Lesson 3

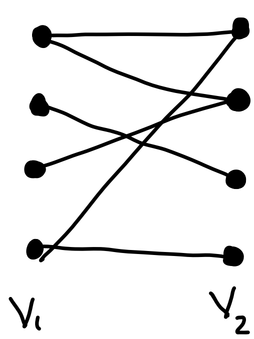
Bipartite Graph

Path Graphs

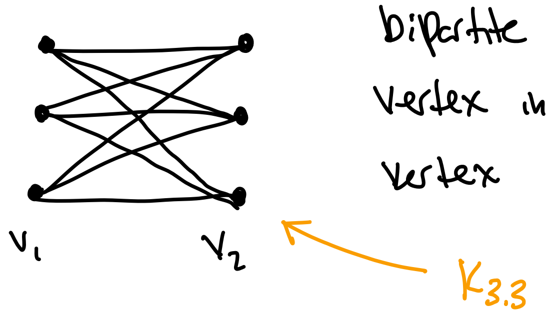
P_2	P_3	P_4
$v = 2$	$v = 3$	$v = 4$
$e = 1$	$e = 2$	$e = 3$

Bipartite Graph

A graph that can be split into two vertex sets V_1 and V_2 such that no two vertices in V_1 are adjacent and no two vertices in V_2 are adjacent



we can notate any bipartite graphs with vertex sets of size a and b as $B_{a,b}$

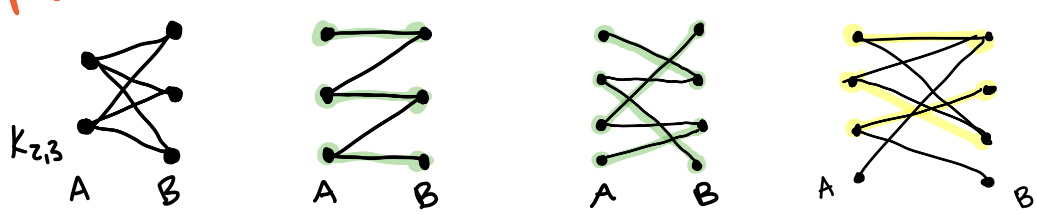


we can also define complete bipartite graphs $K_{a,b}$ where every vertex in V_1 is connected to every vertex in V_2 and vice versa.

The complete bipartite graph $K_{a,b}$ has $v = a+b$ vertices and $e = ab$ edges.

Thus any bipartite graph $B_{a,b}$ has $e \le ab$.

Examples:



A matching in a bipartite graph is a set of edges such that no two edges touch the same vertex.
A perfect matching is a matching such that every vertex is included in some edge.

In other words, a perfect matching is a set of edges that connects each vertex in V_1 to exactly one unique vertex in V_2, thus essentially pairing up the vertices in V_1 with those in V_2.

The second and third graphs above show perfect matchings highlighted in green. The fourth graph shows a matching of size 3.

Logic and proof:

Statement:	$a \Rightarrow b$	if a then b
Converse:	$b \Rightarrow a$	if b then a
Contrapositive:	not $b \Rightarrow$ not a	if not b then not a
inverse:	not $a \Rightarrow$ not b	if not a then not b
iff:	$a \Leftrightarrow b$	a if and only if b

An example of such a statement:

if a shape is a square then a shape is a rectangle.

Notice that the converse is false, as is the inverse.

if a shape is a rectangle, then a shape is a square.

if a shape is NOT a square then a shape is NOT a rectangle.

In general, the converse and/or inverse may or may not be true. However, the contrapositive is always true.

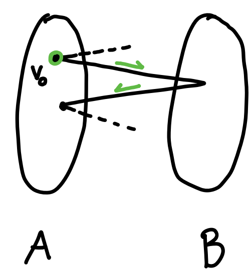
if a shape is NOT a rectangle, then a shape is NOT a square.

We have $a \Leftrightarrow b$ (a if and only if b) when both the statement and its converse are true

So $a \Leftrightarrow b$ means $a \Rightarrow b$ and $b \Rightarrow a$

Claim: If a graph G is bipartite, then it contains no odd cycles. In other words, any cycle is even.

proof: Let G be a bipartite graph with vertex sets A and B. Let C be some cycle in G. Pick some vertex v_0 in C such that $v_0 \in A$. Starting on v_0, walk along C. Since G is bipartite, the vertices will alternate between A and B. Each time we cross from A to B and back, we walk two edges. Since the cycle started in A it must end in A. Thus, we will walk an even number of edges. So, any cycle C will be even.



we have shown: Bipartite $\Rightarrow$ no odd cycles

Is the converse true? Does no odd cycles $\Rightarrow$ bipartite?

Hint/Preview: A graph is called 2-colorable if you can color the vertices with 2 colors such that no two vertices of the same color are adjacent. This is equivalent to being bipartite.