

Assignment-5

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Download all latex-tikz codes from

<https://github.com/spdanda/AI1103/blob/main/Assignment5/Assignment5.tex>

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Let X and Y be independent and identically distributed random variables such that $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

- 1) X and W are independent.
- 2) Y and W are independent.
- 3) Z and W are uncorrelated.
- 4) Z and W are independent.

Solution:

$X, Y \in \{0, 1\} \implies Z \in \{0, 1, 2\}$ and $W \in \{0, 1\}$.

Also,

$$\Pr(X = x) = \begin{cases} \frac{1}{2} & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.1)$$

$$\Pr(Y = y) = \begin{cases} \frac{1}{2} & y \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.2)$$

a)

$$\Pr(W = 0) = \Pr(X = 0, Y = 0) + \Pr(X = 1, Y = 1) \quad (0.0.3)$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \quad (0.0.4)$$

$$= \frac{1}{2} \quad (0.0.5)$$

$$\Pr(W = 1) = 1 - \Pr(W = 0) \quad (0.0.6)$$

$$= \frac{1}{2} \quad (0.0.7)$$

$$\therefore \Pr(W = w) = \begin{cases} \frac{1}{2} & w \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.8)$$

$$\text{Also } E[W] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \quad (0.0.9)$$

$$= \frac{1}{2} \quad (0.0.10)$$

b)

$$\Pr(Z = z) = \Pr(X + Y = z)$$

$$= \sum_{x=0}^z \Pr(X = x) \Pr(Y = z - x) \quad (0.0.11)$$

$$= (2 - |z - 1|) \times \frac{1}{2} \times \frac{1}{2} \quad (0.0.12)$$

$$= \frac{2 - |z - 1|}{4} \quad (0.0.13)$$

$$\therefore \Pr(Z = z) = \begin{cases} \frac{2 - |z - 1|}{4} & z \in \{0, 1, 2\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.14)$$

$$\text{And } E[Z] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} \quad (0.0.15)$$

$$= 1 \quad (0.0.16)$$

Now, for checking each option,

1) Checking if X and W are independent

$$p_1 = \Pr(X = x, W = w) \quad (0.0.17)$$

$$= \Pr(X = x, Y = x \pm w) \quad (0.0.18)$$

$$= \Pr(X = x) \times \Pr(Y = x \pm w) \quad (0.0.19)$$

$$= \frac{1}{2} \times \frac{1}{2} \quad (0.0.20)$$

$$= \begin{cases} \frac{1}{4} & (x \pm w) \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \quad (0.0.21)$$

(only one value for Y is obtained for each case when x and w are substituted)

$$\Pr(X = x) \times \Pr(W = w) = \frac{1}{2} \times \frac{1}{2} \quad (0.0.22)$$

$$= \frac{1}{4} \quad (0.0.23)$$

$$\Pr(X = x) \Pr(W = w) = \Pr(X = x, W = w) \quad (0.0.24)$$

$\implies X$ and W are independent and hence Option 1 is true.

2) Checking if Y and W are independent
Solving of this case is identical to the first option except the variable X is replaced by Y . Hence on solving, you get Y and W are independent.
 \therefore Option 2 is also true.

3) Checking if W and Z are uncorrelated
Uncorrelated random variables: Two variables are said to be uncorrelated if the expected value of their joint distribution is equal to product of the expected values of their respective marginal distributions.

Also, $WZ \in \{0, 1, 2\}$

$$\Pr(WZ = 2) = \Pr(W = 1, Z = 2) = 0 \quad (0.0.25)$$

(\because If $Z = 2, X = Y = 1 \implies W = 0$ i.e., \neq to 1)

$$\Pr(WZ = 0) = \frac{1}{2} \quad (0.0.26)$$

(\because either $W = 0$ or $Z = 0$)

$$\Pr(WZ = 1) = 1 - \Pr(WZ = 0) - \Pr(WZ = 2) \quad (0.0.27)$$

$$= 1 - \frac{1}{2} \quad (0.0.28)$$

$$= \frac{1}{2} \quad (0.0.29)$$

\therefore Expected value of $WZ, E[WZ]$

$$= \sum_{k=0,1,2} k \Pr(WZ = k) \quad (0.0.30)$$

$$= 1 \times \frac{1}{2} + 0 \quad (0.0.31)$$

$$= \frac{1}{2} \quad (0.0.32)$$

$$\text{Also } E[W] \times E[Z] = \frac{1}{2} \times 1 = \frac{1}{2} \quad (0.0.33)$$

$$= E[WZ] \quad (0.0.34)$$

Hence from the above equation W and Z are uncorrelated random variables.

\therefore Option 3 is also true.

4) Let's check for one particular case.

$$\Pr(W = 0 | Z = 0) = 1 \quad (0.0.35)$$

($\because Z = 0 \implies X = Y = 0 \implies W = 0$)

$$\therefore \frac{\Pr(W = 0, Z = 0)}{\Pr(Z = 0)} = 1 \quad (0.0.36)$$

$$\begin{aligned}\implies \Pr(W = 0, Z = 0) &= \Pr(Z = 0) \\ &\neq \Pr(W = 0) \times \Pr(Z = 0) \quad (0.0.37)\end{aligned}$$

$\therefore W$ and Z are not independent random variables

Hence Option 4 is incorrect.

\therefore Answer is Option4.