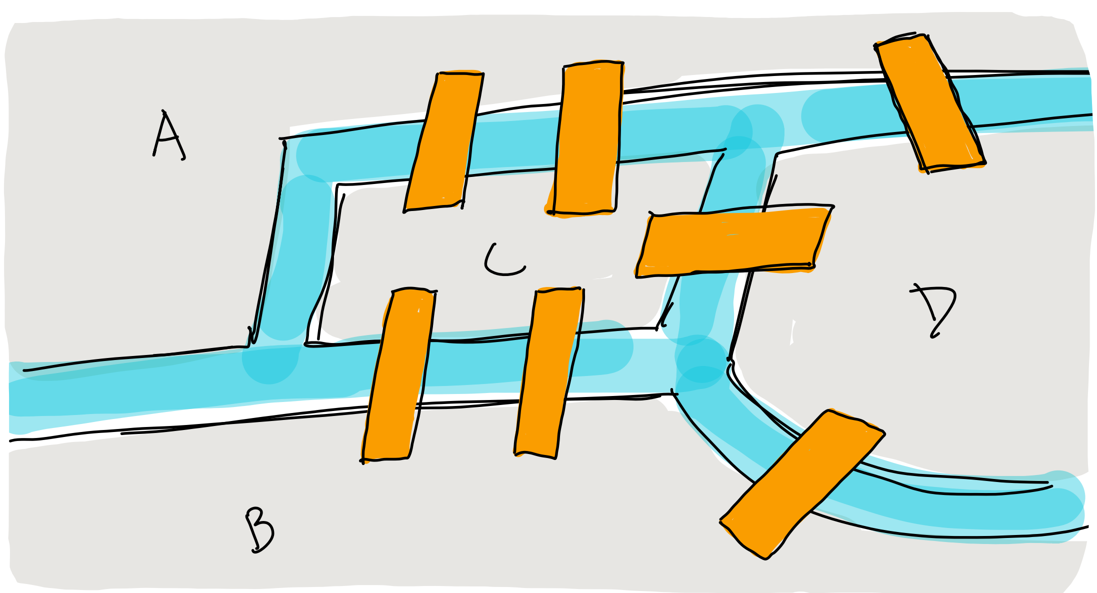
Lesson 1: Graph Theory
The Basics

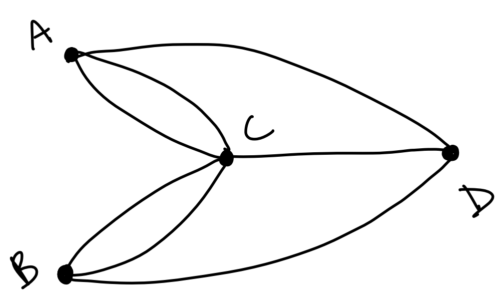
Source: Introduction to graph theory (Richard Trudeau)

The Seven Bridges of Könisberg (Euler)



Goal: Find a way to walk through the city so that you cross each bridge exactly once

↳



The beginning of graph theory...

Solution: Its impossible! why?

Sets and set notation

- a <u>set</u> is a collection of objects called elements
  $S = \{2, x, \text{apple}, \square, \dots\}$
- a set containing no elements is called <u>empty/null</u>
  denoted $\emptyset$ or $\{\}$
- A is a subset of B, denoted $A \subseteq B$ if every element in A is also contained in B
  $\{1, 3, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$
- A equals B if $A \subseteq B$ and $B \subseteq A$.
  $\{1, 2, 3, 4\} = \{1, 2, 3, 4\}$

Graphs

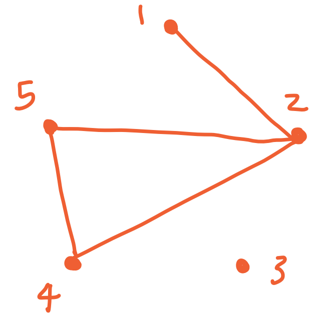
Def: A graph is an object consisting of a <u>vertex set</u> and an <u>edge set</u>

The vertex set is a finite nonempty set.
The edge set consists of two-element subsets of the vertex set

Examples: $G = (V_G, E_G)$

$V_G = \{1, 2, 3, 4, 5\}$

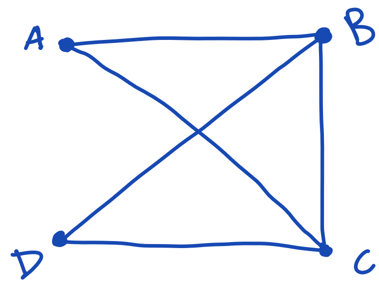
$E_G = \{(1,2), (2,4), (2,5), (4,5)\}$



$H = (V_H, E_H)$

$V_H = \{A, B, C, D\}$

$E_H = \{(A,B), (A,C), (B,C),$
$(B,D), (C,D)\}$



we will let v = # of vertices (also $|V|$)
e = # of edges (also $|E|$)

this notation isn't fixed or standard

sometimes we use e or v to denote a specific vertex or edge, and sometimes $|V| = n$, $|E| = m$

Two vertices x, y are adjacent if they are connected by an edge, i.e. $(x,y) \in E$

The edges (x,y) and (y,z) are adjacent if they connect to a mutual vertex, in this case y

The edge (x,y) is incident to vertices x and y and vice versa

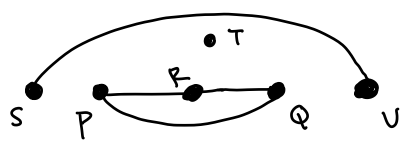
degree: the degree of a vertex x is the number of incident edges, denoted

Graph diagrams

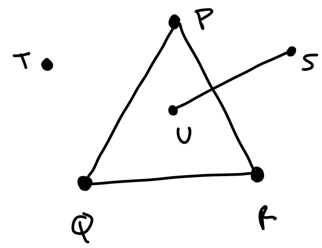
The same graph can be drawn a variety of ways.

$V_G = \{P, Q, R, S, T, U\}$

$E_G = \{\{P,Q\}, \{P,R\}, \{Q,R\}, \{S,U\}\}$



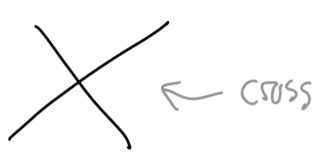
different diagrams, equivalent graphs



Two graphs are **equal** if they have the same vertex and edge sets

Some notes:

- edges may cross each other without a vertex being present



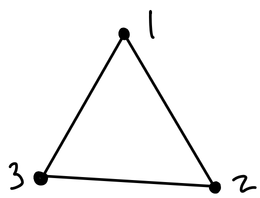
- we are drawing 2D graphs in the plane but graphs can be drawn on other surfaces and in other dimensions.

Common Graphs

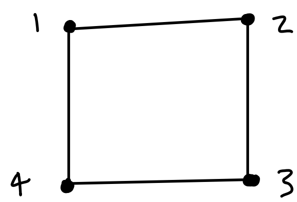
Cyclic graphs: let $v \geq 3$, then the cyclic graph on v vertices is denoted "C_v"

vertex set $V = \{1, 2, \ldots, v\}$

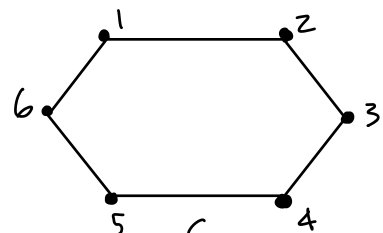
edge set $E = \{\{1,2\}, \{2,3\}, \ldots \{v-1, v\}, \{v, 1\}\}$



C_3

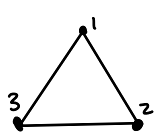


C_4

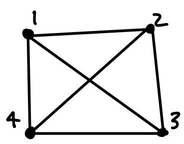


C_6

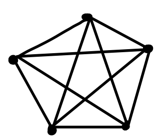
Complete graph: Let v be a positive integer. The complete graph on v vertices is denoted "K_v" with vertex set $\{1, 2, \ldots, v\}$ and all possible edges



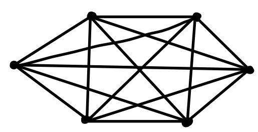
K_3



K_4

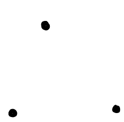


K_5

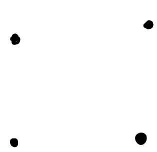


K_6

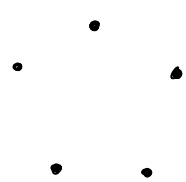
The **null graph N_v** is the graph with the vertex set $\{1, 2, \ldots, v\}$ and no edges



N_3



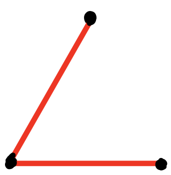
N_4



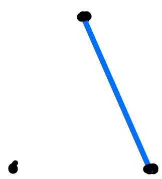
N_5

Complements and Subgraphs

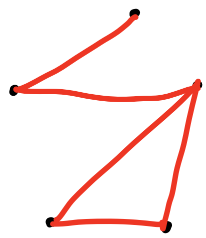
Complement: If G is a graph, then the complement of G, denoted "$\overline{G}$" is a graph having the same vertex set as G; the edge set contains any possible edge not in G.



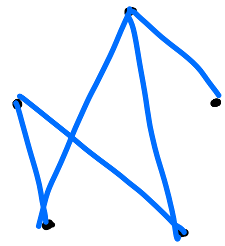
G_1



$\overline{G_1}$



G_2

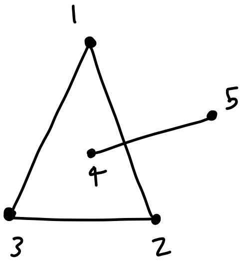


$\overline{G_2}$

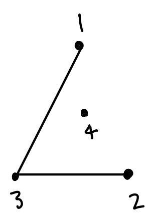
Note: K_v (the complete graph) is the complement of N_v (null)

Subgraph: graph H is a subset of graph G if the vertex set of H is a subset of the vertex set of G and edge set of H $\subseteq$ edge set G.

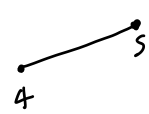
G:



H_1:

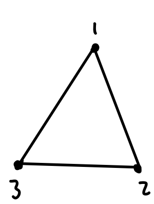


H_2:

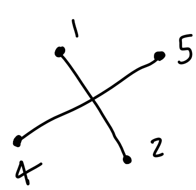


Some subgraphs...

H_3:



H_4:



Isomorphism

If A and B are sets, then a <u>one to one correspondence</u> between A and B is an association of elements of A with elements of B in such a way:

(1) to each element of A there has been associated one element of B

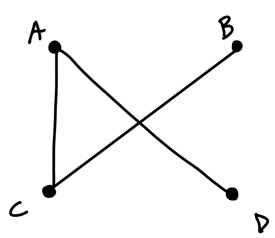
(2) to each element of B there has been associated one element of A

ex: $A = \{1, 2, 3, 4, 5\}$
$B = \{\&, !, ?, \#, \div\}$

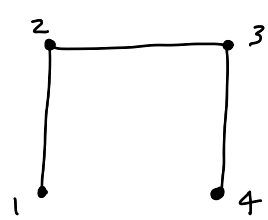
one-to-one correspondence $\longrightarrow$

$1 \leftrightarrow ?$ $\quad 3 \leftrightarrow \&$ $\quad 5 \leftrightarrow !$
$2 \leftrightarrow \#$ $\quad 4 \leftrightarrow \div$

Isomorphic: Two graphs are isomorphic if there exists between their vertex sets a one-to-one correspondence having the property that whenever two vertices are adjacent in either graph, the corresponding to vertices are adjacent in the other graph



$\cong$



$A \longleftrightarrow 2$
$B \longleftrightarrow 4$
$C \longleftrightarrow 3$
$D \longleftrightarrow 1$

Now that we have the terminology...

what do we want to know about graphs?

How can we classify them?

What properties can we identify?