Independent sums of Haar system Hardy spaces Workshop in Analysis and Probability Seminar

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York University

July 30, 2025

Based on joint work with Konstantinos Konstantos.

Overview

Introduction

2 Definitions and main result

Proof

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3 Proof

- Let X be a (real) Banach space.

$$\begin{array}{ccc}
X & \xrightarrow{S} & X \\
B \downarrow & & \uparrow A \\
X & \xrightarrow{T} & X
\end{array}$$

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- Let X be a (real) Banach space.
- Let $\mathcal{B}(X)$ be the set of all bounded linear operators $T\colon X\to X$.

Definition

Let $S, T \in \mathcal{B}(X)$. We say that S factors through T if there are $A, B \in \mathcal{B}(X)$ such that S = ATB:

$$\begin{array}{ccc}
X & \xrightarrow{S} & X \\
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Definition

We say that X has the *primary factorization property* if for every operator $T \in \mathcal{B}(X)$, the identity I_X factors through T or $I_X - T$.

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Primary Banach spaces

- If X has the primary factorization property and $X \sim \ell^p(X)$ for some $1 \leq p \leq \infty$, then X is primary (i.e., $X \sim Y \oplus Z \implies X \sim Y$ or $X \sim Z$).
- Proof idea: If the identity factors through a projection $P \in \mathcal{B}(X)$, then P(X) has a complemented subspace isomorphic to X.

Operator ideals

- Define $\mathcal{M}_X = \{T \in \mathcal{B}(X) : I_X \text{ does not factor through } T\}$
- The set \mathcal{M}_X is an ideal of $\mathcal{B}(X) \iff X$ has the primary factorization property (Dosev-Johnson 2010).
- In that case, \mathcal{M}_X is the *largest ideal* of $\mathcal{B}(X)$

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Definition

We say that a Schauder basis $(e_j)_j$ of X has the factorization property if the identity I_X factors through every operator $T \in \mathcal{B}(X)$ with large diagonal, i.e., $\inf_j |\langle e_j^*, Te_j \rangle| > 0$.

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 - \rightarrow Analogous results + primariness for $\ell^p(Y)$, $1 \le p < \infty$.
- Konstantos, Motakis '25: Both factorization properties hold in the Bourgain-Rosenthal-Schechtman R^p_ω space, $1 (= independent sum of <math>L^p_n$ spaces)
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- Dyadic intervals: $\mathcal{D}=\left\{[0,1),[0,\frac{1}{2}),[\frac{1}{2},1),[0,\frac{1}{4}),[\frac{1}{4},\frac{1}{2}),\dots\right\}$
- $I^+ = \text{left half}, I^- = \text{right half of } I \in \mathcal{D}$
- Define $h_I = \mathbb{1}_{I^+} \mathbb{1}_{I^-}$, $I \in \mathcal{D}$.
- Together with $\mathbb{1}_{[0,1)}$, the Haar system $(h_I)_{I\in\mathcal{D}}$ is a monotone Schauder basis of L^p , $1\leq p<\infty$.
- Rademacher functions: $r_n = \sum_{|K|=2^{-n}} h_K$ for $n \in \mathbb{N}_0$.

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• A Haar system space X is the completion of $H = \operatorname{span}\{\mathbb{1}_{[0,1)}, h_I : I \in \mathcal{D}\}$ under a norm $\|\cdot\|_X$ such that:

- \bullet If $x,y\in H$ and |x| , |y| have the same distribution, then $\|x\|_X=\|y\|_X$
- $\|\mathbb{1}_{[0,1)}\|_X = 1.$
- Examples: L^p , $1 \le p < \infty$, all separable rearrangement-invariant function spaces
- Haar system Hardy space $Y = \text{completion of span}\{h_I : I \in \mathcal{D}\}$ under $\|\cdot\|_X$ or under the square function norm $\|\cdot\|_{\circ}$ given by

$$\left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{\circ} = \left\| \left(\sum_{I \in \mathcal{D}} a_I^2 h_I^2 \right)^{1/2} \right\|_{X}$$

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- Now fix a Haar system Hardy space Y (e.g., H^1).
- Construct **independent** distributional copies (in Y) of the spaces $Y_n = \operatorname{span}\{h_I : |I| \geq 2^{-n}\}, n \in \mathbb{N}_0$ (replacing the functions h_I by blocks $h_I^n = \sum_{K \in \mathcal{B}^n} h_K$).
- Define Y_{ω} as the closed linear span of the functions h_I^n for $n\in\mathbb{N}_0$ and $|I|\geq 2^{-n}$. Then $(h_I^n)_{(n,I)}$ is a monotone Schauder basis of Y_{ω}



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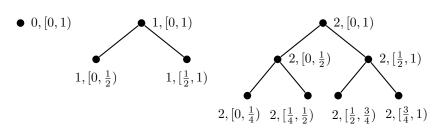
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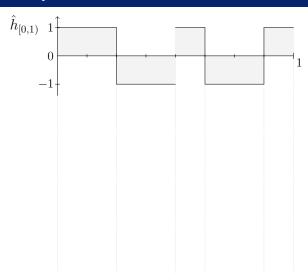


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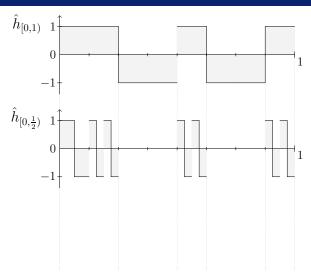


Faithful Haar systems



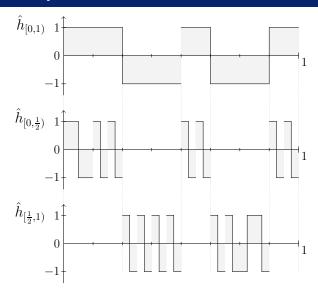
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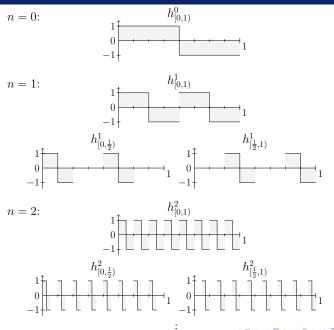
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$$D \approx A_1 T B_1, \quad E = A_2 D B_2, \dots$$

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- Basic proof idea: Step-by-step reduction.

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- Basic proof idea: Step-by-step reduction. Operator $T \to \text{diagonal operator } D \to \ldots \to \text{constant multiple of the}$

$$D \approx A_1 T B_1, \quad E = A_2 D B_2, \dots$$

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• Projectional factorizations: $A_iB_i = I_{Y_{ij}}$

- ullet How are A_i, B_i defined? o faithful Haar system $(b_I^n)_{(n,I)}$
- ullet For every n, construct a finite faithful Haar system $(\hat{h}^n_I)_{|I|>2^{-n}}$ by

$$\hat{h}_I^n = \sum_{K \in \mathcal{B}_I^n} \theta_K^n h_K, \qquad \mathcal{B}_I^n \subset \mathcal{D}, \ \theta_K = \pm 1,$$

and place it in the N(n)th component of Y_{ω} :

$$b_I^n = \sum_{K \in \mathcal{B}_I^n} \theta_K^n h_K^{N(n)}.$$

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$$Bx = \sum_{(n,I)} \frac{\langle h_I^n, x \rangle}{|I|} b_I^n, \qquad Ax = \sum_{(n,I)} \frac{\langle b_I^n, x \rangle}{|I|} h_I^n$$

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 - \rightarrow here: exploit structure of A, B, using conditional expectations $\implies ||A||, ||B|| < 1.$

First step: diagonalization via random faithful Haar systems

$$\hat{h}_I(\theta) = \sum_{K \in \mathcal{B}_I(\theta)} \theta_K h_K, \qquad |I| \ge 2^n,$$

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First step: diagonalization via random faithful Haar systems

- First restrict T to a single component of Y_{ω} , identify it with an operator $T_N \colon Y_N \to Y_N$ (where $N \gg n$).
- Choose signs $\theta = (\theta_K)_{K \in \mathcal{D}} \in \{\pm 1\}^{\mathcal{D}}$ uniformly at random.
- Construct a finite randomized system $(\hat{h}_I(\theta))_I$ by

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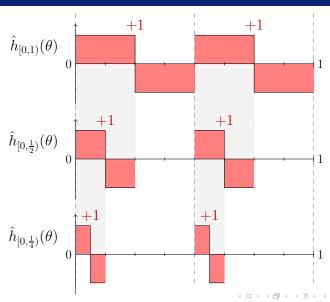
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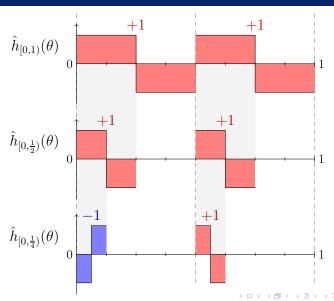
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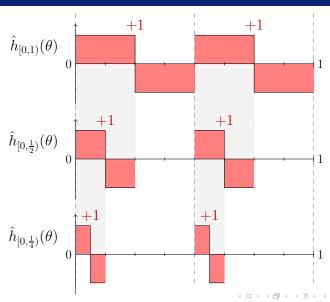
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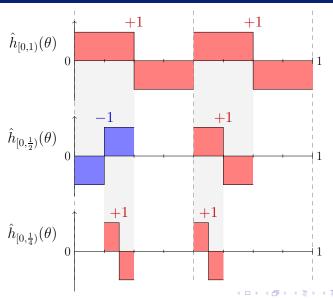
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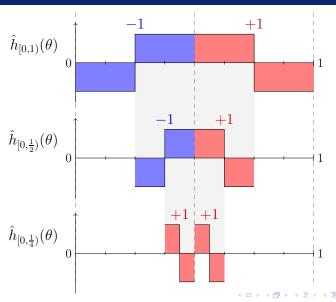




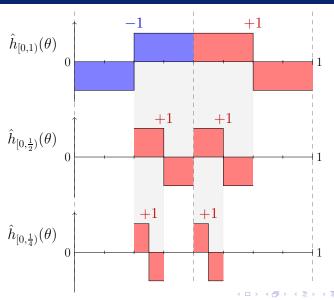
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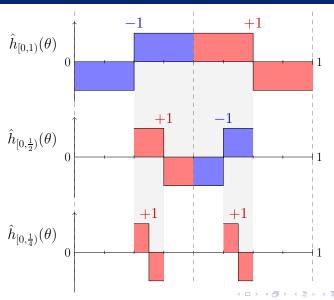


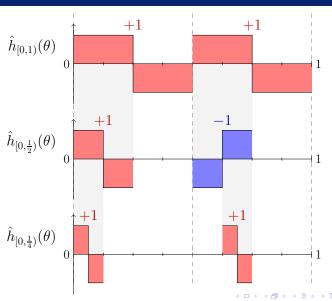


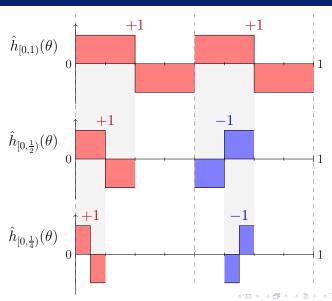


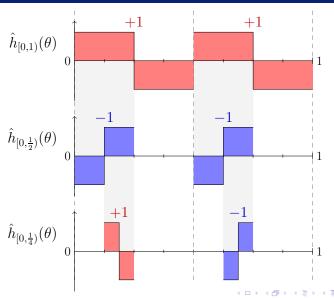
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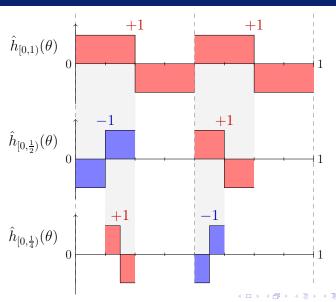


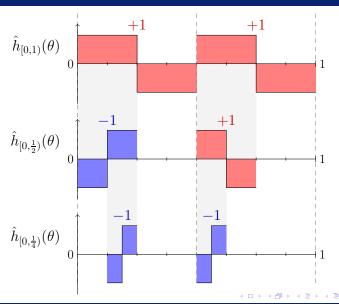




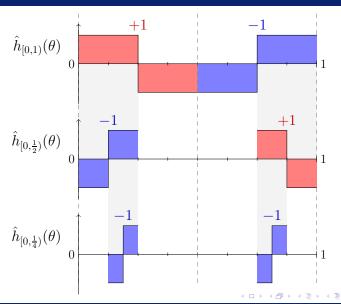




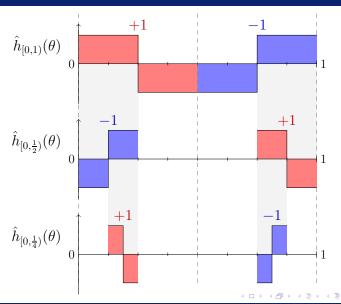




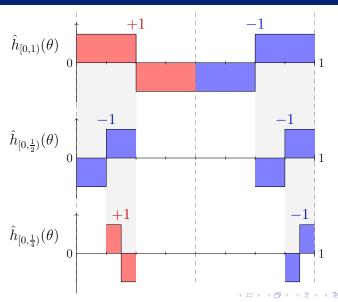
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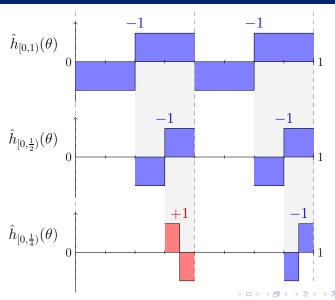


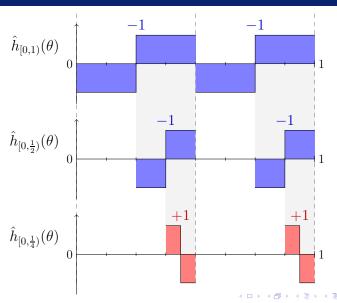
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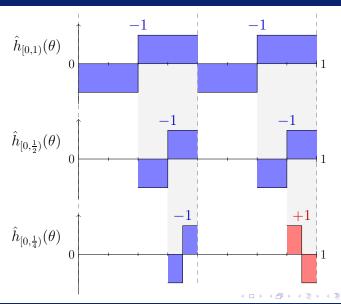
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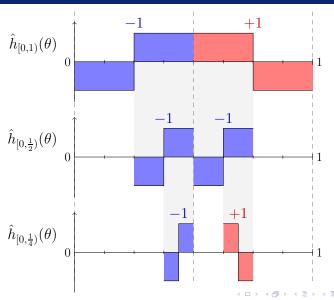


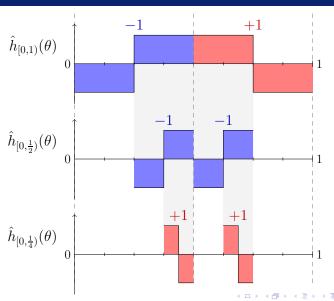


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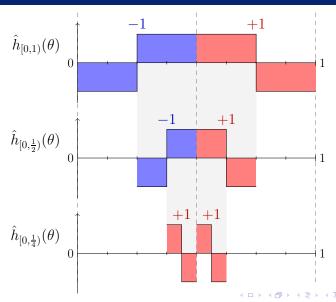


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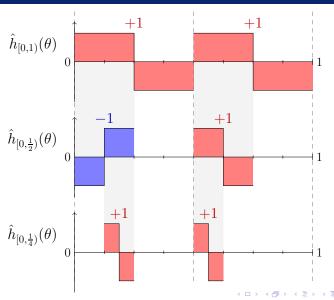


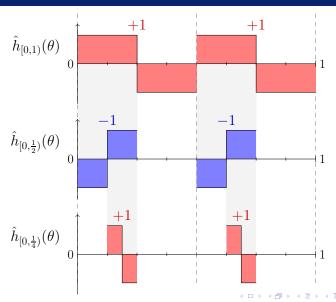


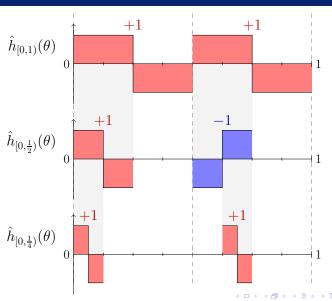
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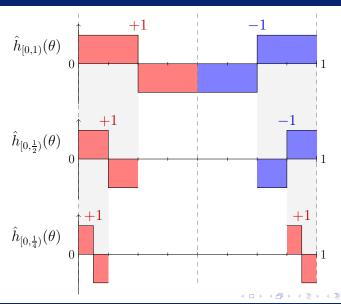


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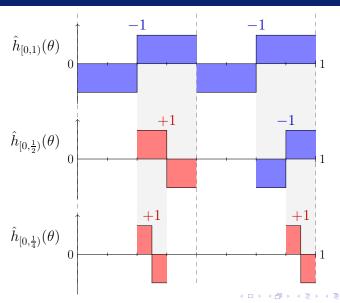




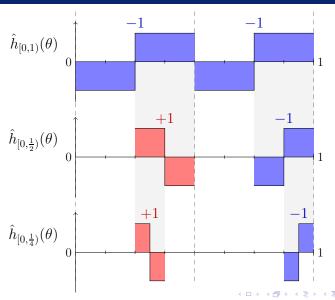


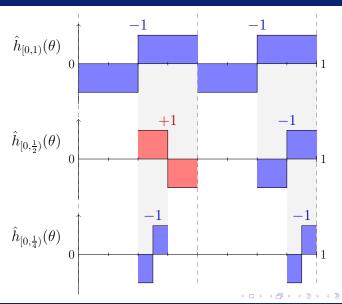


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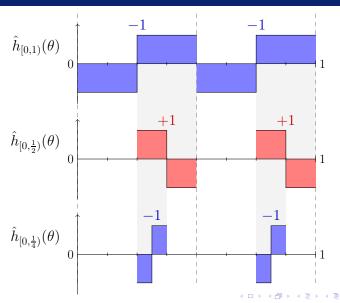


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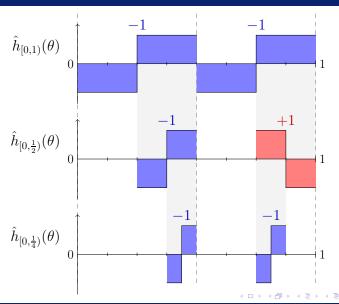




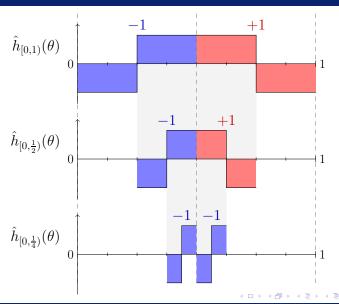
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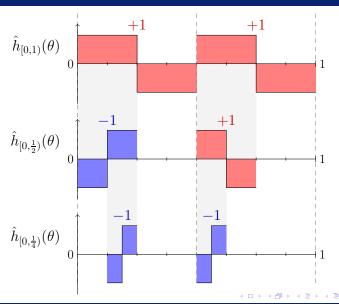
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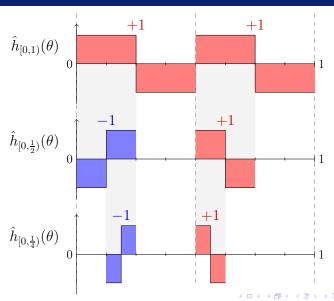
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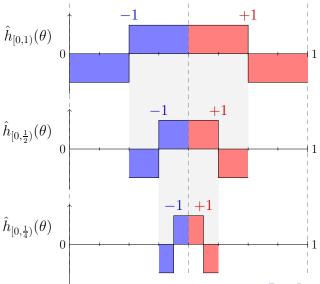


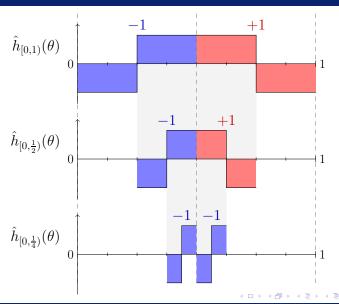
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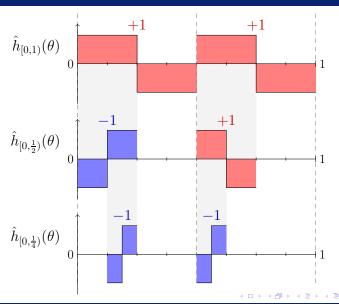
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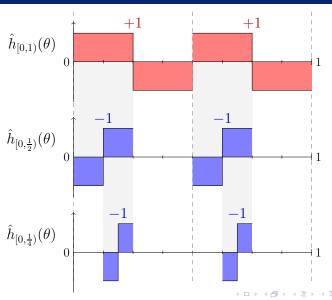


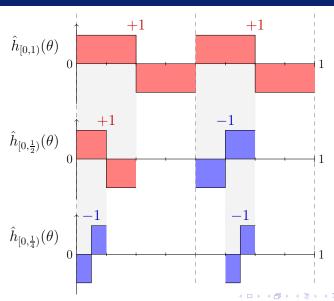


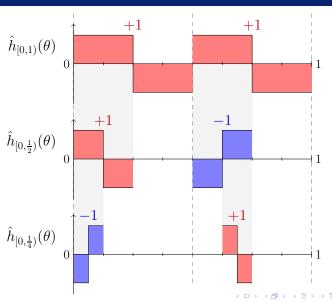
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$$\mathbb{E} X_{I,J} = 0 \text{ for } I \neq J \qquad \text{and} \qquad \mathbb{V} X_{I,J} \leq 3 \|T\|^2 2^{-m/2}$$
 ($m=$ first level used in our construction)

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 ($m = \text{first level used in our construction}$)

- Choose m large \implies for some realization of θ , the system almost diagonalizes T_N .
- ullet Bonus: $\mathbb{E} X_{I,I}$ only depends on $|I| \implies$ level-wise stabilization of T_N .
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$$b_I^n = \sum_{K \in \mathcal{B}_I^n} \theta_K^n h_K^{N(n)},$$

such that $\langle b_I^n, Tb_J^n \rangle$ is small for $I \neq J$. Choose N(n) inductively such that also $\langle b_I^n, Tb_J^m \rangle$ is small for $n \neq m$.

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Second step: Reduction to a scalar operator

- From above, we obtain operators A, B such that ATB is close to a diagonal operator D (and D is stabilized level-wise).
- Stabilize D in every component of Y_{ω} using the pigeonhole principle \to operator C with $Ch_I^n=c_nh_I^n$ for all (n,I)
- Finally: Choose a cluster point c of $(c_n)_n$.
- Then $cI_{Y_{\omega}}$ projectionally factors through T. $\implies Y_{\omega}$ has the primary factorization property.
- Additional reduction step (reduction to positive diagonal) using Gamlen-Gaudet construction $\Rightarrow (h_n^n)_{(n,T)}$ has the factorization property in Y_ω .

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References



Konstantinos Konstantos and Thomas Speckhofer.

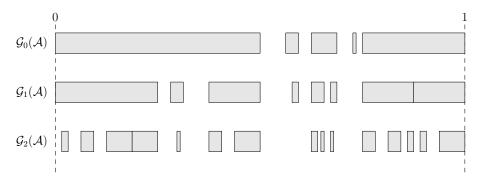
Factorization in independent sums of Haar system Hardy spaces.

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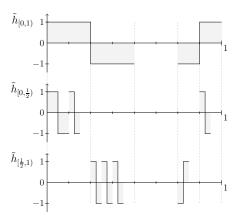
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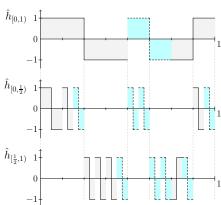
Thank you for your attention!

The Gamlen-Gaudet construction



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