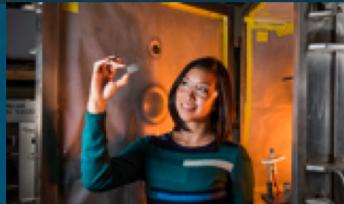
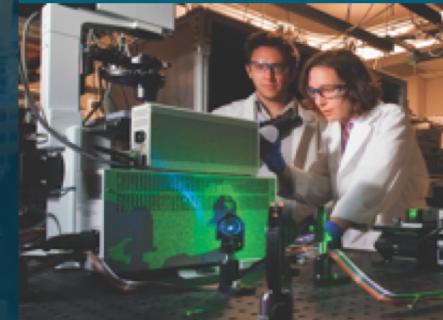


Guest Lecture Stanford ME469: Advection Operators



Sandia
National
Laboratories



PRESENTED BY

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Computational Thermal and Fluid Mechanics

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Advection Operators: Outline

- Classic Analysis on Advection/Diffusion/Time Scalar Transport
- The Role of Diagonal Dominance
- Upwind Methods
- Linear Residual-Based Stabilization
- Nonlinear Residual-Based Stabilization
- Conclusions



Consider a Simple Advection/Diffusion/Time Scalar Transport



- PDE is given by: $\frac{\partial \rho Z}{\partial t} + \frac{\partial \rho u_j Z}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} = 0$

- Integral form: $\int \frac{\partial \rho Z}{\partial t} dV + \int \left[\rho u_j Z - \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} \right] n_j dS = 0$

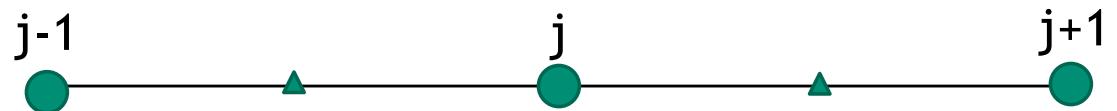
- Discrete form: $\sum_{sip} \frac{(\rho Z^{n+1} - \rho Z^n)}{\Delta t} dV + \sum_{ip} \dot{m} Z_{ip} - \sum_{ip} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} A_j$ Mass flow rate: $\dot{m} = \rho u_j A_j$

- Matrix Form: $(-a_{j-1} \quad a_j \quad -a_{j+1}) \begin{pmatrix} Z_{j-1} \\ Z_j \\ Z_{j+1} \end{pmatrix} = 0$

$$a_{j+1} = \frac{\mu}{Sc} \frac{A}{\Delta x} - \frac{\dot{m}_{j+1/2}}{2} \quad a_{j-1} = \frac{\mu}{Sc} \frac{A}{\Delta x} + \frac{\dot{m}_{j-1/2}}{2}$$

$$\frac{\sum_{i \neq j} |a_{ij}|}{|a_{ii}|} \leq 1$$

$$a_j = a_{j+1} + a_{j-1} + (\dot{m}_{j+1/2} - \dot{m}_{j-1/2}) + \frac{\rho \Delta V}{\Delta t}$$



Diagonal Dominance Check

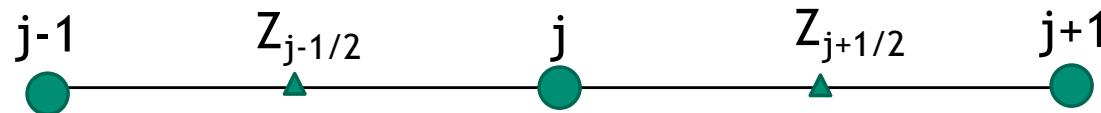


- For a monotonic operator: $DD = \frac{\sum_{i \neq j} |a_{ij}|}{|a_{ii}|} \leq 1$
- Consider $u = 1 \text{ cm/s}$; $\rho = 1.2e-3 \text{ g/cm}^3$; $\mu/\text{Sc} = 1.8e-4 \text{ g/cm-s}$; $\Delta x = 0.1 \text{ cm}$; $\Delta t \gg 1$:
 - $a_{j+1} = 1.2e-5$ $a_{j-1} = 2.4e-5$ $a_j = 3.6e-5$ $DD \leq 1$
 - Stable and monotonic field will be realized; $\text{Re} = 0.67$
- Consider $u = 10 \text{ cm/s}$; $\rho = 1.2e-3 \text{ g/cm}^3$; $\mu/\text{Sc} = 1.8e-4 \text{ g/cm-s}$; $\Delta x = 0.1 \text{ cm}$; $\Delta t \gg 1$:
 - $a_{j+1} = -4.2e-5$ $a_{j-1} = 7.8e-5$ $a_j = 3.605e-5$ $DD > 1$
 - Unstable and lack of scalar monotonicity will be realized; $\text{Re} = 6.7$
 - Standard preconditioner, e.g., SGS, are not expected to converge the linear system
- Mesh spacing for monotonicity must be: $\Delta x \leq \frac{2\mu/\text{Sc}}{\rho u}$ $\text{Pe} = \frac{\rho u \Delta x}{\mu/\text{Sc}} \leq 2$
 - Cell Peclet number, Pe is a cell-Reynolds number based on local properties and mesh spacing
- Consider $u = 10 \text{ cm/s}$; $\rho = 1.2e-3 \text{ g/cm}^3$; $\mu/\text{Sc} = 1.8e-4 \text{ g/cm-s}$; $\Delta x = 0.1 \text{ cm}$; $\Delta t = ?$
 - For diagonal dominance, $\Delta t = 0.0143$ which corresponds to a Courant number of unity $C = \frac{u \Delta t}{\Delta x}$
 - This can limit the ability for some linear solvers to converge the linear system

Upwind Operators



- Upwind advection operators can provide diagonal dominance (at the cost of accuracy)



$$Z_{j+1/2} = \begin{cases} Z_j; u > 0 \\ Z_{j+1}; u < 0 \end{cases} \quad \text{"as the wind blows"}$$

- However, upwind operators, even if higher-order, are numerically diffuse
- Higher-order upwind (second-order in space) can be obtained on an unstructured mesh by use of projected nodal gradients and extrapolation:

$$G_x Z_j = \frac{Z_{j+1} - Z_{j-1}}{2\Delta x} \quad G_x Z_{j-1} = \frac{Z_j - Z_{j-2}}{2\Delta x}$$

for $u > 0$ $Z_{j+1/2} = \alpha \left(Z_j + G_x Z_j \frac{\Delta x}{2} \right) + (1 - \alpha) \left[\frac{1}{2} (Z_j + Z_{j+1}) \right]$

α is a blending parameter

$$Z_{j-1/2} = \alpha \left(Z_{j-1} + G_x Z_{j-1} \frac{\Delta x}{2} \right) + (1 - \alpha) \left[\frac{1}{2} (Z_j + Z_{j-1}) \right]$$

6 Pseudo Higher-order Upwind



- Blending extrapolated upwind with central $(Z_L + Z_R)/2$ provides:

$$\text{for } u > 0 \quad Z_{j+1/2} = \alpha \left(Z_j + G_x Z_j \frac{\Delta x}{2} \right) + (1 - \alpha) \left[\frac{1}{2} (Z_j + Z_{j+1}) \right]$$

$$Z_{j-1/2} = \alpha \left(Z_{j-1} + G_x Z_{j-1} \frac{\Delta x}{2} \right) + (1 - \alpha) \left[\frac{1}{2} (Z_j + Z_{j-1}) \right]$$

$$\text{for } u < 0 \quad Z_{j+1/2} = \alpha \left(Z_{j+1} + G_x Z_{j+1} \frac{\Delta x}{2} \right) + (1 - \alpha) \left[\frac{1}{2} (Z_j + Z_{j+1}) \right]$$

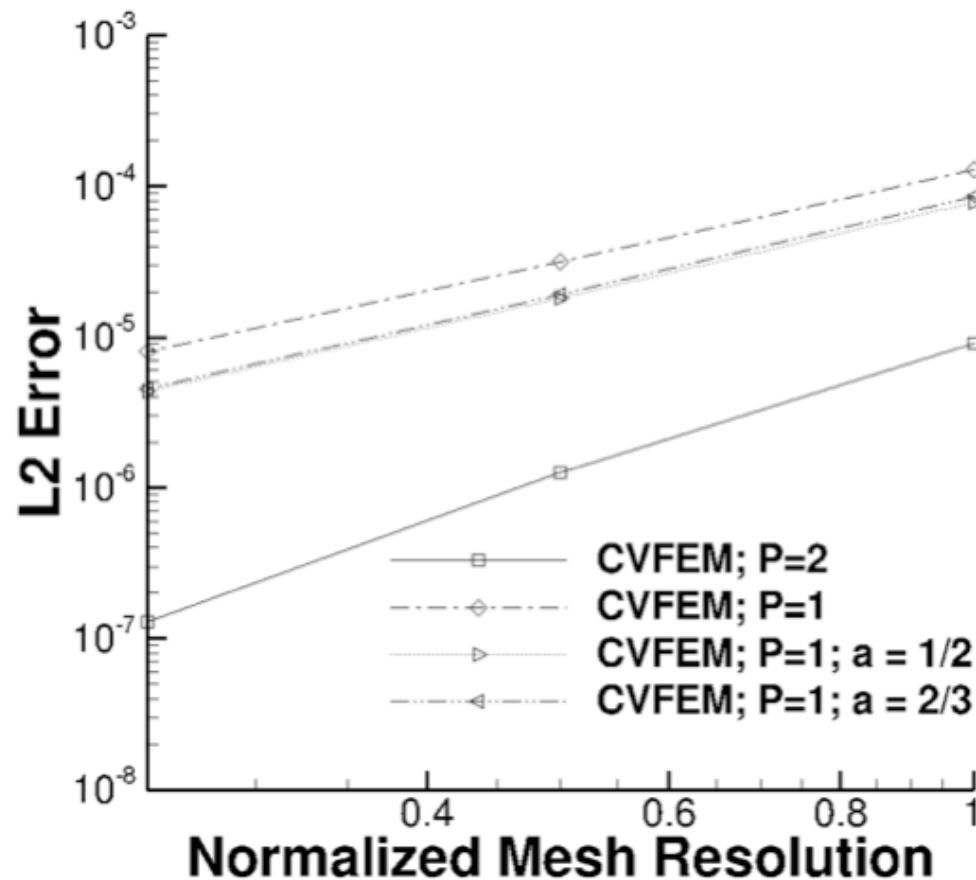
$$Z_{j-1/2} = \alpha \left(Z_j + G_x Z_j \frac{\Delta x}{2} \right) + (1 - \alpha) \left[\frac{1}{2} (Z_j + Z_{j-1}) \right]$$

Resulting
1-D Stencil

j-2	j-1	j	j+1	j+2	α	\dot{m}
1/4	-5/4	+3/4	+1/4	0	1	>0
0	-1/4	-3/4	+5/4	-1/4	1	>0
+1/6	-6/6	+3/6	+2/6	0	1/2	<0
0	-2/6	-3/6	+6/6	-1/6	1/2	<0

- Slope limiting required when gradients are not smooth (Berger et al, 2005, 43rd AIAA)

Pseudo Higher-order Upwind



Caveat: In practice, this method provides only a small shift in accuracy

- Quadrature error dominated



- Is there a better way than either un-stabilized (oscillatory and non-monotonic) and dissipative upwind?
- Consider the model advection/diffusion PDE in the context of a FEM

$$\int w \frac{\partial \rho Z}{\partial t} dV + \int w \frac{\partial \rho u_j Z}{\partial x_j} dV + \int \frac{\partial w}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} dV - \int w \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} n_j dS = 0$$

- Concept: Add a scaled fine-scale residual for stabilization, Streamwise Upwind Petrov Galerkin SUPG (Hughes and Brooks, 1982)

$$+ \sum_{elem} \tau u_k \frac{\partial w}{\partial x_k} \left(\frac{\partial \rho Z}{\partial t} + \frac{\partial \rho u_j Z}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} \right) dV$$

- As noted earlier, CVFEM can be recovered by applying the piecewise-constant test function, w :

$$- \sum_{elem} \tau u_k A_k \left(\frac{\partial \rho Z}{\partial t} + \frac{\partial \rho u_j Z}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} \right)$$

- Godunov's theorem states that a linear stabilization approach is not sufficient to damp out all oscillations

Linear Residual-Based Stabilization for CVFEM



- Is there a better way than either un-stabilized (oscillatory and non-monotonic) and dissipative upwind?
- Consider the model advection/diffusion PDE in the context of a FEM

$$\int w \frac{\partial \rho Z}{\partial t} dV + \int w \frac{\partial \rho u_j Z}{\partial x_j} dV + \int \frac{\partial w}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} dV - \int w \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} n_j dS = 0$$

- Concept: Add a scaled fine-scale residual for stabilization, Streamwise Upwind Petrov Galerkin (Hughes and Brooks, 1982)

$$+ \sum_{elem} \tau u_k \frac{\partial w}{\partial x_k} \left(\frac{\partial \rho Z}{\partial t} + \frac{\partial \rho u_j Z}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} \right) dV$$

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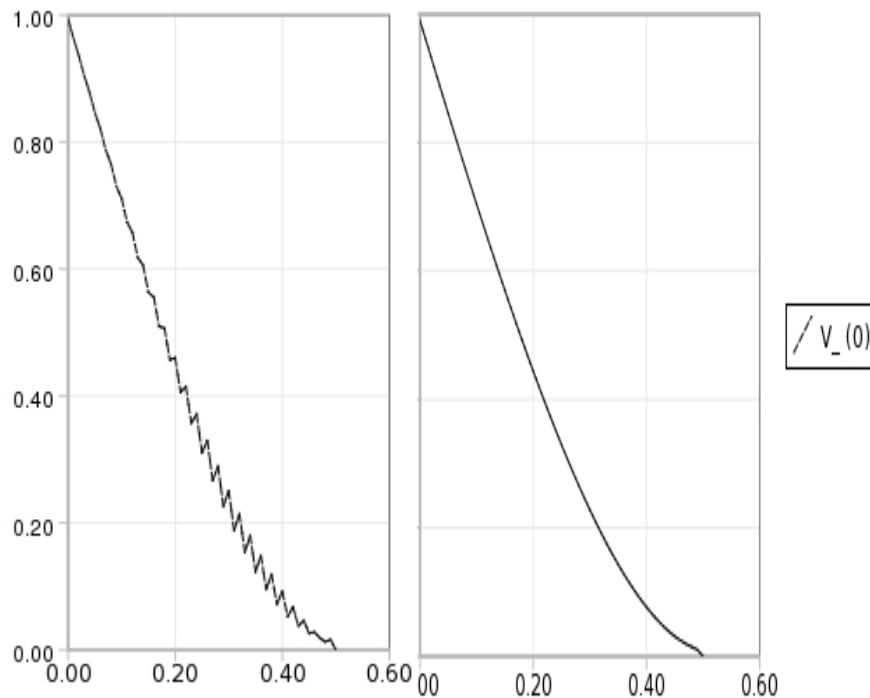
$$- \sum_{elem} \tau u_k A_k \left(\frac{\partial \rho Z}{\partial t} + \frac{\partial \rho u_j Z}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} \right)$$

- Godunov's theorem states that a linear stabilization approach is not sufficient to damp out all oscillations

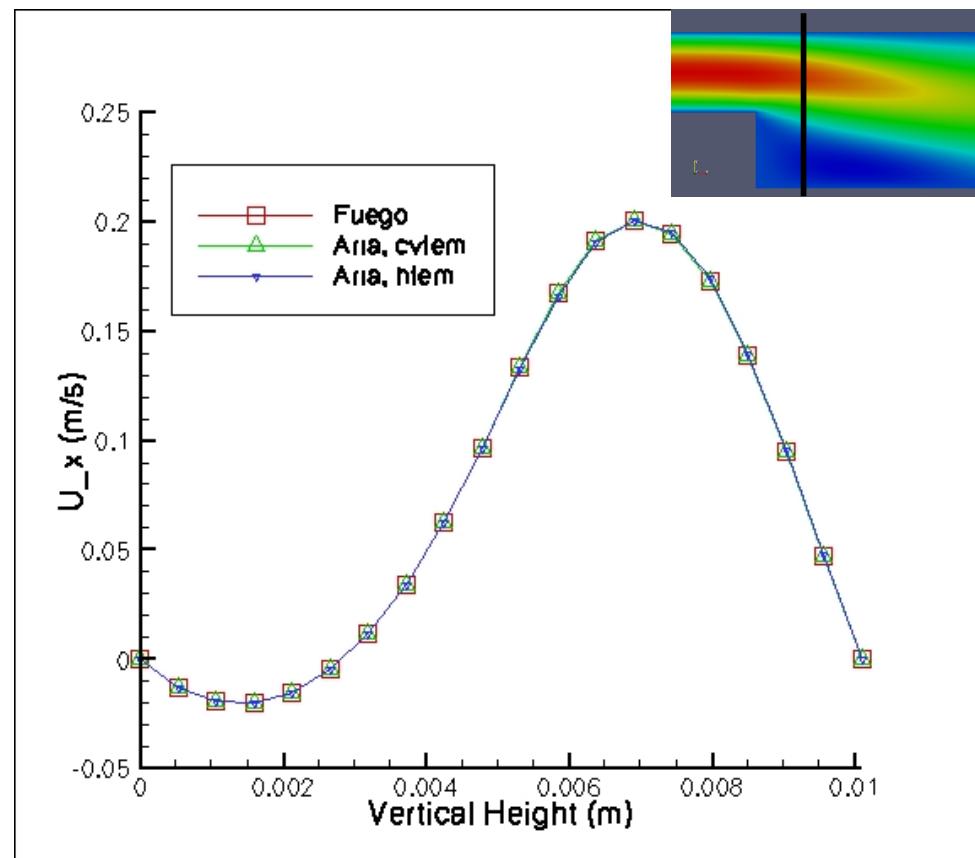
SUPG in Practice (CVFEM and FEM)



- SUPG and analog in CVFEM, Streamwise Upwind Control Volume (SUCV) are effective for low-moderate cell-Peclet numbers



Un-stabilized and SUPG



SUPG/SUCV backstep velocity prediction

Nonlinear Stabilization Approach



- Consider a general transport equation for ϕ
- An artificial viscosity approach is provided with coefficient again related to a fine-scale residual:

$$\sum_e \int_{\Omega} \nu(\mathbf{R}) \frac{\partial w}{\partial x_i} g^{ij} \frac{\partial \phi}{\partial x_j} d\Omega \quad g^{ij} = \frac{\partial x_i}{\partial \xi_k} \frac{\partial x_j}{\partial \xi_k} \quad g_{ij} = \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_k}{\partial x_j}$$

FEM-based DCO (Shakib, 91)

Co-variant and contra-variant metric tensors

$$\nu(\mathbf{R}) = \sqrt{\frac{\mathbf{R}_k \mathbf{R}_k}{\frac{\partial \phi}{\partial x_i} g^{ij} \frac{\partial \phi}{\partial x_j}}} \quad \text{or min....} \nu^{upw} = C_{upw} (\rho^2 u_i g_{ij} u_j)^{\frac{1}{2}}$$

Coefficient based on local residual

Limit the coefficient based on an upwind analogy

- Piecewise-constant test function for CVFEM (with fourth order-form)

$$-\sum_e \int_{\Gamma} \nu(\mathbf{R}) g^{ij} \left(\frac{\partial \phi}{\partial x_j} - G_j \phi \right) n_i dS$$

- Similar concept to Guermond's "Entropy-viscosity" approach, 2013
- Can also form an interesting LES model when the viscosity is a function of the k.e. fine scale residual (Guermond and Larios, 2015)

Advection Operators: Conclusion



- Without diagonal dominance, lack of monotonicity is noted
- Upwind approaches can help, however, comes at the price of numerical accuracy
- Residual-based approaches, which stem from the FEM literature, are useful candidates for CVFEM