Result from Challenging Problem 1

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Download all latex-tikz codes from

https://github.com/spdanda/AI1103/blob/main/Challenging%20Problem1/Result_from_ChallengingProb1/main.tex

Statement :
A non-negative random variable has zero expectation value if and only if it is zero.

Proof :
Let Y be a non-negative random variable defined on the probability space Ω.
Let A_m be the set of all $\omega \in \Omega$ such that the value of Y is greater than or equal to $\frac{1}{m}$ where $m \in N$ i.e.,

$$A_m = \left\{\omega \in \Omega \,:\, Y(\omega) \geq \frac{1}{m}\right\}$$

Given

$$E(Y) = 0 \implies \int_\Omega Y f(y) d_Y = 0 \tag{0.0.1}$$

where $f(y)$ represents the pdf of the random variable Y. Let P denote the total probability i.e.,

$$P = \int_\Omega f(y) d_Y \tag{0.0.2}$$

$$\implies \frac{d_P}{d_Y} = f(y) \tag{0.0.3}$$

$$\implies f(y) d_Y = d_p \tag{0.0.4}$$

$$\therefore E(Y) = \int_\Omega Y d_P = 0 \tag{0.0.5}$$

As Y is non-negative,

$$\int_\Omega Y d_P \geq \int_{A_m} Y d_p \tag{0.0.6}$$

Also in the set A_m, $Y \geq \frac{1}{m}$

$$\therefore \int_{A_m} Y d_p \geq \int_{A_m} \frac{1}{m} d_p \tag{0.0.7}$$

$$= \frac{1}{m} \int_{A_m} d_p \tag{0.0.8}$$

$$= \frac{1}{m} \Pr(A_m) \tag{0.0.9}$$

$\therefore$ From (0.0.6) and (0.0.9)

$$\int_\Omega Y d_P \geq \frac{1}{m} \Pr(A_m) \tag{0.0.10}$$

$$\implies \frac{1}{m} \Pr(A_m) \leq 0 \tag{0.0.11}$$

$$\implies \Pr(A_m) \leq 0 \; (\because m \in N) \tag{0.0.12}$$

$$\implies \Pr(A_m) = 0 \tag{0.0.13}$$

Also,

$$\Pr(Y \neq 0) = \Pr\left(\bigcup A_m\right)$$

$$(\because \text{In the set } A_m,\, Y \geq \frac{1}{m} \implies Y \text{ always} > 0) \tag{0.0.14}$$

$$= \Pr(A_1 + A_2 + A_3 + ...) \tag{0.0.15}$$

$$= 0 \,(\because \Pr(A_m) = 0 \;\forall m \in N) \tag{0.0.16}$$

$$\implies \Pr(Y \neq 0) = 0 \tag{0.0.17}$$

$$\implies \Pr(Y = 0) = 1 - \Pr(Y \neq 0) = 1 \tag{0.0.18}$$

Hence, Probability of the random variable equals to 0 is one i.e., the random variable is always equals to 0 in it’s domain.

Conversely, If Y=0 then

$$\int_{\Omega} Y d_P = 0 \tag{0.0.19}$$

$$\implies E(Y) = 0 \tag{0.0.20}$$

Hence proved.