41)$$

$$(\because Z = 0 \implies X = Y = 0 \implies W = 0)$$

$$\therefore \frac{\Pr(W = 0, Z = 0)}{\Pr(Z = 0)} = 1 \quad (0.0.42)$$

$$\implies \Pr(W = 0, Z = 0) = \Pr(Z = 0)$$

$$\neq \Pr(W = 0) \times \Pr(Z = 0) \quad (0.0.43)$$

∴ W and Z are not independent random variables

Hence Option 4 is incorrect.

∴ Answer is Option 4.