§ Constructions of Brownian motion

Let us recall the definition of a Brownian motion.

Definition (Brownian motion). A stochastic process $W:[0,1] \to \mathbb{R}$ is called a Brownian motion on $[0,1]$, if the following properties hold.

1) $W(0)=0$. a.s.

2). For any integer $k \geq 1$ and reals $0 \leq t_0 < t_1 < t_2 < \cdots t_k \leq 1$, the increments $W(t_k)-W(t_{k-1})$, $W(t_{k-1})-W(t_{k-2})$, ... $W(t_1)-W(t_0)$ are mutually independent.

3) For every $t > s \in [0,1]$, the increment $W(t)-W(s) \sim N(0, t-s)$ is Normal distributed.

4) $t \mapsto W(t)$ is continuous. a.s.,

The Brownian motion (BM) is a stochastic process

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that satisfies certain properties. It is natural to ask whether such a process exist or if it makes sense.

Remark 2. We narrow ourself to $\mathbb{R}$-valued BM on the interval $[0,1]$. The result in this lecture note can generalise to $\mathbb{R}^d$-valued BM on $[0,+\infty)$ ~~be~~ thanks to the followings.

1) We can construct $\mathbb{R}^d$-valued BM via d independent BMs.

2) We can construct a BM W on any $t \in [0,\infty)$ by using the construction on $[0,1]$. For instance, let W_1, $W_2, \dots$ be independent BMs on $[0,1]$, Then the process $W : [0,+\infty) \to \mathbb{R}$ defined by the ~~conce~~ concatenation

$$W(t) := \begin{cases} W_1(t), & t \in [0,1) \\ W_1(1) + W_2(t-1), & t \in [1,2) \\ \vdots & \vdots \\ W_1(1) + \dots W_k(1) + W_{k+1}(t-k), & t \in [k,k+1) \\ \vdots & \end{cases}$$

is a BM.

There are a plethora ways to ~~answer~~ verify that the BM exists. The most straitforward one is by using Kolmogorov extension theorem and Kolmogorov-chentsov (continuity) theorem. The Kolmogorov extension theorem says that if a finite-dimensional distribution satisfies a consistency criterion, then there exists a stochastic process on some probability space that induces the finite-dimensional distribution. By writing down the distribution of $W(t_1)$, $W(t_2)$, ... $W(t_k)$, we can easily verify that the distribution is consistent in terms of any permutation and marginalisation. The Kolmogorov extension theorem thus implies that such a process W exists to satisfy the BM properties 1, 2, and 3. The continuity-path property can be verified by using Kolmogorov-chentsov (continuity) theorem, which we will detail later in the lecture note.

We can also prove the existence more explicitly by directly constructing a BM. The most celebrated construction is arguably the Lévy-Ciesielski construction (also known as the L^2 construction). To elaborate this construction, we need to introduce a few technical prerequisites. as follows.

Defin $L^2 := L^2([0,1])$ be the Hilbert space of square integrable fuctions $f: [0,1] \to \mathbb{R}$ with inner product $\langle f, g \rangle_{L^2} := \int_{[0,1]} f(x) g(x) \, dx$ w.r.t the Lebesgue measure. Denote the norm $\| \cdot \|_{L^2} := \sqrt{\langle \cdot, \cdot \rangle_{L^2}}$. Let $\{\phi_n\}_{n \geq 1}$ be an ~~orthonormor~~ orthonormal system of L^2, viz, $\langle \phi_n, \phi_m \rangle_{L^2} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$ Furthermore, define a probability spae $(\Omega, \Sigma, \mathbb{P})$ and a sequence of Normal random variables $\{A_n \sim N(0,1)\}_{n \geq 1}$ on the space. Define the Hibert space of random variables $L^2(\mathbb{P})$ with inner product $\langle X, Y \rangle_{L^2(\mathbb{P})} := \mathbb{E}[XY]$ and norm $\|X\|_{L^2(\mathbb{P})} := \mathbb{E}[X^2]^{\frac{1}{2}}$.

we then have an ansatz construction

$$W_N(t) := \sum_{n=1}^{N} A_n \langle \mathbb{1}_{[0,t)}, \phi_n \rangle_{L^2} = \sum_{n=1}^{N} A_n \int_0^t \phi_n(s)\,ds,$$

We shall show that the process defined by the limit of W_N as $N \to \infty$ is a ~~Br~~ BM. To do so, we first need to show that the limit $\lim_{N\to\infty} W_N(t)$ is well-defined, that is, the limit exists and converges in $L^2(\mathbb{P})$ for all $t \in [0,1]$. Then, we show that the limit process satisfies the axioms of the BM.

Lemma 3. The limit $W(t) := \lim_{N\to\infty} W_N(t)$ exists and converges in $L^2(\mathbb{P})$ for all $t \in [0,1]$. Moreover, the process $t \mapsto W(t)$ satisfies the axiom 1, 2, and 3 of BM.

proof. Taking the $L^2(\mathbb{P})$-norm on $W_N(t)$ we have

$$\begin{aligned}
\|W_N(t)\|_{L^2(\mathbb{P})}^2 &= \mathbb{E}[W_N(t)^2] \\
&= \mathbb{E}\Big[\Big(\sum_{n=1}^{N} A_n \langle \mathbb{1}_{[0,t)}, \phi_n \rangle_{L^2}\Big)\Big(\sum_{n=1}^{N} A_n \langle \mathbb{1}_{[0,t)}, \phi_n \rangle_{L^2}\Big)\Big] \\
&= \mathbb{E}\Big[\sum_{n,m=1}^{N} A_n A_m \langle \mathbb{1}_{[0,t)}, \phi_n \rangle_{L^2} \langle \mathbb{1}_{[0,t)}, \phi_n \rangle_{L^2}\Big]
\end{aligned}$$

$$= \sum_{n=1}^{N} \langle \mathbb{1}_{[0,t)}, \phi_n \rangle_{L^2}^2 \qquad (\text{independence of } \{A_n\})$$

$$< \sum_{n=1}^{\infty} \langle \mathbb{1}_{[0,t)}, \phi_n \rangle_{L^2}^2 .$$

By Parseval's equation, $\lim_{N\to\infty} \sum_{n=1}^{N} \langle \mathbb{1}_{[0,t)}, \phi_n \rangle_{L^2}^2 = \| \mathbb{1}_{[0,t)} \|_{L^2}^2$ $= t < \infty$. This shows that $\| W_N(t) \|_{L^2(\mathbb{P})}$ is finite hence $W_N(t) \in L^2(\mathbb{P})$. To show that the limit $\lim_{N\to\infty} W_N(t)$ exists and converges, we prove that $\{W_N(t)\}_{N\geq 1}$ is a Cauchy sequence. By definition of Cauchy sequence, we have

$$\lim_{M\to\infty} \sup_{N\geq M} \cancel{\sum_{k=M+1}^{N}} \| W_N(t) - W_M(t) \|_{L^2(\mathbb{P})}^2$$

$$= \lim_{M\to\infty} \sup_{N\geq M} \sum_{k=M+1}^{N} \langle \mathbb{1}_{[0,t)}, \phi_k \rangle_{L^2}^2$$

$$\leq \lim_{M\to\infty} \sup_{N\geq M} \sum_{k=M+1}^{\infty} \langle \mathbb{1}_{[0,t)}, \phi_k \rangle_{L^2}^2 .$$

This shows that $\lim_{M\to\infty} \sup_{N\geq M} \| W_N(t) - W_M(t) \|_{L^2(\mathbb{P})}^2 = 0$. Hence $\{W_N(t)\}_{N\geq 1}$ is a Cauchy sequence. Since $L^2(\mathbb{P})$ is complete, we have the limit of the Cauchy sequence converges in $L^2(\mathbb{P})$.

The first axiom $W(0)=0$ is trivial to prove. To prove axioms 2 and 3, we can leverage the characteristic function which is useful in showing the properties (e.g., distribution and independence) of limits of random variables.

axiom 2): the characteristic function of $W(t)-W(s)$ for $t>s$ is

$$\mathbb{E}[\exp(iz(W(t)-W(s)))]$$

$$=\mathbb{E}[\exp(iz\lim_{N\to\infty}\sum_{n=1}^{N}A_n\langle \mathbb{1}_{[0,t)}-\mathbb{1}_{[0,s)},\phi_n\rangle_{L^2})]$$

bounded convergence theorem

$$=\lim_{N\to\infty}\mathbb{E}[\exp(iz\sum_{n=1}^{N}A_n\langle \mathbb{1}_{(s,t)},\phi_n\rangle_{L^2})].$$

Now, since $\{A_n\}_{n\geq 1}$ are i.i.d. Normal, by the property of characteristic function, the equation above amounts to

$$=\lim_{N\to\infty}\prod_{n=1}^{N}\mathbb{E}[\exp(izA_n\langle \mathbb{1}_{(s,t)},\phi_n\rangle_{L^2})]$$ (recall the c.f. of Normal)

$N(m,v) \Leftarrow\Rightarrow e^{imz-\frac{1}{2}v^2z^2}$

$$=\lim_{N\to\infty}\prod_{n=1}^{N}\exp(-\tfrac{1}{2}\langle \mathbb{1}_{(s,t)},\phi_n\rangle_{L^2}^2 z^2)$$

$$=\exp(\lim_{N\to\infty}\sum_{n=1}^{N}-\tfrac{1}{2}\langle \mathbb{1}_{(s,t)},\phi_n\rangle_{L^2}^2 z^2)=\exp(-\tfrac{1}{2}\|\mathbb{1}_{(s,t)}\|_{L^2}^2 z^2)$$

(t-s) no square

$$=\exp(-\tfrac{1}{2}(t-s)^2z^2)$$

which is the characteristic function of $N(0, t-s)$.

Hence $W(t) - W(s) \sim N(t-s)$.

axiom 3): To show that $W(t) - W(s)$ and $W(v) - W(u)$ are independent for all $s < t < u < v$, we apply Kac's theorem

Essentially, random variables X and Y are independent iff $\varphi_{X,Y}(\alpha, \beta) = \varphi_X(\alpha)\varphi_Y(\beta)$ for all α, β.. We have

$$\mathbb{E}\left[\exp\left(i\alpha(W(t) - W(s)) + i\beta(W(v) - W(u))\right)\right]$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} \mathbb{E}\left[\exp\left(i\alpha A_n \langle \mathbb{1}_{[s,t)}, \phi_n \rangle_{L^2} + i\beta A_n \langle \mathbb{1}_{[u,v)}, \phi_n \rangle_{L^2}\right)\right]$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} \mathbb{E}\left[\exp\left(i\left(\alpha \langle \mathbb{1}_{[s,t)}, \phi_n \rangle_{L^2} + \beta \langle \mathbb{1}_{[u,v)}, \phi_n \rangle_{L^2}\right) A_n\right)\right]$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} \exp\left(-\frac{1}{2}\left(\alpha \langle \mathbb{1}_{[s,t)}, \phi_n \rangle_{L^2} + \beta \langle \mathbb{1}_{[u,v)}, \phi_n \rangle_{L^2}\right)^2\right)$$

$$= \lim_{N\to\infty} \Bigg(\exp\left(-\frac{\alpha^2}{2} \sum_{n=1}^{N} \langle \mathbb{1}_{[s,t)}, \phi_n \rangle_{L^2}^2\right) \exp\left(-\frac{\beta^2}{2} \sum_{n=1}^{N} \langle \mathbb{1}_{[u,v)}, \phi_n \rangle_{L^2}^2\right)$$
$$\underline{\exp\left(-\frac{\alpha\beta}{2} \sum_{n=1}^{N} \langle \mathbb{1}_{[s,t)}, \phi_n \rangle_{L^2} \langle \mathbb{1}_{[u,v)}, \phi_n \rangle_{L^2}\right)} \Bigg)$$

(t-s) and (u-v) no squares

$$= \exp\left(-\frac{1}{2}(t-s)^2\alpha^2\right)\exp\left(-\frac{1}{2}(u-v)^2\beta^2\right)$$

Converges to $\langle \mathbb{1}_{[s,t)}, \mathbb{1}_{[u,v)} \rangle$ $= 0$ by Parseval.

which is the product of the characteristic functions of $W(t) - W(s)$ and $W(v) - W(u)$. Hence, they are independent. $\square$

The Lemma above shows that the process W defined via the limit $\lim_{N\to\infty} W_N(t)$ satisfies the axioms 1, 2, and 3 of BM. We also need to show that it satisfies the path-continuity property in 4). We at least have two ways to do so. One way is by applying the Kolmogorov–Chentsov (continuity) theorem. Essentially, the theorem says that if a process X satisfies.

$$\mathbb{E}[\,|X(t)-X(s)|^{\alpha}\,] \le C\,|t-s|^{\beta+1} \text{ for all } s,t \ge 0,$$

for some positive constants α, β, and C, the X has a version $\tilde{X}$ whose path is a.s. Hölder-continuous of order $\gamma \in (0, \frac{\beta}{\alpha})$. For ~~W,~~ our W, we can verify

$$\mathbb{E}[\,|W(t)-W(s)|^{2n}\,] = C_n\,|t-s|^{n}. \text{ for } n = 0, 1, \ldots$$

Hence, W has a continuous version $\tilde{X}$ of Hölder condition $\frac{n-1}{2n}$ for $n = 2, 3, \ldots$ hence $\gamma_W \in (0, \frac{1}{2})$.

It is also possible to prove the continuity without relying on the Kolmogorov-Chentsov theorem but a clever selection of the orthonormal system $\{\phi_n\}_{n\geq 1}$.

The two most celebrated systems are the Haar-Schauder and Paley-Wiener systems. The basic idea of proving the continuity is that $\{W_N\}_{N\geq 1}$ is a sequence of continuous functions. Hence, to show that the limit $\lim_{N\to\infty} W_N$ is also a continuous function, we can show that the sequence $\{W_N\}_{N\geq 1}$ or a ~~subseq~~ subsequence of it "uniformly" converges to the limit. Essentially, we would like to show that

$$\lim_{N,M\to\infty} \sup_{t\in[0,1]} |W_N(t) - W_M(t)| = 0. \quad \text{a.s.}$$

To do so, define $\Delta_j(t) := W_{2^j}(t) - W_{2^{j-1}}(t)$, $j = 1, 2, \ldots$

then we have

$$\left\| \sup_{t\in[0,1]} \left| W_N(t) - W_M(t) \right| \right\|_{L^p(\mathbb{P})}$$

$$= \left\| \sup_{t\in[0,1]} \left| \sum_{j=M+1}^{N} \left(W_{2^j}(t) - W_{2^{j-1}}(t) \right) \right| \right\|_{L^p(\mathbb{P})}$$

$$\leq \left\| \sum_{j=M+1}^{N} \sup_{t\in[0,1]} \left| W_{2^j}(t) - W_{2^{j-1}}(t) \right| \right\|_{L^p(\mathbb{P})}$$

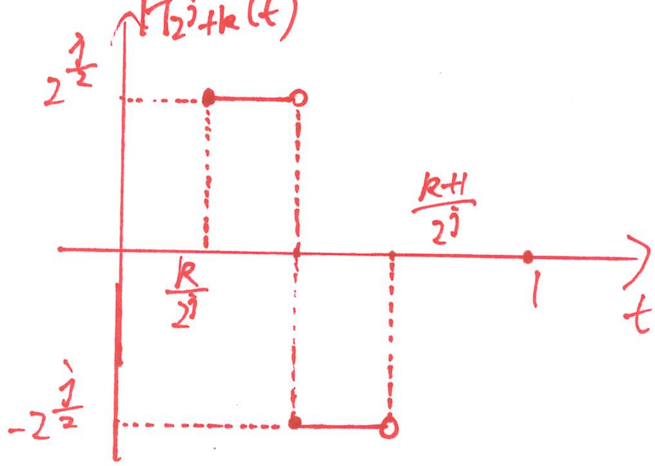
$$\leq \sum_{j=M+1}^{N} \left\| \sup_{t\in[0,1]} |\Delta_j| \right\|_{L^p(\mathbb{P})}$$

Now we need to bring in the selected orthonormal system to find an upper bound of $\sup_{t\in[0,1]} |\Delta_j|$ such that it converges to zero.

Take the Haar-Schauder system for example, for $n = 2^j + k$, $j \geq 0$, $k = 0, 1, \dots 2^j - 1$, the functions

$$H_0(t) = 1$$

$$H_{2^j+k}(t) = \begin{cases} 2^{\frac{j}{2}}, & t \in \left[\frac{k}{2^j}, \frac{2k+1}{2^{j+1}}\right) \\ -2^{\frac{j}{2}}, & t \in \left[\frac{2k+1}{2^{j+1}}, \frac{k+1}{2^j}\right) \\ 0, & \text{otherwise} \end{cases}$$



are the Haar functions and they are orthonormal in L^2.

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under this basis. We can show that

$$\mathbb{E}\Big[\sup_{t\in[0,1)}|\Delta_j(t)|^4\Big]\le 3\sum_{k=0}^{2^j-1}2^{-2j}=3\,2^{-j},\quad j\ge 1.$$

Hence,

choose $p=4$

$$\sum_{j=M+1}^{N}\Big\|\sup_{t\in[0,1)}|\Delta_j(t)|\Big\|_{L^4(\mathbb{P})}$$

$$\le 3^{\frac14}\sum_{j=M+1}^{N}2^{-\frac{j}{4}}=3^{\frac14}2^{-\frac{M+1}{4}}\Big(1+2^{-\frac14}+2^{-\frac24}+\dots 2^{-\frac{\cancel{M+1}\,(N-M+1)}{4}}\Big)$$

$$\xrightarrow{N,M\to\infty}0.$$

Finally Fatou's lemma gives

$$\Big\|\lim_{N,M\to\infty}\sup_{t\in[0,1)}|W_N(t)-W_M(t)|\Big\|_{L^4(\mathbb{P})}\le\lim_{N,M\to\infty}\Big\|\sup_{[0,1)}|W_N(t)-W_M(t)|\Big\|_{L^4(\mathbb{P})}$$

$$=0.\quad \cancel{a.s.}.$$

Hence, $\lim_{N,M\to\infty}\sup_{t\in[0,1)}|W_N(t)-W_M(t)|=0$, a.s., Hence the uniform convergence is true a.s., Finally, we have

$$\overline{W}(t,\omega):=\begin{cases}\lim_{N\to\infty}W_N(t,\omega), & \omega\in\Omega\setminus\Omega_0\\ 0, & \omega\in\Omega_0\end{cases}$$

is a BM, where $\mathbb{P}(\Omega_0)=0$.

We can also prove the uniform convergence by using Paley-Wiener's trigonometric series. $\{\phi_n(t)\} := \{\sqrt{2}\cos(n\pi t)\}_{n\geq 0}$, and

$$W_N(t) = A_0 t + \sum_{n=1}^{N} A_n \sin(n\pi t)\frac{\sqrt{2}}{\pi n}.$$

In this case we have $\mathbb{E}[\sup_{t\in[0,1]} |\Delta_j(t)|^2] \leq C 2^{-\frac{j}{2}}$. It is also worth noting that if we take a formal derivative of W we get

$$\frac{dW(t)}{dt} = A_0 + \sum_{n=1}^{\infty} \sqrt{2} A_n \cos(n\pi t).$$

The Fourier coefficients are $\{\sqrt{2}A_n\}$ which have the same distribution across all the frequencies. This means that the spectrum of $\frac{dW(t)}{dt}$ is white, hence the name white noise.

There are also other construction of BM. For instance, Donsker's random walk. Recall that $W(t) \sim N(0,t)$, we can define a cumsum random variable

$$S_N := \xi_1 + \xi_2 + \cdots \xi_N, \quad \{\xi_n\}_{n=1}^{N} \text{ are i.i.d. } N(0,1).$$

then Let

$$W_N(t) := \frac{1}{\sqrt{N}}\left(S_{\lfloor nt\rfloor} - (nt - \lfloor nt\rfloor)\,\xi_{\lfloor nt\rfloor+1}\right)$$

be a candidate BM. when $N\to\infty$, We can by the central limit theorem check that $W_N(t)$ satisfies the BM axioms 1, 2, and 3 as $N\to\infty$. Donsker 1951 proved that W_N converges to a BM weakly (i.e., in distribution)

In all the mentioned constructions, it is easy to verify the finite-dimensional distribution properties of BM, but difficult to prove the continuity property.

We can also set-up a space of continuous-functions then assign a measure over it to verify that the coordinate random variable has the BM distribution.

More specifically, Let C_0 be a Banach space of continuous functions, viz., $C_0 := \{\omega : [0,1]\to\mathbb{R}$, ω is continuous and $\omega(0)=0\}$ equipped with norm

$$\|\omega\|_\infty := \sup_{t\in[0,1]} |\omega(t)|.$$

Define cylinder subset F_U of C_0 of the form

$$F_U = \{ \omega \in C_0 : (\omega(t_1), \omega(t_2), \ldots \omega(t_n)) \in U \},$$

where $0 < t_1 < t_2 < \cdots t_n \le 1$ and $U \in \mathcal{B}(\mathbb{R}^n)$ the Borel sigma-algebra of $\mathbb{R}^n$. Let $\mathcal{Z}$ be the collection of all such cylider subsets, i.e.,

$$\mathcal{Z} := \{ F_U : U \in \mathcal{B}(\mathbb{R}^n) \}.$$

Then $\mathcal{Z}$ is an algebra (a ring). Now let us define a set function

$$\mu(F_U) := \int_U \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left(-\frac{(u_i - u_{i-1})^2}{2(t_i - t_{i-1})}\right) du_1 du_2 \cdots du_n,$$

where $t_0 = u_0 = 0$. This μ resembles the finite-dimensional distribution of BM. Then Wiener 1923 showed that this function μ is a pre-measure, which is sigma-additive. That is, for any countable disjoint $F_1, F_2, \ldots \in \mathcal{Z}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i).$$

Then by Carathéodory's extension theorem, μ has ~~an~~ a unique extension $\bar{\mu}$ to the sigma-algebra $\sigma(Z)$ generated by Z. Moreover, it turns out that this generated sigma-algebra coincides with the Borel sigma-algebra $\mathcal{B}(C_0)$. ~~the~~ Finally, define a probability space $(C_0, \sigma(Z), \bar{\mu})$ and random variable $W(t,\omega) := \omega(t) \in C_0$, then W is a BM.