Extended Schur Functions and Bases Related by Involutions

Spencer Daugherty \cdot NC State \rightarrow CU Boulder \cdot spencer.page.daugherty@gmail.com

For a composition α , a **shin-tableau** T is a diagram of shape α filled with positive integers such that each row weakly increases from left to right and each column strictly increases from top to bottom.

The Extended Schur Functions [1]

For a composition α , the **extended Schur function** is defined by

$$m{v}_{lpha}^{*}=\sum_{-}x^{T},$$

where the sum runs over shin-tableaux T of shape α . These functions form a basis of QSym, and \mathbf{v}_{λ}^{*} is equal to the Schur function s_{λ} when λ is a partition.

$$\mathbf{v}_{(2,3)}^* = x_1^2 x_2^3 + x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2 + x_1 x_2 x_3 x_4^2 + \cdots$$

1	1		1	1		1	1		1	2		1	3		•
2	2	2	2	2	3	2	3	3	2	3	3	2	4	4	

A standard shin-tableau U of size n has entries $\{1, 2, \ldots, n\}$ each appearing once.

$$Des_{\mathfrak{F}_{\boldsymbol{v}}}(U) = \{i : i+1 \text{ is in a strictly lower row than } i \}$$

$$co_{\mathfrak{F}}(U) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d) \text{ for } Des_{\mathfrak{F}}(U) = \{i_1, \dots, i_d\}$$

For a composition α ,

$$m{v}_{lpha}^* = \sum_{U} F_{co_{m{v}}(U)},$$

where the sum runs over standard shin-tableaux U of shape α .

The **complement** of a composition α , denoted α^c , is the composition obtained from the complement of the set associated with α . The **reverse** of $(\alpha_1, \ldots, \alpha_k)$, denoted α^r , is $(\alpha_k, \ldots, \alpha_1)$. The **transpose** of α is defined by $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$.

Involutions on QSym [4]

For a composition α , define the following involutive automorphisms on QSym:

$$\psi(F_{\alpha}) = F_{\alpha^c}$$
 $\rho(F_{\alpha}) = F_{\alpha^r}$ $\omega(F_{\alpha}) = F_{\alpha^t}$

Row-strict Extended Schur Functions [5]

The **row-strict extended Schur functions** are defined via *row-strict shin-tableaux* which have strictly increasing rows and weakly increasing columns. For a *standard* row-strict shin-tableau U, a *descent* $i \in Des_{\mathfrak{RW}}(U)$ is defined as an entry i such that i+1 is in a weakly higher row.

The set of standard row-strict shin-tableaux is the same as the set of standard shin-tableaux, and for a standard (row-strict) shin-tableau U, we have $co_{\mathbf{w}}(U)^c = co_{\mathfrak{R}\mathbf{w}}(U)$.

For a composition α ,

$$\psi(\mathbf{z}^*) = \Re \mathbf{z}_\alpha^*$$

An Involution on the Schur Functions

The classical involution $\omega: Sym \to Sym$ maps the Schur basis to itself by $\omega(s_{\lambda}) = s_{\lambda'}$ where λ' is the conjugate of λ . Collectively, the involutions ψ , ρ , and ω on QSym serve as an analogue to the classical ω . The extended Schur functions are part of what is essentially a system of four bases that is closed under the involutions ψ , ρ , and ω in QSym. We introduce two new Schur-like bases of QSym that, when paired with the extended Schur and row-strict extended Schur functions, complete this picture.

 ψ and ω on QSym restrict to ω on Sym, and ρ restricts to the identity map on Sym.

Flipped Extended Schur Functions

Let α be a composition and β a weak composition. A *flipped shin-tableau* of shape α and type β is a composition diagram α filled with positive integers that weakly decrease along the rows from left to right and strictly increase along the columns from top to bottom, where each positive integer i appears β_i times. A *standard* flipped tableau S of size n contains the entries $\{1, 2, \ldots, n\}$ each exactly once.

$$Des_{\mathfrak{F}}(S) = \{i : i+1 \text{ is in a strictly lower row than } i \}$$

$$co_{\mathfrak{F}}(S) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d) \text{ for } Des_{\mathfrak{F}}(S) = \{i_1, \dots, i_d\}$$

For a composition α , the **flipped extended Schur function** is defined as

$${\mathfrak F}{f v}_lpha^* = \sum_S F_{co_{{\mathfrak F}{f v}}(S)},$$

where the sum runs over standard flipped shin-tableaux S of shape α .

There is a map flip between standard shin-tableaux U and standard flipped shin-tableaux S such that the descent composition of U is the reverse of the descent composition of flip(U) = S. First, flip U horizontally, then replace each entry i with n - i.

Theorem (D. 2023)

For a composition α ,

$$ho(\mathbf{z}_{\alpha}^*) = \mathfrak{F}\mathbf{z}_{\alpha}^* \text{ and } \omega(\mathfrak{R}\mathbf{z}_{\alpha}^*) = \mathfrak{F}\mathbf{z}_{\alpha}^*.$$

Backward Extended Schur Functions

Let α be a composition and β be a weak composition. A backward shin-tableau of shape α and type β is a composition diagram α filled with positive integers that strictly decrease along the rows from left to right and weakly increase along the columns from top to bottom, where each integer i appears β_i times. A standard backward tableau S of size n contains the entries $\{1, 2, \ldots, n\}$ each exactly once.

$$Des_{\mathfrak{B}}(S) = \{i : i+1 \text{ is in a weakly higher row than } i \}$$

$$co_{\mathfrak{B}}(S) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d) \text{ for } Des_{\mathfrak{B}}(S) = \{i_1, \dots, i_d\}$$

For composition α , the **backward extended Schur function** is defined as

$${\mathfrak B}{m v}_lpha^* = \sum_{C} F_{co_{{\mathfrak B}{m v}}(S)},$$

where the sum runs over standard backward shin-tableaux S of shape α .

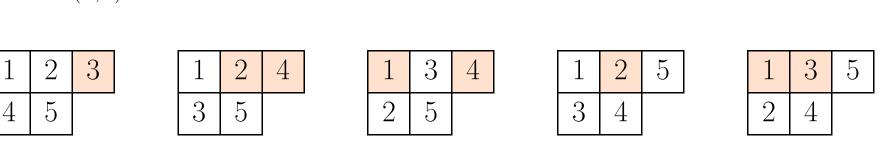
Theorem (D. 2023)

For a composition α ,

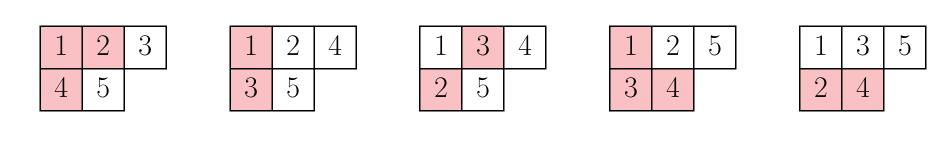
$$\omega(\mathbf{v}_{\alpha}^*) = \mathbf{\mathfrak{Z}}\mathbf{v}_{\alpha^r}^* \quad \text{and} \quad \rho(\mathbf{\mathfrak{R}}\mathbf{v}_{\alpha}^*) = \mathbf{\mathfrak{Z}}\mathbf{v}_{\alpha^r}^* \quad \text{and} \quad \psi(\mathbf{\mathfrak{T}}\mathbf{v}_{\alpha}^*) = \mathbf{\mathfrak{Z}}\mathbf{v}_{\alpha}^*.$$

Backward shin-tableaux are a row-strict version of flipped shin-tableaux. In fact, the set of standard row-strict tableaux and the set of standard backward tableaux are the same but for a tableaux S, we will have $co_{\mathfrak{F}_{\mathbf{v}}}(S) = co_{\mathfrak{F}_{\mathbf{v}}}(S)^{c}$.

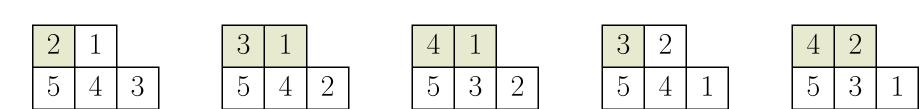
$$\mathbf{z}_{(3,2)}^* = F_{(3,2)} + F_{(2,2,1)} + F_{(1,3,1)} + F_{(2,3)} + F_{(1,2,2)}$$



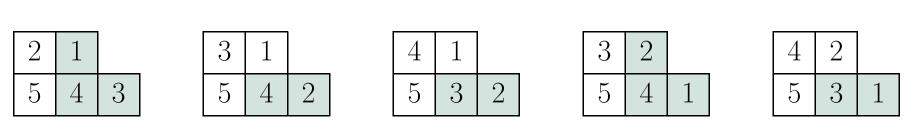
$$\Re \mathbf{v}_{(3,2)}^* = F_{(1,1,2,1)} + F_{(1,2,2)} + F_{(2,1,2)} + F_{(1,2,1,1)} + F_{(2,2,1)}$$



$$\mathfrak{F}_{(2,3)}^* = F_{(2,3)} + F_{(1,2,2)} + F_{(1,3,1)} + F_{(3,2)} + F_{(2,2,1)}.$$



$$\mathfrak{B} \pmb{v}_{(2,3)}^* = F_{(1,2,1,1)} + F_{(2,2,1)} + F_{(2,1,2)} + F_{(1,1,2,1)} + F_{(1,2,2)}$$



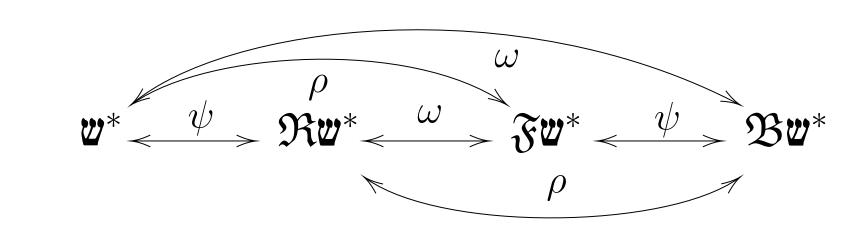
Dual Bases in NSym

Each of our bases is dually paired with a variant of the *shin basis* of NSym. We introduce two of these: the **flipped shin functions** and **backward shin functions**. For each of these bases, we can obtain immediate properties by applying ψ , ρ , or ω to results on the shin functions. We also know the *commutative image* of a flipped or backward shin function. For a partition λ ,

$$\chi(\mathfrak{F}\boldsymbol{v}_{\lambda^r}) = s_{\lambda} \quad ext{and} \quad \chi(\mathfrak{B}\boldsymbol{v}_{\lambda^r}) = s_{\lambda'},$$

and for a composition α that is not the reverse of a partition,

$$\chi(\mathfrak{F}\mathbf{w}_{\alpha}) = \chi(\mathfrak{B}\mathbf{w}_{\alpha}) = 0.$$



[1] S. Assaf and D. Searles. Kohnert polynomials. 2022. [2] J. M. Campbell, K. Feldman, J. Light, P. Shuldiner, and Y. Xu. A Schur-like basis of NSym defined by a Pieri rule. 2014. [3] S. Daugherty. Extended Schur functions and bases related by involutions. Preprint. 2023. [4] K. Luoto, S. Mykytiuk, and S. van Willigenburg. An introduction to quasisymmetric Schur functions. 2013. [5] E. Niese, S. Sundaram, S. van Willigenburg, J. Vega, and S. van Willigenburg. An introduction to quasisymmetric Schur functions. 2022.