Extended Schur Functions and Bases Related by Involutions

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For a composition α, a **shin-tableau** T is a diagram of shape α filled with positive integers such that each row weakly increases from left to right and each column strictly increases from top to bottom.

The Extended Schur Functions [1]

For a composition α, the **extended Schur function** is defined by

$$\mathfrak{w}^*_\alpha = \sum_T x^T,$$

where the sum runs over shin-tableaux T of shape α. These functions form a basis of $QSym$, and $\mathfrak{w}^*_\lambda$ is equal to the Schur function s_λ when λ is a partition.

$$\mathfrak{w}^*_{(2,3)} = x_1^2x_2^3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1x_2x_3x_4^2 + \cdots$$

$$\begin{array}{ccc} 1&1& \\ 2&2&2 \end{array} \quad \begin{array}{ccc} 1&1& \\ 2&2&3 \end{array} \quad \begin{array}{ccc} 1&1& \\ 2&3&3 \end{array} \quad \begin{array}{ccc} 1&2& \\ 2&3&3 \end{array} \quad \begin{array}{ccc} 1&3& \\ 2&4&4 \end{array} \quad \cdots$$

A *standard* shin-tableau U of size n has entries $\{1,2,\dots,n\}$ each appearing once.

$$Des_{\mathfrak{w}}(U) = \{i : i+1 \text{ is in a strictly lower row than } i\}$$

$$co_{\mathfrak{w}}(U) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d) \text{ for } Des_{\mathfrak{w}}(U) = \{i_1, \dots, i_d\}$$

For a composition α,

$$\mathfrak{w}^*_\alpha = \sum_U F_{co_{\mathfrak{w}}(U)},$$

where the sum runs over standard shin-tableaux U of shape α.

The **complement** of a composition α, denoted α^c, is the composition obtained from the complement of the set associated with α. The **reverse** of $(\alpha_1,\dots,\alpha_k)$, denoted α^r, is $(\alpha_k,\dots,\alpha_1)$. The **transpose** of α is defined by $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$.

Involutions on $QSym$ [4]

For a composition α, define the following involutive automorphisms on $QSym$:

$$\psi(F_\alpha) = F_{\alpha^c} \qquad \rho(F_\alpha) = F_{\alpha^r} \qquad \omega(F_\alpha) = F_{\alpha^t}$$

Row-strict Extended Schur Functions [5]

The **row-strict extended Schur functions** are defined via *row-strict shin-tableaux* which have strictly increasing rows and weakly increasing columns. For a *standard* row-strict shin-tableau U, a *descent* $i \in Des_{\mathfrak{Rw}}(U)$ is defined as an entry i such that $i+1$ is in a weakly higher row.

The set of standard row-strict shin-tableaux is the same as the set of standard shin-tableaux, and for a standard (row-strict) shin-tableau U, we have $co_{\mathfrak{w}}(U)^c = co_{\mathfrak{Rw}}(U)$.

For a composition α,

$$\psi(\mathfrak{w}^*) = \mathfrak{Rw}^*_\alpha$$

An Involution on the Schur Functions

The classical involution $\omega : Sym \to Sym$ maps the Schur basis to itself by $\omega(s_\lambda) = s_{\lambda'}$ where λ' is the conjugate of λ. Collectively, the involutions ψ, ρ, and ω on $QSym$ serve as an analogue to the classical ω. The extended Schur functions are part of what is essentially a system of four bases that is closed under the involutions ψ, ρ, and ω in $QSym$. **We introduce two new Schur-like bases of QSym** that, when paired with the extended Schur and row-strict extended Schur functions, complete this picture.

ψ and ω on $QSym$ restrict to ω on Sym, and ρ restricts to the identity map on Sym.

Flipped Extended Schur Functions

Let α be a composition and β a weak composition. A *flipped shin-tableau* of shape α and type β is a composition diagram α filled with positive integers that weakly decrease along the rows from left to right and strictly increase along the columns from top to bottom, where each positive integer i appears β_i times. A *standard* flipped tableau S of size n contains the entries $\{1,2,\dots,n\}$ each exactly once.

$$Des_{\mathfrak{Fw}}(S) = \{i : i+1 \text{ is in a strictly lower row than } i\}$$

$$co_{\mathfrak{Fw}}(S) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d) \text{ for } Des_{\mathfrak{Fw}}(S) = \{i_1, \dots, i_d\}$$

For a composition α, the **flipped extended Schur function** is defined as

$$\mathfrak{Fw}^*_\alpha = \sum_S F_{co_{\mathfrak{Fw}}(S)},$$

where the sum runs over standard flipped shin-tableaux S of shape α.

There is a map *flip* between standard shin-tableaux U and standard flipped shin-tableaux S such that the descent composition of U is the reverse of the descent composition of $flip(U) = S$. First, flip U horizontally, then replace each entry i with $n - i$.

$$(2,2,1) \quad \begin{array}{ccc} 1&2&4 \\ 3&5& \end{array} \to \begin{array}{ccc} &3&5 \\ 1&2&4 \end{array} \to \begin{array}{ccc} 3&1& \\ 5&4&2 \end{array} \quad (1,2,2)$$

Theorem (D. 2023)

For a composition α,

$$\rho(\mathfrak{w}^*_\alpha) = \mathfrak{Fw}^*_{\alpha^r} \text{ and } \omega(\mathfrak{Rw}^*_\alpha) = \mathfrak{Fw}^*_{\alpha^r}.$$

Backward Extended Schur Functions

Let α be a composition and β be a weak composition. A *backward shin-tableau* of shape α and type β is a composition diagram α filled with positive integers that strictly decrease along the rows from left to right and weakly increase along the columns from top to bottom, where each integer i appears β_i times. A *standard* backward tableau S of size n contains the entries $\{1,2,\dots,n\}$ each exactly once.

$$Des_{\mathfrak{Bw}}(S) = \{i : i+1 \text{ is in a weakly higher row than } i\}$$

$$co_{\mathfrak{Bw}}(S) = (i_1, i_2 - i_1, \dots, i_d - i_{d-1}, n - i_d) \text{ for } Des_{\mathfrak{Bw}}(S) = \{i_1, \dots, i_d\}$$

For composition α, the **backward extended Schur function** is defined as

$$\mathfrak{Bw}^*_\alpha = \sum_S F_{co_{\mathfrak{Bw}}(S)},$$

where the sum runs over standard backward shin-tableaux S of shape α.

Theorem (D. 2023)

For a composition α,

$$\omega(\mathfrak{w}^*_\alpha) = \mathfrak{Bw}^*_{\alpha^r} \quad \text{and} \quad \rho(\mathfrak{Rw}^*_\alpha) = \mathfrak{Bw}^*_{\alpha^r} \quad \text{and} \quad \psi(\mathfrak{Fw}^*_\alpha) = \mathfrak{Bw}^*_\alpha.$$

Backward shin-tableaux are a row-strict version of flipped shin-tableaux. In fact, the set of standard row-strict tableaux and the set of standard backward tableaux are the same but for a tableaux S, we will have $co_{\mathfrak{Fw}}(S) = co_{\mathfrak{Bw}}(S)^c$.

$$\mathfrak{w}^*_{(3,2)} = F_{(3,2)} + F_{(2,2,1)} + F_{(1,3,1)} + F_{(2,3)} + F_{(1,2,2)}$$

$$\begin{array}{ccc} 1&2&3 \\ 4&5& \end{array} \quad \begin{array}{ccc} 1&2&4 \\ 3&5& \end{array} \quad \begin{array}{ccc} 1&3&4 \\ 2&5& \end{array} \quad \begin{array}{ccc} 1&2&5 \\ 3&4& \end{array} \quad \begin{array}{ccc} 1&3&5 \\ 2&4& \end{array}$$

$$\mathfrak{Rw}^*_{(3,2)} = F_{(1,1,2,1)} + F_{(1,2,2)} + F_{(2,1,2)} + F_{(1,2,1,1)} + F_{(2,2,1)}$$

$$\begin{array}{ccc} 1&2&3 \\ 4&5& \end{array} \quad \begin{array}{ccc} 1&2&4 \\ 3&5& \end{array} \quad \begin{array}{ccc} 1&3&4 \\ 2&5& \end{array} \quad \begin{array}{ccc} 1&2&5 \\ 3&4& \end{array} \quad \begin{array}{ccc} 1&3&5 \\ 2&4& \end{array}$$

$$\mathfrak{Fw}^*_{(2,3)} = F_{(2,3)} + F_{(1,2,2)} + F_{(1,3,1)} + F_{(3,2)} + F_{(2,2,1)}.$$

$$\begin{array}{ccc} 2&1& \\ 5&4&3 \end{array} \quad \begin{array}{ccc} 3&1& \\ 5&4&2 \end{array} \quad \begin{array}{ccc} 4&1& \\ 5&3&2 \end{array} \quad \begin{array}{ccc} 3&2& \\ 5&4&1 \end{array} \quad \begin{array}{ccc} 4&2& \\ 5&3&1 \end{array}$$

$$\mathfrak{Bw}^*_{(2,3)} = F_{(1,2,1,1)} + F_{(2,2,1)} + F_{(2,1,2)} + F_{(1,1,2,1)} + F_{(1,2,2)}$$

$$\begin{array}{ccc} 2&1& \\ 5&4&3 \end{array} \quad \begin{array}{ccc} 3&1& \\ 5&4&2 \end{array} \quad \begin{array}{ccc} 4&1& \\ 5&3&2 \end{array} \quad \begin{array}{ccc} 3&2& \\ 5&4&1 \end{array} \quad \begin{array}{ccc} 4&2& \\ 5&3&1 \end{array}$$

Dual Bases in $NSym$

Each of our bases is dually paired with a variant of the *shin basis* of *NSym*. We introduce two of these: the **flipped shin functions** and **backward shin functions**. For each of these bases, we can obtain immediate properties by applying ψ, ρ, or ω to results on the shin functions. We also know the *commutative image* of a flipped or backward shin function. For a partition λ,

$$\chi(\mathfrak{Fw}_{\lambda^r}) = s_\lambda \quad \text{and} \quad \chi(\mathfrak{Bw}_{\lambda^r}) = s_{\lambda'},$$

and for a composition α that is not the reverse of a partition,

$$\chi(\mathfrak{Fw}_\alpha) = \chi(\mathfrak{Bw}_\alpha) = 0.$$

$$\begin{tikzcd}
\mathfrak{w}^* \arrow[r, leftrightarrow, "\psi"] \arrow[rr, leftrightarrow, bend left, "\rho"] \arrow[rrr, leftrightarrow, bend left, "\omega"] & \mathfrak{Rw}^* \arrow[r, leftrightarrow, "\omega"] \arrow[rr, leftrightarrow, bend right, "\rho"'] & \mathfrak{Fw}^* \arrow[r, leftrightarrow, "\psi"] & \mathfrak{Bw}^*
\end{tikzcd}$$

[1] S. Assaf and D. Searles. *Kohnert polynomials.* 2022. [2] J. M. Campbell, K. Feldman, J. Light, P. Shuldiner, and Y. Xu. *A Schur-like basis of NSym defined by a Pieri rule. 2014.* [3] S. Daugherty. *Extended Schur functions and bases related by involutions.* Preprint. 2023. [4] K. Luoto, S. Mykytiuk, and S. van Willigenburg. *An introduction to quasisymmetric Schur functions.* 2013. [5] E. Niese, S. Sundaram, S. van Willigenburg, J. Vega, and S. Wang. *Row-strict dual immaculate functions and 0-Hecke modules.* 2022.