Research Statement

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My research focuses on problems in algebraic and enumerative combinatorics. In my thesis, I study Schur-like bases of the quasisymmetric and noncommutative symmetric functions and introduce their colored generalizations in related dual algebras. I like to find combinatorial and algebraic ways to relate different bases to each other and to understand their multiplicative structure. My work outside of my thesis has mainly been on chromatic symmetric functions. In my work on e-positivity problems, I use representation theory to give combinatorial interpretations for the e-positive coefficients of the chromatic symmetric function of certain graphs. I am also taking a structural and bijective approach to the problem of distinguishing trees by their chromatic symmetric functions.

Extended Schur Functions and Bases Related by Involutions. The Schur symmetric functions, defined combinatorially in terms of semistandard young tableaux, form a basis of the symmetric functions with robust combinatorics and a wide variety of applications in representation theory, algebra, and geometry. There has been significant interest in the last decade in developing Schur-like bases for the quasisymmetric and noncommutative symmetric functions. A basis of $NSym$ is Schur-like if the commutative images of its elements indexed by partitions are the Schur functions. Among these bases are the shin functions [3] which are notable in that the commutative image of any element not indexed by a partition is 0. They are dual to the extended Schur functions [1] in $QSym$ which are defined combinatorially by shin-tableaux. In [4], I develop creation operators that allow us to construct certain shin functions and specifically define their expansion into the complete homogeneous basis. This construction allows us to define a Jacobi-Trudi rule which expresses certain shin functions as a matrix determinant. I also define the skew extended Schur functions, relate them to the skew Schur functions, and connect them to the multiplicative structure of the shin functions.

Additionally, I introduce two new bases: the flipped extended Schur functions and the backward extended Schur functions, which are the image of the extended Schur basis under two involutions ρ and ω on $QSym$. These bases relate closely with the row-strict extended Schur functions [10], which are the image of the extended Schur basis under the involution ψ on $QSym$. These involutions take complements, reversals, and transposes on a certain basis of $QSym$ then extend linearly. All three variants of the extended Schur functions are defined over versions of shin-tableaux with different conditions on the order of the entries in their rows and columns. Using these involutions I am able to prove various properties on each basis and their duals in $NSym$. In the future, I hope to find a combinatorial way of understanding the coefficients that appear in the product of two shin functions, in other words, a Littlewood Richardson rule.

Colored Generalizations of Schur-like Bases. In [6], Doliwa defines a colored generalization of $NSym$ that is dual to a generalization of $QSym$ in partially commutative colored variables. In [4] and [5], I define generalizations of a series of bases including the immaculate, shin, dual immaculate, and extended Schur functions. For example, the colored immaculate basis is defined using a colored generalization of the non-commutative Bernstein operators that define the immaculate functions [2] while the colored dual immaculate functions are defined combinatorially by a colored version of immaculate tableaux, which also generalize semistandard Young tableaux. I prove various properties of both bases including a right Pieri rule, Hopf algebra structure, and expansions to and from other bases.

The colored shin functions are defined as the unique functions that satisfy a colored right Pieri rule. Their duals, the colored extended Schur functions, are defined via colored versions of shin-tableaux. I again prove various properties of these two bases including combinatorially definitions for their expansions to and from other bases and a rule for multiplying colored shin functions on the right by colored ribbon functions. In addition to their value as parts of the spaces $NSym_A$ and $QSym_A$, these colored Schur-like bases are interesting because all of their results immediately specialize to results for their analogous bases in $NSym$ and $QSym$. In many cases, the combinatorics of these bases are obscured by cancellation which is significantly reduced by coloring the variables. To complete my thesis, I plan to define colored generalizations of two other Schur-like bases, the Young quasisymmetric Schur and Young non-commutative Schur bases and continue studying properties of all the aforementioned bases[9]. I will also continue to study the effects of the involutions ψ, ρ, and ω on other Schur-like bases and the analogous involutions on their colored generalizations.

e-positivity and Stembridge Codes. A chromatic symmetric function of a graph G with vertices $v_1, \ldots, v_n$ is defined as the sum of monomials $x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}$ for all proper colorings κ of G. Many open problems on chromatic symmetric functions relate to their positive expansions into various bases of the symmetric functions, often the elementary or schur basis. The problem of e-positivity of certain chromatic symmetric functions has close ties to representation theory. In [12], Stembridge defines objects called codes that can be used to combinatorially express the e-coefficients of the chromatic symmetric function of a path. In collaboration with Sheila Sundaram and Kyle Celano, I have defined a similar object called cycle-codes which are similar marked sequences. The Frobenius characteristic of the S_n action on cycle-codes is equivalent to the generating function for the chromatic symmetric functions of cycles. Thus, the e-coefficients of the chromatic symmetric function can be calculated combinatorially using cycle-codes. Moving forward we plan hope to define similar objects for other classes of graphs known to be e-positive.

Distinguishing Trees. Another longstanding open problem is the conjecture that trees are distinguished by their chromatic symmetric functions [11]. The data encoded by a chromatic symmetric function can be determined by the number of stable vertex partitions of T of a given type. We have constructed a bijection from edge partitions of T to stable vertex partitions of T that we hope will allow us to draw an easier comparison between the structural properties of the tree and its chromatic symmetric function. This project is in collaboration with Bryson Kagy and Ian Klein.

(q,t)-Catalan Numbers. The q,t-Catalan numbers are a generalization of the famous Catalan numbers. Their combinatorial definition relies on a pair of statistics on Dyck paths called area and bounce [7]. It is known that $C_n(q,t) = C_n(t,q)$, meaning that area and bounce have a symmetric joint distribution, but there is no combinatorial proof of this fact. In joint work with John Lentfer, we approach the problem by studying pairs of statistics with the same symmetric joint distribution on other Catalan objects such as noncrossing partitions, triangulations of $n+2$-gons, and pattern-avoiding permutations.

Generalize Parking Function Polytopes. Generalized $\mathbf{x}$-Parking functions are defined for a vector $\mathbf{x} = (x_1, x_2, \ldots)$ as sequences of positive integers $(a_1, a_2, \ldots, a_n)$ with a nondecreasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ where $b_i \leq \sum_i x_i$. As part of a group from the 2023 Graduate Research Workshop in Combinatorics led by Andrés R. Vindas Meléndez, I am studying the polytope $\mathfrak{X}_n$ found by taking the convex hull of $\mathbf{x}$-parking functions of length n [8]. We show that these polytopes are generalized permutahedra and polymatroids from which a variety of interesting properties follow. I am also interested in the algebraic and enumerative aspects of generalized parking functions.

References

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