Lesson 9
walks

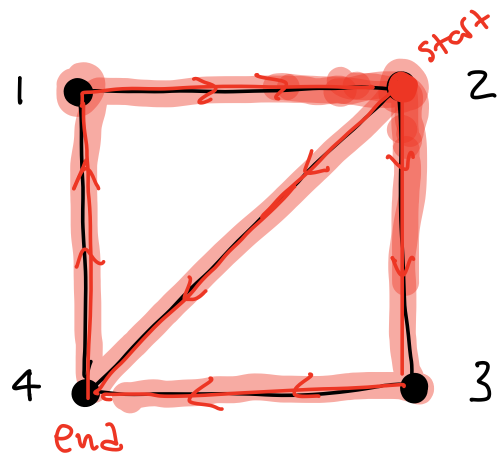
A **walk** in a graph is a segrence $V_1 V_2 V_3 \ldots V_n$ of not necessarily distinct vertices in which V_1 shares an edge to V_2, V_2 has an edge to V_3, etc...

A **closed walk** is a walk that starts and ends on the same vertex.

An **open walk** is a walk that ends on a different vertex than it sterted on

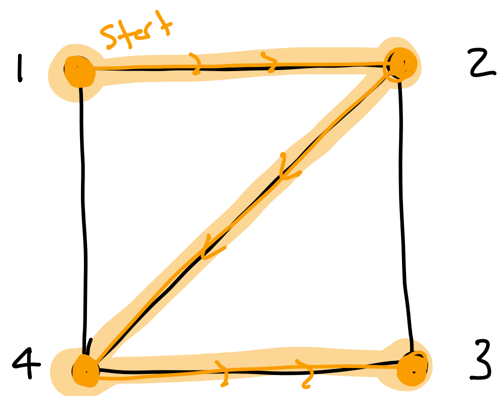
An **euler walk** is a walk that uses every edge exactly once

A **hamilton walk** is a walk that uses every vertex exactly once (potentially with the exception of the first/last being the same)



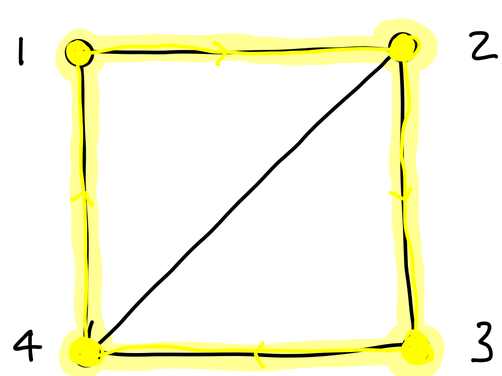
open euler walk

2 4 1 2 3 4



open hamilton walk

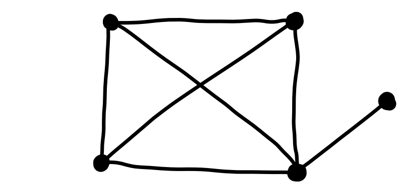
1 2 4 3

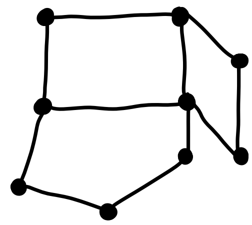


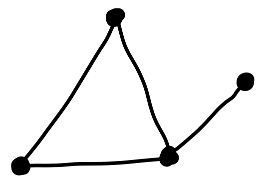
Closed Hamilton walk

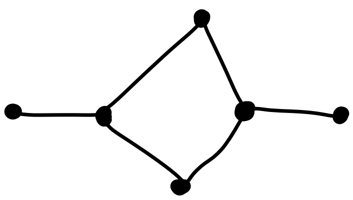
1 2 3 4 1

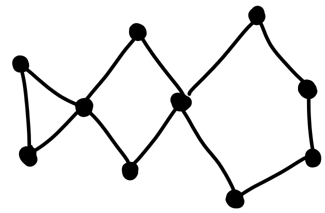
Do the following examples:

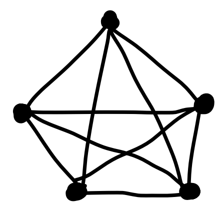




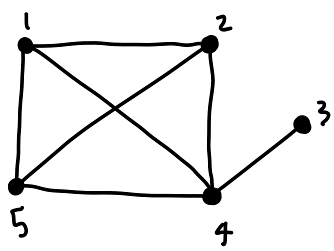


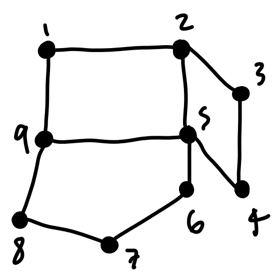


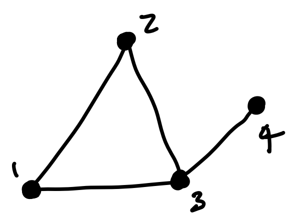




Solutions:







open hamilton

1 5 2 4 3

open euler

2 3 4 5 6 7 8 9 5 2 1 9

closed
open hamilton

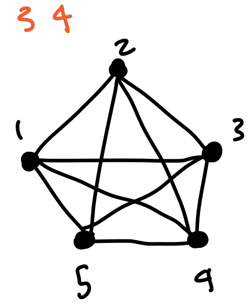
1 2 3 4 5 6 7 8 9 (1)

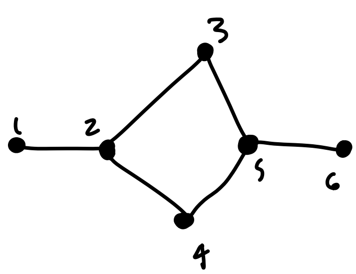
open euler

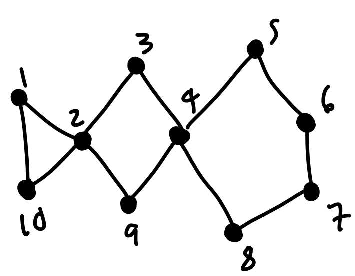
3 1 2 3 4

open hamilton

1 2 3 4







none

closed euler

1 2 3 4 5 6 7 8 4 9 2 10 1

closed euler

1 2 3 4 5 1 3 5 2 4 1

closed
open hamilton

1 2 3 4 5 (1)

For what $v \geq 2$ does K_v have:

(a) closed euler
(b) open euler
(c) closed hamilton
(d) open hamilton

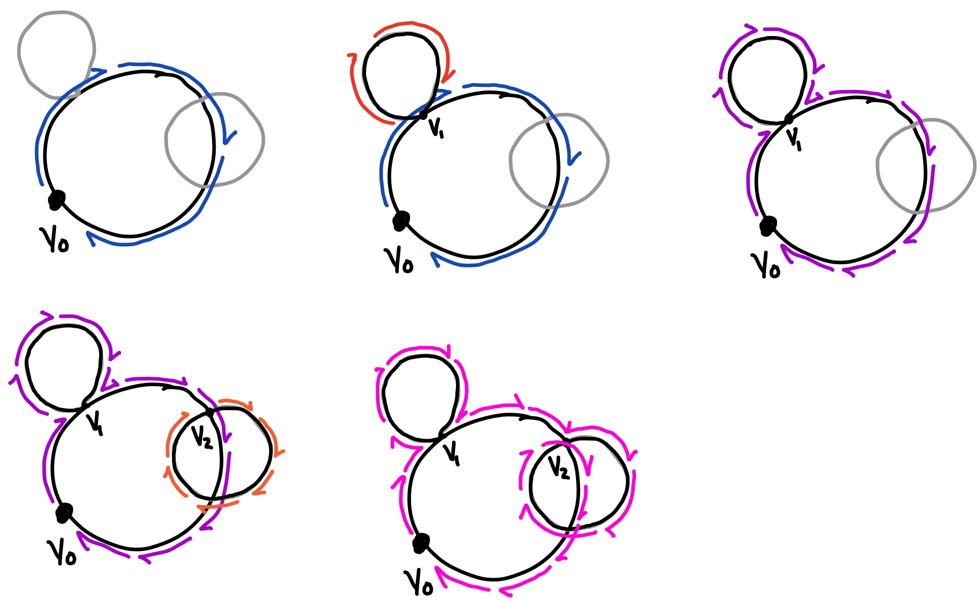
(a) V = odd
(b) $V = 2$
(c) $V \geq 3$
(d) $V \geq 2$

what conditions allow for a euler walk?
what about a hamilton walk?

Theorem: G has a closed euler walk if and only if G is even (aka every vertex has even degree).

Proof: (closed euler $\Rightarrow$ even) Let G be a graph with a closed euler walk. Each time the walk visits a vertex it uses an edge IN and an edge OUT, so 2 edges. Thus there will be an even number of edges to each vertex.

(even $\Rightarrow$ closed euler) Let G be a graph with all even degree vertices. Pick a vertex V_0. Starting at V_0 walk along unused edges until you no longer can to create walk W. This walk will necessarily bring you back to V_0 to be closed. Let V_1 be the first vertex on W with unused edges. Create a walk starting at V_1 the same way as before, called A_1 which will end up closing off at V_1. Splice walk A into walk W at the point V_1. Continue this process until there are no unused edges. W is now an euler walk.



Try this algorithm on an example graph!

Theorem: G has an open euler walk if and only if it has exactly two vertices of odd degree and the rest are even.

Theorem: If the sum of degrees of every pair of vertices in G is at least $V-1$ then G has an open hamilton walk. If the sum of degrees of every pair of vertices in G is at least V then G has a closed hamilton walk.

No proof today!

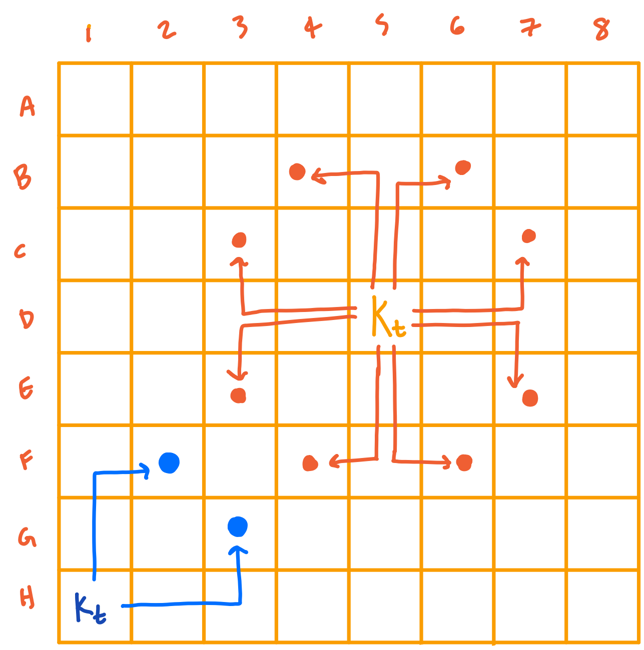
A fun problem to end class...

A knight can move:

- 2 horizontal 1 vertical
- 2 vertical 1 horizontal

Let G be a graph with 64 vertices representing the squares of the chessboard.

Let two vertices be adjacent if a knight can move between them in one move. So for example E3 is adjacent to D5.



Does G have an Euler walk?

Prove that G has a closed Hamilton walk, also called a "knights tour"?