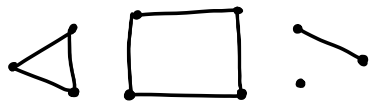
Lesson 8
connectivity

A **connected** graph is a graph where there exists a path between any two vertices.

We refer to each connected 'piece' of a disconnected graph as a **connected component**

$G =$  4 connected components

Up until now, we have only considered connected graphs. Many theorems still hold, others need to be changed slightly.

Eulers formula for disconnected graphs: For a planar graph with k connected components, $v-e+f=k+1$

Proof: let G have connected components $H_1, \dots, H_k$. If we exclude the infinite face we have $v_1-e_1+f_1=1, \dots, v_k-e_k+f_k=1$ for each component. Thus for the total graph we have $v-e+(f-1)=k$, so $v-e+f=k+1$.

Next we want to consider some measure of how connected a connected graph is.

A vertex cut set is a set of vertices that, if removed from the graph, would disconnect it.

An edge cut set is a set of edges that, if removed from the graph would disconnect it.

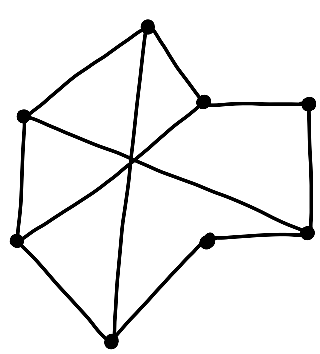
We may also refer to a single cut vertex or cut edge.

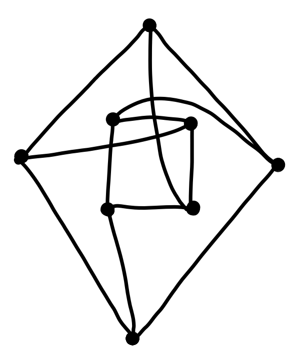
The vertex connectivity $\kappa(G)$ is the size of the minimum vertex cut set

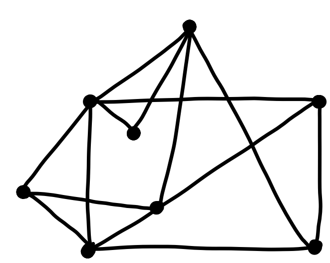
The edge connectivity $\lambda(G)$ is the size of the minimum edge cut set

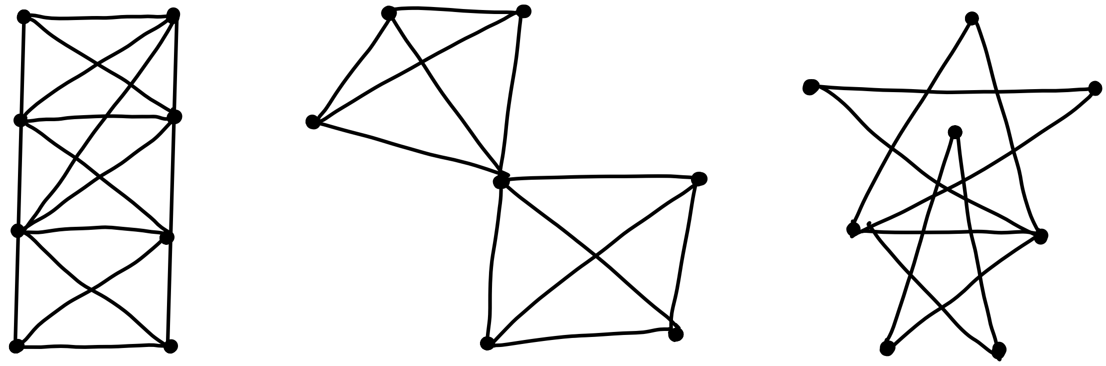
lets try to find some bounds on κ and λ.

First find κ and λ for the following examples:

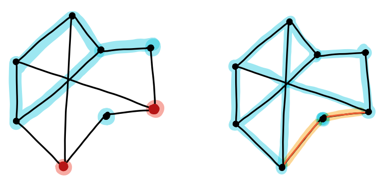




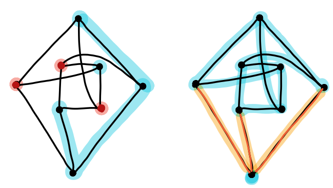




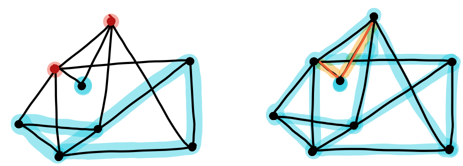
Solutions ... vertex cut set edge cut set whats left



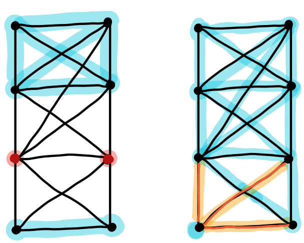
$\kappa(G) = 2$
$\lambda(G) = 2$



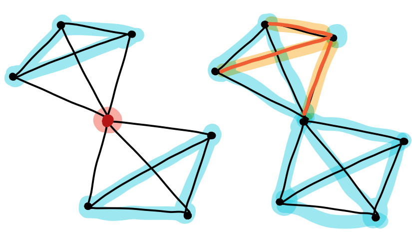
$\kappa(G) = 3$
$\lambda(G) = 3$



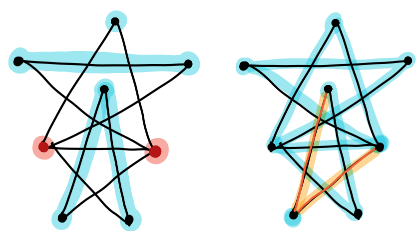
$\kappa(G) = 2$
$\lambda(G) = 2$



$\kappa(G) = 2$
$\lambda(G) = 3$



$\kappa(G) = 1$
$\lambda(G) = 3$



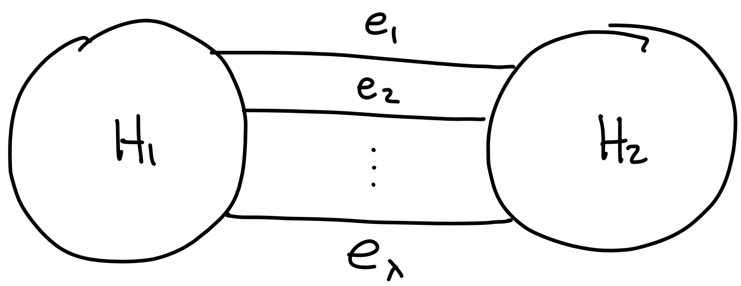
$\kappa(G) = 2$
$\lambda(G) = 2$

What bounds can we put on $\kappa(G)$ and $\lambda(G)$?
Can we relate them to each other?

. . .

claim: $\kappa(G) \le \lambda(G)$

proof: Let G have an edge cutset S_e of size $\lambda(G)$. Where the removal of S_e separates G into connected components H_1 and H_2. Let S_v be a set of vertices such that for edge $(x,y) \in S_e$ with $x \in H_1$ and $y \in H_2$, either $x \in S_v$ or $y \in S_v$. If possible, choose S_v so that $H_1 \not\subseteq S_v$ and $H_2 \not\subseteq S_v$.



case 1: It is possible to choose S_v such that $H_1 \not\subseteq S_v$ and $H_2 \not\subseteq S_v$. In this case we can remove S_v from G to get at least two connected components, since there will exist some vertices from H_1 and some from H_2 that are disconnected. Since S_v has size λ, we have $\kappa(G) \le \lambda(G)$.

case 2: WLOG, say $H_1 \subseteq S_v$. This case would only occur if every there were no vertices $a \in H_1$ and $b \in H_2$ that arent adjacent. If these vertices did exist, S_v would simply consist of every other vertex in H_1 and H_2. So every vertex in H_1 is connected to every vertex in H_2. If H_1 and H_2 have V_1 and V_2 vertices then we have $V = V_1 + V_2$ and $\lambda = V_1 V_2$. So $\lambda(G) \geq V-1$. Observe that $K(G) \leq V-1$ because worst case we remove all but one vertex. Thus $K(G) \leq \lambda(G)$.

claim: $K(G) \le \lambda(G) \le \delta(G)$ where δ is minimum degree

proof: Let V_0 be a vertex of min degree, and E_0 be its set of incident edges. Removing E_0 will disconnect V_0 from the other vertices in G. Thus E_0 is an edge cutset of size $\delta(G)$. So $\lambda(G) \le \delta(G)$

claim: $K(G) \le \lambda(G) \le \frac{2e}{v}$

proof: $\frac{2e}{v}$ is the average degree in G. we know min. degree $\le$ avg degree so $\delta \le 2e/v$. Therefore we have $K(G) \le \lambda(G) \le \frac{2e}{v}$

Mengers Theorem: If G is K-connected, then there are K independent paths between any two vertices.

Proof: complex, but there are many good versions out there! feel free to email me if you'd like to discuss.