Factorization in Haar system Hardy spaces Workshop in Analysis and Probability Seminar

Thomas Speckhofer

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July 23, 2024

Overview

Introduction

Definitions

- Main results
- Proofs

Primary Banach spaces

- ullet Let X be a Banach space.
- Let $\mathcal{B}(X)$ be the set of all bounded linear operators $T\colon X\to X$.
- X is called *primary* if for all spaces Y,Z, we have that $X\sim Y\oplus Z$ implies $Y\sim X$ or $Z\sim X$.
- Examples: c_0 , ℓ^p , L^p $(1 \le p \le \infty)$, H^1 , some rearrangement-invariant function spaces, ...

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- Let $P \in \mathcal{B}(X)$ be a projection. **Goal**: Show that P(X) or $(I_X P)(X)$ has a complemented subspace isomorphic to X. (*)
- If X satisfies (*) and $X \sim \ell^p(X)$ for some $1 \leq p \leq \infty$, then by Pełczyński's decomposition method, X is primary.
- lacktriangle Sufficient for (*): X has the primary factorization property.

Definition

Let $S, T \in \mathcal{B}(X)$. We say that S factors through T if there are $A, B \in \mathcal{B}(X)$ such that S = ATB.

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We say that a Banach space X has the *primary factorization property* if for every $T \in \mathcal{B}(X)$, the identity I_X factors through T or $I_X - T$.

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ullet For a Banach space X, define

$$\mathcal{M}_X = \{ T \in \mathcal{B}(X) : I_X \text{ does not factor through } T \}.$$

- The set \mathcal{M}_X is an ideal of $\mathcal{B}(X) \iff X$ has the primary factorization property (see Dosev-Johnson [1]).
- In that case, \mathcal{M}_X is the unique maximal ideal of $\mathcal{B}(X)$.

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- Dyadic intervals: $\mathcal{D} = \left\{ [0,1), [0,\frac{1}{2}), [\frac{1}{2},1), [0,\frac{1}{4}), [\frac{1}{4},\frac{1}{2}), \dots \right\}$
- $I^+ = \text{left half}, I^- = \text{right half of } I \in \mathcal{I}$
- Define $h_I=\mathbb{1}_{I^+}-\mathbb{1}_{I^-}$, $I\in\mathcal{D}$.
- Put $h_{\varnothing} = \mathbb{1}_{[0,1)}$ and $\mathcal{D}^+ = \mathcal{D} \cup \{\varnothing\}$.
- The Haar system $(h_I)_{I\in\mathcal{D}^+}$ is a *Schauder basis* for L^p , $1\leq p<\infty$
- $D \in \mathcal{B}(L^p)$ is called a *Haar multiplier* if $Dh_I = d_I h_I$ for all $I \in \mathcal{D}^+$ $(d_I \in \mathbb{R}, I \in \mathcal{D})$.

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- A Haar system space X is the completion of $H = \operatorname{span}\{h_I : I \in \mathcal{D}^+\}$ under a norm $\|\cdot\|_X$ such that:
 - If $x,y\in H$ and $|x|,\,|y|$ have the same distribution, then $\|x\|_X=\|y\|_X$.
 - $\|\mathbb{1}_{[0,1)}\|_X = 1$
- Examples: L^p , $1 \le p < \infty$, all separable rearrangement-invariant function spaces
- Let $\mathbf{r} = (r_I)_{I \in \mathcal{D}}$ be a *constant* or *independent* family of random variables uniformly distributed on $\{+1, -1\}$.
- Haar system Hardy space $X(\mathbf{r})$: completion of $\mathrm{span}\{h_I\}_{I\in\mathcal{D}}$ under

$$\left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{X(\mathbf{r})} = \left\| s \mapsto \mathbb{E} \right| \sum_{I \in \mathcal{D}} r_I a_I h_I(s) \left| \right|_X$$
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$$\begin{split} \left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{X(\mathbf{r})} &= \left\| s \mapsto \mathbb{E} \left| \sum_{I \in \mathcal{D}} r_I a_I h_I(s) \right| \right\|_{X} \\ &= \left\| \sum_I a_I h_I \right\|_{X} \text{ or } \sim \left\| \left(\sum_I a_I^2 h_I^2 \right)^{1/2} \right\|_{X}. \end{split}$$

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Now fix a Haar system Hardy space Y.

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Theorem (R. Lechner and T. S. '23)

Suppose that $\|\cdot\|_Y \nsim \|\cdot\|_{L^{\infty}}$ on the dyadic simple functions. Let E be one of the following spaces:

- (i) E=Y
- (ii) $E = \ell^p(Y)$ for some $1 \le p < \infty$
- (iii) $E = \ell^{\infty}(Y)$ if Y is "asymptotically curved" w.r.t. $(h_I)_{I \in \mathcal{D}}$.

Then E has the primary factorization property, and hence, \mathcal{M}_E is the unique maximal ideal of $\mathcal{B}(E)$. In particular, the spaces in (ii) and (iii) are primary.

- Basic idea: Step-by-step reduction.

$$D \approx A_1 T B_1, \quad D^{\text{stab}} = A_2 D B_2, \dots$$

$$\hat{h}_I = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K, \qquad \mathcal{B}_I \subseteq \mathcal{D}, \ \varepsilon_K = \pm 1$$

Proof method

- Basic idea: Step-by-step reduction. Operator T o Haar multiplier D o stable Haar multiplier D^{stab}
- ullet Clearly, the identity factors through cI_Y or $(1-c)I_{Y^{\perp}}$

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 \bullet \mathcal{B}_I are pairwise disjoint and satisfy some compatibility conditions.

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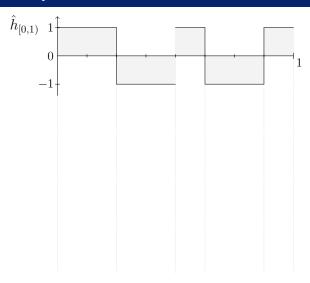
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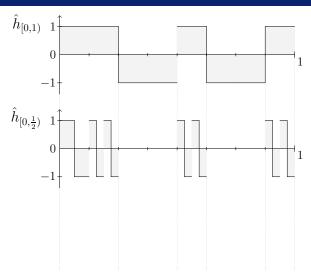
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Faithful Haar systems



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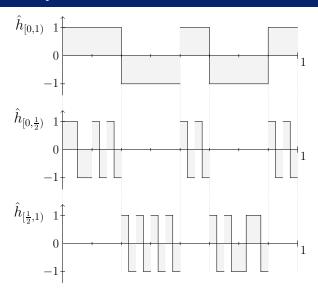
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• Associated operators *A*, *B*:

$$Bx = \sum_{I \in \mathcal{D}} \frac{\langle h_I, x \rangle}{|I|} \hat{h}_I, \qquad Ax = \sum_{I \in \mathcal{D}} \frac{\langle \hat{h}_I, x \rangle}{|I|} h_I$$

Diagonalization

• Let $I, J \in \mathcal{D}$ with "I < J". Given \hat{h}_I , construct \hat{h}_J out of sufficiently high-frequency "building blocks" h_K .

$$\Longrightarrow |\langle \hat{h}_I, T\hat{h}_J
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Stabilization

- Let $D: Y \to Y$ be a Haar multiplier. Put $r_n = \sum_{|K|=2^{-n}} h_K$ and $r_n^{\Gamma} = \mathbb{1}_{\Gamma} \cdot r_n \ (n > 0, \ \Gamma \subset [0, 1)).$

for each dyadic
$$\Gamma\subseteq[0,1),\quad \left(\langle r_n^\Gamma,Dr_n^\Gamma
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• Step 2: Inductively construct a faithful Haar system $(\hat{h}_I)_{I \in \mathcal{D}}$ with "frequencies" in \mathcal{N} and with random signs:

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• \Longrightarrow The entries d_I^{stab} of $D^{\mathrm{stab}} = ADB$ satisfy

$$\mathbb{E} \, d_{I^{\pm}}^{\mathrm{stab}} \approx d_{I}^{\mathrm{stab}},$$

and the variance is small \rightarrow choose a "good" realization of (θ_K) .

• We have $d_{[0,1)}^{
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Conclusion of the proof

- Perturbation argument $\implies cI_Y ADB$ is small
- Combined with diagonalization: $cI_Y \tilde{A}T\tilde{B}$ is small

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Factorization through operators with large diagonal

• An operator $T: Y \to Y$ has large diagonal (w.r.t. the Haar basis) if

$$\inf_{I\in\mathcal{D}}\frac{|\langle h_I, Th_I\rangle|}{|I|} > 0.$$

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Theorem (R. Lechner and T. S. '23)

Let Y be a Haar system Hardy space with $\|\cdot\|_{Y} \not\sim \|\cdot\|_{L^{\infty}}$. Then the identity I_Y factors through all operators $T \in \mathcal{B}(Y)$ with large diagonal, i.e., $(h_I)_{I\in\mathcal{D}}$ has the factorization property in Y.

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• Analogous results for ℓ^p -sums of Haar system Hardy spaces.

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- \bullet First step: Switch to large positive diagonal: $\frac{\langle h_I, Th_I \rangle}{|I|} \geq \delta$ for all I(Gamlen-Gaudet)

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 - Our approach: Stabilization yields $c \geq \delta$. Works in all Haar system Hardy spaces.

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Thank you for your attention!

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References I



D. Dosev and W. B. Johnson.

Commutators on ℓ_{∞} .

Bull. Lond. Math. Soc., 42(1):155-169, 2010.



Richard Lechner, Pavlos Motakis, Paul F. X. Müller, and Thomas Schlumprecht.

Strategically reproducible bases and the factorization property.

Israel J. Math., 238(1):13-60, 2020.

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References II



Richard Lechner and Thomas Speckhofer.

Factorization in Haar system Hardy spaces.

arXiv:2310.10572, Oct. 2023.



E. M. Semenov and S. N. Uksusov.

Multipliers of series in the Haar system.

Sibirsk. Mat. Zh., 53(2):388-395, 2012.

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