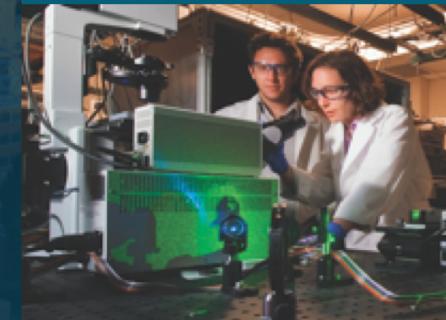


Guest Lecture Stanford ME469: Common low-Mach Discretization Approaches



PRESENTED BY

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Computational Thermal and Fluid Mechanics

Sandia National Laboratories SAND2018-4536 PE



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Common low-Mach Discretization Approaches: Outline



- Cell-centered Finite Volume (FV)
- Edge-based Vertex-Centered (EBVC)
- Finite Element Method (FEM)
- Control-Volume Finite Element Method (CVFEM)
- Staggered arrangement
- Conclusions

Fundamentals of Discretization



- Given a partial differential equation (PDE) and associated volumetric form:

$$\rho C_p \frac{\partial T}{\partial t} - \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} = 0 \quad \int \rho C_p \frac{\partial T}{\partial t} dV - \int \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} dV = 0$$

- Applying Gauss Divergence provides the standard finite volume form:

$$\int \frac{\partial q_j}{\partial x_j} dV = \int q_j n_j dS \quad \int \rho C_p \frac{\partial T}{\partial t} dV - \int \lambda \frac{\partial T}{\partial x_j} n_j dS = 0$$

- We can also multiple PDE by an arbitrary test function, w , and integrate over a volume,

$$\int w \rho C_p \frac{\partial T}{\partial t} dV - \int w \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} n_j dS = 0 \quad \text{Next, integrate by parts and apply Gauss-Divergence:}$$

$$\int w \rho C_p \frac{\partial T}{\partial t} dV + \int \frac{\partial w}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} dV - \int w \lambda \frac{\partial T}{\partial x_j} dS = 0$$

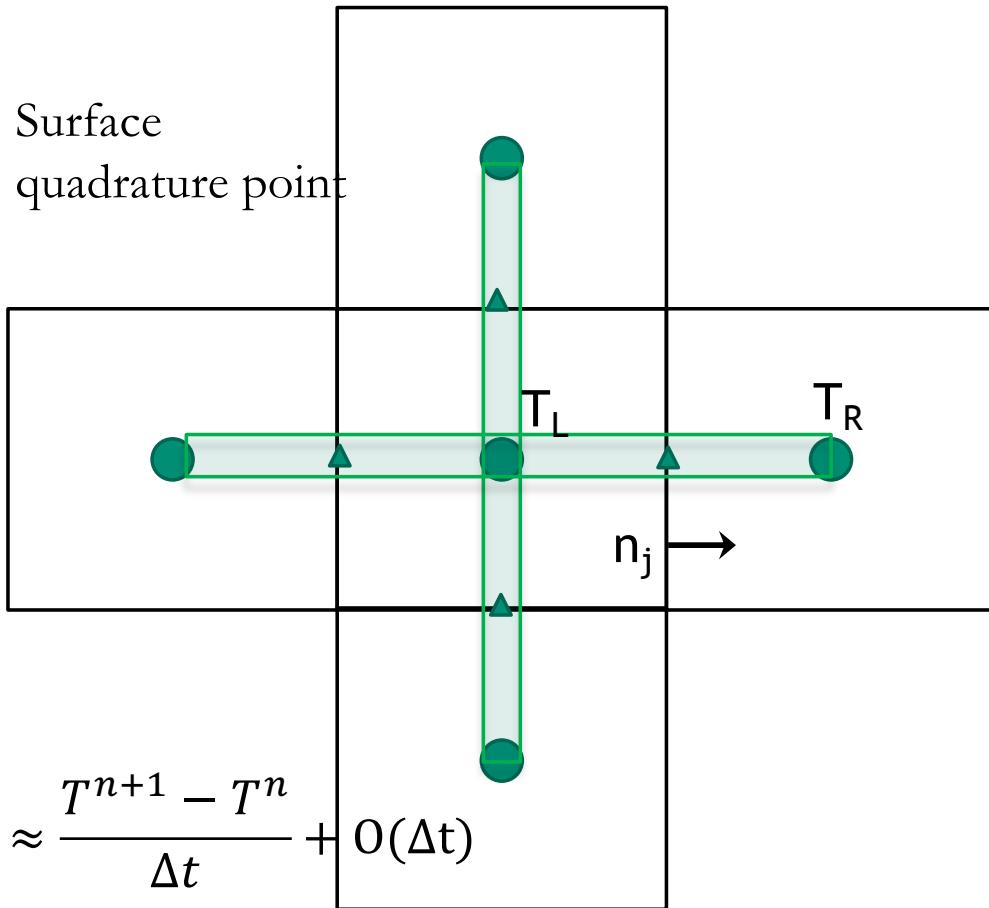
Equal-Order Interpolation Cell-Centered (CC) Finite Volume



- All primitives are collocated at the cell-center of the element with equal-order interpolation
- Classic two-state, “L” and “R” approach provides spatially second-order accuracy



Surface quadrature point



- Consider a simple heat conduction model PDE,

$$\rho C_p \frac{\partial T}{\partial t} - \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} = 0$$

- Integrate over a control volume and use Gauss-Divergence,

$$\int \rho C_p \frac{\partial T}{\partial t} dV - \int \lambda \frac{\partial T}{\partial x_j} n_j dS = 0$$

with: $\frac{\partial T}{\partial x_j} \approx \frac{(T_R - T_L)n_j}{\|\Delta x_j\|} + O(\Delta x)$

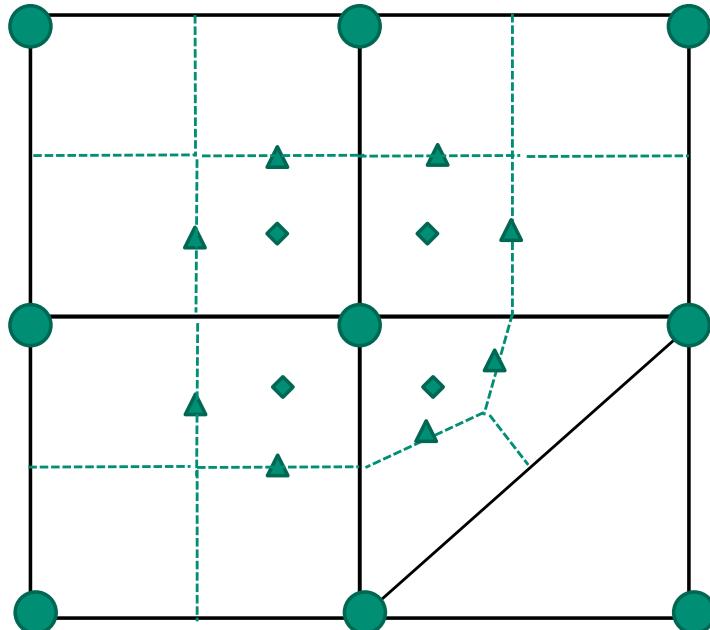
- Iterate element cell-centers for volume contributions (time/source)
- Iterate element faces for surface flux contribution

Equal Order Interpolation Edge-Based Vertex-Centered (EBVC) Finite Volume

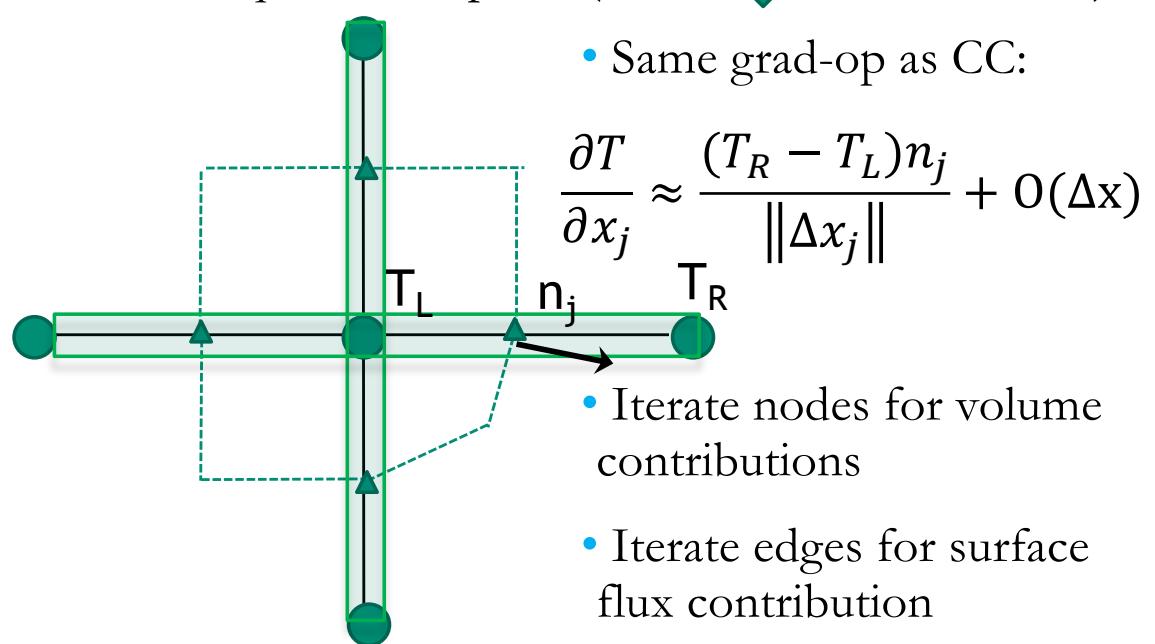
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- All primitives are collocated at the vertices of the elements with equal-order interpolation
- A dual mesh is constructed to obtain flux and volume quadrature locations
- Classic two-state, “L” and “R” approach provides spatially second-order accuracy
 - ▲ Surface quadrature point (area summed to edge)
 - ◆ Volume quadrature point (sub-vol summed to node)



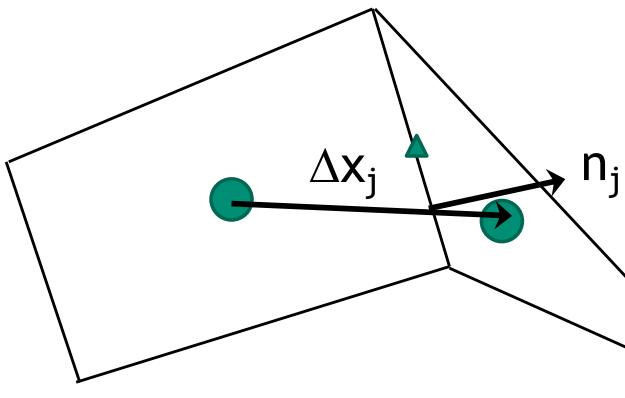
Dual-volume definition



6 Typical Failings for Two-State Discretization Methods



- With two points, only a linear basis can be used.
- Therefore, unstructured CC and EBVC are limited to second-order spatial accuracy
- Non-orthogonality is problematic for gradient-operator



$$\frac{\partial T}{\partial x_j} = G_j T + [(T_R - T_L) - G_k T \Delta x_k] \frac{A_j}{A_l \Delta x_l}$$

With area vector defined by: $A_j = n_j dS$

- Above, $G_j T$ is a projected nodal gradient at the cell-center, or vertex center:
- Non-orthogonality is simply defined as the mis-alignment of the distance vector $G_j T = \frac{\int T A_j}{\int dV}$ between the two "L" and "R" states and the surface normal
- Both edge- and cell centered-based schemes show degraded accuracy on typical production meshes
- Several non-orthogonality approaches are available, for the best source, see Jasak
 - Jasak, "Error analysis and error estimation for the finite volume method with applications to fluid flow", Imperial College Dissertation, 1996

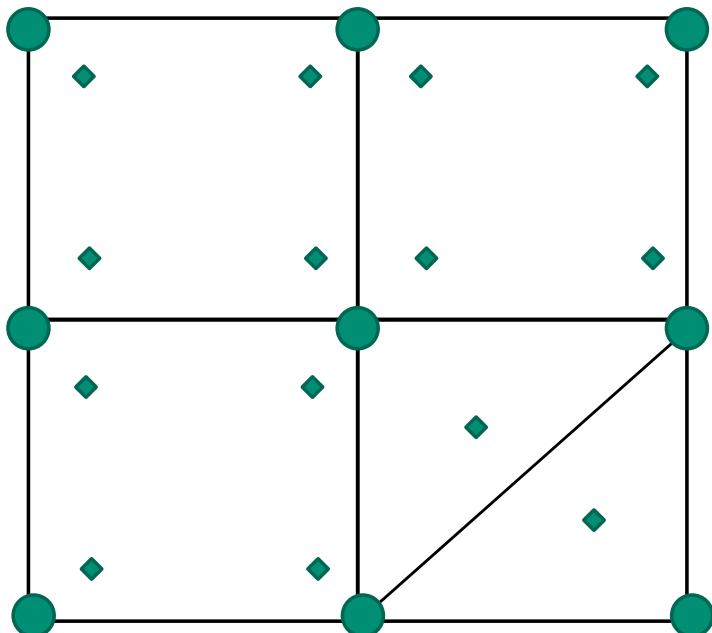
The Finite Element Method (FEM)



- Consider an alternative approach in which a finite element method (FEM) is employed

- Define an underlying nodal basis with the element:

$$T(x_k) \approx \sum_{i=1}^{npe} N_i(x_k) T_i \quad \frac{\partial T(x_k)}{\partial x_j} \approx \sum_{i=1}^{npe} \frac{\partial N_i(x_k)}{\partial x_j} T_i$$



- Gaussian Quadrature is used
- On a -1:1 range, +/- sqrt(3)/3

- Consider a simple heat conduction model PDE,

$$\rho C_p \frac{\partial T}{\partial t} - \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} = 0$$

- Integrate using a test function,

$$\int w \rho C_p \frac{\partial T}{\partial t} dV - \int w \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} dV = 0$$

- Integration-by-parts (with G-D) provides:

$$\int w \rho C_p \frac{\partial T}{\partial t} dV + \int \frac{\partial w}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} dV - \int w \lambda \frac{\partial T}{\partial x_j} n_j dS = 0$$

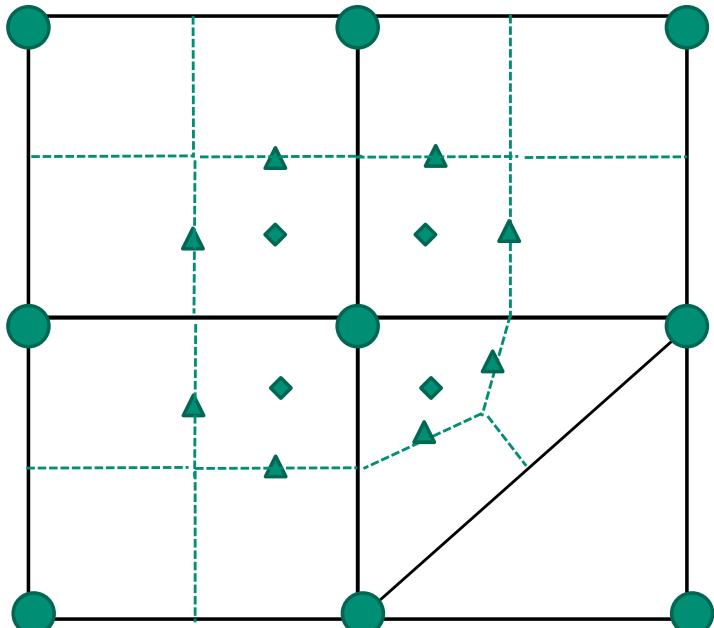
- Iterate element quadrature points
- Note that N can be arbitrary in order (shown here for a linear)

The Hybrid Control-Volume Finite Element Method (CVFEM)



- A combination between the edge-based vertex-centered and FEM is the method known as Control Volume Finite Element ()
- A dual mesh is constructed to obtain flux and volume quadrature locations
- As with FEM, a basis is defined:

$$T(x_k) \approx \sum_{i=1}^{npe} N_i(x_k) T_i \quad \frac{\partial T(x_k)}{\partial x_j} \approx \sum_{i=1}^{npe} \frac{\partial N_i(x_k)}{\partial x_j} T_i$$



Dual-volume definition

- Integration-by-parts over test function w:

$$\int w \rho C_p \frac{\partial T}{\partial t} dV + \int \frac{\partial w}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} dV - \int w \lambda \frac{\partial T}{\partial x_j} n_j dS = 0$$

- However, define a test function, w, as a piece-wise constant function (Heavyside) to be 1 inside the dual volume and 0 outside. Gradient is a Dirac-delta function:

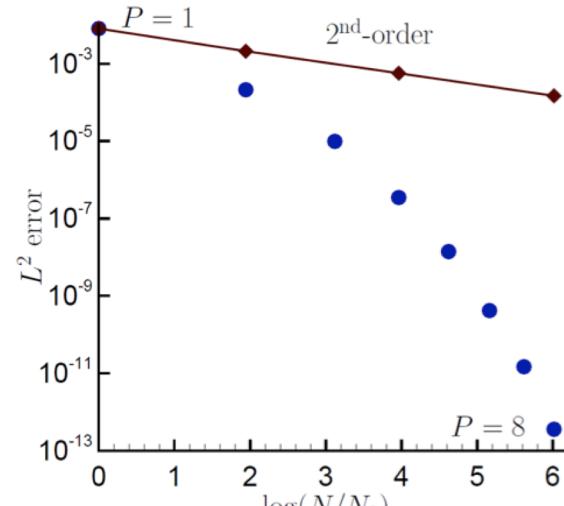
$$\frac{\partial w}{\partial x_j} = -n_j \delta(x_j - xIP_j)$$

- Leading to: $\int \rho C_p \frac{\partial T}{\partial t} dV - \int \lambda \frac{\partial T}{\partial x_j} n_j dS = 0$

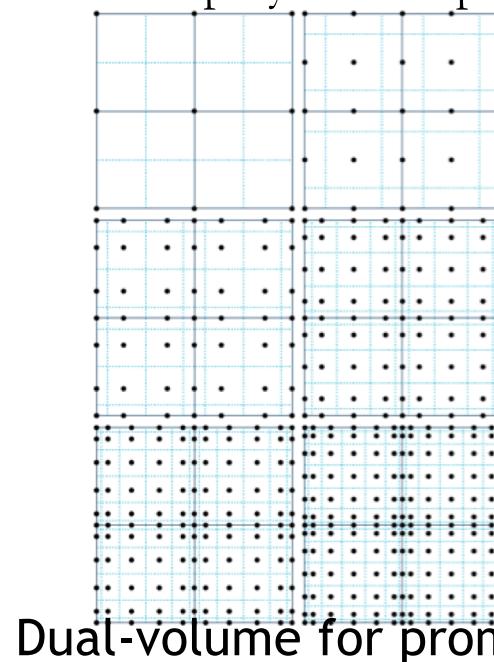
Control-Volume Finite Element Method Attributes



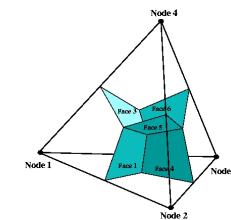
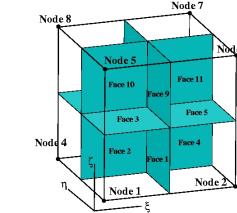
- The CVFEM method, therefore, is a finite volume scheme that is locally conservative, i.e., momentum leaving one dual volume face enters the adjacent dual volume
- However, the gradient operator, like its FEM counterpart is absent of any error due to non-orthogonality
- Since the test-function, w , is different from the underlying basis representation, this method can be considered a Petrov-Galerkin method
- The method can also be promoted in polynomial space



Spectral convergence



Dual-volume for promoted quad4

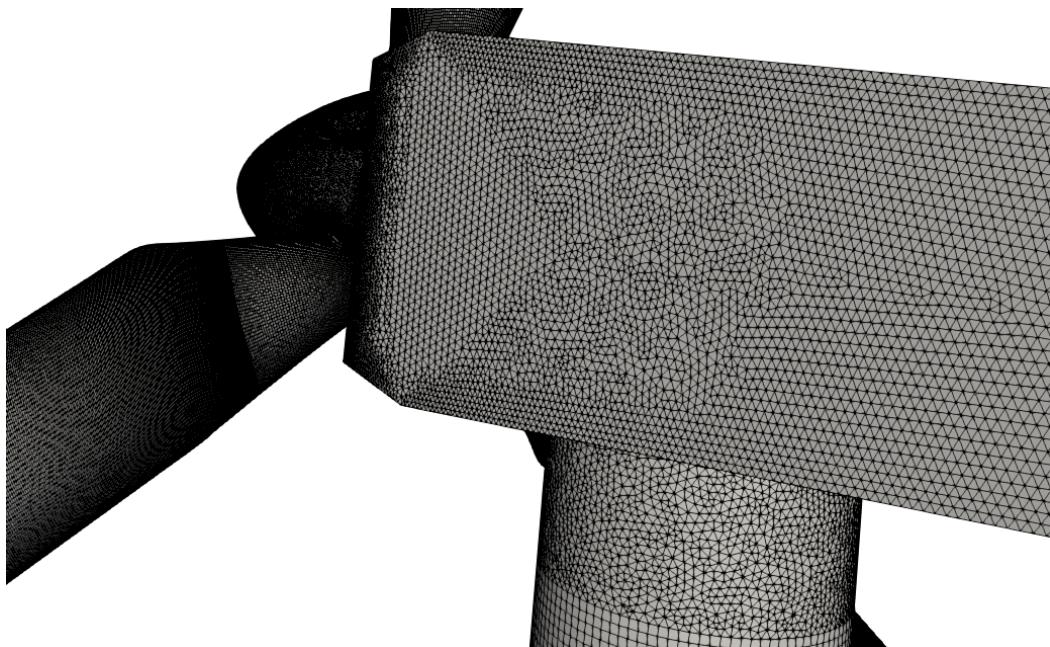


$P=1$ (left) and $P=4$ (right)
Helium plume (VR-density)



Finally, Some Common Unstructured Element Types

- To manage complex geometries, hex, tet, pyramid, and wedge (or prisms) are used
- There are also arbitrary polyhedra methods (CC) in which polyhedra size can be > 10
- Activity: ASC/SNL is focused on reducing meshing time, increasing code usability, and providing a **Next Generation Simulation** Capability



Vestas V27 hybrid mesh

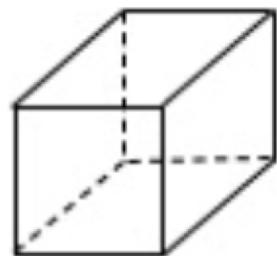


Very complex geometry!

Examples of Various Topologies



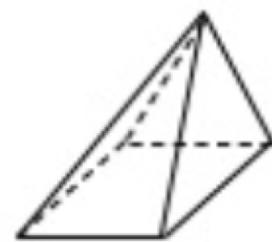
Hex8



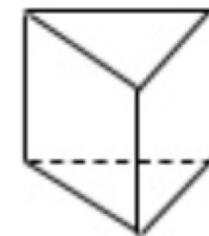
Tet4



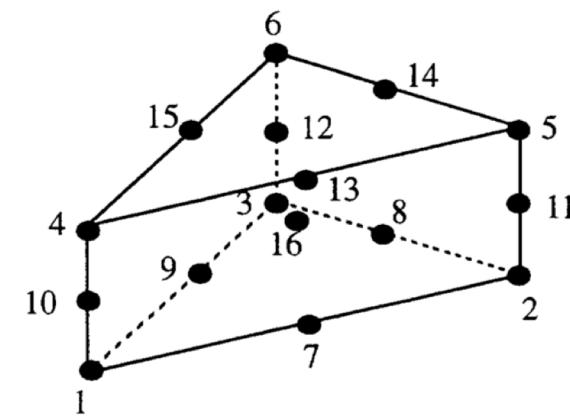
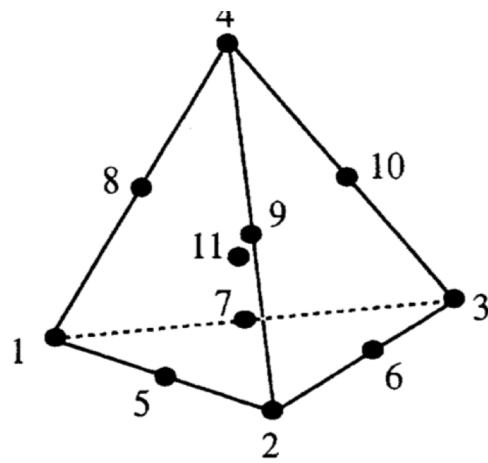
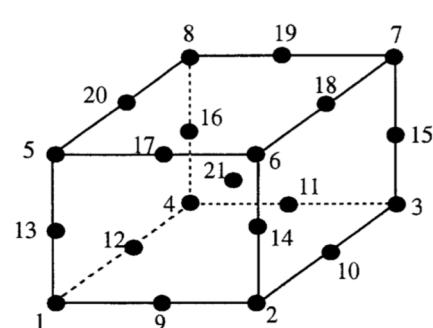
Pyramid5



Wedge6



Arbitrary



Higher-order promoted elements (Hex27, Tet10, Wedge16, Hex64, etc.)

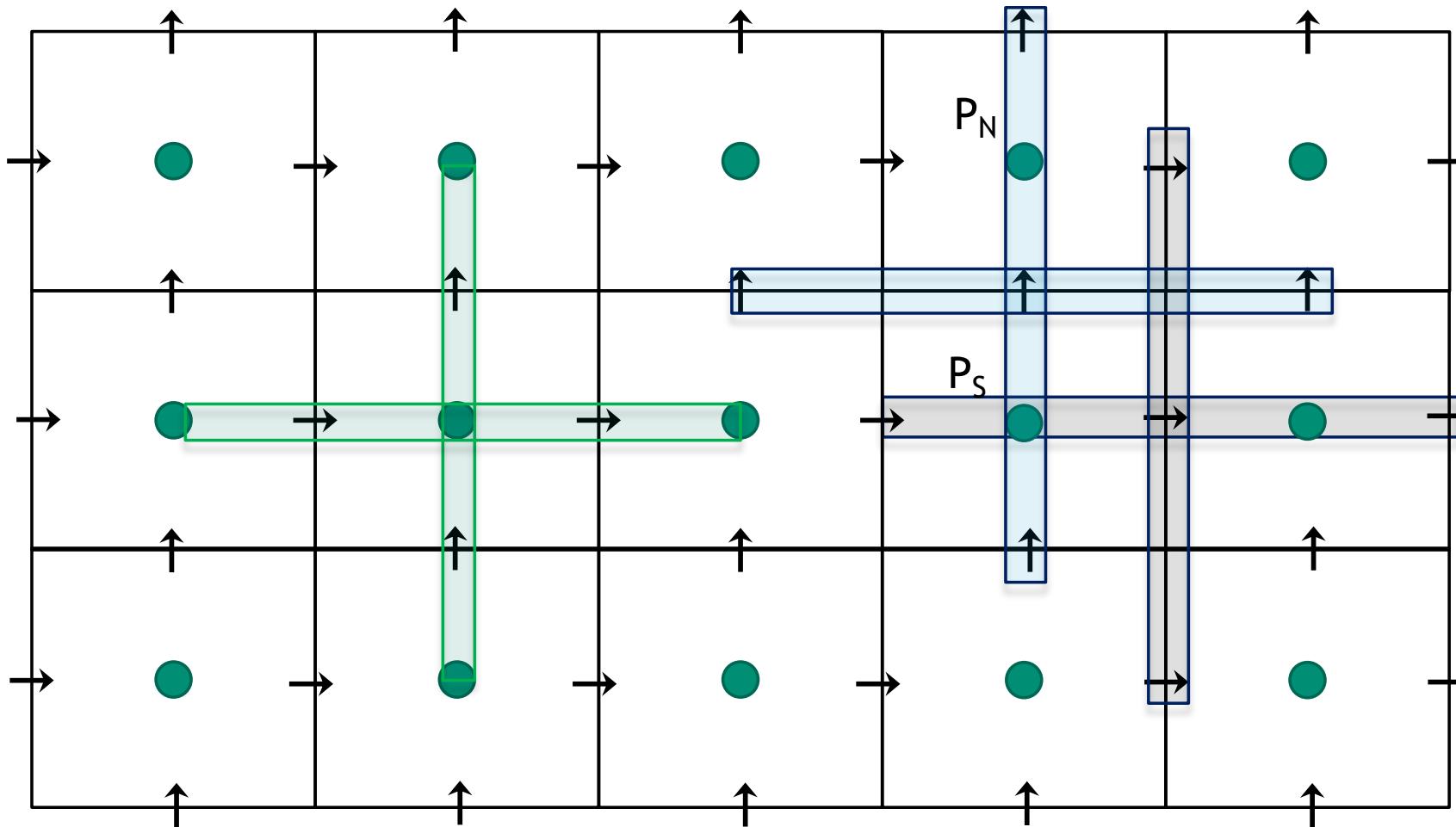
Classic Staggered Finite Volume

Stencil for CC-quantities ●

Stencil for x-velocity →

Stencil for y-velocity ↑

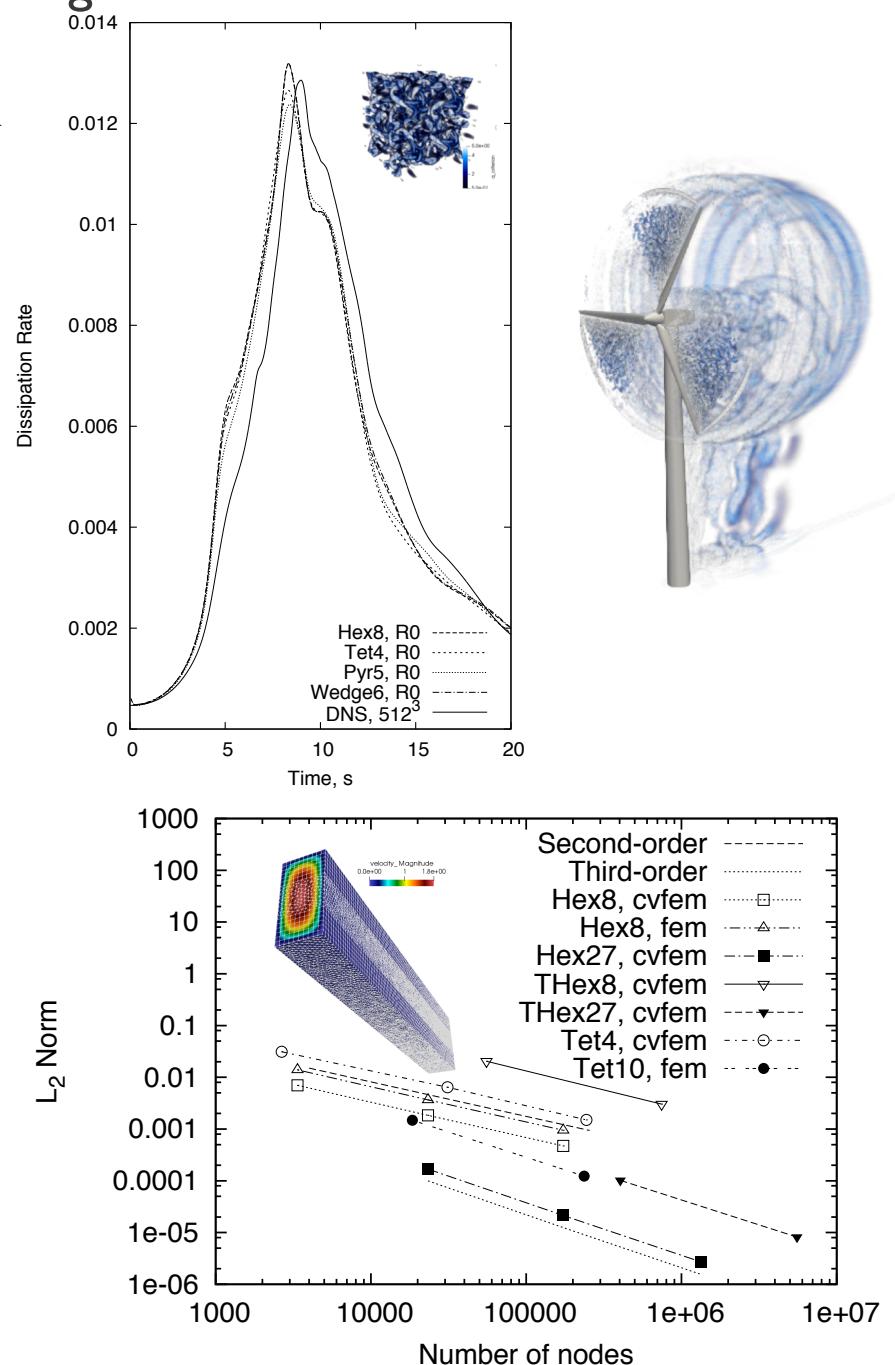
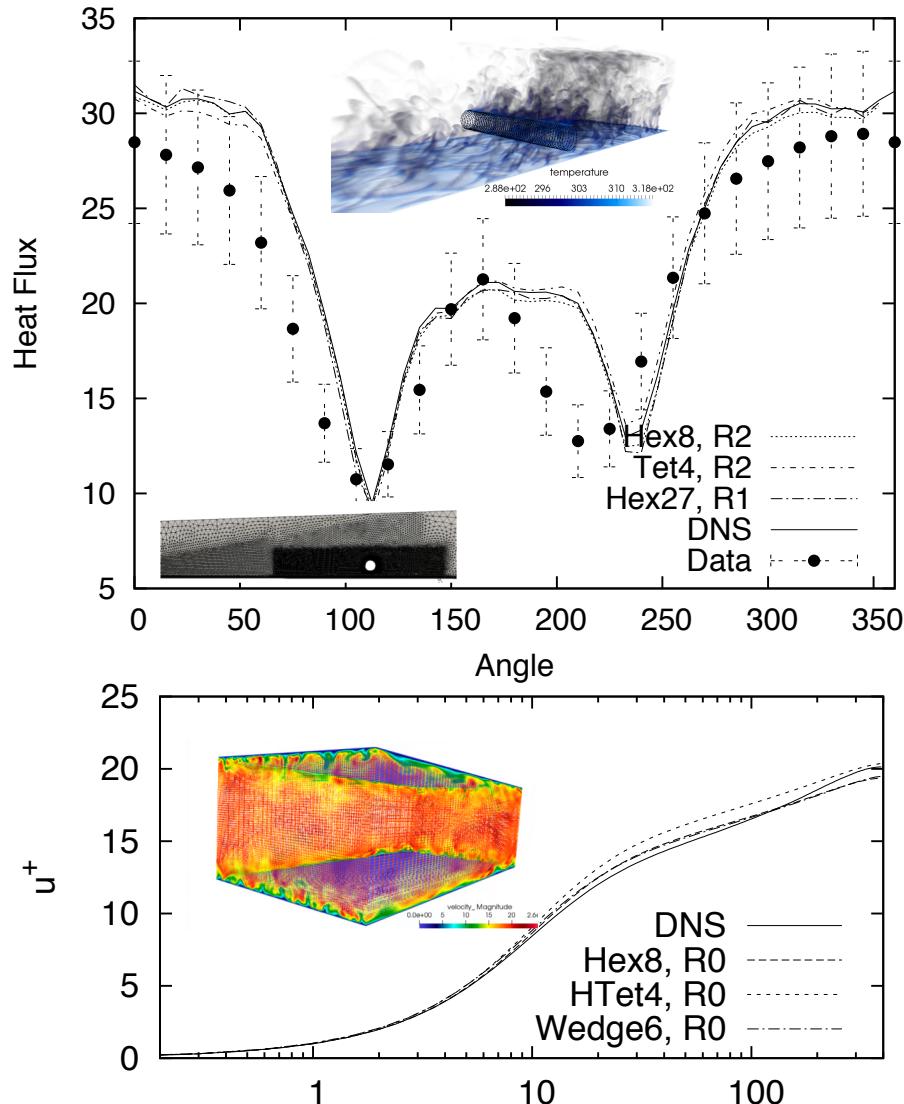
- Velocity degree-of-freedom is staggered relative to pressure and other primitives, e.g., enthalpy, mixture fraction, etc.



Recent Generalized Unstructured Findings



- Domino, et. al, “An assessment of atypical mesh topologies for low-Mach large-eddy simulation”
2019



Attributes of a Staggered Scheme



- By design, non-orthogonality is absent, however, complex geometry will be stair-stepped
- From a fluids perspective, the operators are ideal, i.e., pressure gradient for momentum is compact, e.g., $(P_E - P_W)\Delta x^{-1}$
 - As will be seen in future lecture topics, the skew-adjoint nature of the Divergence operator, **D**, and Gradient operator, **G**, allows for a Laplace operator, **L** = **DG**
- Can be extended to higher-order
- Frequently, meshing complex geometries can be extremely difficult (consider our V27 example)



Common low-Mach Discretization Approaches: Conclusions



- Two-state methods, e.g., cell-centered and EBVC are attractive due to simplicity, however, suffer from non-orthogonality issues in the diffusion operator
- FEM provides a machinery to provide accurate discretizations on non-ideal meshes, however, the same diffusion operator suffers on high-aspect ratio meshes
- CVFEM is a hybrid method that contains the likeable attributes of both FV and FEM (same high-aspect ratio diffusion operator finding)
- Staggered arrangement is well suited for a class of fluid mechanics applications where low-order or simple geometries are found