Assignment-5

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Download all latex-tikz codes from

https://github.com/spdanda/AI1103/blob/main/Assignment5/Assignment5.tex

CSIR-UGC-NET June-2016 Q50 :
Let X and Y be independent and identically distributed random variables such that $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

1) X and W are independent.
2) Y and W are independent.
3) Z and W are uncorrelated.
4) Z and W are independent.

Solution :
$X, Y \in \{0, 1\} \implies Z \in \{0, 1, 2\}$ and $W \in \{0, 1\}$.

Definition 0.1. PMF's for the given random variables X and Y are

$$\Pr(X = x) = \begin{cases} \frac{1}{2} & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

$$\Pr(Y = y) = \begin{cases} \frac{1}{2} & y \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

Definition 0.2. Probability Generating functions(PGF's) for the random variables X and Y are

$$\mathcal{G}_X(z) = E[z^x] \tag{0.0.3}$$

$$= \sum_{i=0}^{1} p_i z^i \tag{0.0.4}$$

$$= \frac{1 + z}{2} \tag{0.0.5}$$

Similarly,

$$\mathcal{G}_Y(z) = \frac{1 + z}{2} \tag{0.0.6}$$

Lemma 0.1. *Generating function for $W = |X - Y|$ where $\mathcal{G}_X(z) = \frac{1 + z}{2}$ and $\mathcal{G}_Y(z) = \frac{1 + z}{2}$ is*

$$\mathcal{G}_W(z) = \frac{1 + z}{2}$$

Proof.

$$\mathcal{G}_W(z) = E\left[z^{|X-Y|}\right] \tag{0.0.7}$$

$$\mathcal{G}_W(z) = E\left[z^{X-Y} |(X > Y)\right] + E\left[z^{Y-X} |(Y > X)\right] + E\left[z^{0} |(X = Y)\right] \tag{0.0.8}$$

$$= \frac{1}{4}z^1 + \frac{1}{4}z^1 + \left(\frac{1}{4}z^0 + \frac{1}{4}z^0\right) \tag{0.0.9}$$

($\because$ Only possibility for (X, Y) such that $X > Y$ is (1,0); Similarly for $Y > X$ and for $X = Y$, two possibilities, (0,0),(1,1))

$$= \frac{1 + z}{2} \tag{0.0.10}$$

□

Definition 0.3. The pmf of W with $\mathcal{G}_W(z) = \frac{1 + z}{2}$ is

$$\Pr(W = w) = \begin{cases} \frac{1}{2} & w \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.11}$$

Expected value of W

$$E[W] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} \tag{0.0.12}$$
$$= \frac{1}{2} \tag{0.0.13}$$

Lemma 0.2. *Generating Function for $Z = X + Y$ is (X and Y being independent)*

$$\mathcal{G}_Z(z) = \mathcal{G}_X(z) \times \mathcal{G}_Y(z)$$

Proof.

$$\mathcal{G}_Z(z) = E[z^{X+Y}] \tag{0.0.14}$$
$$= E[z^X z^Y] \tag{0.0.15}$$
$$= E[z^X] \times E[z^Y] \tag{0.0.16}$$

($\because$ X and Y are independent)

$$= \mathcal{G}_X(z) \times \mathcal{G}_Y(z) \tag{0.0.17}$$

$\square$

$\therefore$ From **Lemma 0.2**;

$$\mathcal{G}_Z(z) = \left(\frac{1+z}{2}\right)^2 \tag{0.0.18}$$
$$= \frac{1 + 2z + z^2}{4} \tag{0.0.19}$$
$$= \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^2 \tag{0.0.20}$$

Definition 0.4. The pmf of Z with $\mathcal{G}_Z(z) = \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^2$ is

$$\Pr(Z = z) = \begin{cases} \frac{1}{4} & z = 0 \\ \frac{1}{2} & z = 1 \\ \frac{1}{4} & z = 2 \end{cases} \tag{0.0.21}$$

Expected value of Z:

$$E[Z] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} \tag{0.0.22}$$
$$= 1 \tag{0.0.23}$$

Now, checking for each option,

1) Checking if X and W are independent

$$p_1 = \Pr(X = x, W = w) \tag{0.0.24}$$
$$= \Pr(X = x, Y = x \pm w) \tag{0.0.25}$$
$$= \Pr(X = x) \times \Pr(Y = x \pm w) \tag{0.0.26}$$
$$= \frac{1}{2} \times \frac{1}{2} \tag{0.0.27}$$
$$= \begin{cases} \frac{1}{4} & (x \pm w) \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.28}$$

(only one value for Y is obtained for each case when x and w are substituted)

$$\Pr(X = x) \times \Pr(W = w) = \frac{1}{2} \times \frac{1}{2} \tag{0.0.29}$$
$$= \frac{1}{4} \tag{0.0.30}$$

$$\Pr(X = x)\Pr(W = w) = \Pr(X = x, W = w) \tag{0.0.31}$$

$\implies$ X and W are independent and hence Option 1 is true.

2) Checking if Y and W are independent
Solving of this case is identical to the first option except the variable X is replaced by Y. Hence on solving, you get Y and W are independent.
$\therefore$ Option 2 is also true.

3) Checking if W and Z are uncorrelated
Uncorrelated random variables: Two variables are said to be uncorrelated if the expected value of their joint distribution is equal to product of the expected values of their respective marginal distributions.

Also, $WZ \in \{0,1,2\}$

$$\Pr(WZ = 2) = \Pr(W = 1, Z = 2) = 0 \tag{0.0.32}$$

($\because$ If $Z = 2$, $X = Y = 1 \implies W = 0$ i.e., $\neq$ to 1)

$$\Pr(WZ = 0) = \frac{1}{2} \tag{0.0.33}$$

($\because$ either $W = 0$ or $Z = 0$)

$$\Pr(WZ = 1) = 1 - \Pr(WZ = 0) - \Pr(WZ = 2) \tag{0.0.34}$$

$$= 1 - \frac{1}{2} \tag{0.0.35}$$

$$= \frac{1}{2} \tag{0.0.36}$$

$\therefore$ Expected value of WZ, $E[WZ]$

$$= \sum_{k=0,1,2} k \Pr(WZ = k) \tag{0.0.37}$$

$$= 1 \times \frac{1}{2} + 0 \tag{0.0.38}$$

$$= \frac{1}{2} \tag{0.0.39}$$

$$\text{Also } E[W] \times E[Z] = \frac{1}{2} \times 1 = \frac{1}{2} \tag{0.0.40}$$

$$= E[WZ] \tag{0.0.41}$$

Hence from the above equation W and Z are uncorrelated random variables.
$\therefore$Option 3 is also true.

4) Let's check for one particular case.

$$\Pr(W = 0 | Z = 0) = 1 \tag{0.0.42}$$

($\because$ $Z = 0 \implies X = Y = 0 \implies W = 0$)

$$\therefore \frac{\Pr(W = 0, Z = 0)}{\Pr(Z = 0)} = 1 \tag{0.0.43}$$

$$\implies \Pr(W = 0, Z = 0) = \Pr(Z = 0) \neq \Pr(W = 0) \times \Pr(Z = 0) \tag{0.0.44}$$

$\therefore$ W and Z are not independent random variables
Hence Option 4 is incorrect.
$\therefore$ Answer is Option4.