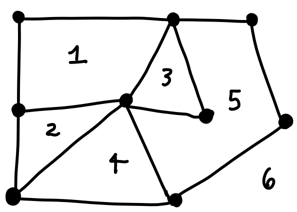
# Lesson 6

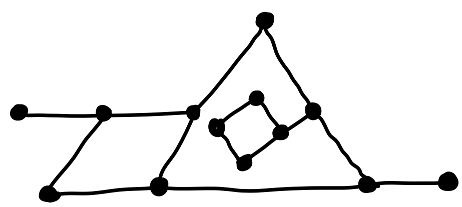
## Eulers formula + planar graphs

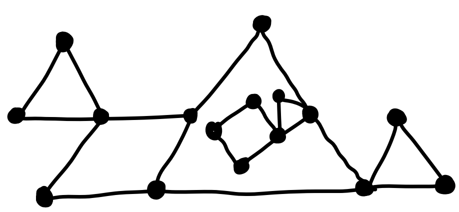
A <u>face</u> of a planar graph is a region of the plane bounded by edges

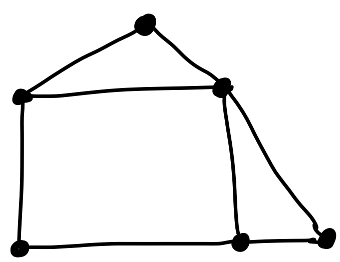
The <u>infinite face</u> is the "exterior" region that stretches off infinitly

How can we relate the numbers of edges, vertices, and faces? Start by counting some examples.



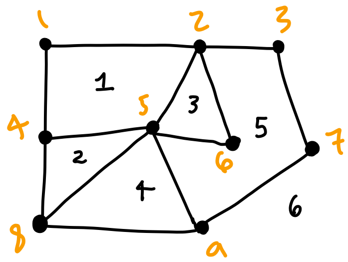





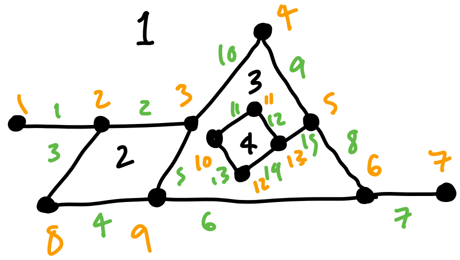


Def: A graph is connected if there exists a path between any two vertices.

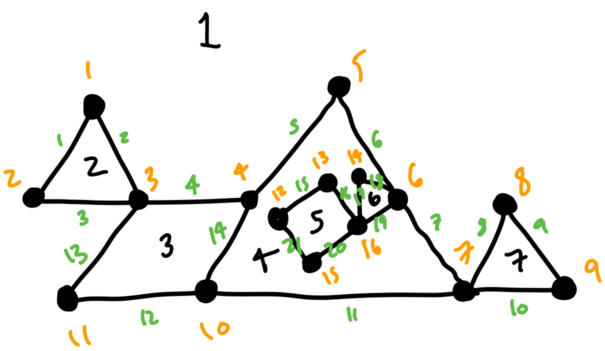
# Answers



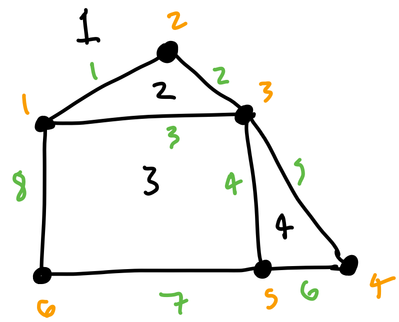
V = 9
e = 13
f = 6



V = 13
e = 15
f = 4

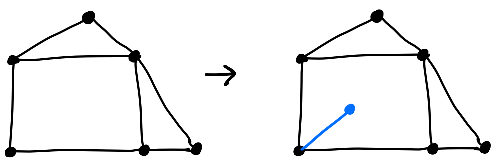


V = 16
e = 21
f = 7



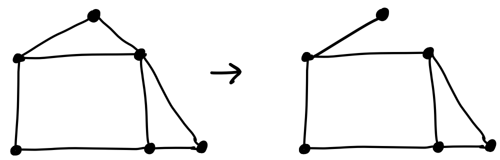
V = 6
e = 8
f = 4

what happens when we add or subtract edges and vertices?



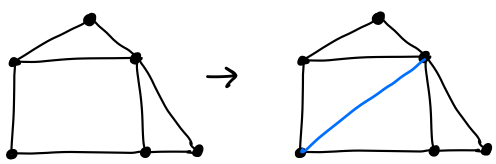
v
e
f

v+1
e+1
f



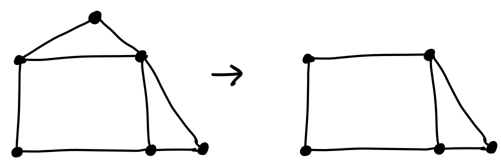
v
e
f

v
e − 1
f − 1



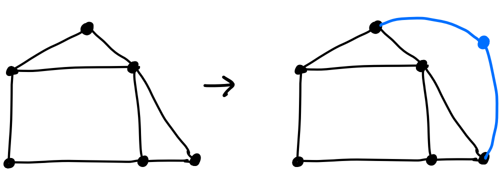
v
e
f

v
e+1
f+1



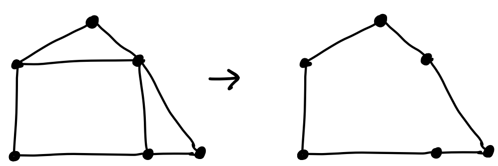
v
e
f

v − 1
e − 2
f − 1



v
e
f

v+1
e+2
f+1



v
e
f

v
e − 2
f − 2

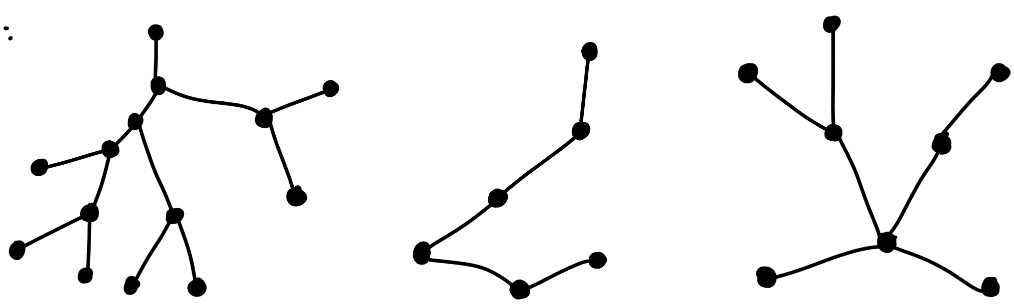
Claim: $3f \leq 2e$

Proof: Let $G$ be a connected planar graph with $v$ vertices, $e$ edges, and $f$ faces. Every face is bounded by at least 3 edges so $3 \leq$ perimeter of face for each face. If we sum the perimeter of every face, we count every edge exactly twice. So summing the above inequality for every face gives us $3f \leq 2e$.

Background for the next thing:

Definition: A tree is a graph with no cycles. Similarly, a graph is a tree if and only if its a planar graph with only one face

examples:



To prove the number of edges in a tree, we'll use proof by induction.

Proof by induction: we prove our claim on various cases based on some number or parameter. for example, # of vertices, # of cycles, # of colors, etc.

Base case: we prove our claim is true for the smallest one or two cases.

Inductive hypothesis: we assume our claim is true for the case $k$.

Inductive Step: we prove that if case $k$ is true, then case $k+1$ is also true.

Since we've shown that case 1 is true and that case $k \Rightarrow$ case $k+1$, this will give us case $1 \Rightarrow$ case 2, then case $2 \Rightarrow$ case 3, then case $3 \Rightarrow$ case 4 etc. So our claim is true for all cases.

Claim: If a connected tree $T$ has $v$ vertices then it has $e = v-1$ edges.

Proof: We will prove by induction on # of vertices.

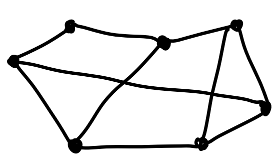
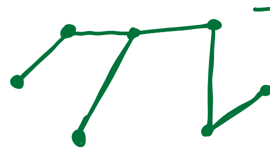
Base Case: Let $v=1$. The only tree on $V=1$ is •
so $e=0=1-1$. Let $V=2$. The only tree on $V=2$ is  so $e=1=2-1$.

Inductive Hyp: Assume that a tree with $v=k$ vertices has $e=k-1$ edges.

Inductive Step: Let $T$ be a tree with $V=k+1$ vertices. We can remove a vertex of degree one $V_0$ and its edge $e_0$ to get a new tree $T'$. $T'$ now has $k$ vertices so by our inductive hypothesis, $T'$ has $k-1$ edges. Since we removed a single edge from $T$ to get $T'$, $T$ must have $k$ edges so $e=k=(k+1)-1$.

So our claim holds for all $v \geq 1$.

Another Fact: Every Graph $G$ has a <u>spanning tree</u> $T$ such that $V_T = V_G$ and $E_T \subseteq E_G$

ex  G  T

# Eulers Formula

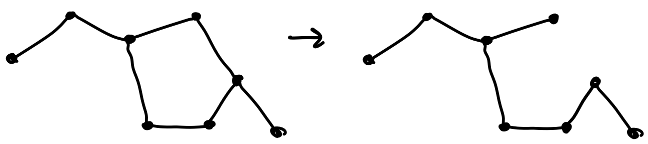
Claim: If $G$ is connected and planar, $v-e+f=2$.

proof: Let $G$ be a connected, planar graph. First we will show that there is a way to remove exacty $f-1$ edges from $G$ to get a spanning tree $T$. We will prove this by induction on the number of faces. Base case: Let $f=1$. Then $G=T$. Let $f=2$. Then $G$ has exactly one cycle. Remove any edge on the cycle to get a spanning tree $T$. Ind Hyp: Assume we have such a spanning tree $T$ for $f=k$. Now, let $G$ have $f=k+1$. Remove one of the edges, say $e_0$, that bounds two different faces to get $G'$, which has $f=k$. There is a set $S$ of $k-1$ edges we can remove from $G'$ to get a spanning tree. $T$. If we remove $S+\{e_0\}$ from $G$, we get $T$ as well which also spans $G$ and has exactly $k$ fewer edges. Thus there is always a way to remove $f-1$ edges to get a spanning tree.

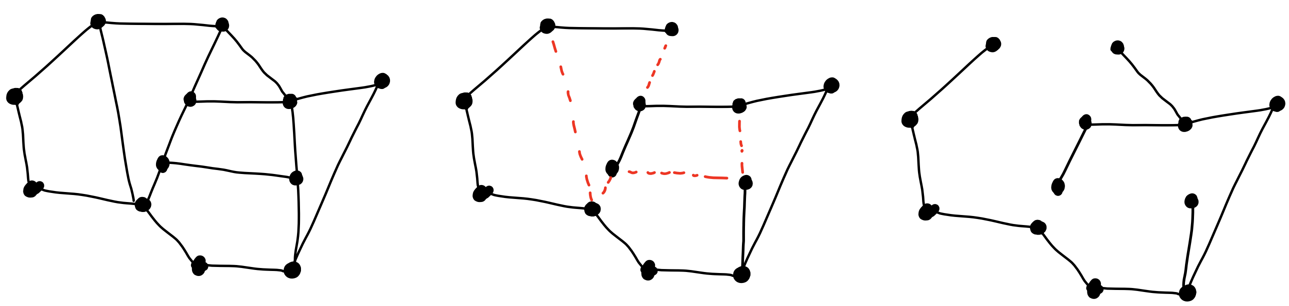
Now, consider $G$ with $v_G$, $e_G$, and $f_G$. we can find a spanning tree $T$ with $v_T=v_G$, $e_T=e_G-(f_G-1)$, and $f_G=1$. Since $T$

is a tree we also know $e_T = V_T - 1 = V_G - 1$.
Since $e_T = V_G - 1$ and $e_T = e_G - (f_G - 1)$ we have
$V_G - 1 = e_G - f_G + 1$ which becomes
$V_G - e_G + f_G = 2$.

Figures: $G$ with $f = 1 \rightarrow T$



Bigger example:



Given $3f \leq 2e$ and $v-e+f=2$ we can find one more formula.

$$3f \leq 2e$$
$$f \leq \frac{2}{3}e$$

$$v-e+f=2$$
$$f=-v+e+2$$

$$-v+e+2 \leq \frac{2}{3}e$$
$$\frac{1}{3}e \leq v-2$$
$$e \leq 3v-6$$

one last theorem...

claim: Every planar graph has a vertex of degree $\leq 5$. That is, $\delta \leq 5$.

proof: First, note that $\delta \cdot v \leq$ total degree. We know total degree $= 2e$ so $\delta \cdot v \leq 2e$. By our inequality above, we can get that $2e \leq 6v-12$. Thus $\delta \cdot v \leq 6v-12$. If $\delta=6$ we would have $6v \leq 6v-12$ which is false. We get the same for any $\delta > 6$. Thus, $\delta \leq 5$.