Lesson 4

Vertex coloring

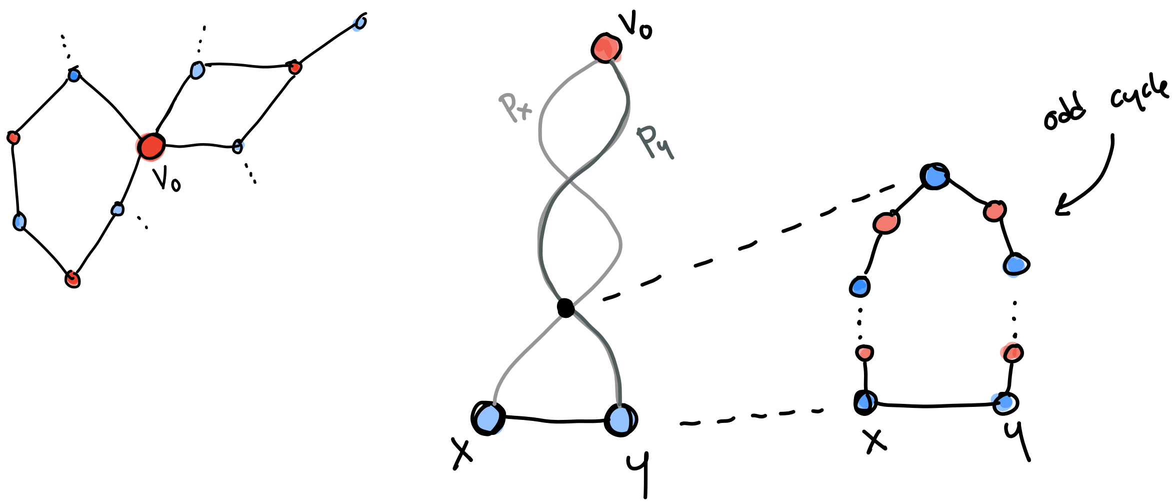
FACT: A graph is bipartite if and only if it is 2-colorable.

Claim: If a graph G has no odd cycles then it is bipartite.

Proof: Let G be a graph with no odd cycles. We will show G is bipartite by proving that it is 2-colorable. Pick some vertex v_0 in G and color it red. Going along the shortest path to v_0, color every vertex an odd distance from v_0 blue, and every vertex an even distance from v_0 red. Assume for contradiction that there are two adjacent vertices x and y of the same color. Let P_x and P_y be the shortest path from v_0 to x any y respectively. Let P_x' and P_y' be the paths P_x and P_y between their final point of intersection and x or y.
P_x' and P_y' go from a single vertex to two vertices of the same color, so they will have the same parity. Thus there is an even number of edges in $P_x' + P_y'$. We can form a cycle C

with p_x', p_y', and edge (x,y). This cycle will have an odd number of edges which contradicts the fact that G has no odd cycles. Thus, there cannot exist two adjacent vertices of the same color. So, G is 2-colorable.

ex)



Since we have proved the converse of our statement from lesson 3, we have shown

G is bipartite if and only if it has no odd cycles

Graph Coloring

A vertex coloring of a graph is an assignment of colors to vertices such that no two vertices of the same color are adjacent.

G is k-colorable if it can be colored with k colors

The chromatic number χ is the minimum number of colors you can color G with.

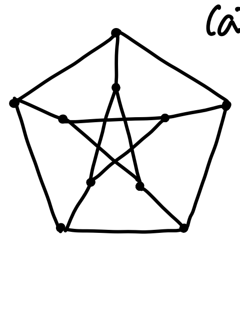
claim: G is k-colorable for all $k \geq \chi$.

claim: if G is k-colorable then G is j-colorable for $j \geq k$.

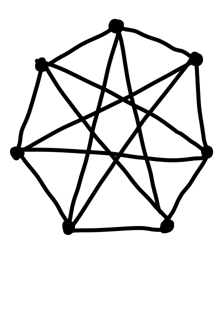
Find the chromatic numbers for C_n, W_n, and K_n.

Find the chromatic numbers for the following:

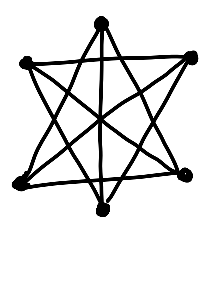
(a)



(b)



(c)

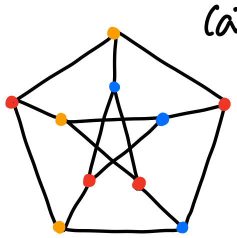


Solutions:

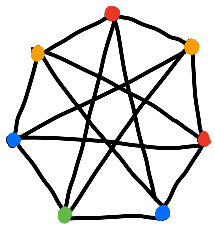
C_v $\quad \chi = \begin{cases} 2 & \text{if } v \text{ even} \\ 3 & \text{if } v \text{ odd} \end{cases}$

W_v $\quad \chi = \begin{cases} 3 & \text{if } v \text{ even} \\ 4 & \text{if } v \text{ odd} \end{cases}$

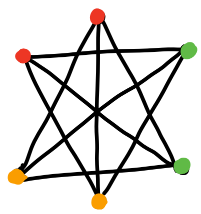
K_v $\quad \chi = v$



$\chi = 3$



$\chi = 4$



$\chi = 3$

Are there upper and lower bounds on chromatic number?

Let G be a graph with v vertices.
Then $\chi_G \leq v$, we could color each vertex a different color.

Let G have subgraph H. Then $\chi_H \leq \chi_G$.

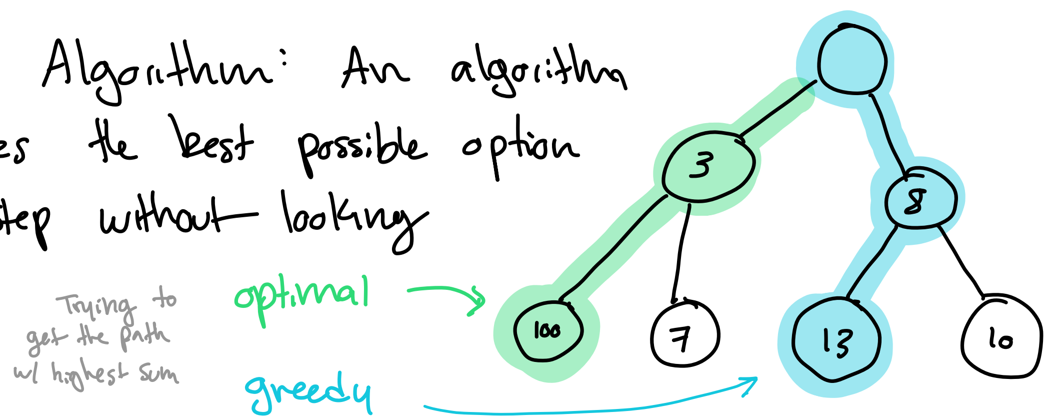
Define the clique # of a graph G as the largest m such that K_m is a subgraph of G.
Then clique # $\leq \chi_G$

Δ_G - the maximum degree in G
δ_G - the minimum degree in G

Claim: $\chi_G \leq \Delta_G + 1$
what we need for the proof:

Def: the <u>neighborhood</u> of a vertex is the set of all vertices it is adjacent to.

Greedy Algorithm: An algorithm that chooses the best possible option at each step without looking ahead.



Greedy Coloring Algorithm:

- order the vertices $\{v_1, v_2, \ldots, v_n\}$ we will use colors $1, 2, 3, \ldots$
- for every vertex (in order), label it with the smallest color not already in use in its neighborhood

Proof (that $\chi_G \leq \Delta + 1$): We claim that the greedy coloring algorithm produces a valid coloring with $\Delta+1$ or less colors. We color each vertex with the lowest color not already in its neighborhood, so we will never create two adjacent vertices of the same color. A vertex can have at most Δ different colors in its neighborhood, in which case we only need one additional color to color the vertex. No vertex is connected to more than Δ vertices so there will always be a color from 1 thru $\Delta+1$ available to use. Thus, our coloring only uses $\Delta+1$ colors.