Най-кратък път в Граф

Лекция 11 по СДА, Софтуерно Инженерство
Зимен семестър 2018-2019г

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От миналият път

Видове графи

- насочен/ненасочен
- с тегла по ребрата/без тегла по ребрата

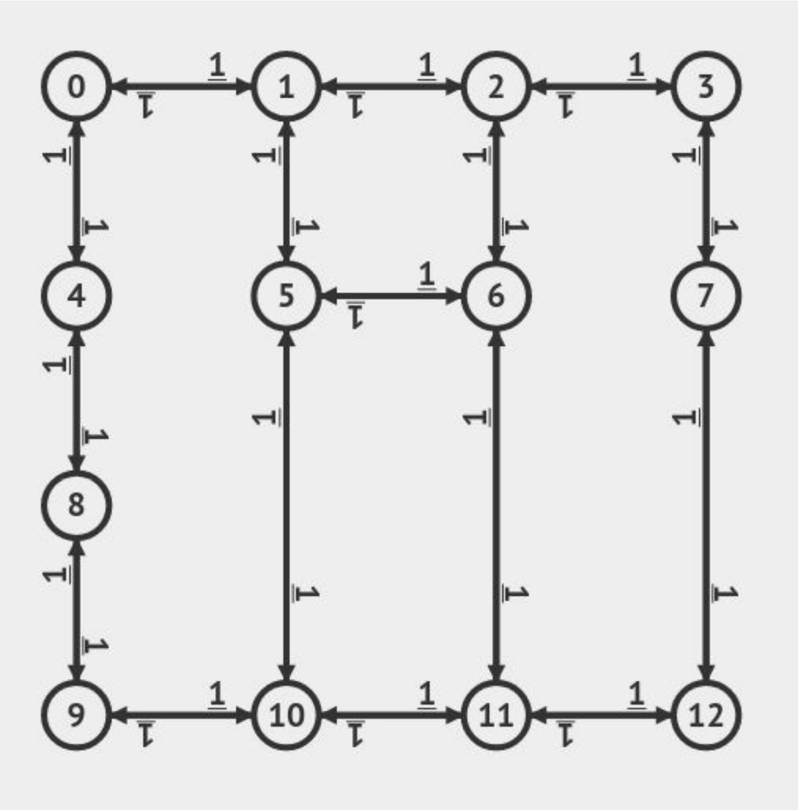
Представяне на графи

- с матрица на съседство
- със списък на съседи

Обхождане на графи

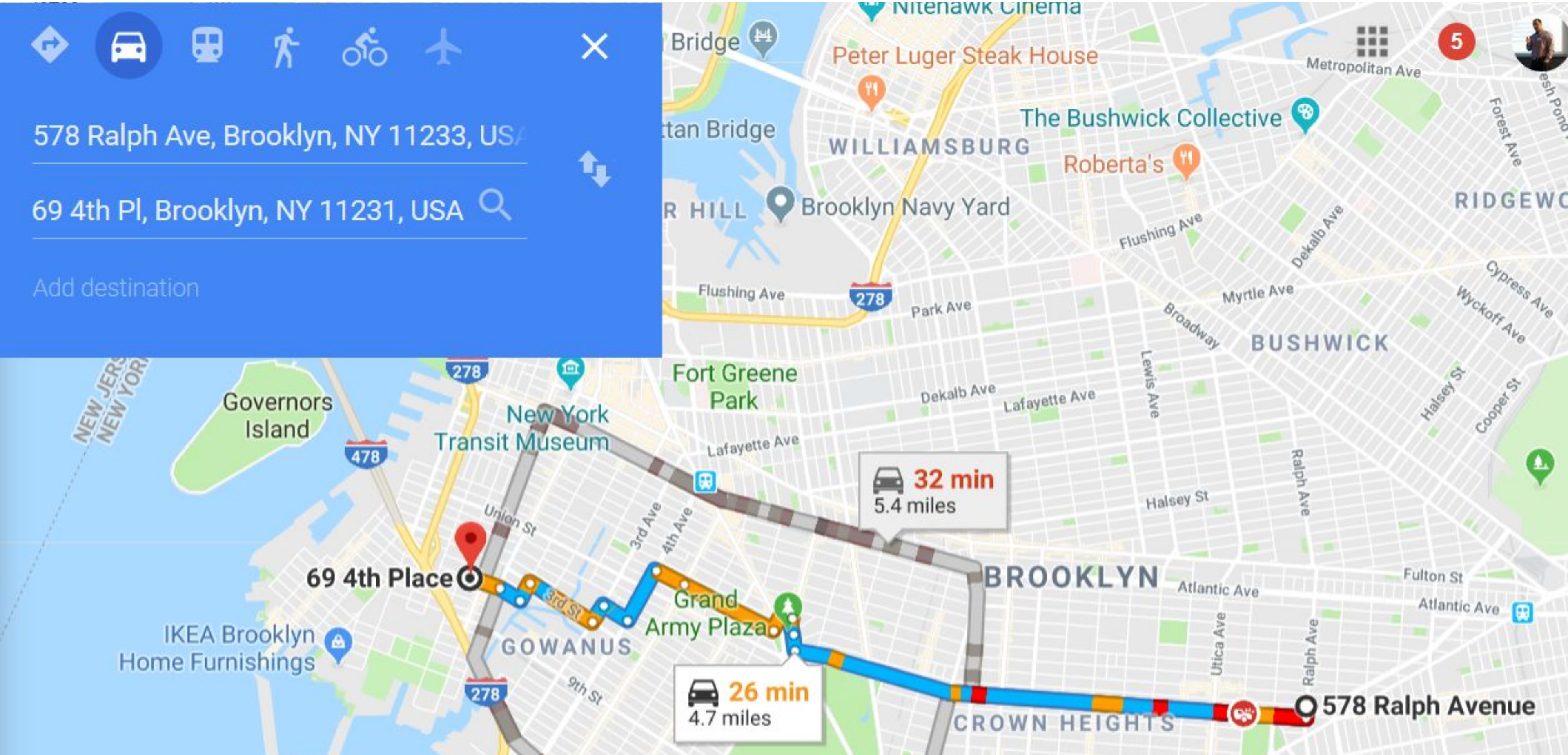
- dfs
- bfs

Минимален път в непретеглен граф



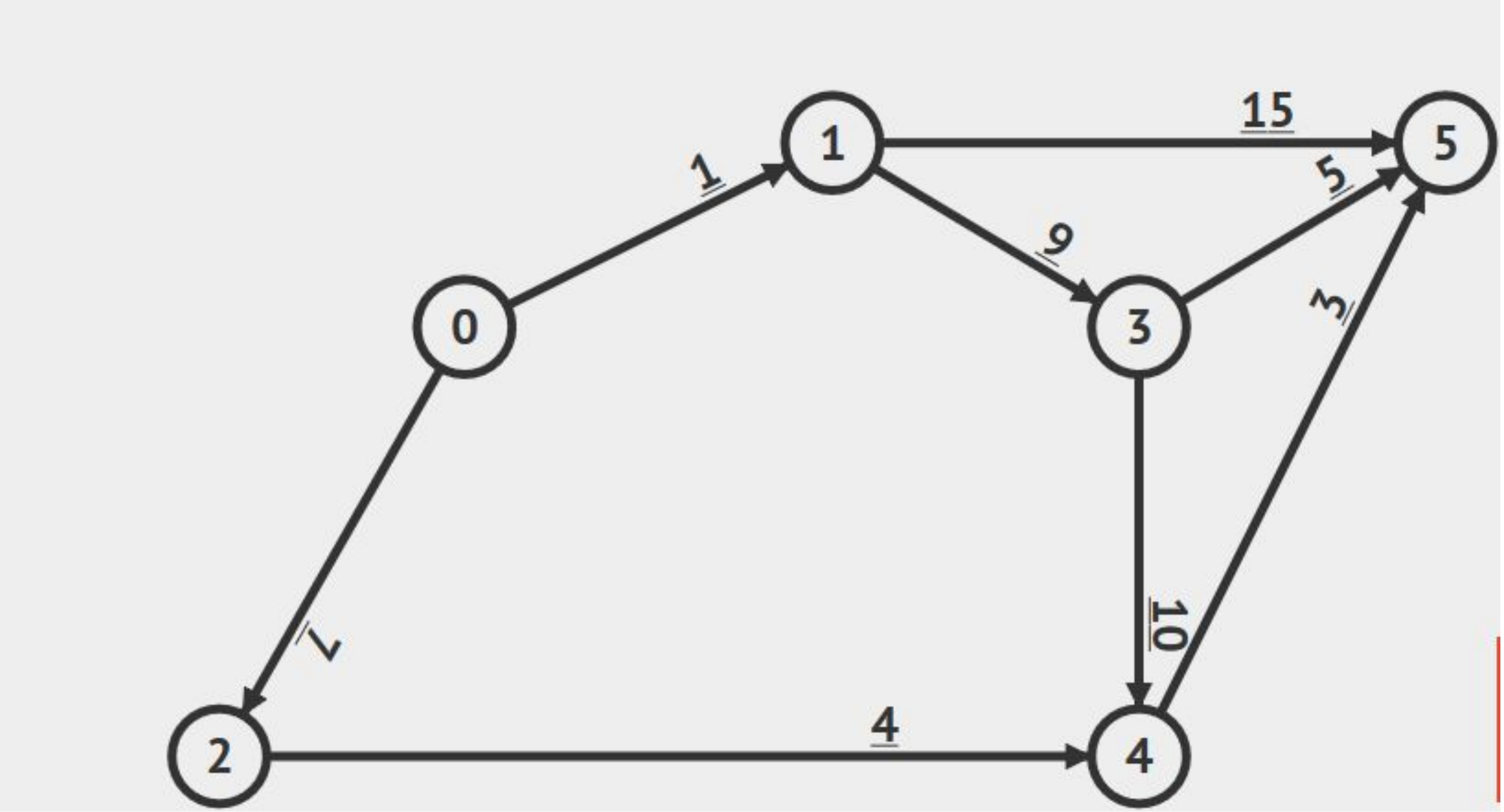
Намира се с bfs!

Минимален път в претеглен граф ?



Задачи

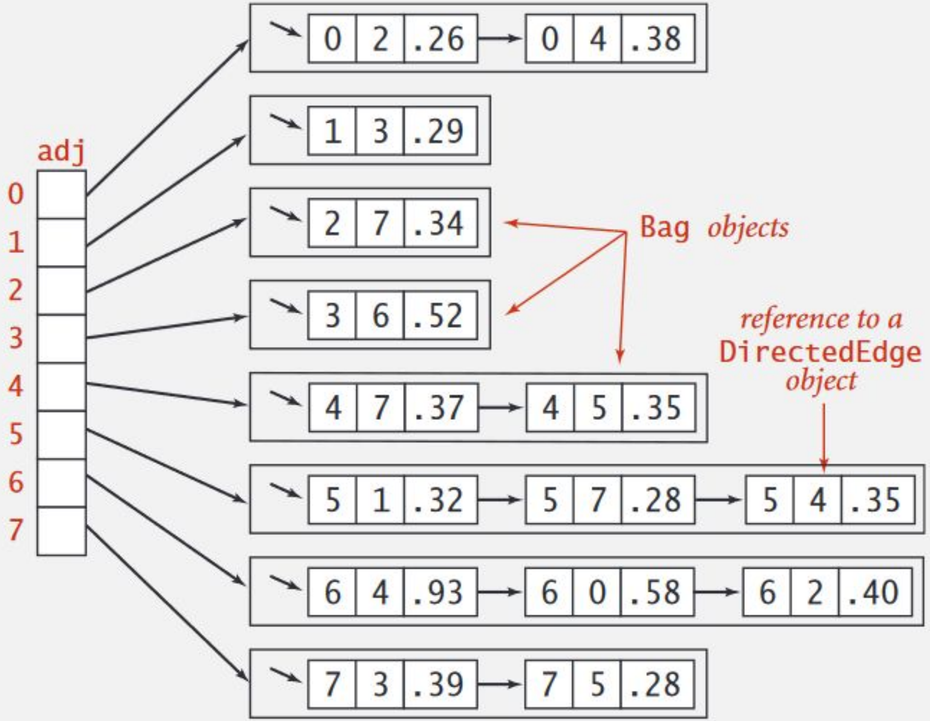
1. Изчисляване на най-къс път от връх до друг връх
2. Изчисляване на най-късите пътища от връх до всички други върхове
3. Изчисляване на най-късите пътища от всеки до всеки връх


0
1
2
3
4
5
1
15
9
5
3
7
10
4

Edge-weighted digraph: adjacency-lists representation

tinyEWD.txt

```
V  8
15  E
4 5  0.35
5 4  0.35
4 7  0.37
5 7  0.28
7 5  0.28
5 1  0.32
0 4  0.38
0 2  0.26
7 3  0.39
1 3  0.29
2 7  0.34
6 2  0.40
3 6  0.52
6 0  0.58
6 4  0.93
```

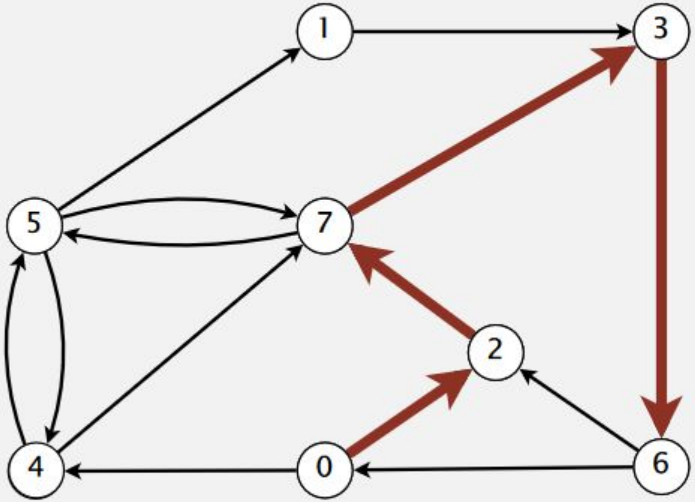


Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from s to t.

edge-weighted digraph

4->5	0.35
5->4	0.35
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



shortest path from 0 to 6

$0 \to 2 \to 7 \to 3 \to 6$

length of path = 1.51

$(0.26 + 0.34 + 0.39 + 0.52\ )$

Алгоритъм на Дийкстра

Основна идея: Подобен на търсене в ширина, но вместо с опашка с приоритетна опашка.

```
function Dijkstra(Graph, source):

    create vertex set Q

    for each vertex v in Graph:             // Initialization
        dist[v] ← INFINITY                  // Unknown distance from source to v
        prev[v] ← UNDEFINED                 // Previous node in optimal path from source
        add v to Q                          // All nodes initially in Q (unvisited nodes)

    dist[source] ← 0                        // Distance from source to source

    while Q is not empty:
        u ← vertex in Q with min dist[u]    // Node with the least distance
                                            // will be selected first
        remove u from Q

        for each neighbor v of u:           // where v is still in Q.
            alt ← dist[u] + length(u, v)
            if alt < dist[v]:               // A shorter path to v has been found
                dist[v] ← alt
                prev[v] ← u

    return dist[], prev[]
```

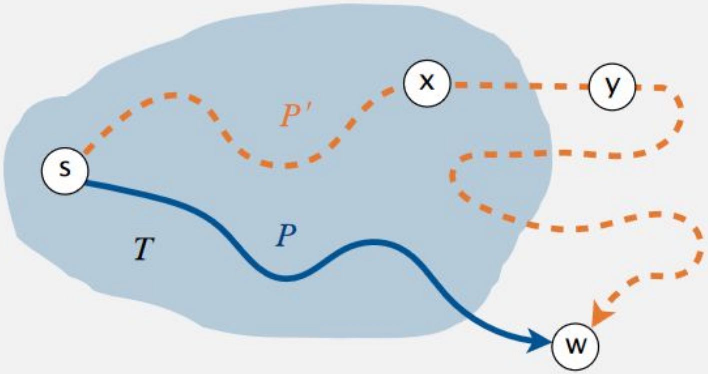
Dijkstra's algorithm: correctness proof

Invariant. For each vertex v in T, `distTo[v]` $= d^*(v)$.

length of shortest $s \rightarrow v$ path

Pf. [by induction on $|T|$]

- Let w be next vertex added to T.
- Let P be the $s \rightarrow w$ path of length `distTo[w]`.
- Consider any other $s \rightarrow w$ path P'.
- Let $x \rightarrow y$ be first edge in P' that leaves T.
- P' is no shorter than P :



by construction

$\text{length}(P) =$ `distTo[w]`

Dijkstra chose w instead of y → $\leq$ `distTo[y]`

relax vertex x → $\leq$ `distTo[x]` $+\ weight(x, y)$

induction → $= d^*(x) + weight(x, y)$

weight are non-negative → $\leq \text{length}(P')$ ∎

What is the order of growth of the running time of Dijkstra's algorithm in the worst case when using a binary heap for the priority queue?

A. $V + E$

B. $V \log V$

C. $E \log V$

D. $E \log E$

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V INSERT, V DELETE-MIN, $\leq E$ DECREASE-KEY.

PQ implementation	INSERT	DELETE-MIN	DECREASE-KEY	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap	$1^{\dagger}$	$\log V^{\dagger}$	$1^{\dagger}$	$E + V \log V$

† amortized

Bottom line.

- Array implementation optimal for complete graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

Algorithm for shortest paths

Variations on a theme: vertex relaxations.

- Bellman–Ford: relax all vertices; repeat $V - 1$ times.
- Dijkstra: relax vertices in order of distance from s.

algorithm	worst-case running time	negative weights †	directed cycles
Bellman-Ford	$E V$	✔	✔
Dijkstra	$E \log V$		✔

Какво следва

- Домашна работа 8 - 17.12.2020 - 05.01.2020
- Лекция 12 - Минимално покриващо дърво
- Домашна работа 9 - 07.01 - 14.01.2020
- Лекция 13 (Ойлеров и Хамилтонов цикъл. NP complete задачи)
  + Контролна работа 6
- Допълнителни задачи за домашна работа 14.01.2020 - 23.01.2020
- Изпит - 24.01.2020