Lesson 2: Graph Theory
Isomorphisms and Properties

Recall: two graphs are isomorphic if we can relabel the vertices of one graph to get two equal graphs.

How do we recognize isomorphic graphs?
What properties are preserved under isomorphism.

properties preserved under isomorphism:

- number of vertices
- number of edges
- distribution of degrees aka degree sequence
- subgraphs

This is to say, if $G \cong H$ the G and H will have the same number of vertices and the same number of edges. G and H will have the same number of vertices of degree d; i.e. if G has two vertices of degree 3 then H must also have exactly two vertices of degree 3. If G has a subgraph F, the H must have a subgraph isomorphic to F. i.e. if G contains $\triangle$ then H must also contain $\triangle$.

If two graphs have differences in any of these properties, they are NOT isomorphic.

However, these properties holding doesn't necessarily mean that two graphs will be isomorphic. To show that, you must describe the one-to-one correspondence that you can use to relabel the graphs.

problems

1. if $v \geq 4$, the wheel graph W_v has $V = \{1, 2, \dots, v\}$ and $E = \{(1,2), (1,3), \dots, (1,v), (2,3), (3,4), \dots, (v-1, v), (v,2)\}$.
   Draw the first 5 wheel graphs.
   Prove a formula for the # of edges in W_v.

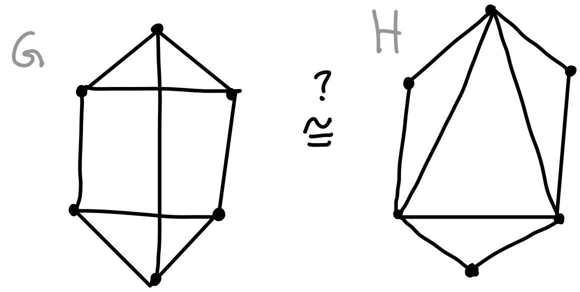
2. Let G be a graph with $v = 6$. Prove that either G or $\bar{G}$ has a subgraph isomorphic to K_3.

3. Prove the sum of degrees of a graph is $2e$.

4. Prove that $C_5 \cong \bar{C}_5$. Then prove that no other cycle graph is isomorphic to its complement.

5. Decide whether each pair of graphs is isomorphic or not, then prove it.

49



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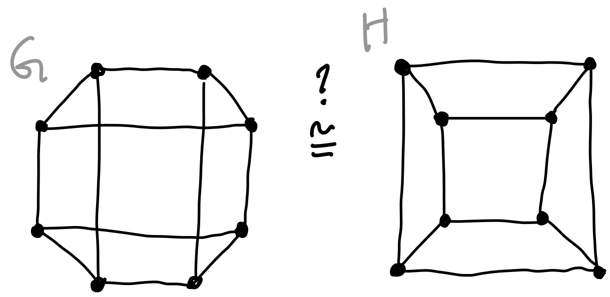

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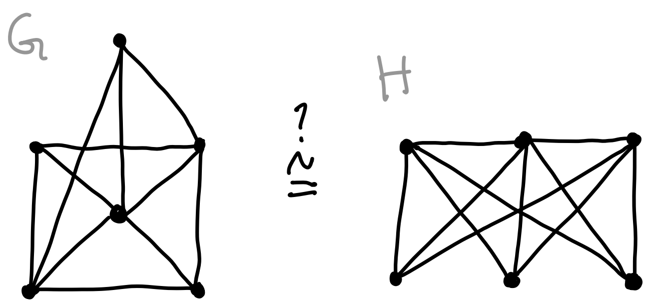

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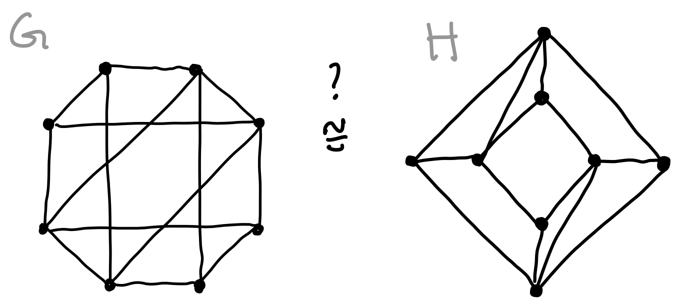
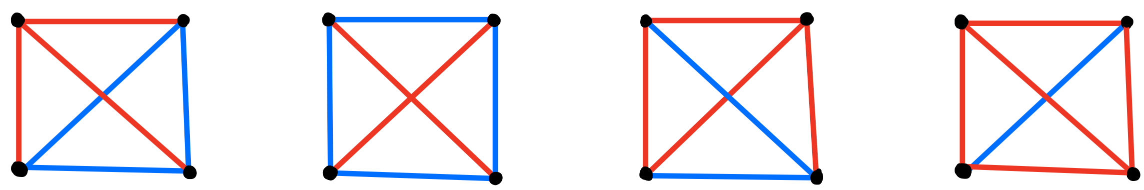


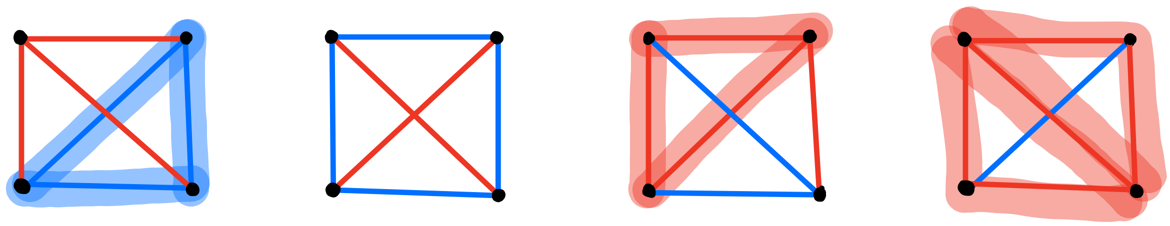
Example problem and proof

Consider the complete graph K_4 Color the edges such that every edge is red or blue

examples of possible colorings:



Notice that some of these edge-colorings contain triangles of entirely red or blue. Other don't.



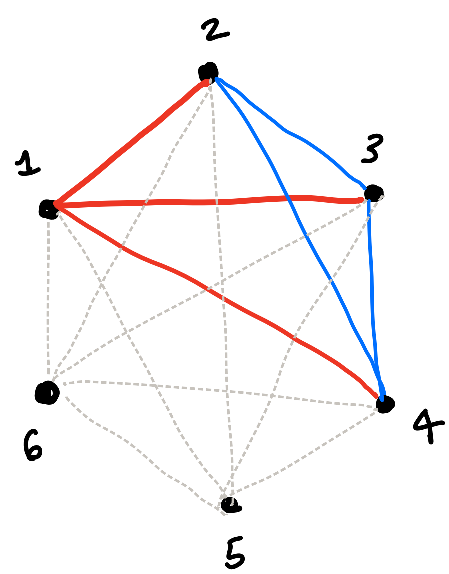
Note that these triangles are really K_3 - the complete graph on three vertices

Goal: Determine the smallest number v such that any such coloring of K_v must contain either a red or blue triangle. Can you prove it?

Answer: $V = 6$

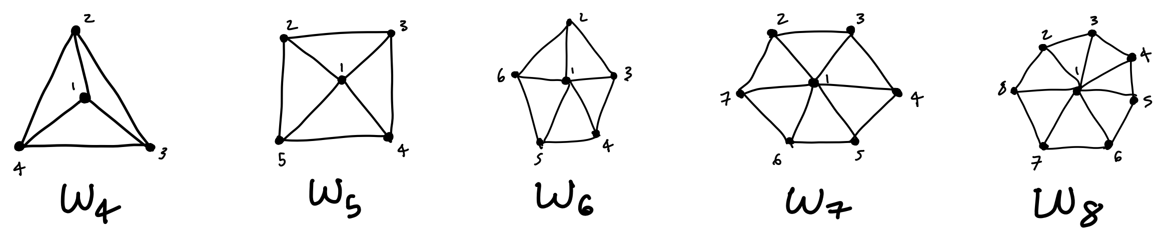
Proof: Given K_6, consider a single vertex, say 1. At least 3 of the edges incident to vertex 1 must be the same color. Without loss of generality, call these edges (1,2), (1,3), and (1,4) and let them be red. If either (2,3), (3,4) or (2,4) is colored red then we would get an all red K_3 ($\Delta 123$, $\Delta 124$, $\Delta 134$ respectively). Thus none can be red. If (2,3), (3,4) and (2,4) are all blue then we have an all blue K_3 on $\Delta 234$. Thus, there must be a K_3 subgraph all of one color.

K_6 is a subgraph of every K_V for $V \geq 6$, so those K_V will also have a K_3 subgraph of all one color.



Solutions

1.



If W_v has v vertices and e edges, $e = 2(v-1)$

proof: Vertex 1 is connected to vertices $2, 3, \ldots, v$ so there are $v-1$ "spokes" on our wheel. The remaining edges form a cycle on the vertices $2, 3, \ldots, v$. A cycle on $v-1$ vertices will have $v-1$ edges.* Thus, we have $2(v-1)$ edges total.

* Prop: a cycle on v vertices has v edges, i.e. $v=e$.

proof: The cycle C_v has vertex set $V = \{1, 2, \ldots, v\}$ and edge set $E = \{(1,2), (2,3), \ldots, (v-1, v), (v, 1)\}$. For each of the vertices 1 through $v-1$ we have the edge $(i, i+1)$. (ex: for vertex 5 we have edge $(5,6)$). So there are $v-1$ of these edges. The only remaining edge to count is $(v,1)$. Thus, $e=v$. Alternatively, every vertex in a cycle has degree 2 so the total degree is $2v$. Using problem 3 we get $2v = 2e$. Thus $v=e$.

2. Consider K_6. If an edge is in G, color it red (in K_6). If an edge is in $\overline{G}$, color it blue (in K_6). This accounts for every edge. By the proof above, this graph will contain either a red or blue K_3. Thus, either G or $\overline{G}$ contains a subgraph isomorphic to K_3.

3. Degree counts the number of edges incident to a vertex. Since each edge is incident to two vertices, each edge contributes 2 to the total degree of the graph. Thus total degree $= 2e$.

4. We have $C_5 = (V, E)$ with

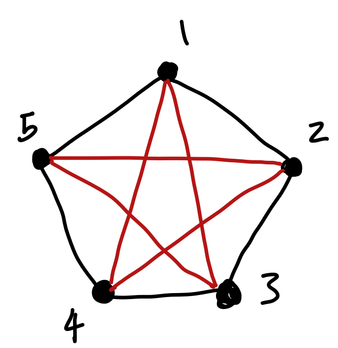
$V = \{1, 2, 3, 4, 5\}$ and

$E = \{(1,2), (2,3), (3,4), (4,5), (5,1)\}$

and $\overline{C_5} = (V, \overline{E})$ with

$V = \{1, 2, 3, 4, 5\}$

$\overline{E} = \{(1,3), (1,4), (2,4), (2,5), (3,5)\}$



relabel $\overline{C_5}$ as follows: $1 \to 1$ $\quad$ $3 \to 2$ $\quad$ $5 \to 3$

$2 \to 4$ $\quad$ $4 \to 5$

Let our relabelled $\overline{C_5}$ be $\overline{C_5}' = (\overline{V}', \overline{E}')$ (prime).
Then, we have

$\overline{V}' = \{1, 4, 2, 5, 3\}$ and

$\overline{E}' = \{(1,2), (1,5), (4,5), (4,3), (2,3)\}$

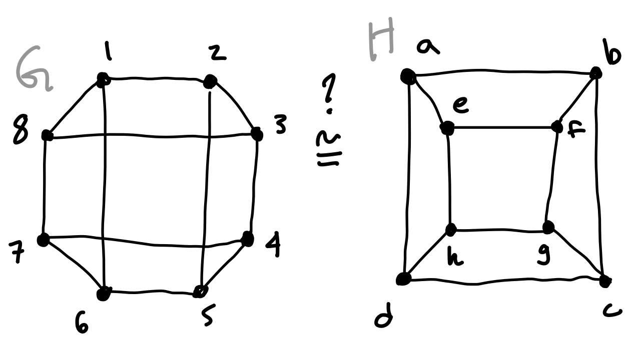
so $\overline{V}' = V$ and $\overline{E}' = E$. Therefore $C_5 \cong \overline{C_5}$.

Consider C_v with $v \neq 5$. Each vertex has degree 2. There will be $(v-3)$ vertices each vertex is not currently adjacent to. Thus, the degree of each vertex in $\overline{C_v}$ will be $(v-3)$. Since $v \neq 5$ we have $(v-3) \neq 2$. Therefore C_v isn't isomorphic to $\overline{C_v}$.

5.

(49) Not isomorphic. G is all vertices of degree 3 and H has no vertices of degree 3.

(50) isomorphic.
Lets label our graphs as shown to the right.



So, $G = (V_G, E_G)$ with

$V_G = \{1,2,3,4,5,6,7,8\}$

$E_G = \{(1,2),(1,6),(1,8),(2,3),(2,5),(3,4)$
$(3,8),(4,5),(4,7),(6,7),(7,8),(8,1)\}$

and $H = (V_H, E_H)$ with

$V_H = \{a,b,c,d,e,f,g,h\}$

$E_H = \{(a,b),(a,e),(a,d),(b,f),(b,c),(c,g)$
$(c,d),(d,h),(e,f),(f,g),(g,h),(h,e)\}$

relabel H to get $H' = (V_H', E_H')$ as follows:

$a \to 1$ $d \to 6$ $g \to 4$
$b \to 2$ $e \to 8$ $h \to 7$
$c \to 5$ $f \to 3$

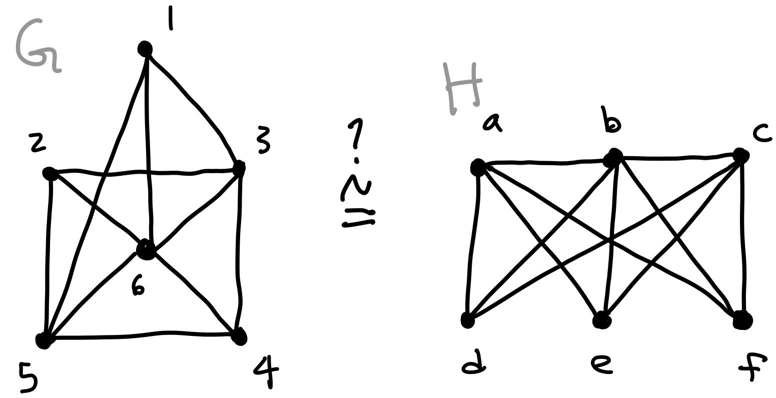
So $V_H' = \{1,2,5,6,8,3,4,7\}$

$E_H' = \{(1,2),(1,8),(1,6),(2,3),(2,5),(5,4),(5,6),$
$(6,7),(8,3),(3,4),(4,7),(7,8)\}$

So we have $V_H' = V_G$ and $E_H' = E_G$.
Thus, $G \cong H$.

(52) label our graphs as:

so we have



$V_G = \{1, 2, 3, 4, 5, 6\}$

$E_G = \{(1,5), (1,6), (1,3), (2,3), (2,6), (2,5), (3,6), (3,4)$
$(4,5), (4,6), (5,6)\}$

$V_H = \{a, b, c, d, e, f\}$

$E_H = \{(a,b), (a,d), (a,e), (a,f), (b,c), (b,d), (b,e), (b,f),$
$(c,d), (c,e), (c,f)\}$

Relabel the vertices of H to H' as follows:

$a \to 5 \quad b \to 6 \quad c \to 3 \quad d \to 2 \quad e \to 1 \quad f \to 4$

so $V_H' = \{5, 6, 3, 2, 1, 4\}$ and
$E_H' = \{(5,6), (5,2), (5,1), (5,4), (6,3), (6,2), (6,1)$
$(6,4), (3,2), (3,1), (3,4)\}$

So $V_H' = V_G$ and $E_H' = E_G$. Thus $G \cong H$.

(59) Not isomorphic. H has a K_3 subgraph and G does not.