# Independent sums of Haar system Hardy spaces

Workshop in Analysis and Probability Seminar

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Based on joint work with Konstantinos Konstantos.

# Overview

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# Introduction

- Let $X$ be a (real) Banach space.
- Let $\mathcal{B}(X)$ be the set of all bounded linear operators $T\colon X \to X$.

**Definition**

Let $S, T \in \mathcal{B}(X)$. We say that $S$ *factors through* $T$ if there are $A, B \in \mathcal{B}(X)$ such that $S = ATB$:

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "B"'] & X \\ X \arrow[r, "T"] & X \arrow[u, "A"'] \end{tikzcd}$$

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We say that $X$ has the *primary factorization property* if for every operator $T \in \mathcal{B}(X)$, the identity $I_X$ factors through $T$ or $I_X - T$.

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# Applications

## Primary Banach spaces

- If $X$ has the primary factorization property and $X \sim \ell^p(X)$ for some $1 \leq p \leq \infty$, then $X$ is primary (i.e., $X \sim Y \oplus Z \implies X \sim Y$ or $X \sim Z$).
- Proof idea: If the identity factors through a projection $P \in \mathcal{B}(X)$, then $P(X)$ has a complemented subspace isomorphic to $X$.

## Operator ideals

- Define $\mathcal{M}_X = \{T \in \mathcal{B}(X) : I_X$ does **not** factor through $T\}$.
- The set $\mathcal{M}_X$ is an ideal of $\mathcal{B}(X) \iff X$ has the primary factorization property (Dosev-Johnson 2010).
- In that case, $\mathcal{M}_X$ is the *largest ideal* of $\mathcal{B}(X)$.

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# Recent results

- *Lechner, S. '25:* Every **Haar system Hardy space** $Y$ (in which $(r_n)_n$ is weakly null) has the primary factorization property, and its Haar basis has the factorization property.
  $\rightarrow$ Analogous results + primariness for $\ell^p(Y)$, $1 \leq p < \infty$.
- *Konstantos, Motakis '25:* Both factorization properties hold in the **Bourgain-Rosenthal-Schechtman $R^p_\omega$ space**, $1 < p < \infty$ ($=$ independent sum of $L^p_n$ spaces)
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# Definitions

- Dyadic intervals: $\mathcal{D} = \{[0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots\}$
- $I^+$ = left half, $I^-$ = right half of $I \in \mathcal{D}$
- Define $h_I = \mathbb{1}_{I^+} - \mathbb{1}_{I^-}$, $I \in \mathcal{D}$.
- Together with $\mathbb{1}_{[0,1)}$, the Haar system $(h_I)_{I\in\mathcal{D}}$ is a monotone Schauder basis of $L^p$, $1 \leq p < \infty$.
- Rademacher functions: $r_n = \sum_{|K|=2^{-n}} h_K$ for $n \in \mathbb{N}_0$.

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- A **Haar system space** $X$ is the completion of $H = \operatorname{span}\{\mathbb{1}_{[0,1)}, h_I : I \in \mathcal{D}\}$ under a norm $\|\cdot\|_X$ such that:
  - If $x, y \in H$ and $|x|$, $|y|$ have the same distribution, then $\|x\|_X = \|y\|_X$.
  - $\|\mathbb{1}_{[0,1)}\|_X = 1$.
- Examples: $L^p$, $1 \leq p < \infty$, all separable rearrangement-invariant function spaces
- **Haar system Hardy space** $Y$ = completion of $\operatorname{span}\{h_I : I \in \mathcal{D}\}$ under $\|\cdot\|_X$ or under the square function norm $\|\cdot\|_\circ$ given by

$$\Big\|\sum_{I\in\mathcal{D}} a_I h_I\Big\|_\circ = \Big\|\Big(\sum_{I\in\mathcal{D}} a_I^2 h_I^2\Big)^{1/2}\Big\|_X.$$

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# Definitions

- Now fix a Haar system Hardy space $Y$ (e.g., $H^1$).
- Construct **independent** distributional copies (in $Y$) of the spaces $Y_n = \operatorname{span}\{h_I : |I| \geq 2^{-n}\}$, $n \in \mathbb{N}_0$ (replacing the functions $h_I$ by blocks $h_I^n = \sum_{K \in \mathcal{B}_I^n} h_K$).
- Define $Y_\omega$ as the closed linear span of the functions $h_I^n$ for $n \in \mathbb{N}_0$ and $|I| \geq 2^{-n}$. Then $(h_I^n)_{(n,I)}$ is a monotone Schauder basis of $Y_\omega$.

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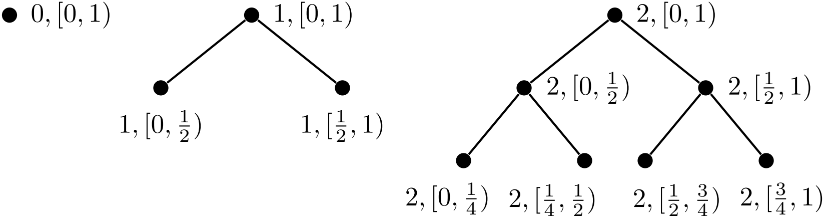
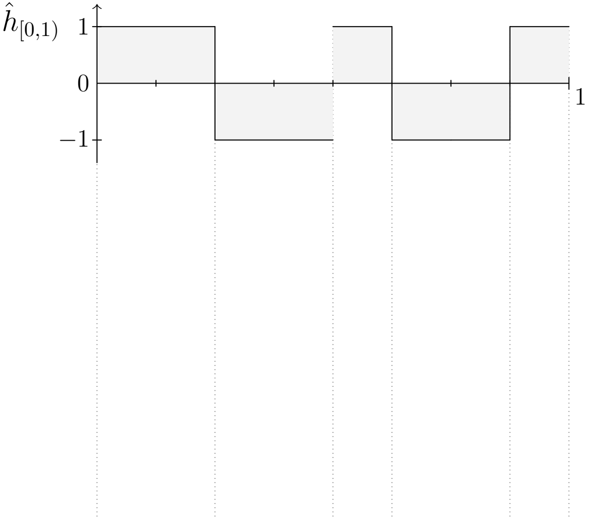


# Faithful Haar systems



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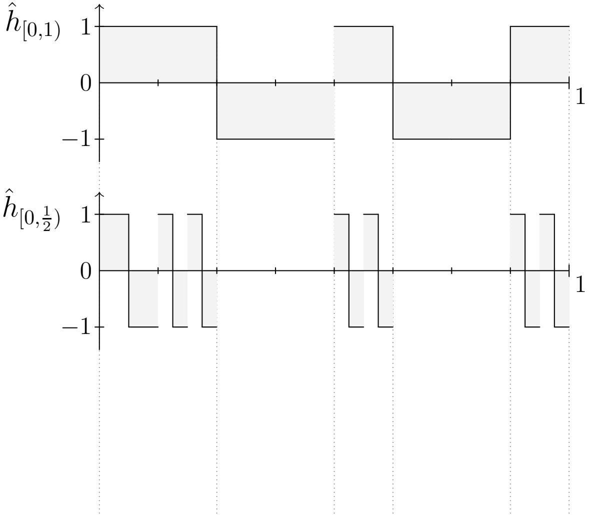

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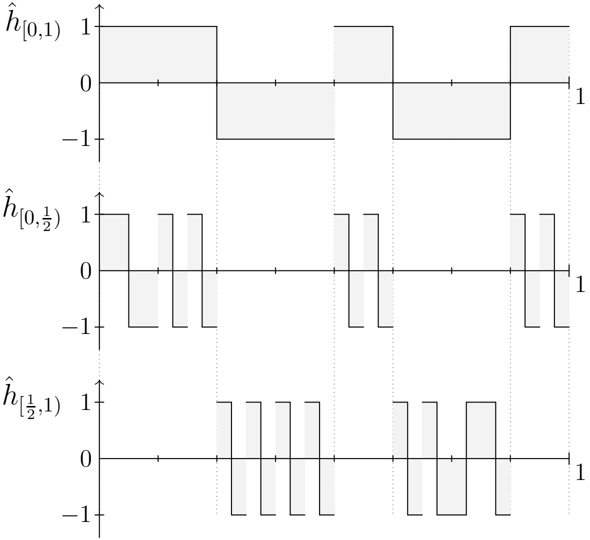
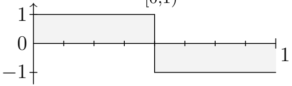


$n = 0$:

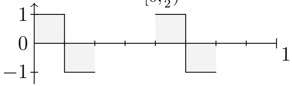
$h^0_{[0,1)}$



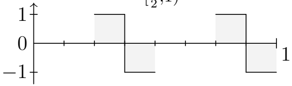

$n = 1$:

$h^1_{[0,1)}$


$h^1_{[0,\frac{1}{2})}$

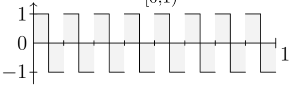



$h^1_{[\frac{1}{2},1)}$

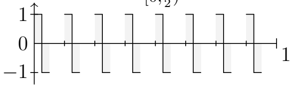



$n = 2$:

$h^2_{[0,1)}$




$h^2_{[0,\frac{1}{2})}$

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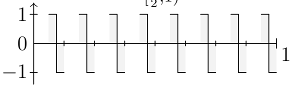


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# Main result

**Theorem (Konstantos and S. '25)**

*Suppose that $(r_n)_n$ is weakly null in a Haar system Hardy space $Y$. Then $Y_\omega$ has the primary factorization property, and its basis $(h_I^n)_{(n,I)}$ has the factorization property. In particular, $\mathcal{M}_{Y_\omega}$ is the largest ideal of $\mathcal{B}(Y_\omega)$.*

- Basic proof idea: Step-by-step reduction. Operator $T \to$ diagonal operator $D \to \ldots \to$ constant multiple of the identity, $cI_{Y_\omega}$
- Clearly, the identity factors through $cI_{Y_\omega}$ or $(1-c)I_{Y_\omega}$.

$$D \approx A_1 T B_1, \quad E = A_2 D B_2, \ \ldots$$

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- How are $A_i, B_i$ defined? $\rightarrow$ *faithful Haar system* $(b_I^n)_{(n,I)}$
- For every $n$, construct a finite faithful Haar system $(\hat{h}_I^n)_{|I| \geq 2^{-n}}$ by

$$\hat{h}_I^n = \sum_{K \in \mathcal{B}_I^n} \theta_K^n h_K, \qquad \mathcal{B}_I^n \subset \mathcal{D},\ \theta_K = \pm 1,$$

and place it in the $N(n)$th component of $Y_\omega$:

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- Associated operators $A, B$ on $Y_\omega$:

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# Proof: Diagonalization and level-wise stabilization

**First step:** ***diagonalization via random faithful Haar systems***

- First restrict $T$ to a single component of $Y_\omega$, identify it with an operator $T_N : Y_N \to Y_N$ (where $N \gg n$).
- Choose signs $\theta = (\theta_K)_{K \in \mathcal{D}} \in \{\pm 1\}^{\mathcal{D}}$ uniformly at random.
- Construct a finite randomized system $(\hat{h}_I(\theta))_I$ by

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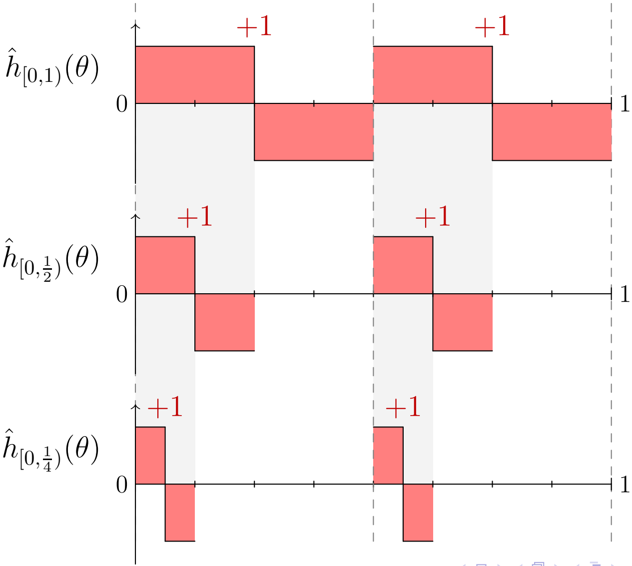

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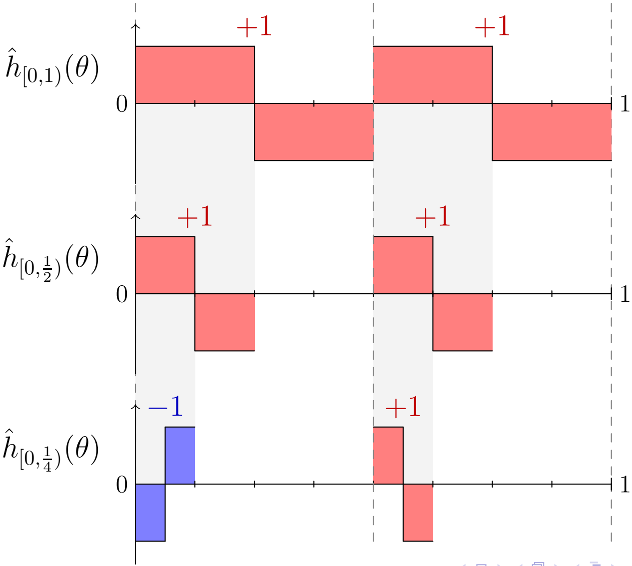

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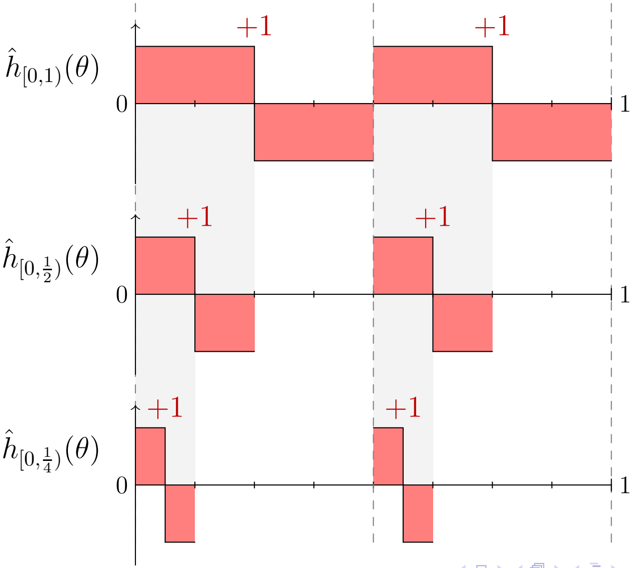

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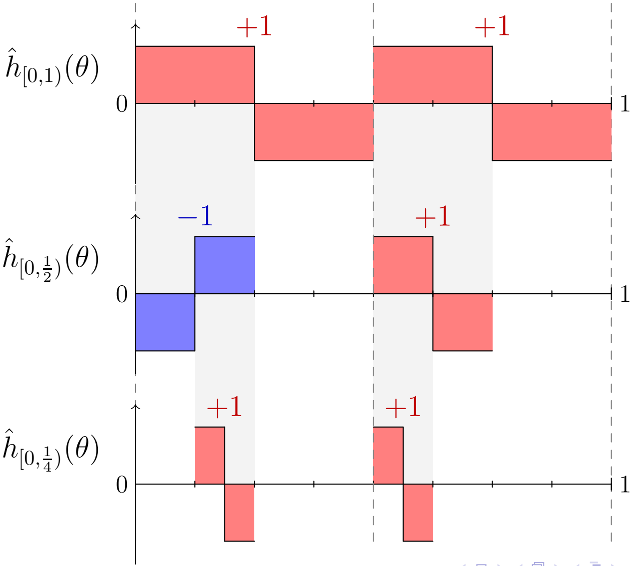

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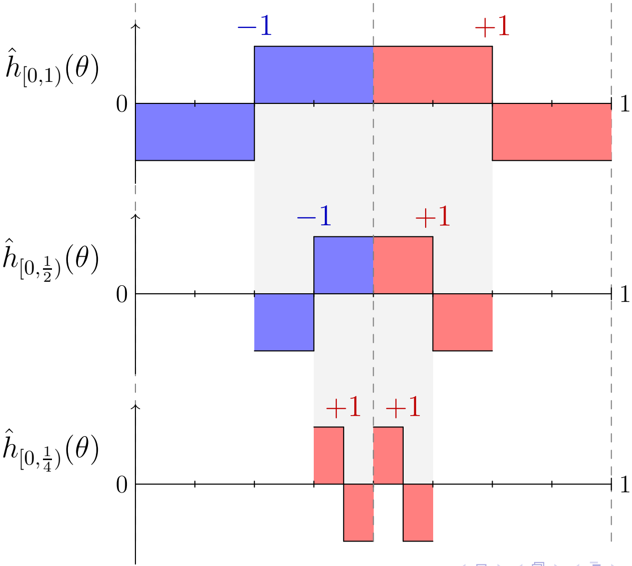

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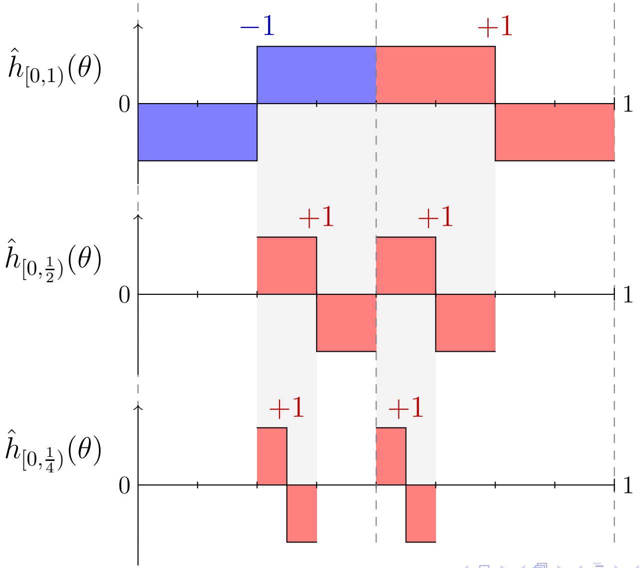

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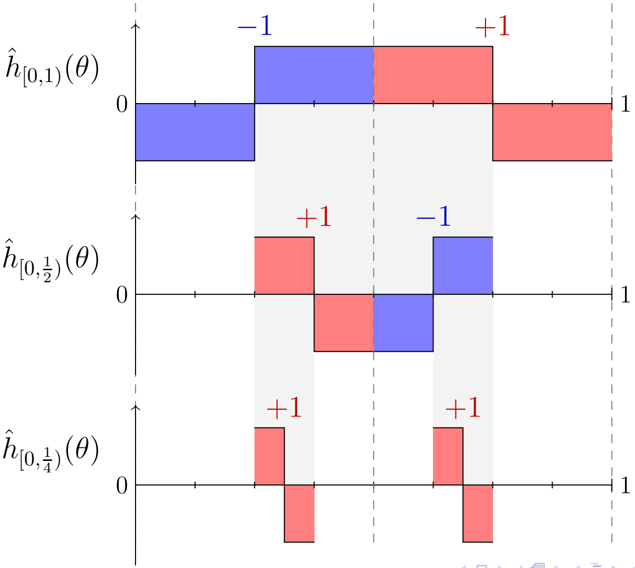

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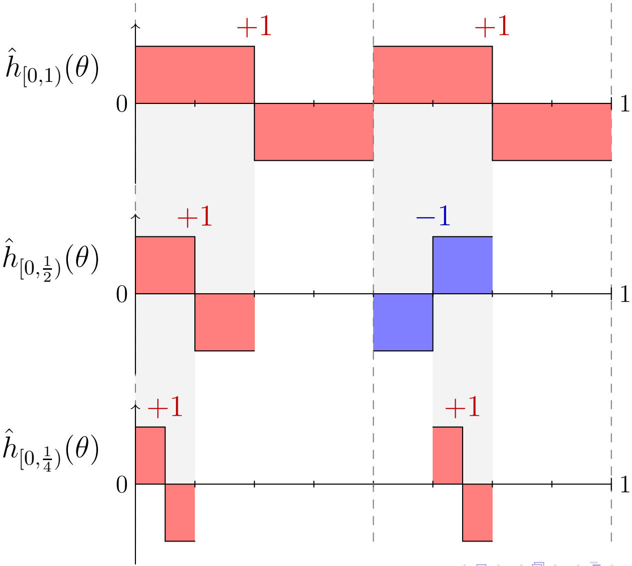

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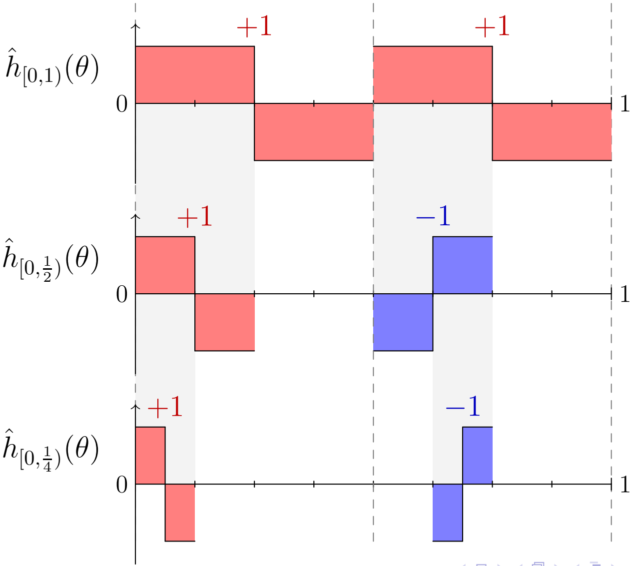

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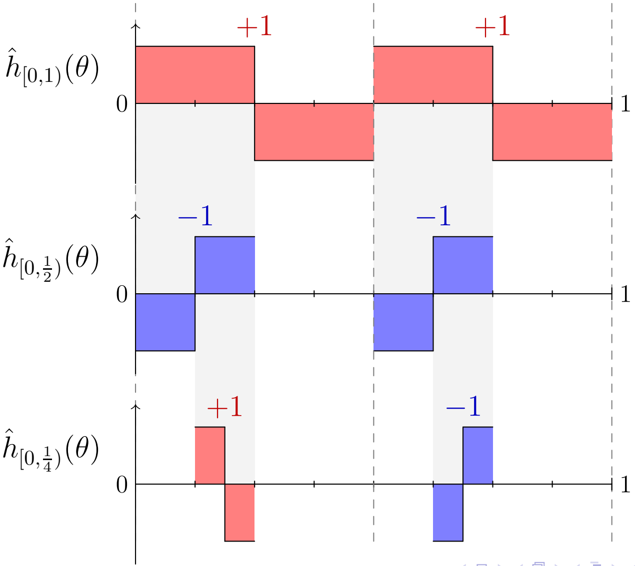

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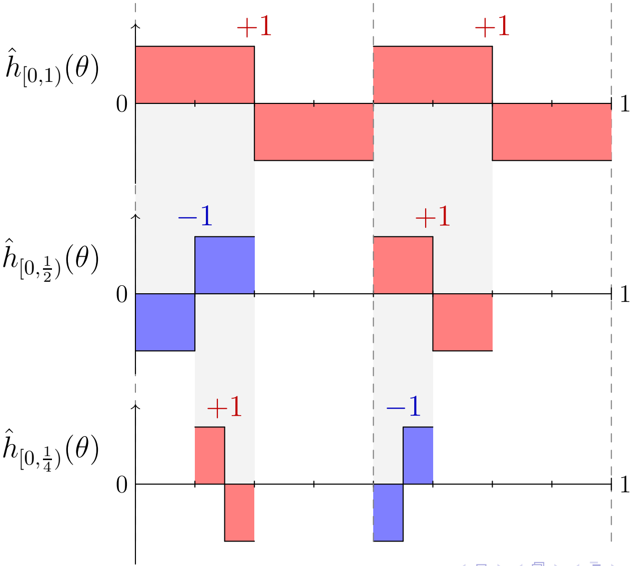

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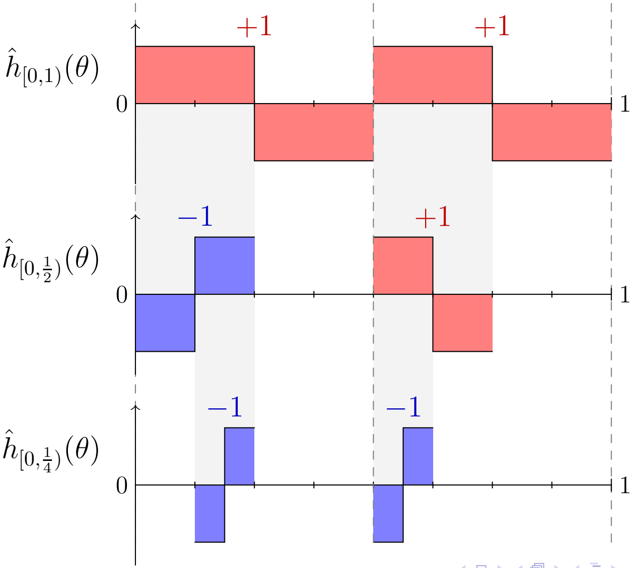

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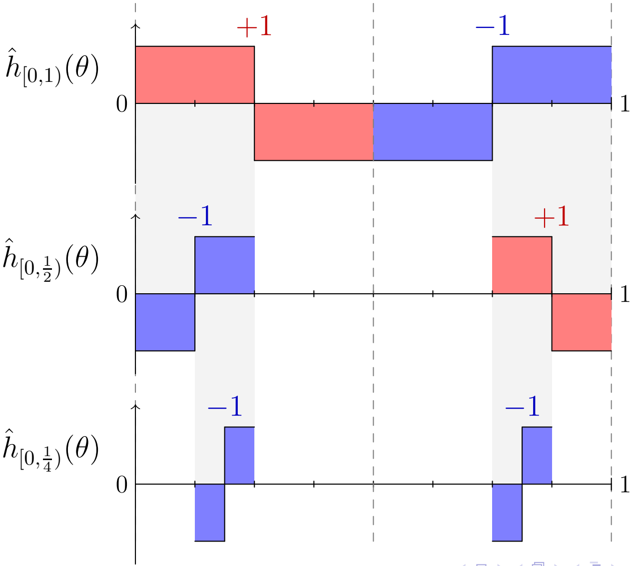

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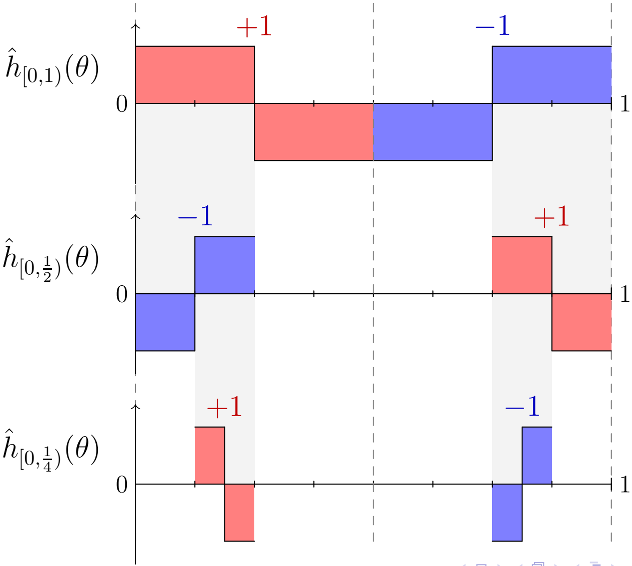

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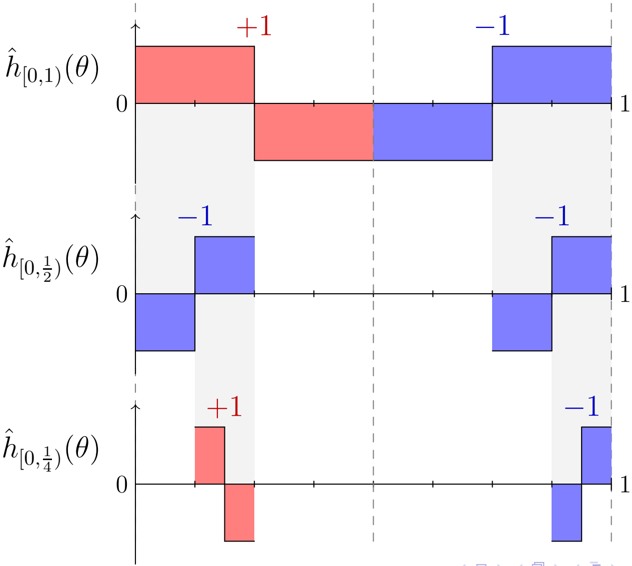

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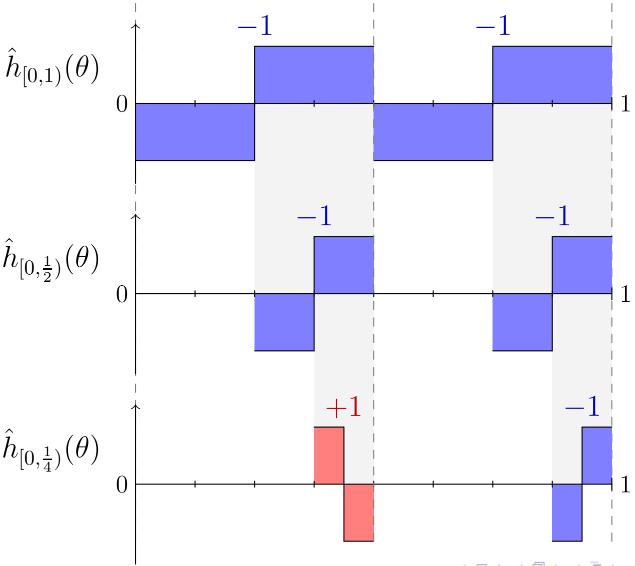

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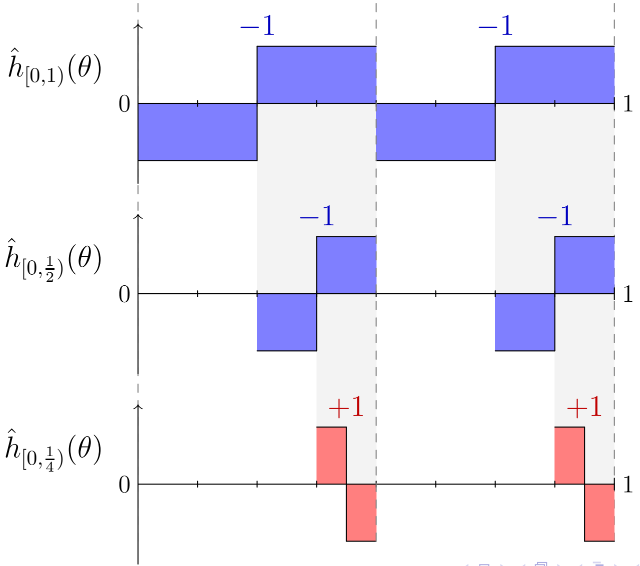

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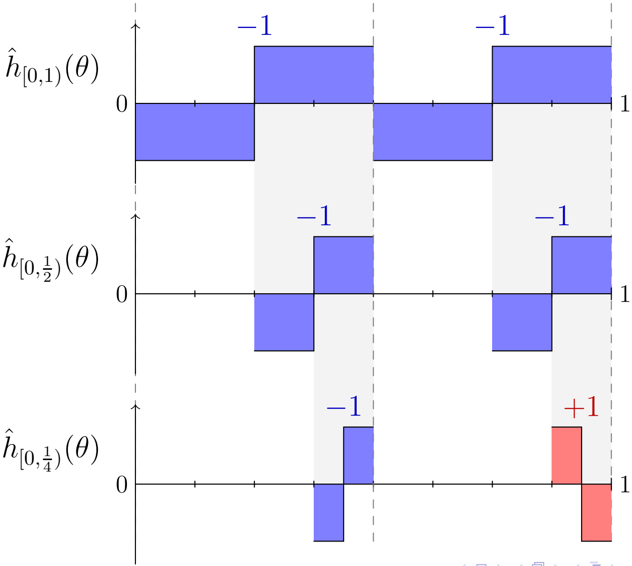

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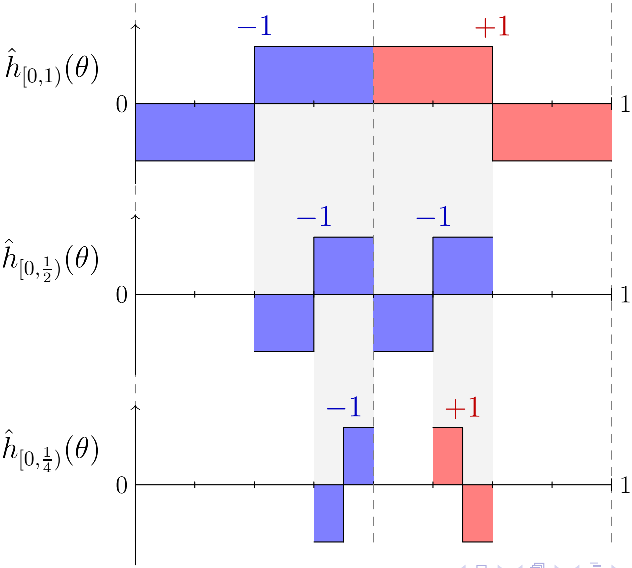

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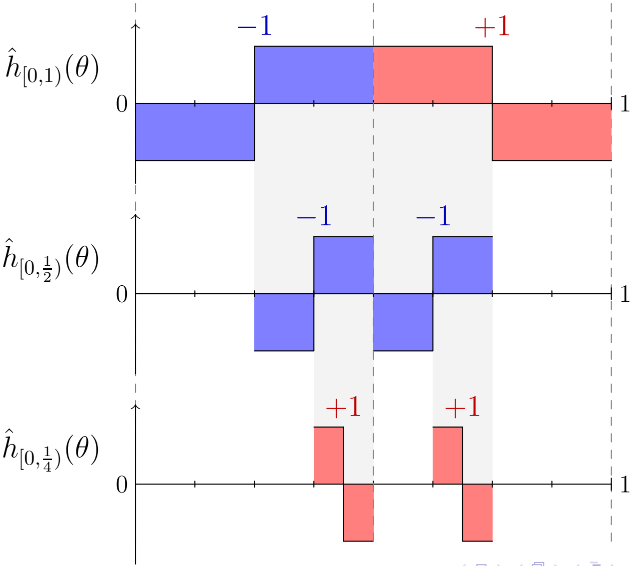

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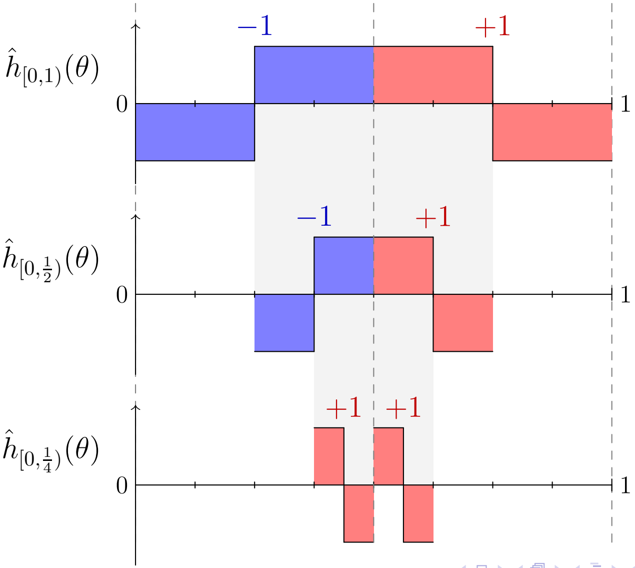

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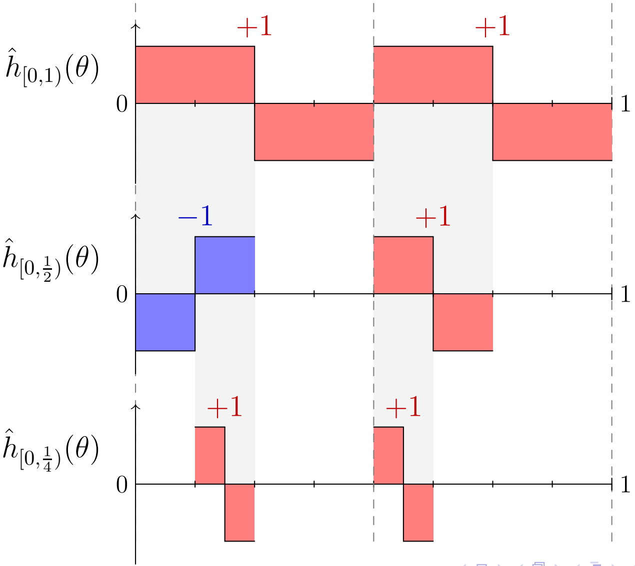

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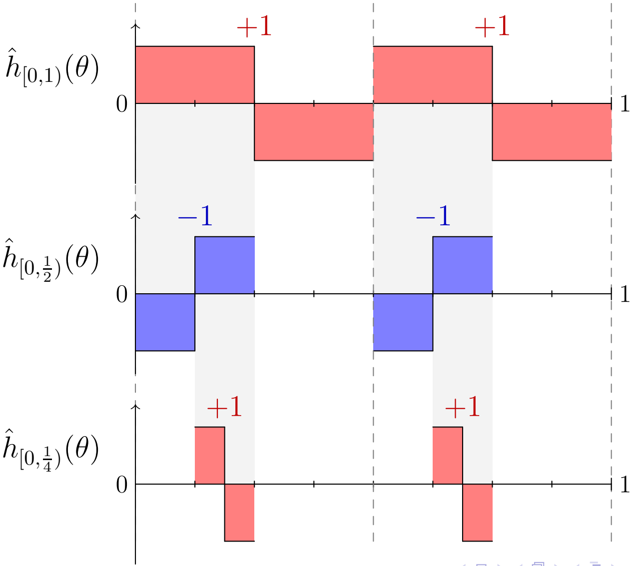

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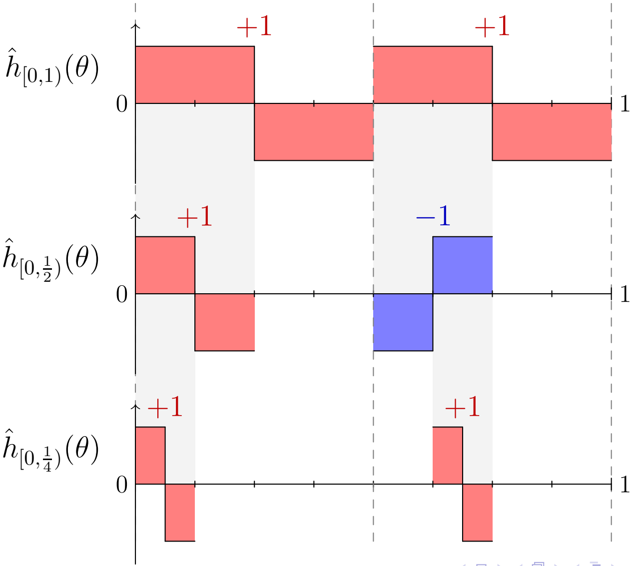

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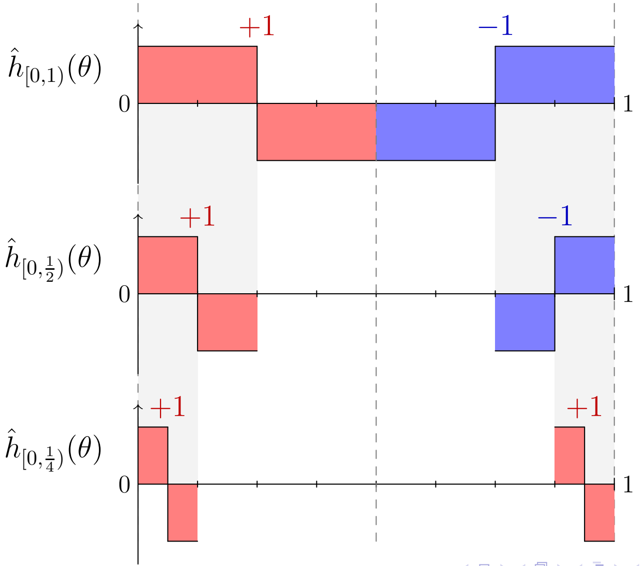

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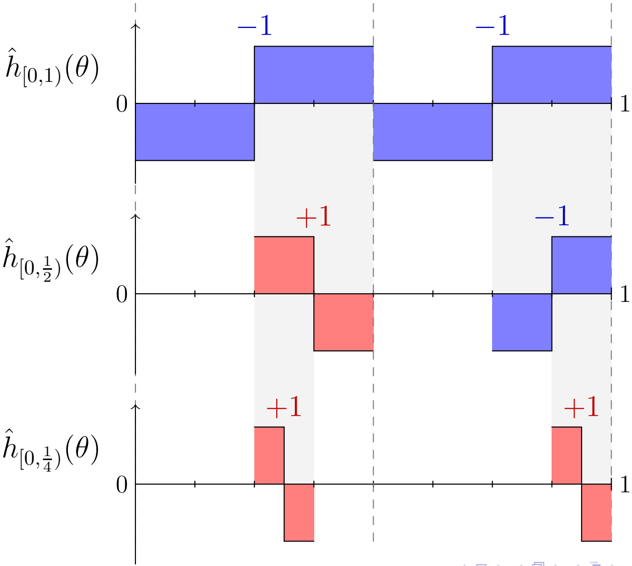

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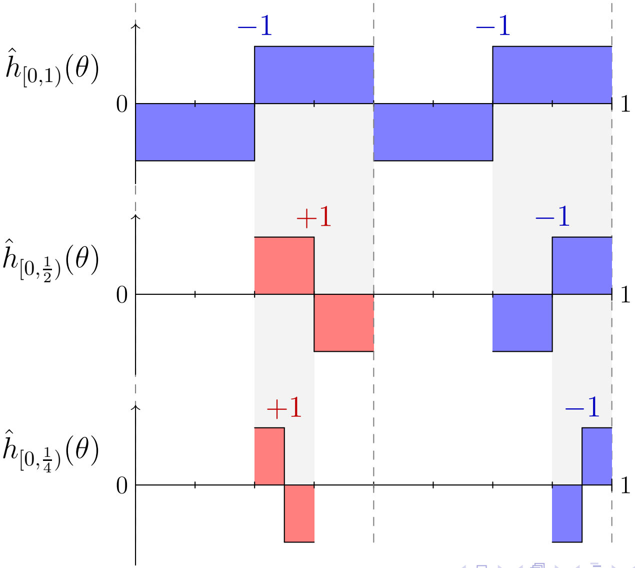

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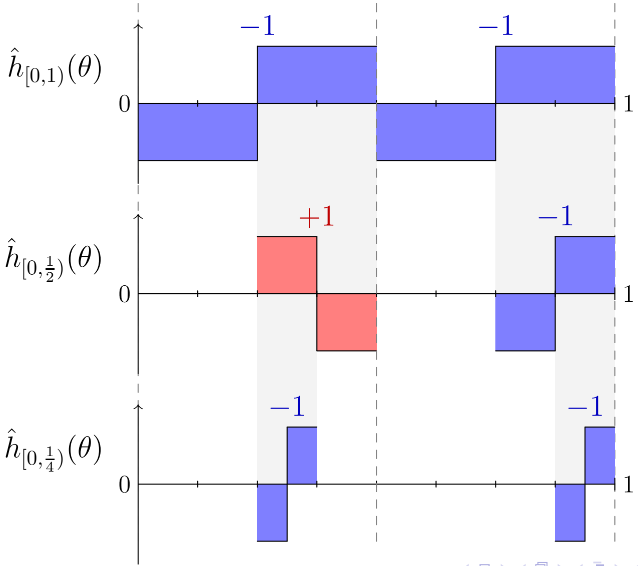

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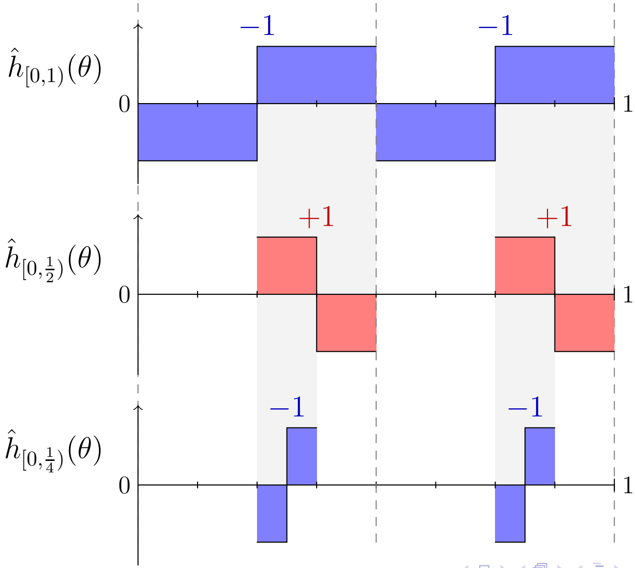

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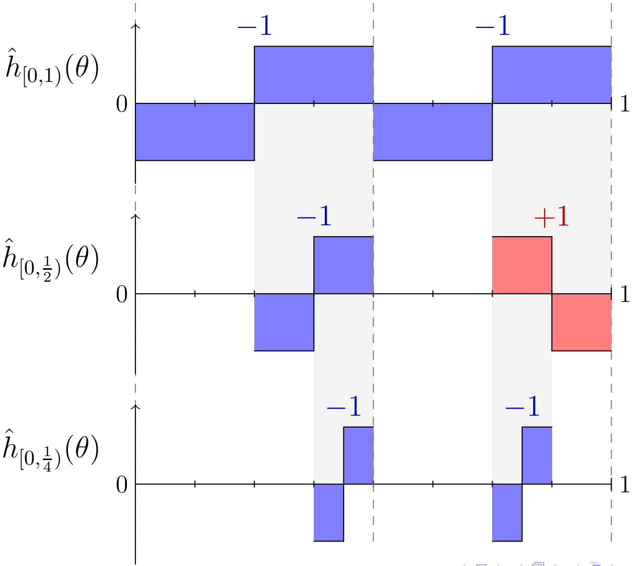

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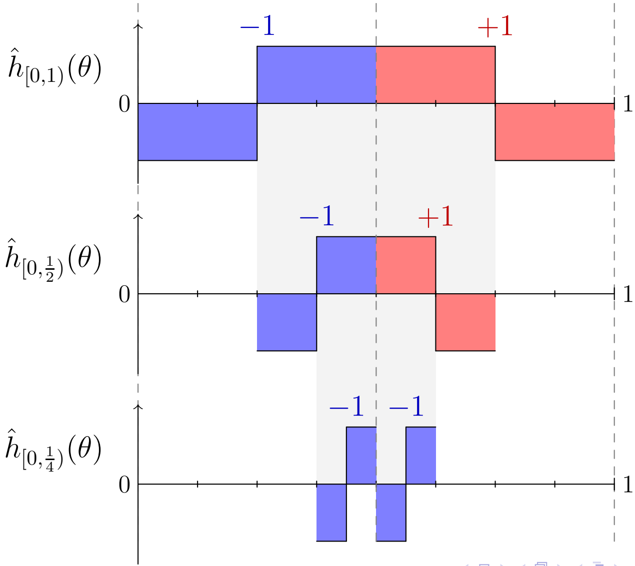

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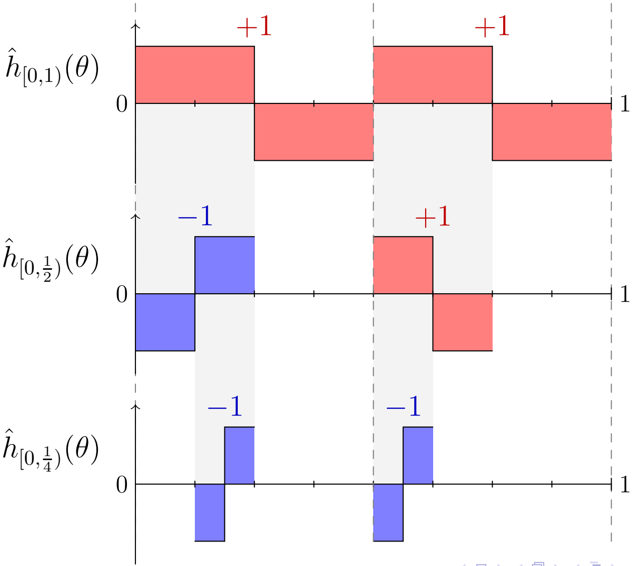

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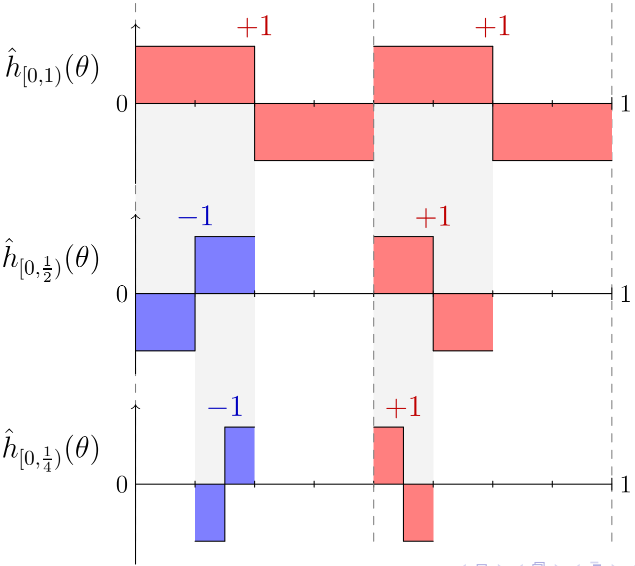

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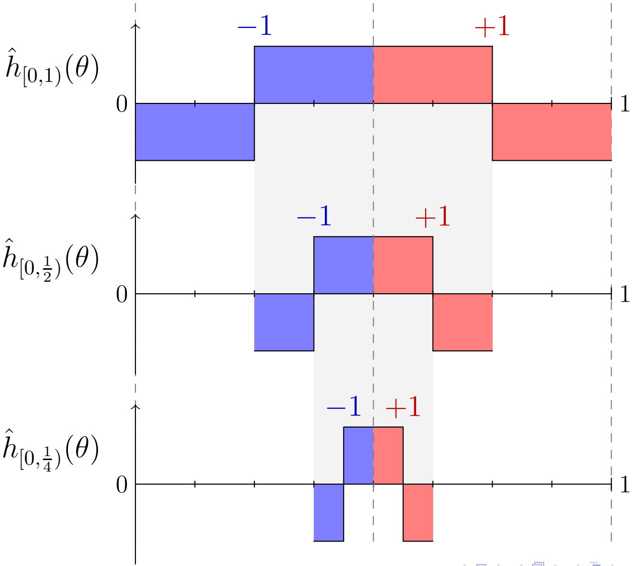

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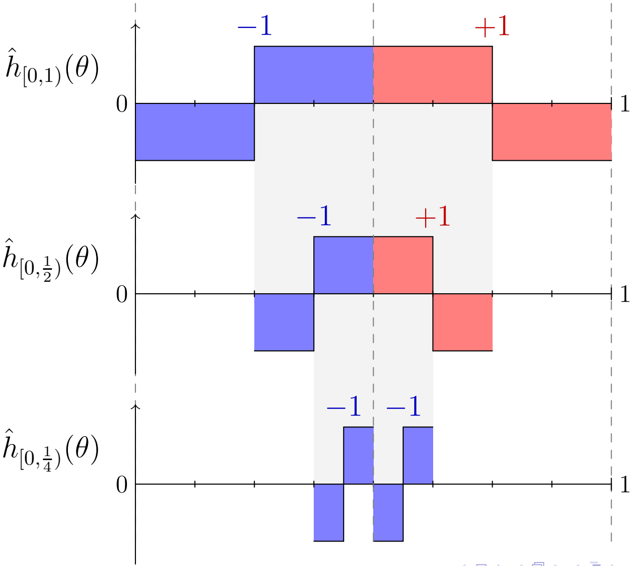

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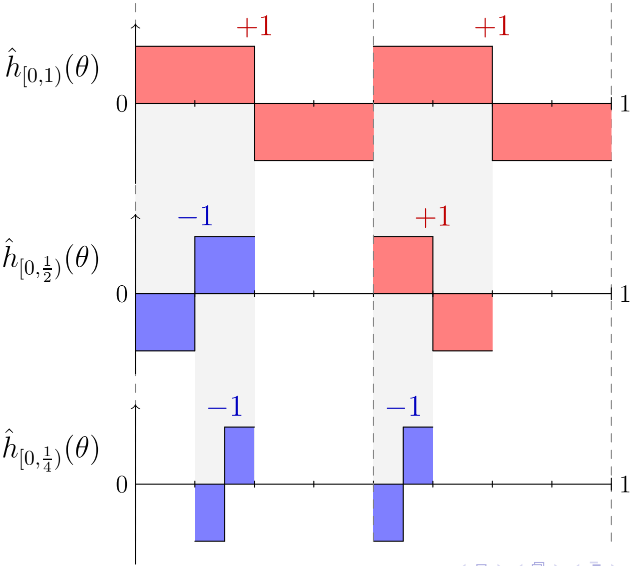

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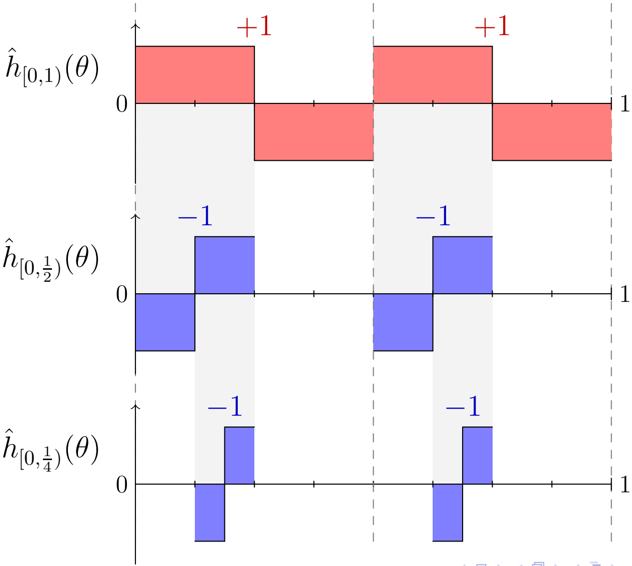

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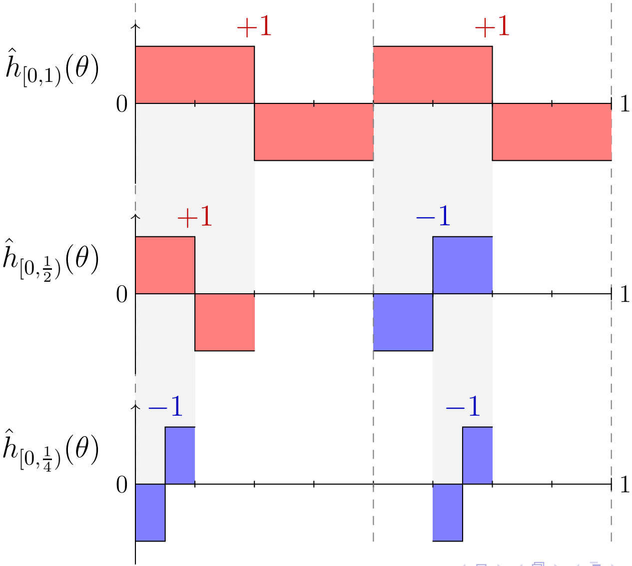

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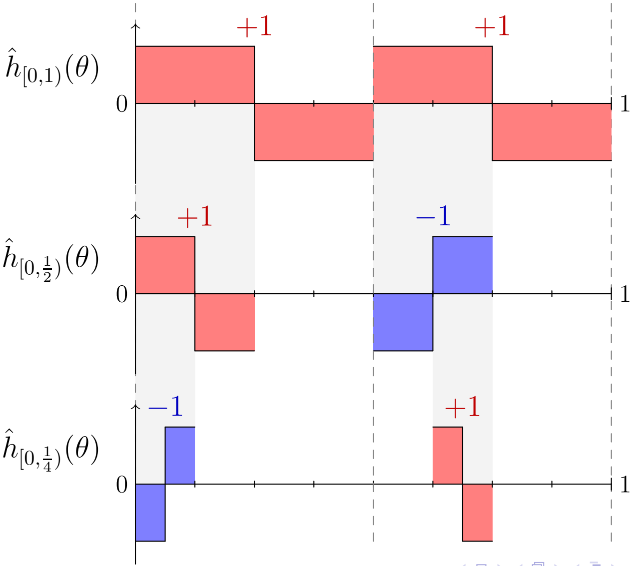

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- The random variables $X_{I,J}(\theta) = \langle \hat{h}_I(\theta), T_N \hat{h}_J(\theta) \rangle$ satisfy

$$\mathbb{E} X_{I,J} = 0 \text{ for } I \neq J \qquad \text{and} \qquad \mathbb{V} X_{I,J} \leq 3\|T\|^2 2^{-m/2}$$

($m =$ first level used in our construction)

- Choose $m$ large $\implies$ for some realization of $\theta$, the system almost diagonalizes $T_N$.
- Bonus: $\mathbb{E} X_{I,I}$ only depends on $|I| \implies$ level-wise stabilization of $T_N$.
- Combine the systems coming from different $n$ to obtain $(b_I^n)_{(n,I)}$,

$$b_I^n = \sum_{K \in \mathcal{B}_I^n} \theta_K^n h_K^{N(n)},$$

such that $\langle b_I^n, T b_J^n \rangle$ is small for $I \neq J$. Choose $N(n)$ inductively such that also $\langle b_I^n, T b_J^m \rangle$ is small for $n \neq m$.

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# Proof: Reduction to a scalar operator

**Second step: Reduction to a scalar operator**

- From above, we obtain operators $A, B$ such that $ATB$ is close to a diagonal operator $D$ (and $D$ is stabilized level-wise).
- Stabilize $D$ in every component of $Y_\omega$ using the pigeonhole principle $\to$ operator $C$ with $Ch_I^n = c_n h_I^n$ for all $(n, I)$
- Finally: Choose a cluster point $c$ of $(c_n)_n$.
- Then $cI_{Y_\omega}$ projectionally factors through $T$.
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- Additional reduction step (reduction to positive diagonal) using Gamlen-Gaudet construction
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# References

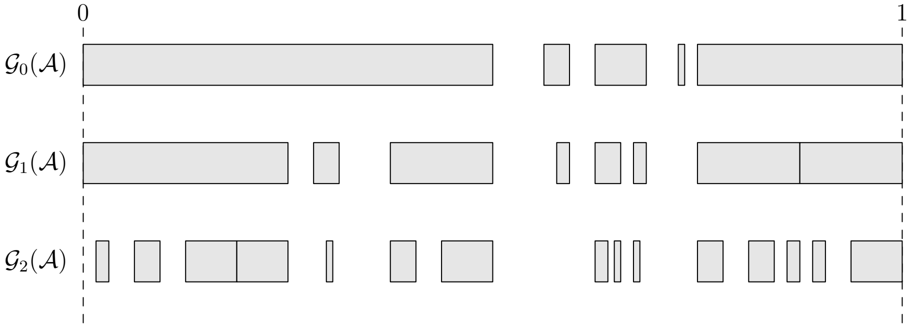
Konstantinos Konstantos and Thomas Speckhofer.

Factorization in independent sums of Haar system Hardy spaces.

Preprint, arXiv:2507.18600, 2025.

# Thank you for your attention!

# The Gamlen-Gaudet construction



# The Gamlen-Gaudet construction