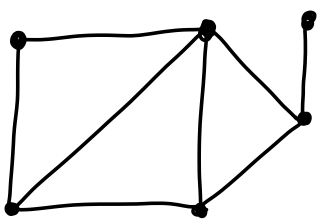
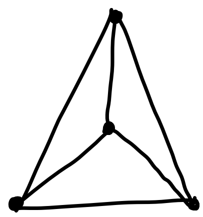
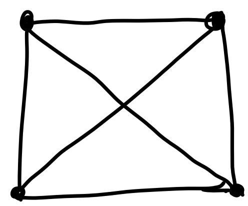
Lesson 5
Planar graphs

A graph is planar if it is isomorphic to a graph that can be drawn in the plane with no edge crossings.

ex)





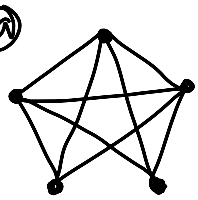


A nonplanar graph is one that cant be drawn without edge crossings.

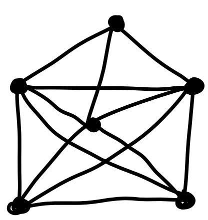
Consider graphs we know, which are planar?
C_n? W_n? How about K_n? $K_{a,b}$?

What about:

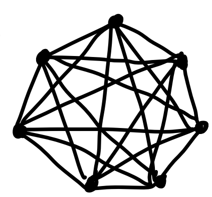
(a)



(b)



(c)



K_7

C_v - planar $\quad$ W_v - planar $\quad$ P_v - planar

K_v planar if $v \leq 4$
non-planar if $v \geq 5$

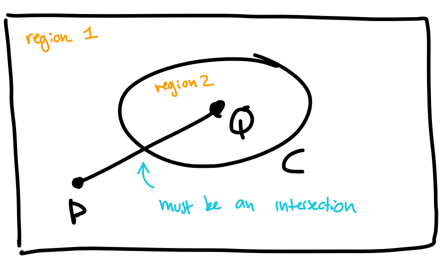
$K_{a,b}$ Planar if $a<3$ or $b<3$
nonplanar if $a \geq 3$ and $b \geq 3$

(a) planar $\quad$ (b) nonplanar $\quad$ (c) nonplanar

We prove graphs are planar by showing a planar drawing. How do we prove something is nonplanar?

Jordan Curve Theorem: If C is a continuous simple closed curve in a plane, then C divides the rest of the plane into two regions. If a point P in one of these regions is joined to a point Q in the other by a continuous curve L, then L intersects C.

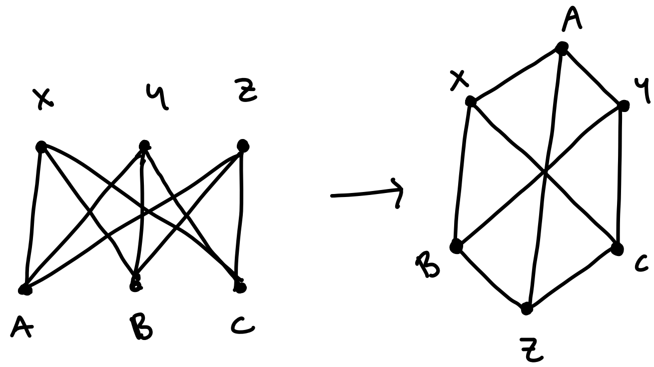
All this actually says... →



We will use this seemingly obvious fact in our proofs that graphs are nonplanar.

Claim: $K_{3,3}$ is nonplanar.

Redraw $K_{3,3}$ from the typical shape to the shape on the right. Let $\mathbf{C}$ denote the curve / cycle on



A - Y - C - Z - B - X - A. Then we also have edges / "curves" $\{A,Z\}$, $\{B,Y\}$, and $\{C,Z\}$. By Jordan Curve Theorem, eac of these edges is either entirely inside or outside $\mathbf{C}$. Now we consider two cases:

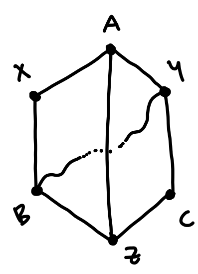
Case 1. There are at least two edges inside C

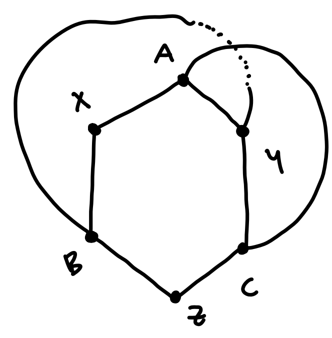
Say that $\{A,Z\}$ and $\{B,Y\}$ are definitely inside $\mathbf{C}$. $\{A,Z\}$ divides $\mathbf{C}$ into two regions, with B and Y in separate regions. By JCT, the edge $\{B,Z\}$ must intersect $\{A,Z\}$.

Case 2. There are at least two edges outside $\mathbf{C}$

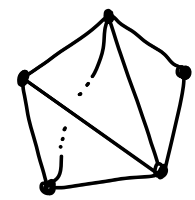
Say that $\{A,Z\}$ and $\{B,Y\}$ are definitely outside $\mathbf{C}$. Then $\{A,Z\}$ divides the area outside $\mathbf{C}$ into two regions, with B and Y laying in separate regions. Thus, $\{B,Y\}$ must intersect $\{A,Z\}$.

Thus it is impossible to draw $K_{3,3}$ in the plane without edge crossings.





Claim: K_5 is nonplanar.



proof: Similar to above
the cases will be case 1: ≥ 3 edges inside and case 2: ≥ 3 edges outside. Put down 2 edges then show theres no way a 3rd edge can be in the same region.

What other graphs are non-planar?

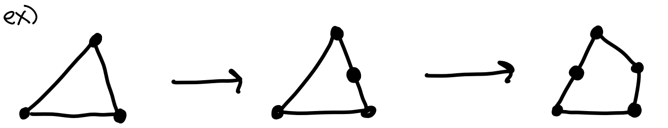
FACT: If H is nonplanar and H is a subgraph of G, then G is nonplanar.

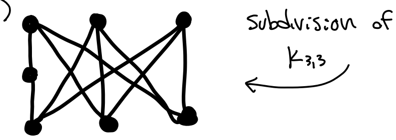
To draw G crossing-free in the plane, we would also have to be able to draw H, but we know we can't. Thus G is also nonplanar.

It follows that any graph with a $K_{3,3}$ or K_5 subgraph is nonplanar. Whats more surprising is that we can use $K_{3,3}$ and K_5 to define <u>ALL</u> nonplanar graphs.

Def: If H is a subgraph of G we say that G is a supergraph of H.

Def: If some new vertices of degree 2 are added to some edges of a graph G, then the resulting graph is called a subdivision (or expansion) of G.

ex) 

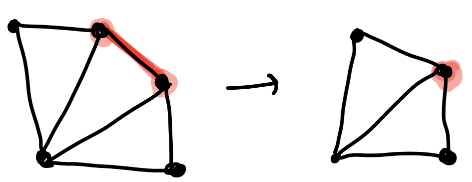
ex) 

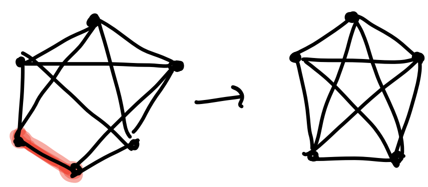
Corollary: Every subdivision of $K_{3,3}$ and K_5 is planar.

Kuratowskis Theorem: Every non-planar graph G has a subdivision/expansion of $K_{3,3}$ or K_5 as a subgraph.

Alternatively, we have an equivalent theorem.

Def: A graph H is a G minor if we can delete edges and vertices and contract edges in H to get G





Wagners Theorem: G is planar if and only if it does not have a $K_{3,3}$ or K_5 minor subgraph.