Lesson 7

coloring planar graphs

Recall:

χ - chromatic #, minimum # of colors needed

δ - minimum degree

Δ - maximum degree

clique # - size of largest complete subgraph

clique # $\leq \chi \leq \Delta + 1$

if G connected, planar:

$v - e + f = 2$

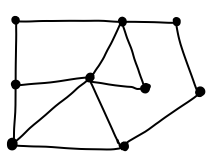
$3f \leq 2e$

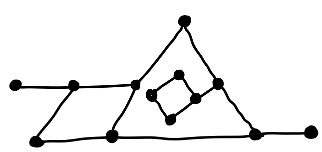
$e \leq 3v - 6$

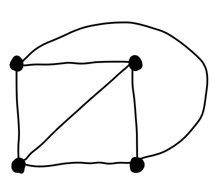
$\delta \leq 5$

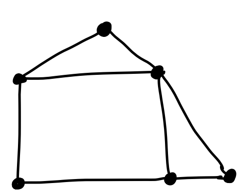
+ inductive proofs, trees, etc.

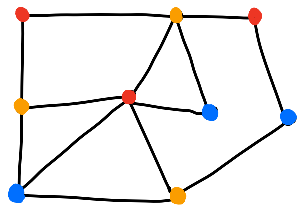
lets start by finding the chromatic # for some planar graphs



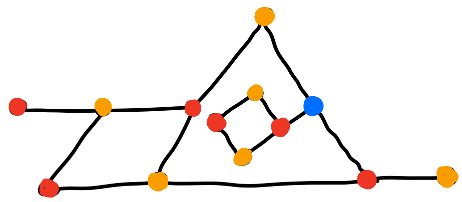




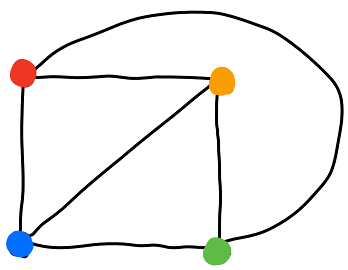




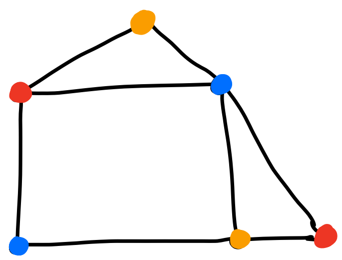
$\chi = 3$



$\chi = 3$



$\chi = 4$



$\chi = 3$

Now try some of your own examples. whats the highest chromatic # you can find?

Assuming you colored correctly, you won't find anything higher than four colors...

Four Color Theorem: Every planar graph has $\chi \leq 4$.

This proof, however, is beyond the scope of our class. Instead we will look at the next best thing.

Six color theorem: Every planar graph has $\chi \leq 6$.

We will proceed by induction on the number of vertices.

Base Case: $v=1$, $v=2$ are trivial

Inductive Hyp: Assume any planar graph with k vertices has $\chi \leq 6$

Inductive Step: Let G have $V=k+1$. Let v_0 be a vertex of degree ≤ 5. $G' = G - v_0$. G' has k vertices so it is 6-colorable. Color G according to G'. v_0 can be adjacent to at most 5 other vertices so we can color it with the remaining color. Thus, G is 6-colorable so $\chi \leq 6$.

Five color theorem: Harder but doable.

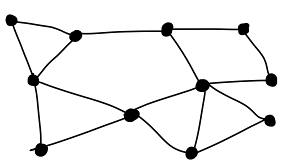
Do some experimenting with edge coloring and face coloring.

Make a statement about face coloring specifically.

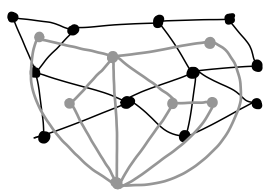
Applications of the four color theorem: Map coloring

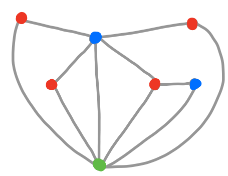
Any "map" (i.e. division of the plane into contiguous regions) can be colored in (the regions) with 4 colors. How? The dual graph!

Given a planar graph G, the dual of G call it D, has a vertex to represent each face of G and edges represent when faces are adjacent. D will also be planar. Since D is 4-colorable on vertices (by 4 color theorem) we have G is 4-face-colorable.

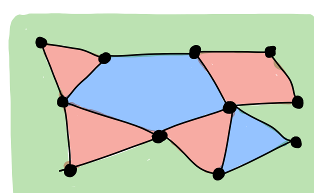


G





D_G



generalizing planarity...

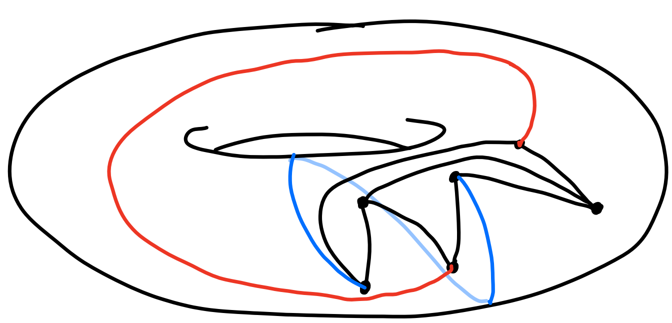
Let S_g be the surface with g holes...

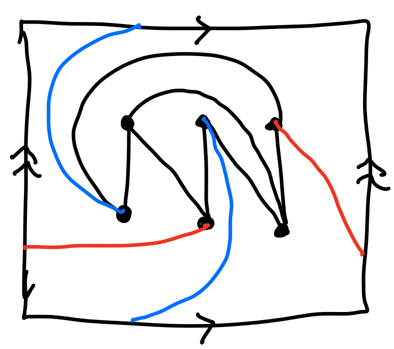
The plane is equivalent to the sphere S_0

The genus of a graph G is the smallest number g such that G can be drawn non-crossing on S_g

Fact: planar graphs are graphs with genus 0

So, we can use genus to put nonplanar graphs into different categories based on when we can draw them crossing free



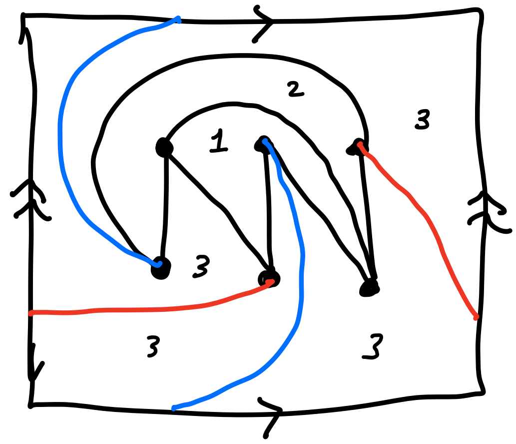


so $K_{3,3}$ has genus $g=1$.

we can count faces here!

$V=6$ $\quad$ $f=3$

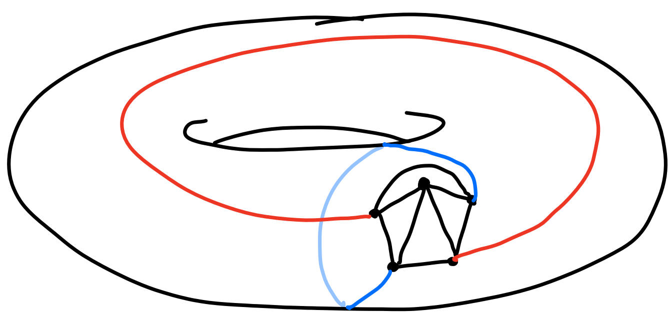
$e=9$ $\quad$ $g=0$

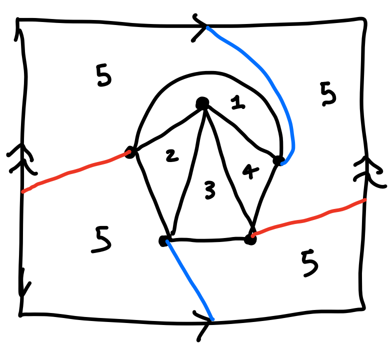


eulers formula doesn't hold in S_1.... is there a diff one?

Try drawing K_5 in S_1 and counting faces.

⋮



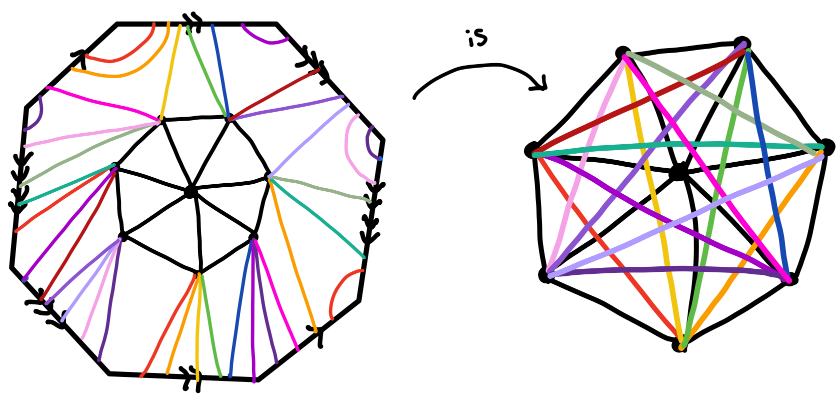


$V=5$ $\quad$ $e=10$ $\quad$ $f=5$ $\quad$ $g=1$

$$V-e+f=2 \qquad g=0$$
$$6-9+3=0 \qquad g=1$$
$$5-10+5=0 \qquad g=1$$

Genus $g = 2$ graphs get pretty complicated:
K_8 embedded on double torus S_2

$V = 8$
$e = 28$
$f = 18$
$g = 2$



So we've seen that:

$V - e + f = 2$ for $g = 0$
$V - e + f = 0$ for $g = 1$
$V - e + f = -2$ for $g = 2$

$\Bigg\}$ which gives...

Euler's second formula: $V - e + f = 2 - 2g$