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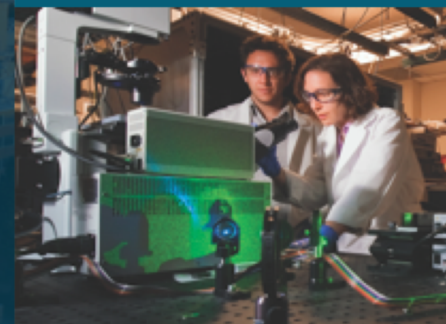

Guest Lecture Stanford ME469: Splitting and Stabilization Errors

PRESENTED BY

Stefan P. Domino

Computational Thermal and Fluid Mechanics

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Splitting and Stabilization Errors: Outline

- Block Matrix and Operator Form
- Approximate Factorization
- Splitting Errors
- Stabilization Errors
- Detailed Code Verification
- Conclusions

Introduction to Block Matrix Form

- Consider the divergence of a vector, **D**F (a scalar)

$$\frac{\partial F_j}{\partial x_j} = 0 \quad \begin{cases} \int \frac{\partial F_j}{\partial x_j} dV = 0 \xrightarrow{\text{(Gauss-Divergence)}} \int F_j n_j \, dS = 0 \\ \int w \frac{\partial F_j}{\partial x_j} d\Omega = 0 \xrightarrow{\text{(Integration-by-parts)}} -\int F_j \frac{\partial w}{\partial x_j} d\Omega + \int w F_j n_j d\,\Gamma = 0 \end{cases}$$

(piecewise-const w)

- Gradient of a scalar, **G**p (a vector)

$$\frac{\partial p}{\partial x_i} = 0 \quad \begin{cases} \int \frac{\partial p}{\partial x_i} dV = 0 \xrightarrow{\text{(Green-Gauss)}} \int p n_i \, dS = 0 \\ \int w \frac{\partial p}{\partial x_i} d\Omega = 0 \xrightarrow{\text{(Integration-by-parts)}} -\int p \frac{\partial w}{\partial x_i} d\Omega + \int w p n_i d\,\Gamma = 0 \end{cases}$$

(piecewise-const w)

- Laplace operator, $\mathbf{L}_\lambda$T

$$\frac{\partial q_j}{\partial x_j} = 0, \quad q_j = -\lambda \frac{\partial T}{\partial x_j} \quad \begin{cases} \int \frac{\partial q_j}{\partial x_j} dV = 0 \xrightarrow{\text{(Gauss-Divergence)}} -\int \lambda \frac{\partial T}{\partial x_j} n_j \, dS = 0 \\ \int w \frac{\partial q_j}{\partial x_j} d\Omega = 0 \xrightarrow{\text{(Integration-by-parts)}} -\int q_j \frac{\partial w}{\partial x_j} d\Omega + \int w q_j n_j d\,\Gamma = 0 \end{cases}$$

(piecewise-const w)

Introduction to Block Matrix Form for Fluids

- Consider the monolithic, uniform density, low-Mach equation system:

$$\frac{\partial u_j}{\partial x_j} = 0$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p_i}{\partial x_i} + S_i$$

that can be written in block form as:

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

- We seek to factorize this system via:

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \approx \begin{bmatrix} A & 0 \\ D & B_1 \end{bmatrix} \begin{bmatrix} I & B_2 G \\ 0 & I \end{bmatrix} \approx \begin{bmatrix} A & A B_2 G \\ D & (B_1 + D B_2 G) \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

the exact factorization can be recovered by defining: $\begin{cases} B_2 = A^{-1} \\ B_1 = -D B_2 G \end{cases}$

- $\mathbf{B_2}$ determines the projection time scale, ideally chosen to approximate $\mathbf{A^{-1}}$
- $\mathbf{B_1}$ controls the projection error, ideally chosen to cancel $\mathbf{BD_2G}$

Introduction to Block Matrix Form

- The approximate factorization

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \approx \begin{bmatrix} A & 0 \\ D & B_1 \end{bmatrix} \begin{bmatrix} I & B_2 G \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AB_2 G \\ D & (B_1 + DB_2 G) \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

- can be written now as two segregated steps:

Momentum and Continuity: $$\begin{bmatrix} A & 0 \\ D & B_1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \quad \left\{ \begin{aligned} A\hat{u} &= f \\ D\hat{u} + B_1 \hat{p} &= 0 \end{aligned} \right.$$

Nodal Projection: $$\begin{bmatrix} I & B_2 G \\ 0 & I \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix} \left\{ \begin{aligned} u^{n+1} &= \hat{u} - B_2 G p^{n+1} = 0 \\ p^{n+1} &= \hat{p} \end{aligned} \right.$$

- This approach seems to be straight forward, however, what errors have we introduced by this procedure of splitting the monolithic (fully coupled) system?

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} + \begin{bmatrix} (I - AB_2) G p^{n+1} \\ -(B_1 + DB_2 G) p^{n+1} \end{bmatrix} \quad \text{Exact iff:} \quad \begin{aligned} B_2 &= A^{-1} \\ B_1 &= -DB_2 G \end{aligned}$$

- In most cases, $\mathbf{B}_2$ is approximately $\mathbf{A}^{-1}$ and a first-order temporal splitting error is noted

Incremental Pressure-Projection without Pressure Stabilization

- Let the inverse of **A**, $\mathbf{A}^{-1}$ be approximated by $\mathbf{B}_2$ as a scalar, $\boldsymbol{\tau}$
- Let $\mathbf{B}_1$ be equal to the scaled Laplace operator, - $\boldsymbol{\tau}\mathbf{L}$

Momentum and Continuity: $$\begin{bmatrix} A & 0 \\ D & -\tau L \end{bmatrix} \begin{bmatrix} \hat{u} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \hat{f} \\ -\tau L p^{n} \end{bmatrix} \quad \left\{ \begin{aligned} A\hat{u} &= \hat{f} = f - Gp^{n} \\ D\hat{u} - \tau L(p^{n+1} - p^{n}) &= 0 \end{aligned} \right.$$

Nodal Projection: $$\begin{bmatrix} I & \tau G \\ 0 & I \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix} + \begin{bmatrix} \tau G p^{n} \\ 0 \end{bmatrix} \quad \left\{ \begin{aligned} u^{n+1} &= \hat{u} - \tau G(p^{n+1} - p^{n}) \\ p^{n+1} &= \hat{p} \end{aligned} \right.$$

- The new splitting and stabilization error is given by:

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} + \begin{bmatrix} (I - A\tau)G(p^{n+1} - p^{n}) \\ \tau(L - DG)(p^{n+1} - p^{n}) \end{bmatrix}$$

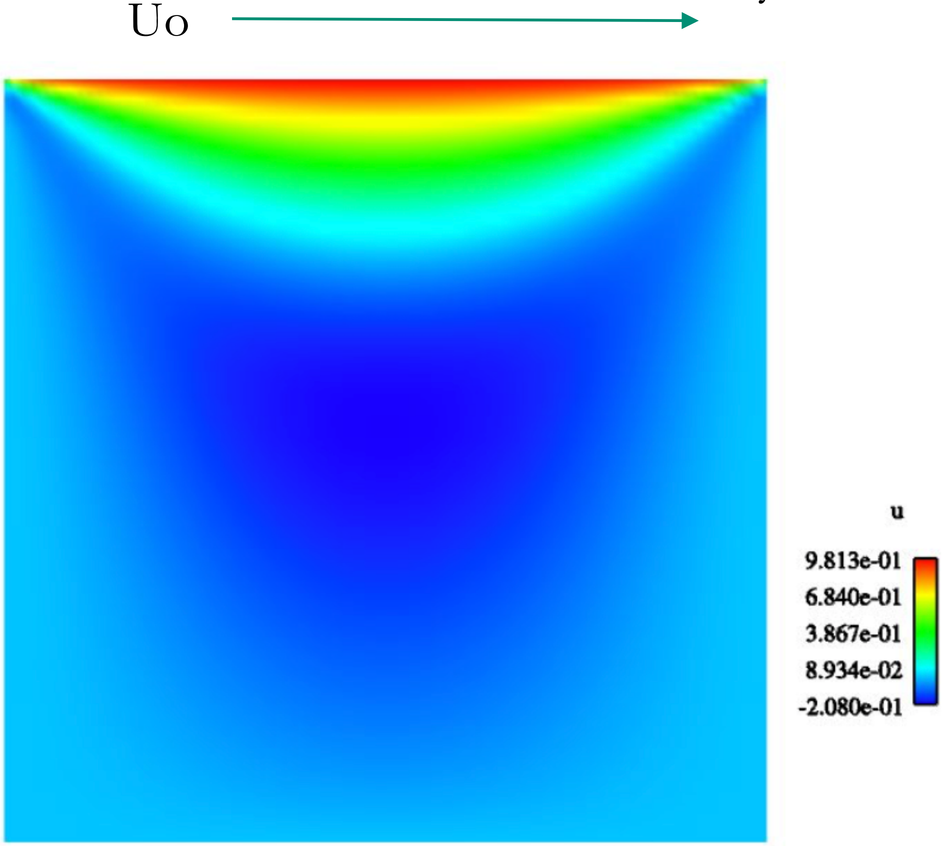
Examples:

- Dwyer (1990)
- Almgren (2000)

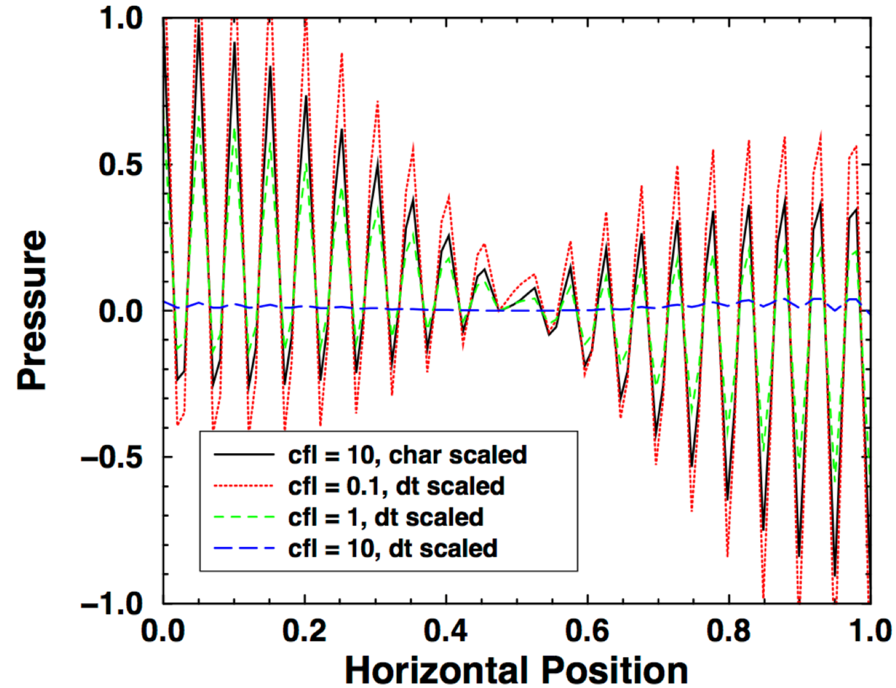
- The above can be shown to demonstrate second-order temporal error (coming)
- A scheme can be designed such that **L** = **DG** (staggered)
- A scheme in which **L** != **DG** (collocated or equal-order) can show that **L-DG** ~ 4^{th}-order pressure stabilization

The Role of Pressure Stabilization (L!=DG) in an Equal-Order Approach

- Consider the classic lid-driven cavity flow with top wall velocity of Uo



lid-driven cavity velocity (u-component)



Unsmoothed pressure field at various Courant numbers

Equal-Order: same basis and interpolation operators for continuity and momentum
Also known as: “collocated”

Incremental Approximate Pressure-Projection with Pressure Stabilization Errors

- Let the inverse of **A**, $\mathbf{A}^{-1}$ be approximated by $\mathbf{B}_2$ as a scalar, **τ (which is ~ time scale)**
- Let $\mathbf{B}_1$ be equal to the scaled Laplace operator, - **τL**

Momentum and Continuity: $$\begin{bmatrix} A & 0 \\ D & -\tau L \end{bmatrix} \begin{bmatrix} \hat{u} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \hat{f} \\ -D\tau G p^n \end{bmatrix} \quad \begin{cases} A\hat{u} = f - Gp^n \\ D\hat{u} = \tau(Lp^{n+1} - DGp^n) \end{cases}$$

Nodal Projection: $$\begin{bmatrix} I & \tau G \\ 0 & I \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix} + \begin{bmatrix} \tau G p^n \\ 0 \end{bmatrix} \quad \begin{cases} u^{n+1} = \hat{u} - \tau G(p^{n+1} - p^n) \\ p^{n+1} = \hat{p} \end{cases}$$

- The new splitting and stabilization error is given by:

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} + \begin{bmatrix} (I - A\tau)G(p^{n+1} - p^n) \\ \tau(L - DG)p^{n+1} \end{bmatrix}$$

Examples:
- Rhie-Chow (1983)
- Peric (1985)

- The above can be shown to hold a second-order temporal error (coming)
- Here, due to equal-order interpolation, i.e., collocation of primitives, **L** != **DG**
- Therefore, **L-DG** ~ 4th-order pressure stabilization (pressure oscillations damped)
- Therefore, pressure-stabilization error remains

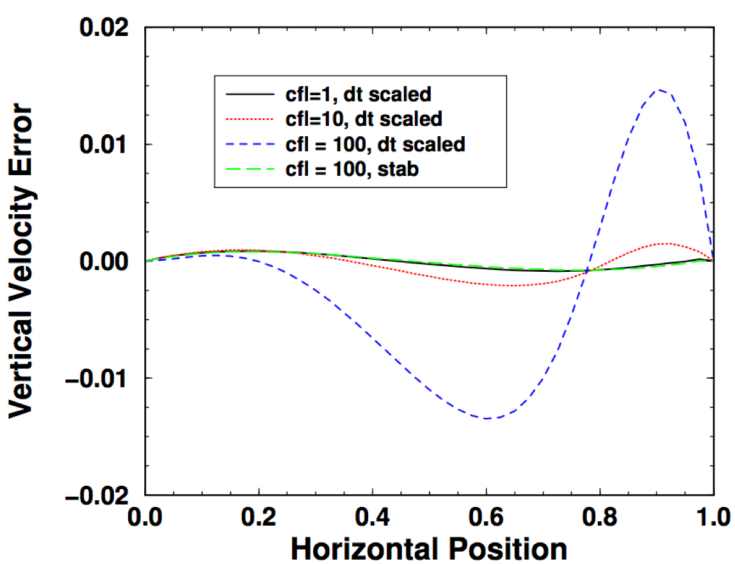
Sensitivity to chosen τ

- Recall that the equal-order pressure stabilization error is given by,

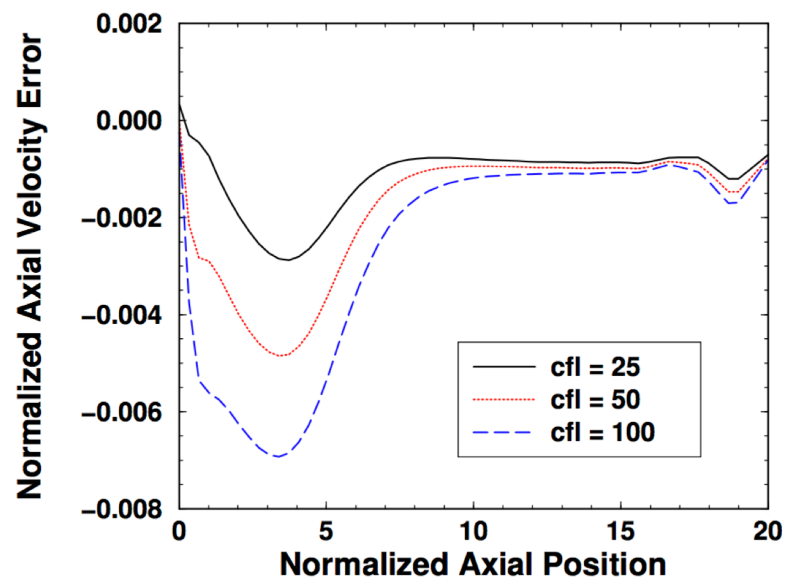
$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix}\begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} + \begin{bmatrix} (I - A\tau)G(p^{n+1} - p^{n}) \\ \tau(L - DG)p^{n+1} \end{bmatrix}$$

Stabilizing effect

- Practical examples of error as a function of $\tau = \Delta t$ for canonical cavity and jet flow; error relative to either characteristic time scale, i.e., $\tau^{C} \sim (u/\Delta x)^{-1}$, or the "stabilized" approach of Soto and Lohner, i.e., $\tau^{C}(L - DG)p^{n+1} + \Delta t L(p^{n+1} - p^{n})$



Driven Cavity



Open Jet

Monolithic Staggered or Equal-order Interpolation (Collocated)

- Note that we need not split the system for a staggered scheme:

$$\begin{bmatrix} A & G \\ D & -\tau L \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ -\tau L p^n \end{bmatrix}$$

0

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tau(L - DG)(p^{n+1} - p^n) \end{bmatrix}$$

- or collocated:

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \approx \begin{bmatrix} A & G \\ D & -\tau L \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ -\tau D G p^n \end{bmatrix}$$

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tau(L - DG)p^{n+1} \end{bmatrix}$$

- Conclusion: Monolithic schemes control splitting error, however, dealing with pressure stabilization is an additional complexity for equal-order methods

The Choice of the B_l can Vary by The Method..

- In Pressure-Stabilized Petrov-Galerkin methods (Hughes et al, 1985):

$$\begin{bmatrix} A & G \\ D & -\tau G w \cdot M \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

- Here, **M** is a fine scale momentum residual and w is the test function for the Finite Element Method
- Note that **M** contains a local pressure gradient which, thereby, provides the pressure stabilization
- The fine-scale momentum residual is evaluated locally at the quadrature point and with mesh refinement reduces at a design-order rate
- With some algebra, one can show that **L-DG** ~ **M**
- For more references, see Magumdar, Numerical Heat Transfer, 1988
- τ can be ~:
  - Simulation time step, i.e., Δt
  - Local advection/diffusion time scale, $\left(\frac{u}{\Delta x} + \frac{v}{\Delta x^2}\right)^{-1}$ - or a more accurate flow-aligned approach
  - Diagonal of A
  - Full inverse of A (Ozawa)

Code Verification To Establish Accuracy

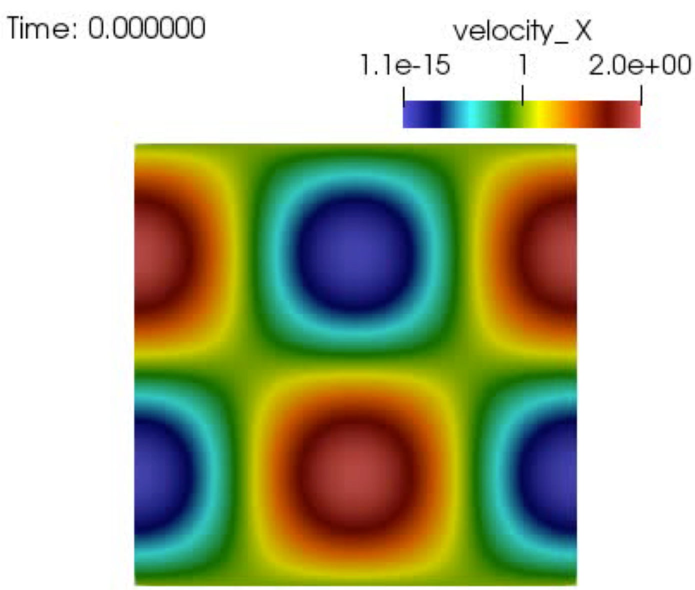
- Consider a two-dimensional transient solution to the incompressible equations of motion:

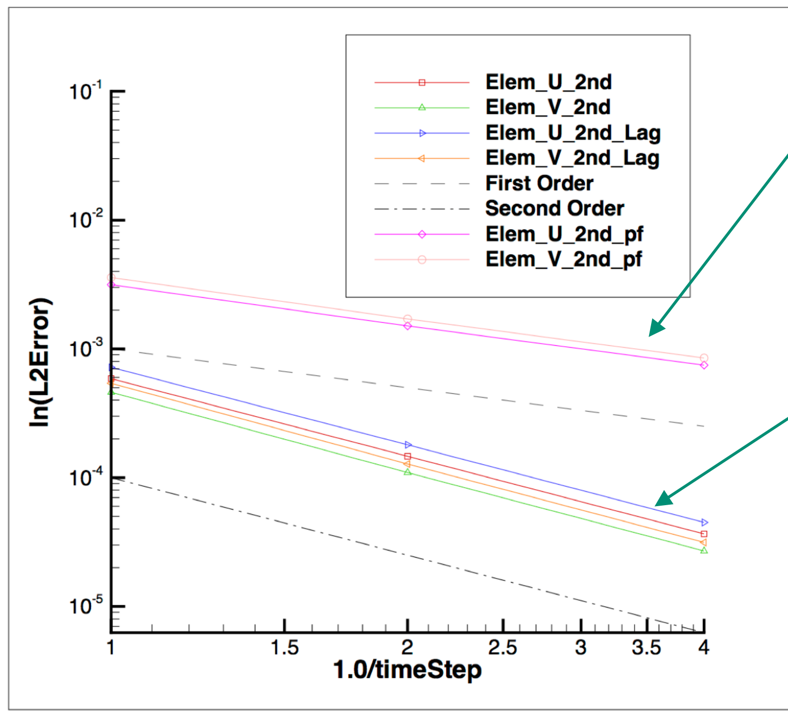
$$u = u^o - \cos\big(\pi(x - u^o t)\big)\sin\big(\pi(y - v^o t)\big)\, e^{-2\omega t}$$

$$v = v^o + \sin\big(\pi(x - u^o t)\big)\cos\big(\pi(y - v^o t)\big)\, e^{-2\omega t}$$

$$\mathrm{p} = -\frac{p^o}{4}\Big[\,\big(\cos\big(2\pi(x - u^o t)\big) + \cos(\,\pi(y - v^o t))\Big]e^{-4\omega t}$$

- This solution is known as the convecting, decaying, Taylor vortex





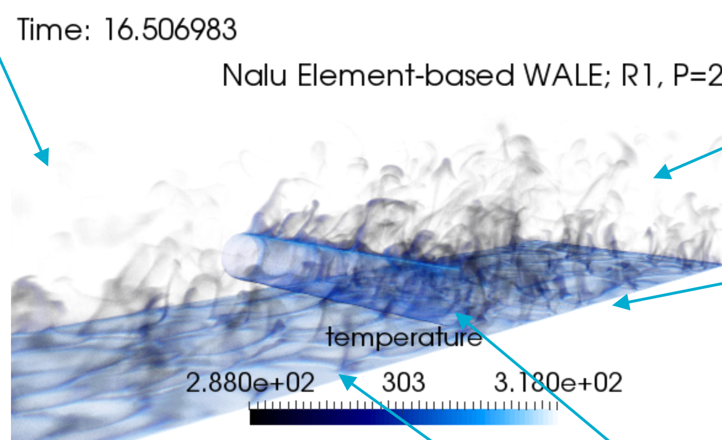
Approximate factorization

Flavors of incremental

A note on boundary conditions

- Typical boundaries include the following (mapping to what we know physically in the system):
  - Inflow:
  - Wall: velocity, temperature, scalar flux, etc., are known ("no-slip")
  - Open: flow leaves or enters the domain with a known "pressure"
  - Symmetry: no-penetration, zero tangential viscous condition
  - Periodic: perfect mapping between nodes at provided surface pairs; provides an infinite domain assumption
  - Non-conformal: static or dynamic/sliding mesh interface in which the surface pair is nonconformal in nature

Symmetry, "flow slips along" (normal stress $pn_i dS$ applied with tangential stress = 0)



Open
(known P and far field temperature)

Inflow
(known T and u_i)

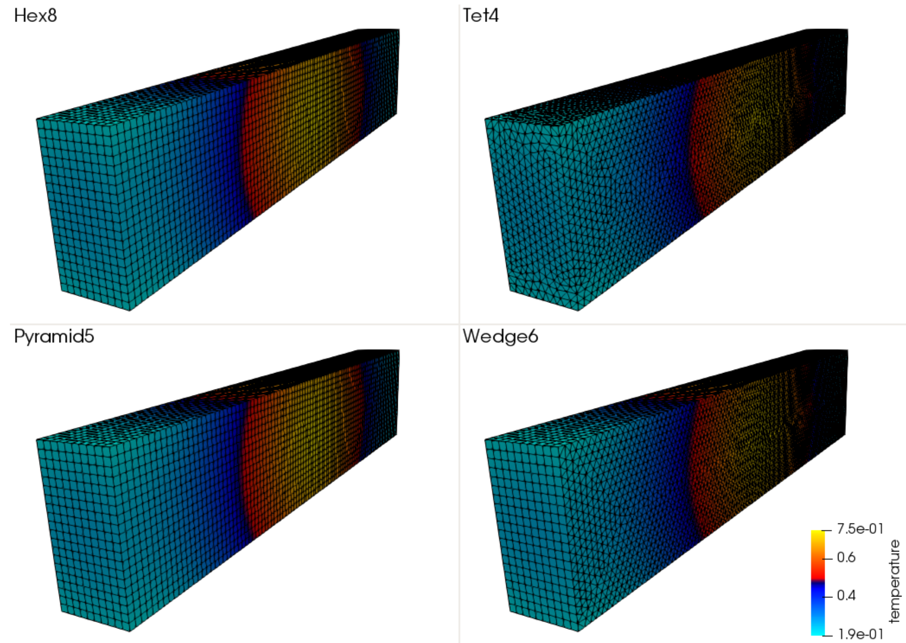
Span-wise periodic

Neumann for p everywhere other than open; SPL

Wall (known T and u_i)

Weak vs Strong

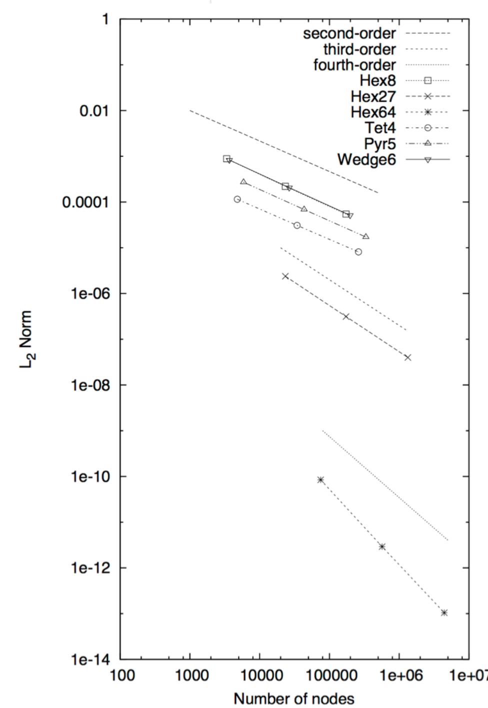
$$T^{mms} = \frac{k}{4}\left(cos(2\pi x) + cos(2\pi y) + cos(2\pi z)\right)$$



- For vertex-based schemes, the degree-of-freedom “lives” at the boundary
- In some cases, e.g., wall, inflow and open, boundary condition can be provided by a Dirichlet bc (AKA, strong implementation of a known field).
- A Dirichlet procedure either condenses the row out of the system or zeroes the row with a unity in the diagonal and the specified value on the RHS
- Consider a simple heat conduction solution in which temperature is known at the boundaries
- It can be shown (see Svard and Nordstrom, JCP, 2008) that a stable and accurate weak BC implementation is provided by:

$$q_n = -\lambda \frac{\partial T}{\partial x_j} n_j + \gamma \frac{\lambda}{L}\left(T - T^{bc}\right)$$

Here, γ is a constant greater than unity



Splitting and Stabilization Errors: Conclusion

- Block matrix and operator form represents a useful construct to analyze coupling and stabilization
- Approximate Factorization is generally $O(\Delta t)$
- With very simple modifications, splitting error is mitigated
- Detailed code verification is a critical tool to both test theoretical understandings in addition to establishing a proper code implementation