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Valuing a European Option with the Heston Model

A thesis present

by

Yuan Yang

to

The School of the Mathematical Sciences

in partial fulfillment of the requirements

for the degree of

Master of Science

in the subject of

Applied and Computational Mathematics

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Abstract

In spite of the Black-Scholes (BS) equation being widely used to price options, this method is based on a hypothesis that the volatility of the underlying is a constant. A number of scholars began to improve the formula, and they proposed to employ stochastic volatility models to predict the behavior of the volatility.

One of the results of the improvement is stochastic volatility models, which replaces the fixed volatility by a stochastic volatility process. The purpose of this dissertation is to adopt one of the famous stochastic volatility models, Heston Model (1993), to price European call options. Put option values can easily obtained by call-put parity if it is needed.

We derive a model based on the Heston model. Then, we compare it with Black-Scholes equation, and make a sensitivity analysis for its parameters.

Contents

Introduction

In modern financial analysis, due to some limitations of Black-Scholes equation, stochastic process theories are prevalent for asset pricing, especially in option pricing. Lots of mathematicians and statisticians are focusing on determining the behavior of the underlying assets in both academic and real-trading market.

Among the variety of financial derivatives, the option is one of the most important financial instruments. An option is define as the right, but not the obligation, to buy (call option) or sell (put option) a specific asset by paying a strike price on or before a specific date. Nowadays, the use of options as instrument of speculation and hedging is so widespread that, in many cases, the number of options traded much surpasses the number of shares available for that corresponding asset. There are mainly four kinds of options, including American option, European option, Asian option, and Barrier option, in current financial markets. In this dissertation, we only focus on pricing a European option, which can only be exercised on the maturity date. Therefore, the call option and put option we mention in this dissertation will stand for European call option and European put option respectively.

Black and Scholes (1973) used the following stochastic differential equation to model the random behavior of the stock:

$$dS = \mu S dt + \sigma S dW$$

Where drift μ, and volatility σ are constants, and dW shows that the process $S(t)$ follows the Wiener process. Based on this equation, we can simply calculate the price of a call option and a put option by a given function. However, one of major assumptions for B-S equation is that the volatility is a constant. In order to get the price more accurately, financial mathematicians have suggested some alternatives, such as stochastic volatility models. Among these mathematicians, Hull and White (1987), Stein-Stein (1991), and Heston(1993) are the most three famous people. Each of them has their own stochastic volatility model. We will introduce the first two models in Chapter 2, and, we will illustrate the Heston model, which was introduced by Steven L. Heston in his dissertation *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options(1993)*, in detail.

We use a numerical method to solve the Heston model by Excel-VBA, and get a new model after optimizing by Excel-Solver. To check the results of our model, we compare them with what derives from Black-Scholes equation. Finally, we do the sensitivity analysis for the Heston model.

Our goal is to:

- Use the Heston method to calculate the call option prices of shares with no dividends.
- Compare the accuracy of the Heston model's results to Black-Scholes equation's results.

We organize this dissertation as follows. Chapter 1 summarizes basic concept and theories. Chapter 2 introduces basic information of the Hull-White model and the Stein-Stein model, and, hopefully, gives readers a general idea on the study of option pricing problem. In Chapter 3, we comprehensively explain the Heston model from its background to its derivation, and we make experiment to examine its parameters. Chapter 4 calibrates a model which is based on the Heston model. Chapter 5 tests the model by comparing to Black-Scholes equation, and then we will make conclusions and describe the direction of future work.

Chapter 1 Basic Concepts

In this chapter, we explain some basic concepts which will be mentioned in this dissertation. It refers to European Call options, Black-Scholes equation, stochastic processes, stochastic volatility, Ornstein-Uhlenbeck processes and CIR process, and Ito's lemma.

1.1 European Call options

A *European call option* is a contract that gives its holder the right, but not the obligation, to buy one unit of a stock for a predetermined strike price *K* on the maturity date *T*.

Example: Consider an investor who buys a European call option with the strike price of \$100 to purchase 100 shares of a certain stock. Suppose that the current stock price is \$98, the maturity date of the option is in 4 months, and the price of an option to purchase one share is \$5. The initial investment is \$500. Since the option is European, the investor can exercise only on the maturity date. If the stock price on is less than \$100, the investor will clearly choose not to exercise. In this case, the loss of the investor is only for the \$500 initial investment. If the stock price is above \$100 on the maturity date, the option might be exercised. The payoff of a call option is shown below,

$$V(S_T) = \begin{cases} S_T - K & if\ S_T > K, \\ 0 & if\ S_T \le K. \end{cases} \tag{1.1}$$

S_T is the price of the underlying asset at maturity time T, and K is the strike price. Thus if the stock is valued at \$104 at the maturity data the profit is \$400 on the initial \$500 investment. If the investor had purchased the stock at \$98 the profit would have been a little over 6%.

1.2 Black-Scholes Equation

Black and Scholes first proposed the Black-Scholes equation in their paper '*The pricing of options and corporate liabilities' (1973)*. It brought a huge change in the financial market, and it was the first time when people knew how to make a price for an option. When the Black-Scholes equation was first published, it was under the following assumptions,

(1) The stock price S follows the stochastic process $dS = \mu S dt + \sigma S dz$, with fixed μ and σ;
(2) Unrestricted short-selling of stock, with full use of short-sale proceeds;
(3) No transactions costs and no taxes;
(4) No dividends are paid during the life of the option;
(5) There are no riskless arbitrage opportunities;
(6) Based on European options
(7) The risk-free rate of interest *r* is constant and same for all maturities
(8) Continuous trading

In order to make a price for a call option on a non-dividend paying stock with the Black-Scholes Equation, we need to know current stock price, strike price, risk-free interest rate, volatility and time to maturity. It is easy to get all above inputs variables in the market except the volatility. For the price of a non-dividend paying call option, the Black-Scholes equation is described as:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{1.2}$$

where,

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

Where S is the stock price at time t, T is the maturity date, K is the strike price, $N(d_2)$ is the cumulative normal distribution, σ is the volatility.

Although Black-Scholes equation is still widespread used in the market, much evidence has shown that the assumption of fixed volatility is not suitable for actual data. Consequently, in this dissertation, we consider the volatility following a stochastic process rather than a constant during the life of a call option.

1.3 Stochastic Processes

This part is mainly based upon *'Options, Futures, and Other Derivatives 7th edition'* (John C. Hull, 2009).

Any variable whose value changes over time in an uncertain way is said to follow a *stochastic process*. Stochastic processes can be classified as discrete time or continuous time. A discrete-time stochastic process is one where the value of the variable can change only at certain fixed points in time such as every hour, whereas a continuous-time stochastic process is one where changes can take place at any time.

A *Markov process* is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant. A *Wiener process*, sometimes known as a *Brownian motion*, is a particular case of a Markov process. We consider that a variable follow a Wiener process if:

I. The change Δz during a small period of time Δt is

$$\Delta z = \epsilon\sqrt{\Delta t}$$

Where ϵ has a standard normal distribution $\phi(0,1)$.

II. The value Δz for any two different short intervals of time, Δt, are independent.

Furthermore, the value Δz has a normal distribution with mean equal zero, standard deviation is $\sqrt{\Delta t}$, and variance is Δt.

A *generalized Wiener process* (John H.,2009) is given by the equation

$$dx = a\,dt + b\,dz \tag{1.3}$$

where a and b is constants.

Equation (1.3) can be considered as a variable, x, for Wiener process adds an expect drift rate a per unit of time and b times volatility. It can be shown that the change in the value of x in any time interval T is normally distribute with mean aT, standard deviation $b\sqrt{T}$, and variance b^2T. Figure 1.1 below is an example of Generalized Wiener process with $a = 0.2$, and $b = 1.2$.

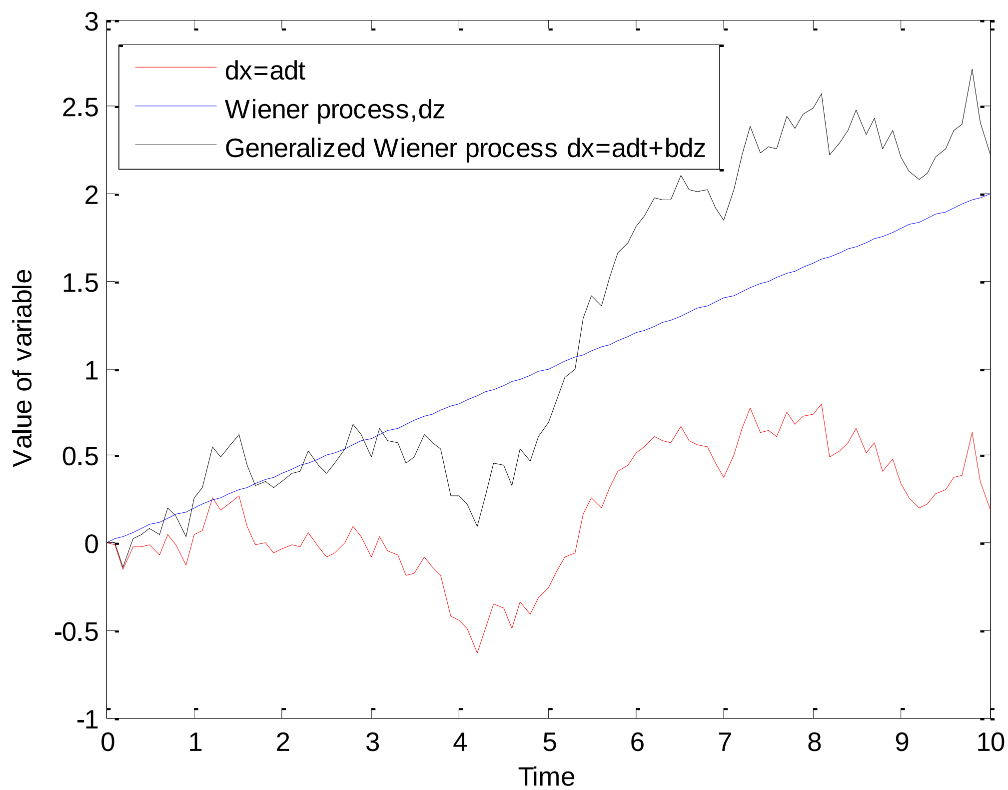

Figure 1.1: Generalized Wiener process with a=0.2 and b=1.2

1.4 Stochastic Volatility

Volatility is a measure for variation of price of a stock over time. *Stochastic volatility* is described as processes in which the return variation dynamics include an unobservable shock that cannot be predicted using current available information. Stochastic volatility models, which let the volatility follow Brownian motion, make the option price much better adapted to the realities of the market.

Generally speaking, we consider that a stock price can be described by a stochastic model if the behavior of the stock price satisfies stochastic differential equation (Fouque, Papanicolaou, Sircar, 2000):

$$\begin{cases} dX_t = \mu X_t dt + f(Y_t) X_t dW_t \\ dY_t = b(Y_t)dt + \sigma(Y_t)dZ_t \end{cases}$$

In the above equation, W and Z are standard one-dimensional Brownian motions. The stock price is modeled by X, while the volatility of the stock is described by the process *f(Y)*. μ is the mean of the stock return, and the volatility process *f(Y)* is interpreted as the standard deviation.

1.5 Ornstein-Uhlenbeck Processes and CIR Processes

The Ornstein-Uhlenbeck (O-U processes) is a stochastic process that, roughly speaking, describes the velocity of a massive Brownian particle under the influence of friction. The process is stationary, Gaussian, and Markovian, and is the only nontrivial process that satisfies these three conditions. (Ornstein, Uhlenbeck, 1930) It is an example of a mean-reverting continuous diffusion process.

Let $q > 0, m \geq 0,$ and $\sigma > 0$, Z_t is a Wiener process and consider the following stochastic differential equation:

$$dY_t = q(m - Y_t)dt + \sigma dZ_t, \quad Y_0 = y_0. \tag{1.4}$$

The process Y is called the O-U processes. Next, we want to derive the solution for an O-U process.

We solve this equation by variation of parameters.

First, suppose

$$f(Y_t, t) = Y_t e^{qt}$$

Then using Ito's lemma we get,

$$df(Y_t, t) = qe^{qt} Y_t dt + e^{qt} dY_t \tag{1.5}$$

Plug (1.4) in (1.5), and get

$$df(Y_t, t) = e^{qt} qmdt + e^{qt} \sigma dZ_t \tag{1.6}$$

Integrating both side from 0 to t, we have

$$Y_t e^{qt} = \int_0^t e^{qs} \sigma dZ_s + m(e^{qt} - 1) + y_0 \tag{1.7}$$

Therefore,

$$Y_t = \sigma \int_0^t e^{q(s-t)} dZ_s + m(1 - e^{-qt}) + y_0 e^{-qt} \tag{1.8}$$

The CIR processes (Cox JC, Ingersoll, Ross, 1985) is defined as a sum of squared Ornstein-Uhlenbeck processes. The stochastic volatility process in the Heston model follows the CIR process. It is a non-negative process, which is expressed as:

$$Y_t = q(m - Y_t)dt + c\sqrt{Y_t} dZ_t, \quad Y_0 = y_0 \tag{1.9}$$

1.6 Ito's Lemma

A random process can be defined by the following equation:

$$x(t) = x(0) + \int_0^t a(x, s) ds + \int_0^t b(x, s) dW(s) \tag{1.10}$$

where function a and b are respectively the instantaneous mean and instantaneous standard deviation. In the financial market, this equation can be thought as the security price at time t is composed of three parts: an initial price $x(0)$, the average change from history data, and an error tern which following Wiener process. Taking the derivative of (1.10), we get

$$dx = a(x,t)dt + b(x,t)dW(t) \tag{1.11}$$

Let $Y(t) = g(t,x)$, the Ito's Lemma is defined as:

$$dY = \left(\frac{\partial g}{\partial t} + a(x,t)\frac{\partial g}{\partial x} + \frac{1}{2}b^2(x,t)\frac{\partial^2 g}{\partial x^2}\right)dt + b(x,t)\frac{\partial g}{\partial x}dW \tag{1.12}$$

Chapter 2 Stochastic Volatility Models

In this section, we first introduce two important concepts, implied volatility and the volatility smile. From the occurrence of the volatility smile, we can easily find one aspect of shortage of the Black-Scholes equation – a constant volatility, and this is one of the stimuli that derives stochastic volatility models. Then, two general and basic stochastic volatility models, Hull-White model (H&W) and Stein-Stein model (S&S), and their application to the option will be explained.

2.1 Implied Volatility and Volatility Smile

2.1.1 Implied Volatility

By the definition of Black-Scholes equation, the parameter which is used to measure the risk is called volatility, and the volatility that corresponds to the market data is called implied volatility. Assuming the market price is V_M and implied volatility is expressed as σ_I, then under the Black-Scholes equation,

$$V_{BS}(S_t, K, T, t, \sigma_I, r) = V_M \tag{2.1}$$

Note that implied volatility is, generally, negatively related to the stock price. We can examine this statement by using the one year S&P 500 Index and the one year VIX index. The VIX index is a popular method, which is calculated and disseminated by Chicago Board Options Exchange (CBOE), to measure the implied volatility of the S&P 500 index. The calculation steps of the VIX index are recorded in *VIX: CBOE Volatility Index*.

We divide the S&P 500 index by 100 so as to see the relationship with the VIX index easily. From Figure 2.1, we can find out that when the S&P 500 index increases, the VIX index decreases, and vice versa, which shows that they are negatively correlated.

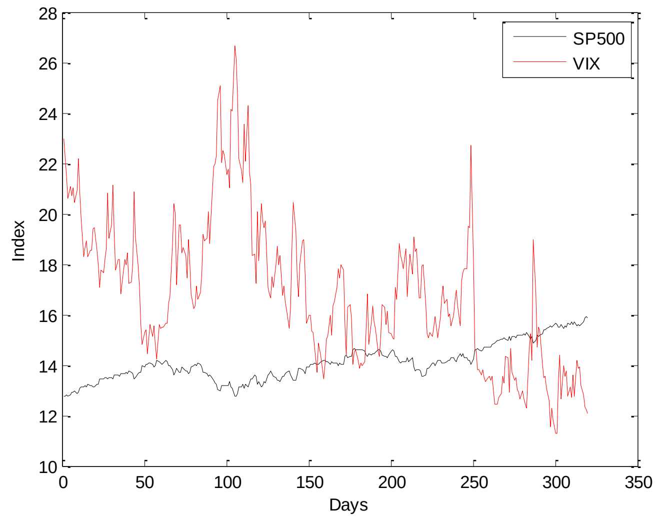

Figure 2.1: S&P500 index against VIX index from 01/03/12 to 04/12/13

2.1.2 Volatility Smile

We derive a graph below explains the relationship between the implied volatility and strike price for the call option of Google. Inc.(GOOG), which will expire at Sep 20, 2013, on the Apr 12, 2013.

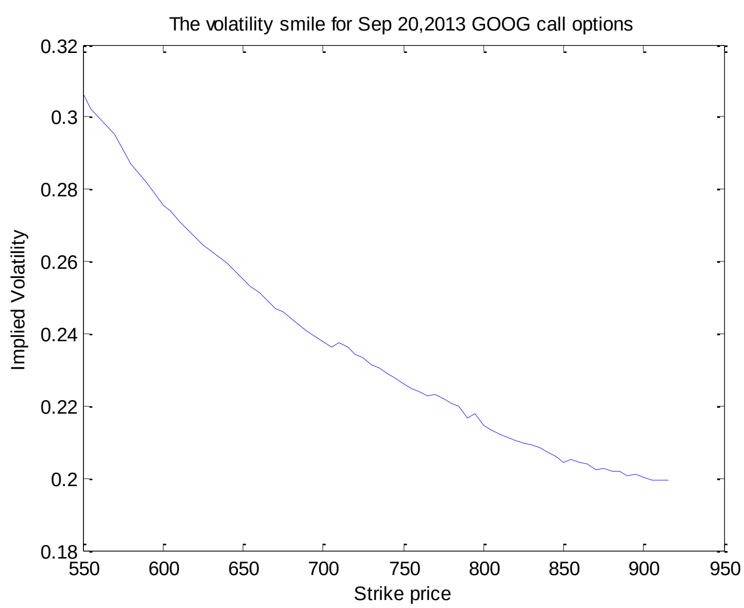

Figure 2.2: The volatility smile for Sep 20, 2013 Google call options

It is a curve, even though there are some jumps, rather than a straight line, which means that the volatility should not be a constant value. Such a phenomenon is call "volatility smile". We have to mention that different underlying assets have different volatility smile graph. For stock options like our example whose graph is down sloping, we usually call it 'volatility skewed' instead of 'volatility smile'; while for FX options, the graph is much more familiar with the term 'smile'.

2.2 Hull-White Model

2.2.1 Hull-White Stochastic Model

Hull and White (1987) started by considering a derivative asset f with a price that depends upon some security price, S, and its instantaneous variance, $V = \sigma^2$, which obeys the following stochastic processes:

$$\begin{cases} dS = \varphi S dt + \sigma S dw \\ dV = \mu V dt + \xi V dz \end{cases} \tag{2.2}$$

The drift term φ is dependent on S, σ, t. The variable μ and ξ depend on $\sigma$and t. The Wiener process dw and dz are correlated, with the correlation coefficient ρ.

2.2.2 Assumptions

The following assumptions are important so as to understand H&W model better:

- S and σ^2 are the only two variables that affect the price of derivative f. Therefore, the risk-free rate, r, must be constant or deterministic.
- The volatility V is uncorrelated with the stock price S.
- The volatility V is uncorrelated with the aggregate consumption, or in other words the volatility has no systematic risk

Systematic risk: A kind of risk that affects the entire market and cannot be avoided through diversification, including interest rates, recession and wars.

2.3 Stein-Stein Model

Stein-Stein model (1991) assumes that the volatility, which reverts to a long-run mean, follows an arithmetic Ornstein-Uhlenbeck process. The supporting Brownian motions, which drive the randomness of the underlying security and volatility, are assumed to be independent.

2.3.1 Stein-Stein Stochastic Model

The Stein-Stein model is defined as follows:

$$\begin{cases} dS = \mu S dt + \sigma S dz_1 \\ d\sigma = -\delta(\sigma - \theta)dt + k dz_2 \end{cases} \tag{2.3}$$

where S is the stock price, σ is the "volatility" of the stock, k, μ, δ and θ are fixed constants and dz_1, dz_2 are independent Wiener processes. The parameter μ in (2.3) is the drift of the stock price, δ is the rate of reversion to long-run mean of the volatility process, k is the volatility-of-volatility parameter, and θ is long-run mean of volatility process. For a wide range of relevant parameter values, the probability for getting $\sigma = 0$ is too small to be considered.

2.3.2 Assumption

The most important assumption for Stein-Stein model is that the volatility which is governed by the arithmetic O-U process has a possibility to be a negative value. Lipton and Sepp (2008) have shown that the negative number does not lead to any computational problems. Since the first and second moments of the asset price should be finite, we need to satisfy the following requirements:

$$E\left[\int_0^t |\mu(t', S(t'), \omega)| dt'\right] < \infty,$$

$$E\left[\int_0^t |\sigma^2(t', S(t'), \omega)| dt'\right] < \infty, \qquad 0 \le t < \infty, \tag{2.4}$$

where the expectation is taken over all possible realizations of ω for the underlying asset.

Under requirements (2.4), Lipton and Sepp solved the stochastic differential equation:

$$dS(t) = \mu(t)S(t)dt + \sigma(t, \omega)S(t)dW_1(t) \tag{2.5}$$

and derive an exponential solution:

$$S(t) = S_0 e^{\int_0^t \mu(t')dt' - \frac{1}{2}I(t) + Y(t)}, \tag{2.6}$$

where

$$I(t) = \int_0^t \sigma^2(t', \omega)dt', \;\; Y(t) = \int_0^t \sigma(t', \omega)dW_1(t'), \tag{2.7}$$

There is no any restrictions on the signs of $Y(t)$ and $\sigma(t)$, which supports Stein-Stein model's main assumption.

2.4 Application to Option Pricing

The expected return of underlying asset is not risk-neutral. Risk adverse investors would require higher expected returns than risk seeking investors when they are in the same risk level. If we want to make a better and deeper analysis for financial derivatives, we need to employ the risk-neutral valuation method, which guarantees that all the parameters involved are not related to investors' risk preference. Assuming the financial market is risk-neutral, the expected return for every investor ought to be risk-free rate which sometimes are considered to be the long-term debt rate in certain country.

In order to eliminate or reduce the effect of risk for pricing financial asset, we need to form a hedge portfolio. Due to the dynamic behavior for security prices, contingent claims must satisfy a PDE that includes a market risk premium for volatility. In 1976, Garman figured out that stochastic volatility option price must satisfy a bivariate fundamental partial differential equation (PDE) in the two state variables, security price and volatility. The fundamental PDE for the security price F is

$$\frac{1}{2}\sigma^2 P^2 F_{pp} + rPF_p - rF + F_t + \frac{1}{2}k^2(v)F_{vv} + [m(v) - \varphi k(v)]F_v = 0, \tag{2.8}$$

where, φ is the market price of volatility risk and r is the instantaneous risk-free rate. And for the aspect of simplicity, we let $v = \sigma^2$,

$$dv = m(v)dt + k(v)dz_2.$$

Assuming there is no arbitrage opportunity in the market, which means $\varphi = 0$, the European call option should be the present value of the option price on maturity date at the risk-free rate. Therefore, the price of option is given by

$$f(S_t, \sigma_t^2, t) = e^{-r(T-t)} \int_{S=K}^{\infty} f(S_t\,, {\sigma_t}^2, T) p(S_T | S_t, \sigma_t^2) dS_T. \tag{2.9}$$

The option price $f(S_t, \sigma_t^2, t)$ is max [0, S-K]. Consequently, (2.7) can be rewritten as

$$f(S_t, \sigma_t^2, t) = e^{-rt} \int_{S=K}^{\infty} (S - K)\, p(S_T | S_t, \sigma_t^2) dS_T \tag{2.10}$$

where T is the maturity time of the option, S_T is stock price at time t, σ_t is the instantaneous standard deviation at time t, and $p(S_T | S_t, \sigma_t^2)$ is the conditional distribution of S_T given the stock price and variance at time t.

Again, note that, in a risk-neutral world, the expected rate of return on S is the risk-free rate, and the expectation of S_T conditional on S_t is $S_t e^{r(T-t)}$.

One of main technical problems for both H&W and S&S is to define what the distribution of the average variance of the underlying asset is, and Moment Generating Function (MGF) may be used. The distribution of the average variance of the underlying asset over its life [0,T] is given by

$$AV = \frac{1}{T}\int_0^T \sigma^2(t)dt \tag{2.11}$$

2.4.1 Algebra Method for the Hull-White model

Using (2.11), the distribution of S_T may be written as

$$p(S_T|\sigma_t^2) = \int g(S_T|AV)h(AV|\sigma_t^2) \tag{2.12}$$

By substituting (2.12) into (2.10), we can rewrite (2.10) as

$$f(S_t, \sigma_t^2, t) = \int_{S=K}^{\infty} C(AV)h(AV|\,\sigma_t^2)dAV \tag{2.13}$$

The $C(AV)$ term is the Black-Scholes price that we mention in Chapter One, see (1.2). h is the conditional density function of AV given the instantaneous variance σ_t^2.The equation (2.14) says that the option price is the Black-Scholes price integrated over the distribution of the mean volatility.

H&W calculate all the moments of AV, while keeping ξ and μ constant, to find the solution of the distribution of AV. Instead of looking for an analytic form, H&W used power series approximation technique to solve the problem in series form.

H&W expanded the option price with volatility AV on its expected value. They used moments for the distribution of AV, and found the following expansion of Taylor series:

$$f(S,\sigma^2) = C(\sigma^2) + \frac{1}{2}\frac{S\sqrt{T-t}N(d_1)(d_1 d_2 - 1)}{4\sigma^3} \times \left[\frac{2\sigma^4\left(e^k - k - 1\right)}{k^2} - \sigma^4\right]$$

$$+\frac{1}{6}\frac{S\sqrt{T-t}N'(d_1)[(d_1d_2-3)(d_1d_2-1)-(d_1^2+d_2^2)]}{8\sigma^5}$$

$$\times\sigma^6\left[\frac{e^{3k}-(9+18k)e^k+(8+24k+18k^2+6k^3)}{3k^3}-\sigma^4\right]+\cdots$$

where

$$k=\xi_2^3(T-t)$$

2.4.2 Algebra Method for Stein-Stein Model

Solving PDE (2.8) to price option is difficult, and it is hard to implement the solution in the most practical way. The method S&S used is based upon Fourier inversion methods, but we can just call this approach relatively straightforward, since it still includes tedious computation.

We, now, give the closed-form solution for the distribution of stock price.

$$P_0(S,t)=(2\pi)^{-1}S^{-3/2}\times\int_{\eta=-\infty}^{\infty}I\left(\left(\eta^2+\frac{1}{4}\right)\frac{t}{2}\right)e^{i\eta logP}d\eta \tag{2.14}$$

$$P(S,t)=e^{-\mu t}P_0(Se^{-\mu t}) \tag{2.15}$$

The new defining variables are shown in the Appendix A.

Note that $P(S,t)$ is a conditional distribution. It relies on the current stock price and current volatility. Therefore, the precise form should be written as $P(S,t|S_0,\sigma_0)$.

Suppose the risk premium is proportional to volatility, the S&S solved the PDE (2.10) by discounting the adjusted future security price dynamics at the risk-free rate which is so-called Feynman-Kac functional. The given density function of future stock price, in general case, is (2.10). By using Fourier inversion methods, S&S found out a closed-form for the distribution of stock price, given by the equation (2.15). With the limitation of MGF over the period t, the variable I in (2.15) must satisfy

$$I(\lambda) = E\left[-\lambda \int_0^t |\sigma^2(t', S(t'), \omega| dt'\right]$$

Chapter 3 The Heston Stochastic VolatilityModel

After the proposal of H&W model and S&S model (see Chapter 2), in 1993, professor Heston from Yale University proposed the Heston stochastic volatility model (Heston model) in his paper 'A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options(1993). Nowadays, the Heston model becomes one of the most widely used stochastic volatility models today. The big advantage of this model is that it provides a closed-form solution for European call options, which can be obtained by call-put parity, when the volatility process is correlated with the spot asset.

3.1A Brief Introduction of the Heston model

Heston (1993) assumes that the process S_t follows a log-normal distribution, and the process V_t follows a Cox-Ingersoll-Ross process (CIR process) (1985). The model is given as:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t \\ dV_t = \varkappa(\theta - V_t)dt + \sigma\sqrt{V_t} dZ_t \\ dW_t dZ_t = \rho dt \end{cases} \tag{3.1}$$

The parameters of (3.1) are shown below:

- μ is the drift coefficient of the stock price
- θ is the long-term mean of variance
- $\varkappa$ is the rate of mean reversion
- σ is the volatility of volatility
- S_t and V_t are the price and volatility process respectively
- To take into account the leverage effect, stock returns and implied volatility are negatively correlated, W_t and Z_t are correlated Wiener process, and the correlation coefficients is ρ

Note that the variance of the CIR process is always positive and if $2\varkappa\theta > \sigma^2$, then it cannot reach zero, and the deterministic part of process V_t is asymptotically stable if $\varkappa > 0$. (Mikhailov & Nogel, 2003)

Here are two assumptions for this dissertation:

- There is no dividend payment.
- The interest rate is a constant, hence μ is a fixed value.

We plot an example of stock price and volatility stochastic process in the Figure 3.1 and Figure 3.2 respectively. (MATLAB code for simulation is available in Appendix B)

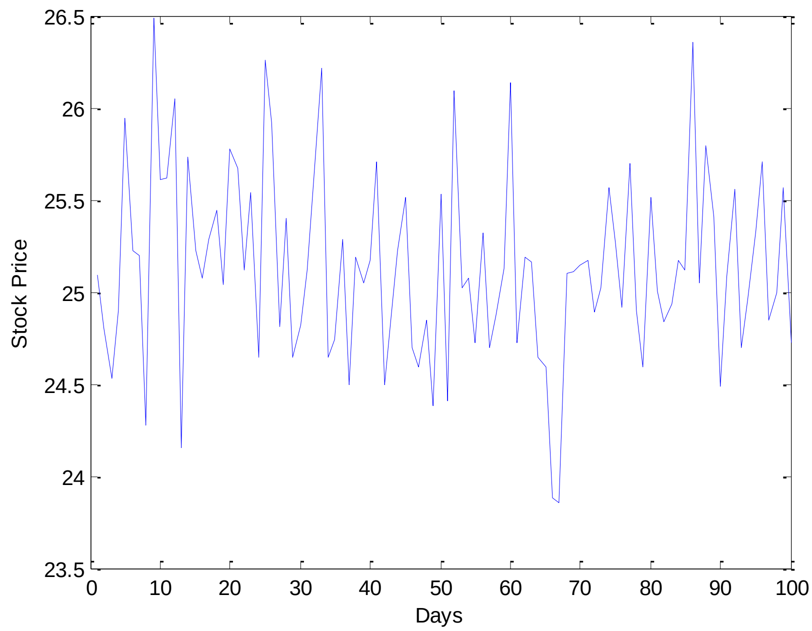

Figure 3.1: Stock price dynamics in the Heston model

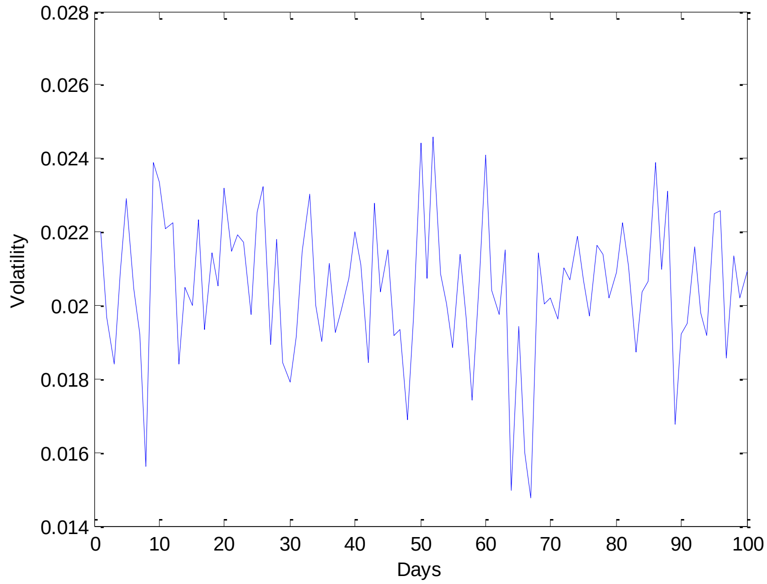

Figure 3.2: Volatility dynamics in the Heston model

3.2 Correlated Heston model

Consider the model described by equation (3.1). It assumes the two Wiener process W_t and Z_t are correlated, with the correlation coefficients ρ. If $\rho \neq 0$, we call equation (3.1) the correlated Heston model.

Assume $\rho \in [-1,1]$ and

$$W_t = \sqrt{1-\rho^2}\widetilde{W} + \rho Z_t, \tag{3.2}$$

where $\widetilde{W}$ is Generalized Wiener process independent of Z_t. In this case, (3.1) can be written as

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{V_t} S_t [\rho dZ_t + \sqrt{1-\rho^2} d\widetilde{W}] \\ dV_t = \varkappa(\theta - V_t)dt + \sigma\sqrt{V_t} dZ_t \\ dW_t dZ_t = \rho dt \end{cases} \tag{3.3}$$

3.3Examine Parameters in the Heston Model

Empirical studies have shown that the log-stock price distribution is non-Gaussian, and it is featured by heavy tails and high peaks.

- Correlation coefficient ρ

The correlation coefficient ρ is described as the correlation between the shock to the stock price(logarithm form) and the shock to the volatility. If $\rho > 0$, then the volatility will increase as the stock price increase. If $\rho < 0$, then the volatility will increase while the stock price decrease. If $\rho = 0$, there is no effect to the skewness of distribution.

The option price given by the Heston model is set as C. Figure 3.3 illustrates the difference between $\rho > 0$, and when $\rho = 0$. $(Difference = C_{\rho>0} - C_{\rho=0})$. Figure 3.4 illustrates the difference between $\rho < 0$, and when $\rho = 0$. $(Difference = C_{\rho<0} - C_{\rho=0})$.

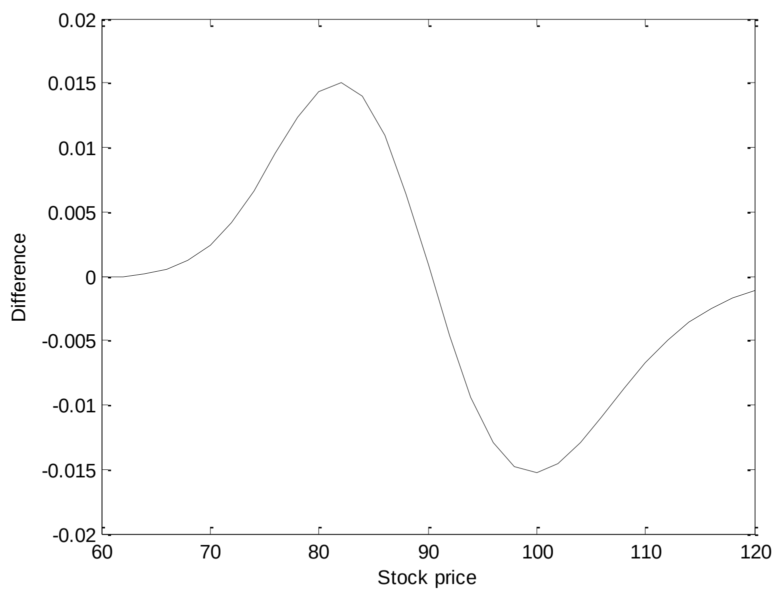

Figure 3.3: The difference between $\rho = 0.5$ and $\rho = 0$

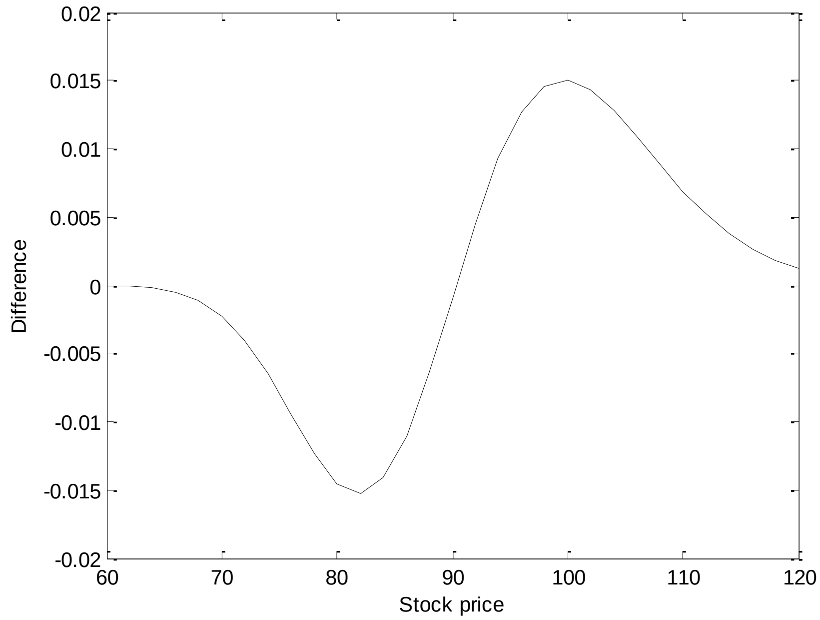

Figure 3.4: The difference between $\rho = -0.5$ and $\rho = 0$

In the Figure 3.3, the difference is positive for the left side and negative for the right side, so the distribution of stock price is negative skewness ; while, in the Figure 3.4, the difference is negative for the left side and positive for the right side, so the distribution of stock price is positive skewness. Therefore, ρ affects the skewness of the stock price distribution. In detailed, positive ρ leads to a fat right tail, and negative ρ results in a fat left tail.

- The volatility of volatility parameter σ

The parameter σ controls the volatility of volatility, and it affects the kurtosis of the probability density distribution. When σ is zero the volatility is deterministic, so the distribution of stock price follows the normal distribution. Otherwise, increasing σ causes the kurtosis to increase. Note that higher σ means that the volatility is more volatile, which states that the market has a greater chance of extreme movements. (Heston, 1993)

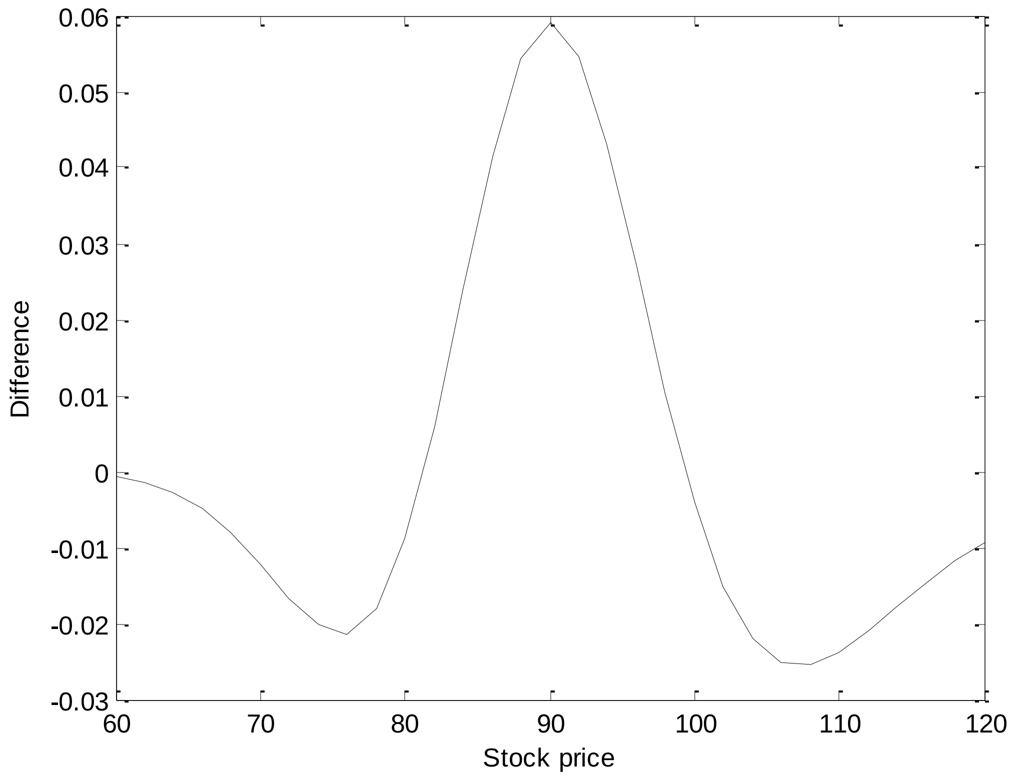

Figure 3.5: The difference on density function between $\sigma = 0.1$ and $\sigma = 0.2$

Figure 3.5 illustrates that the difference ($Difference = C_{\sigma=0.1} - C_{\sigma=0.2}$) is negative at the both end, which means the increase of volatility σ will cause a fat-tail.

- The rate of mean reversion $\varkappa$ and the long-run mean of variance θ

The mean reversion speed $\varkappa$ is considered as representing the degree of "volatility cluster". It defines the how fast the variance process reverting to its long term mean, and it can be found in the real market. Generally speaking, a large price variation is more likely to be followed by a large volatility, and a large volatility more likely follows a small $\varkappa$.

The variance drifts toward a long run mean of θ, with mean-reversion rate $\varkappa$. The mean reversion determines the relative weights of the current variance and the long-run variance on option pricing. When mean reversion is positive, the variance has a steady-state distribution with mean θ. (Cox, Ingersoll, and Ross, 1985)

These two parameters cause the mean reversion property of the Heston model. See the second equation in (3.1), when $\varkappa > 0$, once $V_t > \theta$, the drift term will increase the value of process $\{V_t\}$, and vice versa.

3.4 Advantages and Disadvantage of the Heston Model

Although the Heston model is widely used in the financial researches, it still has its own drawbacks. Here, we would summarize the advantages and disadvantages of the Heston model:

Advantages:

- Provides a closed-form solution for European call option
- Be able to explain the property of stock price when its distribution is non-Gaussian distribution
- Fits the implied volatility surface of the option prices in the market
- Allows the correlation between stock price and volatility to be negative

Disadvantages:

- Hard to find proper parameters to calibrate the stochastic model
- The prices produced by the Heston model are sensitive to the parameters, so the fitness of the model depends on the calibration. (Mikhailov & Nogel, 2003).
- It cannot capture the skew at short maturity as the one given by the market.(Mikhailov & Nogel, 2003).

3.5 The Closed-Form Solution

As we mentioned in Chapter 2, a European call option with strike price K and maturing at time T must satisfy PDE (2.7). Heston (1993) solved the PDE (2.7) not in the direct way but using the method of characteristic functions. He guessed the form of solution should be the same as Black-Scholes formula

$$C(S, v, t, T) = SP_1 - Ke^{-r(T-t)}P_2 \tag{3.4}$$

where the first term is the present value of the spot asset upon optimal exercise, and second term is the present value of the strike-price payment. Both P_1 and P_2 ought to satisfy PDE (2.7). It is convenient to define

$$x = \ln(S) \tag{3.5}$$

Suppose the characteristic functions f_1 and f_2 are known, then P_1 and P_2 can be defined via Fourier Inversion Transformation:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\varphi lnK} f_j(x, v, \tau; \varphi)}{i\varphi}\right] d\varphi, \quad j = 1,2 \tag{3.6}$$

Heston (1993) assumes the characteristic function solution is

$$f_j(x, v, \tau; \varphi_j) = exp\{C_j(\tau; \varphi_j) + D_j(\tau; \varphi_j)v + i\varphi_j x\} \tag{3.7}$$

where

$$C_j(\tau; \varphi_j) = \mu\varphi_j\tau + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\varphi_j i + d_j)\tau - 2ln\left[\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right]\right\}$$

$$D_j(\tau; \varphi_j) = \frac{b_j - \rho\sigma\varphi_j i + d_j}{b_j - \rho\sigma\varphi_j i - d_j}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]$$

and

$$g_j = \frac{b_j - \rho\sigma\varphi_j i + d_j}{b_j - \rho\sigma\varphi_j i - d_j}$$

$$d_j = \sqrt{(\rho\sigma\varphi_j i - b_j)^2 - \sigma^2(2u_j\varphi_j - \varphi_j{}^2)}$$

$$u_1 = 0.5, \;\; u_2 = -0.5, a = \varkappa\theta,$$

$$b_1 = \varkappa + \lambda - \rho\sigma, \;\; b_2 = \varkappa + \lambda$$

The concreted step of derivation for this closed-form is shown in Heston's paper (1993). In his paper, Heston incorporates stochastic interest rates into the option pricing model, but, in our dissertation, we assume the interest rate is a constant.

3.6 The Greeks

The Greeks measure a different dimension to the risk of the option. Each of the Greeks stands for a kind of sensitivity of the option value with respect to a given parameter. The most common of the Greeks includes Delta, Vega, Theta, Rho and Gamma. Here, we only discuss about how to calculate the Delta and the Vega in the Heston model.

3.6.1 The Delta in the Heston Model

The Delta, Δ, of a stock option is the ratio of the shock in the price of the stock option (C) with respect to the shock in the price of the underlying stock (S). It can be express as equation (3.8). We use this Greek to measure the sensitivity of the option to the stock price. And *"Delta Hedging"* is a very common strategy to do the arbitrage and minimize risk of portfolio in the option market.

$$\Delta = \frac{\partial C}{\partial S} \tag{3.8}$$

Suppose the Delta in the Heston model is set to be ΔH, and then we can derive ΔH from equation (3.4). We differentiate the option price $C(S, v, t)$ with respect to the stock price S, and show the steps below:

$$\frac{\partial C(S, v, t, T)}{\partial S} = P_1 + S\frac{\partial P_1}{\partial S} - K\frac{\partial P_2}{\partial S} \tag{3.9}$$

Let's consider $\frac{\partial P_i}{\partial S}$, $i = 1{,}2$. [P_i is refer to equation (3.6)]

$$\frac{\partial P_i}{\partial S} = \frac{1}{\pi}\int_0^\infty Re\left\{\frac{\partial\left[\frac{e^{-i\varphi lnK} f_j(x, v, \tau; \varphi)}{i\varphi}\right]}{\partial S}\right\} d\varphi \tag{3.10}$$

The only term in the equation (3.10) includes S is $f_j(x, v, \tau; \varphi)$ from $x = \ln(S)$. Therefore, we can use chain rule to take the derivative.

$$\frac{\partial P_i}{\partial S} = \frac{1}{\pi}\int_0^\infty Re\left\{\frac{\partial\left[\frac{e^{-i\varphi lnK} f_j(x, v, \tau; \varphi)}{i\varphi}\right]}{\partial f_j(x, v, \tau; \varphi)} \cdot \frac{\partial f_j(x, v, \tau; \varphi)}{\partial S}\right\} d\varphi \tag{3.11}$$

Recall that,

$$f_j(x, v, \tau; \varphi) = exp\{C_j(\tau; \varphi_j) + D_j(\tau; \varphi_j)v + i\varphi x\}$$

Therefore,

$$\frac{\partial P_i}{\partial S} = \frac{1}{\pi}\int_0^\infty Re\left\{\frac{e^{-i\varphi lnK} f_j(x, v, \tau; \varphi)}{i\varphi} \cdot i\varphi \cdot \frac{1}{S}\right\} d\varphi = \frac{1}{\pi}\int_0^\infty Re\left\{\frac{e^{-i\varphi lnK} f_j(x, v, \tau; \varphi)}{S}\right\} d\varphi \tag{3.12}$$

Insert the equation (3.12) into the equation (3.9), we can obtain,

$$\Delta H = P_1 + \frac{S}{\pi}\int_0^\infty Re\left\{\frac{e^{-i\varphi lnK} f_1(x,v,\tau;\varphi)}{S}\right\} d\varphi - \frac{S}{\pi}\int_0^\infty Re\left\{\frac{e^{-i\varphi lnK} f_2(x,v,\tau;\varphi)}{S}\right\} d\varphi \tag{3.13}$$

3.6.2 The Vega in the Heston Model

The Vega, V, measures the sensitivity to volatility, which expresses as the amount of money per stock gain or lose as volatility increases or decreases by 1%. It is the derivative of the option value $(C)$with respect to the volatility of the stock price (σ). We use

$$\mathrm{V} = \frac{\partial C}{\partial v} \tag{3.14}$$

Set the Vega in the Heston model as νH. The derivation of νH is similar the derivation in the section 3.6.1.

$$\mathrm{V} = \frac{\partial C(S,v,t,T)}{\partial v} = S\frac{\partial P_1}{\partial v} - K\frac{\partial P_2}{\partial v} \tag{3.15}$$

The only difference is that Vega contains the term $D_j(\tau;\varphi)$ in equation (3.7). So,

$$\frac{\partial P_i}{\partial v} = \int_0^\infty Re\left\{\frac{e^{-i\varphi lnK} f_i(x,v,\tau;\varphi)}{i\varphi} \cdot D_j(\tau;\varphi)\right\} d\varphi$$

Chapter 4 Option Pricing and Calibration

4.1 Risk-neutralized approach with the Heston Model

For stochastic volatility model, a risk-neutralized method, also called an Equivalent Martingale Measure (EMM), is widely used in the pricing of financial derivatives. It is based on the Girsanov's theorem of asset pricing. The basic way is to set up a new model that replaces the drift μ by the risk-free interest rate and transforms the drift in the volatility equation. In a complete market, the discounted expected value of the future payoff under the unique risk-neutralized measure $\mathbb{q}$. It can be expressed by

$$Option\ Price = \mathbb{E}_t^{\mathbb{q}}\left[e^{r(T-t)}H(T)\right] \tag{4.1}$$

where $H(T)$ is the payoff of the option at time T and r is the risk free rate of interest over the time period $[t, T]$. This approach can be applied to the Heston model particularly. Recall the Heston model in (3.1)

Set

$$d\widetilde{W_t} = dW_t + \frac{\mu - r}{\sigma}$$

$$d\widetilde{Z_t} = dZ_t + \gamma(S, V, t)dt$$

By the definition of Girsanov's theorem, the independent Wiener process $\widetilde{W_t}$ and $\widetilde{Z_t}$ can be expressed under EMM $\mathbb{q}$:

$$\frac{d\mathbb{q}}{d\mathbb{p}} = exp\left\{-\frac{1}{2}\int_0^t (\vartheta_s^2 + \gamma(S, V, s)^2)ds - \right.$$

$$\left.\int_0^t \vartheta_s dW_s - \int_0^t \gamma(S, V, t)dZ_s\right\}$$

where $\mathbb{p}$ is the real market measure. Under measure $\mathbb{q}$, the Heston model (3.3) can be simplified as

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t d\widetilde{W_t} \\ dV_t = \varkappa^*(\theta^* - V_t)dt + \sigma\sqrt{V_t} d\widetilde{Z_t} \\ dW_t dZ_t = \rho dt \end{cases} \tag{4.2}$$

where,

$$\varkappa^* = \varkappa + \lambda, \tag{4.3}$$

$$\theta^* = \frac{\varkappa\theta}{\varkappa+\lambda} \tag{4.4}$$

In this result, λ has effectively been replaced. Note that any acceptable $\gamma(S, V, t)$ can determine different equivalent martingale measure $\mathbb{q}$, since $\gamma(S, V, t)$ is not unique when the volatility is not a constant.

4.2 Numerical Solution for the Heston Model by Excel-VBA

At the time t the price of a European call option, based on Black-Scholes Equation, with time to maturity $(T - t)$ should be formulated as the form of (3.4)

$$C(S, v, t) = SP_1 - KP(t, T)P_2.$$

The option price can be obtained by solving P_1 and P_2 based upon equation (3.5) and equations (3.6).

The most difficult part for obtaining P_1 and P_2, is how to evaluate the complex integrals by using Excel-VBA. Here, we define different kinds of operations of complex number, such as addition, division and square root. We refer to the book 'Hand book of complex variable' (Krantz, 1999), and do the calculation of the real part and image part separately. For the numerical integration method, we employ Simpson's rule, which is relatively accurate and is not hard to program in Excel-VBA.

The input variables for the closed form including stock spot price S, strike price K, risk-free interest rate r, time step which need to do integration $deltax$, and the five parameters , v, $\varkappa$, θ, σ and ρ, that needed to estimate.

Heston (1993) has mentioned that the integrand in equation (3.6) is a smooth function that decays rapidly and presents no difficulty. Thereby, we choose the set of integration between 0 and 100, and the length of each step 0.1.

The Excel-VBA code for this part is available in Appendix C. Next section will illustrate how to estimate parameters for Heston model.

4.3 Model Calibration

There are five parameters, v, $\varkappa$, θ, σ and ρ, needed to be estimated in the Heston model. The change for each parameter will bring a big impact for the correctness (see Section 3.3), so the estimation of parameters becomes very important.

A variety of methods can be chosen. For instance, one can observe the real market data, and use statistic tool to fit data in the Heston model (Ait-Sahila, Kimmel, 2005); Monte Carlo simulation is another famous method to do the calibration (Alexander V.H, 2010). What we selected is another common used in a way that is called an inverse problem, which means that we need to get the data from the real market first, and then estimate parameters by the known data. This kind of problem can be solved by minimizing the error difference between the Heston model prices, which are obtained from section 4.3, and real market price, which can be easily found from the Internet. The expression from the math aspect is shown below.

Assume Ω is a set of realization for the parameters in the Heston model. For a call option that is calculated from the Heston model, the optimization problem can be described as

$$MinS(\Omega) = min(\Omega) \sum_{i=1}^{N} (C_i^H - C_i^M)^2$$

subject to

$$2\varkappa\theta > \sigma^2, -1 < \rho < 1, \varkappa > 0$$

$$0 < \theta < 1, 0 < \sigma < 1,$$

where C_i^H and C_i^M are the i^{th} call option price, respectively, calculated by the Heston model and collected from the real market. N is the number of options that are used to calibrate the model.

The optimization in this dissertation is done by Excel Solver. Excel Solver turns out to be very robust and reliable among all kinds of local optimizers. (Sergei Mikhailov, 2008). The method employed by Excel Solver is *Generalized Reduce Gradient (GRG)* method (see Appendix D).

4.4 Calibration Results

With the help of Excel-VBA and Excel-Solver, we can start to calibrate the model.

European call option on *Google Inc.(GOOG)* shares listed on NASDAQ was used as market data. The data is recorded on Apr 6, 2013, and the data sample is available in the Appendix E. The results are shown below:

correlation coefficients	**volatility of volatility**	**rate of reversion**	**long-run mean**	**variance**	**SqrD sum**
ρ	σ	κ	θ	v	$MinS(\Omega)$
-0.50903932	0.467514601	2.040210844	0.053565543	0.069545829	2.646446663089

Table 4.1: Estimated parameters in the Heston model

Our sample comprises 21 groups of data, which are divided by three sets according to their expiration date. The term *'SqrD sum'* records the difference between our model's price and real market price. It is calculate as follows:

$$SqrD\ sum = \sum (Our\ price - Real\ price)^2$$

It is also the objective cell when the Excel-Solver is working, which means, in order to find a model that fits market data well, *'SqrD sum'* needed to be minimize.

Chapter 5 Comparison

This chapter is going to test the Heston model by contrasting results with the Black-Scholes equation, in order to see which method is superior. Many effects are related to the time-series dynamics of volatility. For example, a higher variance *(v)* raises the prices of all options, just as it does in the Black-Scholes equation (Heston, 1993). Note that this comparison is supposed that the option is under risk-free interest rate and there is no dividend yield.

5.1 Comparison with Black-Scholes Equation

Under the help of Excel-VBA and Excel-Solver, we can start to calibrate the model. The results are listed in Table 4.1, and we name the model as *HestonR.*

A variety of statistic measures can be selected to check the accuracy of the *HestonR*. In this dissertation, a widely used measure, average relative percentage method, is employed. Simply speaking, we check the percent error for each method. The model which has small percent error, obviously, is the better one. The mathematic expression for percent error is shown as follow,

$$Error = \frac{|Experimental\ price - real\ price|}{|real\ price|} \times 100\% \tag{5.1}$$

The results of comparison are shown in Table 5.1:

Expire Date	Real Price	HestonR	Black-Scholes (BS)	HestonR Error	BS Error	Compare
5/17/2013	272.90	273.06	273.0586903	0.05786169%	0.05814960%	0
	193.25	193.18	193.0675695	0.03610552%	0.09440129%	0
	183.45	183.24	183.0758893	0.11475428%	0.20393058%	0
	168.85	168.38	168.1038103	0.27976949%	0.44192463%	0

	159.00	158.52	158.1416471	0.30420343%	0.53984456%	0
	154.10	153.60	153.1698285	0.32175831%	0.60361551%	0
	139.20	138.96	138.3128182	0.16884994%	0.63734328%	0
9/20/2013	388.30	388.28	388.1624688	0.00578358%	0.03541879%	0
	373.40	373.34	373.1680436	0.01670109%	0.06212007%	0
	348.55	348.49	348.1818752	0.01848158%	0.10561607%	0
	333.65	333.61	333.1962973	0.01093122%	0.13598163%	0
	284.35	284.38	283.3546459	0.01191370%	0.35004542%	0
	260.00	260.07	258.5940143	0.02820521%	0.54076374%	0
	235.50	236.07	234.0696598	0.24231659%	0.60736316%	0
1/17/2014	389.70	389.10	388.4217732	0.15389047%	0.32800277%	0
	375.00	374.33	373.4700124	0.17833779%	0.40799669%	0
	350.45	349.83	348.6058202	0.17616437%	0.52623194%	0
	335.70	335.23	333.7395316	0.14070876%	0.58399416%	0
	287.40	287.21	284.7233666	0.06638822%	0.93132686%	0
	263.55	263.69	260.7312452	0.05355543%	1.06953321%	0
	240.10	240.63	237.2742015	0.21982474%	1.17692565%	0

Table 5.1: Price comparison between *HestonR* and BS

In the last column, *'0'* means *'HestonR Error'* is less than *'BS Error'*, which can be said HestonR is better than BS in this case; while, if any *'1'* shows up, which represents that BS is superior to HestonR. From the Table 5.1, there is no doubt that BS is easily beaten by HestonR in this case.

5.2 Sensitivity Analysis

This section will analyze the effect of stochastic volatility model on option pricing. We pay attention to two key parameters, the correlation coefficient ρ and the volatility of volatility σ, and the Delta.

For all cases, we shall consider two situations, including "in the money", which means strike price is less then stock price, and "out of money", which means strike price is larger than stock price. Note that all option price used in this section is calculated from the Heston model (3.4). For the parameters in the Heston model, we follow the calibration steps in the Chapter 4

5.2.1 Correlation Coefficient

Set current stock price from $S = 60$ to $S = 120$, and strike price $K = 90$. The correlation coefficients $\rho = 0.5$ and $\rho = -0.5$ are used. The parameters used in the Heston model are in Table 5.2. The data sample is shown is Table 5.3. In Figure 5.1, we can see the difference with two different ρ. For the comparison, we shall use the Black-Scholes equation with a volatility parameter that match the (square root of the) variance of the spot return over the life of the option (Heston, 1993).

Volatility of volatility	0.01
Rate of reversion	2.931465
Long-run mean	0.001001
Variance	0.028087
Time to maturity (yr)	1
Interest Rate	0.00135

Table 5.2: Parameters used to make the sensitivity analysis

	ρ=0.5			ρ=-0.5		
Stock price	HestonR	BS	Difference	HestonR	BS	Difference
60	0	3.40762E-05	-3.408E-05	0	3.11783E-05	-3.11783E-05
62	0	0.000147688	-0.0001477	0	0.000147856	-0.000147856
64	0.000485122	0.00055525	-7.013E-05	8.30421E-05	0.000555793	-0.000472751
66	0.002153337	0.001834071	0.0003193	0.001124771	0.001835606	-0.000710835
68	0.00643203	0.00538424	0.0010478	0.004116438	0.005388073	-0.001271635
70	0.016484332	0.014195208	0.0022891	0.011861469	0.014203746	-0.002342277
72	0.038104386	0.033930607	0.0041738	0.029866411	0.03394773	-0.004081319
74	0.080864472	0.074172859	0.0066916	0.067691016	0.074204015	-0.006512999
76	0.159077502	0.149473778	0.0096037	0.140101336	0.149525595	-0.009424259
78	0.292152849	0.279733575	0.0124193	0.267484469	0.279812862	-0.012328393
80	0.503960164	0.489483797	0.0144764	0.475053346	0.489596081	-0.014542735
82	0.821032804	0.805913588	0.0151192	0.790687199	0.806061565	-0.015374367
84	1.269785878	1.255874728	0.0139111	1.241709554	1.256057115	-0.014347561
86	1.873264408	1.862467436	0.010797	1.851300799	1.862678631	-0.011377833
88	2.64813166	2.641985735	0.0061459	2.635397676	2.642216456	-0.00681878

90	3.602572475	3.601914195	0.0006583	3.60078708	3.602152902	-0.001365822
92	4.735537476	4.740361874	-0.0048244	4.744734953	4.740596597	0.004138357
94	6.037393088	6.046924937	-0.0095318	6.05606408	6.047145011	0.00891907
96	7.491703065	7.504632382	-0.0129293	7.517263156	7.504829718	0.012433437
98	9.077654482	9.092446581	-0.0147921	9.10705975	9.092616281	0.014443469
100	10.77259506	10.78778322	-0.0151882	10.8029221	10.78792353	0.014998573
102	12.55424473	12.56864089	-0.0143962	12.58310876	12.56875271	0.014356043
104	14.40231879	14.41511722	-0.0127984	14.4280814	14.4152033	0.012878098
106	16.2994844	16.31026755	-0.0107832	16.32126994	16.31033169	0.010938249
108	18.23171582	18.24039244	-0.0086766	18.24929476	18.24043878	0.008855977
110	20.18819618	20.19490687	-0.0067107	20.20180378	20.1949394	0.006864377
112	22.16093659	22.16595641	-0.0050198	22.17108408	22.16597863	0.005105443
114	24.14426504	24.14792208	-0.003657	24.15158008	24.14793688	0.003643205
116	26.13429819	26.13691602	-0.0026178	26.13941048	26.13692563	0.002484847
118	28.1284654	28.13032934	-0.0018639	28.13193789	28.13033545	0.001602441
120	30.12511762	30.12646028	-0.0013427	30.12741508	30.12646407	0.000951012

Table 5.3: The sample data of correlation coefficient sensitivity analysis

Recall that we have let the volatility in Black-Scholes equation match the volatility in the Heston model, so the difference is not caused by the volatility. Therefore, it is the correlation coefficient that results in the difference. As shown in Section 3.3, the different ρ values have impact on the skewness of the stock price distribution. In detailed, a negative ρ value, which means the decrease of volatility follows the increase of stock price, will spread left tail, squeeze right tail and create a fat-left tail distribution.

In Figure 5.3, there is an intersect around $S = 90$, and we have set the strike price $K = 90$. Hence, Figure 5.3 illustrates that, in the case $\rho < 0$, when the option is “out-of- money” ($S < K$), the difference between *HestonR* and Black-Scholes equation is relatively smaller than when the option is “in-the-money”. (The red dashed line is below the black line for left part of Figure 5.3)

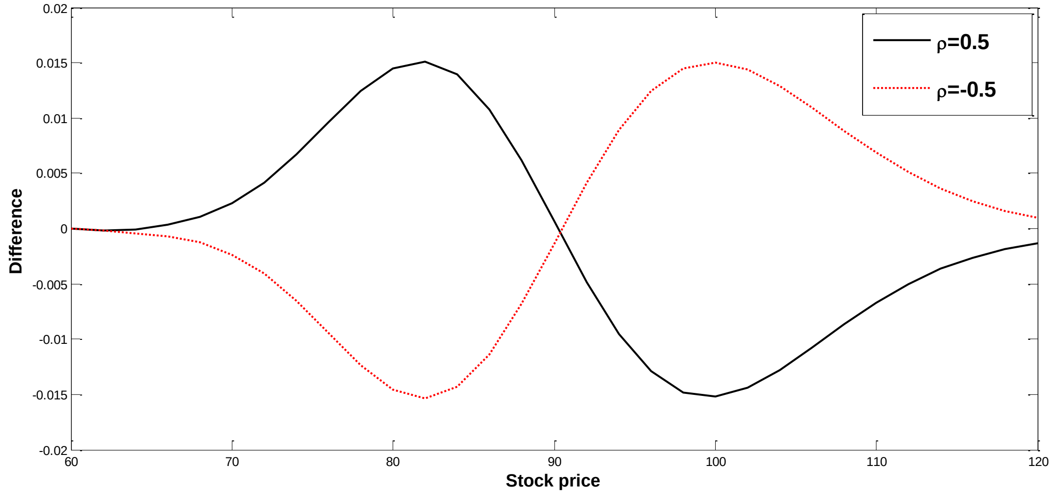

Figure 5.3: Difference when $\rho = 0.5$ and $\rho = -0.5$

Then, see the payoff of call option in Figure 5.4. We know, instinctively, that "out-of-money" options are sensitive to the thickness of the right tail. The Heston model can capture the decrease of the right tail thickness and decrease "out-of-money" option price consequently, while Black-Scholes equation cannot make such a response. When $\rho > 0$, the Heston model still has a better sensitivity than Black-Scholes equation, though the positive correlation leads to opposite outcomes.

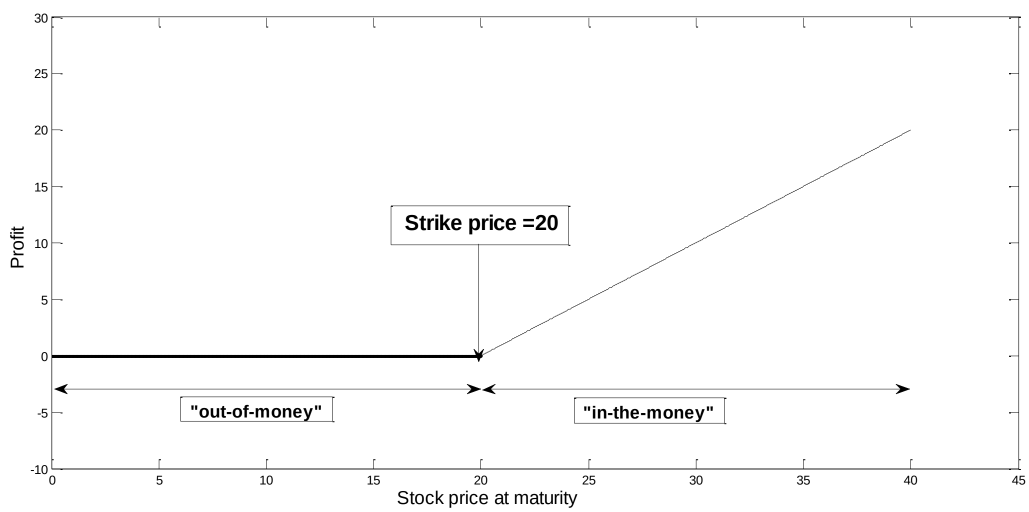

Figure 5.4: An example of call option payoff graph

5.2.2 Volatility of Volatility

We perform the impact of the volatility of volatility σ to call option price in this subsection.

As shown in Section 3.3, the volatility of volatility σ affects the kurtosis of stock distribution. A higher σ increases the kurtosis, and a lower σ decreases the kurtosis. When $\sigma = 0$, the volatility is deterministic, so the continuous compound stock price returns has normal distribution. (Heston, 1993)

The stock price is set from 60 to 120 like the previous experiment, and the strike price $K = 90$. The only changing variable, the volatility of volatility σ, is equal to 0.1 and 0.2 for two cases respectively. The parameters calculated by the steps in the Chapter 4 are shown in the Table 5.4, and the sample data is in the Table 5.5. Note that we let $\rho = 0$ so as to not affect the skewness of PDF.

Correlation coefficient	0
Rate of reversion	3.000000
Long-run mean	0.014794
Variance	0.0001
Time to maturity (yr)	1
Interest Rate	0.00135

Table 5.2: Parameters used to make the sensitivity analysis

	σ=0.1			σ=0.2		
Stock price	HestonR	BS	Difference	HestonR	BS	Difference
60	0	4.35E-05	-4.4E-05	0.000528	4.03E-05	0.000488
62	0.000167	0.000182	-1.5E-05	0.00169	0.000182	0.001507
64	0.00116	0.000666	0.000494	0.004009	0.000666	0.003343
66	0.003568	0.002143	0.001425	0.008555	0.002143	0.006412
68	0.009121	0.006147	0.002974	0.017228	0.006147	0.011081
70	0.021133	0.015877	0.005256	0.033279	0.015877	0.017403
72	0.045456	0.037274	0.008182	0.062036	0.037274	0.024762
74	0.091575	0.080211	0.011363	0.11182	0.080211	0.031609

76	0.173584	0.159454	0.014131	0.194972	0.159454	0.035518
78	0.31065	0.294925	0.015725	0.328714	0.294925	0.033789
80	0.526509	0.510907	0.015602	0.535449	0.510907	0.024542
82	0.84777	0.834053	0.013717	0.841968	0.834053	0.007915
84	1.3011	1.290472	0.010628	1.277153	1.290472	-0.01332
86	1.909794	1.902462	0.007332	1.868218	1.902462	-0.03424
88	2.690545	2.685636	0.004909	2.636228	2.685636	-0.04941
90	3.651191	3.647062	0.004128	3.592222	3.647062	-0.05484
92	4.789973	4.784772	0.005201	4.735313	4.784772	-0.04946
94	6.09637	6.088602	0.007767	6.053405	6.088602	-0.0352
96	7.553145	7.54206	0.011085	7.526101	7.54206	-0.01596
98	9.139008	9.124698	0.014311	9.128615	9.124698	0.003918
100	10.83127	10.81452	0.016753	10.83547	10.81452	0.02095
102	12.60802	12.59001	0.018015	12.62315	12.59001	0.033141
104	14.44964	14.43162	0.01802	14.4716	14.43162	0.03998
106	16.33955	16.32261	0.016941	16.36465	16.32261	0.042038
108	18.26444	18.24935	0.015093	18.28981	18.24935	0.040465
110	20.21405	20.20122	0.012824	20.23777	20.20122	0.03655
112	22.18074	22.17029	0.010445	22.20174	22.17029	0.031442
114	24.15901	24.15082	0.008184	24.17684	24.15082	0.026017
116	26.145	26.13881	0.006183	26.15966	26.13881	0.020852
118	28.13605	28.13154	0.004506	28.14781	28.13154	0.016273
120	30.13038	30.12722	0.003159	30.13963	30.12722	0.01241

Table 5.3: The sample data of volatility of volatility sensitivity analysis

It is known that, since a higher σ increases the kurtosis, which induces two fat-tails, the price in the two end ought to be higher than what is caused by a smaller σ. In the Figure 5.3, the Heston model price is lower than Black-Scholes equation price when the stock price equals the strike price $S = K = 90$, but it has a higher price at both right end and left end. Consequently, the Heston model is able to show the change of thickness, whereas Black-Scholes equation does not have such ability. Additionally, the graph in the Figure 5.3 is nearly symmetrical, so we conclude that the σ has no or a little influence on the skewness of the stock distribution.

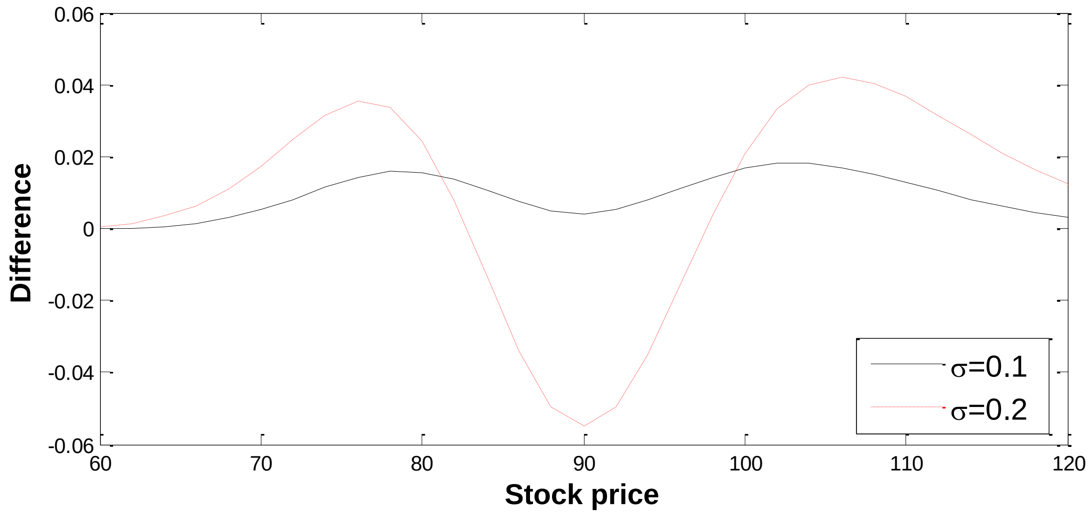

Figure 5.3: Difference when $\sigma = 0.1$ and $\sigma = 0.2$

5.2.3 Delta

We have mentioned the Delta in the Heston model in the Section 3.6.1, and we have the closed-form in equation (3.13). Now using Excel-VBA to obtain the Delta as what we do for pricing option used in the Heston model. The Figure 5.4 shows the difference Delta in the Heston model and Black-Scholes equation against the ratio S/K.

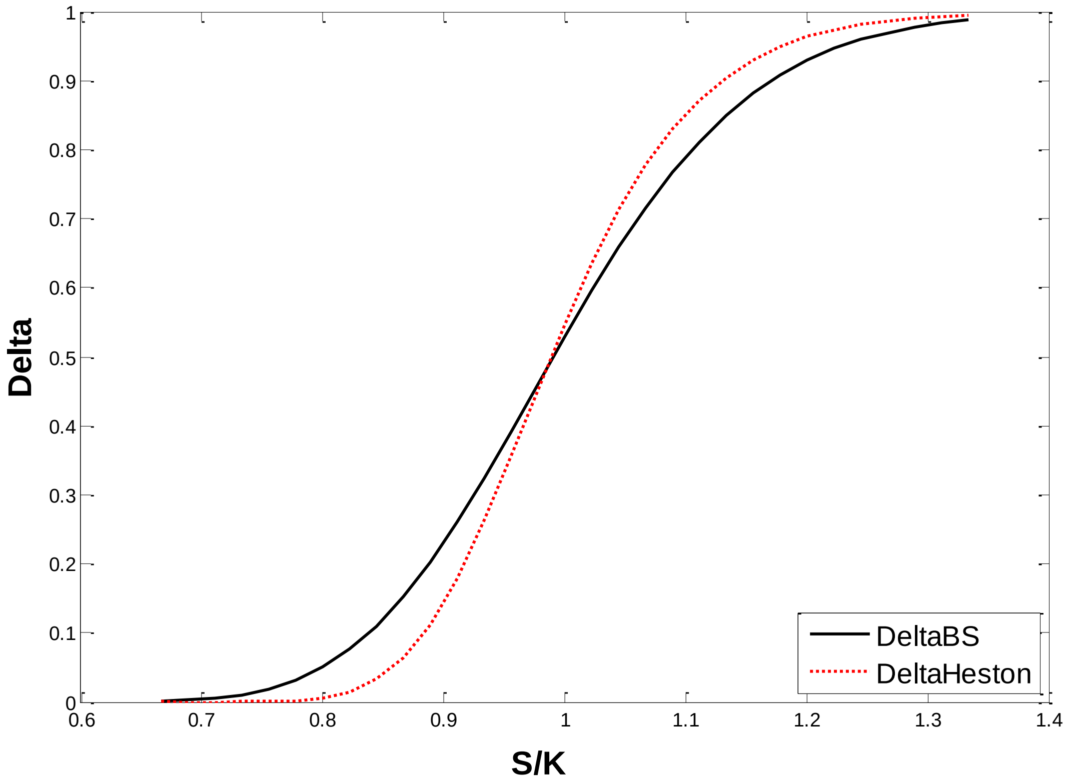

Figure 5.4: The Delta between the Heston model and Black-Scholes equation.

$(\sigma = 0.1, \rho = 0.5, \varkappa = 2.93145, \theta = 0.001001, v = 0.028, \tau = 1, r = 0.00135)$

From the Figure 5.4, the Delta calculated from the Heston model is cheaper than the Delta given by Black-Scholes equation for "in-the-money" options, while, for "out-of-money" options, the Delta in the Heston model is much higher. This result may be useful for delta hedging when an investor considers about his/her portfolio's risk.

Assuming the Heston model and Black-Scholes equation are the only two methods to price options. A risk-averse investor who owns a call option might utilize Black-Scholes equation to calculate the Delta for "in-the-money" options, and use the Heston model to obtain the Delta "out-the-money" options.

Chapter 6 Conclusion

This dissertation is performed as follows.

First of all, we demonstrate the "volatility smile" in option pricing, which shows the weakness of Black-Scholes equation. In order to make up the disadvantage of Black-Scholes equation, then stochastic volatility models are proposed. We simply introduce two common stochastic volatility models including the Hull-White model (1987) and the Stein-Stein model (1991). Furthermore, we emphasize the Heston model (1993) and analyze each parameter. In the next step, we use the Heston model to calibrate an option pricing model. Finally, we compare the Heston model and Black-Scholes equation. In the experiment of sensitivity analysis, we figure out that the correlation coefficient ρ and the volatility of volatility σ in the Heston affect the skewness and kurtosis of the stock distribution respectively.

Appendix

A. Closed-form for Stock Distribution (Stein-Stein stochastic model)

$$A = \frac{-\sigma}{k^2}$$

$$B = \frac{\theta\sigma}{k^2}$$

$$C = \frac{-\lambda}{k^2 t}$$

$$a = (A^2 - 2C)^{\frac{1}{2}}$$

$$b = \frac{-A}{a}$$

$$L = -A - a(\frac{\sinh(ak^2t) + bcosh(ak^2t)}{\cosh(ak^2t) + bsinh(ak^2t)})$$

$$M = B\{\frac{bsinh(ak^2t) + b^2cosh(ak^2t) + 1 - b^2}{cosh(ak^2t) + bsinh(ak^2t)} - 1\}$$

$$N = \frac{a-A}{2a^2}[a^2 - AB^2 - B^2a]k^2t + \frac{B^2[A^2 - a^2]}{2a^3}$$

$$\times\{\frac{(2A+a) + (2A-a)e^{2ak^2t}}{A + a + (a-A)e^{2ak^2t}}$$

$$+ \frac{2AB^2(a^2 - A^2)e^{2ak^2t}}{a^3\left(A + a + (a-A)e^{2ak^2t}\right)}$$

$$-\frac{1}{2}\log\{\frac{1}{2}\left(\frac{A}{a} + 1\right) + \frac{1}{2}\left(1 - \frac{A}{a}\right)e^{2ak^2t}\}$$

$$I = \exp(\frac{L{\sigma_0}^2}{2} + M\sigma_0 + N)$$

The closed-form for stock distribution:

$$S_0(P,t) = (2\pi)^{-1}P^{-3/2} \times \int_{\eta=-\infty}^{\infty} I\left(\left(\eta^2+\frac{1}{4}\right)\frac{t}{2}\right)e^{i\eta logP}d\eta$$

$$S(P,t) = e^{-\mu t}S_0(Pe^{-\mu t})$$

B. Stock Price Simulation

Using Euler-Maruyama method, we discrete stochastic process $\{S_t\}$ and $\{V_t\}$. It leads to the equation following two equations:

$$V_{t=}V_{t-1} + \varkappa(\theta - V_{t-1})dt + \sigma\sqrt{V_{t-1}}dZ_t$$

$$S_t = S_{t-1} + \mu S_{t-1}dt + \sqrt{V_{t-1}}S_{t-1}dt\left[\rho dZ_t + \sqrt{1-\rho^2}dW_t\right]$$

The simulation MATLAB code is shown below:

```
function stochastic( S0,V0,mu,kappa,theta,sigma,delT,rho )

% Simulation of probability density distribution of the stock price with
% the Heston model
times=10000;
for i=1:times
random1=randn(1,times); %get random number from normal distribution
random2=randn(1,times);

S=zeros(1,times);
V=zeros(1,times);
V=V0+kappa*(theta-V0)*delT+sigma*sqrt(V0)*random1*sqrt(delT);
V=abs(V); %avoid negative values of volatility
S=S0+mu*S0*delT+S0*V.^(0.5).*(rho* random1+sqrt(1-rho^2)*
random2)*sqrt(delT);
figure; plot(S)
figure; plot(V)

end
```

C. Excel-VBA Code for the Heston Model Numerical Evaluation

This is the VBA code to solve the closed-form (3.6), and there are four main steps to make the evaluation.

1. *To define the operation of complex number. It needs to be calculated by real part and image part separately (Krantz, 1999).*

'Define a new type - Complex'

```
Type Complex
re As Double
im As Double
End Type
```

'Define real part and image part of a complex number'

```
Function Complex (a As Double, b As Double) As Complex
Complex.re = a
Complex.im = b
End Function
```

'Addition for complex number'

```
Function AddCpx (a As Complex, b As Complex) As Complex
AddCpx.re = a.re + b.re
AddCpx.im = a.im + b.im
End Function
```

'Subtraction for complex number'

```
Function SubCpx(a As Complex, b As Complex) As Complex
SubCpx.re = a.re - b.re
SubCpx.im = a.im - b.im
End Function
```

'Multiplication for complex number'

```
Function MultCpx(a As Complex, b As Complex) As Complex
MultCpx.re = a.re * b.re - a.im * b.im
MultCpx.im = a.re * b.im + a.im * b.re
End Function
```

'Division for complex number'

```
Function DivCpx(a As Complex, b As Complex) As Complex
DivCpx.re = (a.re * b.re + a.im * b.im) / (b.re ^ 2 + b.im ^ 2)
```

```
DivCpx.im = (a.im * b.re - a.re * b.im) / (b.re ^ 2 + b.im ^ 2)
End Function
```

<u>'Square for complex number'</u>

```
Function SqCpx(a As Complex) As Complex
SqCpx.re = a.re ^ 2 - a.im ^ 2
SqCpx.im = 2 * a.re * a.im
End Function
```

<u>'Square root for complex number'</u>

```
Function SqrtCpx(a As Complex) As Complex
w = Sqr(a.re ^ 2 + a.im ^ 2)
u = Atn(a.im / a.re)
SqrtCpx.re = Sqr(w) * Cos(u / 2)
SqrtCpx.im = Sqr(w) * Sin(u / 2)
End Function
```

<u>'Exponential for complex number'</u>

```
Function ExpCpx(a As Complex) As Complex
ExpCpx.re = Exp(a.re) * Cos(a.im)
ExpCpx.im = Exp(a.re) * Sin(a.im)
End Function
```

<u>'Natural log for complex number'</u>

```
Function LnCpx(a As Complex) As Complex
w = (a.re ^ 2 + a.im ^ 2) ^ 0.5
LnCpx.re = Application.Ln(w)
LnCpx.im = Atn(a.im / a.re)
End Function
```

2. *To gain the integrand in (3.6), which is Re* $\left[\frac{e^{-iu\ln K}\varphi_j(x,v,T;u)}{iu}\right]$. *(When* $i = 1$ *)*

<u>'Define the integrand that can yield P1'</u>

```
Function HestonP1(rho As Double, sigma As Double, phi As Double, kappa As Double, theta As Double, tau As Double, K As Double, S As Double, r As Double, v As Double) As Double

Dim w As Double, b1 As Double, d1 As Complex, g1 As Complex
Dim c111 As Complex, c1121 As Complex, c1122 As Complex, c112 As Complex, c113 As Complex
Dim c11 As Complex, CC1 As Complex, d11 As Complex, d12 As Complex, DD1 As Complex
Dim f1 As Complex, a As Complex, b As Complex

u1 = 0.5
w  =  rho * sigma * phi
b1 =  kappa - rho * sigma
d1 =  SqrtCpx(SubCpx(SqCpx(SubCpx(Complex(0, w), Complex(b1, 0))), MultCpx(Complex(sigma
```

```
    ^ 2, 0), SubCpx(Complex(0, 2 * u1 * phi), Complex(phi ^ 2, 0)))))
g1 = DivCpx(SubCpx(AddCpx(Complex(b1, 0), d1), Complex(0, w)), SubCpx(SubCpx(Complex(b1,
    0), d1), Complex(0, w)))
c111   = SubCpx(AddCpx(Complex(b1, 0), d1), Complex(0, w))
c1121 = ExpCpx(MultCpx(d1, Complex(tau, 0)))
c1122 = MultCpx(g1, c1121)
c112   = DivCpx(SubCpx(Complex(1, 0), c1122), SubCpx(Complex(1, 0), g1))
c113   = MultCpx(Complex(2, 0), LnCpx(c112))
c11     = SubCpx(MultCpx(c111, Complex(tau, 0)), c113)
CC1    = AddCpx(Complex(0, r * phi * tau), MultCpx(Complex((kappa * theta) / sigma ^ 2, 0), c11))
d11    = DivCpx(SubCpx(Complex(1, 0), c1121), SubCpx(Complex(1, 0), c1122))
d12    = DivCpx(c111, Complex(sigma ^ 2, 0))
DD1   = MultCpx(d11, d12)
f1       = ExpCpx(AddCpx(AddCpx(CC1, MultCpx(DD1, Complex(v, 0))), Complex(0, phi *
           Application.Ln(S))))
a = ExpCpx(Complex(0, -phi * Application.Ln(K)))
b = DivCpx(MultCpx(a, f1), Complex(0, phi))

HestonP1 = b.re 'Extract the real part of the complex number'
End Function
```

If $i = 2$, then we can simply replace "$u_1 = 0.5$" by " $u_2 = -0.5$", and set "$b_2 = \varkappa$" instead of

"$b_1 = \varkappa - \rho\sigma$".Then rest part of *HestonP2( )* is the same as *HestonP1( )*.

'Define the integrand that can yield P2'

```
Function HestonP2(rho As Double, sigma As Double, phi As Double, kappa As Double, theta As
Double, tau As Double, K As Double, S As Double, r As Double, v As Double) As Double

Dim w As Double, b1 As Double, d1 As Complex, g1 As Complex
Dim c111 As Complex, c1121 As Complex, c1122 As Complex, c112 As Complex, c113 As
Complex
Dim c11 As Complex, CC1 As Complex, d11 As Complex, d12 As Complex, DD1 As Complex
Dim f1 As Complex, a As Complex, b As Complex

u1 = -0.5
w  =  rho * sigma * phi
b1 =  kappa
d1 =  SqrtCpx(SubCpx(SqCpx(SubCpx(Complex(0, w), Complex(b1, 0))), MultCpx(Complex(sigma
    ^ 2, 0), SubCpx(Complex(0, 2 * u1 * phi), Complex(phi ^ 2, 0)))))
g1 = DivCpx(SubCpx(AddCpx(Complex(b1, 0), d1), Complex(0, w)), SubCpx(SubCpx(Complex(b1,
    0), d1), Complex(0, w)))
c111   = SubCpx(AddCpx(Complex(b1, 0), d1), Complex(0, w))
c1121 = ExpCpx(MultCpx(d1, Complex(tau, 0)))
c1122 = MultCpx(g1, c1121)
c112   = DivCpx(SubCpx(Complex(1, 0), c1122), SubCpx(Complex(1, 0), g1))
c113   = MultCpx(Complex(2, 0), LnCpx(c112))
c11     = SubCpx(MultCpx(c111, Complex(tau, 0)), c113)
CC1    = AddCpx(Complex(0, r * phi * tau), MultCpx(Complex((kappa * theta) / sigma ^ 2, 0), c11))
```

```
d11    = DivCpx(SubCpx(Complex(1, 0), c1121), SubCpx(Complex(1, 0), c1122))
d12    = DivCpx(c111, Complex(sigma ^ 2, 0))
DD1    = MultCpx(d11, d12)
f1     = ExpCpx(AddCpx(AddCpx(CC1, MultCpx(DD1, Complex(v, 0))), Complex(0, phi *
         Application.Ln(S))))
a = ExpCpx(Complex(0, -phi * Application.Ln(K)))
b = DivCpx(MultCpx(a, f1), Complex(0, phi))

HestonP2 = b.re 'Extract the real part of the complex number'
End Function
```

3. *Function Simpson ( ) sets up Simpson's Rule.*

'Simpsons Rule Integration'

```
Function Simpson(deltax, y) As Double
n = Application.Count(y)
sum = 0
For t = 1 To (n - 1) / 2
n = t * 2 - 1
sum = sum + (1 / 3) * (y(n) + 4 * y(n + 1) + y(n + 2)) * deltax
Next t
Simpson = sum
End Function
```

4. *Last Step is to price call option using the equation (3.4).*

'Pricing an European call option using Heston model'

```
Function HestonCall(rho As Double, sigma As Double, kappa As Double, theta As Double, tau As
Double, K As Double, S As Double, r As Double, v As Double, deltax As Double) As Double

Dim P11(1001) As Double, P22(1001) As Double
Dim p1 As Double, p2 As Double, phi As Double


Pi = Application.Pi() * 1 'define the value of Pi'

'Using Simpsons Rule to do the integration'
n = 1
For phi = 0.0001 To 100.0001 Step 0.1
P11(n) = HestonP1(rho, sigma, phi, kappa, theta, tau, K, S, r, v)
P22(n) = HestonP2(rho, sigma, phi, kappa, theta, tau, K, S, r, v)
n = n + 1
Next phi

p1 = 0.5 + (1 / Pi) * Simpson(deltax, P11)
p2 = 0.5 + (1 / Pi) * Simpson(deltax, P22)

'Ensure the probability if between 0 and 1'
If p1 < 0 Then p1 = 0
If p1 > 1 Then p1 = 1
If p2 < 0 Then p2 = 0
```

```
If p2 > 1 Then p2 = 1

HestonCallF = S * p1 - K * Exp(-tau * r) * p2

'Ensure price is non-negative'
If HestonCallF < 0 Then HestonCall = 0
If HestonCallF > 0 Then HestonCall = HestonCallF

End Function
```

D. Generalized Reduced Gradient Optimization Method

The GRG method is an algorithm well known in the mathematical programming arena for solving optimization problems. The steps for GRG are shown below:

- Step1: linearizing the non-linear objective and constraint functions at a local solution with Taylor expansion equation.
- Step 2: divides the variable set into two subsets of basic and non-basic variable and the concept of implicit variable elimination to express the basic variable by the non-basic variable
- Step 3: eliminate the constraints and deduce the variable space to only non-basic variables

The mathematic form is:

$$Minf(x): h(x) = 0,$$

subject to

$$L \leq x \leq U,$$

where h has dimension m. The method supposes we can partition $x = (v, w)$ such that:

- v has dimension m and w has dimension n-m
- The value of v are strictly within their bounds: $L < v < U$
- $Grad\nabla_v[h(x)]$ is nonsingular at $x = (v, w)$.

For any w there is a unique value, $v(w)$, such that $h(v(w), w) = 0$, which implies that

$$\frac{dv}{dw} = Grad\nabla_v[h(x)]^{-1} Grad\nabla_w[h(x)]$$

The idea is to choose the direction of the independent variables to be the reduced gradient: $Grad\nabla_w[f(x) - yh(x)]$, where

$$y = \frac{dv}{dw} = Grad\nabla_v[h(x)]^{-1} Grad\nabla_w[h(x)]$$

Then, the step size is chosen and a correction procedure applied to return to the surface $h(x) = 0$

E. The Sample of Market Data Used to Calibrate

European call option for Google Inc. (GOOG)						
Price recorded on Apr 6, 2013						
Expire at May 17,2013						
Days to maturity		*41*				
interest rate (r)	Time to Maturity (τ)	Spot Price(S)	Strike Price(K)	Bid Price	Ask Price	**Market Price**
0.000151644	0.112328767	783.05	510	271.00	274.80	**272.90**
0.000151644	0.112328767	783.05	590	191.50	195.00	**193.25**
0.000151644	0.112328767	783.05	600	181.90	185.00	**183.45**
0.000151644	0.112328767	783.05	615	167.10	170.60	**168.85**
0.000151644	0.112328767	783.05	625	157.20	160.80	**159.00**
0.000151644	0.112328767	783.05	630	152.70	155.50	**154.10**
0.000151644	0.112328767	783.05	645	137.50	140.90	**139.20**
Expire at Sep 20,2013						
Days to maturity		*167*				
interest rate (r)	Time to Maturity (τ)	Spot Price(S)	Strike Price(K)	Bid Price	Ask Price	**Market Price**
0.000617671	0.457534247	783.05	395	386.70	389.90	**388.30**
0.000617671	0.457534247	783.05	410	371.70	375.10	**373.40**
0.000617671	0.457534247	783.05	435	346.90	350.20	**348.55**
0.000617671	0.457534247	783.05	450	332.00	335.30	**333.65**

0.000617671	0.457534247	783.05	500	282.70	286.00	**284.35**
0.000617671	0.457534247	783.05	525	258.40	261.60	**260.00**
0.000617671	0.457534247	783.05	550	233.90	237.10	**235.50**
Expire at Jan 17,2014						
Days to maturity		*286*				
interest rate (r)	Time to Maturity (τ)	Spot Price(S)	Strike Price(K)	Bid Price	Ask Price	**Market Price**
0.001057808	0.783561644	783.05	395	387.60	391.80	**389.70**
0.001057808	0.783561644	783.05	410	373.00	377.00	**375.00**
0.001057808	0.783561644	783.05	435	348.40	352.50	**350.45**
0.001057808	0.783561644	783.05	450	333.80	337.60	**335.70**
0.001057808	0.783561644	783.05	500	285.50	289.30	**287.40**
0.001057808	0.783561644	783.05	525	261.40	265.70	**263.55**
0.001057808	0.783561644	783.05	550	238.30	241.90	**240.10**

Note: Market price is the average of bid price and ask price, which can be expressed as:

$$Market\ price = \frac{bid\ price + ask\ price}{2}$$

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