Séminaire Lotharingien de Combinatoire **XX** (2024)
Article #YY, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Extended Schur Functions and Bases Related by Involutions

Spencer Daugherty*[1]

[1]*Department of Mathematics, North Carolina State University*

Abstract. The extended Schur and shin functions are Schur-like bases of $QSym$ and $NSym$. We define a creation operator and a Jacobi-Trudi rule for certain shin functions and show that a similar Jacobi-Trudi rule does not exist for every shin function. We also define the skew extended Schur functions and relate them to the multiplicative structure of the shin basis. Then, we introduce two new pairs of dual bases that result from applying the ρ and ω involutions to the extended Schur and shin functions. These bases are defined combinatorially via variations on shin-tableaux much like the row-strict extended Schur functions.

Keywords: Schur Functions, QSym, NSym, Tableaux.

1 Introduction

There has been considerable interest over the last decade in studying Schur-like bases of $NSym$ and $QSym$. A basis $\{\mathcal{S}_\alpha\}_\alpha$ of $NSym$ is generally considered *Schur-like basis* if $\chi(\mathcal{S}_\lambda) = s_\lambda$ for any partition λ where $\chi : NSym \to Sym$ gives the commutative image of an element in $NSym$. A Schur-like basis $\{\mathcal{S}^*_\alpha\}_\alpha$ of $QSym$ is informally defined as a basis dual to a Schur-like basis $\{\mathcal{S}_\alpha\}_\alpha$ of $NSym$. These bases are usually defined combinatorially in terms of tableaux that resemble or generalize the semistandard Young tableaux. The canonical Schur-like bases of $NSym$ and $QSym$ are the immaculate basis [3], the Young noncommutative Schur basis [7], and the shin basis [6], as well as the dual immaculate basis, the Young quasisymmetric Schur basis, and the extended Schur functions.

The shin and extended Schur functions, which are dual bases, are unique among the Schur-like bases for having arguably the most natural relationship with the Schur functions. In $NSym$, the commutative image of a shin function indexed by a partition is a Schur function, while the commutative image of any other shin function is 0. In $QSym$, the extended Schur functions indexed by partitions are equal to Schur functions [6]. The goal of this extended abstract is to further the study of these two bases and introduce new, related bases.

*sdaughe@nscu.edu.

1.1 The Shin and Extended Schur Functions

The dual shin functions were introduced by Campbell, Feldman, Light, Shuldiner, and Xu in [6] as the duals to the shin functions and defined independently by Assaf and Searles in [2] as the extended Schur functions, which are the stable limits of polynomials related to Kohnert diagrams. We will use the name 'extended Schur functions' but otherwise retain the notation and terminology of the dual shin functions.

Definition 1.1. Let α and β be compositions of n. A *shin-tableau* of shape α and type β is a labeling of the boxes of the diagram of α by positive integers such that the number of boxes labeled by i is β_i, the sequence of entries in each row is weakly increasing from left to right, and the sequence of entries in each column is strictly increasing from top to bottom.

Example 1.2. The shin-tableaux of shape $(3,4)$ and type $(1,2,1,1,2)$ are

$$\begin{array}{|c|c|c|c}\hline 1 & 2 & 2 & \\ \hline 3 & 4 & 5 & \multicolumn{1}{c|}{5} \\ \hline\end{array} \qquad \begin{array}{|c|c|c|c}\hline 1 & 2 & 3 & \\ \hline 2 & 4 & 5 & \multicolumn{1}{c|}{5} \\ \hline\end{array} \qquad \begin{array}{|c|c|c|c}\hline 1 & 2 & 4 & \\ \hline 2 & 3 & 5 & \multicolumn{1}{c|}{5} \\ \hline\end{array}$$

A shin-tableau is *standard* if each number 1 through n appears exactly once. The *descent set* is defined as $Des_{ש}(U) = \{i : i+1 \text{ is strictly below } i \text{ in } U\}$ for a standard shin-tableau U. Each entry i in $Des_{ש}(U)$ is called a *descent* of U. The *descent composition* of U is defined $co_{ש}(U) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{ש}(U) = \{i_1, \ldots, i_d\}$.

The *shin reading word* of a shin-tableau T, denoted $rw_{ש}(T)$ is the word obtained by reading the rows of T from left to right starting with the bottom row and moving up. To *standardize* a shin-tableau T, replace the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{ש}(T)$, then the 2's, etc.

For a composition α, the extended Schur function is defined as $ש^*_\alpha = \sum_T x^T$ where the sum runs over shin-tableaux of shape α. They have positive expansions into the monomial and fundamental bases in terms of these shin-tableaux [2, 6]. For a composition α, we have

$$ש^*_\alpha = \sum_\beta \mathcal{K}_{\alpha,\beta} M_\beta \qquad \text{and} \qquad ש^*_\alpha = \sum_\beta \mathcal{L}_{\alpha,\beta} F_\beta, \tag{1.1}$$

where $\mathcal{K}_{\alpha,\beta}$ denotes the number of shin-tableaux of shape α and type β, and $\mathcal{L}_{\alpha,\beta}$ denotes the number of standard shin-tableaux of shape α with descent composition β.

Example 1.3. The F-expansion of the extended Schur function $ש^*_{(2,3)}$.

$$ש^*_{(2,3)} = F_{(2,3)} + F_{(1,2,2)} \qquad \begin{array}{|c|c|c}\hline 1 & 2 & \\ \hline 3 & 4 & \multicolumn{1}{c|}{5} \\ \hline\end{array} \qquad \begin{array}{|c|c|c}\hline 1 & 3 & \\ \hline 2 & 4 & \multicolumn{1}{c|}{5} \\ \hline\end{array}$$

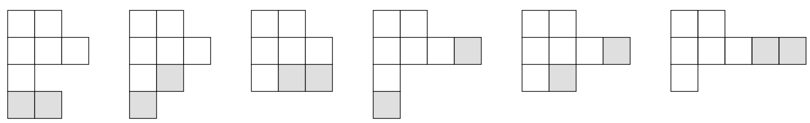
ש is the hebrew character *shin*.

The shin basis of $NSym$ was introduced in [6] by Campbell, Feldman, Light, Shuldiner, and Xu. Let α and β be compositions. Then β is said to differ from α by a *shin-horizontal strip* of size r, denoted $\alpha \subset_r^{\mathfrak{w}} \beta$, provided for all i, we have $\beta_i \geq \alpha_i$, $|\beta| = |\alpha| + r$, and for any $i \in \mathbb{N}$ if $\beta_i > \alpha_i$ then for all $j > i$, we have $\beta_j \leq \alpha_i$. The shin functions are defined recursively based on a right Pieri rule using shin-horizontal strips.

Definition 1.4. The shin basis $\{\mathfrak{w}_\alpha\}_\alpha$ is defined as the unique set of functions $\mathfrak{w}_\alpha$ such that $\mathfrak{w}_\alpha H_r = \sum_{\alpha \subset_r^{\mathfrak{w}} \beta} \mathfrak{w}_\beta$, where the sum runs over all compositions β which differ from α by a shin-horizontal strip of size r.

Intuituvely, the β in the right hand side are given by taking diagrams of α and adding r blocks on the right such that if you add boxes to some row i then no row below i is longer than the original row i. This is referred to as the overhang rule.

Example 1.5. $\mathfrak{w}_{(2,3,1)} H_{(2)} = \mathfrak{w}_{(2,3,1,2)} + \mathfrak{w}_{(2,3,2,1)} + \mathfrak{w}_{(2,4,1,1)} + \mathfrak{w}_{(2,4,2)} + \mathfrak{w}_{(2,5,1)}$



Repeated application of this right Pieri rule yields the expansion of a complete homogeneous noncommutative symmetric function in terms of the shin functions. This expansion verifies that the extended Schur functions and the shin functions are dual bases. This allows us to expand the ribbon functions into the shin basis dually to the expansion of the extended Schur functions expanded into the fundamental basis [2, 6].

$$H_\beta = \sum_{\alpha \geq_\ell \beta} \mathcal{K}_{\alpha,\beta} \mathfrak{w}_\alpha \qquad \text{and} \qquad R_\beta = \sum_{\beta \leq_\ell \alpha} \mathcal{L}_{\alpha,\beta} \mathfrak{w}_\alpha. \tag{1.2}$$

The extended Schur functions have the special property that $\mathfrak{w}_\lambda^* = s_\lambda$. Since the forgetful map χ is dual to the inclusion map from Sym to $QSym$, we have

$$\chi(\mathfrak{w}_\alpha) = \sum_\lambda (\text{coefficient of } \mathfrak{w}_\alpha^* \text{ in } s_\lambda) s_\lambda = \begin{cases} s_\lambda & \text{if } \alpha = \lambda \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

so it follows that $\chi(\mathfrak{w}_\lambda) = s_\lambda$ when λ is a partition and $\chi(\mathfrak{w}_\alpha) = 0$ otherwise.

2 A Creation Operator for Certain Shin Functions

The Schur functions and the immaculate basis can both be defined in terms of creation operators. In fact, the immaculate basis was originally defined in terms of noncommutative bernstein operators. It is using these operators that one can prove various properties of the immaculate basis including the Jacobi-Trudi rule [3], a left Pieri rule [5], a combinatorial interpretation of the inverse Kostka matrix [1], and a partial Littlewood Richardson rule [4]. Here we give similar creation operators for certain shin functions which then allow us to define a Jacobi-Trudi rule. This rule is especially useful because there is currently no other combinatorial way to expand shin functions into the complete homogeneous basis.

Definition 2.1. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $k \geq 1$ and a positive integer m, define the linear operator $\beth_m$ on the complete homogeneous basis by

$$\beth_m(1) = H_m \qquad \text{and} \qquad \beth_m(H_\alpha) = H_{m,\alpha_1,\alpha_2,\ldots} - H_{\alpha_1,m,\alpha_2,\ldots}.$$

Theorem 2.2. *If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $k \geq 1$ and $0 < m < \alpha_1$, then $\beth_m(\text{ш}_\alpha) = \text{ш}_{m,\alpha}$.*

To prove this theorem, we first show inductively that the functions given by $\beth_m(\text{ш}_\alpha)$ satisfy the right Pieri rule that defines the shin functions. Then we show that one can recursively calculate the value of $\beth_m(\text{ш}_\alpha)$ and that it will be $\text{ш}_{(m,\alpha)}$.

These operators allow us to construct shin functions indexed by strictly increasing compositions from the ground up.

Corollary 2.3. *Let $\beta = (\beta_1, \ldots, \beta_k)$ where $\beta_i < \beta_{i+1}$. Then, $\beth_{\beta_1} \cdots \beth_{\beta_k}(1) = \text{ш}_\beta$.*

Example 2.4. Creation operators can be used to build up a shin function as follows.

$$\text{ш}_{(1,3,4)} = \beth_1\beth_3\beth_4(1) = \beth_1\beth_3(H_4) = \beth_1(H_{(3,4)} - H_{(4,3)}) = H_{(1,3,4)} - H_{(1,4,3)} - H_{(3,1,4)} + H_{(4,1,3)}.$$

Using these operators, we can define a Jacobi-Trudi rule to express these same shin functions as matrix determinants. Let $S_k^{\geq}(-1)$ be the set of permutations $\sigma \in S_k$ such that $\sigma(i) \geq i - 1$ for all $i \in [k]$.

Theorem 2.5. *Let $\beta = (\beta_1, \ldots, \beta_k)$ be a composition such that $\beta_i < \beta_{i+1}$ for all i. Then,*

$$\text{ш}_\beta = \sum_{\sigma \in S_k^{\geq}(-1)} (-1)^\sigma H_{\beta_{\sigma(1)}} \cdots H_{\beta_{\sigma(k)}}.$$

Equivalently, ш_β can be expressed as the matrix determinant

$$\text{ш}_\beta = det \begin{vmatrix} H_{\beta_1} & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ H_{\beta_1} & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & H_{\beta_2} & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & H_{\beta_3} & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & 0 & \cdots & H_{\beta_{k-2}} & H_{\beta_{k-1}} & H_{\beta_k} \\ 0 & 0 & 0 & \cdots & 0 & H_{\beta_{k-1}} & H_{\beta_k} \end{vmatrix}$$

$\beth$ is the hebrew character *beth*.

where we use the noncommutative analogue to the determinant obtained by expanding along the first row.

We can show by counterexample that there is not a matrix rule of this form for every shin function, not even those indexed by partitions. It remains open to find a combinatorial or algebraic way of understanding the expansion of the shin basis into the complete homogeneous basis for the general case.

3 Skew Extended Schur Functions

To define skew extended Schur functions, we first use an algebraic approach, and then connect it to tableaux combinatorics. For $F \in QSym$, the operator $F^{\perp}$ acts on elements $H \in NSym$ based on the relation $\langle H, FG\rangle = \langle F^{\perp}H, G\rangle$. For dual bases $\{A_\alpha\}_\alpha$ of $QSym$ and $\{B_\alpha\}_\alpha$ of $NSym$ this expands as $F^{\perp}(H) = \sum_\alpha \langle H, FA_\alpha\rangle B_\alpha$.

Definition 3.1. For compositions α and β with $\beta \subseteq \alpha$, the *skew extended Schur functions* are defined as $\mathfrak{w}^*_{\alpha/\beta} = \mathfrak{w}^{\perp}_\beta(\mathfrak{w}^*_\alpha)$.

By the equation for $F^{\perp}$ above, we can expand $\mathfrak{w}^*_{\alpha/\beta}$ into various bases as follows.

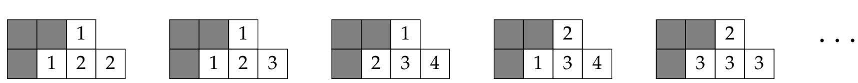
Proposition 3.2. *For compositions α and β with $\beta \subseteq \alpha$, $\mathfrak{w}^*_{\alpha/\beta} = \sum_\gamma \langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha\rangle M_\gamma = \sum_\gamma \langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha\rangle \mathfrak{w}^*_\gamma$. The coefficients $C^\alpha_{\beta,\gamma} = \langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha\rangle$ are also the coefficients that appear when multiplying shin functions, $\mathfrak{w}_\beta \mathfrak{w}_\gamma = \sum_\alpha \langle \mathfrak{w}_\beta \mathfrak{w}_\gamma, \mathfrak{w}^*_\alpha\rangle \mathfrak{w}_\alpha$.*

Using the properties of the forgetful map and the shin basis, we have the following statement about the coefficients that appear in the skew extended Schur functions.

Proposition 3.3. *Let α, β, γ be compositions that are not partitions and let λ, μ, ν be partitions. Then, $C^\nu_{\lambda,\beta} = C^\nu_{\alpha,\mu} = C^\nu_{\alpha,\beta} = 0$ and $C^\nu_{\lambda,\mu} = c^\nu_{\lambda,\mu}$, where $c^\nu_{\lambda,\mu}$ are the usual Littlewood-Richardson coefficients.*

The skew extended Schur functions can also be expressed in terms of skew shin-tableaux.

Proposition 3.4. *For compositions α and β such that $\beta \subseteq \alpha$, $\langle \mathfrak{w}_\beta H_\gamma, \mathfrak{w}^*_\alpha\rangle$ is equal to the number of skew shin-tableau of shape α/β and type γ. Moreover, $\mathfrak{w}^*_{\alpha/\beta} = \sum_T x^T$, where the sum runs over skew shin-tableau T of shape α/β.*



$$\mathfrak{w}^*_{(3,4)/(2,1)} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2x_3x_4 + x_1x_2x_3x_4 + x_2x_3^3 + \dots$$

Skew shin-tableaux of shape λ/μ where λ and μ are partitions are simply skew semistandard Young tableaux. By Proposition 3.4, these skew extended Schur functions are equal to the usual skew Schur functions $\mathfrak{w}^*_{\lambda/\mu} = s_{\lambda/\mu}$.

4 Involutions on *QSym* and *Nsym*

In *QSym*, we will consider three involutions defined on the fundamental basis [7]. These correspond with three involutions in *NSym*, the first of which is an automorphism and the second and third being anti-automorphisms (meaning, for example, $\rho(R_\alpha R_\beta) = \rho(R_\beta)\rho(R_\alpha)$). They each are defined as extensions of involutions on compositions. The *complement* of a composition α is defined $\alpha^c = comp(set(\alpha)^c)$ Here $set((\alpha_1, \ldots, \alpha_k)) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$ where α is a composition of n and $comp(\{s_1, \ldots, s_j\}) = (s_1, s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j)$ where $\{s_1, \ldots, s_j\}$ is a subset of $[n-1]$. The *reverse* of $(\alpha_1, \ldots, \alpha_k)$, denoted α^r, is $(\alpha_k, \ldots, \alpha_1)$. The *transpose* of α is defined $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$. Then our 3 involutions on *QSym* and *NSym* are defined

$$\psi(F_\alpha) = F_{\alpha^c} \qquad \rho(F_\alpha) = F_{\alpha^r} \qquad \omega(F_\alpha) = F_{\alpha^t}$$
$$\psi(R_\alpha) = R_{\alpha^c} \qquad \rho(R_\alpha) = R_{\alpha^r} \qquad \omega(R_\alpha) = R_{\alpha^t}.$$

Note that we will use the same notation for the corresponding involutions on *QSym* and *NSym*. These automorphisms commute and $\omega = \rho \circ \psi = \psi \circ \rho$. When ω and ψ are restricted to *Sym*, they are both equivalent to the classical involution $\omega : Sym \to Sym$ which acts on the Schur functions by $\omega(s_\lambda) = \lambda'$ where λ' is the conjugate of λ. The conjugate of a partition λ is found by flipping the diagram of λ over the diagonal.

We define two new pairs of dual bases in *QSym* and *NSym* by applying ρ and ω to the extended Schur and shin functions. Applying ψ to the extended Schur and shin functions recovers the row strict extended schur and row strict extended shin functions of Niese, Sundaram, van Willigenburg, Vega, and Wang in [8].

$$\psi(\mathfrak{w}^*_\alpha) = \mathfrak{R}\mathfrak{w}^*_\alpha \qquad \rho(\mathfrak{w}^*_\alpha) = \mathfrak{F}\mathfrak{w}^*_{\alpha^r} \qquad \omega(\mathfrak{w}^*_\alpha) = \mathfrak{B}\mathfrak{w}^*_{\alpha^r}$$
$$\psi(\mathfrak{w}_\alpha) = \mathfrak{R}\mathfrak{w}_\alpha \qquad \rho(\mathfrak{w}_\alpha) = \mathfrak{F}\mathfrak{w}_{\alpha^r} \qquad \omega(\mathfrak{w}_\alpha) = \mathfrak{B}\mathfrak{w}_{\alpha^r}$$

We then give combinatorial interpretations of these 2 new pairs of bases in terms of variations on shin-tableaux. While specific definitions are to follow, we can describe intuitively how ψ, ρ, and ω act on the tableaux defining each basis. Recall that shin-tableaux have weakly increasing columns and strictly increasing rows. The ψ map will switch whether the strict increasing/decreasing condition is on rows or columns (the other has a weak increasing/decreasing condition). The ρ map will switch the row condition from increasing to decreasing or vice versa. The ω map will do both. Through this combinatorial interpretation, each of the 4 pairs of dual bases is related to any other by one of the three involutions ψ, ρ, or ω as shown in the figure below.

$$\begin{tikzcd}
\mathfrak{w} \arrow[r, leftrightarrow, "\psi"] \arrow[d, leftrightarrow, "\rho"'] & \mathfrak{Rw} \arrow[d, leftrightarrow, "\rho"] \arrow[dl, leftrightarrow, "\omega"'] \\
\mathfrak{Fw} \arrow[r, leftrightarrow, "\psi"'] & \mathfrak{Bw} \arrow[ul, leftrightarrow, bend right=90, "\omega"']
\end{tikzcd}$$

Figure 1: Mappings between bases.

The table below serves to summarize the tableaux defined over the course of this section. It lists each type of tableaux, the position of $i+1$ relative to i that makes i a descent, the order the boxes appear in the reading word (Left, Right, Top, Bottom), the condition on entries of each row, and the condition on entries in each column.

	Descent	**Reading Word**	**Rows**	**Columns**
Shin	strictly below	L to R, B to T	weakly increasing	strictly increasing
Row-strict	weakly above	L to R, T to B	strictly increasing	weakly increasing
Flipped	strictly below	R to L, B to T	weakly decreasing	strictly increasing
Backward	weakly above	R to L, T to B	strictly decreasing	weakly increasing

We now briefly review row-strict extended Schur and row-strict shin functions but reserve more details for the full paper. Let α be a composition and let β be a weak composition. A *row-strict shin-tableaux* (RSST) of shape α and type β is a filling of the composition diagram of α with positive integers such that each row strictly increases from left to right, each column weakly increases from top to bottom, and each integer i appears β_i times. A *standard* row-strict shin-tableaux (SRSST) with n boxes is one containing the entries 1 through n each exactly once. For a composition α, define the *row strict extended Schur function* as $\mathfrak{Rw}^*_\alpha = \sum_T x^T$, where the sum runs over all row-strict shin-tableaux T of shape α. The row strict shin functions are defined as the duals in $NSym$ to the row strict extended Schur functions in $QSym$.

For a standard row-strict shin-tableau U, the descent set is defined to be $Des_{\mathfrak{Rw}}(U) = \{i : i+1 \text{ is weakly above } i \text{ in } U\}$. Each entry i in $Des_{\mathfrak{Rw}}(U)$ is called a *descent* of U. The *descent composition* of U is defined to be $co_{\mathfrak{Rw}}(U) = (i_1, i_2 - i_1, \ldots, i_d - i_{d-1}, n - i_d)$ for $Des_{\mathfrak{Rw}}(U) = \{i_1, \ldots, i_d\}$. Equivalently, the descent composition is found by counting the number of entries in U (in the order they are numbered) between each descent. Note that the set of standard row-strict shin-tableaux is exactly the same as the set of standard shin-tableaux. Using the framework of standard row-strict shin-tableaux, it is shown in [8] that for a composition α, the row-strict extended Schur function expands into the fundamental basis as $\mathfrak{Rw}^*_\alpha = \sum_U F_{co_{\mathfrak{Rw}}(U)}$, where the sum runs over all standard row-strict shin-tableaux.

Example 4.1. The F-expansion of the row-strict extended Schur function $\mathfrak{Rw}^*_{(2,3)}$.

$$\mathfrak{Rw}^*_{(2,3)} = F_{(1,2,1,1)} + F_{(2,2,1)} \qquad \begin{array}{|c|c|c}\hline 1 & 2 & \\ \hline 3 & 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 1 & 3 & \\ \hline 2 & 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

We can now relate the extended Schur and row-strict extended Schur functions by using ψ on their F-expansions. This relationship follows from the fact that the set of standard tableaux is the same but the definitions of descent sets are in a sense complementary and the map ψ is using complements. For all compositions α, $\psi(\mathfrak{w}^*_\alpha) = \mathfrak{Rw}^*_\alpha$, and $\{\mathfrak{Rw}^*_\alpha\}_\alpha$ is a basis of $QSym$. Additionally, $\psi(\mathfrak{w}_\alpha) = \mathfrak{Rw}_\alpha$ and $\{\mathfrak{Rw}_\alpha\}_\alpha$ is a basis of $NSym$.

4.1 Flipped extended Schur and shin functions.

Let α be a composition and β a weak composition. A *flipped shin-tableau* (FST) of shape α and type β is a composition diagram α filled with positive integers that weakly decrease along the rows and strictly increase along the columns (from top to bottom) where each positive integer i appears β_i times.

Definition 4.2. For a composition α, the *flipped extended Schur function* is defined as $\mathfrak{Fw}^*_\alpha = \sum_T x^T$, where the sum runs over all flipped shin-tableaux T of shape α.

A *standard* flipped shin-tableau (SFST) of shape α is one containing the entries 1 through n each exactly once. For a standard flipped shin-tableau S, the *descent set* is defined as $Des_{\mathfrak{Fw}}(S) = \{i : i+1 \text{ is strictly below } i \text{ in } S\}$. Each entry i in $Des_{\mathfrak{Fw}}(S)$ is called a *descent* of S. The *descent composition* of S is defined $co_{\mathfrak{Fw}}(S) = comp(Des_{\mathfrak{Fw}}(S))$. Define $flip(S)$ to be the tableau U obtained by flipping S horizontally (in other words, reversing the order of the rows of S) and then replacing each entry i with $n - i$. It is easy to see that the map $flip$ is an involution between the set of standard shin-tableaux and the set of standard flipped shin-tableaux.

$$flip\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \right) = \begin{array}{|c|c|c}\hline 4 & 1 & \\ \hline 5 & 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

The *flipped shin-reading word* of a flipped shin-tableau T, denoted $rw_{\mathfrak{Fw}}(T)$ is the word obtained by reading the rows of T from right to left starting with the top row and moving down. To *standardize* a flipped shin-tableau T, replace the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{\mathfrak{Fw}}(T)$, then the 2's, etc.

Proposition 4.3. *For a composition α, $\mathfrak{Fw}^*_\alpha = \sum_S F_{co_{\mathfrak{Fw}}(S)}$, where the sum runs over standard flipped shin-tableau U of shape α.*

Example 4.4. The F-expansion of the flipped extended Schur function $\mathfrak{F}\mathfrak{w}^*_{(3,2)}$.

$$\mathfrak{F}\mathfrak{w}^*_{(3,2)} = F_{(3,2)} + F_{(2,2,1)} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \\ \cline{1-2}\end{array}$$

The descent composition of a standard shin-tableaux U is the reverse of the descent composition of the standard flipped shin-tableaux given by $flip(U)$. Using this fact, we show that the flipped extended Schur functions are the image of the extended Schur functions under ρ.

Theorem 4.5. *For a composition α, $\rho(\mathfrak{w}^*_\alpha) = \mathfrak{F}\mathfrak{w}^*_{\alpha^r}$, and $\{\mathfrak{F}\mathfrak{w}_\alpha\}_\alpha$ is a basis of QSym.*

The flipped extended Schur basis is not equivalent to the extended Schur basis or the row-strict extended Schur basis. To see, for example, that the flipped extended Schur functions are not equivalent to the extended Schur functions, observe that there is no composition β where $\mathfrak{w}^*_\beta = \mathfrak{F}\mathfrak{w}^*_{(3,2)}$ from the example above. The only standard shin-tableaux with descent composition $(3,2)$ are tableaux of shape $(3,2)$ or shape $(4,1)$ meaning $\mathfrak{w}^*_{(3,2)}$ and $\mathfrak{w}^*_{(4,1)}$ are the only extended schur functions in which $F_{(3,2)}$ will appear, but neither of them equal $\mathfrak{F}\mathfrak{w}^*_{(3,2)}$.

Definition 4.6. For a composition α, the *flipped shin function* is defined as $\mathfrak{F}\mathfrak{w}_\alpha = \rho(\mathfrak{w}_{\alpha^r})$.

By the invariance of ρ under duality, we have that the flipped shin functions are the dual basis to the flipped extended Schur functions, that is $\langle \mathfrak{F}\mathfrak{w}_\alpha, \mathfrak{F}\mathfrak{w}^*_\beta \rangle = \delta_{\alpha,\beta}$.

Let $\mathcal{K}^{\mathfrak{F}\mathfrak{w}}_{\alpha,\beta}$ be the number of flipped shin-tableaux of shape α and type β. Let $\mathcal{L}^{\mathfrak{F}\mathfrak{w}}_{\alpha,\beta}$ be the number of flipped shin-tableaux with shape α and descent composition β. Then,

$$\mathfrak{F}\mathfrak{w}^*_\alpha = \sum_\beta \mathcal{K}^{\mathfrak{F}\mathfrak{w}}_{\alpha,\beta} M_\beta = \sum_\beta \mathcal{L}^{\mathfrak{F}\mathfrak{w}}_{\alpha,\beta} F_\beta \qquad \text{and} \qquad H_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{F}\mathfrak{w}}_{\alpha,\beta} \mathfrak{F}\mathfrak{w}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{F}\mathfrak{w}}_{\alpha,\beta} \mathfrak{F}\mathfrak{w}_\alpha.$$

By applying ρ, we can translate many of the results on the shin functions to the flipped shin functions.

Theorem 4.7. *For compositions α, β and a positive integer m,*

1. $H_m \mathfrak{F}\mathfrak{w}_\alpha = \sum_{\alpha^r \subset^{\mathfrak{w}}_m \beta^r} \mathfrak{F}\mathfrak{w}_\beta.$

2. $H_\beta = \sum_\alpha \mathcal{K}_{\alpha^r,\beta^r} \mathfrak{F}\mathfrak{w}_\alpha$ *and* $R_\beta = \sum_\alpha \mathcal{L}_{\alpha^r,\beta^r} \mathfrak{F}\mathfrak{w}_\alpha.$

3. $\mathfrak{F}\mathfrak{w}^*_{\lambda^r} = s_\lambda$. *Also, $\chi(\mathfrak{F}\mathfrak{w}_{\lambda^r}) = s_\lambda$ and $\chi(\mathfrak{F}\mathfrak{w}_\alpha) = 0$ when α^r is not a partition.*

4. *Let γ be a composition such that $\gamma_i > \gamma_{i+1}$ for all $1 \le i \le \ell(\gamma)$. Then,*

$$\mathfrak{F}\mathfrak{w}_\gamma = \sum_{\sigma \in S_{\ell(\gamma)}} (-1)^\sigma H_{\gamma_{\sigma(1)}} \cdots H_{\gamma_{\sigma(\ell(\gamma))}},$$

where the sum runs over $\sigma \in S_{\ell(\gamma)}$ such that $\sigma(i) \le i+1$ for all $i \in [\ell(\gamma)]$.

4.2 Backward extended Schur and shin functions.

Let α be a composition and β be a weak composition. A *backward shin-tableau* (BST) of shape α and type β is a filling of the diagram of α with positive integers such that the entries in each row are strictly decreasing from left to right and the entries in each column are weakly increasing from top to bottom where each integer i appears β_i times. These are essentially a row-strict version of the flipped shin-tableaux.

Definition 4.8. For a composition α, the *backward extended Schur function* is defined as $\mathfrak{Bw}^*_\alpha = \sum_T x^T$, where the sum runs over all backward shin-tableaux T of shape α.

A backward shin-tableau (SBST) of shape α is standard if it includes the entries 1 through n each exactly once. For a standard backward shin-tableau S, the *descent set* is defined to be $Des_{\mathfrak{Bw}}(S) = \{i : i+1 \text{ is weakly above } i \text{ in } S\}$. Each entry i in $Des_{\mathfrak{Bw}}(S)$ is called a *descent* of S. Then, we define the descent composition of S to be $co_{\mathfrak{Bw}}(S) = comp(Des_{\mathfrak{Bw}}(S))$. Equivalently, the descent composition is found by counting the number of entries in S (in the order they are numbered) between each descent. Note that the set of standard backward shin-tableaux is exactly the same as the set of standard flipped shin-tableaux.

The *backward shin reading word* of a shin-tableau T, denoted $rw_{\mathfrak{Bw}}(T)$ is the word obtained by reading the rows of T from right to left starting with the bottom row and moving up. We can *standardize* a standard backward shin-tableaux as follows. Given a BSST T, its *standardization* is the SBSST obtained by replacing the 1's in T with $1, 2, \ldots$ in the order they appear in $rw_{\mathfrak{Bw}}(T)$, then the 2's, then 3's, etc.

Proposition 4.9. *For a composition α, $\mathfrak{Bw}^*_\alpha = \sum_S F_{co_{\mathfrak{Bw}}(S)}$, where the sum runs over standard backward shin-tableaux S.*

Example 4.10. The F-expansion of the flipped extended Schur function $\mathfrak{Bw}^*_{(3,2)}$.

$$\mathfrak{Bw}^*_{(3,2)} = F_{(1,1,2,1)} + F_{(1,2,2)}$$

3	2	1
5	4	

4	2	1
5	3	

The descent compositions of backward shin tableaux of shape α^r are complementary to the descent compositions of flipped tableaux of shape α^r, thus $\psi(\mathfrak{Fw}_{\alpha^r}) = \mathfrak{Bw}^*_{\alpha^r}$. Given that $\mathfrak{Fw}^*_{\alpha^r} = \rho(\mathfrak{w}^*_\alpha)$ and $\psi \circ \rho = \omega$, we have the following.

Theorem 4.11. *For a composition α, $\omega(\mathfrak{w}^*_\alpha) = \mathfrak{Bw}^*_{\alpha^r}$ and $\{\mathfrak{Bw}^*_\alpha\}_\alpha$ is a basis of $QSym$.*

The backward extended Schur basis is not equivalent to the extended Schur basis, the row-strict extended Schur basis, or the flipped extended Schur basis. Again, it is simple to check that there exist backward extended Schur functions that are not elements in the extended Schur, row-strict extended Schur, or flipped extended Schur bases. Like with

ψ and ρ, it follows from the dual definitions of ω in *NSym* and *QSym* that ω is invariant under duality. Thus, the backward extended Schur functions are dual to the backward shin functions when defined as follows.

Definition 4.12. For a composition α, define the backward shin function $\mathfrak{B}\mathfrak{w}_\alpha = \omega(\mathfrak{w}_{\alpha^r})$.

Let $\mathcal{K}^{\mathfrak{B}\mathfrak{w}}_{\alpha,\beta}$ be the number of backward shin-tableaux of shape α and type β. Let $\mathcal{L}^{\mathfrak{B}\mathfrak{w}}_{\alpha,\beta}$ be the number of standard backward shin-tableaux with shape α and descent composition β. The expansions of the backward extended Schur functions into the monomial and fundamental bases follow those of the extended Schur functions.

$$\mathfrak{B}\mathfrak{w}^*_\alpha = \sum_\beta \mathcal{K}^{\mathfrak{B}\mathfrak{w}}_{\alpha,\beta} M_\beta = \sum_\beta \mathcal{L}^{\mathfrak{B}\mathfrak{w}}_{\alpha,\beta} F_\beta \qquad \text{and} \qquad H_\beta = \sum_\alpha \mathcal{K}^{\mathfrak{B}\mathfrak{w}}_{\alpha,\beta} \mathfrak{B}\mathfrak{w}_\alpha \quad \text{and} \quad R_\beta = \sum_\alpha \mathcal{L}^{\mathfrak{B}\mathfrak{w}}_{\alpha,\beta} \mathfrak{B}\mathfrak{w}_\alpha.$$

Now, we can apply ω the the various results on the shin and extended Schur bases to find analogous results on the backward shin and backward extended Schur bases.

Theorem 4.13. *For compositions α, β and a positive integer m,*

1. $E_m \mathfrak{B}\mathfrak{w}_\alpha = \displaystyle\sum_{\alpha^r \subset^{\mathfrak{w}}_m \beta^r} \mathfrak{B}\mathfrak{w}_\beta.$

2. $E_\beta = \displaystyle\sum_\alpha \mathcal{K}_{\alpha^r,\beta^r} \mathfrak{B}\mathfrak{w}_\alpha$ *and* $R_\beta = \displaystyle\sum_\alpha \mathcal{L}_{\alpha^r,\beta^t} \mathfrak{B}\mathfrak{w}_\alpha$

3. $\mathfrak{B}\mathfrak{w}^*_{\lambda^r} = s_{\lambda'}$. *Also, $\chi(\mathfrak{B}\mathfrak{w}_{\lambda^r}) = s_{\lambda'}$ and $\chi(\mathfrak{B}\mathfrak{w}_\alpha) = 0$ when α^r is not a partition.*

4. *Let γ be a composition such that $\gamma_i > \gamma_{i+1}$. Then,*

$$\mathfrak{B}\mathfrak{w}_\gamma = \sum_{\sigma \in S_{\ell(\gamma)}} (-1)^\sigma E_{\gamma_{\sigma(1)}} E_{\gamma_{\sigma(2)}} \cdots E_{\gamma_{\sigma(\ell(\gamma))}}$$

where the sum runs over $\sigma \in S_{\ell(\gamma)}$ such that $\sigma(i) \le i+1$ for all $i \in [\ell(\gamma)]$.

5 Future work

There are a variety of open questions on the extended Schur functions (and related bases), many regarding their multiplicative structure. It would also be interesting to find combinatorial rules for the expansions of these bases into the other Schur-like bases such as the (dual) immaculate functions. Additionally, the row-strict extended Schur functions were first introduced while studying 0-Hecke submodules so it is likely that there is similar work to be done for the flipped and backward extended Schur functions.

Acknowledgements

The author thanks Laura Colmenarejo and Sarah Mason for their guidance and support and Kyle Celano for his thoughtful feedback. We also thank Mike Zabrocki for helpful answers about creation operators and John Campbell for sharing useful Sage code.

References

[1] E. E. Allen and S. K. Mason. "A combinatorial interpretation of the noncommutative inverse Kostka matrix". 2022. arXiv:2207.05903.

[2] S. Assaf and D. Searles. "Kohnert polynomials". *Exp. Math.* **31**.1 (2022), pp. 93–119. DOI.

[3] C. Berg, N. Bergeron, F. Saliola, L. Serrano, and M. Zabrocki. "A lift of the Schur and Hall-Littlewood bases to non-commutative symmetric functions". *Canad. J. Math.* **66**.3 (2014), pp. 525–565. DOI.

[4] C. Berg, N. Bergeron, F. Saliola, L. Serrano, and M. Zabrocki. "Multiplicative structures of the immaculate basis of non-commutative symmetric functions". *J. Combin. Theory Ser. A* **152** (2017), pp. 10–44. DOI.

[5] N. Bergeron, J. Sánchez-Ortega, and M. Zabrocki. "The Pieri rule for dual immaculate quasi-symmetric functions". *Ann. Comb.* **20**.2 (2016), pp. 283–300. DOI.

[6] J. M. Campbell, K. Feldman, J. Light, P. Shuldiner, and Y. Xu. "A Schur-like basis of NSym defined by a Pieri rule". *Electron. J. Combin.* **21**.3 (2014), Paper 3.41, 19. DOI.

[7] K. Luoto, S. Mykytiuk, and S. van Willigenburg. *An introduction to quasisymmetric Schur functions*. SpringerBriefs in Mathematics. Hopf algebras, quasisymmetric functions, and Young composition tableaux. Springer, New York, 2013, pp. xiv+89. DOI.

[8] E. Niese, S. Sundaram, S. van Willigenburg, J. Vega, and S. Wang. "Row-strict dual immaculate functions and 0-Hecke modules". *Sém. Lothar. Combin.* **86B** (2022), Art. 6, 12.