Constructive Analysis in Condensed Type Theory

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1 Introduction

This paper outlines foundational aspects of Condensed Mathematics and its formulation within a Condensed Type Theory. We begin by motivating the need for condensed sets through examples, highlighting certain desirable categorical constructions that are absent or ill-behaved in the category of topological abelian groups. The category of condensed sets is introduced as a framework designed to rectify these deficiencies.

The primary objective of this project is to investigate mathematical structures, particularly the real numbers and their analytic properties, through the lens of Condensed Type Theory. The axioms of this type theory are outlined in Section 5. Within this framework, we define the Dedekind real numbers and construct a total order upon them. A key goal is to explore the extent to which classical theorems of real analysis can be proven constructively, potentially leveraging axioms such as Dependent Choice. Specifically, we aim to investigate Brouwer's continuity principle, which posits that all functions from real numbers to real numbers are continuous. We will also discuss results that have been formalized in the Lean4 proof assistant and outline parts that are currently under formalization.

2 Background: Topological Spaces and Their Limitations

Topological spaces provide a general framework for notions of "nearness" or "continuity." Different aspects of nearness are captured by various topological properties. For instance, homotopy theory explores connectedness and path-connectedness, while the concept of convergence of sequences or nets deals with points becoming arbitrarily close.

However, when algebraic structures are endowed with topologies, the interplay can be subtle. The category of topological groups, and specifically topological abelian groups (**TopAb**), while natural, lacks some of the robust categorical properties needed for advanced homological algebra or for a smooth theory of "spaces" in certain arithmetic contexts. This motivates the search for alternative frameworks.

3 Abelian Categories

To understand the desired properties lacking in **TopAb**, we first recall the definition of an abelian category. An abelian category $\mathcal{A}$ is a category where hom-sets are abelian groups and notions like kernels, cokernels, images, and coimages behave well, allowing for a robust theory of exact sequences.

Definition 3.1. *A category $\mathcal{C}$ is **pre-additive** if for any two objects $X, Y \in \mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group, and composition of morphisms is bilinear: $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$ and $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$.*

Example 3.1. *The category $\mathbf{Mod}(\mathbb{Z})$ (or **Ab**, the category of abelian groups) is a pre-additive category. More generally, $\mathbf{Mod}(R)$ for any ring R is pre-additive.*

Definition 3.2. *A **zero object** 0 in a category $\mathcal{C}$ is an object that is both initial and terminal. In a pre-additive category, this means $\mathrm{Hom}_{\mathcal{C}}(X, 0)$ and $\mathrm{Hom}_{\mathcal{C}}(0, X)$ are trivial groups for any X, and the unique map $X \to 0 \to Y$ is the zero morphism.*

Definition 3.3. *In a pre-additive category $\mathcal{C}$ with a zero object, the **biproduct** of two objects X_1, X_2, denoted $X_1 \oplus X_2$, is an object equipped with morphisms $p_j : X_1 \oplus X_2 \to X_j$ (projections) and $i_j : X_j \to X_1 \oplus X_2$ (injections) for $j = 1, 2$, such that $p_j \circ i_k = \delta_{jk} \cdot \mathrm{id}_{X_k}$ (where δ_{jk} is the Kronecker delta) and $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{X_1 \oplus X_2}$. If biproducts exist, then $X_1 \oplus X_2$ serves as both the product $X_1 \times X_2$ and the coproduct $X_1 \sqcup X_2$.*

Definition 3.4. *A pre-additive category $\mathcal{C}$ is an **additive category** if it has a zero object and all finite biproducts (i.e., biproducts for any pair of objects).*

Remark. *An equivalent condition for item 3 in your original definition of an additive category: In a pre-additive category with finite products and finite coproducts, there's a canonical morphism $r : X_1 \sqcup X_2 \to X_1 \times X_2$ defined by $p_j \circ r \circ i_k = \delta_{jk} \cdot \mathrm{id}_{X_k}$. The category is additive if this canonical morphism is always an isomorphism. If a pre-additive category has all finite products and a zero object, it automatically has all finite coproducts, and they form biproducts, making it additive.*

Definition 3.5. *Let $\mathcal{C}, \mathcal{C}'$ be pre-additive categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is **additive** if for all $X, Y \in \mathcal{C}$, the map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ (given by $f \mapsto F(f)$) is a homomorphism of abelian groups.*

Lemma 3.2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be additive categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is additive if and only if it preserves finite biproducts.*

Remark. *The statement "F is an additive functor if F is fully faithful" is incorrect. For example, the inclusion of a full subcategory might be fully faithful but not preserve biproducts if they differ in the subcategory versus the larger category. Additivity is about preserving the algebraic structure on Hom-sets.*

Example 3.3. *The category* $\mathbf{Mod}(R)$ *of* R*-modules and* $\mathbf{Ban}_{\mathbb{C}}$ *of complex Banach spaces with continuous linear maps are both additive categories.*

In an additive category $\mathcal{C}$:

- The **kernel** of a morphism $f : X \to Y$ is an object $\ker(f)$ with a morphism $k : \ker(f) \to X$ such that $f \circ k = 0$, and for any $k' : Z \to X$ with $f \circ k' = 0$, there's a unique $u : Z \to \ker(f)$ such that $k \circ u = k'$. This is equivalent to $\ker(f)$ being the pullback of f along the zero map $0 \to Y$.
- The **cokernel** of $f : X \to Y$ is an object $\operatorname{coker}(f)$ with a morphism $q : Y \to \operatorname{coker}(f)$ such that $q \circ f = 0$, and for any $q' : Y \to Z$ with $q' \circ f = 0$, there's a unique $v : \operatorname{coker}(f) \to Z$ such that $v \circ q = q'$. This is the dual notion to kernel.

Definition 3.6. *An additive category* $\mathcal{A}$ *is* ***abelian*** *if:*

1. *Every morphism has a kernel and a cokernel.*
2. *For every morphism* f*, the canonical map* $\operatorname{coim}(f) \to \operatorname{im}(f)$ *is an isomorphism. (Where* $\operatorname{coim}(f) := \operatorname{coker}(\ker(f) \to X)$ *and* $\operatorname{im}(f) := \ker(Y \to \operatorname{coker}(f))$*).*

Equivalently, every monomorphism is a kernel of its cokernel, and every epimorphism is a cokernel of its kernel.

Example 3.4. $\mathbf{Mod}(R)$ *is an abelian category. However,* $\mathbf{Ban}_{\mathbb{C}}$ *is not abelian: for instance, not every continuous linear monomorphism between Banach spaces has a closed image (which would be required for it to be a kernel in the categorical sense, as kernels are often expected to be "embeddings" respecting the structure). More precisely, a continuous linear injective map* $f : X \to Y$ *is a monomorphism. Its cokernel is* $Y/\overline{f(X)}$*. If* $f(X)$ *is not closed, then* f *is not the kernel of* $Y \to Y/f(X)$ *(this quotient might not even be Banach if* $f(X)$ *isn't closed). The issue is often that* $f : X \to Y$ *can be injective and have dense image but not be surjective, and thus* $Y \to \operatorname{coker}(f)$ *is* $Y \to 0$*, so* f *would be an epimorphism, but it's not necessarily an isomorphism. Morphisms in* $\mathbf{Ban}_{\mathbb{C}}$ *are generally not "strict" in the sense required for an abelian category.*

Abelian categories provide a good setting for homological algebra because exact sequences behave well.

4 Motivation for Condensed Mathematics

Consider the category **Top** of topological spaces and continuous maps. Many objects of interest carry both algebraic and topological structures:

1. Analytic objects: $\mathbb{R}$ or $\mathbb{C}$-valued functions, L^p spaces. Groups like $\mathbb{R}^n$ under addition with the Euclidean topology.

2. Algebraic objects: Abelian groups with the discrete topology (e.g., $\mathbb{Z}_{disc}$). Groups with p-adic topologies (e.g., $\mathbb{Z}_p, \mathbb{Q}_p$).

Let's focus on topological abelian groups (**TopAb**).

Example 4.1 (Failure of **TopAb** to be Abelian)**.** *Consider the sequence* $0 \longrightarrow \mathbb{R}_{disc} \xrightarrow{i} \mathbb{R}_{std} \longrightarrow C \longrightarrow 0$*, where* $\mathbb{R}_{disc}$ *is* $\mathbb{R}$ *with the discrete topology,* $\mathbb{R}_{std}$ *is* $\mathbb{R}$ *with the standard topology, and* i *is the identity map on underlying sets (which is continuous). If this were a short exact sequence in an abelian category,* C *would be* $\mathrm{coker}(i)$*. The universal property of* $\mathrm{coker}(i)$ *means that for any* $D \in \mathbf{TopAb}$*,* $\mathrm{Hom}_{\mathbf{TopAb}}(C, D) \cong \{\phi \in \mathrm{Hom}_{\mathbf{TopAb}}(\mathbb{R}_{std}, D) \mid \phi \circ i = 0\}$*. Since* i *maps* $\mathbb{R}_{disc}$ *to* $\mathbb{R}_{std}$*,* $\phi \circ i = 0$ *means* $\phi(x) = 0$ *for all* $x \in \mathbb{R}$*. As* $\mathbb{R}_{disc}$ *(as a set) is just* $\mathbb{R}$*, this means* ϕ *must be the zero map. Thus,* $\mathrm{Hom}_{\mathbf{TopAb}}(C, D) = \{0\}$ *for all* D*. By the Yoneda Lemma, this implies* $C \cong 0$ *(the trivial group). If* $C \cong 0$*, then the sequence implies* $i : \mathbb{R}_{disc} \to \mathbb{R}_{std}$ *is an isomorphism in* **TopAb***. However,* i *is a continuous bijection but its inverse (identity map from* $\mathbb{R}_{std}$ *to* $\mathbb{R}_{disc}$*) is not continuous. So* i *is not a homeomorphism, hence not an isomorphism in* **TopAb***. This shows that* i *is a monomorphism and an epimorphism (since its cokernel is 0), but not an isomorphism. Categories where this can happen are called "balanced," but* **TopAb** *is not, and this is a symptom of it not being abelian. The cokernel object* $\mathbb{R}_{std}/\mathbb{R}_{disc}$ *in* **TopAb** *is the trivial group, which is not what one might intuitively expect if* $\mathbb{R}_{disc}$ *were a "normal subgroup object."*

The category **CHaus** of compact Hausdorff spaces has better properties in some respects (e.g., quotients by closed equivalence relations are well-behaved). However, it excludes many important spaces, such as infinite discrete spaces (e.g., $\mathbb{Z}_{disc}$) or non-compact spaces like $\mathbb{R}_{std}$.

Previous attempts to find a "better" category of topological spaces, like Johnstone's topological topos, have had their own limitations. For example, the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$ for a topological topos $\mathcal{E}$ might not preserve infinite products, which is something we'd like to have. Condensed mathematics presents a new perspective.

5 Condensed Sets

The core idea of condensed mathematics is to define a space not by its internal point-set topology directly, but by how it interacts with well-behaved compact Hausdorff spaces, specifically profinite sets.

Definition 5.1. *A* ***profinite set*** *is a topological space that is an inverse limit of finite discrete sets. Equivalently, it is a compact, Hausdorff, and totally disconnected space. Let* **ProFin** *be the category of profinite sets and continuous maps.*

Definition 5.2. *A **condensed set** is a sheaf $T : (\mathbf{ProFin})^{\mathrm{op}} \to \mathbf{Set}$. This means T is a functor satisfying:*

- *$T(\emptyset) = *$ (a terminal object, i.e., a singleton set).*
- *For any finite disjoint union (coproduct) $S_1 \sqcup S_2$ of profinite sets, the canonical map $T(S_1 \sqcup S_2) \to T(S_1) \times T(S_2)$ is a bijection.*
- *For any surjective map $p : S' \twoheadrightarrow S$ of profinite sets (which is an epimorphism in **ProFin** and corresponds to a covering family if S' is a disjoint union of S'_i mapping to S), the sequence*

$$T(S) \longrightarrow T(S') \rightrightarrows T(S' \times_S S')$$

*is an equalizer diagram in **Set**. ($S' \times_S S'$ is the kernel pair of p).*

*The category of condensed sets is denoted **CondSet**.*

Example 5.1. *Let Q be any topological space. We can associate a condensed set $\underline{Q}$ to it, defined by*

$$\underline{Q}(S) := \mathrm{Hom}_{\mathbf{Top}}(S, Q)$$

for any profinite set S. The functoriality and sheaf conditions follow from properties of continuous maps and profinite sets. If Q is itself a profinite set, then $\underline{Q} \cong \mathrm{Hom}_{\mathbf{ProFin}}(-, Q)$, which is representable.

A **condensed abelian group** is a condensed set A such that $A(S)$ is an abelian group for every $S \in \mathbf{ProFin}$, and for every map $f : S_1 \to S_2$ in **ProFin**, the map $A(f) : A(S_2) \to A(S_1)$ is a group homomorphism. The category **CondAb** of condensed abelian groups is an abelian category with excellent properties (e.g., it has enough injectives and projectives).

6 Condensed Type Theory

We now sketch a type theory designed to internalize concepts from condensed mathematics.

6.1 Predicates and Subuniverses

We introduce two predicates on types:

1. $isCHaus$: Type → Prop (intended to mean the type is compact Hausdorff)
2. $isODisc$: Type → Prop (intended to mean the type is "overt discrete")

An overt space X is one where the predicate "is U inhabited?" for $U \subseteq X$ open is well-behaved constructively (e.g., for any Y, and open $U \subseteq X \times Y$, the set $\{y \in Y \mid \exists x \in X, (x, y) \in U\}$ is open in Y). For discrete spaces, overtness often implies decidable equality and that any proposition about its elements is overt.

We define subuniverses:

1. $CHaus := \{A : \text{Type} \mid isCHaus(A)\}$
2. $ODisc := \{A : \text{Type} \mid isODisc(A)\}$

These subuniverses are postulated to be closed under certain type constructions:

6.2 Natural Numbers

$\mathbb{N}$ is overt and discrete.

$$\frac{}{isODisc(\mathbb{N})}\ \text{(N-ODisc)}$$

6.3 Equality (Identity Types)

If X is CHaus (resp. ODisc), then the identity type $a = b$ for $a, b : X$ is CHaus (resp. ODisc). Since identity types are propositions, this typically means they are discrete.

$$\frac{isCHaus(X) \qquad a, b : X}{isCHaus(a =_X b)}\ \text{(Eq-CHaus)}$$

$$\frac{isODisc(X) \qquad a, b : X}{isODisc(a =_X b)}\ \text{(Eq-ODisc)}$$

(Note: $a =_X b$ is a type, its elements are proofs of equality. Its $isCHaus/isODisc$ property usually means it's **0** or **1** in a CHaus/ODisc way).

6.4 Finite Types (Products of Unit Type)

Let $[n]$ denote a canonical finite type with n elements (e.g., $\text{Fin}(n)$).

$$\frac{n : \mathbb{N}}{isCHaus([n])}\ \text{(Fin-CHaus)}$$

$$\frac{n : \mathbb{N}}{isODisc([n])}\ \text{(Fin-ODisc)}$$

6.5 Equalisers (as Sigma Types)

The equaliser of $f, g : X \to Y$ is $\Sigma_{x:X}(f(x) =_Y g(x))$.

$$\frac{isCHaus(X) \qquad isCHaus(Y) \qquad f, g : X \to Y}{isCHaus(\Sigma_{x:X}(f(x) =_Y g(x)))}\ \text{(Eql-CHaus)}$$

$$\frac{isODisc(X) \qquad isODisc(Y) \qquad f, g : X \to Y}{isODisc(\Sigma_{x:X}(f(x) =_Y g(x)))}\ \text{(Eql-ODisc)}$$

6.6 Quotients

Given a type X and an equivalence relation $\sim$ on X.

$$\frac{isCHaus(X) \qquad \forall x, y : X,\ isCHaus(x \sim y)}{isCHaus(X/\sim)} \text{ (Quot-CHaus)}$$

(The condition $isCHaus(x \sim y)$ means the relation is "closed".)

$$\frac{isODisc(X) \qquad \forall x, y : X,\ isODisc(x \sim y)}{isODisc(X/\sim)} \text{ (Quot-ODisc)}$$

6.7 Isomorphism (Equivalence of Types)

$$\frac{isCHaus(Y) \qquad X \simeq Y}{isCHaus(X)} \text{ (Iso-CHaus)}$$

$$\frac{isODisc(Y) \qquad X \simeq Y}{isODisc(X)} \text{ (Iso-ODisc)}$$

6.8 Sigma Formation (Dependent Pair Types)

$$\frac{isCHaus(X) \qquad \forall x : X,\ isCHaus(Y(x))}{isCHaus(\Sigma_{x:X} Y(x))} \text{ (Σ-CHaus)}$$

$$\frac{isODisc(X) \qquad \forall x : X,\ isODisc(Y(x))}{isODisc(\Sigma_{x:X} Y(x))} \text{ (Σ-ODisc)}$$

6.9 Pi Formation Rules (Dependent Function Types)

$$\frac{isODisc(X) \qquad \forall x : X,\ isCHaus(Y(x))}{isCHaus(\Pi_{x:X} Y(x))} \text{ (Π-CHaus-ODisc)}$$

$$\frac{isCHaus(X) \qquad \forall x : X,\ isODisc(Y(x))}{isODisc(\Pi_{x:X} Y(x))} \text{ (Π-ODisc-CHaus)}$$

6.10 Collection Axioms

These assert a form of projectivity or "smallness" condition. Let $X : \text{Type}_a$, $Y : \text{Type}_b$ (potentially in different universes). Axiom (Coll-CHaus):

$$\frac{X : \text{Type}_a \qquad isCHaus(Y) \qquad f : X \twoheadrightarrow Y \text{ (surjective)}}{\exists (Y' : \text{Type}_c), isCHaus(Y'), \exists (g : Y' \twoheadrightarrow Y), \exists (s : Y' \to X),\ \text{s.t. } f \circ s = g}$$

(This says any CHaus type Y which is a quotient of X is also a quotient of some CHaus type Y' that maps into X compatibly.)

Axiom (Coll-ODisc): (Similar for ODisc)

$$\frac{X : \text{Type}_a \qquad isODisc(Y) \qquad f : X \twoheadrightarrow Y \text{ (surjective)}}{\exists (Y' : \text{Type}_c), isODisc(Y'), \exists (g : Y' \twoheadrightarrow Y), \exists (s : Y' \to X),\ \text{s.t. } f \circ s = g}$$

6.11 Axiom F (Factorization)

Functions from CHaus to ODisc factor through a finite type.

$$\frac{isCHaus(X) \qquad isODisc(Y) \qquad f : X \to Y}{\exists(n : \mathbb{N}), \exists(\alpha : X \to [n]), \exists(\beta : [n] \to Y),\ \text{s.t. } \beta \circ \alpha = f}$$

6.12 Axiom G (Generation)

Any type X can be "covered" by CHaus types indexed by an ODisc type. For any $X : \mathrm{Type}_a$, there exists $I : \mathrm{Type}_a$ such that $isODisc(I)$, and a type family $K : I \to \mathrm{Type}_a$ such that for all $i : I$, $isCHaus(K(i))$, and there is a surjective map $f : (\Sigma_{i:I} K(i)) \twoheadrightarrow X$.

6.13 Axiom SC (Scott Continuity / Finite Support)

Functions from certain Pi types into finite types depend on finitely many coordinates. Given $Y : \mathrm{Type}_a$ with $isODisc(Y)$, and $X : Y \to \mathrm{Type}_b$ such that for all $y : Y$, $isCHaus(X(y))$. Let $P = \Pi_{y:Y} X(y)$. For any function $f : P \to [n]$ (where $[n]$ is a finite type), there exists a finite subtype $Y_{fin} \subseteq Y$ (e.g. given by $F : [m] \to Y$ for some $m : \mathbb{N}$) and a function $f' : (\Pi_{y:Y_{fin}} X(y)) \to [n]$ such that for any $x : P$, $f(x) = f'(x|_{Y_{fin}})$. (Your formulation was close; this is a more standard way to state finite support).

7 Real Numbers in Condensed Type Theory

Our axioms state $isODisc(\mathbb{N})$. We define integers $\mathbb{Z}$ from $\mathbb{N}$ (e.g., as pairs of naturals modulo an equivalence relation, or $\mathbb{N} \sqcup \mathbb{N}$ for positive/negative). $\mathbb{Z}$ will also be ODisc. Rationals $\mathbb{Q}$ are defined as pairs (num, den) where $num : \mathbb{Z}$, $den : \mathbb{N}^{>0}$ (positive naturals), with num and den coprime (or via equivalence classes of pairs (a, b) with $b \neq 0$). $\mathbb{Q}$ is ODisc.

A Dedekind real number x is a pair of predicates (L_x, U_x) on $\mathbb{Q}$, $L_x, U_x : \mathbb{Q} \to \text{Prop}$, satisfying:

1. **Boundedness**: $\exists q_1, q_2 \in \mathbb{Q}, L_x(q_1) \wedge U_x(q_2)$.
2. **Downward Closure of L**: $\forall q_1, q_2 \in \mathbb{Q}, (L_x(q_2) \wedge q_1 < q_2) \implies L_x(q_1)$.
3. **Upward Closure of U**: $\forall q_1, q_2 \in \mathbb{Q}, (U_x(q_1) \wedge q_1 < q_2) \implies U_x(q_2)$.
4. **Roundedness of L**: $\forall q \in \mathbb{Q}, L_x(q) \implies \exists q' \in \mathbb{Q}, q < q' \wedge L_x(q')$.
5. **Roundedness of U**: $\forall q \in \mathbb{Q}, U_x(q) \implies \exists q' \in \mathbb{Q}, q' < q \wedge U_x(q')$.
6. **Locatedness**: $\forall q_1, q_2 \in \mathbb{Q}, q_1 < q_2 \implies L_x(q_1) \vee U_x(q_2)$.
7. **Disjointness**: $\forall q \in \mathbb{Q}, \neg(L_x(q) \wedge U_x(q))$.

Let $\mathbb{R}_{Ded}$ denote the type of Dedekind real numbers. When we write $r \in \mathbb{R}_{Ded}$, we mean $r = (L_r, U_r)$. Order: $x < y \iff \exists q \in \mathbb{Q}, U_x(q) \wedge L_y(q)$.

Lemma 7.1 (Negation)**.** *For $x \in \mathbb{R}_{Ded}$, its negation $-x$ is defined by $L_{-x}(q) \iff U_x(-q)$ and $U_{-x}(q) \iff L_x(-q)$.*

Lemma 7.2. *If $x \in \mathbb{R}_{Ded}$, then $x = -(-x)$.*

Theorem 7.3. *For any $x : \mathbb{R}_{Ded}$ and $q : \mathbb{Q}$, the proposition $L_x(q)$ is ODisc (i.e., $isODisc(L_x(q))$ holds). Similarly for $U_x(q)$.*

Proof Sketch. This typically follows from $\mathbb{Q}$ being ODisc and the definition of $L_x(q)$ (and $U_x(q)$) involving quantifiers over $\mathbb{Q}$ and comparisons, which are assumed to preserve ODisc status for propositions. For instance, if $P(q')$ is ODisc, then $\exists q' > q, P(q')$ can be ODisc under suitable axioms for ODisc types (closure under existential quantification if the domain is ODisc and the predicate is ODisc).

In order to give a more detailed proof, we would need a precise formulation of ODisc propositions and how they handle quantifiers over ODisc types like $\mathbb{Q}$. $isODisc(P)$ for a proposition P means $P \vee \neg P$ holds and that P itself can be treated as an ODisc type (e.g., mapping to $\mathbf{1}$ or $\mathbf{0}$ which are ODisc).

The constructive nature here means that $L_x(q)$ should be a decidable predicate. $\square$

8 Progress

We have established that $\mathbb{R}_{Ded}$ forms an ordered field. This involved defining addition, multiplication, and their respective identities and inverses, and proving properties related to the order relations $<$ and $\leq$ (where $x \leq y \iff \neg(y < x)$). Equality $x = y$ is $x \leq y \wedge y \leq x$. Intervals and basic interval arithmetic have also been developed.

9 Problem: Constructive Continuity

We aim to prove a version of Brouwer's continuity principle, typically stating that any function $f : \mathbb{R} \to \mathbb{R}$ is continuous. As a step, we often prove that functions on compact intervals are uniformly continuous. We assume $isCHaus([0, 1])$ as an axiom (or a theorem derived from $\mathbb{R}_{Ded}$ being CHaus and $[0, 1]$ being a closed subset). Let $I = [-1, 1]$. We assume $isCHaus(I)$.

Let U be an open subset of I such that $0 \in U$. (Here, $U : I \to$ OProp where OProp is the type of overt propositions; $x \in U$ means $U(x)$ holds). Define a family of subsets $W(j)$ for $j \in \mathbb{N}^+ \sqcup \{*\}$:

- For $n \in \mathbb{N}^+$, $W(n) := \{r \in I \mid r < -1/n \vee r > 1/n\}$.
- $W(*) := U$.

Each $W(j)$ is open in I. We want to show that I is covered by a finite subfamily of these $W(j)$'s.

9.1 First Claim: The Family Covers I

Lemma 9.1. $\bigcup_{j \in \mathbb{N}^+ \sqcup \{*\}} W(j) = I$. *(Meaning $\forall r \in I, \exists j \in \mathbb{N}^+ \sqcup \{*\}, r \in W(j)$).*

Proof. Let $r \in I$. Assume for contradiction that $r \notin W(j)$ for all $j \in \mathbb{N}^+ \sqcup \{*\}$. Then $r \notin W(*)$, so $r \notin U$. Also, for any $n \in \mathbb{N}^+$, $r \notin W(n)$. This means $\neg(r < -1/n \vee r > 1/n)$. So, for all $n \in \mathbb{N}^+$, $r \not< -1/n$ AND $r \not> 1/n$. This implies $-1/n \leq r \leq 1/n$ for all $n \in \mathbb{N}^+$. As $n \to \infty$, $1/n \to 0$ and $-1/n \to 0$. In constructive real numbers, $\forall n \in \mathbb{N}^+, (-1/n \leq r \leq 1/n)$ implies $r = 0$. (Proof of $r = 0$: Suppose $r > 0$. Then $\exists k \in \mathbb{N}^+$ such that $1/k < r$. This contradicts $r \leq 1/k$. So $\neg(r > 0)$, i.e. $r \leq 0$. Suppose $r < 0$. Then $\exists k \in \mathbb{N}^+$ such that $r < -1/k$. This contradicts $-1/k \leq r$. So $\neg(r < 0)$, i.e. $r \geq 0$. Since $r \leq 0$ and $r \geq 0$, we have $r = 0$.) So, $r = 0$. But we are given that $0 \in U$, so $0 \in W(*)$. This contradicts our assumption that $r \notin W(j)$ for any j. Thus, the assumption is false, and the family $\{W(j)\}$ covers I. $\square$

9.2 Second Claim: Finite Subcover Exists

We now use the axioms of Condensed Type Theory to show there's a finite M such that $I = U \cup \bigcup_{n=1}^{M} W(n)$.

Let $S := \mathbb{N}^+ \sqcup \{*\}$. By $isODisc(\mathbb{N})$ and $isODisc([1])$ (for $\{*\}$), S is ODisc. Consider the type $X_{cover} := \Sigma_{j:S} W(j)$. The first component projection is $p_1 : X_{cover} \to S$. The second component gives context: if $(j, x) \in X_{cover}$, then $x \in W(j)$. There is a canonical map $u : X_{cover} \to I$ given by $u(j, x) := x$. By Lemma 9.1, u is surjective. So, $u : (\Sigma_{j:S} W(j)) \twoheadrightarrow I$. We have $isCHaus(I)$. We are in the situation of the Collection Axiom (Coll-CHaus). Let $X_{univ} = \Sigma_{j:S} W(j)$. (Note: X_{univ} is not necessarily CHaus itself, as S is infinite ODisc and $W(j)$ are open, not necessarily CHaus). The Collection Axiom (Coll-CHaus) states: Given X_{some} : Type, $isCHaus(Y_{CH})$, $f : X_{some} \twoheadrightarrow Y_{CH}$, then $\exists (Y'_{CH} : \text{Type}), isCHaus(Y'_{CH}), \exists (g : Y'_{CH} \twoheadrightarrow Y_{CH}), \exists (s : Y'_{CH} \to X_{some})$, s.t. $f \circ s = g$.

Let's apply Axiom G first to I. Axiom G states that for I (which is CHaus, or we can consider it as a general type for a moment), there exists an ODisc type I_{idx} and a family $K : I_{idx} \to \text{Type}$ with each $K(i)$ CHaus, and a surjection $\pi : \Sigma_{i:I_{idx}} K(i) \twoheadrightarrow I$. Since I is already CHaus, we can take $I_{idx} = [1]$ (a singleton ODisc type), $K(*) = I$, so $\Sigma K(i) = I$, and $\pi = id_I$. This fits Axiom G trivially.

We have the surjective map $u : \Sigma_{j:S} W(j) \twoheadrightarrow I$. Here, $Y = I$ is CHaus. $X = \Sigma_{j:S} W(j)$. By the Collection Axiom (Coll-CHaus) for $u : X \twoheadrightarrow I$: There exists a type X' : Type such that $isCHaus(X')$, and there exist maps:

- $p : X' \twoheadrightarrow I$ (surjective)
- $r : X' \to \Sigma_{j:S} W(j)$

such that $u \circ r = p$.

$$\begin{tikzcd}
\Sigma_{j:S} W(j) \arrow[r, two heads, "u"'] & I \\
X' \arrow[u, "r"] \arrow[ur, two heads, "p"'] &
\end{tikzcd}$$

Now consider the map $q_{idx} : \Sigma_{j:S} W(j) \to S$, which projects to the index j. Compose r with q_{idx}: $q_{idx} \circ r : X' \to S$. We have:

- $isCHaus(X')$
- $isODisc(S)$ (since $S = \mathbb{N}^+ \sqcup \{*\}$ and $\mathbb{N}$ is ODisc)

By Axiom F (Factorization), the map $q_{idx} \circ r : X' \to S$ must factor through a finite type $[k]$ for some $k : \mathbb{N}$. So there exist $\alpha : X' \to [k]$ and $\beta : [k] \to S$ such that $\beta \circ \alpha = q_{idx} \circ r$.

$$\begin{tikzcd}
& {[k]} \arrow[dr, "\beta"] & \\
X' \arrow[ur, "\alpha"] \arrow[rr, "q_{idx} \circ r"'] & & S
\end{tikzcd}$$

Let $S_{fin} := \text{Im}(\beta) = \{\beta(i) \mid i \in [k]\} \subseteq S$. S_{fin} is a finite subset of S. So, for any $x' \in X'$, $(q_{idx} \circ r)(x')$ lands in S_{fin}. Let $y \in I$. Since $p : X' \twoheadrightarrow I$ is surjective, there exists $x' \in X'$ such that $p(x') = y$. Then $u(r(x')) = p(x') = y$. Let $r(x') = (j_0, x_0) \in \Sigma_{j:S} W(j)$. So $j_0 = q_{idx}(r(x'))$ and $x_0 \in W(j_0)$. We know $j_0 \in S_{fin}$. And $u(r(x')) = x_0 = y$. So $y = x_0 \in W(j_0)$ for some $j_0 \in S_{fin}$. This means that for any $y \in I$, $y \in \bigcup_{j \in S_{fin}} W(j)$. Therefore, $I = \bigcup_{j \in S_{fin}} W(j)$. Since S_{fin} is a finite subset of $\mathbb{N}^+ \sqcup \{*\}$, this establishes that I is covered by a finite number of the $W(j)$'s. If $\{*\} \in S_{fin}$, then U is in the finite subcover. Let $N_{max} = \max(\{n \mid n \in S_{fin} \cap \mathbb{N}^+\} \cup \{0\})$. Then $I = W(*) \cup \bigcup_{n=1, n \in S_{fin}}^{N_{max}} W(n)$ (if $W(*)$ is needed) or $I = \bigcup_{n=1, n \in S_{fin}}^{N_{max}} W(n)$ (if $W(*)$ is not needed). In either case, it's a finite subcover.

Claim 1. *$I = \bigcup_{j \in S_{fin}} W(j)$, where S_{fin} is a finite subset of $S = \mathbb{N}^+ \sqcup \{*\}$.*

Proof. As derived above: 1. The family $\{W(j) \mid j \in S\}$ covers I, meaning the map $u : \Sigma_{j:S} W(j) \twoheadrightarrow I$ is surjective. 2. By the Collection Axiom (Coll-CHaus), since I is CHaus, there exists a CHaus type X' and maps $p : X' \twoheadrightarrow I$ (surjective) and $r : X' \to \Sigma_{j:S} W(j)$ such that $u \circ r = p$. 3. Let $q_{idx} : \Sigma_{j:S} W(j) \to S$ be the projection to the index. The composite $q_{idx} \circ r : X' \to S$ is a map from a CHaus type to an ODisc type (S). 4. By Axiom F, $q_{idx} \circ r$ factors as $X' \xrightarrow{\alpha} [k] \xrightarrow{\beta} S$ for some finite type $[k]$. 5. Let $S_{fin} = \mathrm{Im}(\beta) \subseteq S$. S_{fin} is finite. 6. For any $y \in I$: Since p is surjective, $\exists x' \in X'$ such that $p(x') = y$. Then $y = p(x') = u(r(x'))$. Let $r(x') = (j_0, x_0)$, where $j_0 \in S$ and $x_0 \in W(j_0)$. Then $u(r(x')) = x_0$. So $y = x_0$. The index j_0 is $q_{idx}(r(x'))$. Since $q_{idx}(r(x')) = \beta(\alpha(x'))$, we have $j_0 \in S_{fin}$. So $y = x_0 \in W(j_0)$ for some $j_0 \in S_{fin}$. 7. Thus, $\forall y \in I, y \in \bigcup_{j \in S_{fin}} W(j)$. This means $I = \bigcup_{j \in S_{fin}} W(j)$. This completes the proof that I has a finite subcover from the original family $\{W(j)\}_{j \in S}$. □

This result is a key step (Heine-Borel property for $I = [-1, 1]$ shown via these axioms) towards proving continuity properties. For example, to show $f : I \to \mathbb{R}$ is uniformly continuous, one might cover I by small open balls $B(x_i, \delta_i)$ such that f varies by less than ϵ on $B(x_i, \delta_i)$, then extract a finite subcover. The argument structure you've outlined is indeed how one might use these abstract axioms (CHaus, ODisc, Collection, Factorization) to recover classical compactness arguments.