Research Statement

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My research focuses on problems in algebraic and enumerative combinatorics. In my thesis, I study Schur-like bases of the quasisymmetric and noncommutative symmetric functions and introduce their colored generalizations in two similar dual algebras. I like to find combinatorial and algebraic ways to relate different bases to each other and to understand their multiplicative structure. My work outside of my thesis has mainly been on chromatic symmetric functions. In my work on e-positivity problems, I use representation theory to give combinatorial interpretations for the coefficients in the chromatic symmetric functions of certain graphs. I am also taking a structural and bijective approach to the problem of distinguishing trees by their chromatic symmetric functions.

1. Thesis Project: Schur-like bases and their colored generalizations

1.1. Schur-like bases of $QSym$ and $NSym$.

The Schur symmetric functions form a basis of the symmetric functions with robust combinatorics and a wide variety of applications. One way to define them is with *semi-standard Young tableaux*. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, a semistandard Young tableaux of shape λ is a left-aligned diagram of boxes such that row i (from the top) has λ_i boxes where each box is filled with a positive integer such that every row is weakly increasing left to right and every column is strictly increasing top to bottom. For a partition λ, the Schur symmetric function is defined as $s_\lambda = \sum_T x^T$, where the sum runs over all semistandard Young tableaux T of shape λ with entries in $\mathbb{Z}_{>0}$. Given a tableau T, the monomial x^T has a factor x_i for each time i appears in T. For instance,

$$s_{(2,2)} = x_1^2x_2 + x_1^2x_2x_3 + x_1^2x_3^2 + 2x_1x_2^2x_3 + x_2^2x_3^2 + \ldots$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}\quad \ldots$$

The Schur functions can be defined in numerous other ways including via a matrix determinant, known as the Jacobi-Trudi formula, or constructively with linear operators. One of their most important properties is that they are the characters of irreducible representations of the general linear group [53]. Additionally, the product of two Schur functions is given by $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$, where $c^\lambda_{\mu,\nu}$ are the famous *Littlewood Richardson coefficients*. These coefficients, and the Schur functions themselves, appear in representation theory, algebraic geometry, linear algebra, and of course combinatorics. The Schur functions are widely studied and lay at the center of many open problems in combinatorics, including Schur positivity problems, the Schur plethysm problem, and problems on the Kroenecker coefficients of Schur functions [14, 25, 37, 40, 49, 54].

The quasisymmetric functions introduced by Gessel [28] and the non-commutative symmetric functions introduced by Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon [27] are generalizations of the symmetric functions with rich theory and importance in algebraic combinatorics. Their algebras, QSym and NSym, are dual Hopf algebras that also appear in representation theory, algebraic geometry, and category theory [1, 35, 43, 45]. A significant amount of work has been done to find quasisymmetric or non-commutative analogues of symmetric function objects, specifically of the Schur basis. This includes the development of a variety of *Schur-like bases* such as the immaculate functions, the shin functions, the quasisymmetric Schur functions, the extended Schur functions, and the row-strict immaculate functions [6, 11, 12, 32, 42, 47].

A Schur-like basis of $NSym$ is one in which the commutative image of any element indexed by a partition is the Schur function indexed by that partition. Schur-like bases in $QSym$ are the bases dual to the aforementioned bases of $NSym$ and tend to have combinatorial definitions in terms of tableaux that are closely related to semistandard Young tableaux. The tableaux below are examples of the types of tableaux associated with the dual immaculate, extended Schur, and Young quasisymmetric Schur bases, respectively.

$$\begin{array}{|c|c|c|c}\cline{1-3} 1&3&4&\\\hline 2&2&5&\multicolumn{1}{c|}{6}\\\hline\end{array}\qquad \begin{array}{|c|c|c|c}\cline{1-3} 1&1&3&\\\hline 2&3&4&\multicolumn{1}{c|}{5}\\\hline\end{array}\qquad \begin{array}{|c|c|c|c}\cline{1-3} 1&1&4&\\\hline 2&3&3&\multicolumn{1}{c|}{5}\\\hline\end{array}$$

dual immaculate   extended Schur   Young quasi-Schur

With any Schur-like basis, the properties of the Schur functions we would like to find analogues for include a matrix determinant formula, various types of multiplicative rules, Hopf algebraic structure, and representation theoretic applications. The primary goal of my thesis is to continue developing the theory of these Schur-like bases and to define their colored analogues in $QSym_A$ and $NSym_A$, which are generalizations of $QSym$ and $NSym$ introduced by Doliwa [20] in 2021.

1.2. **Colored generalizations of Schur-like bases in $QSym_A$ and $NSym_A$.** The Hopf algebra $NSym$ has a natural generalization called $NSym_A$ that stems from $NSym$'s relationship with the algebra of rooted trees. Given an alphabet A, $NSym_A$ is defined as the algebra with noncommutative generators H_w for all words w in A. Its dual algebra, $QSym_A$ is defined dually using partially commutative colored variables. For a color $a \in A$, define the set of infinite colored variables $x_a = \{x_{a,1}, x_{a,2}, \ldots\}$ and let $x_A = \cup_{a\in A} x_a$. These variables are assumed to be partially commutative in the sense that variables only commute if the second indices are different. That is, for $a, b \in A$,

$$x_{a,i}x_{b,j} = x_{b,j}x_{a,i} \text{ for } i \neq j \qquad \text{and} \qquad x_{a,i}x_{b,i} \neq x_{b,i}x_{a,i} \text{ if } a \neq b.$$

As a result, every monomial in variables $x_{a,i}$ can be uniquely re-ordered so that the sequence of the second indices of the variables is weakly increasing, at which point any first indices sharing the same color can be combined into a single word. Every monomial has a sentence $(w_1, \ldots, w_m)$ defined by its re-ordered, combined form $x_{w_1,j_1} \cdots x_{w_m,j_m}$ where $j_1 < \ldots < j_m$. $QSym_A$ is the set of formal power series where the coefficients of monomials indexed by the same sentence are equal. These generalizations extend the study of the relationship between symmetric functions and integrable systems to a non-commutative setting which is of growing interest in mathematical physics [19, 36]. Additionally, the Hopf algebra of rooted trees has various applications in the field of symbolic computation [29].

In addition to the inherent value in expanding the theory of these spaces, any results on bases in $QSym_A$ and $NSym_A$ specialize immediately to results on their analogous bases in $QSym$ and $NSym$ because they are isomorphic in the case that A is an alphabet of just one letter. In many cases, the combinatorics behind relationships in $QSym$ and $NSym$ can be obscured by cancellation which is significantly reduced when coloring the variables.

In [17], I generalize the immaculate and dual immaculate bases of Berg, Bergeron, Saliola, Serrano, and Zabrocki [11] to the colored immaculate ($\mathfrak{S}_I$) and colored dual immaculate ($\mathfrak{S}^*_I$) bases in $QSym_A$ and $NSym_A$. The colored dual immaculate functions are defined combinatorially in terms of *colored immaculate tableaux*. Tthe tableaux below are associated with monomials $x_{(a,b,cb)}$, $x_{(a,cb,b)}$, and $x_{(a,cb,b)}$ respectively.

$$\mathfrak{S}^*_{(ab,cb)} = x_{ab,1}x_{cb,2} + x_{ab,1}x_{c,2}x_{b,3} + x_{a,1}x_{cbb,2} + 2x_{a,1}x_{cb,2}x_{b,3} + x_{a,1}x_{c,2}x_{bb,3} + x_{a,1}x_{b,2}x_{cb,3} + \ldots$$

$$\begin{array}{|c|c|}\hline a,1 & b,1 \\ \hline c,2 & b,2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline a,1 & b,1 \\ \hline c,2 & b,3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline a,1 & b,2 \\ \hline c,2 & b,2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline a,1 & b,2 \\ \hline c,2 & b,3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline a,1 & b,3 \\ \hline c,2 & b,2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline a,1 & b,3 \\ \hline c,2 & b,3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline a,1 & b,2 \\ \hline c,3 & b,3 \\ \hline\end{array}$$

The colored immaculate functions are defined as their dual basis, but can be defined equivalently with generalizations of the non-commutative Bernstein operators from [11]. These operators allow us to construct the colored immaculate functions as an alternating sum that uses a secondary operator in $NSym$ defined to be dual to multiplication in $QSym$.

Theorem 1.1. [17] *The colored immaculate basis can be defined with creation operators as:*

$$\mathfrak{S}_{(v_1,\ldots,v_h)} = \mathbb{B}_{v_1}\mathbb{B}_{v_2}\cdots\mathbb{B}_{v_h}(1) \quad \textit{where} \quad \mathbb{B}_v = \sum_u \sum_Q (-1)^{\ell(Q)} H_{v\cdot u} \sum_{S\succeq Q} M^{\perp}_S.$$

where $M^{\perp}$ is an operator defined to be dual to right-multiplication by monomial functions in $QSym$.

I prove various properties of both bases including a right Pieri rule, their Hopf algebra structure, and expansions to and from the colored monomial, colored fundamental, colored complete homogeneous, and colored ribbon bases. My expansion of the colored fundamentals into the colored dual immaculate specializes to a new combinatorial interpretation that allows for fairly straightforward computation of coefficients in the non-colored case as well. The skew colored dual immaculate functions also prove to have an interesting combinatorial relationship with the structure constants of the colored immaculate functions. The row-strict colored immaculate and row-strict colored dual immaculate functions are defined similarly and yield similar results through the use of an involution on sentences that sends each sentence to its complement.

1.3. **The shin and extended Schur functions.** In [16], I study the shin and extended Schur bases, denoted $\{\boldsymbol{\mathfrak{w}}_\alpha\}_\alpha$ and $\{\boldsymbol{\mathfrak{w}}^*_\alpha\}_\alpha$, of Campbell, Feldman, Light, Shuldiner, and Xu [12] and Assaf and Searles [7]. The extended Schur bases are defined combinatorially over *shin-tableaux* which generalize Young tableaux to composition shapes. Their dual, the shin basis, is defined as the unique set of functions that satisfies a multiplicative property called a right Pieri rule. I show that certain cases of the shin functions can be built using creation operators which leads to a rule that expresses the shin functions in terms of the complete homogeneous noncommutative symmetric functions as a matrix determinant.

Theorem 1.2. [16] *For a composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ and $m \in \mathbb{N}$, define the linear operator $\sqsupset_m(H_\gamma) = H_{m,\gamma_1,\gamma_2,\ldots} - H_{\gamma_1,m,\gamma_2,\ldots}$. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $m \in \mathbb{N}$ where $m < \alpha_1$,*

$$\boldsymbol{\mathfrak{w}}_{m,\alpha} = \sqsupset_m(\boldsymbol{\mathfrak{w}}_\alpha).$$

Moreover, when $\beta = (\beta_1, \ldots, \beta_k)$ is a composition where $\beta_i < \beta_{i+1}$ then,

$$\boldsymbol{\mathfrak{w}}_\beta = \sqsupset_{\beta_1} \cdots \sqsupset_{\beta_k}(1) = \sum_\sigma (-1)^\sigma H_{\beta_{\sigma(1)}} \cdots H_{\beta_{\sigma(k)}},$$

where the sum runs over all permutations σ such that $\sigma(i) \geq i-1$ for all $i \in [k]$.

I also show via counterexample that there is not a matrix with entries of the form $[H_{b_{(i,j)}}]_{1 \leq i,j \leq n}$ where $b_{(i,j)} \in \mathbb{Z}$ such that $\boldsymbol{\mathfrak{w}}_{(2,2,2,1)} = det[H_{n_{(i,j)}}]$. Thus there is not a matrix determinant rule of this type for shin functions in general, nor specifically those indexed by partitions.

Next, I study the image of the extended Schur basis under the involutions ψ, ρ, and ω on $QSym$ that take complements, reversals, and transposes on a certain basis of $QSym$ then extend linearly. The image of the extended Schur functions under each involution is a distinct basis defined via variants of the shin-tableaux. The *row-strict extended Schur functions*, which are the image of the extended Schur functions under ψ, were introduced in [46] by Niese, Sundaram, van Willigenburg, Vega, and Wang. I introduce the *flipped extended Schur functions* and the *backward extended Schur functions* which are the images of the extended Schur functions under the involutions ρ and ω respectively, and prove various properties of all three bases and their duals by applying ψ, ρ, and ω to properties of the shin and extended Schur functions.

Theorem 1.3. [16] *Define a flipped shin-tableaux of shape α as a diagram of α filled with positive integers such that the rows are weakly decreasing left to right and the columns are strictly increasing top to bottom. For a composition α, define the flipped extended Schur function $\mathfrak{F}\boldsymbol{\mathfrak{w}}^*_\alpha = \sum_T x^T$ where the sum runs over all flipped shin-tablaeux T of shape α. Then,*

$$\rho(\boldsymbol{\mathfrak{w}}^*_\alpha) = \mathfrak{F}\boldsymbol{\mathfrak{w}}^*_{\alpha^r}.$$

Moreover, $\{\mathfrak{F}\boldsymbol{\mathfrak{w}}_\alpha\}_\alpha$ is a basis of $QSym$.

Finally, I define colored generalizations of the shin and extended Schur functions in $NSym_A$ and $QSym_A$. The colored extended Schur functions are defined using a colored generalization of shin-tableaux and the colored shin functions are defined as their duals. I am able to generalize various properties from the original bases including, for example, a multiplication rule for the colored ribbon functions.

Theorem 1.4. [16] *For sentences I and J,*

$$\boldsymbol{\mathfrak{w}}_I R_J = \sum_K \sum_S \boldsymbol{\mathfrak{w}}_K,$$

summing over certain sentences K such and certain skew standard colored shin-tableau S of shape K/I.

1.4. **Future Work.** As part of my thesis, I am working on defining colored generalizations for the Young quasisymmetric Schur and Young noncommutative symmetric Schur functions of Luoto, Mykytiuk, and van Willigenburg [42]. I will also extend my work on the dual immaculate functions by studying their images under ρ and ω. Another more difficult problem I would like to work on over time would be to find a combinatorial rule the product of two shin or extended Schur functions.

I am also interested in studying the commutative image of $NSym_A$ which is a colored generalization of the symmetric functions one might call Sym_A. I plan to study Sym_A as a Hopf algebra and hope to define colored Schur functions to generalize the Schur symmetric functions. These colored Schur functions should be the commutative images of my colored immaculate and colored shin bases.

2. Chromatic symmetric functions

The *chromatic symmetric functions* (CSF) were introduced by Stanley in [52] as a generalization of the classic chromatic polynomial on graphs. For a simple graph G, let $\kappa : V \to \mathbb{N}$ be a proper coloring of G if $\kappa(v) \neq \kappa(u)$ whenever v and u are connected by an edge. The chromatic symmetric function of G is

$$X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa_{v_2}} \cdots x_{\kappa(v_{n+1})},$$

where the sum ranges over all proper colorings κ of G. Common topics involving chromatic symmetric functions are distinguishing graphs [15], deletion contraction principles [48], expansion into various bases [21], refinement into chromatic quasisymmetric functions [50], and algebraic applications for various classes of graphs [51].

2.1. **Distinguishing Trees.** A well-known open conjecture by Stanley states that trees are distinguished by their chromatic symmetric functions [52]. Steady proof has been made on this conjecture over time and it has been checked computationally for trees of up to 28 vertices [3, 8, 22, 26, 34, 44, 48]. Recent work on this problem includes a proof for trees of diameter 5 or less [2] and a paper I especially enjoyed that proves a rooted variant of this conjecture [41]. In collaboration with Bryson Kagy and Ian Klein, we develop and study a bijection between the set of edge partitions and the set of stable vertex partitions for a tree T. We hope this will allow us to link the structural aspects of the tree to its chromatic symmetric function.

2.2. ***e*-Positivity and Stembridge Codes.** Another major open problem on chromatic symmetric functions is the Stanley-Stembridge e-positivity conjecture [57], which states that the chromatic symmetric function of the incomparability graph of a (3+1)-free poset expands positively into the elementary basis of the symmetric function. This problem has been widely studied with significant progress and partial results [21, 30]. This problem, and e-positivity problems in general, have strong ties to representation theory in that it should be possible to express any e-positive chromatic symmetric function as the Frobenius characteristic of some S_n action. In [58], Stembridge defines objects called *codes* that are in bijection with permutations. The generating function for the Frobenius character of the S_n action on these codes turns out to be a q-analogue of the generating function for the chromatic symmetric function of a path (under the ω involution). This gives a combinatorial interpretation of the coefficients of the e-expansion of the path graph CSF.

In collaboration with Sheila Sundaram and Kyle Celano, I have defined a similar object for the cycle graph that gives us a combinatorial interpretation for its e-coefficients. A *cycle-code* (β, g) is a pairing between a non-negative sequence of integers β (with some restrictions) and a marking function g that decorates the sequence. Define the *index* of a cycle-code (β, g) be $ind(\beta, g) = \sum_{j=0}^{max(\beta)} g(j)$. For $n = 4$, the cycle-codes are

$$\begin{array}{cccccccccccc}
\dot{0}\hat{0}00 & \dot{0}0\hat{0}0 & \dot{0}00\hat{0} & 0\dot{\hat{0}}00 & 0\dot{0}\hat{0}0 & 0\dot{0}0\hat{0} & 0\hat{0}\dot{0}0 & 00\dot{\hat{0}}0 & 000\dot{\hat{0}} & 0\hat{0}0\dot{0} & 00\hat{0}\dot{0} & 000\dot{\hat{0}} \\
\dot{0}\hat{0}1\hat{1} & 0\dot{\hat{0}}1\hat{1} & \dot{0}1\hat{0}\hat{1} & 01\dot{\hat{0}}\hat{1} & 1\dot{0}\hat{0}\hat{1} & 10\dot{\hat{0}}\hat{1} & \dot{0}1\hat{1}\hat{0} & 01\hat{1}\dot{\hat{0}} & 1\dot{0}\hat{1}\hat{0} & 10\hat{1}\dot{\hat{0}} & 1\hat{1}\dot{0}\hat{0} & 1\hat{1}0\dot{\hat{0}}
\end{array}$$

The action of S_n on cycle-codes essentially moves the entries around without changing the marking function. Let $\mathbb{C}\mathcal{CC}_n$ be the representation corresponding to this action.

Theorem 2.1. [13] *For all $n \geq 3$, $ch(\mathbb{C}\mathcal{CC}_n) = X_{C_n}$. Moreover,*

$$[e_\lambda]X_{C_n} = \sum_{sort(\mu)=\lambda} |\{g : \ g \text{ is a marking function for codes of type } \mu\}|$$

where $[e_\lambda]X_{C_n}$ denotes the coefficient of the term e_λ in the chromatic symmetric function of the cycle, X_{C_n}.

Interestingly, these Frobenius characters are also closely related to the cohomology of the toric variety of the permutohedron and other polytopes. I am interested in developing analogues of Stembridge codes for other e-positive graphs, specifically for classes of unit interval graphs.

3. q, t-Catalan combinatorics

The *Catalan numbers* are a widely studied sequence known to enumerate a vast number of combinatorial objects, 214 of which can be found in [55]. One of their many generalizations are called the q, t-Catalan numbers, which were originally developed by Garsia and Haiman in the context of Macdonald polynomials and diagonal harmonics [24]. The combinatorial definition of the q, t-Catalan number, developed by Haglund, relies on popular Catalan objects called Dyck paths [31]. A Dyck path D of size n is a sequence of steps $(0,1)$ and $(1,0)$, North and East steps respectively, that form a path from $(0,0)$ to (n,n) that never goes below the line $y=x$. There are three equidistributed statistics on Dyck paths called *area*, *bounce*, and *dinv* that can be used in pairs to define the *q,t-Catalan numbers* as

$$C_n(q,t) = \sum_{D \in Dyck_n} q^{area(D)} t^{bounce(D)} = \sum_{D \in Dyck_n} q^{area(D)} t^{dinv(D)}.$$

It is known that $C_n(q,t) = C_n(t,q)$ [23] but finding a bijective proof of this symmetry property is an open problem [38]. We are approaching this problem by enumerating the q,t-Catalan numbers in terms of statistics on other Catalan objects. This process should lead to interesting new statistics and illuminate the combinatorics of these objects and their connections.

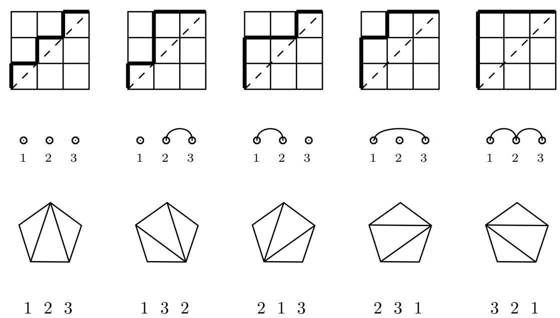


Figure 1. Dyck paths, noncrossing partitions, triangulations, and 312-avoiding permutations.

Loehr took this approach in [39] and was able to enumerate the q,t-Catalan numbers in terms of coninversion and major index on two new subsets of permutations counted by the Catalan numbers. In his master's thesis [5], Ammar enumerates the q,t-Catalan numbers in terms of Murasaki diagrams and rooted plane trees by studying the effect of known bijections on area, bounce, and dinv. Bandlow and Killpatrick [9] and Stump [59] define new bijections between Dyck paths and pattern-avoiding permutations that map one of our desired statistic to inversion number, but the symmetrically paired statistic is not studied. Thus far, we have enumerated the q,t-Catalan numbers in terms of statistics on noncrossing partitions, triangulations of $n+2$-gons, standard Young tableaux of shape (n,n). For example, let T be a triangulation of an $n+2$-gon P with vertices 1 through $n+2$ and diagonals d_1 through d_{n-1} given by $d_i = (v_i, u_i)$ where $v_i > u_i$ are the vertices connected by d_i. Define the *diag* statistic on T as

$$diag(T) = \sum_{i=1}^{n-1} v_i - (i+2).$$

For a vertex v, let $deg_{\leq}(v)$ be the number of diagonals from v to a vertex smaller than v. Define the *dinv* statistic on T as

$$dinv(T) = \sum_{i=3}^{n+2} |\{j : \sum_{i<k\leq j} deg_{\leq}(k) = j-i\}| + |\{j : \sum_{i<k\leq j} deg_{\leq}(k) = j-i+1\}|.$$

Proposition 3.1. [18] *The q,t-Catalan number $C_n(q,t)$ is expressed over triangulations of $n+2$-gons as*

$$C_n(q,t) = \sum_{T \in \mathcal{T}_{n+2}} t^{diag(T)} q^{dinv(T)}.$$

We have also used the FindStat.org database to identify a variety of statistics that have the same distribution as area, bounce, and dinv, including some symmetric pairs. We will continue to expand our set of statistics and Catalan objects and search for bijections to prove symmetry combinatorially. This project is joint work with John Lentfer.

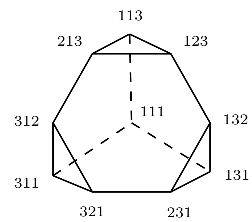
4. Parking Functions and Polytopes

Parking functions are defined as sequences of positive integers $(a_1, a_2, \ldots, a_n)$ with a nondecreasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ such that $b_i \leq i$. Parking functions appear in a variety of settings including hashing and linear programming, noncrossing partitions, hyperplane arrangements, and chip firing [60]. In addition to the problems described here, I am interested in the purely enumeration and algebraic combinatorics of (generalized) parking functions, especially connections to trees and Catalan numbers.

111 112 121 211 113 131 311 122 212 221 123 132 213 231 312 321

In [56], Stanley posed questions about the number of vertices, number of faces, volume, and number of lattice points of the polytope PF_n found by taking the convex hull of all parking functions of length n. These initial questions were studied in [4] and [33], the latter of which extends certain results to the generalized parking function polytope $\mathfrak{X}_n$.

Together with a group from the 2023 Graduate Research Workshop in Combinatorics led by Andrés R. Vindas Meléndez, I am continuing this work on parking function polytopes by studying PF_n and $\mathfrak{X}_n$ as generalized permutahedra and polymatroids, and the interesting properties that follow [10]. For example, I proved that PF_n is a y-generalized permutahedron which means it admits quadratic triangulation and is Erhart-positive. There are also enumerative questions to be examined, such as a conjectured bijection between facets in a unimodular triangulation of PF_n and a class of $(0, 1)$-matrices with row sum equal to 2 and positive permanents.



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