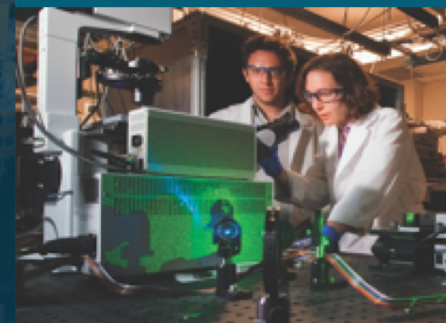
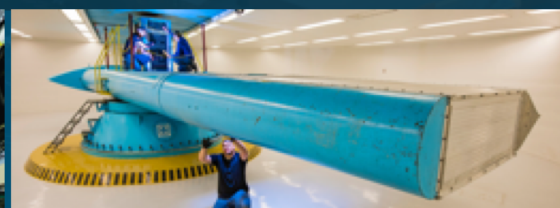
Guest Lecture Stanford ME469: Introduction to the low-Mach Number Approximation





PRESENTED BY

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Computational Thermal and Fluid Mechanics

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Introduction to the low-Mach Number Approximation: Outline

- Variable density, non-isothermal equation set
- DOF vs Equation count
- Non-dimensional form
- Asymptotic Expansion
- Derived low-Mach Equation Set
- The Role of Pressure
  - Motion and Thermodynamic
- Conclusions

Consider a Variable Density, non-Isothermal Fluid Flow System

- Consider the variable density (non-isothermal) equations of motion (momentum and continuity) with energy transport

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \rho u_j H}{\partial x_j} - \frac{\partial u_i \tau_{ij}}{\partial x_j} = -\frac{\partial q_j}{\partial x_j}$$

2+nDim

+1

$$\rho = \frac{pM}{RT} \qquad \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u}{\partial x_k}\delta_{ij} \qquad q_j = -\lambda\frac{\partial T}{\partial x_j}$$

$$E = H - \frac{p}{\rho} \qquad H = h + \frac{1}{2}u_i \qquad h = \int_{T^o}^{T} C_p dT$$

- See Paolucci (1982) or Baum (1978) for the low-Mach pedigree
- Number of Equations = (3+nDim) = Number of unknowns

- DOF
  - Density, ρ
  - Pressure, p
  - Velocity, u_i
  - Total energy, E
    - Total enthalpy, H
    - Static enthalpy, h
    - Temperature, T
- Properties:
  - Viscosity, μ
  - Specific heat, C_p
  - Thermal conductivity, λ
- Constitutive Relationships
  - Ideal gas law (+1)
  - Newtonian stress, τ_{ij}
  - Heat flux vector, q
  - Total enthalpy, H
  - Static enthalpy, h
  - CpdT

Dimensionless Form

- Non-dimensionalized by characteristic velocity and length scale

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j}{\partial \bar{x}_j} = 0$$

$$\frac{\partial \bar{\rho}\bar{u}_i}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{u}_i}{\partial \bar{x}_j} + \frac{1}{\gamma M_a^2}\frac{\partial \bar{p}}{\partial \bar{x}_i} = \frac{1}{R_e}\frac{\partial \bar{\tau}_{ij}}{\partial \bar{x}_j}$$

$$\frac{\partial \bar{\rho}\bar{h}}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{h}}{\partial \bar{x}_j} = -\frac{1}{P_r R_e}\frac{\partial \bar{q}_j}{\partial \bar{x}_j} + \frac{\gamma-1}{\gamma}\frac{\partial \bar{p}}{\partial \bar{t}} + \frac{\gamma-1}{\gamma}\frac{M_a^2}{R_e}\frac{\partial \bar{u}_i\bar{\tau}_{ij}}{\partial \bar{x}_j} - \frac{\gamma-1}{\gamma} M_a^2 \left(\frac{\partial \bar{\rho}\bar{u}_k\bar{u}_k}{\partial \bar{x}_j} + \frac{\partial \bar{\rho}\bar{u}_j\bar{u}_k\bar{u}_k}{\partial \bar{x}_j}\right)$$

$$R_e = \frac{\rho_\infty U_\infty L}{\mu_\infty} \qquad P_r = \frac{C_{p\infty}\mu_\infty}{\lambda_\infty} \qquad M_a = \sqrt{\frac{U_\infty^2}{\gamma R T_\infty / M}}$$

- Note, as the Mach number approaches zero, the viscous work and kinetic energy terms become negligible
- However, the momentum equation notes a singular due to the pressure gradient term
- Conclusions: In the limit of zero Mach number, the momentum equation is not well defined and, in fact, singular

Exploration of the Pressure Singularity

- To explore the singularity, write each DOF as an asymptotic series:

$$\bar{p} = \bar{p}_o + \bar{p}_1\epsilon + \bar{p}_2\epsilon^2 + \dots$$

$$\bar{u}_i = \bar{u}_{o,i} + \bar{u}_{1,i}\epsilon + \bar{u}_{1,i}\epsilon^2 + \dots$$

$$\bar{T} = \bar{T}_o + \bar{T}_1\epsilon + \bar{T}_2\epsilon^2 + \dots$$

- The resulting zeroth-order equations are as follows:

$$\frac{\partial \bar{\rho}_o}{\partial \bar{t}} + \frac{\partial \bar{\rho}_o \bar{u}_{o,j}}{\partial \bar{x}_j} = 0$$

$$\frac{\partial \bar{\rho}_o \bar{u}_{o,i}}{\partial \bar{t}} + \frac{\partial \bar{\rho}_o \bar{u}_{o,j} \bar{u}_{o,i}}{\partial \bar{x}_j} + \frac{1}{\gamma M_a^2}\left(\frac{\partial \bar{p}_o}{\partial \bar{x}_i} + \epsilon \frac{\partial \bar{p}_1}{\partial \bar{x}_i}\right) = \frac{1}{R_e}\frac{\partial \bar{\tau}_{o,ij}}{\partial \bar{x}_j}$$

$$\frac{\partial \bar{\rho}_o \bar{h}_o}{\partial \bar{t}} + \frac{\partial \bar{\rho} \bar{u}_{o,j} \bar{h}_o}{\partial \bar{x}_j} = -\frac{1}{P_r R_e}\frac{\partial \bar{q}_{o,j}}{\partial \bar{x}_j} + \frac{\gamma - 1}{\gamma}\frac{\partial \bar{p}_o}{\partial \bar{t}}$$

- In order for the zeroeth-order momentum equation to be well conditioned in the limit of zero Mach number, $\frac{\partial \bar{p}_o}{\partial \bar{x}_i}$ must be spatially zero with $\epsilon = \gamma M_a^2$
- p_o is the constant-in-space, possibly variable-in-time thermodynamic pressure, p^t
- p_1 is the variable is space pressure, which is also known as the "motion pressure", p^m
  - Recall, this is simply a perturbation about the pressure, $\bar{p} = \bar{p}_o + \bar{p}_1\epsilon$

The Final low-Mach Number Equation Set

- In dimensional form, the low-Mach system is as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} = \frac{-\partial p^m}{\partial x_i}$$

$$\frac{\partial \rho h}{\partial t} + \frac{\partial \rho u_j h}{\partial x_j} = -\frac{\partial q_j}{\partial x_j} + \frac{\partial p^t}{\partial t}$$

2+nDim

$$\rho = \frac{p^t M}{RT} \qquad \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u}{\partial x_k}\delta_{ij}$$

$$q_j = -\lambda\frac{\partial T}{\partial x_j} \qquad h = \int_{T^o}^{T} C_p dT$$

- Ramifications?
  - We have effectively filtered out the acoustics, i.e., the wave speed is infinitely fast
  - Continuity is met over any finite length instantly
  - DOF/Equation system is: ρ, p^m, u_i, h; p^t is a constant for an open domain
  - In practice, a functional form for the motion pressure is derived:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} - \frac{\partial}{\partial x_j}\tau\frac{\partial}{\partial x_j}\Delta p^m = 0 \quad \text{- or} \quad \frac{\partial}{\partial x_i}\left(\frac{\partial \rho u_i}{\partial t} + \cdots\right)$$

Pressure Poisson Equation (PPE)

- DOF
  - Density, ρ
  - Pressure, p^m
  - Velocity, u_i
  - Total energy, E
    - Total enthalpy, H
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- Properties:
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Introduction to the low-Mach Number Approximation: Conclusions

- Variable density, non-isothermal equation set allow for all length and time scales for a given physics
- The non-dimensionalized form of the momentum equation is singular in the limit as Mach number approaches zero
- low-Mach equations are derived by non-dimensionalization and asymptotic expansion
- For a low-Mach flow, the density and pressure are not coupled by an equation of state
- The constant-in-space thermodynamic pressure is used in the EOS
- The low-Mach procedure filters the acoustics, which can not be of prime interest in the flow