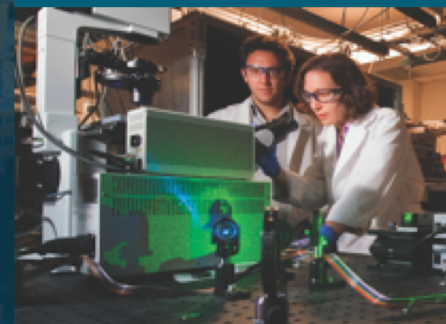
Guest Lecture Stanford ME469: Common low-Mach Discretization Approaches



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Common low-Mach Discretization Approaches: Outline

- Cell-centered Finite Volume (FV)
- Edge-based Vertex-Centered (EBVC)
- Finite Element Method (FEM)
- Control-Volume Finite Element Method (CVFEM)
- Staggered arrangement
- Conclusions

Fundamentals of Discretization

- Given a partial differential equation (PDE) and associated volumetric form:

$$\rho C_p \frac{\partial T}{\partial t} - \frac{\partial}{\partial x_j}\lambda\frac{\partial T}{\partial x_j} = 0 \qquad \int \rho C_p \frac{\partial T}{\partial t} dV - \int \frac{\partial}{\partial x_j}\lambda\frac{\partial T}{\partial x_j} dV = 0$$

- Applying Gauss Divergence provides the standard finite volume form:

$$\int \frac{\partial q_j}{\partial x_j} dV = \int q_j n_j \, dS \qquad \int \rho C_p \frac{\partial T}{\partial t} dV - \int \lambda\frac{\partial T}{\partial x_j} n_j \, dS = 0$$

- We can also multiple PDE by an arbitrary test function, w, and integrate over a volume,

$$\int w\rho C_p \frac{\partial T}{\partial t} dV - \int w\frac{\partial}{\partial x_j}\lambda\frac{\partial T}{\partial x_j} n_j \, dS = 0$$

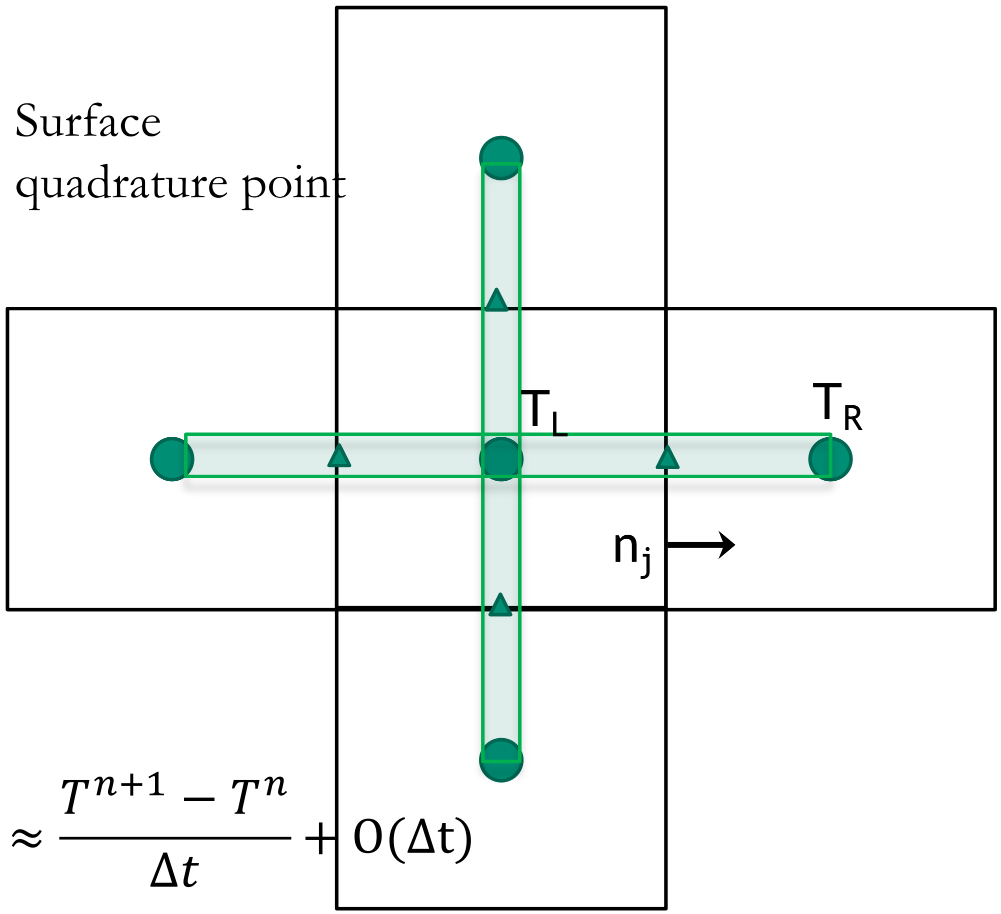
Next, integrate by parts and apply Gauss-Divergence:

$$\int w\rho C_p \frac{\partial T}{\partial t} dV + \int \frac{\partial w}{\partial x_j}\lambda\frac{\partial T}{\partial x_j} dV - \int w\lambda\frac{\partial T}{\partial x_j} dS = 0$$

Equal-Order Interpolation Cell-Centered (CC) Finite Volume

- All primitives are collocated at the cell-center of the element with equal-order interpolation
- Classic two-state, “L” and “R” approach provides spatially second-order accuracy

▲ Surface quadrature point



$$\frac{\partial T}{\partial t} \approx \frac{T^{n+1} - T^{n}}{\Delta t} + O(\Delta t)$$

- Consider a simple heat conduction model PDE,

$$\rho C_p \frac{\partial T}{\partial t} - \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} = 0$$

- Integrate over a control volume and use Gauss-Divergence,

$$\int \rho C_p \frac{\partial T}{\partial t} dV - \int \lambda \frac{\partial T}{\partial x_j} n_j \, dS = 0$$

with: $$\frac{\partial T}{\partial x_j} \approx \frac{(T_R - T_L) n_j}{\|\Delta x_j\|} + O(\Delta x)$$

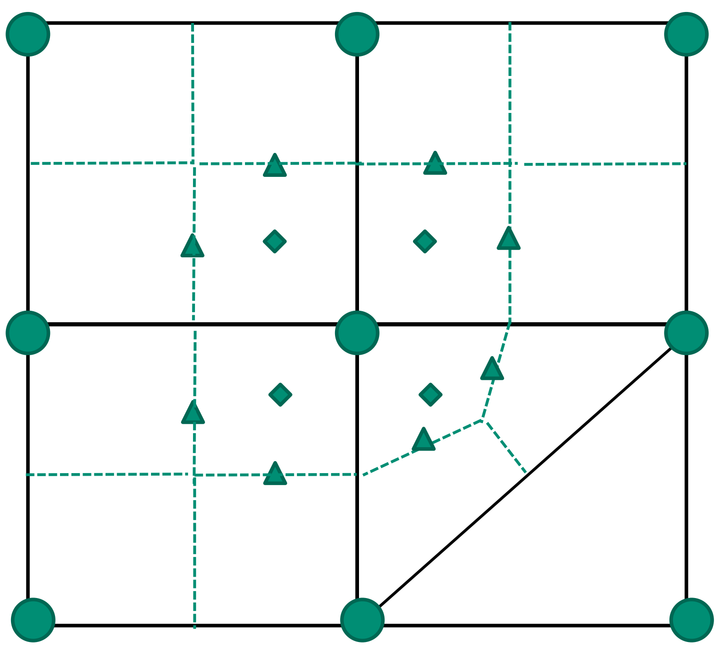
- Iterate element cell-centers for volume contributions (time/source)
- Iterate element faces for surface flux contribution

Equal Order Interpolation Edge-Based Vertex-Centered (EBVC) Finite Volume

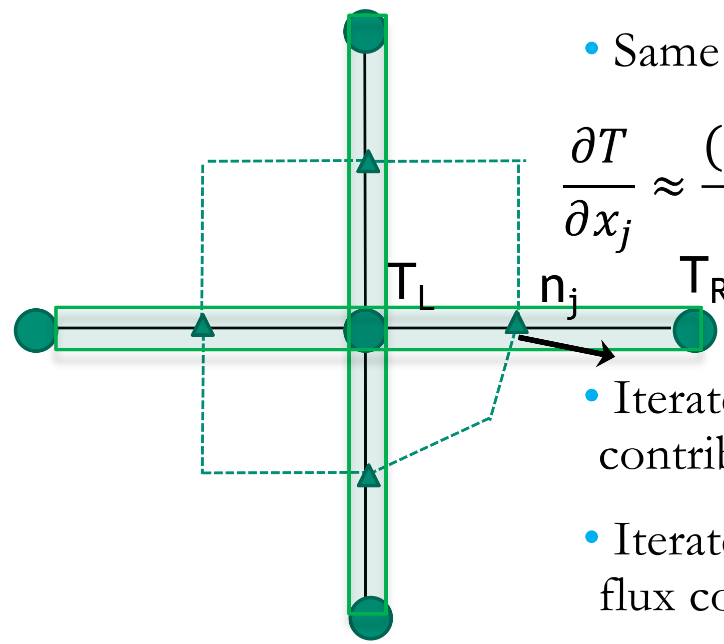
- All primitives are collocated at the vertices of the elements with equal-order interpolation
- A dual mesh is constructed to obtain flux and volume quadrature locations
- Classic two-state, "L" and "R" approach provides spatially second-order accuracy

▲ Surface quadrature point (area summed to edge)

◆ Volume quadrature point (sub-vol summed to node)



Dual-volume definition



Edge-based stencil

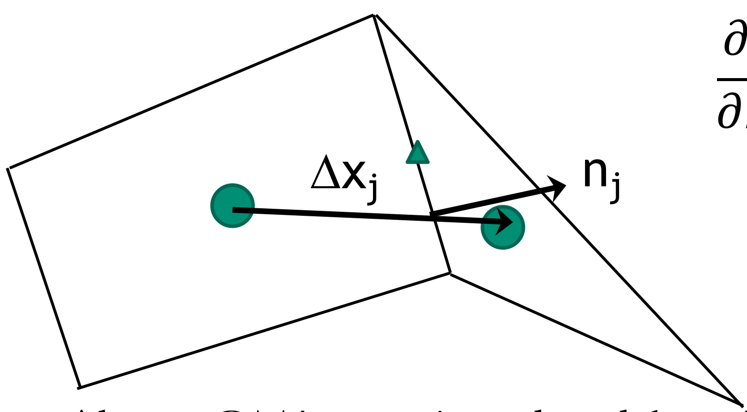
- Same grad-op as CC:

$$\frac{\partial T}{\partial x_j} \approx \frac{(T_R - T_L)n_j}{\|\Delta x_j\|} + O(\Delta \mathrm{x})$$

- Iterate nodes for volume contributions
- Iterate edges for surface flux contribution

Typical Failings for Two-State Discretization Methods

- With two points, only a linear basis can be used.
- Therefore, unstructured CC and EBVC are limited to second-order spatial accuracy
- Non-orthogonality is problematic for gradient-operator



$$\frac{\partial T}{\partial x_j} = G_j T + \left[(T_R - T_L) - G_k T \Delta x_k\right] \frac{A_j}{A_l \Delta x_l}$$

With area vector defined by: $A_j = n_j dS$

- Above, $G_j T$ is a projected nodal gradient at the cell-center, or vertex center:
- Non-orthogonality is simply defined as the mis-alignment of the distance vector between the two "L" and "R" states and the surface normal

$$G_j T = \frac{\int T A_j}{\int dV}$$

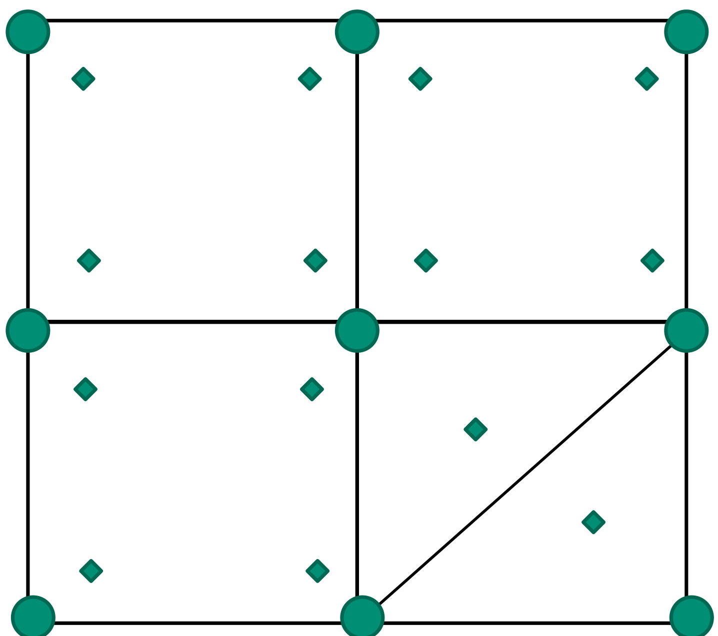
- Both edge- and cell centered-based schemes show degraded accuracy on typical production meshes
- Several non-orthogonality approaches are available, for the best source, see Jasak
  - Jasak, "Error analysis and error estimation for the finite volume method with applications to fluid flow", Imperial College Dissertation, 1996

The Finite Element Method (FEM)

- Consider an alternative approach in which a finite element method (FEM) is employed

- Define an underlying nodal basis with the element:

$$T(x_k) \approx \sum_{i=1}^{npe} N_i(x_k) T_i \qquad \frac{\partial T(x_k)}{\partial x_j} \approx \sum_{i=1}^{npe} \frac{\partial N_i(x_k)}{\partial x_j} T_i$$



- Gaussian Quadrature is used

- On a -1:1 range, +/- sqrt(3)/3

- Consider a simple heat conduction model PDE,

$$\rho C_p \frac{\partial T}{\partial t} - \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} = 0$$

- Integrate using a test function,

$$\int w \rho C_p \frac{\partial T}{\partial t} dV - \int w \frac{\partial}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} dV = 0$$

- Integration-by-parts (with G-D) provides:

$$\int w \rho C_p \frac{\partial T}{\partial t} dV + \int \frac{\partial w}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} dV - \int w \, \lambda \frac{\partial T}{\partial x_j} n_j dS = 0$$

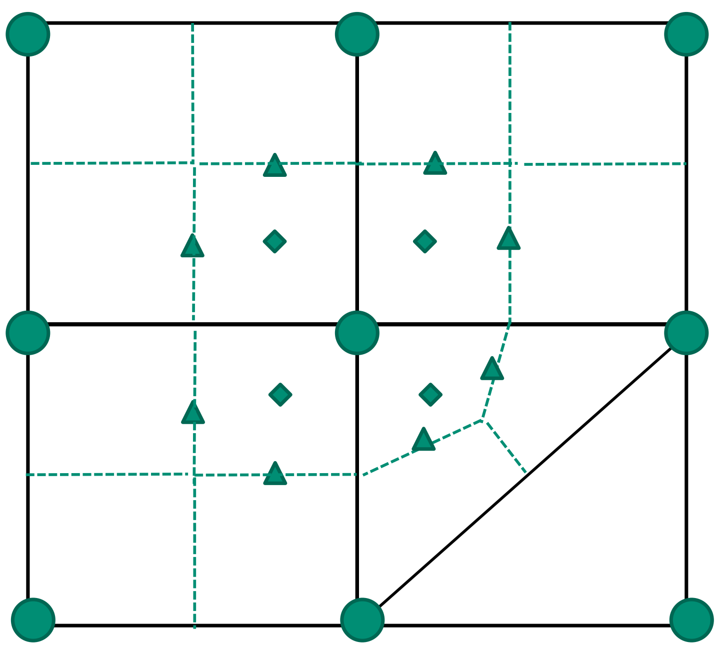
- Iterate element quadrature points

- Note that N can be arbitrary in order (shown here for a linear)

The Hybrid Control-Volume Finite Element Method (CVFEM)

- A combination between the edge-based vertex-centered and FEM is the method known as Control Volume Finite Element ()
- A dual mesh is constructed to obtain flux and volume quadrature locations
- As with FEM, a basis is defined:

$$T(x_k) \approx \sum_{i=1}^{npe} N_i(x_k) T_i \qquad \frac{\partial T(x_k)}{\partial x_j} \approx \sum_{i=1}^{npe} \frac{\partial N_i(x_k)}{\partial x_j} T_i$$



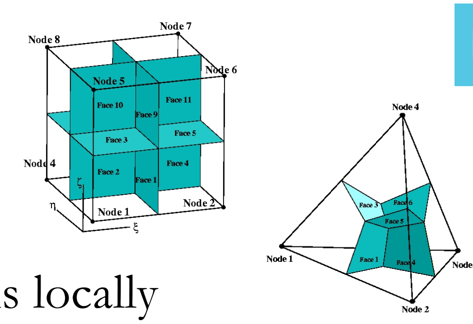
Dual-volume definition

- Integration-by-parts over test function w:

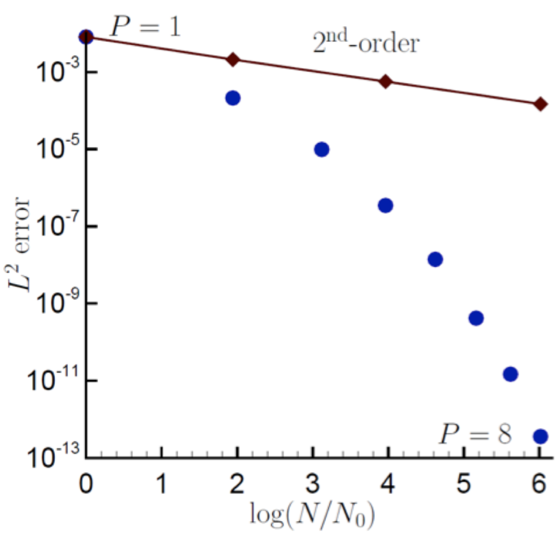
$$\int w\rho C_p \frac{\partial T}{\partial t} dV + \int \frac{\partial w}{\partial x_j} \lambda \frac{\partial T}{\partial x_j} dV - \int w\,\lambda \frac{\partial T}{\partial x_j} n_j dS = 0$$

- However, define a test function, w, as a piece-wise constant function (Heavyside) to be 1 inside the dual volume and 0 outside. Gradient is a Dirac-delta function: $\frac{\partial w}{\partial x_j} = -n_j \delta(x_j - xIP_j)$
- Leading to: $\int \rho C_p \frac{\partial T}{\partial t} dV - \int \lambda \frac{\partial T}{\partial x_j} n_j\, dS = 0$

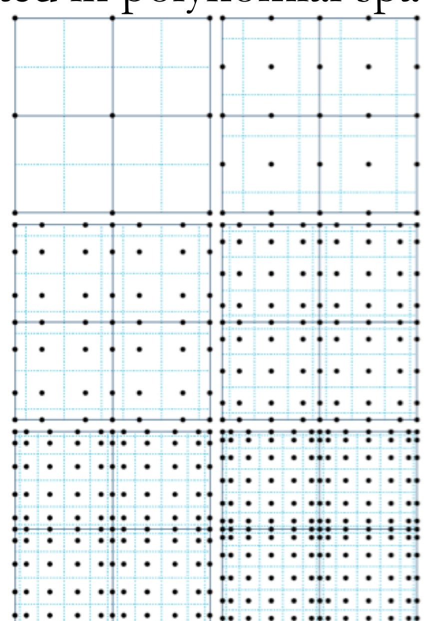
Control-Volume Finite Element Method Attributes



- The CVFEM method, therefore, is a finite volume scheme that is locally conservative, i.e., momentum leaving one dual volume face enters the adjacent dual volume
- However, the gradient operator, like its FEM counterpart is absent of any error due to non-orthogonality
- Since the test-function, w, is different from the underlying basis representation, this method can be considered a Petrov-Galerkin method
- The method can also be promoted in polynomial space



Spectral convergence

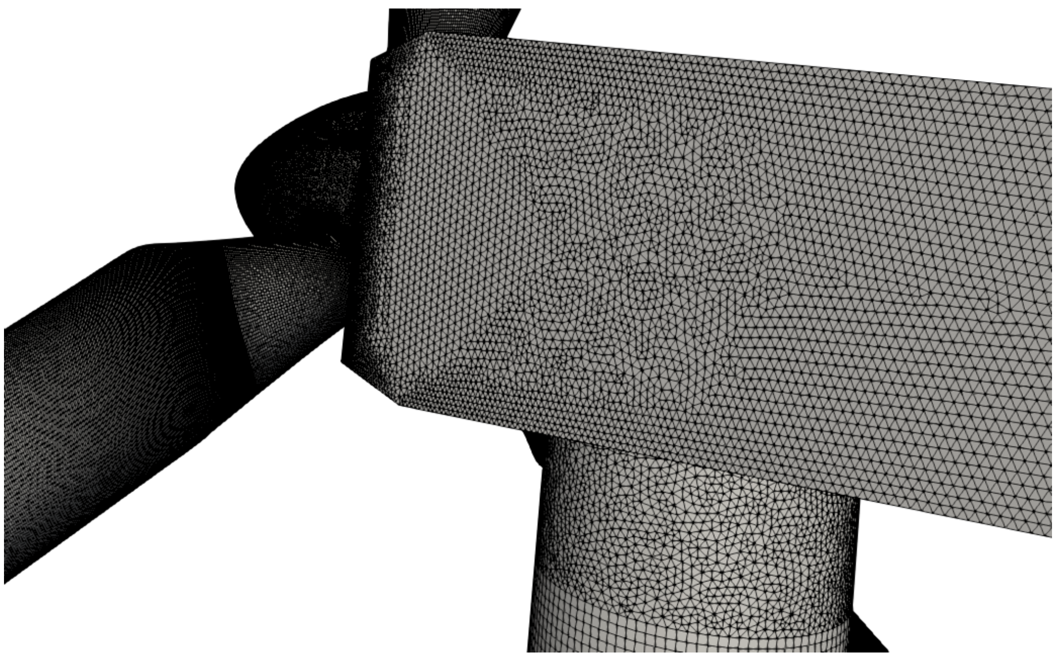


Dual-volume for promoted quad4

P=1 (left) and P=4 (right)
Helium plume (VR-density)

Finally, Some Common Unstructured Element Types

- To manage complex geometries, hex, tet, pyramid, and wedge (or prisms) are used
- There are also arbitrary polyhedra methods (CC) in which polyhedra size can be > 10
- Activity: ASC/SNL is focused on reducing meshing time, increasing code usability, and providing a **Next Generation Simulation** Capability

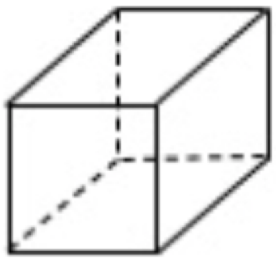


Vestas V27 hybrid mesh

Very complex geometry!

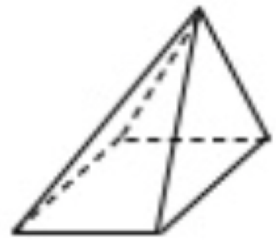
Examples of Various Topoliogies

Hex8



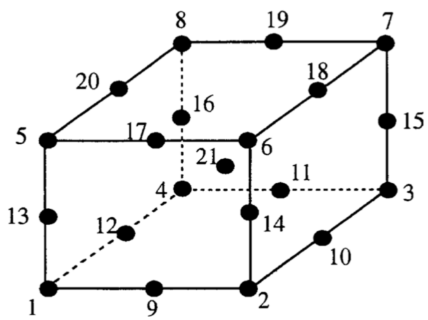
Tet4

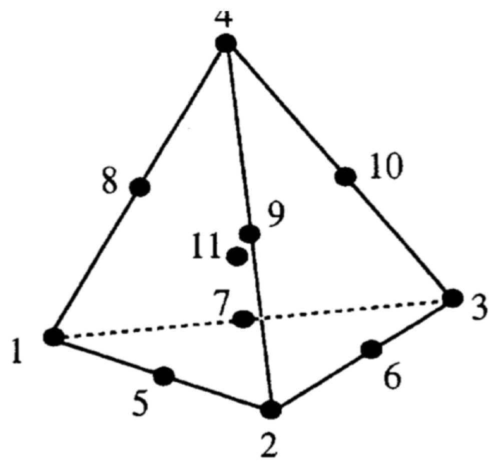
Pyramid5

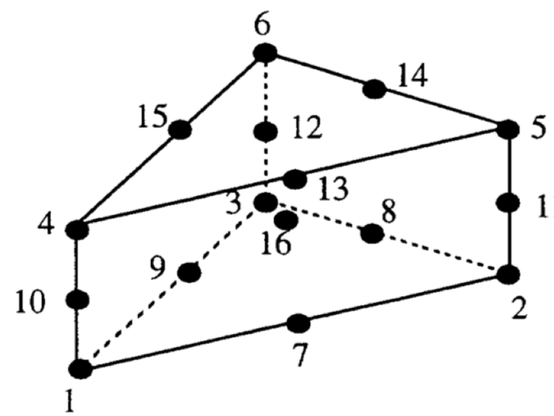


Wedge6

Arbirtrary

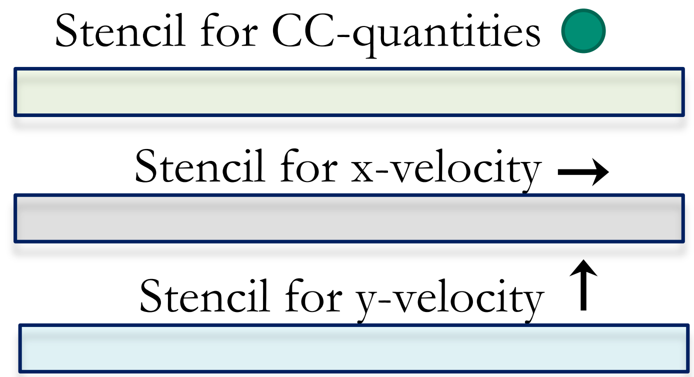




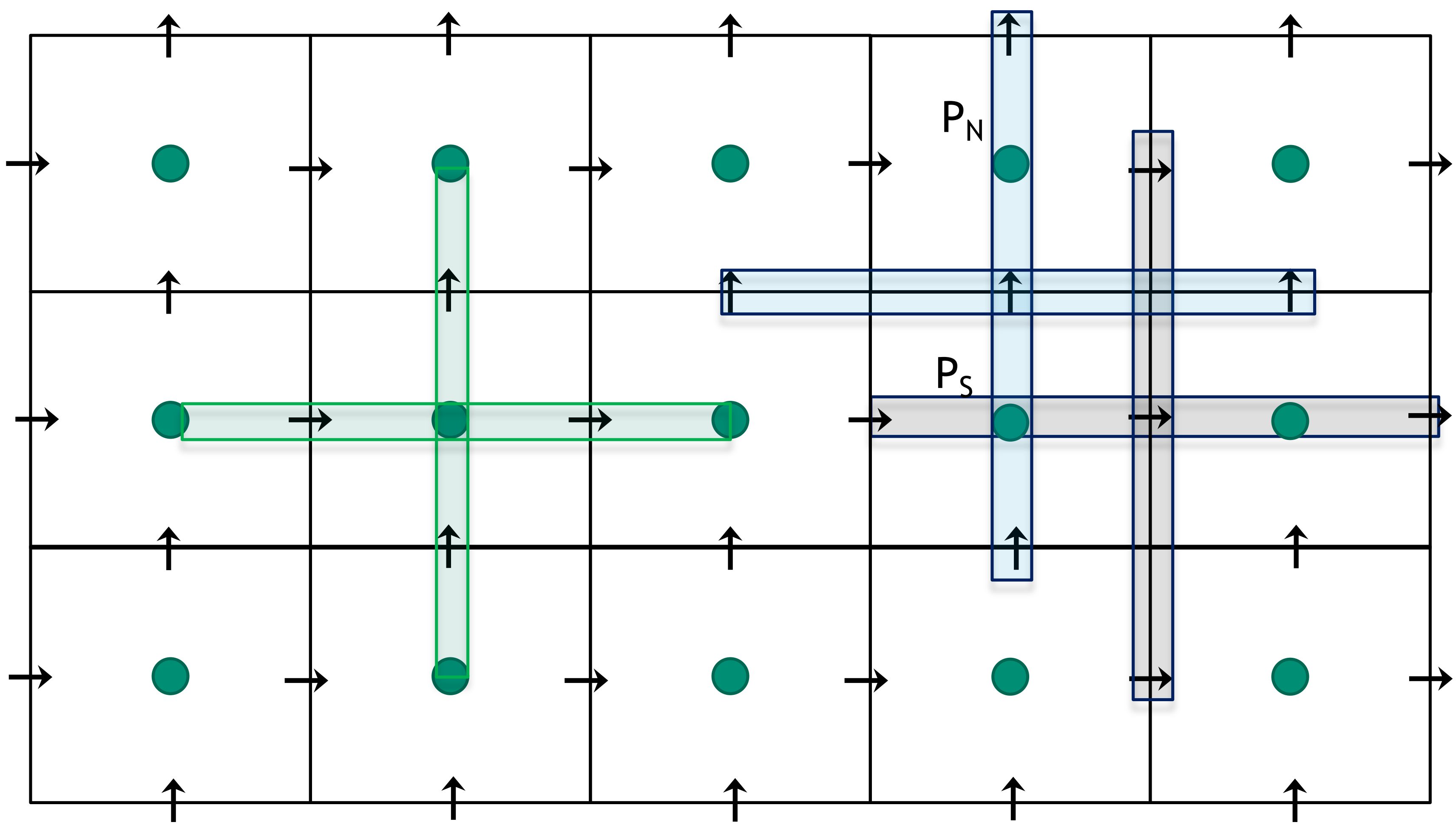


Higher-order promoted elements (Hex27, Hex64, etc)

Classic Staggered Finite Volume



- Velocity degree-of-freedom is staggered relative to pressure and other primitives, e.g., enthalpy, mixture fraction, etc.



Attributes of a Staggered Scheme

- By design, non-orthogonality is absent, however, complex geometry will be stair-stepped
- From a fluids perspective, the operators are ideal, i.e., pressure gradient for momentum is compact, e.g., $(P_E - P_W)\Delta x^{-1}$
  - As will be seen in future lecture topics, the skew-adjoint nature of the Divergence operator, **D**, and Gradient operator, **G**, allows for a Laplace operator, **L** = **DG**
- Can be extended to higher-order
- Frequently, meshing complex geometries can be extremely difficult (consider our V27 example)

Common low-Mach Discretization Approaches: Conclusions

- Two-state methods, e.g., cell-centered and EBVC are attractive due to simplicity, however, suffer from non-orthogonality issues in the diffusion operator
- FEM provides a machinery to provide accurate discretizations on non-ideal meshes, however, the same diffusion operator suffers on high-aspect ratio meshes
- CVFEM is a hybrid method that contains the likeable attributes of both FV and FEM (same high-aspect ratio diffusion operator finding)
- Staggered arrangement is well suited for a class of fluid mechanics applications where low-order or simple geometries are found