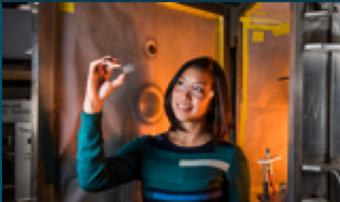
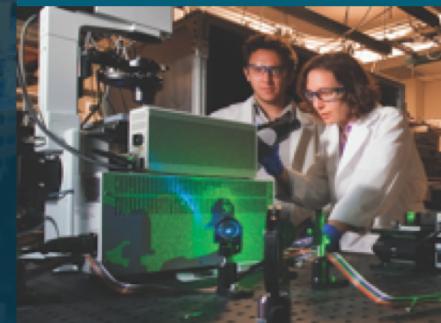




Guest Lecture Stanford ME469: Introduction to the low-Mach Number Approximation



PRESENTED BY

Stefan P. Domino

Computational Thermal and Fluid Mechanics

Sandia National Laboratories SAND2018-4536 PE



Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

Introduction to the low-Mach Number Approximation: Outline



- Variable density, non-isothermal equation set
- DOF vs Equation count
- Non-dimensional form
- Asymptotic Expansion
- Derived low-Mach Equation Set
- The Role of Pressure
 - Motion and Thermodynamic
- Conclusions

3

Consider a Variable Density, non-Isothermal Fluid Flow System



- Consider the variable density (non-isothermal) equations of motion (momentum and continuity) with energy transport

$$\begin{aligned}
 \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} &= 0 \\
 \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} &= -\frac{\partial p}{\partial x_i} \\
 \frac{\partial \rho E}{\partial t} + \frac{\partial \rho u_j H}{\partial x_j} - \frac{\partial u_i \tau_{ij}}{\partial x_j} &= -\frac{\partial q_j}{\partial x_j}
 \end{aligned}
 \quad \left. \right\} 2+n\text{Dim}$$

+1

$$\rho = \frac{pM}{RT} \quad \tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u}{\partial x_k} \delta_{ij} \quad q_j = -\lambda \frac{\partial T}{\partial x_j}$$

$$E = H - \frac{p}{\rho} \quad H = h + \frac{1}{2} u_i \quad h = \int_{T^o}^T C_p dT$$

- See Paolucci (1982) or Baum (1978) for the low-Mach pedigree
- Number of Equations = (3+nDim) = Number of unknowns

- DOF

- Density, ρ
- Pressure, p
- Velocity, u_i
- Total energy, E
 - Total enthalpy, H
 - Static enthalpy, h
 - Temperature, T

- Properties:

- Viscosity, μ
- Specific heat, C_p
- Thermal conductivity, λ

- Constitutive Relationships

- Ideal gas law (+1)
- Newtonian stress, τ_{ij}
- Heat flux vector, q
- Total enthalpy, H
- Static enthalpy, h
- $C_p dT$

Dimensionless Form



- Nondimensionalization is characteristic velocity and length scale

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial \bar{\rho} \bar{u}_j}{\partial \bar{x}_j} = 0$$

$$\frac{\partial \bar{\rho} \bar{u}_i}{\partial \bar{t}} + \frac{\partial \bar{\rho} \bar{u}_j \bar{u}_i}{\partial \bar{x}_j} + \frac{1}{\gamma M_a^2} \frac{\partial \bar{p}}{\partial \bar{x}_i} = \frac{1}{R_e} \frac{\partial \bar{\tau}_{ij}}{\partial \bar{x}_j}$$

$$\frac{\partial \bar{\rho} \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{\rho} \bar{u}_j \bar{h}}{\partial \bar{x}_j} = - \frac{1}{P_r R_e} \frac{\partial \bar{q}_j}{\partial \bar{x}_j} + \frac{\gamma-1}{\gamma} \frac{\partial \bar{p}}{\partial \bar{t}} + \frac{\gamma-1}{\gamma} \frac{M_a^2}{R_e} \frac{\partial \bar{u}_i \bar{\tau}_{ij}}{\partial \bar{x}_j} - \frac{\gamma-1}{\gamma} M_a^2 \left(\frac{\partial \bar{\rho} \bar{u}_k \bar{u}_k}{\partial \bar{x}_j} + \frac{\partial \bar{\rho} \bar{u}_j \bar{u}_k \bar{u}_k}{\partial \bar{x}_j} \right)$$

$$R_e = \frac{\rho_\infty U_\infty L}{\mu_\infty} \quad P_r = \frac{C_p \infty \mu_\infty}{\lambda_\infty} \quad M_a = \sqrt{\frac{U_\infty^2}{\gamma R T_\infty / M}}$$

- Note, as the Mach number approaches zero, the viscous work and kinetic energy terms become negligible
- However, the momentum equation notes a singular due to the pressure gradient term
- Conclusions: In the limit of zero Mach number, the momentum equation is not well defined and, in fact, singular

Exploration of the Pressure Singularity



- To explore the singularity, write each DOF as an asymptotic series:

$$\bar{p} = \bar{p}_o + \bar{p}_1 \epsilon + \bar{p}_2 \epsilon^2 + \dots$$

$$\bar{u}_i = \bar{u}_{o,i} + \bar{u}_{1,i} \epsilon + \bar{u}_{2,i} \epsilon^2 + \dots$$

$$\bar{T} = \bar{T}_o + \bar{T}_1 \epsilon + \bar{T}_2 \epsilon^2 + \dots$$

- The resulting zeroth-order equations are as follows:

$$\frac{\partial \bar{p}_o}{\partial \bar{t}} + \frac{\partial \bar{p}_o \bar{u}_{o,j}}{\partial \bar{x}_j} = 0$$

$$\frac{\partial \bar{p}_o \bar{u}_{o,i}}{\partial \bar{t}} + \frac{\partial \bar{p}_o \bar{u}_{o,j} \bar{u}_{o,i}}{\partial \bar{x}_j} + \frac{1}{\gamma M_a^2} \left(\frac{\partial \bar{p}_o}{\partial \bar{x}_i} + \epsilon \frac{\partial \bar{p}_1}{\partial \bar{x}_i} \right) = \frac{1}{R_e} \frac{\partial \bar{\tau}_{o,ij}}{\partial \bar{x}_j}$$

$$\frac{\partial \bar{p}_o \bar{h}_o}{\partial \bar{t}} + \frac{\partial \bar{p} \bar{u}_{o,j} \bar{h}_o}{\partial \bar{x}_j} = - \frac{1}{P_r R_e} \frac{\partial \bar{q}_{o,j}}{\partial \bar{x}_j} + \frac{\gamma-1}{\gamma} \frac{\partial \bar{p}_o}{\partial \bar{t}}$$

- In order for the zeroth-order momentum equation to be well conditioned in the limit of zero Mach number, $\frac{\partial \bar{p}_o}{\partial \bar{x}_i}$ must be spatially zero with $\epsilon = \gamma M_a^2$
- p_o is the constant-in-space, possibly variable-in-time thermodynamic pressure, p^t
- p_1 is the variable in space pressure, which is also known as the “motion pressure”, p^m
 - Recall, this is simply a perturbation about the pressure, $\bar{p} = \bar{p}_o + \bar{p}_1 \epsilon$

The Final low-Mach Number Equation Set



- In dimensional form, the low-Mach system is as follows:

$$\begin{aligned}
 \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} &= 0 \\
 \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} &= \frac{-\partial p^m}{\partial x_i} \\
 \frac{\partial \rho h}{\partial t} + \frac{\partial \rho u_j h}{\partial x_j} &= -\frac{\partial q_j}{\partial x_j} + \frac{\partial p^t}{\partial t} \\
 \rho = \frac{p^t M}{RT} \quad \tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u}{\partial x_k} \delta_{ij} \\
 q_j = -\lambda \frac{\partial T}{\partial x_j} \quad h = \int_{T^o}^T C_p dT
 \end{aligned}$$

2+nDim

- Ramifications?

- We have effectively filtered out the acoustics, i.e., the wave speed is infinitely fast
 - Continuity is met over any finite length instantly
 - DOF/Equation system is: $\rho, p^m, u_i, h; p^t$ is a constant for an open domain
 - In practice, a functional form for the motion pressure is derived:
- $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} - \frac{\partial}{\partial x_j} \tau \frac{\partial}{\partial x_j} \Delta p^m = 0 \quad \text{or} \quad \frac{\partial}{\partial x_i} \left(\frac{\partial \rho u_i}{\partial t} + \dots \right)$$

- DOF
 - Density, ρ
 - Pressure, p^m
 - Velocity, u_i
 - Total energy, E
 - Total enthalpy, H
 - Static enthalpy, h
 - Temperature, T
- Properties:
 - Viscosity, μ
 - Specific heat, C_p
 - Thermal conductivity, λ
- Constitutive Relationships
 - Ideal gas law
 - Newtonian stress, τ_{ij}
 - Heat flux vector, q
 - Total enthalpy, H
 - Static enthalpy, h
 - $CpdT$

Pressure Poisson Equation
(PPE)

Introduction to the low-Mach Number Approximation: Conclusions



- Variable density, non-isothermal equation set allow for all length and time scales for a given physics
- The non-dimensional form of the momentum equation is singular in the limit as Mach number approaches zero
- low-Mach equations are derived by nondimensionalization and asymptotic expansion
- For a low-Mach flow, the density and thermodynamic pressure are not strongly coupled by an equation of state, hence, stiffness arises as $\text{Ma} \rightarrow 0$
- The constant-in-space thermodynamic pressure is used in the EOS
- The low-Mach procedure filters the acoustics, which can not be of prime interest in the flow