Embedded control of a DC-DC converter

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1 Introduction

2 Model dynamics

2.1 Main circuit

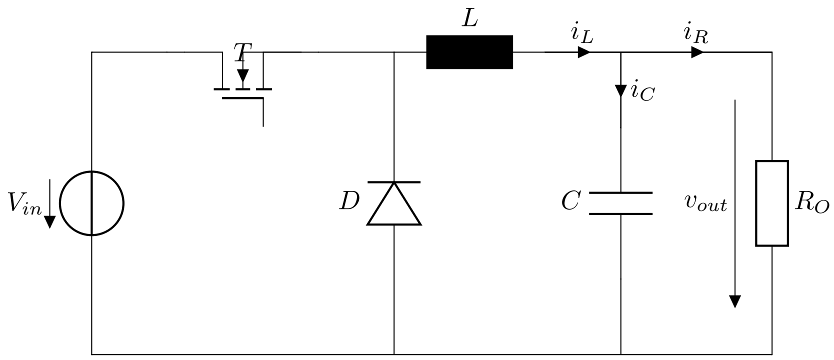

Figure 1

The system has 2 states :

$$\begin{aligned} x_1 &= i_L \\ x_2 &= v_{out} \end{aligned} \tag{1}$$

The input u of the system is the duty cycle of the PWM signal applied between the gate and the source of the MOSFET. The model depicted above is not perfect because it only considers ideal components. To get closer to the real world, some parasitic components are added :

1. R_L : Resistance in serie with the inductor
2. R_C : Resistance in serie with the capacitor
3. R_{on} : Resistance between the drain and the source of the MOSFET
4. V_j : Voltage drop across the diode when it is direct polarized

The system operates in continuous conduction mode. That means that the current through the inductor never reaches 0. Thus, there are 2 operatiing states during one switching period :

1. The switch is closed : the switching node is connected to the input voltage
2. The switch is open : the switching node is connected to (ground - voltage accross the diode)

2.2 Closed switch

When the switch is closed, the diode is reverse polarized and can be replaced by an open circuit :

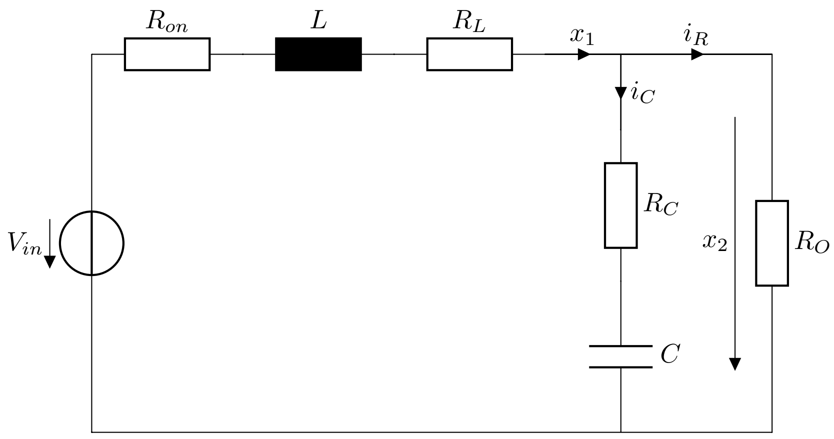

Figure 2

For x_1, we have :

$$\begin{gathered} V_{in} - x_2 = (R_{on} + R_L)x_1 + L\dot{x}_1 \\ \Rightarrow \dot{x}_1 = -\tfrac{R_{on}+R_L}{L}x_1 - \tfrac{1}{L}x_2 + \tfrac{V_{in}}{L} \end{gathered} \tag{2}$$

For x_2, we have :

$$\begin{gathered} x_2 = R_C i_c + \tfrac{1}{C}\int i_c dt \\ \Rightarrow \dot{x}_2 = R_C\dot{i}_C + \tfrac{1}{C}i_c \\ \Rightarrow \dot{x}_2 = R_C(\dot{x}_1 - \dot{i}_R) + \tfrac{1}{C}(x_1 - i_R) \\ \Rightarrow \dot{x}_2 = R_C\big(\dot{x}_1 - \tfrac{1}{R_O}\dot{x}_2\big) + \tfrac{1}{C}(x_1 - \tfrac{1}{R_O}x_2) \\ \Rightarrow \dot{x}_2 = R_C\dot{x}_1 - \tfrac{R_C}{R_O}\dot{x}_2 + \tfrac{1}{C}x_1 - \tfrac{1}{R_O C}x_2 \\ \Rightarrow \tfrac{R_O+R_C}{R_O}\dot{x}_2 = R_C\dot{x}_1 + \tfrac{1}{C}x_1 - \tfrac{1}{R_O C}x_2 \\ \Rightarrow \dot{x}_2 = \tfrac{R_C R_O}{R_C+R_O}\dot{x}_1 + \tfrac{R_O}{(R_C+R_O)C}x_1 - \tfrac{1}{(R_C+R_O)C}x_2 \end{gathered} \tag{3}$$

By using equation (3) in equation (4), we get :

$$\begin{gathered} \dot{x}_2 = \tfrac{R_C R_O}{R_C+R_O}\Big(- \tfrac{R_{on}+R_L}{L}x_1 - \tfrac{1}{L}x_2 + \tfrac{V_{in}}{L}\Big) + \tfrac{R_O}{(R_C+R_O)C}x_1 - \tfrac{1}{(R_C+R_O)C}x_2 \\ \Rightarrow \dot{x}_2 = \Big(- \tfrac{R_C R_O(R_{on}+R_L)}{(R_C+R_O)L} + \tfrac{R_O}{(R_C+R_O)C}\Big)x_1 - \Big(\tfrac{R_C R_O}{(R_C+R_O)L} + \tfrac{1}{(R_C+R_O)C}\Big)x_2 + \tfrac{R_C R_O V_{in}}{(R_C+R_O)L} \\ \Rightarrow \dot{x}_2 = \tfrac{-R_C R_O(R_{on}+R_L)C+R_O L}{(R_C+R_O)LC}x_1 - \tfrac{R_C R_O C+L}{(R_C+R_O)LC}x_2 + \tfrac{R_C R_O V_{in}}{(R_C+R_O)L} \end{gathered} \tag{4}$$

2.3 Opened switch

When the switch is opened, the diode is direct polarized so the circuit can be simplified like this :

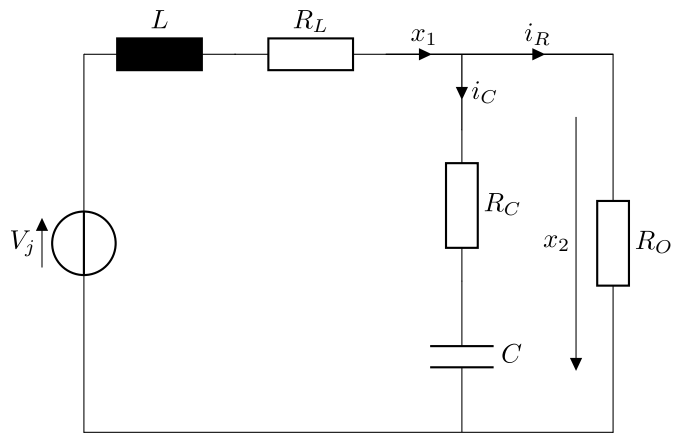

Figure 3

For x_1, we have :

$$\begin{aligned} -V_j - x_2 &= L\dot{x}_1 + R_L x_1 \\ \Rightarrow \dot{x}_1 &= -\tfrac{R_L}{L}x_1 - \tfrac{1}{L}x_2 - \tfrac{V_j}{L} \end{aligned} \tag{5}$$

For x_2, the equation is still :

$$\begin{aligned} &\Rightarrow \dot{x}_2 = \tfrac{R_C R_O}{R_C+R_O}\dot{x}_1 + \tfrac{R_O}{(R_C+R_O)C}x_1 - \tfrac{1}{(R_C+R_O)C}x_2 \\ &\Rightarrow \dot{x}_2 = \tfrac{R_C R_O}{R_C+R_O}\Big(-\tfrac{R_L}{L}x_1 - \tfrac{1}{L}x_2 - \tfrac{V_j}{L}\Big) + \tfrac{R_O}{(R_C+R_O)C}x_1 - \tfrac{1}{(R_C+R_O)C}x_2 \\ &\Rightarrow \dot{x}_2 = \Big(-\tfrac{R_C R_O R_L}{(R_C+R_O)L} + \tfrac{R_O}{(R_C+R_O)C}\Big)x_1 - \Big(\tfrac{R_C R_O}{(R_C+R_O)L} + \tfrac{1}{(R_C+R_O)C}\Big)x_2 - \tfrac{R_C R_O V_j}{(R_C+R_O)L} \\ &\Rightarrow \dot{x}_2 = \tfrac{-R_C R_O R_L C + R_O L}{(R_C+R_O)LC}x_1 - \tfrac{R_C R_O C + L}{(R_C+R_O)LC}x_2 - \tfrac{R_C R_O V_j}{(R_C+R_O)L} \end{aligned} \tag{6}$$

2.4 Combination

The input of the system is the duty cycle u. For deriving the dynamics, we take the average behaviour of $\dot{x}_1$ and $\dot{x}_2$. Thus, we have :

$$\begin{gathered}\dot{x}_1 = \dot{x}_1(closed)u + \dot{x}_1(opened)(1-u) \\ \Rightarrow \dot{x}_1 = u\Big(-\tfrac{R_{on}+R_L}{L}x_1 - \tfrac{1}{L}x_2 + \tfrac{V_{in}}{L}\Big) + (1-u)\Big(-\tfrac{R_L}{L}x_1 - \tfrac{1}{L}x_2 - \tfrac{V_j}{L}\Big) \\ \Rightarrow \dot{x}_1 = -\tfrac{R_L}{L}x_1 - \tfrac{1}{L}x_2 - \tfrac{R_{on}}{L}x_1 u + \tfrac{V_{in}+V_j}{L}u - \tfrac{V_j}{L}\end{gathered} \tag{7}$$

$$\begin{gathered}\dot{x}_2 = \dot{x}_2(closed)u + \dot{x}_2(opened)(1-u) \\ \Rightarrow \dot{x}_2 = u\left(\tfrac{-R_C R_O(R_{on}+R_L)C+R_O L}{(R_C+R_O)LC}x_1 - \tfrac{R_C R_O C+L}{(R_C+R_O)LC}x_2 + \tfrac{R_C R_O V_{in}}{(R_C+R_O)L}\right)+ \\ (1-u)\left(\tfrac{-R_C R_O R_L C+R_O L}{(R_C+R_O)LC}x_1 - \tfrac{R_C R_O C+L}{(R_C+R_O)LC}x_2 - \tfrac{R_C R_O V_j}{(R_C+R_O)L}\right) \\ \Rightarrow \dot{x}_2 = u\Big(\tfrac{-R_C R_O(R_{on}+R_L)C+R_O L}{(R_C+R_O)LC} - \tfrac{-R_C R_O R_L C+R_O L}{(R_C+R_O)LC}\Big)x_1 + u\Big(-\tfrac{R_C R_O C+L}{(R_C+R_O)LC} + \tfrac{R_C R_O C+L}{(R_C+R_O)LC}\Big)x_2 \\ +u\Big(\tfrac{R_C R_O V_{in}}{(R_C+R_O)L} + \tfrac{R_C R_O V_j}{(R_C+R_O)L}\Big) + \tfrac{-R_C R_O R_L C+R_O L}{(R_C+R_O)LC}x_1 - \tfrac{R_C R_O C+L}{(R_C+R_O)LC}x_2 - \tfrac{R_C R_O V_j}{(R_C+R_O)L} \\ \Rightarrow \dot{x}_2 = \tfrac{-R_C R_O R_L C+R_O L}{(R_C+R_O)LC}x_1 - \tfrac{R_C R_O C+L}{(R_C+R_O)LC}x_2 + \tfrac{R_C R_O(V_{in}+V_j)}{(R_C+R_O)L}u - \tfrac{R_C R_O R_{on}}{(R_C+R_O)L}x_1 u - \tfrac{R_C R_O V_j}{(R_C+R_O)L}\end{gathered} \tag{8}$$

2.5 Linearization

From the equations above, we see that there are bilinearities in the system. That means that there is at least one term where two system variables are multiplied together (e.g. $x_1 u$). We also see that the system is affine because $\dot{x}$ depends on at least one term that does not contain any system variable. At this point we have :

$$\dot{x} = \begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = f(x_1, x_2, u) = \begin{pmatrix}f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u)\end{pmatrix} \tag{9}$$

f is affine and contains bilinearities. If we linearize it around the equilibrium point using the first order Taylor serie we have :

$$\dot{x} = f(x, u) \approx f(x_{eq}, u_{eq}) + J_x^f(x_{eq}, u_{eq})(x - x_{eq}) + J_u^f(x_{eq}, u_{eq})(u - u_{eq}) \tag{10}$$

Where J_x^f and J_u^f are respectively the jacobians of f with respect to x and u.
Since $\dot{x}_{eq} = f(x_{eq}, u_{eq}) = 0$, we have :

$$\dot{x} = f(x, u) \approx A\Delta x + B\Delta u \tag{11}$$

Where $\Delta x = x - x_{eq}$, $\Delta u = u - u_{eq}$, $A = J_x^f(x_{eq}, u_{eq})$ and $B = J_u^f(x_{eq}, u_{eq})$.

The affine term has disappeared and the equation above is linear.

Now it is time to compute the steady state values. Because we want to linearize the system around the equilibrium point, we set $\dot{x}_1$ and $\dot{x}_2$ to zero.

Thus we have :

$$
\begin{aligned}
0 &= -\frac{R_L}{L} x_{1eq} - \frac{1}{L} x_{2eq} - \frac{R_{on}}{L} x_{1eq} u + \frac{V_{in}+V_j}{L} u_{eq} - \frac{V_j}{L} \\
0 &= \frac{-R_C R_O R_L C + R_O L}{(R_C+R_O)LC} x_{1eq} - \frac{R_C R_O C + L}{(R_C+R_O)LC} x_{2eq} + \frac{R_C R_O (V_{in}+V_j)}{(R_C+R_O)L} u_{eq} - \frac{R_C R_O R_{on}}{(R_C+R_O)L} x_{1eq} u_{eq} - \frac{R_C R_O V_j}{(R_C+R_O)L}
\end{aligned}
\tag{12}
$$

As we want that $x_{2eq} = 5V$, we have 2 unknowns and 2 equations. After solving the equation, we get :

$$
\begin{aligned}
x_{1eq} &= \frac{1}{R_O} x_{2eq} \\
u_{eq} &= \frac{R_O V_j + (R_L + R_O) x_{2eq}}{R_O (V_{in}+V_j) - R_{on} x_{2eq}}
\end{aligned}
\tag{13}
$$

Once we have these values, we can compute the matrices A and B :

$$
A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_{1eq}, x_{2eq}, u_{eq}) & \frac{\partial f_1}{\partial x_2}(x_{1eq}, x_{2eq}, u_{eq}) \\ \\ \frac{\partial f_2}{\partial x_1}(x_{1eq}, x_{2eq}, u_{eq}) & \frac{\partial f_2}{\partial x_2}(x_{1eq}, x_{2eq}, u_{eq}) \end{pmatrix} = \begin{pmatrix} -\frac{R_L + R_{on} u_{eq}}{L} & -\frac{1}{L} \\ \\ \frac{-R_C R_O R_L C + R_O L - R_C R_O R_{on} C u_{eq}}{(R_C+R_O)LC} & -\frac{R_C R_O C + L}{(R_C+R_O)LC} \end{pmatrix}
$$

and $B = \begin{pmatrix} \frac{\partial f_1}{\partial u}(x_{1eq}, x_{2eq}, u_{eq}) \\ \\ \frac{\partial f_2}{\partial u}(x_{1eq}, x_{2eq}, u_{eq}) \end{pmatrix} = \begin{pmatrix} \frac{V_{in} + V_j - R_{on} x_{1eq}}{L} \\ \\ \frac{R_C R_O (V_{in} + V_j - R_{on} x_{1eq})}{(R_C+R_O)L} \end{pmatrix}$