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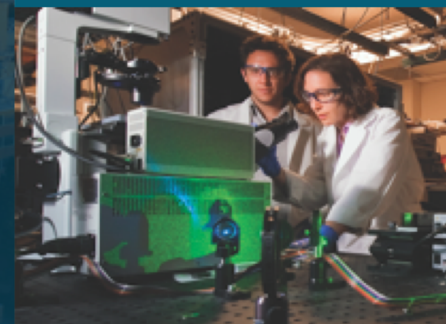
Guest Lecture Stanford ME469: Advection Operators



PRESENTED BY

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Computational Thermal and Fluid Mechanics

Sandia National Laboratories

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Advection Operators: Outline

- Classic Analysis on Advection/Diffusion/Time Scalar Transport
- The Role of Diagonal Dominance
- Upwind Methods
- Linear Residual-Based Stabilization
- Nonlinear Residual-Based Stabilization
- Conclusions

Consider a Simple Advection/Diffusion/Time Scalar Transport

- PDE is given by: $\dfrac{\partial \rho Z}{\partial t} + \dfrac{\partial \rho u_j Z}{\partial x_j} - \dfrac{\partial}{\partial x_j}\dfrac{\mu}{Sc}\dfrac{\partial Z}{\partial x_j} = 0$

- Integral form: $\displaystyle\int \frac{\partial \rho Z}{\partial t} dV + \int \left[\rho u_j Z - \frac{\mu}{Sc}\frac{\partial Z}{\partial x_j}\right] n_j \, dS = 0$

- Discrete form: $\displaystyle\sum_{sip} \frac{(\rho Z^{n+1} - \rho Z^{n})}{\Delta t} dV + \sum_{ip} \dot{m} Z_{ip} - \sum_{ip} \frac{\mu}{Sc}\frac{\partial Z}{\partial x_j} A_j$

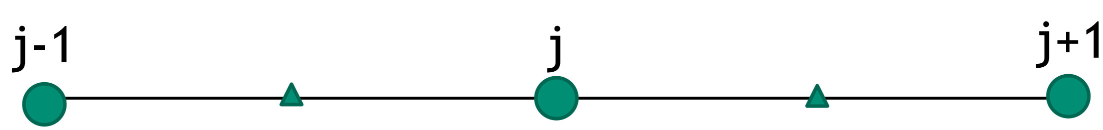
Mass flow rate:
$\dot{m} = \rho u_j A_j$

- Matrix Form: $\begin{pmatrix} -a_{j-1} & a_j & -a_{j+1} \end{pmatrix} \begin{pmatrix} Z_{j-1} \\ Z_j \\ Z_{j+1} \end{pmatrix} = 0$

$$a_{j+1} = \frac{\mu}{Sc}\frac{A}{\Delta x} - \frac{\dot{m}_{j+1/2}}{2} \qquad a_{j-1} = \frac{\mu}{Sc}\frac{A}{\Delta x} + \frac{\dot{m}_{j-1/2}}{2}$$

$$\frac{\sum_{i \neq j} |a_{ij}|}{|a_{ii}|} \leq 1$$

$$a_j = a_{j+1} + a_{j-1} + \left(\dot{m}_{j+1/2} - \dot{m}_{j-1/2}\right) + \frac{\rho \Delta V}{\Delta t}$$



Diagonal Dominance Check

- For a monotonic operator: $DD = \frac{\sum_{i \neq j} |a_{ij}|}{|a_{ii}|} \leq 1$

- Consider u = 1 cm/s; ρ = 1.2e-3 g/cm3; μ/Sc = 1.8e-4 g/cm-s; Δx = 0.1 cm; ; Δt >> 1:
  - a_{j+1}=1.2e-5 a_{j-1}=2.4e-5 a_j = 3.6e-5 $DD \leq 1$
  - Stable and monotonic field will be realized; Re = 0.67

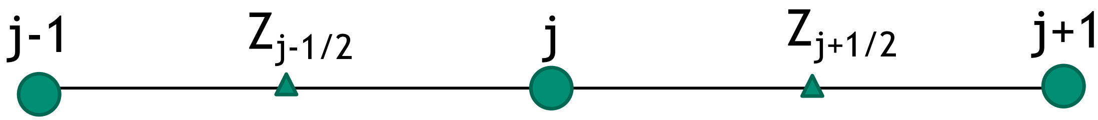
- Consider u = 10 cm/s; ρ = 1.2e-3 g/cm3; μ/Sc = 1.8e-4 g/cm-s; Δx = 0.1 cm; Δt >> 1:
  - a_{j+1}=-4.2e-5 a_{j-1}=7.8e-5 a_j = 3.605e-5 $DD > 1$
  - Unstable and lack of scalar monotonicity will be realized; Re = 6.7
  - Standard preconditioner, e.g., SGS, are not expected to converge the linear system

- Mesh spacing for monotonicity must be: $\Delta x \leq \frac{2\mu/Sc}{\rho u}$ $\mathrm{Pe} = \frac{\rho u \Delta x}{\mu/Sc} \leq 2$
  - Cell Peclet number, Pe is a cell-Reynolds number based on local properties and mesh spacing

- Consider u = 10 cm/s; ρ = 1.2e-3 g/cm3; μ/Sc = 1.8e-4 g/cm-s; Δx = 0.1 cm; Δt =?
  - For diagonal dominance, Δt = 0.0143 which corresponds to a Courant number of unity $C = \frac{u\Delta t}{\Delta x}$
  - This can limit the ability for some linear solvers to converge the linear system

Upwind Operators

- Upwind advection operators can provide diagonal dominance (at the cost of accuracy)



$$Z_{j+1/2} = \begin{cases} Z_j; u > 0 \\ Z_{j+1}; u < 0 \end{cases}$$

"as the wind blows"

- However, upwind operators, even if higher-order, are numerically diffuse
- Higher-order upwind (second-order in space) can be obtained on an unstructured mesh by use of projected nodal gradients and extrapolation:

$$G_x Z_j = \frac{Z_{j+1} - Z_{j-1}}{2\Delta x} \qquad G_x Z_{j-1} = \frac{Z_j - Z_{j-2}}{2\Delta x}$$

for u>0

$$Z_{j+1/2} = \propto \left(Z_j + G_x Z_j \frac{\Delta x}{2} \right) + (1 - \propto) \left[\frac{1}{2} (Z_j + Z_{j+1}) \right]$$

$$Z_{j-1/2} = \propto \left(Z_{j-1} + G_x Z_{j-1} \frac{\Delta x}{2} \right) + (1 - \propto) \left[\frac{1}{2} (Z_j + Z_{j-1}) \right]$$

α is a blending parameter

Higher-order Upwind

- Blending extrapolated upwind with central $(Z_L+Z_R)/2$ provides:

for u>0
$$Z_{j+1/2} = \propto \left(Z_j + G_x Z_j \frac{\Delta x}{2}\right) + (1 - \propto)\left[\frac{1}{2}(Z_j + Z_{j+1})\right]$$
$$Z_{j-1/2} = \propto \left(Z_{j-1} + G_x Z_{j-1} \frac{\Delta x}{2}\right) + (1 - \propto)\left[\frac{1}{2}(Z_j + Z_{j-1})\right]$$

for u<0
$$Z_{j+1/2} = \propto \left(Z_{j+1} + G_x Z_{j+1} \frac{\Delta x}{2}\right) + (1 - \propto)\left[\frac{1}{2}(Z_j + Z_{j+1})\right]$$
$$Z_{j-1/2} = \propto \left(Z_j + G_x Z_j \frac{\Delta x}{2}\right) + (1 - \propto)\left[\frac{1}{2}(Z_j + Z_{j-1})\right]$$

Resulting 1-D Stencil

j-2	j-1	j	j+1	j+2	α	$\dot{m}$
1/4	-5/4	+3/4	+1/4	0	1	>0
0	-1/4	-3/4	+5/4	-1/4	1	>0
+1/6	-6/6	+3/6	+2/6	0	1/2	<0
0	-2/6	-3/6	+6/6	-1/6	1/2	<0

- Slope limiting required when gradients are not smooth (Berger et al, 2005, 43rd AIAA)

Linear Residual-Based Stabilization

- Is there a better way that either un-stabilized (oscillatory and non-monotonic) and dissipative upwind?
- Consider the model advection/diffusion PDE in the context of a FEM

$$\int w \frac{\partial \rho Z}{\partial t} dV + \int w \frac{\partial \rho u_j Z}{\partial x_j} dV + \int \frac{\partial w}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} dV - \int w \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} n_j dS = 0$$

- Concept: Add a scaled fine-scale residual for stabilization, Streamwise Upwind Petrov Galerkin SUPG (Hughes and Brooks, 1982)

$$+ \sum_{elem} \tau u_k \frac{\partial w}{\partial x_k} \left(\frac{\partial \rho Z}{\partial t} + \frac{\partial \rho u_j Z}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} \right) dV$$

- As noted earlier, CVFEM can be recovered by applying the piecewise-constant test function, w:

$$- \sum_{elem} \tau u_k A_k \left(\frac{\partial \rho Z}{\partial t} + \frac{\partial \rho u_j Z}{\partial x_j} - \frac{\partial}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} \right)$$

- Godunuv's theorem states that a linear stabilization approach is not sufficient to damp out all oscillations

Linear Residual-Based Stabilization for CVFEM

- Is there a better way that either un-stabilized (oscillatory and non-monotonic) and dissipative upwind?
- Consider the model advection/diffusion PDE in the context of a FEM

$$\int w \frac{\partial \rho Z}{\partial t} dV + \int w \frac{\partial \rho u_j Z}{\partial x_j} dV + \int \frac{\partial w}{\partial x_j} \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} dV - \int w \frac{\mu}{Sc} \frac{\partial Z}{\partial x_j} n_j dS = 0$$

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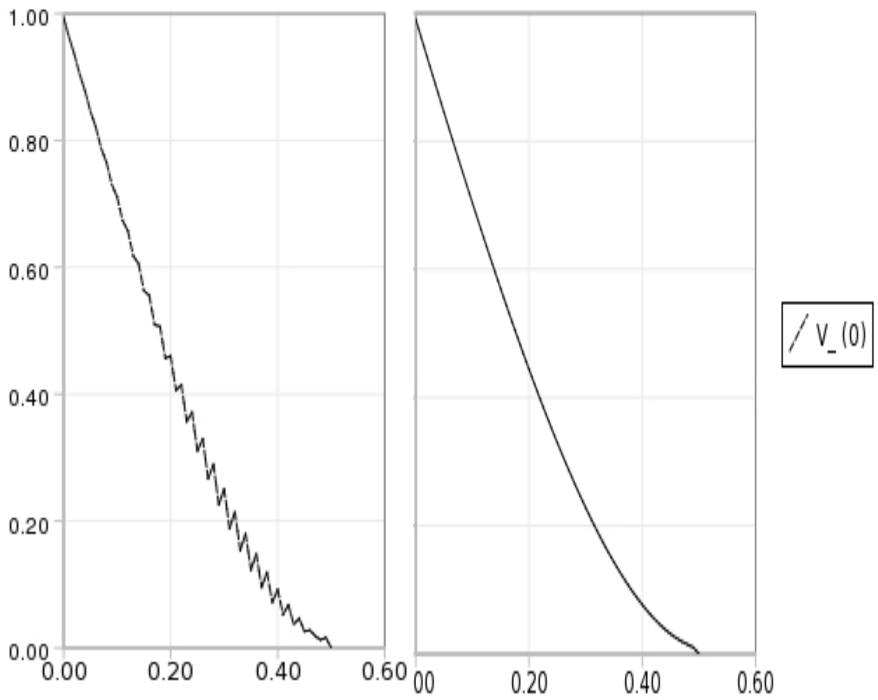
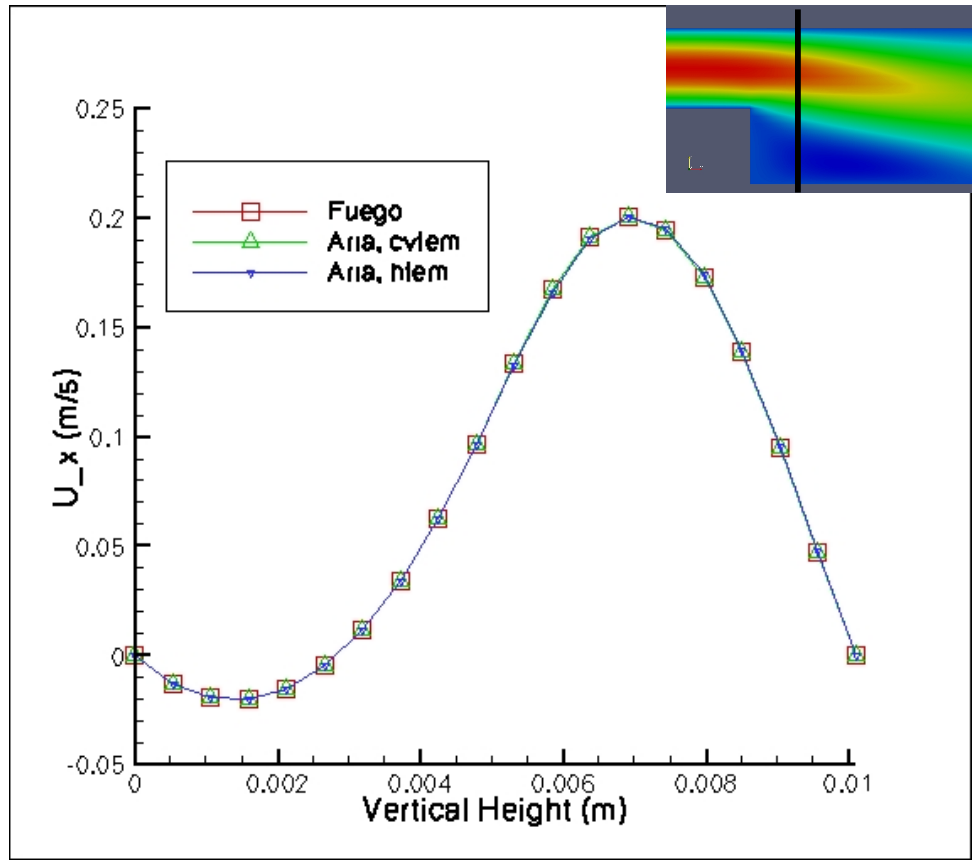
SUPG in Practice (CVFEM and FEM)

- SUPG and analog in CVFEM, Streamwise Upwind Control Volume (SUCV) are effective for low-moderate cell-Peclet numbers



Un-stabilized and SUPG



SUPG/SUCV backstep velocity prediction

Nonlinear Stabilization Approach

- Consider a general transport equation for ϕ
- An artificial viscosity approach is provided with coefficient again related to a fine-scale residual:

$$\sum_e \int_\Omega \nu(\mathbf{R}) \frac{\partial w}{\partial x_i} g^{ij} \frac{\partial \phi}{\partial x_j} d\Omega \qquad g^{ij} = \frac{\partial x_i}{\partial \xi_k} \frac{\partial x_j}{\partial \xi_k} \qquad g_{ij} = \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_k}{\partial x_j}$$

FEM-based DCO (Shakib, 91) — Co-variant and contra-variant metric tensors

$$\nu(\mathbf{R}) = \sqrt{\frac{\mathbf{R_k R_k}}{\frac{\partial \phi}{\partial x_i} g^{ij} \frac{\partial \phi}{\partial x_j}}}$$ or min….. $$\nu^{upw} = C_{upw} (\rho^2 u_i g_{ij} u_j)^{\frac{1}{2}}$$

Coefficient based on local residual — Limit the coefficient based on an upwind analogy

- Piecewise-constant test function for CVFEM (with fourth order-form)

$$-\sum_e \int_\Gamma \nu(\mathbf{R}) g^{ij} (\frac{\partial \phi}{\partial x_j} - G_j \phi) n_i dS$$

- Similar concept to Guermond's "Entropy-viscosity" approach, 2013
- Can also form an interesting LES model when the viscosity is a function of the k.e. fine scale residual (Guermond and Larios, 2015)

Advection Operators: Conclusion

- Without diagonal dominance, lack of monotonicity is noted
- Upwind approaches can help, however, comes at the price of numerical accuracy
- Residual-based approaches, which stem from the FEM literature, are useful candidates for CVFEM